2020

Higgs Triples And Ruled Surfaces

Michail Gerapetritis
University of Pennsylvania

Higgs Triples And Ruled Surfaces

Abstract

Aiming to understand complexes of coherent sheaves on algebraic Poisson surfaces and the associated deformation quantizations and moduli problems, we begin our study by examining the case of ruled surfaces over a smooth projective curve X, namely the Poisson surface will be $S=\mathbb{P}(\mathcal{O}\oplus\omega)$, where ω is the canonical line bundle of X. Fixing a vector bundle $F\to X$, after revisiting the background technology of \textsl{spectral data and Higgs bundles} we aim to encode $(D,\,F)$-framed sheaves on S as a form of \textsl{extended Higgs data} [Chapter 3], i.e. Higgs triples, as introduced by A. Minets , and F-prolonged Higgs bundles. We present our first main result, demonstrating the correspondence between pure F-prolonged Higgs bundles on X and $(D,\,F)$-framed torsion free sheaves on S, globally generated along the fibers of the natural projection. Moreover, exploring the close relation between the two types of extended Higgs data, we aim to place them in the context of perverse coherent sheaves on X and examine the stability of the Higgs data as a polynomial stability in the sense of Bayer \cite{bayer}. So, using the polynomial stability given by the dual to the large volume perversity, we recover the notion of stability for Higgs triples as introduced by Minets, but also derive a stability condition for (pure) F-prolonged Higgs bundles, so the stable objects correspond to Huybrechts-Lehn stable $(D,\, F)$-framed torsion free sheaves.

Degree Type

Dissertation

Degree Name

Doctor of Philosophy (PhD)

Graduate Group

Mathematics

First Advisor

Tony Pantev

Keywords

algebraic geometry, Deformation quantization, Higgs Bundles, nonabelian hodge theory, Perverse sheaves, Poisson surfaces

Subject Categories

Mathematics

HIGGS TRIPLES AND RULED SURFACES

Michail Gerapetritis

A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2020

Supervisor of Dissertation

____

Tony Pantev, Class of 1939 Professor of Mathematics

Graduate Group Chairperson

____

Julia Hartmann, Professor of Mathematics

Dissertation Committee:
Jonathan Block, Professor of Mathematics
Ron Donagi , Professor of Mathematics
Tony Pantev, Class of 1939 Professor of Mathematics

Acknowledgments

First of all, I would like to thank my immediate family and close friends for the emotional support throughout the program. I would also like to thank the allies in the Math office, Monica Pallanti, Paula Scarborough, Reshma Tanna and Robin Toney, who by always being available and taking care of all the administrative issues, allowed me to focus on my research. I could not help but thank Jules Corrado and her empowering community, for providing solid ground under my feet throughout the last two years of my program, and Donna Carnow for constantly encouraging high morale throughout the challenging period of the COVID-19 pandemic lockdown.

But most of all, I would like to thank my advisor Tony Pantev, for being a constant source of wisdom and inspiration, and for being as helpful as an advisor can be. Without his help and guidance, this dissertation would be far from its current form.

During the preparation of this thesis the author was supported by NSF grant DMS 1901876, and by Simons HMS Collaboration grant # 347070

ABSTRACT

HIGGS TRIPLES AND RULED SURFACES

Michail Gerapetritis

Tony Pantev

Contents

Chapter 1

Introduction

Aiming to understand complexes of coherent sheaves on algebraic Poisson surfaces and the associated deformation quantizations and moduli problems, we begin our study by examining the case of ruled surfaces over a smooth projective curve X, namely the Poisson surface will be $S = \mathbb{P}(\mathcal{O} \oplus \omega)$, where ω is the canonical line bundle of X. Fixing a vector bundle $F \to X$, after revisiting the background technology of *spectral data and Higgs bundles* [Chapter 2] we aim to encode (D, F)-framed sheaves on S as a form of *extended Higgs data* [Chapter 3], i.e. Higgs triples, as introduced by A. Minets [Min18], and F-prolonged Higgs bundles. We present our first main result 3.2.2, demonstrating the correspondence between pure F-prolonged Higgs bundles on X and (D, F)-framed torsion free sheaves on S, globally generated along the fibers of the natural projection. Moreover, exploring the close relation between the two types of extended Higgs data, we aim to place them in the context of perverse coherent sheaves on X and examine the stability of the Higgs data as a polynomial stability in the sense of Bayer [Bay09]. So, using the polynomial stability given by the dual to the large volume perversity, we recover the notion of stability for Higgs triples as introduced by Minets, but also derive a stability condition for (pure) F-prolonged Higgs bundles, so the stable objects correspond to Huybrechts-Lehn stable $(D,\ F)$-framed torsion free sheaves.

Chapter 2

Spectral data and Higgs bundles

Our goal is to study complexes of coherent sheaves on algebraic Poisson surfaces, and to find concrete geometric descriptions of their deformation quantizations and the associated moduli problems. As a case study we focus on the important case where the background Poisson surface is a geometrically ruled surface over a smooth projective curve. In this setting, we study natural deformations of the spectral construction, which allow us to recast the problem as a more tractable deformation problem for decorated sheaves on the base curve.

2.1 Higgs bundles

2.1.1 Review of spectral covers

The *spectral construction* is a geometric implementation of a common strategy in mathematics, where one uses a duality operation to simplify a given problem. A classical example of this strategy is the Fourier transform of functions on a locally compact abelian group, which gives a concrete way of converting between continuous and discrete data.

The *spectral cover construction* is a similar duality operation, which in a nutshell replaces a linear operator by its spectrum.

Simplest setup: Let V be a finite dimensional $\mathbb{C}$-vector space and let $\theta : V \to V$ be an endomorphism. When θ is generic (=diagonalizable) one can describe θ via its spectral data, i.e. by giving:

- the eigenvalues of θ;
- the decomposition of V into a direct sum of θ-eigenlines;
- a matching between eigenvalues and eigenlines.

If $\dim_{\mathbb{C}} V = n$ this means that we are specifying n complex numbers

$$\lambda_1, \ldots, \lambda_n \in \mathbb{C} \quad (= \text{ spectrum of } \theta)$$

and to each such number we are prescribing a line $L_i \subset V$, so that $L_1 \oplus \ldots \oplus L_n = V$.

The spectral covers appear when we let this picture vary in families. If $\theta_x : V \to V$ is a family of endomorphisms parameterized by $x \in X$, then by repeating the construction for each x we get a subvariety $\overline{X} \subset X \times \mathbb{C}$, where

$$\overline{X} = \{(x, \lambda) \mid \lambda \text{ is an eigenvalue of } \theta_x\}$$

If all θ_x have distinct eigenvalues we also get a family of eigenlines $L_{(x,\lambda)}$ parameterized by the points of $\overline{X}$.

Definition 2.1.1. The space $\overline{X}$ is called the *spectral cover* corresponding to the family $\{\theta_x\}_{x \in X}$. Under the genericity assumption $\overline{X}$ is an unramified n-sheeted cover of X and it carries a line bundle consisting of all eigenlines of the θ_x's.

Remark 2.1.2. The set of data $(\overline{X} \to X, L \to \overline{X})$ completely reconstructs the family $\{\theta_x\}_{x \in X}$.

The correspondence $\{\theta_x\}_{x \in X} \leftrightarrow (\overline{X}, L)$ is not very useful under this stringent genericity assumption. In practice, one needs to deal with θ_x which have repeated eigenvalues. In this case $\overline{X} \to X$ becomes ramified over $x \in X$ and the fibers of

$L \to \overline{X}$ may jump at the ramification points. It is easy to see that L always has the structure of a coherent sheaf on $\overline{X}$, which can fail to be invertible or even torsion free along the ramification locus of $\overline{X} \to X$.

Important special case: Allow θ_x to have multiple eigenvalues but require that there is exactly one Jordan block per eigenvalue. Such an endomorphism of V is called *regular*. It carries a single eigenline per eigenvalue. In particular, if all θ_x, $x \in X$ are regular we get again a line bundle $L \to \overline{X}$ on the spectral cover $\overline{X}$.

More invariantly, consider the polynomial map

$$\begin{array}{rrcl} h: & \mathrm{End}(V) & \longrightarrow & \mathbb{C}^n \\ & \theta & \longmapsto & (a_1(\theta), \ldots, a_n(\theta)), \end{array}$$

where the $a_i(\theta)$'s are the coefficients

$$\det(t \cdot \mathrm{id}_V - \theta) = t^n + a_1(\theta) t^{n-1} + \ldots + a_n(\theta),$$

of the characteristic polynomial of θ.

The spectrum of θ depends only on $h(\theta)$ and so $\overline{X}$ is just the pullback via the map

$$\begin{array}{rcl} X & \longrightarrow & \mathbb{C}^n \\ \theta_x & \longrightarrow & h(\theta_x). \end{array}$$

of the obvious cover

$$\overline{\mathbb{C}}^n \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n,$$

given by $t^n + a_1 t^{n-1} + \ldots + a_n = 0$ in the coordinates $(a_1, \ldots, a_n; t) \in \mathbb{C}^n \times \mathbb{C}$.

The fibers of $h : \mathrm{End}(V) \to \mathbb{C}^n$ are invariant under conjugation action of $GL(V)$ and in fact

$$\mathbb{C}^n = \mathrm{End}(V)//GL(V)$$

is the GIT quotient.

Remark 2.1.3. The orbits of the $GL(V)$-action on $\mathrm{End}(V)$ are not all closed and so the natural topology on the set of orbits $\mathrm{End}(V)/GL(V)$ will not be Hausdorff. To remedy that one looks for a space $\mathrm{End}(V)//GL(V)$ parameterizing the closures of $GL(V)$-orbits in $\mathrm{End}(V)$.

For a general (regular and semisimple) θ in $\mathrm{End}(V)$ the $GL(V)$-orbit is closed and in a neighborhood of such θ the quotients

$$\mathrm{End}(V)/GL(V) \quad \text{and} \quad \mathrm{End}(V)//GL(V)$$

coincide.

When θ is arbitrary, then $\overline{GL(V) \cdot \theta}$ contains a unique closed and a unique open orbit. The closed one is the orbit of a semisimple (diagonalizable) endomorphism and the open one is the orbit of a regular endomorphism. This leads to two interpretations for $\mathrm{End}(V)//GL(V)$: either as the space parameterizing semisimple endomorphisms modulo conjugation, or as the space parameterizing all regular endomorphisms modulo conjugation.

Both interpreations are useful but the one for which the eigenlines vary continuously or algebraically is the interpretation via regular endomorphisms.

Example 2.1.4. Let $\dim_{\mathbb{C}}(V) = 2$. Use $SL(V)$ instead of $GL(V)$. Then we have $h : SL(V) \to \mathbb{C}$, $h(\theta) = \det\theta$, and if $\det\theta \neq 0$, then θ is regular and semisimple. If $\det\theta = 0$, then θ is nilpotent and then

$$h^{-1}(0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \coprod \left\{ SL(V) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

To make the naive spectral cover construction useful in applications one needs to extend it in two ways:

(i) Allow for abitrary groups G, not only for $GL(V)$;

(ii) Allow for twisted versions of θ.

For **(i)**: Instead of looking at elements $\theta \in \mathrm{End}(V)$ we take elements $\theta \in \mathfrak{g}$ for $\mathfrak{g} := \mathrm{Lie}(G)$ of some complex semi-simple group G. The spectral cover construction in this case is somewhat subtler, since it has to reflect the complexity of the group G. We will not discuss this part of the story but many deep results can be found in [Don95], [DG02], [Don98].

For **(ii)**: The 'twisting' of the $\{\theta_x\}_{x\in X}$ can be achieved in two ways. Firstly, one can allow for the vector space V to vary with the point $x \in X$. This is easily realized by replacing $X \times V$ by a non-trivial vector bundle E on X. In this setup the family of endomorphisms, naturally, should be viewed as a section $\theta \in \Gamma(X, \mathrm{End}(E))$. Secondly, one can allow for θ to have nontrivial coefficients in some coefficient object K.

The freedom of choosing K is essential in the applications. Since the elements in K can be thought of as the matrix coefficients of θ, it is natural to require that K has an abelian group structure. Possible natural choices for K are: a vector bundle on X, a family of affine tori on X, a family of abelian varieties on X or more generally a family of commutative group stacks over X.

We will see examples of most of these choices later on and will relate them to moduli of complexes and dualities. The simplest choice is to take K to be a vector bundle. This leads to the classical notion of a Higgs bundle as it was developed in the works of Hitchin [Hit87a, Hit87b] and Simpson [Sim92].

2.1.2 Higgs bundles with linear coefficients

Let X be a a complex algebraic variety and let K be a fixed algebraic vector bundle of rank n on X. Consider a vector bundle $E \to X$ of rank r and an $\mathcal{O}_X$-linear map

$$\theta : E \to E \otimes K.$$

We would like to take the 'spectrum' of θ and recast the data (E, θ) in terms of a spectral cover C of X possibly decorated with some additional structure (e.g. a coherent sheaf).

Remark 2.1.5. The spectrum may not be well defined for a general θ. Indeed, if we trivialize K locally on X, i.e. if we choose a local frame $K_{|V} \cong \mathbb{C}^n \otimes \mathcal{O}_V$, then we see that locally θ comprises n endomorphims

$$\theta_{|V} = (\theta_1, \ldots, \theta_n), \text{ with } \theta_i \in \Gamma(V, \mathrm{End}(E)).$$

We can apply the naive spectral construction to each θ_i but the collection of spectral covers we will get this way will depend on the trivialization of K.

To fix that one may look only at θ's for which all the θ_i's behave in the same way e.g. are simultaneously diagonalizable. More generally we can require that $[\theta_i, \theta_j] = 0$ for all i, j, i.e. that the θ_i's generate a commutative subalgebra in $\mathrm{End}(E)$. The latter condition is clearly equivalent to requiring that

$$\theta \wedge \theta = 0 \in \Gamma(X, \mathrm{End}(E) \otimes \wedge^2 K).$$

This motivates the following

Definition 2.1.6. A K-valued Higgs bundle on an algebraic variety X is a pair $(E, \theta : E \to E \otimes K)$ satisfying $\theta \wedge \theta = 0$. Similarly one defines a Higgs coherent sheaf on X.

Observe that for a Higgs bundle (E, θ), the Higgs field θ can be interpreted as a map $K^\vee \otimes E \to E$ and so generates an action $\boldsymbol{T} K^\vee \otimes E \to E$ of the sheaf of tensor algebras $\boldsymbol{T}K^\vee := \oplus_i (K^\vee)^{\otimes i}$ on E. The condition $\theta \wedge \theta = 0$ is equivalent to saying that this action descends to an action

$$S^\bullet K^\vee \otimes E \to E$$

of the symmetric algebra $S^\bullet K^\vee$ on E.

This fact admits a geometric interpretation. Consider the total space $S := \mathrm{tot}(K)$ of the vector bundle K. Let $p : S \to X$ be the natural projection. Then p is an affine map and

$$p_*\mathcal{O}_S = S^\bullet K^\vee, \qquad S = \mathrm{Spec}(S^\bullet K^\vee).$$

In particular, a quasi-coherent sheaf $\mathcal{E}$ on S is the same thing as a quasi-coherent sheaf $E(= p_*\mathcal{E})$ on X together with a $S^\bullet K^\vee$-action.

Note that since p is affine, an $S^\bullet K^\vee$-module E which is coherent as a sheaf on X will correspond to a coherent sheaf $\mathcal{E}$ on S which is *finite* over X. This proves the following

Lemma 2.1.7. *There is an equivalence of categories*

$$p_* : \left\{ \begin{matrix} \textit{quasi-coherent} \\ \textit{sheaves on } S \end{matrix} \right\} \widetilde{\to} \left\{ \begin{matrix} \textit{Sheaves of } S^\bullet K^\vee\textit{-modules on} \\ X, \textit{ quasi-coherent as sheaves} \\ \textit{of } \mathcal{O}_X\textit{-modules} \end{matrix} \right\}$$

which restricts to an equivalence

$$p_* : \left\{ \begin{matrix} \textit{coherent sheaves on } S, \\ \textit{finite over } X \end{matrix} \right\} \widetilde{\to} \left\{ \begin{matrix} \textit{coherent Higgs sheaves} \\ \textit{on } X \end{matrix} \right\}.$$

This is the K*-valued spectral correspondence.* It converts spectral data (= coherent sheaves on $\mathrm{tot}(K)$ whose support is finite over X) to Higgs data (= K-twisted families of endomorphisms parametrized by X).

Remark 2.1.8.

- The Higgs sheaf (E, θ) corresponding to a sheaf $\mathcal{E}$ on S can be described explicitly: $E = p_*\mathcal{E}$ is the pushforward of $\mathcal{E}$, $\theta : E \to E \otimes K$ is the pushforward of $\mathcal{E} \stackrel{\otimes\lambda}{\to} \mathcal{E} \otimes p^*K$ where, $\lambda \in \Gamma(\mathrm{tot}(K), p^*K)$ is the tautological section.

- If a sheaf $\mathcal{E}$ on S corresponds to a Higgs bundle (E, θ) of rank r, then the spectral cover for (E, θ) is defined as the subscheme $\mathrm{Supp}(\mathcal{E}) \subset S$ which maps onto X and is finite of degree r over X. It is given explicitly as the zero locus of the section

$$\det(\lambda \cdot \mathrm{id} - p^*\theta) \in \Gamma(S, p^*S^rK).$$

- When K is the trivial line bundle on X, then $S = X \times \mathbb{C}$ and we recover the previous definition of a spectral cover for a family of endomorphisms.

Example 2.1.9. Let (X, g) be a compact Kähler manifold with g real-analytic Kähler metric, and let $K = \Omega^1_X$ the holomorphic cotangent bundle of S. The total space $X = \mathrm{tot}(K)$ of K carries a holomorphic symplectic form - the exterior derivative $\Omega = d\lambda$ of the tautological one form λ on X. It is known [Fei01, Kal99] that a tubular neighborhood of the zero section $X \subset S$ of K supports a unique hyper-Kähler metric which is compatible with Ω and restricts to g on X. Thus S is a non-compact, non-complete physicists Calabi-Yau manifold which can be taken as a geometric background for describing suppersymmetric quantum field theories.

The B-branes on S are coherent sheaves on S with compact support, i.e. coherent sheaves $\mathcal{E}$ on S which are finite over X. By the spectral correspondence one can describe the moduli space of such $\mathcal{E}$ as the moduli space of Higgs bundles $(E, \phi : E \to E \otimes \Omega^1_X)$ on X. If the Chern classes of $\mathcal{E}$ are chosen so that $c_1(E) = 0$ and $c_2(E) = 0$, then all such Higgs bundles correspond to flat holomorphic bundles on X or representations of $\pi_1(X)$ by Simpson's non-abelian Hodge theorem [Sim97]. This gives a concrete description of a component of the moduli space of B-branes on S and the flat bundle interpretation gives a canonical perturbative deformation quantization of these branes in the direction of the symplectic form Ω.

2.1.3 Higgs bundles and Koszul duality

In order to understand better the quantization of the spectral construction for Higgs bundles on a smooth complex space X it is useful to recast the spectral construction as a filtered Koszul duality for families of algebras over X.

Let as before K be a fixed coefficient bundle and let $S = \mathrm{tot}(K)$ be its total space. Consider the sheaf of algebras $A = S^{\bullet}K^{\vee}$ on X with the natural filtration induced from the grading. The filtered quadratic dual algebra [Pos93, PoPo05] is the trivial dg algebra $A^! = (\wedge^\bullet K, 0, 0)$:

$$A^! = \left(\mathcal{O}_S \overset{0}{\to} K \overset{0}{\to} \wedge^2 K \overset{0}{\to} \dots \overset{0}{\to} \wedge^n K\right).$$

Every quasi-coherent sheaf $\mathcal{E}$ on S can be viewed as a filtered module over A and so corresponds by filtered quadratic duality [Pos93, PoPo05] to a dg module $\mathcal{E}^!$ over $(\wedge^\bullet K, 0, 0)$. Explicitly we have

$$\mathcal{E}^! = \left(E \overset{\wedge\theta}{\to} E \otimes K \overset{\wedge\theta}{\to} \dots \overset{\wedge\theta}{\to} E \otimes \wedge^n K\right),$$

where (E, θ) is the corresponding Higgs sheaf. Note that even though the differential in the quadratic dual algebra $A^!$ is trivial, we still can have a non-trivial differential for the module.

The Koszul reinterpretation of the spectral construction is useful because it provides a way to deform the spectral correspondence in interesting directions. Next we explore various commutative and non-commutative deformations of the spectral construction.

2.2 Deformations of the spectral construction

Fix a smooth complex variety X and a (coefficient) vector bundle K of rank n on X. Let $p : S = \mathrm{tot}(K) \to X$ be the total space of K. The spectral correspondence establishes an equivalence between the categories comprising the following types of geometric data

(Spectral data) Coherent sheaves $\mathcal{E} \in \mathrm{Coh}(S)$ which are finite over X.

(K-valued Higgs data) Coherent sheaves $E \in \mathrm{Coh}(S)$ equipped with a Higgs field θ, i.e. an $\mathcal{O}_X$-linear K-valued endomorphism

$$\theta : E \to E \otimes K$$

satisfying $\theta \wedge \theta = 0$.

As explained above this correspondence is a special case of filtered Koszul duality [Pos93, PoPo05]:

- View the spectral sheaf $\mathcal{E} \in \mathrm{Coh}(S)$ as a module over the sheaf of algebras $S^\bullet K^\vee$ over X, i.e. replace $\mathcal{E}$ with the equivalent data

$$(E := p_*\mathcal{E} \in \mathrm{Coh}(X)) + (S^\bullet K^\vee - \text{action on } E).$$

- View the Higgs sheaf (E, θ) as a dg module

$$E \stackrel{\wedge\theta}{\to} E \otimes K \stackrel{\wedge\theta}{\to} \ldots \stackrel{\wedge\theta}{\to} E \otimes \wedge^n K$$

  over the dga $\mathcal{O}_S \stackrel{0}{\to} K \stackrel{0}{\to} \ldots \stackrel{0}{\to} \wedge^n K$.

- Use filtered Koszul duality to convert modules over the filtered quadratic algebra $S^\bullet K^\vee$ and dg modules over the dg algebra $(\wedge^\bullet K, 0)$.

Remark 2.2.1. **(i)** The Koszul reformulation of the spectral correspondence has the adavntage of exhibiting both the Higgs and the spectral data in a manifestly deformable form. Indeed, by deforming the structures on $S^{\bullet}K^{\vee}$ and $(\wedge^{\bullet}K, 0)$ so that the Koszul duality still holds, we can obtain a new kind of spectral duality between the deformed module structures.

(ii) Note that there are three possible ways in which we can perturb the structure of $S^{\bullet}K^{\vee}$ so that the resulting algebra will still be filtered quadratic. Clearly $S^{\bullet}K^{\vee}$ is a filtered quadratic algebra of the most trivial type: it is commutative, augmented and the filtration is split.

Thus when we start deforming the product structure on $S^{\bullet}K^{\vee}$ we can perform the deformation so that:

- the product becomes non-commutative;
- the augmentation ceases to be an algebra morphism;
- the filtration is not split anymore.

Similarly we can deform the curved dg algebra structure on $(\wedge^{\bullet}K, 0, 0)$ so that:

- the product becomes non-commutative;
- the differential becomes non-zero;
- the curvature becomes non-zero.

An interesting feature of Koszul duality is that the duality transformation mixes the different types of deformations. Here are specific examples of this phenomenon.

Example 2.2.2. **(filtered commutative deformations)** There are natural deformations of $S^\bullet K^\vee$ as a filtered commutative algebra which is filtered quadratic and has associated graded isomorphic to $S^\bullet K^\vee$. (In particular the deformation will be filtered Koszul.)

Indeed, let $\omega \in H^1(X, K)$. Then ω determines a deformation of the variety $p : S \to X$, namely the total space $S_\omega := \mathrm{tot}(K_\omega)$ of the affine bundle $K_\omega \to X$ corresponding to the class ω.

Concretely ω corresponds to an extension

$$(\omega) \qquad\qquad 0 \to K \to F_\omega \stackrel{\pi_\omega}{\to} \mathcal{O}_X \to 0,$$

and $K_\omega = \pi_\omega^{-1}(1)$ (note that $K = \pi_\omega^{-1}(0)$). Let $p_\omega : S_\omega \to X$ be the natural projection and let

$$S_\omega^\bullet K^\vee := p_{\omega *}\mathcal{O}_{X_\omega}$$

be the sheaf of commutative algebras of functions along the fibers of p_ω.

The fact that S_ω has no section means that $S^\bullet_\omega K^\vee$ is filtered but not graded. Geometrically $S_\omega = \mathbb{P}(F_\omega) - \mathbb{P}(K)$, and in fact

$$p_{\omega*}\mathcal{O}_{S_\omega} = p_{\omega*}\mathcal{O}_{\mathbb{P}(F_\omega)}(\infty \cdot \mathbb{P}(K)).$$

Thus $S^\bullet_\omega K^\vee$ is filtered by "order of poles along the divisor $\mathbb{P}(K) \subset \mathbb{P}(F_\omega)$ at infinity".

Algebraically we have

$$S^\bullet_\omega K^\vee = S^\bullet F^\vee_\omega / \langle 1_{S^\bullet F^\vee_\omega} - 1 \rangle,$$

and so

$$F^i S^\bullet_\omega K^\vee \cong S^i F^\vee_\omega$$

and the filtration is given by the natural maps

$$\mathcal{O}_S \overset{\cdot \pi^\vee_\omega}{\hookrightarrow} F^\vee_\omega \overset{\cdot \pi^\vee_\omega}{\hookrightarrow} S^2 F^\vee_\omega \overset{\cdot \pi^\vee_\omega}{\hookrightarrow} S^3 F^\vee_\omega \overset{\cdot \pi^\vee_\omega}{\hookrightarrow} \dots$$

This implies that the algebra $S^\bullet_\omega K^\vee$ is filtered quadratic and that $\mathrm{gr}_F(S^\bullet_\omega K^\vee) = S^\bullet K^\vee$. In particular the filtered quadratic dual of $S^\bullet_\omega K^\vee$ will be a sheaf $\wedge^\bullet_{\boldsymbol{\omega}} \boldsymbol{K}$ of curved dga whose underlying sheaf of graded algebras is $\wedge^\bullet K = (S^\bullet K^\vee)^!$. Note that the modules over the filtered algebra $S^\bullet_\omega K^\vee$ are just the quasi-coherent sheaves on the deformed space S_ω. So the filtered quadratic duality will convert the B-branes on S_ω into Higgs-like objects on X.

It is not hard to describe the curved dga $\wedge^\bullet_{\boldsymbol{\omega}} \boldsymbol{K}$ and the corresponding modules explicitly. Let $\{U_i\}$ be a Čech cover of X w.r.t. which $\omega \in H^1(X, K)$ is represented by a Čech cocycle

$$\{\omega_{ij}\} \in Z^1(\{U_i\}, K).$$

Now for each i the restricted fibration

$$S_{\omega|U_i} \to U_i$$

has a section which gives rise to a natural splitting of $F_\omega^\vee \twoheadrightarrow K^\vee$ over U_i. Thus we can repeat the previous construction over each U_i. In this way we get the sheaf of curved dga

$$\wedge_{\boldsymbol{\omega}}^\bullet \boldsymbol{K} := \coprod_i (\wedge^\bullet K, 0, 0)_{|U_i} / \sim,$$

where $(\wedge^\bullet K_{|U_i}, 0, 0)$ and $(\wedge^\bullet K_{|U_j}, 0, 0)$ are glued to each other via the isomorphism of curved dga given by the pair $(\mathrm{id}, \omega_{ij})$.

Note that this pair indeed gives a well defined automorphism of the curved dga $(\wedge^\bullet K_{|U_{ij}}, 0, 0)$:

- $[\omega_{ij}, \xi] = 0$ for all $\xi \in \wedge^\bullet K$ since $\wedge^\bullet K$ is supercommutative;
- $\omega_{ij} \wedge \omega_{ij} = 0$ since $\omega_{ij} \in K$.

Thus $\wedge_{\boldsymbol{\omega}}^\bullet \boldsymbol{K}$ is a sheaf of curved dga which is a twisted form of (i.e. is locally isomorphic to) $(\wedge^\bullet K, 0, 0)$. Filtered quadratic duality converts $S_\omega^\bullet K^\vee$ modules into $\wedge_{\boldsymbol{\omega}}^\bullet \boldsymbol{K}$ curved dg modules or *ω-twisted K-valued Higgs sheaves* on X. Unfortunately, as the following lemma shows, the classical spectral sheaves turn out to be obstructed in this direction. That is: a B-brane on S can deform to S_ω only if it is quasi-coherent or if the deformation direction vanishes on its support.

Lemma 2.2.3. There are no *classical spectral sheaves in* S_ω, *i.e. sheaves* $\mathcal{E} \to S_\omega$ *which are coherent and finite over* X, *as long as* $\omega \neq 0$.

In fact if $\mathcal{E}$ *is a classical spectral sheaf on* S_ω, *then*

$$\omega_{|\operatorname{supp}(p_{\omega *}\mathcal{E})} = 0.$$

Proof. Explicitly an ω-twisted Higgs sheaf is a quasi-coherent sheaf $E \to X$ equipped with a collection of local K-valued Higgs fields

$$\theta_i : E_{|U_i} \to E_{|U_i} \otimes K_{|U_i}, \quad \theta_i \wedge \theta_i = 0,$$

so that

$$\theta_i - \theta_j = \omega_{ij} \cdot \mathrm{id}_E \text{ on } U_{ij}.$$

In particular. if $(E, \{\theta_i\})$ is an ω-twisted Higgs bundle of rank r, then

$$\frac{1}{r}\operatorname{tr}(\theta_i) - \frac{1}{r}\operatorname{tr}(\theta_j) = \omega_{ij},$$

i.e. $\omega = 0 \in H^1(X, K)$. □

The notion of a twisted Higgs sheaf on S resembles a lot the notion of a sheaf twisted by a class $\alpha \in H^2(X, \mathcal{O}_X^\times)$.

Recall that if $\{U_i\}$ is a Čech cover of X w.r.t. which $\alpha \in H^2(S, \mathcal{O}_S^\times)$ is represented by a cocycle

$$\{\alpha_{ijk}\} \in Z^2(\{U_i\}, \mathcal{O}^\times),$$

then an α-twisted sheaf on S is a collection (F_i, g_{ij}), where:

- F_i is a coherent sheaf on U_i;

- $g_{ij} : F_{i|U_{ij}} \widetilde{\to} F_{j|U_{ij}}$ are sheaf isomorphisms satisfying the twisted cocycle condition

$$g_{ij} \circ g_{jk} \circ g_{ki} = \alpha_{ijk} \cdot \mathrm{id}\,.$$

From the viewpoint of the geometry of stacks, twisted sheaves appear as ordinary sheaves. More precisely: the element $\alpha \in H^2(X, \mathcal{O}_X^{\times})$ classifies an algebraic (or analytic) $\mathcal{O}^{\times}$-gerbe ${}_{\alpha}X$ on X and an α-twisted sheaf is simply a special kind of sheaf on ${}_{\alpha}X$.

The ω-twisted Higgs sheaves admit a similar interpretation as sheaves on a gerbe. Before we spell this out we need to recast the ordinary (untwisted) Higgs sheaves as sheaves on some geometric object. Such an incarnation of Higgs sheaves was proposed and studied by Simpson [Sim97, Sim02]:

For a vector bundle $N \to S$, let $\widehat{N} \to S$ denote the formal completion of N along the zero section. Then $\widehat{N}$ is a formal group scheme and we can consider the formal stack

$$X_N := [X/\widehat{N}] = B\widehat{N}.$$

A sheaf on X_N is simply a $N^{\vee}$-valued Higgs sheaf on X [Sim97, Sim02]. Similarly one may consider the algebraic stack $BN = [X/N]$. Sheaves on BN correspond to nilpotent $N^{\vee}$-valued Higgs sheaves on X.

Going back to the ω-twisted Higgs sheaves, note that the Leray spectral sequence applied to the map $X_{K^\vee} \to X$ allows us to view $\omega \in H^1(X, K)$ as an element in $H^2(X_{K^\vee}, \mathcal{O})$. In particular, ω gives rise to an $\mathcal{O}$-gerbe ${}_\omega X_{K^\vee}$ on the formal stack $X_{K^\vee}$.

In these terms, the ω-twisted Higgs sheaves on X become simply sheaves on the gerbe ${}_\omega X_{K^\vee}$. This gives yet another (stacky) interpretation of the spectral data living on the variety S_ω.

Comments:

- The correspondence

  $$\left\{\begin{array}{l}\text{quasi-coherent sheaves on the} \\ \text{affine bundle } S_\omega\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{quasi-coherent sheaves on the } \mathcal{O}\text{-} \\ \text{gerbe } {}_\omega X_{K^\vee}\end{array}\right\}$$

  is another instance of a mathematical duality similar to the spectral correspondence. In this case it is the Pontryagin duality for commutative group stacks.

- It is not hard to describe the gerbe ${}_\omega X_{K^\vee}$ explicitly. Indeed, the short exact sequence of vector bundles

  $$0 \to K \to F_\omega \to \mathcal{O}_X \to 0,$$

  gives rise to a short exact sequence of commutative group stacks

  $$0 \to B\widehat{\mathcal{O}_X} \to B\widehat{F_\omega^\vee} \to X_{K^\vee} \to 0,$$

which in turn can be viewed as an $\mathcal{O}$-gerbe on $X_{K^\vee}$. This is precisely the gerbe ${}_\omega X_{K^\vee}$.

It is also instructive to note here that this gerbe can be naturally identified with the stack of homomorphisms $\underline{\mathrm{Hom}}_{\mathrm{cgs}}(F_\omega, B\mathcal{O}_X)$, where both F_ω and $B\mathcal{O}_X$ are both viewed as commutative group stacks over X.

Example 2.2.4. **(non-commutative deformations)** Let $h \in H^0(X, \wedge^2 K)$ be any element. Again, since $\wedge^\bullet K$ is supercommutative we have

$$[h, \xi] = 0$$

for all $\xi \in \wedge^\bullet K$. Thus $(\wedge^\bullet, 0, h)$ is again a curved dg algebra over X. The dg modules over this algebra are the h-curved Higgs bundles, i.e. the pairs

$$(E, \theta : E \to E \otimes K), \quad \text{such that } \theta \wedge \theta = h \cdot \mathrm{id}_E\,.$$

It is not hard to describe the Koszul dual objects. The section $h \in H^0(X, \wedge^2 K)$ gives an extension of $K^\vee$ as a sheaf of Lie algebras. Explicitly consider the vector bundle

$$L_h := \mathcal{O}_X \cdot c \oplus K^\vee$$

with c being a dummy variable. Now h defines a Lie bracket on L_h given by

$$\begin{aligned}
&[c, c] = 0 \\
&[c, a] = 0, \ \text{for all } a \in K^\vee \\
&[a, b] = \langle h, a \wedge b \rangle \cdot c, \ \text{for all } a, b \in K^\vee.
\end{aligned}$$

By construction $(L_h, [\ ,\ ])$ is a sheaf of nilpotent Lie algebras on X with an $\mathcal{O}_X$-linear Lie bracket.

Moreover $\mathcal{O} \cdot c \subset L_h$ is a central ideal and we have a short exact sequence of Lie algebra sheaves

$$0 \to \mathcal{O} \cdot c \to L_h \to K^\vee \to 0$$

where both $\mathcal{O} \cdot c$ and $K^\vee$ are taken to be commutative. With this notation we can identify the Koszul dual

$$(\wedge^\bullet K, 0, h)^! = U(L_h)/\langle 1_{U(L_h)} - c\rangle =: U_h$$

with the enveloping algebra of the central extension L_h taken with its natural filtration.

Note that again

$$\mathrm{gr}_F\, U_h = S^\bullet K^\vee$$

and so the h-curved Higgs bundles have a dual interpretation as *non-commutative branes*, i.e. modules over U_h.

Here we think of U_h as "$p_{h*}\mathcal{O}_{S_h}$" of a non-commutative deformation

$$p_h : S_h \to X$$

of the map $p : S \to X$.

More generally one can check that if $(\wedge^\bullet K, d, h)$ is an arbitrary curved dg algebra having $\wedge^\bullet K$ as the underlying graded algebra, then

$$d^\vee : \wedge^2 K^\vee \to K^\vee$$

will be a Lie bracket on $K^\vee$ and $h \in H^0(X, \wedge^2 K)$ is a two cocycle for the Lie algebra $(K^\vee, d^\vee)$. Let

$$0 \to \mathcal{O} \cdot c \to L_{h,d} \to K^\vee \to 0$$

be the corresponding central extension as sheaves of Lie algebras. Then

$$(\wedge^\bullet K, d, h)^! = U(L_{h,d})/\langle 1_{U(L_{h,d})} - c\rangle =: U_{h,d}$$

as a filtered algebra and so can be thought of as a non-commutative deformation of $p : S \to X$ again.

It is instructive to note that if $h = 0$ then $(\wedge^\bullet K, d, 0)$ is the Cartan-Eilenberg complex of $L_{0,d}$ with trivial coefficients. That is:

$$(\wedge^\bullet K, d, 0) = C^\bullet(K^\vee, \mathcal{O}_X).$$

Similarly, for any module M over the Lie algebra $(K^\vee, d^\vee)$ we have that M viewed as a module over $U_{0,d}$ is a filtered quadratic module and that

$$M^! = C^\bullet(K^\vee, M)$$

is the Cartan-Eilenberg complex computing the cohomology of $(K^\vee, d^\vee)$ with coefficients in M. Thus this type of quantization of spectral data produces a concurrent deformation of Cartan-Eilenberg complexes.

An important special case of the previous construction is when $K = \Omega^1_X$. Then we have a natural dg algebra

$$(\Omega^\bullet_X, d) = (\wedge^\bullet \Omega^1_X, d, 0)$$

- the *holomorphic de Rham complex* on X.

Recall that the modules over $(\Omega^\bullet_X, d)$ are precisely the quasi-coherent sheaves equipped with a flat connection, i.e. are the analytic (algebraic) $\mathcal{D}_X$-modules. This fact is a manifestation of filtered Koszul duality since

$$\mathcal{D}_X = (\wedge^\bullet \Omega^1_X, d, 0)^!$$

is exactly the Koszul dual filtered algebra.

Therefore flat bundles on S can be interpretted dually as non-commutative spectral covers, i.e. as $\mathcal{D}_X$-modules. The latter should be thought of as sheaves on a non-commutative deformation of $p : S \to X$ whose structure sheaf pushes forward to $\mathcal{D}_X$.

Variant: Taking a deormation of Ω^1_X to a ω-twisted cotangent bundle and introducing a curvature $h \in \Omega^2_X$ one gets (similarly to **(i)** and **(ii)**) a ω-twisted curved dg algebra

$$(\Omega^\bullet_X, d, h)_\omega$$

which is Koszul dual to a sheaf $(\mathcal{D}_X)_{h,\omega}$ of twisted differential opertors on X. This gives yet another non-commutative deformation of $p : S \to X$. The nonabelian Hodge correspondence for this deformation was established and studied in [Ga-Ra17].

2.3 Framed sheaves on ruled Poisson surfaces

In this section we introduce our main objects of study - coherent complexes on Poisson surfaces that have unobstructed deformations in the non-commutative direction. When the Poisson surface is a ruled surface the requisite complexes turn out to be generalized spectral data and we can study them by applying an appropriate version of the spectral correspondence and studying the corresponding Higgs data on the base curve.

In this analysis we are guided by the following

Heuristics: Non-commutative deformations of a scheme S should be thought of as deformations of the abelian category $\mathrm{Coh}(S)$ or more generally of the triangulated category $D^b(S)$.

Bondal observed [Bon93] that the typical infinitesimal deformations of $D^b(S)$ come from deformations of the identity functor on $D^b(S)$ which in turn can be computed as the second Hoschchild cohomology of S, i.e.

$$\begin{aligned}\mathrm{Ext}^2_{D^b(S)}(\mathbb{I},\mathbb{I}) &= \mathrm{Hom}_{D^b(S)}(\mathbb{I},\mathbb{I}[2]) \\ &= HH^2(S).\end{aligned}$$

Assume for simplicty that S is smooth. Then it is known by the Gerstenhaber-Shack theorem [GerSch88, Swan96] that $HH^2(S)$ has a Hodge type decomposition:

$$HH^2(S) = H^2(\mathcal{O}_S) \oplus H^1(T_S) \oplus H^0(\wedge^2 T_S).$$

The different pieces in this decomposition have different meaning:

$H^1(T_S)$ parameterizes ordinary geometric deformations of S.

$H^0(\wedge^2 T_S)$ parameterizes deformations of the product on the algebra $\mathcal{O}_S$ to some associative product.

$H^2(\mathcal{O}_S)$ parameterizes deformations of S in the stacky direction, i.e. deformations as an $\mathcal{O}_S^\times$-gerbe.

Remark 2.3.1. The passage to moduli can mix different types of deformations. In particular if we have a moduli problem on S for which the discrete data is fixed so that it deforms unobstructedly in any of the three directions above, then the moduli space may have commutative deformations which are interprettable as moduli spaces of the same type of data on a non-commutative or stacky deformation of S

.

A large class of essential examples of this phenomenon are provided by moduli spaces of framed sheaves.

General setup: Let S be a complex surface which has some non-commutative deformations (at least infinitesimally). This means that S must be a Poisson surface.

Let $\lambda \in H^0(S, \wedge^2 T_S)$ be a fixed Poisson structure. The moduli space M of sheaves on S with framing along $\lambda = 0$ will have some additional deformations M_t (not corresponding to deformations of S) and we would like to identify M_t with the moduli space of framed sheaves on the non-commutative deformation of S in the direction of λ.

Example 2.3.2. **(a)** (Nekrasov-Schwartz [NekSch98], Kapustin-Kuznetsov-Orlov [KKO01]) Let $S = \mathbb{P}^2$ with the Poisson structure $\lambda = f^3$ where $f \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ is the equation of some line $L \subset S$.

Consider the moduli space

$$M = \left\{ \left(E, \varphi : E_{|L} \to \mathcal{O}_L^{\oplus r}\right) \middle| \begin{array}{l} E \text{ is a rank } r \text{ vector bundle on } S \\ \text{with } c_1(E) = 0 \text{ and } c_2(E) = k. \end{array} \right\}$$

of framed sheaves on $S = \mathbb{P}^2$. Using the ADHM construction Donaldson showed [Don84] that M is isomorphic to the moduli space of $SU(r)$ instantons of instanton numbers k on S^4.

In particular M is hyper-Kähler and has a natural twistor deformation

$$\begin{array}{ccccc} M & = & \mathcal{M}_0 & \subset & \mathcal{M} \\ & & \downarrow & & \downarrow \\ & & 0 & \in & \mathbb{P}^1 \end{array}$$

Moreover the restriction of the twistor family $\mathcal{M}$ to the open subset $\mathbb{C}^\times = \mathbb{P}^1 - \{0, \infty\}$ is a holomorphically trivial family, i.e.

$$\mathcal{M}_{|\mathbb{C}^\times} \cong \mathcal{M}_1 \times \mathbb{C}^\times,$$

where $\mathcal{M}_1$ is the fiber of $\mathcal{M} \to \mathbb{P}^1$ over $1 \in \mathbb{P}^1$.

Now Nekrasov-Shwartz and Kapustin-Kuznetsov-Orlov prove [NekSch98, KKO01]

Theorem $\mathcal{M}_1$ *is isomorphic to the moduli space of pairs* (E, φ)*, where:*

- E *is a rank* r *vector bundle on the non-commutative* $\mathbb{P}^2_\lambda$ *defined by* λ*, having* $c_1(E) = 0$ *and* $c_2(E) = k$*;*
- φ *is a framing of* E *along the commutative line* $L \subset \mathbb{P}^2_\lambda$*.*

(b) (Baranovsky-Ginzburg-Kuznetsov [BGK02]) Let P be a projective variety and let A be a graded algebra defining P, i.e. $P = \mathrm{Proj}(A)$. Recall that by Serre's theorem the category $\mathrm{Coh}(P)$ of coherent sheaves on P is equivalent to the quotient category

$$\mathrm{qgr}(A) := \frac{(\text{finitely generated graded } A\text{-modules})}{(\text{finite dimensional graded } A\text{-modules})}.$$

Consider now a finite group Γ acting on P and let $A \sharp \mathbb{C}\Gamma$ be the smash product of algebras with the grading in which $\mathbb{C}\Gamma$ is placed in degree zero.

The algebra $A \sharp \mathbb{C}\Gamma$ is a graded non-commutative algebra and when it happens to be regular of dimension d, then we can view it as the algebra of homogeneous functions on some non-commutative d-dimensional projective scheme $\mathrm{Proj}(A \sharp \mathbb{C}\Gamma)$.

Remark 2.3.3. Recall that a locally finite dimensional graded algebra $A = \oplus_{i \geq 0} A_i$ is called *regular of dimension d* if:

- A_0 is a semisimple $\mathbb{C}$-algebra;
- A global homological dimension d, i.e. d is the minimal integer such that
$$\mathrm{Ext}^d_{A-\mathrm{Mod}}(M, N) = 0, \text{ for all } M, N \in A - \mathrm{Mod};$$
- A is Noetherian of polynomial growth, i.e. we can find integers $m, n > 0$ so that
$$\dim_{\mathbb{C}} A_i \leq m i^n, \text{ for } i \gg 0.$$
- A is Gorenstein of type (d, ℓ), i.e.
$$\mathrm{Ext}^i_{A-\mathrm{Mod}}(A_0, A) = \begin{cases} A_0(\ell), & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly we can make sense of the category $\mathrm{Coh}(\mathrm{Proj}(A \sharp \mathbb{C}\Gamma))$ of coherent sheaves on the non-commutative scheme $\mathrm{Proj}(A \sharp \mathbb{C}\Gamma)$ as the quotient category

$$\mathrm{qgr}(A \sharp \mathbb{C}\Gamma) := \frac{(\text{finitely generated graded } A \sharp \mathbb{C}\Gamma\text{-modules })}{(\text{finite dimensional graded } A \sharp \mathbb{C}\Gamma\text{-modules })}.$$

The next step is to consider deformations of $A \sharp \mathbb{C}\Gamma$ as a graded associative algebra. These deformations will give rise to new non-commutative projective varieties.

The non-commutative projective variety $\mathrm{Proj}(A \sharp \mathbb{C}\Gamma)$ defined in a categorical fashopn above is a non-commutative variety of a very special kind - its non-commutativity is coming from twisting the multiplication in the homogeneous algebra of functions on the ordinary projective variety $P = \mathrm{Proj}(A)$ by the action of the group of authomorphisms $\Gamma \subset \mathrm{Aut}(P)$. Such non-commutative varieties are called *twisted projective varieties.* An interesting feature of these varieties is that their geometry (e.g. their sheaf theory) can also be interpreted by using quotient stacks rather than non-commutative spaces. Indeed, it is straighforward to check that the category $\mathrm{qgr}(A \sharp \mathbb{C}\Gamma)$ of coherent sheaves on $\mathrm{Proj}(A \sharp \mathbb{C}\Gamma)$ is equivalent to the category of coherent sheaves on the stack quotient $[P/\Gamma]$ or equivalently to the category of Γ-equivariant coherent sheaves on P.

In particular, if Γ acts freely, then $S = [P/\Gamma]$ is an ordinary variety and so the sheaf theory on $\mathrm{Proj}(A \sharp \mathbb{C}\Gamma)$ is indistinguishable from the sheaf theory on a commutative variety. This setup also shows that we can deform the category of coherent sheaves on a projective variety to a category which admits two different interpretations: as the category of sheaves on a stack and as the category of sheaves of a twisted projective variety.

(c) (Nevins-Stafford) Let again $S = \mathbb{P}^2$ but now take $\lambda \in H^0(\mathbb{P}^2, \mathcal{O}(3))$ to be the equation of a smooth cubic $\Sigma \subset S$.

Consider the Hilbert schemes

$$\mathrm{Hilb}^k(S), \qquad \mathrm{Hilb}^k(S - \Sigma)$$

of k-points on S and $S - \Sigma$ respectively. As usual we can identify $\mathrm{Hilb}^k(S)$ with the moduli space of stable torsion free rank one sheaves on S having $c_1 = 0$ and $c_2 = k$. Similarly $\mathrm{Hilb}^k(S - \Sigma)$ admits an interpretation as the moduli of framed rank one torsion free sheaves on S with framing $\mathcal{O}_\Sigma$ along Σ.

From the work of Artin-Tate-Van den Bergh [ATvdB90] it is known that the infinitesimal non-commutative deformation of S given by λ can be integrated to an actual non-commutative deformation (given by a Sklyanin algebra) as long as we choose an automorphism $s : \Sigma \to \Sigma$ of the elliptic curve Σ.

In the remainder of the section we focus on the case of Poisson ruled surfaces. To set things up we fix the following notation:

X - a smooth compact complex curve of genus $g > 1$.

S - the projective bundle $S := \mathbb{P}(\mathcal{O}_X \oplus \omega_X) \xrightarrow{\pi} X$.

$\mathcal{O}_S(1)$ - the relative hyperplane bundle of S over X, normalized by the condition $\pi_* \mathcal{O}_S(1) \cong \mathcal{O}_X \oplus \omega_X^{-1}$.

D - the divisor $D \subset X$ at infinity corresponding to the line $\mathcal{O}_X \subset \mathcal{O}_X \oplus \omega_X$.

Note that the surface S has a canonical Poisson structure. Indeed, observe that

- The map $\pi_{|D} : D \to X$ is an isomorphism and $S = D \coprod \mathrm{tot}(\omega_X)$.
- We have $\mathcal{O}_S(1) = \mathcal{O}_S(D)$ and $\mathcal{O}_S(D)_{|D} \cong \omega_X^{-1}$ under the identification $\pi : D \widetilde{\to} S$.
- From the Euler sequence on $\pi : S \to X$ we get that the canonical class ω_S satisfies $\omega_X \cong \mathcal{O}_S(-2D)$. Thus $\omega_S^{-1} = \wedge^2 T_X$ has a unique (up to scale) section λ with divisor $\mathrm{div}(\lambda) = 2D$. In particular (S, λ) is a Poisson surface.

Let now $F \to X$ be a fixed vector bundle of rank n on D and let $\mathcal{F} = \mathbf{i}_* F \in \mathrm{Coh}(S)$, where $\mathbf{i} : D \hookrightarrow S$ denote the natural inclusion. Now we have the following standard definition (see e.g. [HuLe95a, HuLe95b, Sa11])

Definition 2.3.4. (a) An $\mathcal{F}$-framed sheaf on S is a pair (E, φ) where E is a coherent sheaf on S and $\varphi : E \to \mathcal{F}$ is a map of coherent sheaves.

(b) A (D, F)-framed sheaf on S is an $\mathcal{F}$-framed sheaf (E, φ) such that E is locally free in a neighborhood of D, and $\varphi_{|D} : E_D \to F$ is an isomorphism.

Similarly we define $\mathcal{F}$-framed and (D, F)-framed sheaves on the non-commutative deformation of S in the direction of λ. Note that in this non-commutative deformation the curve $D \subset S$ stays commutative and persists as a commutative divisor in all fibers of the deformation family. Therefore the framing by $\mathcal{F}$ transfers verbatim to all these non-commutative deformations.

Remark 2.3.5. Suppose F is the zero bundle, then a (D, F)-framed sheaf is simply a coherent sheaf on S whose support is contained in $\mathrm{tot}(\omega)$, and

(i) By the spectral correspondence the category of such sheaves is equivalent to the category of Higgs coherent sheaves on X.

(ii) Simpson's stability theorem shows that there is $a \gg 0$ so that a Higgs sheaf on X is stable if and only if the corresponding spectral sheaf is stable with respect to the polarization $H = D + a \cdot f$ on S.

(iii) Semistable spectral sheaves corresponding to Higgs sheaves of degree zero quantize to deformation quantization modules which correspond to flat bundles on X.

Our goal is to understand the moduli spaces of (D, F)-framed sheaves and their non-commutative deformations for arbitrary framings F. For this we will need to extend the spectral construction and describe the extended Higgs data that will correspond to (D, F)-framed sheaves on S and its non-commutative deformations.

Chapter 3

Extended Higgs data

As before, we fix a smooth complex projective curve X, and $F \to X$ a vector bundle of rank n on X. Write ω for the canonical line bundle of X. There are various ways to encode (D, F)-framed sheaves on the Poisson surface $S = \mathbb{P}(\mathcal{O} \oplus \omega)$ in some kind of extended Higgs data. We will focus on a specific kind which we call *F-prolonged Higgs bundles*, and which are tailor made for analyzing non-commutative deformations of the corresponding (D, F)-framed sheaves and for analyzing extensions of the non-abelian Hodge theorem. It turns out that the *F-prolonged Higgs bundles* are closely related to another type of extended Higgs data - the so called *Higgs triples*, introduced recently by Alexandre Minets [Min18] for the study of the cohomological Hall algebra for sheaves on S.

We begin by introducing Higgs triples and F-prolonged Higgs bundles, and discuss their relationship.

3.1 Higgs triples

Definition 3.1.1. ([Min18]) Fix X a smooth complex projective curve and $F \to X$ a vector bundle of rank n on X. A *Higgs triple* is the triple of data $(\mathcal{E}, \alpha, \vartheta)$ that consists of

- a coherent sheaf $\mathcal{E} \in \mathrm{Coh}(X)$,
- a map of coherent sheaves $\alpha : F \to \mathcal{E}$, viewed as a two-term complex sitting in degrees 0 and 1,
- a morphism of complexes $\vartheta : \mathcal{E} \to \left[\begin{smallmatrix} F \\ \downarrow \\ \mathcal{E} \end{smallmatrix}\right][1] \otimes \omega$, in $\mathcal{D}^b(X)$.

Remark 3.1.2. Since $\mathrm{Coh}(X)$ has homological dimension one, every object in $\mathcal{D}^b(X)$ is formal (see e.g. [Kap98]) and so we can view $\vartheta \in \mathbb{E}\mathrm{xt}^1(\mathcal{E}, \left[\begin{smallmatrix} F \\ \downarrow \\ \mathcal{E} \end{smallmatrix}\right] \otimes \omega)$ as a pair $\vartheta = (\vartheta_e, \vartheta_h)$, with $\vartheta_e \in \mathrm{Ext}^1(\mathcal{E}, \ker \alpha \otimes \omega)$ and $\vartheta_h \in \mathrm{Hom}(\mathcal{E}, \operatorname{coker} \alpha \otimes \omega)$.

Minets has introduced [Min18] the following notion of stability for the Higgs triples.

Definition 3.1.3. A Higgs triple is called *stable* if there is no subsheaf $\mathcal{E}' \subset \mathcal{E}$ such that:

- $\mathrm{Im}\, \alpha \subset \mathcal{E}'$

- $a(\vartheta) \in \mathrm{Im}(b)$ where a, b are the maps below, induced by the inclusion $\mathcal{E}' \subset \mathcal{E}$:

$$\mathbb{E}\mathrm{xt}^1(\mathcal{E}, (F \to \mathcal{E}) \otimes \omega) \xrightarrow{a} \mathbb{E}\mathrm{xt}^1(\mathcal{E}', (F \to \mathcal{E}) \otimes \omega) \xleftarrow{b} \mathbb{E}\mathrm{xt}^1(\mathcal{E}', (F \to \mathcal{E}') \otimes \omega).$$

Remark 3.1.4. We can unpack this definition of stability to make it look similar to familiar notions of stability of decorated bundles or complexes.

(i) As explained in Remark 3.1.2 the data of the Higgs triple $(\mathcal{E}, \alpha, \vartheta)$ are equivalent to a quadruple $(\mathcal{E}, \alpha, \vartheta_e, \vartheta_h)$ with

$$\vartheta_e \in \mathrm{Ext}^1(\mathcal{E}, \ker\, \alpha), \text{ and } \vartheta_h \in \mathrm{Hom}(\mathcal{E}, \mathrm{coker}\ \alpha\, \otimes \omega).$$

In these terms the stability of $(\mathcal{E}, \alpha, \vartheta)$ is equivalent to the condition that there is no subsheaf $\mathcal{E}' \subset \mathcal{E}$ satisfying:

- $\mathcal{E}' \supset \mathrm{Im}\ \alpha$, and
- $\vartheta_h(\mathcal{E}') \subset \mathcal{E}' / \mathrm{Im}\ \alpha$.

(ii) The stability condition in Definition 3.1.3 can also be reformulated as the condition that the Higgs triple $(\mathcal{E}, \alpha, \vartheta)$ has no non-trivial subtriples $(\mathcal{E}', \alpha', \vartheta') \subset (\mathcal{E}, \alpha, \vartheta)$. In other words, there are no triples $(\mathcal{E}', \alpha', \vartheta')$ such that:

- $\mathcal{E}' \subset \mathcal{E}$.

- The diagram $\begin{tikzcd} F \arrow[r, equal] \arrow[d, "\alpha'"'] & F \arrow[d, "\alpha"] \\ \mathcal{E}' \arrow[r, hook] & \mathcal{E} \end{tikzcd}$ commutes, i.e. the map $\begin{bmatrix} F \\ \downarrow \\ \mathcal{E}' \end{bmatrix} \to \begin{bmatrix} F \\ \downarrow \\ \mathcal{E} \end{bmatrix}$ is an inclusion of complexes of coherent sheaves.

- $\vartheta : \mathcal{E} \to \begin{bmatrix} F \\ \downarrow \\ \mathcal{E} \end{bmatrix} [1] \otimes \omega$ is induced from $\vartheta' : \mathcal{E}' \to \begin{bmatrix} F \\ \downarrow \\ \mathcal{E}' \end{bmatrix} [1] \otimes \omega$ via post-composition of ϑ' with the inclusion of complexes $\begin{bmatrix} F \\ \downarrow \\ \mathcal{E}' \end{bmatrix} \to \begin{bmatrix} F \\ \downarrow \\ \mathcal{E} \end{bmatrix}$.

From this point of view, this stability condition looks very restrictive as it forbids the existence of all subobjects rather than the subobjects of appropriately defined smaller slope. Minets defines this stability condition in general but uses it only for triples for which $\mathcal{E}$ is pure of dimension zero, i.e. a torsion sheaf on X. As explained in [Min18], these are the triples which can be rewritten as (D, F)-framed sheaves while the rest of the Higgs triples do not seem to have any relationship to framed sheaves on S.

In the next section we will analyze this stability condition in more detail and will show that for triples with torsion $\mathcal{E}$, Minets stability condition is simply a polynomial stability condition in the sense of Bayer [Bay09].

3.2 F-prolonged Higgs bundles

Definition 3.2.1. Again, fix X a smooth projective complex curve, and $F \to X$ a vector bundle on X. An *F-prolonged Higgs bundle* is a triple $(V,\, \xi,\, \rho)$ consisting of

- a vector bundle V on X,

- an extension $\xi \in \mathrm{Ext}^1(F, V)$, i.e.

$$(\xi): \ 0 \to V \to W \to F \to 0$$

- a morphism $\rho : V \to W \otimes \omega$.

We say that an F-prolonged Higgs bundle (V, ξ, ρ) is *pure* if it does not contain any ordinary Higgs bundle, i.e. if there is no subsheaf $0 \neq V' \subset V$ such that $\rho(V') \subset V' \otimes \omega \subset W \otimes \omega$.

Our first main result is a generalized spectral correspondence which reinterprets F-prolonged Higgs bundles as certain (D, F)-framed sheaves on S. To state the result properly recall that $\pi : S \to X$ denotes the natural projection.

Theorem 3.2.2. *There is a natural equivalence of categories:*

$$\begin{pmatrix} \text{(D,F)-framed torsion free} \\ \text{sheaves on S, globally} \\ \text{generated along the fibers} \\ \text{of π} \end{pmatrix} \underset{\mathfrak{s}}{\overset{\mathfrak{p}}{\rightleftarrows}} \begin{pmatrix} \text{pure F-prolonged Higgs} \\ \text{bundles on X} \end{pmatrix}.$$

Proof. Let us first describe the functor $\mathfrak{p}$. Suppose (E, φ) is (D, F)-framed sheaf on S, such that E is torsion free and globally generated along the fibers of π. By definition, E being globally generated along the fibers of π, means that the natural evaluation map

$$\mathrm{ev} : \pi^*\pi_*E \to E$$

is surjective. Since E is assumed torsion free, we have that π_*E is torsion free and since X is a smooth curve, it follows that π_*E is locally free. Let $\ker(\mathrm{ev}) \subset \pi^*\pi_*E$ be the kernel of the evaluation map. Then $\ker \mathrm{ev}$ is a saturated subsheaf in the locally free sheaf $\pi^*\pi_*E$ and hence is reflexive. Since S is a smooth surface, this implies that $\ker(\mathrm{ev})$ is locally free as well. In other words, we have a short exact sequence on S

$$0 \longrightarrow \ker(\mathrm{ev}) \longrightarrow \pi^*\pi_*E \xrightarrow{\mathrm{ev}} E \longrightarrow 0,$$

in which the first two terms are locally free and the last is torsion free.

Applying $R\pi_*$ to this sequence and using the projection formula identifications $\pi_*\pi^*A = A \otimes \pi_*\mathcal{O} = A$ and $R^1\pi_*\pi^*A = A \otimes R\pi_*\mathcal{O} = 0$, we get a long exact sequence of derived images:

$$0 \longrightarrow \pi_* \ker(\mathrm{ev}) \longrightarrow \pi_*E \xrightarrow{\mathrm{id}} \pi_*E \longrightarrow R^1\pi_* \ker(\mathrm{ev}) \longrightarrow 0.$$

Therefore $\pi_* \ker(\mathrm{ev}) = R^1\pi_* \ker(\mathrm{ev}) = 0$. In particular, the vector bundle $\ker(\mathrm{ev})$ is semistable of slope (-1) on each fiber of π. Hence by the see-saw theorem

$$\ker(\mathrm{ev}) = \pi^*A \otimes \mathcal{O}(-D)$$

for some vector bundle A on X. So we have a short exact sequence

$$0 \longrightarrow \pi^*A \longrightarrow (\pi^*\pi_*E)(D) \longrightarrow E(D) \longrightarrow 0,$$

which upon pushforward to X yields the sequence

$$0 \longrightarrow A \longrightarrow \pi_* E \otimes (\mathcal{O} \oplus \omega^{-1}) \longrightarrow \pi_* E(D) \longrightarrow 0.$$

Tensoring with ω we get a sequence

$$0 \longrightarrow A \otimes \omega \xrightarrow{\rho \oplus i} \pi_* E \otimes (\omega \oplus \mathcal{O}) \longrightarrow (\pi_* E(D)) \otimes \omega \longrightarrow 0. \tag{3.2.1}$$

Now set

$$V := A \otimes \omega \quad \text{and} \quad W := \pi_* E.$$

With this notation the two components of the first map in (3.2.1) become maps

$$i : V \to W \quad \text{and} \quad \rho : V \to W \otimes \omega.$$

Also by construction the map i can be identified with the restriction to D of the natural inclusion

$$\pi^* A(-D) \hookrightarrow \pi^* W. \tag{3.2.2}$$

Since the inclusion (3.2.2) fits in the short exact sequence

$$0 \longrightarrow \pi^* A(-D) \longrightarrow \pi^* W \longrightarrow E \longrightarrow 0,$$

and (E, φ) is (D, F)-framed, it follows that $i : V \to W$ fits in a short exact sequence

$$0 \longrightarrow V \xrightarrow{i} W \longrightarrow E_{|D} \longrightarrow 0,$$

which via φ is identified with a short exact sequence

$$0 \longrightarrow V \xrightarrow{i} W \longrightarrow F \longrightarrow 0. \tag{ξ}$$

We set $\mathfrak{p}(E, \varphi) = (V, \xi, \rho)$. This assignment manifestly respects maps of framed sheaves and so gives a functor from torsion free (D, F)-framed sheaves to F-prolonged Higgs triples.

To construct the functor $\mathfrak{s}$, start with an F-prolonged Higgs bundle (V, ξ, ρ) on X. Taken together the two maps

$$\begin{aligned} i: &\quad V \longrightarrow W, \\ \rho: &\quad V \longrightarrow W \otimes \omega, \end{aligned}$$

give an injective map of locally free sheaves on X:

$$\rho \oplus i : V \longrightarrow W \otimes (\omega \oplus \mathcal{O})$$

which by adjunction gives a homomorphism of locally free sheaves on S:

$$\pi^* V \longrightarrow \pi^*(W \otimes \omega)(D). \tag{3.2.3}$$

Lemma 3.2.3. *The essential image of the functor $\mathfrak{p}$ is contained in the category of pure F-prolonged Higgs bundles on X.*

Proof. Let (E, φ) be a torsion free (D, F)-framed sheaf on S which is globally generated alpng the fibers of π. Let $(V, \xi, \rho) = \mathfrak{p}(E, \varphi)$. We need to show that the F-prolonged sheaf (V, ξ, ρ) is pure. Suppose $V' \subset V$ is such that $\rho(V') \subset V' \otimes \omega$. Then the image of v' under the map

$$\rho \oplus i : V \to (W \otimes \omega) \oplus W$$

is contained in $V' \otimes (\omega \oplus \mathcal{O})$. therefore by adjunction it follows that the image of the subsheaf $\pi^* V' \subset \pi^* V$ under the natural map

$$\pi^* V \to \pi^*(W \otimes \omega)(D)$$

is contained in the subsheaf $\pi^*(V' \otimes \omega)(D)$. Thus, if we set

$$\mathcal{N}' = \pi^*(V' \otimes \omega)(D)/\pi^* V',$$

we get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^* V' \arrow[r] \arrow[d] & \pi^*(V' \otimes \omega)(D) \arrow[r] \arrow[d] & \mathcal{N}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^* V \arrow[r] & \pi^*(W \otimes \omega)(D) \arrow[r] & E(D) \otimes \pi^*\omega \arrow[r] & 0.
\end{tikzcd}$$

Restricting this diagram to D we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V' \arrow[r, "id"] \arrow[d] & V' \arrow[r] \arrow[d] & \mathcal{N}'_{|D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \arrow[r] & W \arrow[r] & F \arrow[r] & 0.
\end{tikzcd}$$

Hence $\mathcal{N}'_{|D} = 0$, i.e. $\mathcal{N}'$ is torsion and $\mathrm{supp}(\mathcal{N}') \cap D = \varnothing$. But $\mathcal{N}' \subset E(D) \otimes \pi^*\omega$ and, by assumption, E is torsion free. Hence $\mathcal{N}'$ must be the zero sheaf on S. On the other hand, by construction $\mathcal{N}'$ is the spectral data sheaf for the usual Higgs bundle $\rho : V' \to V' \otimes \omega$. Since, as explained in chapter 2, the spectral correspondence is an equivalence we conclude that $V' = 0$ which proves the lemma. $\square$

This completes the construction of the desired functor $\mathfrak{p}$. For future reference, let us also note that from the definition of the functor $\mathfrak{p}$ we have a natural identification $V = \pi_* E(-D)$. Indeed since (E, φ) is (D, F)-framed we have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E(-D) \arrow[r] & E \arrow[r, "\varphi"] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0. \\ & & & \mathbf{i}_* F & \end{tikzcd}$$

Applying $R\pi_*$ to this sequence and taking into account that E being globally generated on the fibers implies $R^1\pi_* E(-D) = 0$, we get a long exact sequence of derived images:

$$\begin{tikzcd} 0 \arrow[r] & \pi_* E(-D) \arrow[r] & \pi_* E \arrow[r] & \pi_* \mathcal{F} \arrow[r] \arrow[d, equal] & R^1\pi_* E(-D). \arrow[d, equal] \\ & & & F & 0 \end{tikzcd}$$

But the map $W = \pi_* E \to F$ is precisely the quotient map in the sequence (ξ). Hence $\pi_* E(-D) = \ker[W \to F] = V$.

Next we will construct the functor $\mathfrak{s}$. Let (V, ξ, ρ) be an F-prolonged Higgs bundle. Together the maps $i : V \subset W$ and $\rho : V \to W \otimes \omega$ give an injective map of locally free sheaves on X:

$$\rho \oplus i : V \longrightarrow W \otimes (\omega \oplus \mathcal{O}).$$

By adjunction, this map corresponds to a map of locally free sheaves on S:

$$\pi^* V \longrightarrow \pi^*(W \otimes \omega)(D). \tag{3.2.4}$$

But after restricting to D the map (3.2.3) becomes the injective map

$$i : V \to W$$

and hence the kernel of (3.2.4) must be a torsion sheaf whose support is disjoint from D. Since this kernel is contained in the locally free sheaf π^*V, it must be zero, and so the map (3.2.4) must be injective. Tensoring with $\pi^*\omega^{-1}(-D)$ we get an injective map of sheaves $\pi^*(V \otimes \omega^{-1})(-D) \to \pi^*W$ and denoting its cokernel by E we get a short exact sequence

$$0 \longrightarrow \pi^*(V \otimes \omega^{-1})(-D) \longrightarrow \pi^*W \longrightarrow E \longrightarrow 0. \tag{3.2.5}$$

Furthermore by construction E is locally free near D and the restriction $E_{|D}$ is naturally identified with the cokernel of the map $i : V \otimes \omega^{-1} \otimes \omega \to W$, i.e with the vector bundle F, i.e. $(E, E \to \mathbf{i}_*F)$ is a (D, F)-framed sheaf on S.

Let $T \subset E$ be the torsion subsheaf of E and let $W_1 \subset \pi^*W$ be the preimage of T. In other words W_1 is the saturation of $\pi^*(V \otimes \omega^{-1})(-D)$ in π^*W. Since W_1 is a saturated subsheaf in a locally free sheaf on a smooth surface it must be locally free. For every $x \in X$ let $\mu_{\max}(x)$ be the slope of the maximal destabilizing sheaf in $W_{1|\pi^{-1}(x)}$. But by construction we have an inclusion of sheaves

$$\pi^*(V \otimes \omega^{-1})(-D) \subset W_1 \subset \pi^*W,$$

and so for every x we must have

$$W_{1|\pi^{-1}(x)} \cong \mathcal{O}(-1)^{\oplus k_x} \oplus \mathcal{O}^{\oplus \ell_x}$$

for some integers $k_x, \ell_x > 0$ such that $k_x + \ell_x = r = \operatorname{rank} V$. By Shatz's semicontinuity theorem [Sh77, LePo97] for the variation of Harder-Narasimhan polygons in families, it follows that there is an integer $\ell > 0$ so that $\ell_x = \ell$ for all x in a dense Zariski open set of X. Thus $V_1 = \pi_* W_1$ will be a rank ℓ vector bundle on X. Now consider the natural map

$$W_1 \to \mathbf{i}_* \mathbf{i}^* W_1.$$

Since the support of T does not intersect D it follows that $\mathbf{i}^* W_1 = W_{1|D} = V$. Pushing down the short exact sequence

$$0 \longrightarrow W_1(-D) \longrightarrow W_1 \longrightarrow \mathbf{i}_* V \longrightarrow 0$$

via π we get a sequence on X:

$$\begin{array}{ccccccc} 0 \longrightarrow & \pi_* W_1(-D) & \longrightarrow & \pi_* W_1 & \longrightarrow & V & \\ & \| & & \| & & & \\ & 0 & & V_1 & & & , \end{array}$$

i.e. we get a natural inclusion $V_1 \subset V$. But now by pushing down the inclusions

$$\pi^* V_1 \subset \pi^* V \subset (W_1 \otimes \pi^* \omega)(D) \subset \pi^* (W \otimes \omega)(D)$$

we get that $\rho(V_1) \subset V_1 \otimes \omega$, which implies that (V, ξ, ρ) can not be pure.

Hence for a pure (V, ξ, ρ) we can set $\mathfrak{s}(V, \xi, \rho)$ to be the torsion free (D, F)-framed sheaf $(E, E \to \mathbf{i}_* F)$. From the short exact sequence (3.2.5) it is immediate that E is globally generated along the fibers of π, and the construction is clearly functorial for maps of F-prolonged bundles. Also, from the construction it is immediate that the two functors are inverses of each other, which completes the proof of the theorem.

$\square$

Remark 3.2.4. **(i)** By carefully tracing the constructions used in the proof of the previous theorem, one can check that the functors $\mathfrak{p}$ and $\mathfrak{s}$ extend to give an equivalence between the category of all (D, F)-framed sheaves that are globally generated along the fibers of π and the category of all F-prolonged coherent Higgs sheaves on X. We will not be needing this equivalence so we did not include the necessary refinements of the arguments.

(ii) Minets has shown [Min18, Theorem 7.14] that there is a natural isomorphism between the moduli stack of stable Higgs triples $(\mathcal{E}, \alpha, \vartheta)$ with $\mathcal{E}$ of pure dimension zero and the moduli stack of torsion free (D, F)-framed sheaves on S which are semistable of slope 0 on the general fibers of π.

Repeating the constructions of the previous theorem in families shows that the stack of pure F-prolonged Higgs bundles on X is equivalent to the stack of torsion free (D, F)-framed sheaves on S which are globally generated on every fiber of π. So both [Min18, Theorem 7.14] and Theorem 3.2.2 can be viewed as extensions of the classical spectral correspondence for Higgs bundles. Still, at least two aspects of this story remain unsatisfactory. First the relationship between these two generalizations of the spectral construction remains unclear even though both constructions interpret framed sheaves in terms of extended HIggs data. Secondly, it is not immediately clear what the correct stability condition on the F-prolonged bundles should be. That is - it is unclear which stability for F-prolonged bundles will correspond to the natural Huybrechts-Lehn stability [HuLe95a, HuLe95b] of framed sheaves on S.

Remark 3.2.5. As a first pass at understanding the puzzles of Remark 3.2.4**(ii)** we observe that Minets's notion of a Higgs triple and our notion of an F-prolonged bundle formally describe the same type of data. Indeed, if (V, ξ, ρ) is an F-prolonged bundle, then the extension $\xi \in \mathrm{Ext}^1(F, V) = \mathrm{Hom}(F, V[1])$ corresponds to a distinguished triangle

$$W \to F \xrightarrow{\xi} V[1] \to W[1]$$

in $\mathcal{D}^b(X)$. So under the identification $W = \mathrm{cone}(\xi)[-1]$, the morphism ρ becomes a map

$$\rho : V \to \mathrm{cone}(\xi)[-1] \otimes \omega,$$

hence

$$\rho[1] : V[1] \to \mathrm{cone}(\xi) \otimes \omega.$$

Thus the data $(V,\ \xi,\ \rho)$ is equivalent to the data $(\mathcal{E} := V[1],\ \alpha := \xi,\ \vartheta := \rho[1])$. Observe that this is formally the same type of data as Minets's Higgs triple, since

$$\mathrm{cone}(\alpha) = \begin{bmatrix} F \\ \downarrow \\ \mathcal{E} \end{bmatrix} [1].$$

This suggests that one should search for a relationship between Higgs triples and F-prolonged Higgs bundles in the derived category of X, and that the correct stability notion for the F-prolonged Higgs bundles should also become visible there

In the next chapter we will resolve both puzzles of Remark 3.2.4**(ii)** by showing that both Higgs triples of pure dimension zero and pure F-prolonged Higgs bundles are Higgs triples of perverse coherent sheaves on X, and that their stability is the corresponding polynomial stability in the sense of Bayer [Bay09].

Chapter 4

Polynomial stability of triples

4.1 The large volume perversity and its dual

We begin by recalling some standard facts about coherent perverse sheaves and polynomial stability, following [ArBez10, Kash94, Bay09].

Definition 4.1.1. A function $p : \{0, 1, \ldots, n\} \to \mathbb{Z}$ is called a *perversity function* if p is monotone decreasing and $\bar{p} : \{0, 1, \ldots, n\} \to \mathbb{Z}$, called the *dual perversity*, given by $\bar{p}(d) = -d - p(d)$ is also monotone decreasing.

Let X be a smooth projective variety of dimension n. Given a perversity function, we can consider the filtration of $\operatorname{Coh} X$ by abelian categories

$$\mathcal{A}^{p,\leq k} = \{\mathcal{F} \in \operatorname{Coh} X : \ p(\dim \operatorname{supp} \mathcal{F}) \geq -k\}.$$

Theorem 4.1.2 ([ArBez10], [Kash94])**.** *If p is a perversity function, then the fol-*

lowing pair defines a bounded t-structure on $\mathcal{D}^b(X)$*:*

$$\mathcal{D}^{p,\leq 0} = \{\mathcal{F} \in \mathcal{D}^b(X) :\ H^{-k}(\mathcal{F}) \in \mathcal{A}^{p,\leq k},\ \textit{for all } k \in \mathbb{Z}\}$$

$$\mathcal{D}^{p,\geq 0} = \{\mathcal{F} \in \mathcal{D}^b(X) :\ \mathrm{Hom}(A,\, \mathcal{F}) = 0,\ \textit{for all } A \in \mathcal{A}^{p,\leq k}[k+1],\, k \in \mathbb{Z}\}$$

Objects in the heart $\mathcal{A}^p = \mathcal{D}^{p,\leq 0} \cap \mathcal{D}^{p,\geq 0}$ *are called perverse coherent sheaves.*

Now if X is a smooth projective variety of dimension n, we have two natural perversity functions for X:

- The *large volume perversity* $p : \{0, 1, \ldots, n\} \to \mathbb{Z}$ given by $p(d) = -[\frac{d}{2}]$, and
- The *dual to large volume perversity* $\bar{p} : \{0, 1, \ldots, n\} \to \mathbb{Z}$, $\bar{p}(d) = -d - p(d)$.

We are specifically interested in the case when X is a curve. In this case we have

- The *large volume perversity* $p : \{0, 1\} \to \mathbb{Z}$, given by $p(0) = p(1) = 0$, and
- The *dual to large volume perversity* $\bar{p} : \{0, 1\} \to \mathbb{Z}$, given by $\bar{p}(0) = 0, \bar{p}(1) = -1$.

Let's trace the abelian category construction for these perversity functions.

Proposition 4.1.3. *Let* X *be a smooth compact curve. Then the large volume perversity* p *and its dual* $\bar{p}$ *define the following abelian categories of perverse coherent sheaves:*

- $\mathcal{A}^p(X) = \mathrm{Coh}(X)$

- $\mathcal{A}^{\bar{p}}(X) = \{\mathcal{F}_{-1} \to \mathcal{F}_0 : \ \mathcal{F}_{-1}$ *is torsion free, and* $\mathrm{coker}(\mathcal{F}_{-1} \to \mathcal{F}_0)$ *is torsion*$\}$

Proof. For the large volume perversity, we have that

$$\mathcal{A}^{p,\leq k} = \{\mathcal{F} \in \mathrm{Coh}\, X : \ 0 \geq -k\} = \begin{cases} \mathrm{Coh}\, X, & \text{if } k \geq 0 \\ 0, & \text{if } k < 0 \end{cases}$$

As a result,

$$\begin{aligned} \mathcal{D}^{p,\leq 0} &= \{\mathcal{F} \in \mathcal{D}^b(X) : \ H^{-k}(\mathcal{F}) \in \mathcal{A}^{p,\leq k}, \text{ for all } k \in \mathbb{Z}\} \\ &= \{\mathcal{F} \in \mathcal{D}^b(X) : \ H^{-k}(\mathcal{F}) = 0 \text{ for all } k < 0\} \\ &= \{\mathcal{F} \in \mathcal{D}^b(X) : \ H^{k}(\mathcal{F}) = 0 \text{ for all } k > 0\} \\ &= \mathcal{D}^{\leq 0}(X) \end{aligned}$$

Now, since

$$\mathcal{A}^{p,\leq k}[k+1] = \begin{cases} \mathrm{Coh}\, X[k+1], & \text{if } k \geq 0 \\ 0, & \text{if } k < 0 \end{cases}$$

we have that

$$\mathcal{D}^{p,\geq 0} = \{\mathcal{F} \in \mathcal{D}^b(X) : \ H^{k}(\mathcal{F}) = 0 \text{ for all } k < 0\}$$

and hence

$$\mathcal{A}^p = \mathrm{Coh}\, X \subset \mathcal{D}^b(X).$$

Now for the dual to the large volume perversity, $\bar{p}$, we get that:

$$\mathcal{A}^{\bar{p},\leq k} = \{\mathcal{F} \in \mathcal{C}\wr\langle\, X : \ \bar{p}(\dim \operatorname{supp} \mathcal{F}) \geq -k\} = \begin{cases} \operatorname{Coh} X & \text{if } k \geq 1 \\ \\ \operatorname{Tor} \operatorname{Coh} X, & \text{if } k = 0 \\ \\ 0, & \text{if } k < 0 \end{cases}$$

so

$$\begin{aligned} \mathcal{D}^{\bar{p},\leq 0} &= \{\mathcal{F} \in \mathcal{D}^b(X) : \ H^{-k}(\mathcal{F}) \in \mathcal{A}^{\bar{p},\leq k}, \text{ for all } k \in \mathbb{Z}\} \\ &= \left\{ \mathcal{F} \in \mathcal{D}^b(X) : \ H^{-k}(\mathcal{F}) = \begin{cases} 0, & \text{if } k < 0 \\ \\ \text{torsion}, & \text{if } k = 0 \end{cases} \right\} \\ &= \left\{ \mathcal{F} \in \mathcal{D}^b(X) : \ H^{k}(\mathcal{F}) = \begin{cases} 0, & \text{if } k > 0 \\ \\ \text{torsion}, & \text{if } k = 0 \end{cases} \right\} \\ &= \{\cdots \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to 0 : \ \operatorname{coker}(\mathcal{F}_{-1} \to \mathcal{F}_0) \text{ is torsion}\} \end{aligned}$$

Now since

$$\mathcal{A}^{\bar{p},\leq k}[k+1] = \begin{cases} \operatorname{Coh} X[k+1] & \text{if } k \geq 1 \\ \\ \operatorname{Tor} \operatorname{Coh} X[1], & \text{if } k = 0 \\ \\ 0, & \text{if } k < 0 \end{cases}$$

we get that

$$\mathcal{D}^{\bar{p},\geq 0} = \{0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \cdots \ : \ \mathcal{F}_{-1} \text{ is torsion free}\}.$$

As a result

$$\mathcal{A}^{\bar{p}} = \{\mathcal{F}_{-1} \to \mathcal{F}_0 : \ \mathcal{F}_{-1} \text{ is torsion free, and } \operatorname{coker}(\mathcal{F}_{-1} \to \mathcal{F}_0) \text{ is torsion}\} \subset \mathcal{D}^b(X).$$

This completes the proof of the proposition. $\square$

It is also instructive to note that these two abelian categories are related (see also [ArBez10, Bay09]) via the Serre duality functor:

Lemma 4.1.4. *Let $\mathbb{D} : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ be the Serre duality functor, i.e. the functor given by*

$$\mathbb{D}(\mathcal{F}) := \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}, \omega_X[1]).$$

Then $\mathcal{A}^{\bar{p}}(X) = \mathbb{D}\mathcal{A}^p(X)$.

Proof. Since $\mathcal{D}^b(X)$ is of homological dimension 1, every object in the category is quasi-isomorphic to the sum of its cohomology. Thus we can describe $\mathcal{A}^{\bar{p}}(X)$ alternatively, as

$$\mathcal{A}^{\bar{p}}(X) = \{\mathcal{F} \in \mathcal{D}^b(X) : \ \mathcal{F} \simeq K[1] \oplus Q, \ K \text{ locally free, and } Q \text{ torsion}\}.$$

But if $\mathcal{F} \in \mathcal{A}^p(X) = \mathrm{Coh}\, X$, then (non-canonically) $\mathcal{F} \simeq T_{\mathcal{F}} \oplus V_{\mathcal{F}}$, where $V_{\mathcal{F}}$ is locally free, and $T_{\mathcal{F}}$ is torsion. Then, applying the Serre duality functor gives $\mathbb{D}\mathcal{F} = \mathbb{D}T_{\mathcal{F}} \oplus \mathbb{D}V_{\mathcal{F}}$. But $\mathbb{D}V_{\mathcal{F}} = V_{\mathcal{F}}^{\vee} \otimes \omega_X[1]$. Also, if we resolve $T_{\mathcal{F}}$ by locally free sheaves

$$0 \to A \to B \to T_{\mathcal{F}} \to 0,$$

then in $\mathcal{D}^b(X)$ we have that $T_{\mathcal{F}} \simeq \begin{bmatrix} A^{-1} \\ \downarrow \\ B^0 \end{bmatrix}$, so

$$\mathbb{D}T_{\mathcal{F}} = \begin{bmatrix} B^\vee \\ \\ \downarrow \\ \\ A^\vee \end{bmatrix} \otimes \omega_X[1] = \begin{bmatrix} B^\vee \otimes \omega_X \\ \\ \downarrow \\ \\ A^\vee \otimes \omega_X \end{bmatrix}.$$

Now, $B^\vee \to A^\vee$ is an injective homomorphism of locally free sheaves of the same rank. As a result, the cokernel $Q = \operatorname{coker}(B^\vee \to A^\vee)$ is a torsion sheaf. This implies that $\mathbb{D}T_{\mathcal{F}} = Q \otimes \omega_X$ and hence $\mathbb{D}\mathcal{A}^p(X) \subset \mathcal{A}^{\bar{p}}(X)$.

It is also obvious from the calculation that the Serre duality functor is essentially surjective, since if $\mathcal{F} = K[1] \oplus Q$, then if we resolve Q by locally free sheaves

$$0 \to M \to N \to Q \to 0$$

and take $T = \operatorname{coker}(N^\vee \to M^\vee) \otimes \omega_X^{-1}$, then $\mathcal{E} = K \otimes \omega_X^{-1} \oplus T$ will satisfy $\mathbb{D}\mathcal{E} = \mathcal{F}$. Now, since the Serre duality functor $\mathbb{D} : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is fully faithful, and $\mathcal{A}^b(X) \subset \mathcal{D}^b(X)$ is a full subcategory, we get that $\mathbb{D} : \mathcal{A}^p(X) \to \mathcal{A}^{\bar{p}}(X)$ is fully faithful and essentially surjective, and hence an equivalence. $\square$

Remark 4.1.5. Let A, $B \in \mathcal{A}^{\bar{p}}(X)$. Then a map $f : A \to B$ in $\mathcal{A}^{\bar{p}}$ corresponds to a triple of maps

- $f_{-1} : \ker a \to \ker b$
- $f_0 : \operatorname{coker} a \to \operatorname{coker} b$

- $f_{0-1} : \operatorname{coker} a[-1] \to \operatorname{coker} b[-1]$

This map is a monomorphism in $\mathcal{A}^{\bar{p}}$ if for any pair of maps

$$\begin{pmatrix} g_{-1} & g_{0-1} \\ 0 & g_0 \end{pmatrix}, \begin{pmatrix} h_{-1} & h_{0-1} \\ 0 & h_0 \end{pmatrix} : \ker c[1] \oplus \operatorname{coker} c \to \ker a[1] \oplus \operatorname{coker} a$$

such that

$$\begin{cases} f_{-1}g_{-1} = f_{-1}h_{-1} \\ f_0 g_0 = f_0 h_0 \\ f_{-1}g_{0-1} + f_{0-1}g_0 = f_{-1}h_{0-1} + f_{0-1}h_0 \end{cases}$$

we have that $g = h$. But if this holds, f_{-1} and f_0 are injective. Conversely, if f_0 and f_{-1} are injective, then for every pair of g, h satisfying the three conditions above, from the first two conditions, we get that $g_{-1} = h_{-1}$ and $g_0 = h_0$, and by the third one (and the injectivity of f_{-1}), we get that $g_{0-1} = h_{0-1}$. As a result, the map $f : A \to B$ is injective if and only if it induces injective maps on cohomology. Some useful instances of this fact are the following:

(i) If $\mathcal{E} \simeq 0 \to \mathcal{E}_0 \in \mathcal{A}^{\bar{p}}(X)$ is a torsion sheaf, viewed as an object in $\mathcal{A}^{\bar{p}}(X)$, then $\mathcal{E}' \to \mathcal{E}$ is injective if and only if $\mathcal{E}' \simeq 0 \to \mathcal{E}'_0$ and $\mathcal{E}'_0 \subset \mathcal{E}_0$.

(ii) If $\mathcal{E} \simeq \mathcal{E}_{-1} \to 0 \in \mathcal{A}^{\bar{p}}(X)$ is a vector bundle in degree -1, viewed as an object in $\mathcal{A}^{\bar{p}}(X)$, then $\mathcal{E}' \to \mathcal{E}$ is injective if and only if $\mathcal{E}' \simeq \mathcal{E}'_{-1} \to 0$ and $\mathcal{E}'_{-1} \subset \mathcal{E}_{-1}$.

4.2 Polynomial stability conditions

Following Bayers [Bay09], for a smooth projective variety X, one can utilize perversity functions, to define polynomial valued stability conditions given by polynomial valued charge

$$Z : K(X) \simeq K(\mathcal{D}^b(X)) \to \mathbb{C}[m].$$

In order to define such central charge, a quadruple of data $\Omega = (\eta,\, \rho,\, p,\, U)$ is required. Namely, the ingredients are

- an ample class $\eta \in A^1(X)_{\mathbb{R}}$, i.e. a Weil divisor $\eta \in A^1(X)_{\mathbb{R}}$ such that $\eta^d \cdot \alpha > 0$ for every effective class $\alpha \in A_d(X)$,
- a stability vector $\rho = (\rho_0,\, \rho_1,\, \ldots,\, \rho_n) \in (\mathbb{C}^*)^{n+1}$, i.e. a vector of non-zero complex numbers such that $\frac{\rho_d}{\rho_{d+1}}$ is in the open upper half-plane, for each $d \in \{0,\, 1\, \ldots,\, n-1\}$,
- a perversity function p associated to ρ, i.e. a perversity function p such that $(-1)^{p(d)}\rho_d$ is the semi-closed upper half-plane, for each $d \in \{0, \ldots,\, n\}$,
- a unipotent operator $U \in A^\bullet(X)_{\mathbb{C}}$, i.e. $U = 1 + N$, where N is concentrated in positive degrees.

Given the data $\Omega = (\eta,\, \rho,\, p,\, U)$ the central charge is defined as

$$Z_\Omega : K(X) \to \mathbb{C}[m], \quad Z_\Omega(\mathcal{F})(m) = \int_X \sum_{d=0}^{n} \rho_d\, \eta^d m^d \cdot \mathrm{ch}(\mathcal{F}) \cdot U.$$

4.2.1 The large volume stability

Now let's fix again our smooth compact curve X. Then the large volume perversity function is associated to the stability vector $\rho = (-1, i)$. So by fixing an ample class $\eta \in A^1(X)_{\mathbb{R}}$ and the unipotent operator $e^{-\beta}\sqrt{\operatorname{td} X}$, where $\beta \in A^1(X)_{\mathbb{R}}$ is an arbitrary class, we get a polynomial valued central charge $Z_\Omega : \mathcal{A}^p(X) \to \mathbb{C}[m]$

$$\begin{aligned} Z_\Omega(\mathcal{F})(m) &= \int_X \sum_{d=0}^{1} \rho_d\, \eta^d m^d \cdot \operatorname{ch}(\mathcal{F}) \cdot e^{-\beta}\sqrt{\operatorname{td} X} \\ &= -\int_X e^{-\beta - mi\eta} \operatorname{ch}(\mathcal{F}) \cdot \sqrt{\operatorname{td} X} \end{aligned}$$

Now, since $e^{-\beta - mi\eta} = 1 - \beta - mi\eta$ and

$$\begin{aligned} -\operatorname{ch}(\mathcal{F}) \cdot \sqrt{\operatorname{td} X} &= (\operatorname{rk}\mathcal{F} + c_1(\mathcal{F}))(1 - \tfrac{1}{4}\omega) \\ &= \operatorname{rk}\mathcal{F} + c_1(\mathcal{F}) - \frac{\operatorname{rk}\mathcal{F}}{4}\omega \end{aligned}$$

we can explicitly compute the central charge

$$\begin{aligned} Z_\Omega(\mathcal{F})(m) &= -\int_X \left(\operatorname{rk}\mathcal{F} + (-\beta \operatorname{rk}\mathcal{F} + c_1(\mathcal{F}) - \tfrac{\operatorname{rk}\mathcal{F}}{4}\omega - i\operatorname{rk}\mathcal{F}\eta m)\right) \\ &= -\int_X \left((\operatorname{rk}\mathcal{F}(-\beta - \tfrac{\omega}{4}) + c_1(\mathcal{F})) - i\operatorname{rk}\mathcal{F} \cdot \eta m\right) \\ &= \operatorname{rk}\mathcal{F}(\deg\beta + \tfrac{1}{2}(g-1)) - c_1(\mathcal{F}) + i\operatorname{rk}\mathcal{F} \cdot \deg\eta \cdot m \end{aligned}$$

Remark 4.2.1. Let's look at a few special choices for the classes η, $\beta \in A^1(X)_{\mathbb{R}}$

(i) If we choose η to be of degree 1, i.e a generator of the positive cone in $H^2(X, \mathbb{Z})$, and we set $\beta = \frac{1}{4}\omega$, then

$$Z(\mathcal{F})(m) = -\chi(\mathcal{F}) + i\mathrm{rk}\,\mathcal{F}\cdot m.$$

So if we look at the η-Hilbert polynomial of $\mathcal{F}$,

$$\chi(\mathcal{F}(m\cdot\eta)) = \chi(\mathcal{F}) + m\cdot\deg\eta\cdot\mathrm{rk}\,\mathcal{F},$$

then the central charge is $\rho_0\chi(\mathcal{F}) + \rho_1 m\mathrm{rk}\,\mathcal{F}$.

(ii) If we take $\deg\eta = 1$ and $\beta = -\frac{1}{4}\omega$, then

$$Z(\mathcal{F})(m) = -c_1(\mathcal{F}) + i\mathrm{rk}\,\mathcal{F}\cdot m$$

which is a polynomial version of Bridgeland's central charge on curves.

(iii) In fact for any value of β we see that the argument of $\phi \in (-1, 0]$ of the central charge is of the form $-\operatorname{arccot}(\mu(\mathcal{F}) + \mathrm{const})/\pi$. where $\mu(\mathcal{F})$ is the Mumford slope of $\mathcal{F}$. Thus a coherent sheaf is stable for the large volume stability if and only if it is a Mumford stable vector bundle on X.

4.2.2 The dual large volume stability

Consider the parity operator $P : A_\bullet(X) \to A_\bullet(X)$, which acts on $A_d(X)$ by $(-1)^{n-d}$ (where n is the dimension of the smooth projective variety X). Given a quadruple of data $\Omega = (\eta, \rho, p, U)$ for a polynomial central charge Z_Ω, we can define the dual data $\Omega^\vee = (\eta, \rho^\vee, \bar{p}, U^\vee)$, where

- the dual stability vector is given by $\rho_d^\vee = (-1)^{D+d}\bar{\rho}_d$,
- $U^\vee = (-1)^D \mathrm{ch}(\omega_X)^{-1} P(\bar{U})$
- D is such that $\omega_X|_{X^{\mathrm{smooth}}}$ is the shift of a line bundle by D,

and the dual central charge $Z^\vee$ is defined by the usual formula, but now, with respect to the data $\Omega^\vee$, i.e. $Z^\vee = Z_{\Omega^\vee} : K(X) \to \mathbb{C}[m]$.

Now, applying this for the case of a smooth compact curve X, and Ω the data of the large volume stability, we get

$$\rho_d^\vee = (-1)^{1+d}\bar{\rho}_d, \quad \text{i.e. } \rho_0^\vee = 1,\ \rho_1^\vee = -i$$

and

$$\begin{aligned}
U^\vee &= -\mathrm{ch}(\omega[1])^{-1} P(e^{-\beta}\sqrt{\mathrm{td}\, X}) \\
&= \mathrm{ch}(\omega)^{-1} P(e^{-\beta}\sqrt{\mathrm{td}\, X}) \\
&= (1-\omega) P(1 + (-\beta - \tfrac{1}{4}\omega)) \\
&= (1-\omega)(-1 + (-\beta - \tfrac{1}{4}\omega)) \\
&= -1 + (\omega - \beta - \tfrac{1}{4}\omega) \\
&= -1 + (\tfrac{3}{4}\omega - \beta)
\end{aligned}$$

Now the central charge $Z^\vee : K(X) \to \mathbb{C}[m]$ defined by $\Omega^\vee$ is a polynomial stability condition on $\mathbb{D}\mathcal{A}^p(X) = \mathcal{A}^{\bar{p}}(X)$. By definition:

$$\begin{aligned} Z^\vee(\mathcal{F})(m) &= \int_X \sum_{d=0}^{1} \rho_d^\vee \eta^d m^d \operatorname{ch}(\mathcal{F}) \cdot U^\vee \\ &= \rho_0^\vee(\operatorname{ch}(\mathcal{F}) \cdot U^\vee) + \rho_1^\vee m(\eta \cdot (\operatorname{ch}(\mathcal{F})U^\vee)_0) \\ &= \operatorname{ch}(\mathcal{F}) \cdot U^\vee - mi\deg \eta(\operatorname{ch}(\mathcal{F}) \cdot U^\vee)_0 \end{aligned}$$

But

$$\begin{aligned} \operatorname{ch}(\mathcal{F}) \cdot U^\vee &= (\operatorname{rk}\mathcal{F} + c_1(\mathcal{F}))(-1 + (\tfrac{3}{4}\omega - \beta)) \\ &= -\operatorname{rk}\mathcal{F} + (\operatorname{rk}\mathcal{F}(\tfrac{3}{4}\omega - \beta) - c_1(\mathcal{F})) \end{aligned}$$

So

$$Z^\vee(\mathcal{F})(m) = -c_1(\mathcal{F}) + \operatorname{rk}\mathcal{F} \cdot \deg(\tfrac{3}{4}\omega - \beta) + mi\operatorname{rk}\mathcal{F}.$$

Examining a few interesting cases we see familiar central charge functions:

(i) for $\beta = \frac{1}{4}\omega$ we get

$$Z^\vee(\mathcal{F})(m) = -c_1(\mathcal{F}) + \operatorname{rk}\mathcal{F} \cdot (g-1) + mi\operatorname{rk}\mathcal{F},$$

(ii) for $\beta = -\frac{1}{4}\omega$ we get

$$Z^\vee(\mathcal{F})(m) = -c_1(\mathcal{F}) + \operatorname{rk}\mathcal{F} \cdot (2g-2) + mi\operatorname{rk}\mathcal{F},$$

(iii) for $\beta = \frac{3}{4}\omega$ we get

$$Z^\vee(\mathcal{F})(m) = -c_1(\mathcal{F}) + mi\operatorname{rk}\mathcal{F}.$$

Again, different choices of β modify the corresponding just by a shift by a constant and so the stability corresponding to this central charge is independent of the value of β.

4.2.3 Recovering stability of Higgs triples

Let $(\mathcal{E}, \alpha, \vartheta)$ be a Higgs triple with $\mathcal{E}$ pure of dimension zero, i.e. a Higgs triple for which $\mathcal{E}$ is a torsion sheaf on X. Then both $\mathcal{E}$ and cone(α) are objects in $\mathcal{A}^{\bar{p}}$, i.e. are perverse coherent sheaves for the dual large volume perversity, and $\vartheta : \mathcal{E} \to \text{cone}\, \alpha \otimes \omega$ is a map of perverse coherent sheaves. This suggests that stability for Higgs triples (at least the ones for which $\mathcal{E}$ is torsion may be related to the polynomial stability for Higgs triples of perverse coherent sheaves. To make a precise statement we introduce the following notion.

Definition 4.2.2. A *Higgs triple of perverse coherent sheaves* (for the dual large volume perversity is a triple $(\mathcal{E}, \alpha, \vartheta)$, where

- $\mathcal{E} \in \mathcal{A}^{\bar{p}}$ is a perverse coherent sheaf.
- $\alpha : F \to \mathcal{E}$ is a map in $\mathcal{D}^b(X)$, such that $\text{cone}(\alpha) \in \mathcal{A}^{\bar{p}}$ is perverse coherent.
- $\vartheta : \mathcal{E} \to \text{cone}(\alpha) \otimes \omega$ is a map of perverse coherent sheaves.

Now note that Bayer's central charge for the dual large volume perversity gives rise to a natural stability of Higgs triples of perverse coherent sheaves. Concretely let ϕ

be the argument of the normalized central charge $Z^\vee(E)(m) = -c_1(E) + im\mathrm{rk}(E)$.

Definition 4.2.3. We say that a Higgs triple $(\mathcal{E}, \alpha, \vartheta)$ of perverse coherent sheaves is *stable* if for every non-trivial subtriple $(\mathcal{E}', \alpha', \vartheta') \subset (\mathcal{E}, \alpha, \vartheta)$ we have that $\phi(\mathcal{E}') < \phi(\mathcal{E})$.

Remark 4.2.4. Here a Higgs subtriple of perverse coherent sheaves is a triple $(\mathcal{E}', \alpha', \vartheta')$ such that $\mathcal{E}' \subset \mathcal{E}$, and $\mathrm{cone}(\alpha') \subset \mathrm{cone}(\alpha)$ are monomorphisms of perverse coherent sheaves, such that the diagram

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{\vartheta'} & \mathrm{cone}(\alpha') \otimes \omega \\ \cap & & \cap \\ \mathcal{E} & \xrightarrow[\vartheta]{} & \mathrm{cone}(\alpha) \otimes \omega \end{array}$$

commutes in $\mathcal{A}^{\bar{p}}$

With these notions in place we now have the following comparison statement:

Lemma 4.2.5. *Suppose $(\mathcal{E}, \alpha, \vartheta)$ is a Minets's Higgs triple with $\mathcal{E}$ torsion. View $(\mathcal{E}, \alpha, \vartheta)$ as a Higgs triple of perverse coherent sheaves. Then $(\mathcal{E}, \alpha, \vartheta)$ is stable in the sense of Minets if and only if it is stable as a Higgs triple of perverse coherent sheaves, i.e. in the sense of Definition 4.2.3.*

Proof. For any non-zero Higgs triple with $\mathcal{E}$ torsion we have $\mathrm{rk}\,\mathcal{E} = 0$ and $c_1(\mathcal{E}) = \mathrm{length}(\mathcal{E}) > 0$. Hence $Z^\vee(\mathcal{E})(m) = c_1(\mathcal{E})e^{\pi i}$, and therefore the polynomial slope function for the stability is $\phi(\mathcal{E}) = 1 \in \mathbb{C}[m]$. In particular, $(\mathcal{E},\ \alpha,\ \vartheta)$ is stable iff it has no no-nzero substriples. $\square$

Remark 4.2.6. Note that if $(\mathcal{E}, \alpha, \vartheta)$ is Minets's triple with a non-torsion $\mathcal{E}$, then in general $(\mathcal{E}, \alpha, \vartheta)$ will not be a perverse coherent HIggs triple since in general cone(α) will not be perverse coherent. Hence it is unclear how to compare polynomial stability and Minets's stability of general triples.

4.2.4 Stability of F-prolonged Higgs bundles

As we remarked at the end of the previous chapter, F-prolonged Higgs bundles are formally similar to the Minets's Higgs triples. This formal similarity can be made more precise. Indeed, if (V, ξ, ρ) is an F-prolonged bundle, then the equivalent triple

$$(\mathcal{E} = V[1], \alpha = \xi, \vartheta = \rho[1])$$

is tautologically a Higgs perverse coherent triple. So we could define stability naively and call (V, ξ, ρ) stable if $(V[1], \xi, \rho[1])$ is stable as a Higgs perverse coherent triple. Explicitly, this naive stability means that for every F-prolonged subbundle[1]

[1] (V', xi', ρ') is an F-prolonged subbundle of (V, ξ, ρ) if $V' \subset V$, the extension ξ' is a subextension of ξ, and ρ' and ρ are compatible. The last two conditions mean that we have a commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & W' & \longrightarrow & F & \longrightarrow & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \longrightarrow & V & \longrightarrow & W & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

(V', ξ', ρ') we have $\mu(V') < \mu(V)$, where μ denotes the Mumford slope.

However, it turns out that this naive stability is too crude and is incompatible with the Huybrechts-Lehn stability of the corresponding (D, F)-framed sheaves on S. It turns out that the correct notion of stability requires a different way of associating Higgs perverse coherent triples to F-prolonged Higgs bundles.

To explain this construction we will need to introduce some notation.

Definition 4.2.7. Given an F-prolonged Higgs bundle (V, ξ, ρ) and a positive integer $e > 0$ define the *e-th companion Dolbeault complex* of (V, ξ, ρ) as the two term complex

$$\mathsf{C}_e^\bullet = \left[\, \mathsf{C}_e^{-1} \xrightarrow{\Psi_e} \mathsf{C}_e^0 \,\right],$$

where:

- $\mathsf{C}_e^{-1} = V \otimes \left(\oplus_{i=0}^{e-1} \omega^{\otimes i}\right)$.
- $\mathsf{C}_e^0 = W \otimes \left(\oplus_{i=1}^{e-1} \omega^{\otimes i}\right)$.

---

and

$$\begin{array}{ccc} V' & \xrightarrow{\rho'} & W' \otimes \omega \\ \cap & & \cap \\ V & \xrightarrow{\rho} & W \otimes \omega \end{array}$$

- $\Psi_e = \begin{pmatrix} \rho & i & 0 & \cdots & 0 \\ 0 & \rho & i & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \rho & i \end{pmatrix}$

Note that by definition we have a short exact sequence of complexes of amplitude $[-1, 0]$:

$$0 \longrightarrow \begin{matrix} V \otimes \omega^e \\ \downarrow \\ W \otimes \omega^e \end{matrix} \xrightarrow{\alpha_e} \mathsf{C}^\bullet_{e+1} \longrightarrow \mathsf{C}^\bullet_e \longrightarrow 0.$$

Since the complex

$$\begin{matrix} V \otimes \omega^e & \to & W \otimes \omega^e \\ -1 & & 0 \end{matrix}$$

is quasi-isomorphic to $F \otimes \omega^e$ we can view the map α_e as a map $\alpha_e : F \otimes \omega^e \to \mathsf{C}^\bullet_{e+1}$ in the derived category $\mathcal{D}^b(X)$. Furthermore form the short exact sequence we get a canonical identification

$$\mathsf{C}^\bullet_e = \operatorname{cone}(\alpha_e).$$

Furthermore we have a natural map of complexes

$$\Theta_e : \mathsf{C}^\bullet_{e+1} \to \mathsf{C}^\bullet_e \otimes \omega$$

for which Θ_e^{-1} projects $V \otimes (\oplus_{i=0}^e \omega^i)$ onto $V \otimes (\oplus_{i=1}^e \omega^i)$, and similarly Θ_e^0 projects $W \otimes (\oplus_{i=1}^e \omega^i)$ onto $V \otimes (\oplus_{i=2}^e \omega^i)$.

Now suppose that (V, ξ, ρ) is an F-prolonged bundle, whose rank is a multiple of $n = \operatorname{rank} F$, i.e. suppose that

$$\operatorname{rank} V = ne, \quad \text{for some integer } e > 0.$$

Then for this e we have

$$\operatorname{rank} \mathsf{C}_0^{-1} = ne^2, \quad \operatorname{rank} \mathsf{C}_0^0 = n(e+1)(e-1),$$

$$\operatorname{rank} \mathsf{C}_{0+1}^{-1} = ne(e+1), \quad \operatorname{rank} \mathsf{C}_{0+1}^0 = ne(e+1),$$

Hence $\mathsf{C}_{e+1}^\bullet$ and $\mathsf{C}_e^\bullet$ will be perverse coherent sheaves if and only if Ψ_{e+1} is injective. So if Ψ_{e+1} is injective, the triple

$$\left(\mathsf{C}_{e+1}^\bullet, \alpha_e, \Theta_e\right)$$

is a Higgs perverse coherent triple. With this notation we are now ready to define stability of F-prolonged bundles.

Definition 4.2.8. Let (V, ξ, ρ) be a pure F-prolonged Higgs bundle. Assume $\operatorname{rank} V = ne$. We say that (V, ξ, ρ) is stable if the differential Ψ_{e+1} in the companion Dolbeault complex $\mathsf{C}_{e+1}^\bullet$ is injective and if the companion Higgs perverse coherent triple $(\mathsf{C}_{e+1}^\bullet, \alpha_e, \Theta_e)$ is stable as a Higgs perverse coherent triple, i.e. is stable in the sense of Definition 4.2.3.

With this definition we now have the following

Theorem 4.2.9. *Suppose (V, ξ, ρ) is a pure F-prolonged HIggs bundle and let $\operatorname{rank} V = ne$. Then (V, ξ, ρ) is stable if and only if the corresponding torsion free (D, F)-framed sheaf (E, φ) is semistable of slope e on the general fiber of π.*

Proof. Suppose (V, ξ, ρ) is stable. Then Ψ_{e+1} is injective and the companion Higgs coherent triple $(\mathsf{C}^{\bullet}_{e+1}, \alpha_e, \Theta_e)$ is stable.

Since Ψ_{e+1} is injective it follows that $\mathsf{C}^{\bullet}_{e+1}$ is quasi isomorphic to coker Ψ_{e+1}. But Ψ_{e+1} is an injective morphism of locally free sheaves of the same rank and so coker Ψ_{e+1} is a torsion sheaf. In particular the Higgs perverse coherent triple $(\mathsf{C}^{\bullet}_{e+1}, \alpha_e, \Theta_e)$ is isomorphic to a stable Minets's triple with torsion sheaf. By [Min18, Lemma 7.16] we now conclude that the $(D, F \otimes \omega^e)$ framed sheaf $(\widetilde{E}, \widetilde{\varphi})$ corresponding to $(\mathsf{C}^{\bullet}_{e+1}, \alpha_e, \Theta_e)$ via [Min18, Theorem 7.14] is semistable of slope zero on the general fiber of π. But by the see-saw theorem we know that the (D, F) framed sheaf (E, φ) corresponding to (V, ξ, ρ) via Theorem 3.2.2 satisfies $(E, \varphi) = (\widetilde{E}, \widetilde{\varphi}) \otimes \mathcal{O}_S(eD)$. Therefore (E, φ) is semistable of slope e on the general fiber of π.

Conversely, if (E, φ) is the framed sheaf corresponding to (V, ξ, ρ), and if we assume it is semistable of slope e on the general fiber, then the framed sheaf $(\widetilde{E}, \widetilde{\varphi}) = (E, \varphi) \otimes \mathcal{O}(-eD)$ is semistable of slope 0 on the general fiber. Again by [Min18, Lemma 7.16] it follows that the Higgs triple $(\widetilde{E}, \tilde{\xi}, \tilde{\vartheta})$ corresponding to $(\widetilde{E}, \widetilde{\varphi})$ has $\widetilde{E}$ torsion and is a stable Higgs triple. But by the see-saw theorem this triple is the companion triple of (V, ξ, ρ) and thus Ψ_{e+1} must be injective (since Ψ_{e+1} is torsion and the companion triple must be stable. □

Remark 4.2.10. Assume for simplicity that F is a stable vector bundle on X. The reasoning of [Min18, Proposition 7.10] applies verbatim to show that if we choose $\delta \gg 0$ and $a > 2g - 2 + \delta$, then every (D, F)-framed torsion free sheaf which is globally generated along the fibers is $(D + af, \delta)$ stable in the sense of Huybrechts and Lehn [HuLe95b]. As an immediate corollary we then get that the stack of pure stable F-prolonged Higgs bundles of rank ne is represented by a quasi-projective variety.

Bibliography

[ArBez10] D. Arinkin, R. Bezrukavnikov. Perverse coherent sheaves. Mosc. Math. J. 10 (2010), no. 1, 329, 271. arXiv:math.AG/0005152.

[ATvdB90] M. Artin, J. Tate, M. Van den Bergh. Some algebras associated to automorphisms of elliptic curves. The Grothendieck Festschrift, Vol. I, 3385, Progr. Math., 86, Birkhuser Boston, Boston, MA, 1990.

[BGK02] V, Baranovsky, V. Ginzburg, A. Kuznetsov. Quiver varieties and a non-commutative $\mathbb{P}^2$. Compositio Math. 134 (2002), no. 3, 283318.

[Bay09] A. Bayer. Polynomial Bridgeland stability conditions and the large volume limit, Geom. Topol. 13 (2009), no. 4, 23892425, arXiv:math.AG/0712.1083

[Bon93] A. Bondal. Non-commutative deformations and Poisson brackets on projective spaces. preprint of Max-Planck-Institut fur Mathematik, MPI/93-67.

[DG02] R. Donagi and D. Gaitsgory. The gerbe of Higgs bundles. *Transform. Groups*, 7(2):109–153, 2002.

[Don95] R. Donagi. Spectral covers. In *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, volume 28 of *Math. Sci. Res. Inst. Publ.*, pages 65–86. Cambridge Univ. Press, Cambridge, 1995.

[Don98] R. Donagi. Taniguchi lectures on principal bundles on elliptic fibrations. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 33–46. World Sci. Publishing, River Edge, NJ, 1998.

[Fei01] B. Feix. Hyperkähler metrics on cotangent bundles. *J. Reine Angew. Math.*, 532:33–46, 2001.

[Ga-Ra17] A. Garca-Raboso. A twisted nonabelian Hodge correspondence. J. Differential Geom. 107 (2017), no. 3, 455518.

[Don84] S. Donaldson. Instantons and geometric invariant theory. Comm. Math. Phys. 93 (1984), no. 4, 453460.

[GerSch88] M. Gerstenhaber, S.D. Schack Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (Il Ciocco, 1986), 11264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 247,

[Hit87a] N. Hitchin. The self-duality equations on a Riemann surface. Proc. London Math. Soc. (3) 55 (1987), no. 1, 59126.

[Hit87b] N. Hitchin. Stable bundles and integrable systems. Duke Math. J. 54 (1987), no. 1, 91114.

[HuLe95a] D. Huybrechts, M. Lehn. Stable pairs on curves and surfaces. J. Algebraic Geom. 4 (1995), no. 1, 67104.

[HuLe95b] D. Huybrechts, M. Lehn. Framed modules and their moduli. Internat. J. Math. 6 (1995), no. 2, 297324.

[Kal99] D. Kaledin. A canonical hyperkähler metric on the total space of a cotangent bundle. In *Quaternionic structures in mathematics and physics (Rome, 1999)*, pages 195–230 (electronic). Univ. Studi Roma "La Sapienza", Rome, 1999.

[Kap98] M. Kapranov Heisenberg doubles and derived categories. J. Algebra, 202(2):712744, 1998.

[KKO01] A. Kapustin, A. Kuznetsov, and D. Orlov. Noncommutative instantons and twistor transform. *Comm. Math. Phys.*, 221(2):385–432, 2001.

[LePo97] J. Le Potier. Lectures on Vector Bundles. Cambridge Studies in Advanced Mathematics, 54. Cambridge University Press, 1997.

[Kash94] M. Kashiwara. t-structures on the derived categories of holonomic $\mathcal{D}$-modules and coherent $\mathcal{O}$-modules. Mosc. Math. J., 4(4):847868, 981, 2004. arXiv:math.AG/0302086.

[Min18] A. Minets. Cohomological Hall algebras for higgs torsion sheaves, moduli of triples and sheaves on surfaces. preprint, 2018, 51. arXiv:math.RT/1801.01429v3.

[NekSch98] N. Nekrasov, A. Schwarz. Instantons on noncommutative $\mathbb{R}^4$, and $(2,0)$ superconformal six-dimensional theory. Comm. Math. Phys. 198 (1998), no. 3, 689703.

[Pos93] L. Positselski. Nonhomogeneous quadratic duality and curvature. *Funktsional. Anal. i Prilozhen.*, 27(3):57–66, 96, 1993.

[PoPo05] A. Polishchuk and L. Positselski. Quadratic algebras. University Lecture Series, 37. American Mathematical Society, Providence, RI, 2005. xii+159 pp.

[Sa11] F. Sala Some topics in the geometry of framed sheaves and their moduli spaces. Ph.D. Thesis, SISSA/ University of Lille, 103 pp.

[Sh77] S.S. Shatz The decomposition and specialization of algebraic families of vector bundles. Compositio Math. 35 (1977), no. 2, 163187.

[Sim92] C. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, 75:5–95, 1992.

[Sim97] C. Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 217–281. Amer. Math. Soc., Providence, RI, 1997.

[Sim02] C. Simpson. Algebraic aspects of higher nonabelian Hodge theory. In *Motives, polylogarithms and Hodge theory, Part II (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 417–604. Int. Press, Somerville, MA, 2002.

[Swan96] R. Swan. Hochschild cohomology of quasiprojective schemes. J. Pure Appl. Algebra 110 (1996), no. 1, 5780.