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Abstract

We prove in a discrete setting that if for all test functions, t , there is a \mathbf{B} measurable test function, s , such that $E_p(t) = E_p(s)$ for all $p \in P$ then some subfield of \mathbf{B} is sufficient for P .

Keywords

sufficiency, test function sufficiency

Disciplines

Business | Statistics and Probability

ON A THEOREM OF MORIMOTO CONCERNING SUFFICIENCY
FOR DISCRETE DISTRIBUTIONS¹

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We prove in a discrete setting that if for all test functions, t , there is a \mathbf{B} measurable test function, s , such that $E_p(t) = E_p(s)$ for all $p \in P$ then some subfield of \mathbf{B} is sufficient for P .

The purpose of this note is to call attention to the fact that the conclusion of Theorem 5 of Morimoto (1972) can be strengthened. In the following we use the notation and definitions of Morimoto (1972). In particular, P is a family of discrete distributions on a set X and all subsets of X are measurable. We assume as in Morimoto (1972) that $p(A) = 0$ for all $p \in P$ implies $A = \emptyset$.

The strengthened version of Morimoto's Theorem 5 is as follows:

THEOREM. *Let \mathbf{B} be a σ -field such that for any test function $t(x)$ there is a \mathbf{B} measurable test function $s(x)$ with $E(t(x)|p) = E(s(x)|p)$ for all $p \in P$. Then \mathbf{B} contains a sufficient subfield, i.e., $\mathbf{B} > \mathbf{B}(\mathbf{M})$.*

(Morimoto's conclusion is that $\mathbf{T}(\mathbf{B}) > \mathbf{M}$ which implies that \mathbf{B} is pairwise sufficient, but not that \mathbf{B} is sufficient.)

PROOF. Write $P = \{p_\omega : \omega \in \Omega\}$ where Ω is a well-ordered set. The collection of subsets (\mathbf{M}) defined in Morimoto (1972) may be rewritten using a transfinite induction as

$$(1) \quad \mathbf{M} = \{T_{\omega_i} : i = 1, 2, \dots, I_\omega \leq \infty\}$$

where

$$(2) \quad p_\omega(x) > 0 \quad \text{for all } x \in \bigcup_{i=1}^{I_\omega} T_{\omega_i} \quad \text{and} \quad P_{\omega'}(\bigcup_{\omega \leq \omega'} \bigcup_{i=1}^{I_\omega} T_{\omega_i}) = 1,$$

and the sets T_{ω_i} are mutually disjoint. (The statement $I_\omega \leq \infty$ above is intended to mean that the index set $\{i\}$ is countable, but possibly infinite.)

Now, $V \in \mathbf{B}(\mathbf{M})$ if and only if V may be written

$$V = \bigcup_{\omega \in \Omega_0} \bigcup_{i \in I(\omega)} T_{\omega_i} = \bigcup_{i=1}^{\infty} \bigcup_{\omega \in \Omega_i} T_{\omega_i}$$

where $\Omega_j \subset \Omega$, $j = 0, 1, \dots$. In order to prove that $\mathbf{B} > \mathbf{B}(\mathbf{M})$ it therefore suffices to prove that any set of the form

$$Q = \bigcup_{\omega \in \Omega'} T_\omega$$

satisfies $Q \in \mathbf{B}$, where $\Omega' \subset \Omega$ and $T_\omega = T_{\omega, i(\omega)}$.

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As a consequence of the characterization described in (7) of Morimoto (1972) and of our definition (2) each set T_ω may be written in the form

$$(3) \quad T_\omega = \{x : p_\omega(x) > 0, p_\xi(x) = 0 \quad \forall \xi < \omega, \\ \text{and } p_\omega(x) = k_i p_{\omega_i}(x); i = 1, 2, \dots, I_\omega \leq \infty\}$$

where $k_i > 0$ and $\omega_i \geq \omega$. (Note that the indices ω_i depend on ω , although this is not indicated by the notation.) After defining $0/0 = 0$ we may rewrite (3) as

$$(4) \quad T_\omega = \{x : p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) = 1 \quad \forall \{\alpha_i\} \ni \alpha_i \geq 0, \sum_{i=1}^{I_\omega'} \alpha_i = 1 \\ \text{and } p_\xi(x) = 0 \quad \forall \xi < \omega\}.$$

We now prove

LEMMA. *There exists a vector (α_1, \dots) in the simplex defined in (4) which satisfies*

$$(5) \quad x \in T_\omega \iff p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) = 1 \quad \text{and} \quad p_\xi(x) = 0 \quad \forall \xi < \omega.$$

PROOF. Consider the metric space, M , consisting of points $(\alpha_1, \dots, \alpha_i, \dots)$, $i = 1, \dots, I_\omega'$, satisfying $0 \leq \alpha_i \leq \max(2^{-i}, 2^{-i}k_i^{-1})$; and with metric, ρ , given from the sup (L_∞) norm: $\rho(\alpha, \beta) = \sup\{|\alpha_i - \beta_i| : i = 1, \dots, I_\omega'\}$. This is a complete metric space.

If $x \notin \bigcup_{i=1}^{I_\omega'} T_{\omega_i}$ then either $p_\xi(x) = 0$ for all $\xi < \omega$ or $p_\omega(x) = 0$ so that the r.h.s. of (5) cannot hold true.

For given $x \in D_\omega = \bigcup_{i=1}^{I_\omega'} T_{\omega_i} - T_\omega$ let S_x denote the set of points in M which satisfy $p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i = \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x)$. For $x \in D_\omega$ $p_\omega(x) > 0$ and there is some index, ω_i^x , say, such that $p_\omega(x) \neq k_i p_{\omega_i^x}(x)$. It follows that if $\beta \in S_x$ and β' satisfies $\beta'_i = \beta_i$ for $i \neq \omega_i^x$ and $\beta'_{\omega_i^x} \neq \beta_{\omega_i^x}$ then $\beta' \notin S_x$. Hence the interior of S_x is empty.

The fact that $\alpha_i \leq \max(2^{-i}, 2^{-i}k_i^{-1}p_{\omega_i}^{-1}(x))$ and the dominated convergence theorem lead to the conclusion that S_x is closed in M . Since D_ω is countable the Baire category theorem may then be invoked to establish the existence of a point $\alpha' \in M$ such that $p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha'_i \neq \sum_{i=1}^{I_\omega'} \alpha'_i k_i p_{\omega_i}(x)$, for all $x \in D_\omega$. The vector $\alpha'' = \alpha' / \sum_{i=1}^{I_\omega'} \alpha'_i$ is in the simplex described in (4) and satisfies the conclusion of (5). \square

Fix any vector in the simplex which satisfies (5). Then $T_\omega = R_\omega - S_\omega$ where

$$R_\omega = \{x : p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) \geq 1 \quad \text{and} \quad p_\xi(x) = 0 \quad \forall \xi < \omega\} \subset \bigcup_{i=1}^{I_\omega'} T_{\omega_i},$$

and

$$S_\omega = \{x : p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) > 1 \quad \text{and} \quad p_\xi(x) = 0 \quad \forall \xi < \omega\}.$$

Note that if $p_\omega(x) > 0$ and $p_\xi(x) > 0$ for some $\xi < \omega$ then $p_\omega(x) / l p_\xi(x) < 1$ for l sufficiently large. Since $\{x : p_\omega(x) > 0\}$ is countable we may thus rewrite R_ω as

$$R_\omega = \{x : p_\omega(x) / [\sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) + \sum_{j=1}^{J_\omega} l_j p_{\xi_j}(x)] \geq 1\}$$

where the ξ_j satisfy $\xi_j < \omega$ and the l_j are suitable positive constants and $0 \leq J_\omega \leq \infty$. S_ω has a similar expression. Hence

$$(6) \quad Q = \bigcup_{\omega \in \Omega'} (R_\omega - S_\omega).$$

Since $S_\omega \subset R_\omega$ for all $\omega \in \Omega'$ and since for $\omega \neq \omega' (\bigcup_{i=1}^{I_\omega} T_{\omega i}) \cap (\bigcup_{i=1}^{I_{\omega'}} T_{\omega' i}) = \emptyset$ we have $R_{\omega'} \cap R_\omega = \emptyset$ for all $\omega, \omega' \in \Omega'$ with $\omega' \neq \omega$. The expression (6) may thus be rewritten as

$$(7) \quad Q = \bigcup_{\omega \in \Omega'} R_\omega - \bigcup_{\omega \in \Omega'} S_\omega.$$

To prove the theorem it therefore suffices to show that any set of the form $\bigcup_{\omega \in \Omega'} R_\omega$ or $\bigcup_{\omega \in \Omega'} S_\omega$ is an element of \mathbf{B} .

Let $W' = \bigcup_{\omega \in \Omega'} A(p_\omega) = \{x : p_\omega(x) > 0 \text{ for some } \omega \in \Omega'\}$.

Let $t = \chi_{\bigcup_{\omega \in \Omega'} R_\omega}$ and let s be the \mathbf{B} measurable function generated by the hypothesis of the theorem such that $E(s|p) = E(t|p)$ for all $p \in P$. $t' = t$ is the essentially unique test function relative to the measure given by $p_\omega + \sum \alpha_i k_i p_{\omega_i} + \sum l_j p_{\xi_j}$ maximizing $E(t'|p_\omega)$ subject to the side conditions

$$E(t'|p_{\omega_i}) \leq E(t|p_{\omega_i}) \quad \text{and} \quad E(t'|p_{\xi_j}) \leq E(t|p_{\xi_j});$$

$$i = 1, \dots, I'_\omega, j = 1, \dots, J_\omega.$$

Hence $s(x) = t(x)$ for all $x \in W'$.

Since $t(x) = 0$ for $x \notin W'$ we thus have $s(x) \geq t(x)$ for all $x \in X$. Suppose $s(x) > t(x)$ for some $x \in X$. Then, for some $p_\omega, p_\omega(x) > 0$ and $E(s|p_\omega) > E(t|p_\omega)$, a contradiction.

It follows that $s(x) = t(x)$ for all x so that $\bigcup_{\omega \in \Omega'} R_\omega \in \mathbf{B}$. Similarly $\bigcup_{\omega \in \Omega'} S_\omega \in \mathbf{B}$ (use the fact that \mathbf{B} is a σ -field). As described above this proves that $\mathbf{B} > \mathbf{B}(\mathbf{M})$, which is the desired conclusion.

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