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# Diffusion of New Products in Heterogeneous Populations: Incorporating Stochastic Coefficients

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**Abstract**

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- changes in the market potential over time (Mahajan & Peterson 1978);
- complimentarity, substitutability, contigent & independent relations of the new product with other brands in the market place (Peterson & Mahajan 1978);
- spatial diffusion pattern (Mahajan & Peterson 1979);
- varying word-of-mouth effects (Easingwood, Mahajan & Muller 1983);
- various marketing mix effects including the effect of price on both innovation and imitation coefficients (Robinson and Lakhani 1975) or advertising effect on the innovation coefficient (Horsky and Simon 1983).
- competitive effects (Eliashberg & Jeuland 1982, Fershtman, Mahajan and Muller 1983)


**Disciplines**

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**Comments**

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DIFFUSION OF NEW PRODUCTS IN HETEROGENEOUS POPULATIONS:

INCORPORATING STOCHASTIC COEFFICIENTS*

by

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## I. Introduction

Diffusion models have had a major impact on the literature and practice of marketing science. Following the pioneering work of Bass (1969), which suggested a deterministic model for homogeneous populations, the basic diffusion model has been extended to incorporate:

- changes in the market potential over time (Mahajan & Peterson 1978);
- complimentarity, substitutability, contingent & independent relations of the new product with other brands in the market place (Peterson & Mahajan 1978);
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These extensions and especially incorporation of competition and marketing mix effects, the dominant current trend in the basic diffusion model, ~~is~~ are responsive to management needs. Yet, it is quite surprising that the diffusion research tradition has ignored stochastic modeling considerations. The desirability of a stochastic perspective is especially vital given the (a) uncertainty inherent in all marketing operations as evident by the rapidly changing consumer tastes, unpredictable competitive activities, technology and other environmental conditions, and (b) the heterogeneity of any target adopting population as evident in the growing literature on market segmentation. More specifically, empirical support for the heterogeneity in the diffusion context is offered by Robertson, Fraser & Wind (1972). Although this issue has been addressed by

Jeuland (1979, 1981), his focus is more on parsimonious models and not the long-term forecasting implications and estimation issues, which are the focus of this paper. In addition , to our knowledge, the proposed modeling approach is different from other efforts.

The objective of this paper is to propose a stochastic extension to the traditional deterministic diffusion models. In particular, a random-coefficients diffusion model is proposed and developed (section 2), its estimation discussed (section 3) and its comparison with the conventional deterministic model is presented based on the results of computer simulations which varied the values of the coefficients of innovation and imitation (section 4). The paper concludes with a brief discussion of the managerial implications of the results and proposed directions for future research.

## II. Derivation of the Model - A Stochastic Differential Equations Approach

For an homogeneous population, we assume the following diffusion model:

$$\dot{N}(t) = \frac{dN(t)}{dt} = (p + q\frac{N(t)}{M})(M - N(t)) + \tilde{\delta}(t) \qquad (2.1)$$

where:

$N(t)$ is the cumulative number of adopters at time t
$M$ is the total market potential (ceiling)
$p$ is the coefficient of innovation ($p>o$)
$q$ is the coefficient of imitation ($q>o$)
$\tilde{\delta}(t)$ is a normally distributed model's error term.

This model has been proposed for marketing applications by Bass (1969), and has been tested empirically and found to be useful for a variety of durable goods (Dodds, 1973); Schmittlein and Mahajan (1982).

Defining $X(t) = N(t)/M$ as the cumulative market penetration[1] at time t, then:

$$\dot{X}(t) = \frac{dX(t)}{dt} = (p + qX(t))(1-X(t)) + \tilde{\varepsilon}(t) \qquad (o \leq X(t) \leq 1) \qquad (2.2)$$

In the sequel we shall omit occasionally the time domain for notational simplicity.

To incorporate heterogeneity into our model, we adopt the following standard statistical assumptions. Both $\tilde{p}$ and $\tilde{q}$ are assumed to be stochastic coefficients such that:

$$\tilde{p} = p + \tilde{\varepsilon}_p \qquad (2.3)$$

$$\tilde{q} = q + \tilde{\varepsilon}_q \qquad (2.4)$$

[1] In this paper we shall focus on market penetration as the managerial statistic of interest, thereby assuming that the total market potential can be estimated through an independent method. An alternative viewpoint is that the market potential is determined judgementally as the target market, a procedure which is done often in practice.

The three disturbance terms, $(\tilde{\varepsilon}_p, \tilde{\varepsilon}_q, \tilde{\varepsilon})$, are assumed to be multivariate normally distributed with mean values vector $\underline{o}$, and variance - covariance matrix:

$$\begin{bmatrix} \sigma_p^2 & \rho\sigma_p\sigma_q & o \\ \rho\sigma_p\sigma_q & \sigma_q^2 & o \\ o & o & \sigma^2 \end{bmatrix} \qquad (2.5)$$

A continuous time random process, $\underline{w}(t)$, is said to be a zero-mean white-noise normal process if $\underline{w}(t)$ has a multivariate normal distribution

with $\quad E[\underline{w}(t)] = \underline{o}$ (2.6)

where $\quad E[\ ]$ denotes expectation operator,

and $\quad Cov[\underline{w}(t), \underline{w}(\tau)] = \underline{o}$ for $t \neq \tau$, (2.7)

where $\quad Cov[\ ]$ denotes the covariance operator.

Another continuous time stochastic process, $\underline{u}(t)$, Wiener process, is defined as the integral of the white-noise normal process, that is,

$$\underline{u}(t) = \int_o^t \underline{w}(\eta)d\eta \text{ and } d\underline{u}(t) = \underline{w}(t)dt. \qquad (2.8)$$

It is also multivariate normally distributed (Sage and Melsa, 1971).

In our model, the white-noise normal process,

$$\underline{w}(t) = \begin{bmatrix} \varepsilon_p \\ \varepsilon_q \\ \varepsilon \end{bmatrix} \qquad (2.9)$$

is assumed to be stationary, and the random vector

$$d\underline{u}(t) = \begin{bmatrix} \varepsilon_p dt \\ \varepsilon_q dt \\ \varepsilon dt \end{bmatrix} \qquad (2.10)$$

has a mean values vector

$$E[d\underline{u}(t)] = \underline{o} \qquad (2.11)$$

and variance - covariance matrix[2] (Sage and Melsa, 1971)

$$E[(\underline{du}(t)(\underline{du}(t))^T] = \begin{bmatrix} \sigma_p^2 dt & \rho\sigma_p\sigma_q dt & o \\ \rho\sigma_p\sigma_q\ dt & \sigma_q^2 dt & o \\ o & o & \sigma^2 dt \end{bmatrix} \qquad (2.12)$$

Finally, it can also be shown (Sage and Melsa, 1971) that:

$$\mathrm{Cov}[X(t)\underline{du}(t)] = \mathrm{Cov}[X^2(t)\underline{du}(t)] = \mathrm{Cov}[X^3(t)\underline{du}(t)] = \underline{o} \qquad (2.13)$$

That is, the stochastic variates generated by the Wiener prcesses are independent of $X(t)$, $X^2(t)$ and $X^3(t)$ at any point of time.

Having introduced the stochastic coefficients, we can rewrite now equation (2.2) as:

$$dX = [p + qX(1-X) - pX + \tilde{\varepsilon}_q X(1-X) + \tilde{\varepsilon}_p(1-X) + \tilde{\varepsilon}]dt \qquad (2.14)$$

This equation can be written more generally as:

$$dX = g(X,t)dt + \sum_{i=1}^{3} a_i(X)du_i \qquad (2.15)$$

where $g(X,t) = p + qX(1-X) - pX$

$a_1(X) = 1 - X$

$a_2(X) = X - X^2$

$a_3(X) = 1$

Euqation (2.15) is a stochastic differential equation. In order to solve it for the probability density function of $\tilde{X}(t)$, one needs to obtain the Fokker-Planck partial differential equation whose solution is possible only under narrow circumstances where g( ) is linear in X and the coefficients of $du_i$ (i = 1,3) do not depend upon X at all (Sage and Melsa, 1971). Assuming that $\tilde{X}(t)$ can be approximated by a normal distribution, we shall proceed now to determine the dynamics of the mean $\mu = E[X(t)]$ and the variance $V = V[X(t)]$ of this distribution. Fundamental to this derivation is the Ito differential

[2]Note that the variance - covariance of $\underline{du}(t)$ is proportional to dt rather than $(dt)^2$.

rule (see, for a concise illustration, Kamien and Schwartz, 1981).

Taking expectations on both sides of (2.15) and recalling (2.13) and (2.11) we obtain:

$$E[dX] = dE[X] = [p + (q-p)\ E[X] - qE[X^2]]dt \qquad (2.16)$$

which can be rewritten as an ordinary differential equation:

$$\frac{d\mu}{dt} = p + (q-p)\mu - q(\mu^2 + V), \qquad (2.17)$$

where $\mu$ and V denote the mean value and the variance of X(t), respectively. Thus, we need to generate an additional differential equation for V.

Let

$$\tilde{y} = F(\tilde{X}) = \tilde{X}^2 \qquad (2.18)$$

Expanding by Taylor series implies the following:

$$dy = \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2}(dX)^2 = 2XdX + (dX)^2, \qquad (2.19)$$

Ito Theorem uses the following multiplication rules (see Sage and Melsa, 1971)

$$(du_i)(du_j) = \sigma_{ij}, \ (du_i)^2 = \sigma_i^2, \qquad (2.20)$$

where: $\sigma_{ij}$ = Cov $[du_i, du_j]$ and $\sigma_i^2$ = Var $[du_i]$
Var [ ] denotes the variance operator.

$$(du_i)(dt) = 0, \qquad (2.21)$$

$$(dt)^2 = 0. \qquad (2.22)$$

But,

$$(dX)^2 = (g(X)dt + \sum_{i=1}^{3} a_i(X)du_i)^2 \qquad (2.23)$$

Using (2.20) - (2.22) and (2.12), it can be shown that

$$(dX)^2 = \sum_{i=1}^{3} a_i^2(X)\,\mathrm{Var}[du_i]\,dt + 2\rho\sigma_p\sigma_q a_1(X)a_2(X)\,dt \qquad (2.24)$$

Hence, substituting (2.15) and (2.24) in (2.19) we obtain

$$dy = [2Xg(X) + \sum_{i=1}^{3} a_i^2(X)\,\mathrm{Var}[du_i] + 2\rho\sigma_p\sigma_q a_1(X)a_2(X)]\,dt + 2X\sum_{i=1}^{3} a_i(X)\,du_i. \qquad (2.25)$$

Taking expectations on both sides of (2.25) and dividing by dt, we obtain again through (2.13) and (2.11) that:

$$\frac{dE[X^2]}{dt} = 2E[pX + (q-p)X^2 - qX^3] + \sigma_p^2 E[1-2X + X^2] + \sigma_q^2 E[X^2-2X^3 + X^4] + \sigma^2 + 2\rho\sigma_p\sigma_q E[X-2X^2 + X^3] \qquad (2.26)$$

Hence,

$$\frac{dE[X^2]}{dt} = A + B\mu + CE[X^2] + DE[X^3] + KE[X^4] \qquad (2.27)$$

where

$$\begin{aligned} A &= \sigma_p^2 + \sigma^2 \\ B &= 2p - 2\sigma_p^2 + 2\rho\sigma_p\sigma_q \\ C &= 2q - 2p + \sigma_p^2 + \sigma_q^2 - 4\rho\sigma_p\sigma_q \\ D &= -2q - 2\sigma_q^2 + 2\rho\sigma_p\sigma_q \\ K &= \sigma_q^2 \end{aligned} \qquad (2.28)$$

Note that Equation (2.27) is given in terms of non-central moments. To rewrite it in terms of central moments, we need to use the following relationships:

$$\begin{aligned} E[X^2] &= V + \mu^2 \\ E[X^3] &= E[(X - \mu)^3] + 3\mu V + \mu^3 \\ E[X^4] &= E[(X - \mu)^4] + 4\mu E[(X - \mu)^3] + 6\mu^2 V + \mu^4 \end{aligned} \qquad (2.29)$$

We shall also use the fact that for a normal distribution (the distribution assumed on X(t)):

$$E[(X-\mu)^3] = o \text{ and } E[(X-\mu)^4] = 3V^2. \tag{2.30}$$

Substituting (2.29) in (2.27) and observing that

$$\frac{dE[X^2]}{dt} = \frac{d}{dt}(V+\mu^2) = \frac{dV}{dt} + 2\mu\frac{d\mu}{dt} \tag{2.31}$$

we obtain

$$\frac{dV}{dt} = A + B\mu + C(V+\mu^2) + D(3\mu V + \mu^3) + K(3V^2 + 6\mu^2 V + \mu^4) -$$

$$- 2\mu[p + (q-p)\mu - q(\mu^2 + V)]. \tag{2.32}$$

Substituting from (2.28) and rearranging terms we obtain the second ordinary differential equation:

$$\begin{aligned}\frac{dV}{dt} = \; & \sigma_p^2 + \sigma^2 + 2\sigma_p(\rho\sigma_q - \sigma_p)\mu + (\sigma_p^2 + \sigma_q^2 - 4\rho\sigma_p\sigma_q)\mu^2 + \\ & 2\sigma_q(\rho\sigma_p - \sigma_q)\mu^3 + \sigma_q^2\mu^4 + 2(3\rho\sigma_p\sigma_q - 3\sigma_q^2 - 2q)\mu V + \\ & 6\sigma_q^2\mu^2 V + (2q - 2p + \sigma_p^2 + \sigma_q^2 - 4\rho\sigma_p\sigma_q)V + 3\sigma_q^2 V^2\end{aligned} \tag{2.33}$$

Equation (2.33) needs to be solved simultaneously with

$$\frac{d\mu}{dt} = p + (q-p)\mu - q\mu^2 - qV, \tag{2.17}$$

for some initial conditions $\mu(t_o)$ and $V(t_o)$.

It is interesting to note that becuase of the nonlinearity in X in Equation (2.2), the last term in Equation (2.17) represents the discrepancy in the predicted evolution of the mean penetration between our model, and a model that assumes no heterogenity and solves Equation (2.2) deterministically.

Suppose, for a moment, that the two approaches yield identical estimated parameters; $\hat{p}$, $\hat{q}$. Since both q and V are positive, the mean penetration curve, E[X] (long-range forecasting), predicted by a deterministic solution of Equation (2.2) (i.e., assuming no heterogeneity and a very small model's error term) will always be overestimated. The magnitude of the discrepancy depends, of course, on $\hat{q}$, the imitation coefficient, as well as on the other parameters that affect V. The one-step-ahead forecast, $E[\frac{dX}{dt}|X]$ (short-term forecast), however, will be the same in both cases. These issues will be further investigated and illustrated later. At this point, we turn to a disucssion of estimating the model's parameters.

## III. Estimation Issues

The model of concern here can be written as:

$$\frac{dX(t)}{dt} = \tilde{p}(1-X(t)) + \tilde{q}X(t)(1-X(t)) + \tilde{\varepsilon}(t) \qquad (o \leq t) \qquad (3.1)$$

Approximating the continuous model by its discrete version, we shall rewrite the time domain as a subscript to emphasize more clearly its discrete nature and obtain:

$$\Delta X_t = \tilde{p}(1-X_t) + \tilde{q}X_t(1-X_t) + \tilde{\varepsilon}_t \qquad (t=1,\ 2,\ \ldots,\ T) \qquad (3.2)$$

Defining

$$Y_t = \Delta X_t \qquad (3.3)$$

$$Z_{1t} = 1 - X_t \qquad (3.4)$$

$$Z_{2t} = X_t(1-X_t) \qquad (3.5)$$

$$\tilde{\beta}_1 = \tilde{p} \qquad (3.6)$$

$$\tilde{\beta}_2 = \tilde{q} \qquad (3.7)$$

We obtain the following multiple regression model:

$$\tilde{Y}_t = \tilde{\beta}_1 Z_{1t} + \tilde{\beta}_2 Z_{2t} + \tilde{\varepsilon}_t \quad (t=1,\ \ldots,\ T) \qquad (3.8)$$

This model which is known as a random regression coefficients model has received much attention in the statistical literature (Rao, 1965; Theil 1971; Maddala, 1977; Pfeffermann, 1982). Random coefficients regression models can be developed under a variety of assumptions which may lead to different estimation procedures. Hence, it is important to interpret the underlying conditions generating the random coefficients model. In our case, it is assumed to the vector $\tilde{\underline{\beta}}$ is drawn once from a multivariate normal distribution and is used repeatedly to generate the observations. The disturbance term is assumed to be drawn from a stationary distribution, every time an observation is generated.

As we have shown in (2.33) and (2.17), six parameters are needed to be estimated: p, q, $\sigma_p$, $\sigma_q$, $\rho$, and $\sigma$. The estimated paramters can be used to describe the evolution of the mean, $E[X(t)]$, and thus to make a long-term forecasting. If, however, one is concerned only with short-term forecasting, $E[\Delta X_t | X_t]$, then it can be readily seen from (3.2) and by recalling that $E[\tilde{\varepsilon}_p] = E[\tilde{\varepsilon}_q] = E[\tilde{\varepsilon}] = o$, that only p and q need to be estimated. We shall describe now estimation procedures for all six parameters.

Rewriting the model in matrix notation we obtain:

$$\underset{(TX1)}{\underline{Y}} = \underset{(TX2)}{\underline{Z}} \; \underset{(2X1)}{\tilde{\underline{\beta}}} + \underset{(TX1)}{\tilde{\underline{\varepsilon}}} \tag{3.9}$$

with

$$E[\tilde{\underline{\varepsilon}}] = \underline{o} \qquad E[\underline{\tilde{\varepsilon}}\underline{\tilde{\varepsilon}}^T] = \sigma^2 \underline{I} \tag{3.10}$$

$$E[\tilde{\underline{\beta}}] = \underline{\Omega} = \begin{bmatrix} p \\ q \end{bmatrix} \qquad E[(\underline{\beta} - \underline{\Omega})(\underline{\beta} - \underline{\Omega})^T] = \underline{\Delta} = \begin{bmatrix} \sigma_p^2 & \rho\sigma_p\sigma_q \\ \rho\sigma_p\sigma_q & \sigma_q^2 \end{bmatrix} \tag{3.11}$$

$$\mathrm{Cov}[\underline{\tilde{\varepsilon}\tilde{\beta}}^T] = \underline{o}$$

We can rewrite the regression structure (3.9) in the form

$$\tilde{\underline{Y}} = \underline{Z\Omega} + \underline{Z}(\tilde{\underline{\beta}} - \underline{\Omega}) + \tilde{\underline{\varepsilon}} = \underline{Z\Omega} + \tilde{\underline{\theta}} \tag{3.12}$$

where from assumptions (3.10) and (3.11)

$$E[\tilde{\underline{\theta}}] = \underline{o} \text{ and } E[\underline{\tilde{\theta}\tilde{\theta}}^T] = \underline{Z\Delta Z}^T + \sigma^2 \underline{I} \tag{3.13}$$

This model exhibits heteroskedastic error structure and thus requires a Generalized Least Squares (GLS) approach for obtaining the Minimum Mean Square Linear Unbiased Estimator (MMSLUE) of $\underline{\Omega}$. It has been shown, however, by Rao (1965, Lemma 3) that due to the $\sigma^2\underline{I}$ term in the error structure, the GLS estimator of any linear combination of $\underline{\Omega}$, $\underline{w}^T\underline{\Omega}$, is the same as the Ordinary Least Square (OLS). That is,

$$\underline{w}^T\hat{\underline{\Omega}} = \underline{w}^T(\underline{Z}^T\underline{Z})^{-1}\underline{Z}^T\underline{Y} \tag{3.14}$$

This result is true regardless what $\underline{\Delta}$ is. The variance of the estimator $\underline{w}^T\underline{\Omega}$ depends on $\underline{\Delta}$ and is given by

$$\mathrm{Var}\,[\underline{w}^T\hat{\underline{\Omega}}] = \underline{w}^T\underline{\Delta}\underline{w} + \underline{w}^T\sigma^2(\underline{Z}^T\underline{Z})^{-1}\underline{w} \tag{3.15}$$

Hence, in order to estimate $E[\tilde{\underline{\beta}}] = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}$, one needs to employ an OLS estimation approach to the regression model described in (3.9).

In order to estimate $\sigma_p$, $\sigma_q$, $\sigma_{pq} = \rho\sigma_p\sigma_q$, and $\sigma$ once $\underline{\Omega}$ has been estimated, we shall follow a procedure illustrated by Theil (1971). From (3.8) we know that

$$\varepsilon_t = Y_t - \hat{\Omega}_1 Z_{1t} - \hat{\Omega}_2 Z_{2t} = Y_t - \underline{Z}_t^T\hat{\underline{\Omega}} \tag{3.16}$$

Hence, from (3.15)

$$\mathrm{Var}\,[\varepsilon_t] = \mathrm{Var}\,[\underline{Z}_t^T\hat{\underline{\Omega}}] = \underline{Z}_t^T\underline{\Delta}\underline{Z}_t + \underline{Z}_t^T\sigma^2(\underline{Z}^T\underline{Z})^{-1}\underline{Z}_t \tag{3.17}$$

Equation (3.17) can also be written as:

$$\mathrm{Var}\,[\varepsilon_t] = Z_{1t}^2\sigma_p^2 + Z_{2t}^2\sigma_q^2 + 2Z_{1t}Z_{2t}\sigma_{pq} + \sigma^2\phi_t \tag{3.18}$$

where

$\phi_t = \underline{Z}_t^T(\underline{Z}^T\underline{Z})^{-1}\underline{Z}_t = \phi(\underline{Z}_t, \underline{Z})$ is some function of the independent variables at time t only, as well as the independent variables at times 1, 2, ..., t...., T.

Since $E[\varepsilon_t] = o$, Equation (3.18) implies that

$$E[\varepsilon_t^2] = Z_{1t}^2\sigma_p^2 + Z_{2t}^2\sigma_q^2 + 2Z_{1t}Z_{2t}\sigma_{pq} + \sigma^2\phi_t. \tag{3.19}$$

We can write (3.19) in the form

$$\varepsilon_t^2 = \sigma_p^2 Z_{1t}^2 + \sigma_q^2 Z_{2t}^2 + \sigma_{pq} 2Z_{1t}Z_{2t} + \sigma^2\phi_t + f_t \tag{3.20}$$

where

$$E[f_t] = 0. \tag{3.21}$$

It can also be shown (Theil, 1971, p. 624) that

$$\text{Var}[f_t] = 2(E[\varepsilon_t^2])^2. \tag{3.22}$$

Given that $Z_{1t}$, $Z_{2t}$ and $\phi_t$ are known, the formulation (3.20) suggests that $\sigma_p^2$, $\sigma_q^2$, $\sigma_{pq}$ and $\sigma^2$ can be estimated by running a regression of $\varepsilon_t^2$ on $Z_{1t}$, $Z_{2t}$ and $\phi_t$. It should be noted though that $f_t$ is a heteroskedastic error term. An estimation procedure for such a regression model has been developed and applied by Theil and Mennes (1959). (It is illustrated in Theil, 1971, p. 246). As noted by them, it appears that a rather considerable number of observations is needed to estimate $\sigma_p$, $\sigma_q$, $\sigma_{pq}$, and $\sigma$ with reasonable precision.

## IV. Some Illustrative Simulation Results

As we have shown in equations (2.17) and (2.33), the long-term stochastic coefficients model's prediction depends upon the nature of the randomness surrounding the coefficients (i.e., $\sigma_p$, $\sigma_q$ and $\rho$). As noted earlier, it is quite clear from (2.17) that the discrepancy between long-term forecasting based on stochastic coefficients and a fully deterministic model (i.e., a model that assumes a perfectly homogeneous population) is determined by the magnitude of q, the imitation coefficient, as well as by the variance, V, which in turn depends on $\sigma_p$, $\sigma_q$, $\rho$ and $\sigma$. In this section we further investigate the nature of these discrepancies.

Having established the fact that both models (deterministic and stochastic) will yield the same OLS estimated coefficeints, $\hat{p}$ and $\hat{q}$, we have chosen to investigate possible forecasting discrepancies for different values of $\hat{p}$ (innovation coefficient) and $\hat{q}$ (the imitation coefficient). OLS estimated parameters for $\hat{p}$ and $\hat{q}$ reported in the literature (Bass (1969), Dodds (1973) Easingwood, Mahajan & Muller (1983) and Schmittlein and Mahajan (1982)) are in the range of .000 - .070 for $\hat{p}$ and .150-2.742 for $\hat{q}$. Table 1 summarizes these findings. Given, however, the limited numbers of products for which the $\hat{p}$ and $\hat{q}$ parameters have been established, and the lack of a conceptual boundary for their values, we decided to simulate results for $\hat{p}$ ranging systematically from .01 to .16 and $\hat{q}$ from .25 to 4.00. We begin our analysis, however, with a specific real world example of a medical product whose parameters have been estimated to be: $\hat{p}$=.0572 and $\hat{q}$=1.7888 (Schmittlein and Mahajan, 1982).

In order to focus on the impact of heterogeneity, we assume that $\sigma \ll \sigma_p$, and we solve the two nonlinear differential equations (2.17) and (2.33) with initial conditions $\mu(o) = V(o) = o$, for various values of $\sigma_p$, $\sigma_q$ and $\rho$. This was done using the program DGEAR, one of the IMSL computer subroutines. Solution to the deterministic differential equation

$$\frac{dX}{dt} = (p + qX)\ (1-X) \tag{4.1}$$

can be obtained through integration and shown to be:

TABLE 1

The Reported Ranges of Innovation Coefficient (p)

and Innovation Coefficient (q)

| Study | Products | Ranges of p | Ranges of q | Range of ratio of q over p |
|---|---|---|---|---|
| Bass (1969) | 11 consumer durables | .005876-.028632 | .17110-.65410 | 9-82 |
| Dodds (1973) | color & cable TV | .005447-.008875 | .44168-.83687 | 51-154 |
| Schmittlein & Mahajan (1982) | 3 Durable goods and 2 Radiology products | .020000-.069620 | .32480-1.7888 | 14-31 |
| Easingwood, Mahajan & Muller (1983)[3] | 5 Consumer durables | .000021-.018380 | .15000-2.74200 | 8-130, 571 |

3 Their model assumes that the coefficient of imitation, q, varies over time, and the values reported in Table 1 represent rough approximations only.

$$X(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} . \qquad (4.2)$$

Defining the coefficient of variation for the two model's parameters as

$$k_p = \frac{\sigma_p}{p} \qquad (4.3)$$

and

$$k_q = \frac{\sigma_q}{q},$$

we have examined the effect of different degrees of heterogeneity (captured through the magnitudes of $k_p$ and $k_q$), and the correlation between the model's coefficients, upon the prediction of the stochastic coefficients model, in comparison to the deterministic model. More specifically, $k_p$ and $k_q$ were set equal to 0.1, 0.2 and 0.3, for each of the following values of $\rho$: -0.95, -0.20, 0. +0.20, +0.95. The two models' forecasts were examined at time intervals equal to 0.1. Table 2 shows the prediction obtained by the deterministic model (equation (4.2)). Table 3 illustrates some interesting comparative results.

The first two columns of Table 3 correspond to the coefficients of variation of $\tilde{p}$ and $\tilde{q}$, respectively. The third column shows the correlation coefficient between $\tilde{p}$ and $\tilde{q}$. The fourth column shows the maximum discrepancy (in percentage) between the penetration predicted by deterministic and the stochastic model, whereas the fifth column demonstrates the time at which this maximum discrepancy occurs.

One advantage of the stochastic coefficients model is the feasibility of constructing interval estimates around the mean, $\mu$. As shown in Table 4,

TABLE 2

Predictions Obtained by the Deterministic Model ($\hat{p}$=.0572, $\hat{q}$=1.7888)

| t | X(t) |
|---|---|
| 0.10000 | 0.00626 |
| 0.20000 | 0.01368 |
| 0.30000 | 0.02245 |
| 0.40000 | 0.03280 |
| 0.50000 | 0.04495 |
| 0.60000 | 0.05917 |
| 0.70000 | 0.07572 |
| 0.80000 | 0.09487 |
| 0.90000 | 0.11687 |
| 1.00000 | 0.14195 |
| 1.10000 | 0.17030 |
| 1.20000 | 0.20200 |
| 1.30000 | 0.23707 |
| 1.40000 | 0.27536 |
| 1.50000 | 0.31661 |
| 1.60000 | 0.36040 |
| 1.70000 | 0.40618 |
| 1.80000 | 0.45325 |
| 1.90000 | 0.50084 |
| 2.00000 | 0.54814 |
| 2.10000 | 0.59437 |
| 2.20000 | 0.63881 |
| 2.30000 | 0.68087 |
| 2.40000 | 0.72007 |
| 2.50000 | 0.75611 |
| 2.60000 | 0.78880 |
| 2.70000 | 0.81881 |
| 2.80000 | 0.84414 |
| 2.90000 | 0.86703 |
| 3.00000 | 0.88698 |
| 3.10000 | 0.90427 |
| 3.20000 | 0.91914 |
| 3.30000 | 0.93187 |
| 3.40000 | 0.94271 |
| 3.50000 | 0.95192 |
| 3.60000 | 0.95970 |
| 3.70000 | 0.96627 |
| 3.80000 | 0.97180 |
| 3.90000 | 0.97645 |
| 4.00000 | 0.98035 |
| 4.10000 | 0.98361 |
| 4.20000 | 0.98635 |
| 4.30000 | 0.98863 |
| 4.40000 | 0.99053 |
| 4.50000 | 0.99211 |

TABLE 3

Comparison Between Deterministic and Stochastic Models' Predictions ($\hat{p}$=.0572, $\hat{q}$=1.7888)

| $k_p$ | $k_q$ | $\rho$ | $\Delta_{max}$ (%) | $t_{\Delta max}$ | max $k_x$ | $t_{max\ k_x}$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | -0.95 | 0.298 | 2.5 | 0.28998 | 0.1 |
| 0.1 | 0.2 | -0.95 | 1.232 | 2.5 | 0.26684 | 0.1 |
| 0.1 | 0.3 | -0.95 | 2.998 | 2.5 | 0.31696 | 1.4 |
| 0.2 | 0.1 | -0.95 | 0.495 | 2.4 | 0.60561 | 0.1 |
| 0.2 | 0.2 | -0.95 | 1.207 | 2.5 | 0.58162 | 0.1 |
| 0.2 | 0.3 | -0.95 | 2.743 | 2.5 | 0.55950 | 0.1 |
| 0.3 | 0.1 | -0.95 | 0.986 | 2.3 | 0.92186 | 0.1 |
| 0.3 | 0.2 | -0.95 | 1.479 | 2.4 | 0.89841 | 0.1 |
| 0.3 | 0.3 | -0.95 | 2.826 | 2.5 | 0.87701 | 0.1 |
| 0.1 | 0.1 | -0.20 | 0.474 | 2.4 | 0.31219 | 0.1 |
| 0.1 | 0.2 | -0.20 | 1.594 | 2.5 | 0.31313 | 0.1 |
| 0.1 | 0.3 | -0.20 | 3.602 | 2.5 | 0.35717 | 1.3 |
| 0.2 | 0.1 | -0.20 | 0.852 | 2.4 | 0.62719 | 0.1 |
| 0.2 | 0.2 | -0.20 | 1.945 | 2.4 | 0.62584 | 0.1 |
| 0.2 | 0.3 | -0.20 | 3.958 | 2.5 | 0.62738 | 0.1 |
| 0.3 | 0.1 | -0.20 | 1.535 | 2.3 | 0.94315 | 0.1 |
| 0.3 | 0.2 | -0.20 | 2.629 | 2.4 | 0.94172 | 0.1 |
| 0.3 | 0.3 | -0.20 | 4.696 | 2.5 | 0.94309 | 0.1 |
| 0.1 | 0.1 | 0 | 0.521 | 2.4 | 0.31785 | 0.1 |
| 0.1 | 0.2 | 0 | 1.692 | 2.5 | 0.32435 | 0.1 |
| 0.1 | 0.3 | 0 | 3.765 | 2.5 | 0.36793 | 1.3 |
| 0.2 | 0.1 | 0 | 0.948 | 2.4 | 0.63282 | 0.1 |
| 0.2 | 0.2 | 0 | 2.147 | 2.4 | 0.63711 | 0.1 |
| 0.2 | 0.3 | 0 | 4.293 | 2.5 | 0.64427 | 0.1 |
| 0.3 | 0.1 | 0 | 1.683 | 2.3 | 0.94874 | 0.1 |
| 0.3 | 0.2 | 0 | 2.943 | 2.4 | 0.95293 | 0.1 |
| 0.3 | 0.3 | 0 | 5.221 | 2.5 | 0.95992 | 0.1 |
| 0.1 | 0.1 | 0.20 | 0.574 | 2.4 | 0.32341 | 0.1 |
| 0.1 | 0.2 | 0.20 | 1.796 | 2.5 | 0.33521 | 0.1 |
| 0.1 | 0.3 | 0.20 | 3.930 | 2.5 | 0.37842 | 1.3 |
| 0.2 | 0.1 | 0.20 | 1.044 | 2.4 | 0.63840 | 0.1 |
| 0.2 | 0.2 | 0.20 | 2.351 | 2.4 | 0.64819 | 0.1 |
| 0.2 | 0.3 | 0.20 | 4.634 | 2.5 | 0.66071 | 0.1 |
| 0.3 | 0.1 | 0.20 | 1.831 | 2.3 | 0.95430 | 0.1 |
| 0.3 | 0.2 | 0.20 | 3.259 | 2.4 | 0.96400 | 0.1 |
| 0.3 | 0.3 | 0.20 | 5.758 | 2.5 | 0.97646 | 0.1 |
| 0.1 | 0.1 | 0.95 | 0.751 | 2.4 | 0.34346 | 0.1 |
| 0.1 | 0.2 | 0.95 | 2.167 | 2.5 | 0.37309 | 0.1 |
| 0.1 | 0.3 | 0.95 | 4.559 | 2.5 | 0.41674 | 1.2 |
| 0.2 | 0.1 | 0.95 | 1.408 | 2.4 | 0.65889 | 0.1 |
| 0.2 | 0.2 | 0.95 | 3.127 | 2.4 | 0.68810 | 0.1 |
| 0.2 | 0.3 | 0.95 | 5.953 | 2.5 | 0.71901 | 0.1 |
| 0.3 | 0.1 | 0.95 | 2.393 | 2.4 | 0.97486 | 0.1 |
| 0.3 | 0.2 | 0.95 | 4.478 | 2.4 | 1.00442 | 0.1 |
| 0.3 | 0.3 | 0.95 | 7.882 | 2.5 | 1.03606 | 0.1 |

TABLE 4

Predictions Obtained by the Stochastic Model

($\hat{p}$=.0572, $\hat{q}$=1.7888, $k_p$=0.1, $k_q$=0.1, $\rho$=0.95)

| $t$ | $\mu$ | $V$ | $k=\sqrt{\frac{V}{\mu}}$ |
|---|---|---|---|
| 0.10000 | 0.00624 | 0.00000 | 0.34346 |
| 0.20000 | 0.01365 | 0.00001 | 0.26351 |
| 0.30000 | 0.02241 | 0.00003 | 0.23203 |
| 0.40000 | 0.03274 | 0.00005 | 0.21577 |
| 0.50000 | 0.04487 | 0.00009 | 0.20642 |
| 0.60000 | 0.05906 | 0.00014 | 0.20042 |
| 0.70000 | 0.07556 | 0.00022 | 0.19638 |
| 0.80000 | 0.09463 | 0.00033 | 0.19325 |
| 0.90000 | 0.11652 | 0.00049 | 0.19041 |
| 1.00000 | 0.14143 | 0.00070 | 0.18763 |
| 1.10000 | 0.16955 | 0.00098 | 0.18457 |
| 1.20000 | 0.20096 | 0.00132 | 0.18094 |
| 1.30000 | 0.23563 | 0.00173 | 0.17655 |
| 1.40000 | 0.27343 | 0.00219 | 0.17128 |
| 1.50000 | 0.31410 | 0.00269 | 0.16511 |
| 1.60000 | 0.35722 | 0.00319 | 0.15800 |
| 1.70000 | 0.40225 | 0.00364 | 0.15002 |
| 1.80000 | 0.44853 | 0.00402 | 0.14127 |
| 1.90000 | 0.49536 | 0.00427 | 0.13190 |
| 2.00000 | 0.54196 | 0.00438 | 0.12210 |
| 2.10000 | 0.58761 | 0.00434 | 0.11207 |
| 2.20000 | 0.63163 | 0.00415 | 0.10200 |
| 2.30000 | 0.67343 | 0.00384 | 0.09206 |
| 2.40000 | 0.71256 | 0.00345 | 0.08245 |
| 2.50000 | 0.74869 | 0.00301 | 0.07332 |
| 2.60000 | 0.78164 | 0.00256 | 0.06478 |
| 2.70000 | 0.81134 | 0.00213 | 0.05689 |
| 2.80000 | 0.83783 | 0.00173 | 0.04970 |
| 2.90000 | 0.86123 | 0.00139 | 0.04321 |
| 3.00000 | 0.88174 | 0.00109 | 0.03742 |
| 3.10000 | 0.89957 | 0.00084 | 0.03227 |
| 3.20000 | 0.91499 | 0.00064 | 0.02773 |
| 3.30000 | 0.92823 | 0.00049 | 0.02377 |
| 3.40000 | 0.93954 | 0.00037 | 0.02035 |
| 3.50000 | 0.94918 | 0.00027 | 0.01737 |
| 3.60000 | 0.95735 | 0.00020 | 0.01481 |
| 3.70000 | 0.96426 | 0.00015 | 0.01264 |
| 3.80000 | 0.97009 | 0.00011 | 0.01065 |
| 3.90000 | 0.97499 | 0.00007 | 0.00888 |
| 4.00000 | 0.97911 | 0.00006 | 0.00787 |
| 4.10000 | 0.98257 | 0.00005 | 0.00733 |
| 4.20000 | 0.98546 | 0.00004 | 0.00645 |
| 4.30000 | 0.98788 | 0.00003 | 0.00525 |
| 4.40000 | 0.98989 | 0.00001 | 0.00387 |
| 4.50000 | 0.99158 | 0.00001 | 0.00257 |

for example, a typical evolution of Var [X(t)] is such that it begins at a very small value, achieves a maximum at some point of time, and finally declines toward zero again. The user of the model should, however, be aware of the coefficient of variation of X(t), $k_x = \frac{\sqrt{V_x}}{\mu_x}$ . Recalling that we are assuming normal distribution for X(t), if $k_x$ exceeds 0.33 at a certain point of time, that implies that $3\sigma_x > \mu_x$, which means that there exists a positive probability that X(t) may be either less than zero or greater than one. This is inconsistent with the fact that both X(t) and $\mu$ should be between zero and one. Thus, statements based on confidence intervals will be valid only whenever $\sigma_x << \mu_x$ ($k_x < 0.33$). It should be noted, though, that most of our simulations show that $k_x$ is monotonically decreasing in time. Referring back to Table 3, the sixth column shows the maximum coefficient of variation for the market penetration under the various conditions, whereas the last column demonstrates the time at which this maximum occurs.

The results in Table 3 are quite illustrative. It appears that the discrepancies between the two models' long-run forecasts increase systematically as $\sigma_p$, $\sigma_q$ and $\rho$ increase. For example, for $\sigma_p = 0.3p$, $\sigma_q = 0.3q$ and $\rho = 0.95$, the discrepancy in prediction between the two models becomes 7.88%. Of course, a larger value of q will amplify this discrepancy even more. The large discrepancies occur, in general, towards the mid-point of the time horizon, whereas the forecasts are basically identical near the end points of the time horizon. (Recall that the variance of the stochastic model approaches zero at the beginning and toward the end of period).

To investigate these phenomena further, the values of $\hat{p}$ and $\hat{q}$ were changed systematically. Figures 1 and 2 illustrate the two forecasts for the following cases:

Figure 1

A Comparison of Deterministic and Stochastic Models for:

p=.02 and q=.25-2.50 when $\sigma$ p=.3p; $\sigma$ q=.3q and R=.95

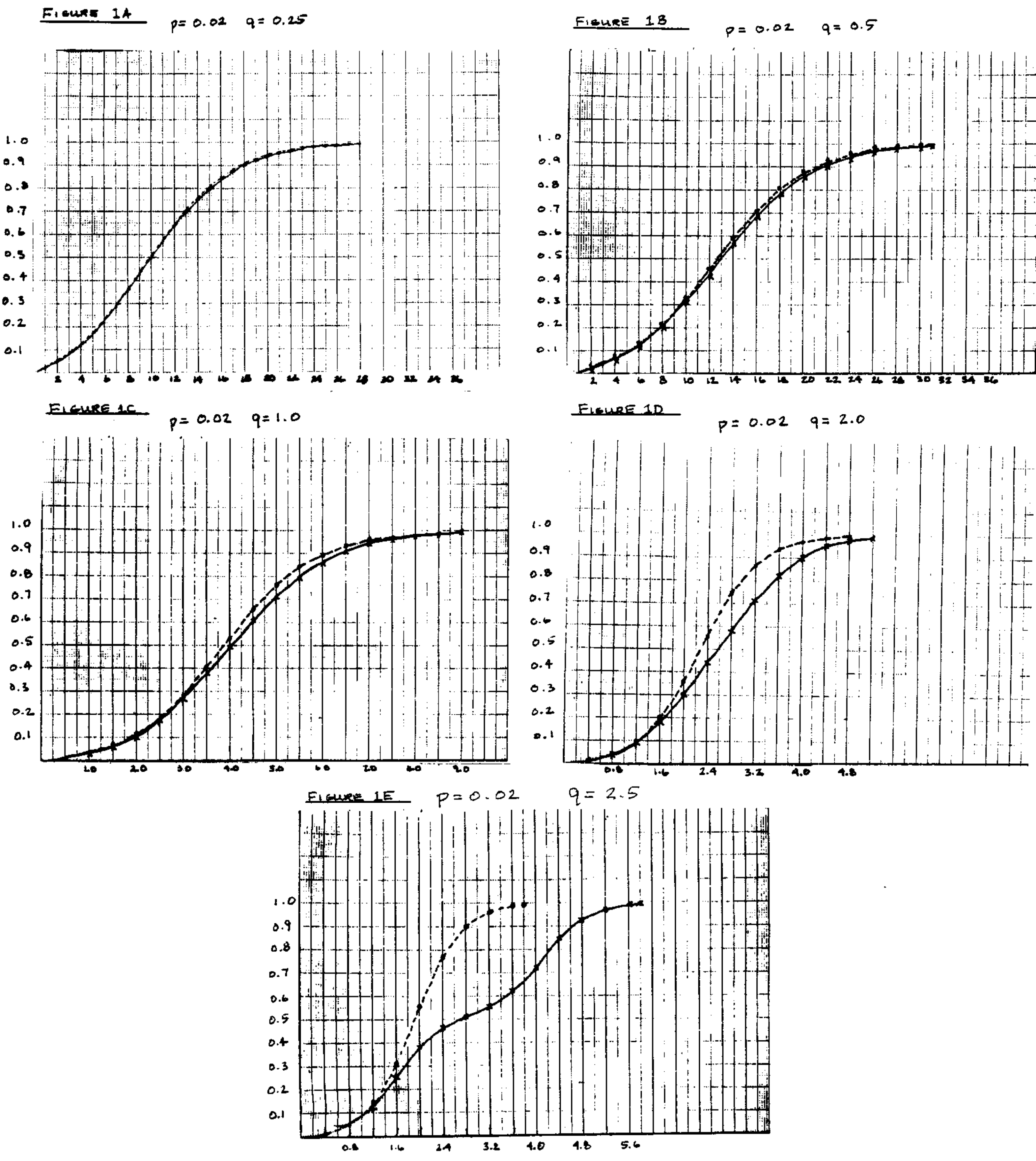


Figure 2

A Comparison of Deterministic and Stochastic Models for:

p=.01-.16 and q=.25-4.0 when $\sigma$p=.3p; $\sigma$q=.3q and R=.95

FIGURE 2A

p= 0.01 q = 0.25

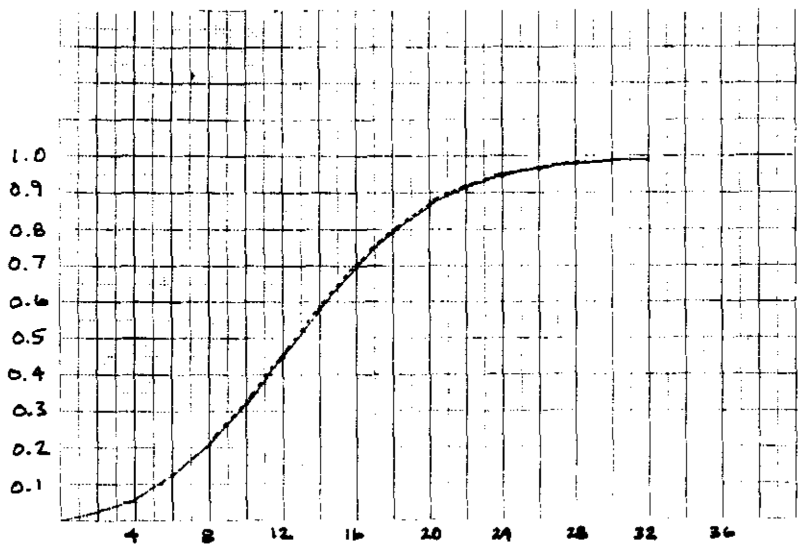



FIGURE 2B

p= 0.02 q= 0.50

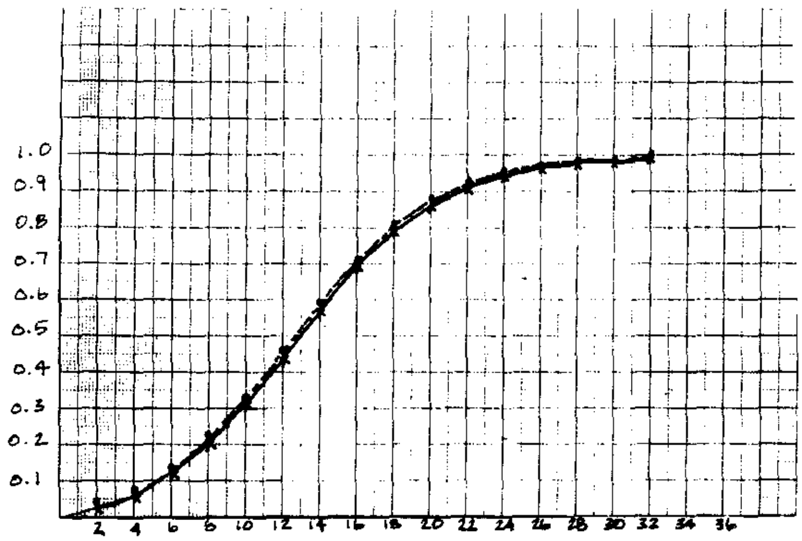



FIGURE 2C

p= 0.04 q= 1.0

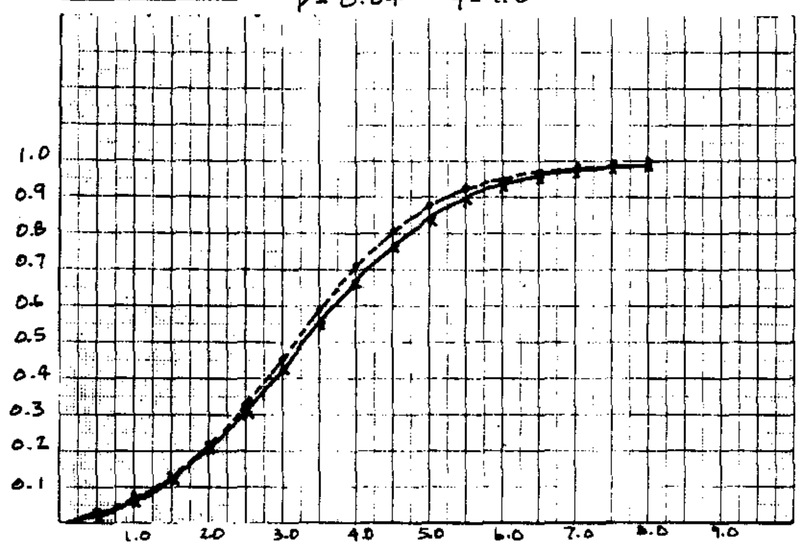



FIGURE 2D

p= 0.08 q= 2.0

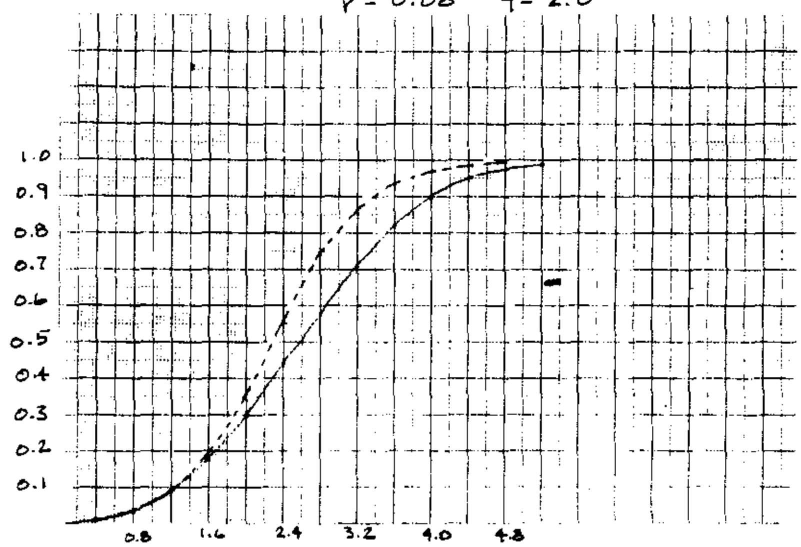
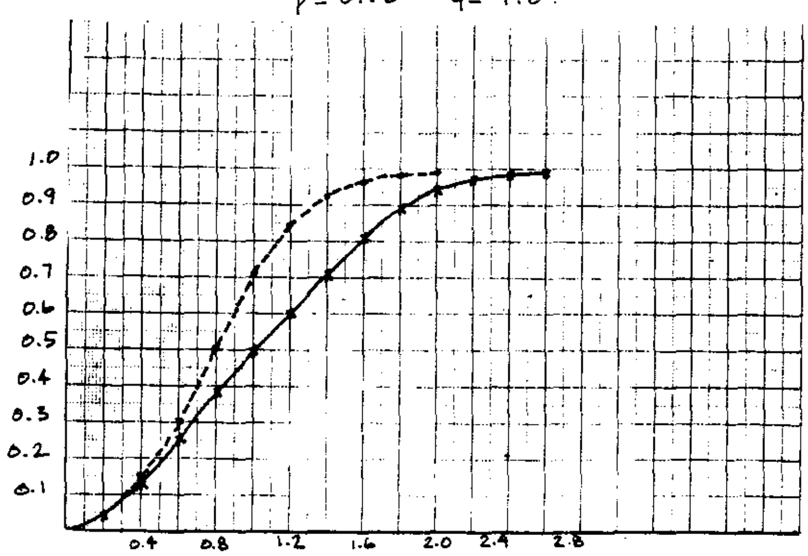



FIGURE 2E

p= 0.16 q= 4.0



| Case | $\hat{p}$ Coefficient of Innovation | $\hat{q}$ Coefficient of Imitation | Figure/Panel | |
|---|---|---|---|---|
| 1 | .02 | .25 | 1 | A |
| 2 | .02 | .50 | | B |
| 3 | .02 | 1.00 | | C |
| 4 | .02 | 2.00 | | D |
| 5 | .02 | 2.50 | | F |
| 6 | .01 | .25 | 2 | A |
| 7 | .02 | .50 | | B |
| 8 | .04 | 1.00 | | C |
| 9 | .08 | 2.00 | | D |
| 10 | .16 | 4.00 | | E |

Common to the 10 cases were $\sigma_p = .3p$ $\sigma_q = .3q$ and $\rho = .95$. The original intention was to simulate the case of $\hat{p}=.02$ and $\hat{q}=4.00$. Yet for a $\hat{q}$ exceeding 2.50 (when $\hat{p}$ is .02) the variance explodes, leading to a negative mean, and hence, to an invalid solution. The reported results are therefore for the case of $\hat{p}=.02$ and $\hat{q}=2.50$. This finding highlights the importance of the ratio between $\hat{q}$ and $\hat{p}$ as well as the magnitude of $\hat{q}$. Too large a ratio of imitators to innovators on top of large magnitudes for the imitation coefficient may lead to infeasible results. Simulating different ratios from 100 to 200 with increments of 5 suggests that the critical ratio for the case of $\hat{p} = .02364$ $\hat{q} = 2.6$ is $\hat{q}/\hat{p} = 110$. This appears to indicate that for extreme cases (large imitation coefficients and stochasticity) the model may not be appropriate. We note, however, that the simulation results reported here are still for coefficients with values that are well within those reported in the literature.

An examination of the figures suggest the following conclusions:

1. With the exception of the two most extreme cases of $\hat{p}=.02$ and $\hat{q}=2.5$ (Figure 1 Panel E) and $\hat{p}=.16$ and $\hat{q}=4.0$ (Figure 2 Panel E) the forecasts of the deterministic and stochastic models are basically identical near the end points of the time horizon.

2. In the moderate cases reported in Figure 1 Panels A, B and Figure 2 Panels A, B the two models produce basically identical forecasts.

3. The discrepancy in the forecasts becomes severe in the mid-time range as the value of $\hat{q}$ is increased beyond 1.0. This can be clearly seen in the last two columns of Table 5.

4. The discrepancy between the two models may even be more severe as the ratio between $\hat{q}$ an $\hat{p}$ increases. This can be seen from a comparison of Figures 1/D and 2/D, and more explicitly in Figure 3.

5. An examination of Table 5 suggests that as the ratio between $\hat{q}$ and $\hat{p}$ increases (cases 1 through 5):
   - # of periods to reach 99% penetration tends to decrease for both the stochastic and deterministic models (col 1).
   - the discrepency between the two models tends to start earlier (col 2) and lasts longer (col 3).
   - the maximum discrepency occurrs in general sooner (col 4) and its magnitude increases (col 5).

6. These results hold also for the case in which $\hat{p}$ and $\hat{q}$ increase but the ratio of $\hat{q}$ over $\hat{p}$ is held constant (at 25). It should be noted, though, that the conclusions discussed above may depend quite heavily upon the specific values of $\hat{p}$ and $\hat{q}$.

TABLE 5

Summary Comparisons of the Ten Stochastic Diffusion Models with Deterministic Models

| | Case | | | -1- # of periods For 99% Cumulative Penetration | | -2- % of periods Before the Discrepancy (0.5%) Starts | % of -3- Periods with Discrepancy Greater than 0.5% | Time -4- At which Maximum Discrepancy Occurrs | -5- Maximum Discrepanc |
|---|---|---|---|---|---|---|---|---|---|
| | $\hat{p}$ | $\hat{q}$ | $\hat{q}/\hat{p}$ | S[4] | D[4] | | | | |
| 1. | 0.02 | 0.25 | 12.5 | 27.0 | 27 | 40.7% | 22.2% | 13.0 | 0.7% |
| 2. | 0.02 | 0.50 | 25 | 16.0 | 16 | 31.2% | 50.0% | 8.0 | 1.8% |
| 3. | 0.02 | 1.00 | 50 | 9.0 | 8.5 | 27.8% | 66.7% | 5.0 | 4.7% |
| 4. | 0.02 | 2.00 | 100 | 5.4 | 4.6 | 25.9% | 70.4% | 3.0 | 16.2% |
| 5. | 0.02 | 2.50 | 125 | 5.8 | 3.8 | 20.7% | 75.9% | 3.2 | 40.6% |
| 6. | 0.01 | 0.25 | 25 | 31.0 | 31 | 38.7% | 32.3% | 16 | 0.9% |
| 7. | 0.02 | 0.50 | 25 | 16.0 | 16 | 31.2% | 50.0% | 8.0 | 1.8% |
| 8. | 0.04 | 1.00 | 25 | 8.0 | 8 | 25.0% | 68.8% | 4.0 | 3.7% |
| 9. | 0.08 | 2.00 | 25 | 4.2 | 3.8 | 19.0% | 76.2% | 2.2 | 8.1% |
| 10. | 0.16 | 4.00 | 25 | 2.5 | 1.9 | 16.0% | 80.0% | 1.2 | 24.9% |

[4] S indicates the stochastic model and D indicates the deterministic model.

## V. Conclusions and Implications

A diffusion model incorporating stochastic coefficients, which is appropriate for heterogeneous populations, was proposed and its estimation discussed. A comparison of this model with the conventional deterministic diffusion model suggests that the deterministic model would overstate the forecast of adoptor population in the mid-time range. Such systematic "bias" of the deterministic model can have significant detrimental effect on management confidence in the reliance on deterministic diffusion models. The simulation findings get added significance given the conceptual attractiveness of a stochastic based diffusion model.

In any diffusion process one can identify two sources of uncertaintly --parametric and structural uncertainty. Parametric uncertainty seeks to study the model behavior due to parameters misspecifications, error in measurement and estimation, --the model error term-- and most importantly the heterogenity of the given population. It is in fact a form, or randomization of the trajectory, that we could expect if there was no uncertainty and the population was homogenous.

Structural uncertainty arises from the probabilistic relationship between components in the diffusion process. For example, how do innovators emerge? How do they affect other innovators and imitators? Similarly, what are the interactions between adopters and potential adopters? Can these relationships be characterized with certainty, or be expressed in terms of probabilities of potential interaction, and probabilities that such interactions will lead to an increase in adoption.

Thus, uncertainty may enter at many points, within the parameters of the diffusion model or by altering the structure of the process. In this paper we restrict ourselves to parametric uncertainties, i.e., uncertainty about the parameters of innovation and imitation due to heterogenous population and a model error term. Subsequent research is needed to deal with structural uncertainty.

In addition to further work on the uncertainty aspects of diffusion, new stochastic diffusion models should be developed to integrate and generalize the work on the dynamic aspects of diffusion, i.e., an extension of deterministic diffusion models with dynamic parameters. Furthermore, still needed are incorporations through stochastic optimal control approaches of marketing mix variables (Tapeiro 1983) and competitive considerations. Such extensions offer exciting challenges to researchers in the diffusion, stochastic modeling and marketing strategy areas helping to increase the managerial relevance of diffusion models.

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