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Mesons and flavor on the conifold

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We explore the addition of fundamental matter to the Klebanov-Witten field theory. We add probe D7-branes to the $\mathcal{N} = 1$ theory obtained from placing D3-branes at the tip of the conifold and compute the meson spectrum for the scalar mesons. In the UV limit of massless quarks we find the exact dimensions of the associated operators, which exhibit a simple scaling in the large-charge limit. For the case of massive quarks we compute the spectrum of scalar mesons numerically.

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I. INTRODUCTION

The gauge theory/string theory correspondence [1–3] furnishes a powerful set of tools for understanding gauge theories at strong coupling by performing computations in a dual string theory at weak coupling. However, the correspondence is only well understood in systems where the string background is highly symmetric and nearly flat, but we expect that the duals to many interesting gauge theories (such as large-N QCD or SQCD) will not have these properties. It is therefore an interesting challenge to study less symmetric string backgrounds, and, in particular, to study backgrounds with reduced supersymmetry.

One interesting class of models arises from compactifications of string theory on noncompact Calabi-Yau manifolds with D3-branes at conical singularities, which generically give rise to $\mathcal{N} = 1$ gauge theories with product gauge groups and bifundamental matter. These models are attractive for several reasons. They possess minimal supersymmetry and are therefore closer to realistic gauge theories than the well-studied $\mathcal{N} = 4$ case; also, they lead to conformal field theories where the quantum conformal invariance is not obvious by inspection of the field theory (but where the supergravity dual makes conformal invariance manifest.) Perhaps the most striking feature of these theories is that one can break conformal invariance in a controlled way by adding fluxes through cycles of the Calabi-Yau geometry, which induce renormalization group (RG) flow and confinement at low energies.

However, one missing element of these models is fundamental matter. Aside from being experimentally important, fundamentals give rise to many interesting things such as the phase structure of super-Yang-Mills theory in the infrared. In confining theories the fundamentals of course do not appear as asymptotic states but are instead confined in mesons and baryons.

In this paper we study the mesonic fluctuations of a particular set of mesons in the conifold theory of Klebanov and Witten [4]. This theory is interesting for its relative simplicity and also because its nonconformal version flows to a theory very similar to $\mathcal{N} = 1$ pure glue theory in the infrared. Moreover all metrics for the corresponding supergravity solutions are known, allowing explicit computations. The mesons which we study arise as fluctuations on D7-branes which are embedded in the string background. The fundamental fields come from strings connecting the stack of D3-branes to the D7-branes. In the usual decoupling limit, the 3-7 strings and 3-3 strings, which describe the gauge theory, have a dual description in terms of the closed strings and 7-7 strings. The closed strings are the usual glueballs of the strongly coupled field theory while the open 7-7 strings are naturally identified with the mesons.

We will compute the spectrum of operator dimensions, which, as we will see, can be done exactly for a large portion of the states, and we will study the effect of giving masses to the quarks (which requires numerical work).

The paper is organized as follows. In Sec. II we review the geometry of the conifold. In Sec. III we discuss adding flavor to the Klebanov-Witten field theory by the addition of probe D7-branes. In Sec. IV we compute the spectrum for scalar mesons. In the case of massive quarks, we compute the mass spectrum numerically, but in the massless case (corresponding to the UV limit of the gauge theory) we obtain the spectrum analytically. In Sec. V we discuss our results.

II. REVIEW OF THE CONIFOLD

In this section we briefly review the geometry of the conifold in order to fix notation. Useful references are [4–9].

The conifold is a noncompact Calabi-Yau threefold, defined by the equation

$$z_1 z_2 - z_3 z_4 = 0 \tag{1}$$

in $\mathbf{C}^4$. Because Eq. (1) is invariant under an overall real rescaling of the coordinates, this space is a cone, whose base is the Einstein space $T^{1,1}$ [4,5]. The metric on the

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conifold may be cast in the form [5]

$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2, \tag{2}$$

where

$$ds_{T^{1,1}}^2 = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) \tag{3}$$

is the metric on $T^{1,1}$. Here ψ is an angular coordinate which ranges from 0 to 4π, while (θ_1, ϕ_1) and (θ_2, ϕ_2) parametrize two $\mathbf{S}^2$s in the standard way. This form of the metric shows that $T^{1,1}$ is a $U(1)$ bundle over $\mathbf{S}^2 \times \mathbf{S}^2$.

These angular coordinates are related to the z_i variables by

$$\begin{aligned} z_1 &= r^{3/2}e^{i/2(\psi-\phi_1-\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \\ z_2 &= r^{3/2}e^{i/2(\psi+\phi_1+\phi_2)}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}, \\ z_3 &= r^{3/2}e^{i/2(\psi+\phi_1-\phi_2)}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \\ z_4 &= r^{3/2}e^{i/2(\psi-\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}. \end{aligned} \tag{4}$$

It is also sometimes helpful to consider a set of "homogeneous" coordinates A_a, B_b where a, $b = 1, 2$, in terms of which the z_i are

$$z_1 = A_1B_1, \qquad z_2 = A_2B_2, \tag{5}$$

$$z_3 = A_1B_2, \qquad z_4 = A_2B_1. \tag{6}$$

With this parametrization the z_i obviously solve the defining equation of the conifold.

We may also parametrize the conifold in terms of an alternative set of complex variables w_i, given by

$$\begin{aligned} z_1 &= w_1 + iw_2, & z_2 &= w_1 - iw_2, \\ z_3 &= -w_3 + iw_4, & z_4 &= w_3 + iw_4. \end{aligned} \tag{7}$$

The conifold equation may now be written as

$$\sum w_i^2 = 0 \tag{8}$$

and we identify the $T^{1,1}$ base of the cone as the intersection of the conifold with the surface

$$\sum |w_i|^2 = r^3. \tag{9}$$

$T^{1,1}$ described in this way is explicitly invariant under $SO(4) \simeq SU(2) \times SU(2)$ rotations of the w_i coordinates and under an overall phase rotation. Thus the symmetry group of $T^{1,1}$ is $SU(2) \times SU(2) \times U(1)$.

An important fact about $T^{1,1}$ is that it has Betti numbers b_2, $b_3 = 1$. The corresponding two-cycle and three-cycle may be expressed in terms of harmonic differential forms:

$$\omega_2 = \tfrac{1}{2}(\Omega_{11} - \Omega_{22}), \tag{10}$$

$$\omega_3 = \zeta \wedge \omega_2. \tag{11}$$

In this paper we will consider D7-branes in the model of Klebanov and Witten [4]. This model is a particularly simple $\mathcal{N} = 1$ gauge/gravity dual, obtained by placing a stack of N D3-branes near a conifold singularity. The branes source the Ramond-Ramond (RR) 5-form flux and warp the geometry:

$$ds_{10}^2 = h(r)^{-1/2}dx_\mu dx^\mu + h(r)^{1/2}(dr^2 + r^2 ds_{T^{1,1}}^2), \tag{12}$$

$$h(r) = 1 + \frac{L^4}{r^4}, \tag{13}$$

$$g_s F_5 = d^4x \wedge dh^{-1} + \star(d^4x \wedge dh^{-1}), \tag{14}$$

$$L^4 = \tfrac{27}{4}\pi g_s N \alpha'^2. \tag{15}$$

Hereafter, we specialize to the near-horizon limit $r/L \ll 1$, and set $L = 1$ for convenience. It may be easily restored by dimensional analysis at any point.

The dual field theory has gauge group $SU(N) \times SU(N)$ and matter fields $A_{1,2}$, $B_{1,2}$ which transform in the bifundamental color representations $(\mathbf{N}, \bar{\mathbf{N}})_c$ and $(\bar{\mathbf{N}}, \mathbf{N})_c$. The theory also has a superpotential

$$W = \lambda\, \mathrm{Tr}(A_iB_jA_kB_l)\epsilon^{ik}\epsilon^{jl}. \tag{16}$$

By solving the F-term equations for this superpotential, we obtain supersymmetric vacua for arbitrary diagonal $A_{1,2}$ and $B_{1,2}$, so that the moduli space of the field theory is precisely that of N D3-branes placed on the conifold.

III. ADDING FLAVOR

In this section we review the procedure of adding flavor branes to AdS/CFT in general and make several useful comments on adding flavor to the Klebanov-Witten field theory both in terms of the bulk geometry and the dual field theory. This general procedure was first pointed out in [10–12] and was exploited in the $\mathrm{AdS}_5 \times S^5$ case in [13]. Some other examples of flavored theories with probe branes have been studied in [14–26].

One way to add flavor to AdS/CFT is to take a system of D3-branes and then to add D7-branes which fill the four x^μ directions and four of the six transverse dimensions [11]. In flat space such a configuration of branes is clearly supersymmetric. As usual there is an $\mathcal{N} = 4$ $SU(N)$ SYM theory living on the D3-branes. Strings with one end on a D3-brane and one end on a D7-brane couple to the fields of the D3-brane gauge theory as quarks.

For AdS/CFT purposes we can now take the supergravity approximation in which D3-branes are replaced by an $AdS_5 \times S^5$ geometry with Ramond-Ramond flux, while we retain the D7-branes as probes which fill the five AdS directions and which wrap a topologically trivial 3-cycle of the internal 5-manifold (for example an S^3 submanifold of the S^5 of $AdS_5 \times S^5$). The triviality of the 3-cycle guarantees that the brane carries no net charge and will not introduce any tadpoles. On the other hand, topological triviality also suggests that one might be able to shrink the S^3 and slip it off of the S^5, naively in contradiction with the flat space picture of D3 and D7-branes. It turns out that subtleties of the AdS geometry play a key role in ensuring stability. The mass eigenvalues of modes controlling the D-brane slipping off the 3-cycle are negative, but are above the Breitenlohner-Freedman bound [27], so that the 7-brane embedding is stable.

In the flat space picture, if the D3-branes and D7-branes intersect then the quarks are massless, and if the D3-branes and D7-branes are separated then the quarks are massive. This translates nicely into the AdS picture in the following way. A D7-brane which intersects the D3-branes in flat space gets mapped to a D7-brane which fills the whole AdS space and wraps a three-sphere of constant size in the S^5. On the other hand, a D7-brane separated from the stack of D3-branes maps to a D7-brane which wraps an S^3 with some asymptotic size at large AdS radius, but this S^3 shrinks to zero size at some finite radius (which is possible because of the topological triviality). In the 5-dimensional AdS space the D7-brane appears to fill out the radial direction up to some minimal radius where it "ends."

It is interesting of course to consider theories with branes in spaces which are not flat. The basic picture of D3 and D7 branes contributing gauge fields and quarks will not change, but many details are different. For simplicity throughout this paper we specialize to the case of a single D7-brane. If the number of D3-branes is large then the D7 backreaction can be systematically neglected and it is appropriate to treat the D7-brane as a probe, which we do throughout this paper. Inclusion of backreaction effects in other geometries has been explored in [10,28,29].

Let us consider D7-branes embedded in the geometry of the conifold by the equation $z_1 = \mu$. In terms of the standard coordinate system,

$$z_1 = r^{3/2} e^{i/2(\psi - \phi_1 - \phi_2)} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}$$

so the embedding equation gives two conditions, one on the magnitude of z_1 and one on the phase:

$$r_0 = \left(\frac{|\mu|}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}\right)^{2/3}, \tag{17}$$

$$\psi_0 = \phi_1 + \phi_2 + \text{const.} \tag{18}$$

This embedding can be explicitly shown to be supersymmetric by considering the κ-symmetry on the world volume of the brane [30]. A slightly different embedding equation was studied in the warped deformed conifold by [31].

It was proposed in [32] that the embedding $z_1 = \mu$ leads to fields, summarized in Table I and a superpotential of the form

$$W = W_{\text{flavors}} + W_{\text{masses}}, \tag{19}$$

$$\begin{aligned} W_{\text{flavors}} &= h\tilde{q}A_1Q + gqB_1\tilde{Q}, \\ W_{\text{masses}} &= \mu_1 q\tilde{q} + \mu_2 Q\tilde{Q}. \end{aligned} \tag{20}$$

To relate this superpotential to the D7-brane geometry, let us probe the space with a single D3-brane, which corresponds to giving some expectation values to A_1 and B_1. One then finds that the theory on this probe has a massless flavor when $A_1B_1 = \mu_1\mu_2/(gh)$, which is exactly of the form of the embedding equation $z_1 = \mu$. Part of the motivation for this superpotential was a comparison with a type IIA brane construction [33] where a D6-brane splits on an NS5-brane, contributing two flavor branes and correspondingly two sets of flavors. For the type IIB picture, in the massless limit of the field theory, this corresponds nicely to the presence of two solution branches of $z_1 = 0$, namely $\theta_1 = 0$ and $\theta_2 = 0$. If the quarks are massless there is an $SU(K) \times SU(K)$ flavor symmetry, where K is the number of probe D7-branes. If the quarks are massive then the two branches of the D7-branes connect and the flavor symmetry is broken down to the diagonal $SU(K)$.

An alternative perspective is to suppose that one of the masses μ_i is larger than the other and then integrate out the associated flavors. Then one obtains a quartic superpotential of the form

$$W = q(A_1B_1 - \mu)\tilde{q} \tag{21}$$

which again produces the appropriate massless locus for a D3-brane probe. Our probe calculations will show that this quartic superpotential is consistent with adding D7-branes with massive flavors (and then with the limit where we take the masses to zero.) Of course, because we believe the quarks can be massive the consistency was virtually guaranteed. However, taking the limit $\mu \to 0$ and setting $\mu = 0$ are different things, and it is unclear whether the theory corresponding to the cubic superpotential can be realized or not.

TABLE I. Representation structure of the added $\mathcal{N} = 1$ flavors.

Field	$SU(N_c) \times SU(N_c)$	$SU(K) \times SU(K)$
q	$(\mathbf{N}, \mathbf{1})$	$(\mathbf{K}, \mathbf{1})$
$\tilde{q}$	$(\bar{\mathbf{N}}, \mathbf{1})$	$(\mathbf{1}, \mathbf{K})$
Q	$(\mathbf{1}, \mathbf{N})$	$(\mathbf{1}, \bar{\mathbf{K}})$
$\tilde{Q}$	$(\mathbf{1}, \bar{\mathbf{N}})$	$(\bar{\mathbf{K}}, \mathbf{1})$

IV. SCALAR MESONS

In this section, we compute the dimension and mass spectra of the scalar mesons. As discussed in the introduction, in the probe and decoupling limits the 7-7 strings are identified with the mesons in the dual field theory. We will thus be able to extract the mass spectrum of the spin $= 0$ mesons and their conformal dimension in the UV limit by studying the 7-7 strings.

The semiclassical dynamics of this D7-brane are captured by the Dirac-Born-Infeld action

$$S_{\mathrm{DBI}} = \tau_7 \int d^8\xi \sqrt{-\det_{ij}(g_{MN} + F_{MN})\frac{\partial y^M}{\partial \xi^i}\frac{\partial y^N}{\partial \xi^j}} + \frac{g_s\tau_7}{2}\int C_4 \wedge F_2 \wedge F_2, \tag{22}$$

where ξ^i are coordinates on the D7-brane. We will compute the spectrum of fluctuations for the D7-branes using this action.

Let us consider the fluctuations of scalar modes alone, with all D7-brane gauge fields turned off. Then the DBI action is simply the world volume of the 7-brane. Let us choose as coordinates on the brane eight of the spacetime coordinates: $(x^\mu, \theta_1, \theta_2, \phi_1, \phi_2)$. The fluctuations can be described by setting

$$r = r_0(\theta_i)(1 + \chi(x^\mu, \theta_i, \phi_j)), \tag{23}$$

$$\psi = \psi_0(\phi_i) + 3\eta(x^\mu, \theta_i, \phi_j). \tag{24}$$

The unperturbed induced metric takes the form

$$g_{MN} = \begin{pmatrix} r_0^2\eta_{\mu\nu} & 0 & 0 \\ 0 & g_{\theta_i\theta_j} & 0 \\ 0 & 0 & g_{\phi_i\phi_j} \end{pmatrix}, \tag{25}$$

$$g_{\theta_i\theta_j} = \begin{pmatrix} \frac{1}{6} + \frac{1}{9}\cot^2\frac{\theta_1}{2} & \frac{1}{9}\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} \\ \frac{1}{9}\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} & \frac{1}{6} + \frac{1}{9}\cot^2\frac{\theta_2}{2} \end{pmatrix}, \tag{26}$$

$$g_{\phi_i\phi_j} = \begin{pmatrix} \frac{1}{6}\sin^2\theta_1 + \frac{1}{9}(1+\cos\theta_1)^2 & \frac{1}{9}(1+\cos\theta_1)(1+\cos\theta_2) \\ \frac{1}{9}(1+\cos\theta_1)(1+\cos\theta_2) & \frac{1}{6}\sin^2\theta_2 + \frac{1}{9}(1+\cos\theta_2)^2 \end{pmatrix}. \tag{27}$$

One expands about this metric via the matrix identity

$$\sqrt{\det A + \delta A} = \sqrt{\det A}\left(1 + \tfrac{1}{2}\mathrm{Tr}A^{-1}\delta A + \tfrac{1}{8}(\mathrm{Tr}A^{-1}\delta A)^2 - \tfrac{1}{4}\mathrm{Tr}A^{-1}\delta A A^{-1}\delta A + \ldots\right). \tag{28}$$

The terms first order in the fluctuations χ and η turn out to be total derivatives, as is necessary for our embedding to be a solution of the equations of motion. The quadratic order fluctuations lead to an action of the form

$$\begin{aligned} S = \tau_7 \int & d^4x d\theta_1 d\theta_2 d\phi_1 d\phi_2 \sqrt{\det g_0}\frac{1}{C}\Big[\frac{1}{2}g_0^{ab}\partial_a\chi\partial_b\chi \\ &+ \frac{1}{2}g_0^{ab}\partial_a\eta\partial_b\eta + \frac{4}{\sin^2\frac{\theta_2}{2}}\chi\partial_{\phi_2}\eta - \frac{2}{C\sin^2\frac{\theta_2}{2}} \\ &\times\left(\cot\frac{\theta_1}{2}\partial_{\theta_1} + \cot\frac{\theta_2}{2}\partial_{\theta_2}\right)\chi\partial_{\phi_2}\eta + \frac{4}{\sin^2\frac{\theta_1}{2}}\chi\partial_{\phi_1}\eta \\ &- \frac{2}{C\sin^2\frac{\theta_1}{2}}\left(\cot\frac{\theta_1}{2}\partial_{\theta_1} + \cot\frac{\theta_2}{2}\partial_{\theta_2}\right)\chi\partial_{\phi_1}\eta\Big] \end{aligned} \tag{29}$$

with

$$C = 1 + \frac{2}{3}\cot^2\frac{\theta_1}{2} + \frac{2}{3}\cot^2\frac{\theta_2}{2}. \tag{30}$$

A. The UV/massless limit

One interesting issue to explore is the spectrum of dimensions of the mesonic operators in the UV field theory. Taking this high-energy limit, we may consistently ignore the masses of the quarks in comparison with the energy scale of the field theory. In the gravity dual, this high-energy limit corresponds to computing near the boundary of AdS, and taking the limit $\mu \to 0$.

Examining (17) we see that there are two ways to go near the boundary: $\theta_1 \to 0$ or $\theta_2 \to 0$. Which one we choose will determine which side of the conifold we are on near the boundary. In the current setup, the physics is symmetric between exchange of θ_1 and θ_2 so we will simply choose the $\theta_1 \to 0$ limit. We can compute the dimensions of the operators in the field theory by examining the scaling of the 7-7 strings near the boundary.

We define $r_0 = \mu^{2/3}e^{-\beta/3}$ (now a good coordinate because of the conformal invariance), and $\cos^2\frac{\theta_2}{2} = x$. Defining the linear combination of fields

$$\Phi^\pm = \chi \pm i\eta \tag{31}$$

we find that the equations for $\Phi^\pm$ are two fully decoupled partial differential equations. This equation is solved by a separation of variables ansatz

$$\Phi^\pm = \rho^\pm(x)e^{k\beta/3}e^{im_1\phi_1 + im_2\phi_2}. \tag{32}$$

The equations of motion for the scalar reduce to ordinary differential equations for the functions $\rho(x)^\pm$ which take the form

$$\begin{aligned} \Big((1-x)\frac{\partial}{\partial x}x\frac{\partial}{\partial x} &+ \frac{1}{6}k(k-1) - \frac{m_1^2}{8}\frac{3-x}{1-x} + \frac{m_1m_2}{2(1-x)} \\ &- \frac{1}{4}\frac{m_2^2}{x(1-x)} \mp \frac{m_1}{4}\frac{1+x}{1-x} \pm \frac{m_2}{2(1-x)}\Big)\rho^\pm(x) = 0. \end{aligned} \tag{33}$$

This equation has singularities only at $x = 0, 1, \infty$ and is therefore of hypergeometric type. To see this explicitly we define new functions by rescaling the $\rho^\pm$ by factors of x and $(1-x)$, which allow us to remove the terms in (33) proportional to $\frac{1}{x}$ and $\frac{1}{1-x}$. Explicitly, we write

$$\rho^{\pm}(x) = x^p (1-x)^q f^{\pm}(x) \tag{34}$$

for which the equation of motion becomes

$$\bigg[x(1-x)\frac{\partial^2}{\partial x^2} + (1+2p - x(1+2p+2q))\frac{\partial}{\partial x} + \frac{(q^2 - q + \frac{1}{4} - \frac{1}{4}(m_1 - m_2 \pm 1)^2)}{1-x} + \frac{(p^2 - \frac{m_2^2}{4})}{x} + \frac{1}{6}k(k-1) - \frac{1}{4}\bigg(\frac{m_1^2}{2} \mp m_1\bigg) - (p+q)^2\bigg] f^{\pm}(x) = 0. \tag{35}$$

Appropriate choice of the parameters p and q eliminates the terms proportional to $1/x$ and $1/(1-x)$. The wave functions are given by

$$f^{\pm}(x) = {}_2F_1(-\alpha, \alpha + 2(p+q), 1+2p; x) \tag{36}$$

which are regular when α is a non-negative integer; it turns out that the original ρ are also regular with this condition over the range $0 \le x \le 1$ $(0 \le \theta_2 \le \pi)$ which encompasses our domain. We also find that there are two possible values of k:

$$k_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 3m_1^2 \mp 6m_1 + 24(\alpha + p + q)^2}, \tag{37}$$

$$k_2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 + 3m_1^2 \mp 6m_1 + 24(\alpha + p + q)^2}. \tag{38}$$

To be painfully explicit, we exhibit the solutions for f^+ (the f^- are straightforwardly related). It is clear that there are always two choices of p and q which do the trick; to make regularity transparent we will always choose p and q to be positive. We then have four cases:

(i) $m_2 \ge 0$, $m_1 \ge m_2$: We choose $p = m_2/2$ and $q = 1 + \frac{1}{2}(m_1 - m_2)$, so that $f^+ = {}_2F_1(-\alpha, 2 + m_1 + \alpha, 1 + m_2; x)$. The wave function is regular if α is a non-negative integer (negative α gives irregular or redundant solutions) and so the two values of k are quantized to be

$$k = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 3m_1^2 - 6m_1 + 24\bigg(\alpha + \frac{m_1}{2} + 1\bigg)^2}. \tag{39}$$

(ii) $m_2 \ge 0$, $m_1 < m_2$: Again we choose $p = m_2/2$, but now to make regularity obvious we take $q = (m_2 - m_1)/2$, such that $q > 0$. Now $f^+ = {}_2F_1(-\alpha, 2m_2 - m_1 + \alpha, 1 + m_2; x)$, again with α a non-negative integer. The quantized values of k are

$$k = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 3m_1^2 - 6m_1 + 24\bigg(\alpha + m_2 - \frac{m_1}{2}\bigg)^2}. \tag{40}$$

(iii) $m_2 < 0$, $m_1 \ge m_2$: Now we choose $p = -m_2/2$ and $q = 1 + \frac{1}{2}(m_1 - m_2)$, finding that $f^+ = {}_2F_1(-\alpha, 2 + m_1 - 2m_2 + \alpha, 1 - m_2; x)$, with α a non-negative integer. The allowed values of k are

$$k = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 3m_1^2 - 6m_1 + 24\bigg(\alpha + \frac{m_1}{2} - m_2 + 1\bigg)^2}. \tag{41}$$

(iv) $m_2 < 0$, $m_1 < m_2$: Now we choose $p = -m_2/2$ and $q = (m_2 - m_1)/2$, finding that $f^+ = {}_2F_1(-\alpha, \alpha - m_1, 1 - m_2; x)$, with α a non-negative integer. The allowed values of k are

$$k = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 3m_1^2 - 6m_1 + 24\bigg(\alpha - \frac{m_1}{2}\bigg)^2}. \tag{42}$$

To find the dimensions of the operators, we recall that $e^{k\beta/3} \sim r^{-k}$. In the AdS/CFT correspondence a minimal massless scalar field dual to an operator of dimension Δ scales as $r^{-\Delta}$ for its normalizable part and $r^{\Delta-4}$ for its non-normalizable part. However, by examining (29) we see that the kinetic terms for these scalars are not canonically normalized, which means that the possible scalings at infinity are modified to $r^{-\Delta+p}$ and $r^{\Delta-4+p}$ for some p. Using the values for k_1 and k_2 we have $2\Delta - 4 = k_1 - k_2$, which one can compute straightforwardly.

The dimensions are mostly complicated irrational numbers (reminiscent of the closed string spectrum on $T^{1,1}$ [8,34]) but a few features of the spectrum stand out. The lowest mode has $m_1 = m_2 = \alpha = 0$, and is simply a constant; it can be assigned dimension $5/2$ or $3/2$. From the earlier discussion of the massive field theory, it is natural to choose dimension $3/2$ and associate this mode with the operator $q\tilde{q}$. Note also that the mode with $m_1 = m_2 = 1$ and $\alpha = 0$ has dimension 3, appropriate for a superpotential term—we identify this mode with the operator $qA_2B_2\tilde{q}$. If added to the superpotential, this operator would change the D7-brane embedding from $z_1 = \mu$ to $z_1 + \epsilon z_2 = \mu$.

For large m_1, all the dimensions scale as $\Delta \sim 3m_1/2$. This is consistent with identification of the corresponding gauge theory operators as

$$q(AB)(AB)\ldots(AB)\tilde{q}, \tag{43}$$

where, ignoring the q fields, each insertion of (AB) should increase the dimension by $3/2$ and the relevant $SU(2)$ charge (associated with m_1) by one unit. Unlike the case of baryonic operators on the conifold, where one finds an exact scaling $\Delta \sim 3N/4$ [35], we see that the mesons only exhibit a simple scaling with the charge in a large-charge limit. For small charges there are boundary effects due to the quarks which, at least in the large-N limit, are completely calculable here. This behavior should also be contrasted with the case of flavors added to $\mathcal{N} = 4$ super-YM theory, where the meson dimensions were pure integers.

If we take instead the limit of large α, we see that the dimensions scale as $\Delta \sim \sqrt{6}\alpha$. It would be interesting to find an explanation for this curious scaling in the field theory.

B. The mass spectra

Having obtained the conformal dimensions for the mesons in the conformal limit, we would like to compute their spectrum by solving the full differential equation without taking any simplifying limits. Unfortunately, we will find that the equation is not amenable to analytic solution and so we will have to appeal to numerical methods. We will display selected results from several cases that are illustrative of the general behavior.

We will again find that the linear combination of fields (31) decouples the equations of motion. Rewriting the action (29) in terms of $\Phi^{\pm}$ and varying gives

$$\frac{1}{\sqrt{-g_0}}\partial_a\left(\frac{1}{C}\sqrt{-g_0}g_0^{ab}\partial_b\Phi^{\pm}\right) \pm 3i\left[\frac{1}{\sqrt{-g_0}}\partial_{\theta_i}(\sqrt{-g_0}\gamma^{\theta_i\phi_j}) - \gamma^{\phi_j}\right]\partial_{\phi_j}\Phi^{\pm} = 0, \quad (44)$$

where the a, b indices run over the x^μ *and* the $\theta_{1,2}$, $\phi_{1,2}$, and

$$\gamma^{\theta_i\phi_j} = \frac{4(1+\cos\theta_j)}{C^2\sin^2\theta_j}\partial_{\theta_i}\ln r_0(\theta_i,\theta_j), \quad (45)$$

$$\gamma^{\phi_j} = \frac{8(1+\cos\theta_j)}{3C\sin^2\theta_j}. \quad (46)$$

The inverse components of the metric are straightforward to find from the form given in (25)–(27). Since ∂_{x^μ} and ∂_{ϕ_i} are Killing vectors we can write

$$\Phi^{\pm} = \psi^{\pm}(\theta_1,\theta_2)e^{ik\cdot x}e^{im_1\phi_1 + im_2\phi_2}. \quad (47)$$

We find that (44) becomes

$$-\partial_{\theta_i}\left(\frac{1}{C}\sqrt{-g_0}g_0^{\theta_i\theta_j}\partial_{\theta_j}\psi^{\pm}\right) + \frac{1}{C}\sqrt{-g_0}g_0^{\phi_i\phi_j}m_im_j\psi^{\pm} \pm 3[\partial_{\theta_i}(\sqrt{-g_0}\gamma^{\theta_i\phi_j}) - \sqrt{-g_0}\gamma^{\phi_j}]m_j\psi^{\pm} = -\frac{\sqrt{-g_0}}{Cr_0^2(\theta_1,\theta_2)}k^2\psi^{\pm}. \quad (48)$$

Note that for massive modes $k^2 = k_\mu k^\mu < 0$ since it must be timelike. Using simple Kaluza-Klein arguments we see that the mass of the mesons in the dual field theory is $M^2 = -k^2$. It is evident from the form of this equation that the only difference between the equation for ψ^+ and for ψ^- is in a term proportional to m_j. Thus, we can choose to solve for ψ^+ without loss of generality. This equation cannot be solved analytically. In addition, we have found no simple way to separate the equation in the θ_1, θ_2 directions and so we must use a numerical approach to solving the partial differential equation.

The numerical approach

Although the equations we would like to solve are linear, the lack of separability forces us to seek a numerical solution. The equations are elliptic, which allows us to use finite-element analysis and the Arnoldi algorithm as implemented in the PDE Toolbox of Matlab to solve for the mass eigenvalues, $-k^2$. (It is worth noting that the method of lines used by Mathematica's NDSolve cannot be used for elliptic PDEs.) We will use a mesh with 2779 nodes and 5392 triangles for all problems involved. The θ_is range from 0 to π. Because we already know $\theta_i \to 0$ corresponds to going near the boundary, and we want normalizable modes, we place Dirichlet boundary conditions at $\theta_{1,2} = 0$. We must also demand regularity at $\theta_{1,2} = \pi$, which corresponds to placing Neumann boundary conditions at $\theta_{1,2} = \pi$.

We will first examine the simplest case, when $m_1 = m_2 = 0$. Setting $\mu^{-4/3} = .02$ we solve (48) for the first 50 eigenvalues. The eigenvalues break up into different series corresponding to the number of nodes in the (θ_1, θ_2) plane. In Figs. 1 and 2 we display the first two such series. Higher series have similar behavior. The + signs denote actual mass eigenvalues, while the solid lines are best-fit lines.

We will also find similar behavior for modes with $m_{1,2} \neq 0$. In Fig. 3 we display the zero node modes for the case $m_1 = 1$, $m_2 = 2$ with $\mu^{-4/3} = 2$ (note: changing μ just changes the eigenvalues by an overall scaling, as expected since it merely scales the mass gap for the quarks). We find

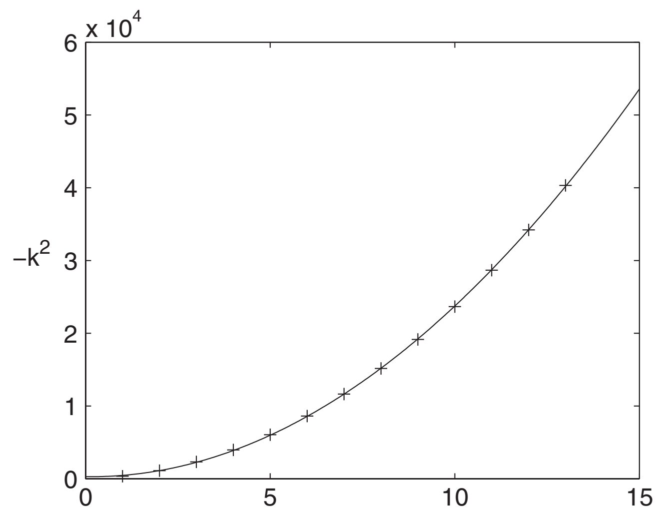

FIG. 1. Mass eigenvalues versus eigenvalue number for the zero node modes with $m_1 = m_2 = 0$, $\mu^{-4/3} = .02$. +s denote actual eigenvalues and the solid line is a best-fit line with equation $-k^2 = 240n^2 - 68n + 290$. The masses are measured in units given by the inverse AdS radius.

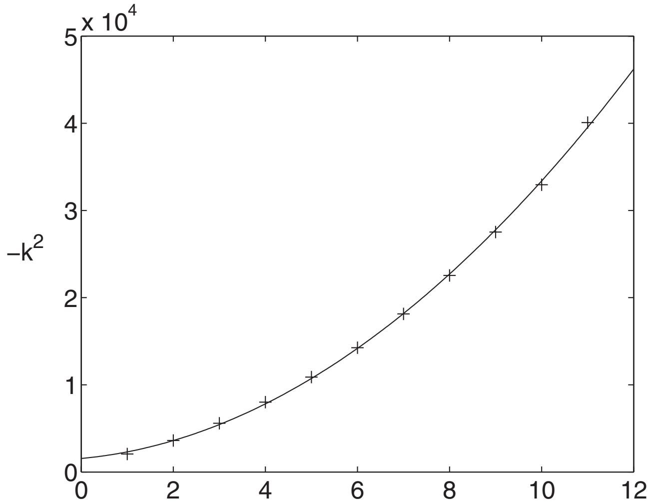

FIG. 2. Mass eigenvalues versus eigenvalue number for the one node modes with $m_1 = m_2 = 0$, $\mu^{-4/3} = .02$. +s denote actual eigenvalues and the solid line is a best-fit line with equation $-k^2 = 270n^2 + 490n + 1500$. The masses are measured in units given by the inverse AdS radius.

similar behavior for other values of $m_{1,2}$ and different number of nodes.

For all the cases we see that in the large n limit we have $M \sim n\mu^{2/3}$ as we would expect. Restoring L by dimensional analysis, and using $m_q \sim \frac{\mu^{2/3}}{2\pi\alpha'}$ we find the mass gap for the lightest meson is

$$M_{\text{gap}} \sim \frac{m_q}{\sqrt{g_s N}}. \tag{49}$$

Therefore, in the supergravity regime where $g_s N \gg 1$ we find the meson mass is much smaller than the quark mass.

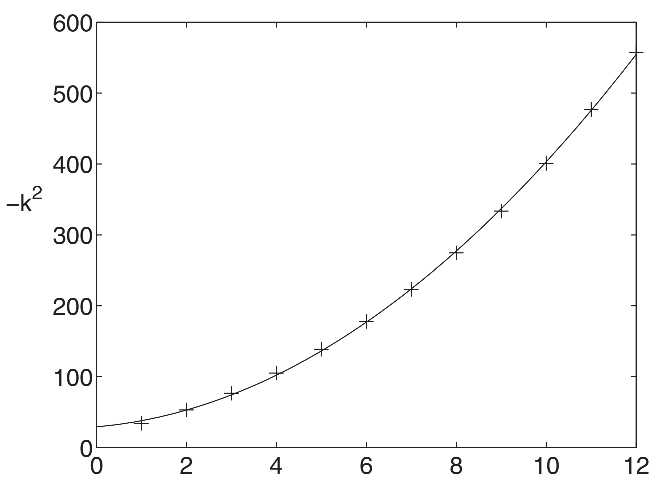

FIG. 3. Mass eigenvalues versus eigenvalue number for the zero node modes with $m_1 = 1$, $m_2 = 2$, $\mu^{-4/3} = 2$. +s denote actual eigenvalues and the solid line is a best-fit line with equation $-k^2 = 3.2n^2 + 5.4n + 29$. The masses are measured in units given by the inverse AdS radius.

At large t'Hooft coupling we find that the binding energy of the mesons almost completely cancels the rest energy of the quarks. This is similar to the situation in $AdS_5 \times S^5$ [13].

V. DISCUSSION

In this paper we have computed the spectrum of mesons in an $\mathcal{N} = 1$ field theory corresponding to fluctuations in the position of a holomorphically embedded D7-brane. In the limit of nearly massless quarks, the field theory is classically conformal, and also conformal at large-N, and the spectrum turns out to be computable exactly, where the dimensions in general are complicated irrationals.

There are a few operators for which the exact results are simple. Among these are the lowest mode, corresponding to a mass term, with dimension $3/2$, and a mode corresponding to a Bogomol'nyi-Prasad-Sommerfield (BPS) fluctuation of the D7-brane, with dimension 3. The existence of these operators suggests that a consistent superpotential for our flavored theory is

$$W = qA_1B_1\tilde{q}. \tag{50}$$

It would be interesting to study the Klebanov-Strassler theory [36] obtained at the end of the duality cascade with the addition of 3-form flux with this superpotential.

In the strictly massless limit, $z_1 = 0$, it is possible to relax our embedding condition slightly. Specifically, with a nonzero mass we imposed a relation between the azimuthal coordinates, $\psi - \phi_1 - \phi_2 = 0$. However, when the mass is zero this condition need not apply; it would be nice to see what relaxing this condition would mean for the field theory (in particular, whether it is possible to realize the cubic superpotential discussed in Sec. III.)

The appearance of irrational dimensions is not surprising, in light of similar results for the glueball spectrum of the conifold [8,34]. However, this feature of the meson spectrum differs from the $\mathcal{N} = 4$ case, where the meson dimensions were pure integers. In particular, we do not find a tower of states with spacing $3/2$, except in the large R-charge limit; more precisely, in this limit the spectrum is of the form $3k/2 + O(1/J)$. It might be possible to compute these $1/J$ corrections in a plane-wave limit, or perhaps in some other formalism. It would be interesting if such a comparison with our exact results were possible.

We have also numerically computed the spectrum for the case of massive quarks. In the large $g_s N$ limit the meson mass gap is significantly smaller than the quark masses. We have uncovered a relatively simple quadratic scaling behavior for the meson masses. It would be nice to find, either with analytical or more numerical work, the exact functional dependence on n, m_1, m_2, etc.

All of our calculations have been in the probe limit and further studies of the backreaction would be interesting, especially for the Klebanov-Strassler deformed conifold theory. However, it may still be possible to learn things from further study of probe theories. In particular, it would be interesting to study the dynamics of nontrivial classical field configurations in the D7-brane world volume. Such fields would correspond to dissolved D3-branes or anti-D3-branes. The antibrane case is particularly interesting, as it would break supersymmetry along the lines of the KKLT scenario [37],[1] but with the possibility for some moduli to be fixed by the D7-brane. We leave these suggestions for the future.

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