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DANTE DECRYPTED: MUSICA UNIVERSALIS IN THE TEXTUAL ARCHITECTURE OF THE ‘COMMEDIA’

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For seven centuries scholars have speculated about the structural design of Dante’s *Commedia* but remain perplexed by the poem’s comprehensive architecture. This study undertakes a strictly empirical quantitative analysis of Dante’s *magnum opus* to address this lacuna. The outcome of this analysis enumerates the correspondence between the foundational rationale of the *Commedia’s* textual architecture and both physical and metaphysical concepts of Ptolemaic cosmology and Pythagorean principles of harmony and proportion as described by Boethius. The poem manifests a musically and mathematically meticulous design conceptualized as *musica universalis* and expressed as *musica instrumentalis* that echoes Paschal and Marian plainchant. With an analytical synthesis of three components—Beatrice’s mathematical identity, the Trinitary ontology of the *terza rima*, and the quantitative properties of the *Commedia’s canto* lengths and their frequency of occurrence—this study decrypts Dante’s comprehensive architectural design of a poem whose structural harmony continues to be felt by readers today.

**Keywords:** Trinity, *Scutum fidei*, *Terza rima*, Architecture, Boethius, Pythagoras, Ptolemy

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In his imposingly titled *The Mind of Dante*, Natalino Sapegno remarks the sense of “profound unity of conception and artistic expression” that the reader of Dante’s *Commedia* enjoys. Sapegno’s is but one in a chorus of voices of critical readers who, though hewing closely to a traditionally fragmentary approach to the poem, acknowledge and extoll the poem’s compositional unity, and echo assertions like those by De Sanctis or by Vallone that the...
Commedia is “l’opera più saldamente unitaria d’ogni tempo” but stop just short of examining the specific elements that account for the reader’s intuition of this unity.

Proponents have for generations affirmed their unambiguous faith in the Commedia’s “unity of conception,” hypothesizing that it is the result of Dante’s predetermined design for the poem and often citing as exemplary the structurally congruous characteristics of the poem’s internal memory, namely its consistently ternary organization, from the gross structure down to its prosody, and its coordinated placement of thematically correlated discourses in corresponding textual regions across the three cantiche. The narrator himself appears to provide grounds for these hypotheses in the Terrestrial Paradise when he declares that limited space (“S’io avessi, lettor, più lungo spazio / da scrivere”) compels him to brevity (“A descriver lor forme più non spargo / rime, lettor; ch’altra spesa mi stringe, / tanto ch’a questa non posso esser largo”). Nevertheless, debate persists regarding the scope and subtleties of the poem’s design. Some argue that the quantitatively coherent placement of structural, textual, and thematic patterns, such as the canto length symmetry at the center of the poem publicized by Singleton and subsequently expounded on by Pegis is simply aleatory, and that inferring hermeneutic intention from them is merely the result of their advocate’s apophenic zeal. Missing from this debate is the empirical analysis of the poem’s quantitative properties necessary to scrutinize the significance of the narrator’s remarks or to articulate an objectively demonstrable architectural design.

This study addresses this lacuna in the scholarship of the Commedia by examining the scope and detail of the poem’s textual architecture and interrogating the rationale for its design within the context of Dante’s poetics, with attention to contemporaneous semiotics. Driving this analysis are elements like the conspicuous patterns of compound dispositional symmetries throughout the text,

3 Aldo Vallone, Strutture e modulazioni nella Divina commedia (Florence: Leo S. Olschki, 1990), 172.
4 Purgatorio 33.136–141; Purg. 29.97–99.
5 Charles Singleton, “The Poet’s Number at the Center,” MLN 80, no. 1 (1965).
like the one exemplified in Figure 1, that challenge the curious reader to examine and possibly articulate the underlying structural design.

Figure 1. Concentric recursive symmetries at the midpoint of the *Commedia*

The most granular element of symmetry in this illustration appears in the verse scansion of the line at the midpoint of the poem §7117 (*Purg.* 17.125) in which Virgil instructs the pilgrim to turn his intellect from retrospectively contemplating the first half of the journey, which now lies behind them, to anticipating what is yet to come on the path ahead. The dispositinal and quantitative symmetry at the center of the *Commedia* is thus reiterated in the narrative symmetry of the surrounding *terzine* (*Purg.* 17.112–139). This congruity is further underscored by the central location of the pilgrim’s itinerary (the fourth terrace: Sloth) and echoed in the quantitative symmetry of the surrounding *canto* lengths from *Purg.* 14 to *Purg.* 20, as well as the global symmetry of the *cantiche*.

Equally compelling are the thematic congruencies that appear throughout the poem that establish and reinforce the correlation of form and signification. A closer look at the narrative structure of the same area of *Purgatorio* outlined in the figure above demonstrates this phenomenon. As illustrated in Figure 2, the twin primacy of both the living example of the Virgin (“che a Cristo più si somiglia”)\(^8\) and the sanctifying teachings of her son (“Filio di Dio e di Maria”)\(^9\) is meticulously woven into the structure of the pilgrim’s itinerary throughout the realm of penitence with symmetrically harmonious periodicity.

\(^{8}\) *Paradiso* 32.85–86.
\(^{9}\) *Par.* 23.136–137.
Figure 2. Maria and the Beatitudes in the symmetrical structure of *Purgatorio*.

For just as the pilgrim’s arrival in each terrace is greeted with an allusion to the Virgin Mary embodying its contrary virtue, so he is delivered of each terrace by its angel custodian intoning the related Beatitude from Jesus’s Sermon on the Mount. The location of each Marian episode relative to its corresponding Beatitude in Figure 2 shows that the pilgrim’s gradually accelerating pace of travel as he approaches the center of the text is mirrored by a reciprocally decelerating pace as he completes the latter half of the climb. Concurrently, a nine- *canto* distance separates the pilgrim’s slumber at the end of each day on the mountain, with each waking dawn marked by an account of his self-revelatory dream starting at the threshold of *Purgatorio* in Canto 9 and ending with the climb’s completion in Canto 27.

These are but two examples of the kind of well-ordered, complex patterns in the poem’s structural disposition that scholars have contributed to documenting with greater attention since Singleton published “The Poet’s Number at the Center.” That an architectural blueprint providing an objective, universal context for such patterns has remained inconspicuous to readers and undiscovered by scholars for seven hundred years requires a radical reexamination of both narrative clues and quantitative features of the poem and imposes an unimpeachable standard of analysis on such a study. The outcome of this analysis reveals that the poem’s structural harmony and “unity of conception” are in fact the consequence of a mathematically meticulous and musically coherent design.

The following preview will provide not only an overview of the guiding premise for this analysis, but also a summary of this
study’s findings before it delves into the technical details of the methods used to decrypt the *Commedia*.

...FICTION RHETORICA MUSICAQUE POITA

The catalogue for the intellectual source material and interpretive context necessary to discover and decrypt the quantitative blueprint for Dante’s *magnum opus* is furnished by the poet himself not only in the *Commedia*, but also in his remarks on the art of poetry. The *Commedia*’s very itinerary across a geocentric Ptolemaic cosmos is significant: The poem’s structure is regulated by the same ordering principles of a conceptual model of a universe wrought by a creator who “tempers and atunes” the harmony of the heavens ("l’armonia che temperi e discerni"). With quantitative properties based on the same Pythagorean principles used to describe the order of the Ptolemaic universe and prosody based on a popular contemporaneous iconography of the Trinity, Dante composes a poetic signifier of the *infinite* and the *universal* in a way that is “profundly conceived” and “majestically articulated” to serve as the stage for the pilgrim’s journey from the *selva oscura* to his final revelatory vision.

Dante articulates the rigour and urgency of developing and executing a poetic design worthy of treating such lofty themes in the *De vulgari eloquentia*. The poet’s “exertion of the intellect, dedicated study of technique, and immersion in knowledge” make thorough use of the quantitative ordering processes available...
through the mathematical liberal arts Musica and Arithmetica in clearly unconventional and apparently unprecedented ways.

The rules of music applied to the poem’s design extend beyond the rhythmical and numerical organization of words and syllables. As a result, the seemingly casual definition of composed vernacular poetry as “nichil aliud est quam fictio rhetorica musicaque poita”\(^{13}\) becomes imbued with a theretofore unprecedented degree of rational precision.

The intellectual authority of the poet’s exacting rationality is founded in Classical works of natural and metaphysical philosophy and icons of the quadrivium, the liberal mathematical arts that concern a speculation on the order of the cosmos from the most basic, Arithmetic, with increasing degrees of complexity through Music, Geometry and Astronomy.\(^{14}\) Dante cites each of these authorities throughout the Commedia using anthropomorphic metonymy, thus representing their intellectual bodies of work as human characters who populate the universe of the pilgrim’s journey. In this way the pilgrim, and by extension the reader, encounters many of the key representatives of Dante’s intellectual universe among the virtuous pagans in the nobile castello allegorizing the artes liberales. Hence, amidst the sighing figures of Limbo whose metonymic function is brought to bear in this study we find “Socrate e Platone” (Inf. 4.134), symbols of Plato’s dialogues of which the poet

\(^{13}\) DVE 2.4.2. As Patrick Boyd notes in his Grove Music Online article, “Dante’s scattered remarks about the relationship of poetry and music are often quoted, and often misrepresented. ‘Poesis … nichil aliud est quam fictio rethovenica musicaque poita,’ […] wrongly rendered (with an earlier reading of ‘posita’ or ‘composita’ for ‘poita’) as ‘poetry is a rhetorical fiction set to music,’ making the musical setting a condition of poetry. A better translation might be: ‘poetry is simply a work of imagination [fictio] composed or made [poita, from Gk. poein] according to the rules of rhetoric and music.’ Good prose is rhetorica poita; so the musical organization of words is certainly that which distinguishes poetry from prose. But ‘musica’ is here used both in a precise and limited technical sense (as governing the rules of rhythm) and in a general sense which allowed Dante to speak of his craft as ‘harmonizing words’ (Conv. 2.13.23; DVE 2.8.5). To bring words into harmony is to organize the sequence of syllables rhythmically and numerically so that they form lines of verse with a fixed number of syllables and certain cadences (musica poita in the technical sense). It is also to temper the harsher and smoother sounds of words (scrupulously defined in DVE 2.7.4–6) so that they will combine to form a structure that is pleasing and appropriate to the meaning (DVE 2.1; Rime 103.1–2; Inf. 32.31–33). Further, it is to bind the lines of verse into groups of three, four or more by rhyme, thus creating the larger metrical units that make up the constituent parts of the stanza in a canzone or ballata, or the quatrains and tercets of a sonnet.” [Emphasis added]. Patrick Boyd, “Dante Alighieri,” Grove Music Online (http://www.oxfordmusiconline.com/ 28 November 2018).

specifically mentions the *Timaeus* in both *Par.* 4 and *Convivio* 3.5.6. Greek Geometry is represented by “Euclide geométra” (*Inf.* 4.142), author of the *Elements*, the definitive compendium on the fundamentals of geometry in the Western tradition, and “Tale” (*Inf.* 4.137) whose inscribed angle theorem Dante cites to specify that when prompted, Solomon did not ask “se del mezzo cerchio far si puote triangol si ch'un retto non avesse” (*Par.* 13.101–102). Sharing a verse with Euclid, the figure of “Tolomeo” (*Inf.* 4.142) metonymically represents the *Harmonics*, a three-volume treatise in which Claudius Ptolemy enumerates the mathematics of music. Turning heavenward to the communion of wise souls in the Sphere of the Sun, we encounter Boethius himself (*Par.* 10.124–129). This particular metonymic citation is especially significant in this study’s analysis because both the *De institutione arithmetica* and *De institutione musica* spell out the established mathematical conception of music in the Middle Ages with a clarity that illuminates the musico-mathematical rationale for the design of the *Commedia’s* quantitative structure.

For early in his *De institutione musica*, Boethius relates a legend in which he explains the properties of music as a mathematical art. It is a well known legend that tells of how Pythagoras was inspired to apprehend the Music of the Spheres upon hearing the hammered anvils in a blacksmith’s workshop ring out at intervals of the octave, the fifth, the fourth, and the second.\(^\text{15}\) The mathematical ratios of these musical intervals expressed the very harmonies of the cosmos that are described by Plato in the *Timaeus* (32B–C, 35–36) and that were widely circulated and elaborated upon throughout Antiquity and the Middle Ages, and especially in Ptolemy’s *Harmonica* (Bk. 3).

Ptolemy, for his part, distinguishes the abstract characteristics of music as mathematical art from music as performable craft by spelling out their different modes of appeal: Where the performable craft appeals to the material perception of “hearing,” the mathematical abstraction appeals to “reason,” or intellectual discernment.\(^\text{16}\) In other words, the distinction between sensory perception

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\(^\text{16}\) “καὶ κριτήρια μὲν ἁμονίας ἀκοῆ καὶ λόγου, οὗ κατὰ τόν αὐτόν δὲ τρόπου, ἀλλ’ ἡ μὲν ἀκοὴ παρὰ τὴν ὑλήν καὶ τὸ πάθος, ὁ δὲ λόγος παρὰ τὸ ἐλίδος καὶ τὸ αἴτιον, ὅτι καὶ καθόλου τῶν μὲν αἰσθήσεων ἱδίων ἔστι τὸ τοῦ μὲν συνεγγυς εὐρετικόν, τοῦ δὲ ἀκριβοὺς παραδεκτικόν, τοῦ δὲ λόγου τὸ τοῦ μὲν συνεγγυς παρα- δεκτικόν, τοῦ δ’ ἀκριβούς εὐρετικόν.” Book 1.1 (“The criteria in
and intellectual perception may be read as that between inference and deduction; feeling or intuiting on the one hand, and knowing or understanding on the other. Ptolemy is concerned with distinguishing between what comes to be known in Boethian terms as *musica instrumentalis*—which “consists of proper and pleasing combinations of sounds”—produced as musical performance—and *musica mundana*, “the final metaphor for proper proportion.” Inherent to *musica mundana* is the “proper ordering of elements in the corporeal universe.” Even though knowledge of the *musica mundana* cannot be acquired by perception, it can be deduced by intellection—by observing the universal order it manifests in the material world, hence the designation of this abstraction as *musica universalis*.

After close examination and analysis, this study reveals how Music as Mathematical Art is key to discerning the architectonic principles of Dante’s *Commedia*. This *musica universalis* is defined by the Ancients as a voiceless, soundless concept used to articulate abstract, dynamic quantities and proportions and to develop a functional, figurative vocabulary for conceptualizing the Order of the Cosmos. Underscoring the intellectual tradition at the root of this concept Pier Vincenzo Mengaldo explains to the reader of the *De vulgari eloquentia* that “la connessione con la musica, da intendersi, come nel *Convivio*, quale intrinseca strutturazione secondo leggi musicali e non quale semplice destinazione a una melodia, è in particolare della tradizione boeziana.” Indeed, the *canto* lengths of the poem evidence a meticulously rational adherence to the fundamentals of harmonic proportions enumerated by Boethius in his two treatises. The *Commedia*’s quantitative framework thus decrypted reveals an additional dimension of significance to the poet’s choice of technical nomenclature whereby verses are referred to as *note*, each of the poem’s one hundred chapters as *canto*, and each of its three books as *cantica*.

Harmonics are hearing and reason, but not in the same way: hearing is the criterion for matter and condition, while reason is the criterion for form and cause. This is because, generally speaking, discovering what is approximate and accepting what is exact are characteristic of perception, while accepting what is approximate and discovering what is exact are characteristic of reason.” Claudius Ptolemy, *Harmonics*, trans. Jon Solomon (Leiden: Brill, 2000), 3–4.


19 See Lino Pertile, “‘Cantica’ nella tradizione medievale e in Dante,” in *Rivista di storia e letteratura religiosa* 28 (1992): 389–412 for a philological and hermeneutic exposition on the technical nomenclature of the poem, along with the preceding article,
The principal cipher, hidden in plain sight, is Beatrice’s mathematical identity as the square of the Trinity declared by Dante in VN 29.3, in conjunction with the poem’s symbolic representation of the Trinity, the terza rima. Beatrice’s mathematical identity is the key to discovering the rationale for one of the fundamental quantitative features of the poem which has to date confounded scholars: namely, the thirteen distinct canto lengths variously allotted to the poem’s one hundred canti, the frequencies of occurrence of each of these thirteen canto lengths, and their distribution across the poem. When the poem’s quantitative structure is analyzed in terms of Pythagorean and Ptolemaic cosmic order and Boethian musical mathematics, its fundamental quadrivial logic becomes clear, further underscoring the cosmomimetic nature of Dante’s compositional design. By this quadrivial logic, each of the thirteen distinct canto lengths corresponds mathematically to a musical note on the Guidonian hexachord. This in turn makes it possible to encode the quantitative structure of the entire poem as an easily accessible melody, allowing its performer to hold the mathematical information accounting for all 14,233 verses of the poem in a brief musical mnemonic.

Another hermeneutically striking discovery born of this analysis is that this mathematically encoded melody mimics a popular plainchant specific to Paschal and Marian feasts recorded in manuscripts of secular liturgy in central Italy during Dante’s time. Correlating the quantitative structure of the Commedia specifically to liturgical feasts that are thematically congruous with the pilgrim’s journey is significant for two immediately pertinent reasons: First, it means that Dante effectively translates the mathematically abstract design of his Commedia’s cosmomimetic structure conceptualized as musica universalis into an audibly concrete and mnemonically practical piece of musica instrumentalis. Secondly, it shows how, in doing this, the poet is able to tie the Commedia’s quantitative structure to a liturgical melody that uniquely signifies not only the Virgin Mary, the Queen of Heaven by whose intercession the pilgrim undertakes and completes his journey, but also Eastertide, the period in the liturgical calendar during which the journey takes place. Through the musica instrumentalis synthesized from the mathematical musica universalis, Dante further underscores the twin primacy of Mother and Son in the structural hermeneutics of the Commedia.


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…TUTTE LE CARTE ORDITE…

The quantitative blueprint of the Commedia that emerges in this analysis will serve scholars who wish to put the congruous patterns of disposition described in various fragmentary studies of the poem’s structure into a unifying context. Furthermore, making the poem’s rational design transparent sheds light on some of the poet’s own references to said design. Consider, for example, the narrator’s extradiegetic allusion, in direct address to his reader, to the limited space available to him in the closing passages of Purgatorio:

S’io avessi, lettor, più lungo spazio
da scrivere, i’ pur cantere’ in parte
lo dolce ber che mai non m’avria sazio;
ma perché piene son tutte le carte
ordite a questa cantica seconda,
non mi lascia più ir lo fren de l’arte. (Purg. 33.136–141)

Masked by a thin veil of narrative reticentia, this passage nonetheless appears to affirm the stringent demands of a meticulous design to which the poet must adhere. It has traditionally been read by commentators as a casual reference to the uniformity of the poem’s gross proportions among the three cantiche. Yet the narrator’s concerns about the limited availability of textual space in this particular address to the reader is explicitly foreshadowed by another in Purg. 29, when the narrator interrupts his demonstration of the Bible’s internal exegetic mechanism using the winged creatures of Ezechiel and John’s visions to declare that he is compelled to tell of other things and cannot expend his rime on further details:

A descriver lor forme più non spargo
rime, lettor; ch’altra spesa mi strigne,
tanto ch’a questa non posso esser largo; (Purg. 29.97–99)

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21 Referring his reader to Ezechiel’s divergent text for more details about the winged creatures, the narrator then reiterates that John’s description will confirm what he has just related. Contrary to the interpretation of critics dismayed by what they see as Dante’s exaggerated estimation of his own power to arbitrate a textual inconsistency in the Bible, the passage is less about the narrator’s authority than it is about John’s. For drawing attention to the difference between these two ostensibly identical elements in the Old and New Testament iterations of a revelatory vision dramatically illustrates the difference between revelation in prophecy as seen in Ezechiel’s Old Testament text and revelation in fact as seen in John’s New Testament Apocalypse.
Here the commentators are all in agreement that the poet of the *Commedia* is alluding to a fixed quota of verses which he must devote to treating predetermined matters. Such a fixed quota, metonymically reasserted as “tutte le carte / ordite a questa cantica seconda” in turn implies a compositional framework that is at least measured in a preset quantity of pages. Given the absence of information about the size or the rule dimensions of Dante’s “carte,” to say nothing of the varied capacities of the pages of the myriad editions of the poem that have been published throughout its history, the only stable units of measurement available for quantitative analysis of the *Commedia* are those to which the narrator explicitly alludes:

\[\text{. . . mi conven far versi} \]
\[\text{e dar matera al ventesimo canto} \]
\[\text{de la prima canzon . . . (Inf. 20.21–23)} \]

Starting therefore with the *versi*, this study first draws upon the Trinitary ontology of the poem’s *terza rima* based on its correspondence to the *scutum fidei* symbol for the Athanasian Creed to explore its radical function in the architecture of the text. The second part of this analysis specifies gradually how the *quadrivial* foundation of the *canto* lengths are contingent upon the particular dynamics of the *terza rima* in conjunction with the mathematical symbol of Beatrice.\(^{22}\) Using the vocabulary of *musica* and *arithmetica* defined by Boethius, the analysis demonstrates how applying Beatrice’s number, nine, to a simple arithmetic algorithm reveals the correlation between each of the poem’s thirteen *canto* lengths and a specific musical note across the three different registers of the *hexachordum durum*, #1, #4, and #7 in the musical scales popularized by Guido d’Arezzo’s mnemonic device for teaching singers.\(^ {23}\)

\(^{22}\) See Stefano Mengozzi, *The Renaissance Reform of Medieval Music Theory: Guido of Arezzo between Myth and History* (Cambridge: Cambridge University Press, 2010), 2–5, and Karol Berger, “The Guidonian Hand,” in *The Medieval Craft of Memory: An Anthology of Texts and Pictures*, ed. Mary Carruthers and Jan M. Ziolkowski (Philadelphia: University of Pennsylvania Press, 2002). In her discussion about solmization, Margaret Bent emphasizes the fundamentally mnemonic function of the Guidonian hexachord: “Hexachords are inter-valvically identical, and would have been pointless as tools for classification; pieces are not “in” hexachords, as they maybe conceived in keys or classified by modes. The only point of hexachords lies in their flexible mutual relationship. [. . .] Hexachords are means of articulating decisions made on other grounds; they are descriptive, not prescriptive, as Cross agrees (1990: 181). The practical and pedagogical language of solmisation is the principal or only means by which theoretical writers can specify and name semitone inflections in the

\(^{23}\) Adoyo: DANTE DECRYPTED

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DANTE’S *TERZA RIMA* AND THE TRINITY

Among the structural elements of the *Commedia* that seamlessly integrate form and signification, the Trinitary symbolism of the poem’s ternary architecture is the most axiomatic. The poem’s distinctive rhyme scheme, in particular—variously emulated by poets and extensively contemplated by scholars—has for so long been seen to represent the Trinity without examination throughout the life of the *Commedia* that the empirical foundation for this interpretation remains unexplained by the commentary tradition. Even in his essay “The Significance of the *Terza Rima*,” John Freccero acknowledges that although the *terza rima* “has seemed too obviously to represent the Trinity,” it may be read as an inherently fallacious axiom since “everything represents God in this poem; the abstraction is so remote as to be meaningless.”24 Some scholars have even gone as far as to question the viability of the Trinitary symbolism of the *terza rima* because of what they consider to be an important structural inconsistency. The rhyme scheme begins each *canto* with the pattern $A_2 B_1 A_3 | B_2 C_1 B_3 | C_2 D_1 C_3 \ldots$ and continues through an unbroken series of tercets concatenated by interwoven triad rhymes that conclude with $\ldots X_2 Y_1 X_3 | Y_2 Z_1 Y_3 | Z_2$. In his analysis, Freccero affirms that the seemingly “arbitrary” duality of the opening and closing rhymes, $A_2A_3$ and $Z_1Z_2$, has provoked some disquiet among theorists concerned that the *rime rilevate* subvert the form’s triadic integrity, making it seemingly “incompatible with the idea that it might represent the Trinity.”25 In light of these concerns, it is incumbent upon any analysis of the poem’s architecture to interrogate this hermeneutic axiom and, if possible, articulate the concrete foundation for correlating between the *terza rima* and the Trinity in order to ensure the integrity of the study’s outcome. To do so, one need only follow Gorni’s suggestion and approach the poem on its own terms: “la poesia reclama sempre un’adesione, prima che un giudizio. Bisogna farsi un po’ gli apostoli del testo che si legge, che si è scelto di leggere.”26

In that spirit, this analysis accepts the *Commedia’s* representations of the Trinity, which not only observe the exacting criteria

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25 Ibid., 261.
for mathematical language and geometric iconography, but also succeed in dramatizing the doctrine of the Trinity. The clearest such example is the poem’s purely quantitative characterization of a creator who contains and permeates all creation:

Quell’uno e due e tre che sempre vive  
e regna sempre in tre e ’n due e ’n uno,  
non circunscritto, e tutto circunsctre . . . (Par. 14.28–30)

When asked later by Saint Peter to articulate his faith, the pilgrim Dante echoes the doctrine of the Trinity in plainly orthodox terms:

[. . .] Io credo in uno Dio  
solo ed eterno, che tutto ’l ciel move,  
non moto, con amore e con disio;  
[. . .]  
e credo in tre persone eterne, e queste  
credo una essenza si una e si trina,  
che soffera congiunto ‘sono’ ed ‘este’. (Par. 24.130–141)

Both these representations pithily synthesize the Trinitary doctrine as it is articulated in the Athanasian Creed:

we venerate one God in the Trinity, and the Trinity in oneness; neither confounding the persons, nor dividing the substance; for there is one person of the Father, another of the Son, (and) another of the Holy Spirit [. . .] and nevertheless there are not three eternals, [. . .] thus the Father is God, the Son is God, (and) the Holy Spirit is God; and yet there are not three gods, but there is one God. 27

27 The complete text enumerating the doctrine of the Trinity in the Athanasian Creed is hereby abridged to highlight the passages immediately pertinent to this study’s observations about the structure of the *terza rima*. “Fides autem catholica haec est: ut unum Deum in Trinitate, et Trinitatem in unitate veneremur. Neque confundentes personas, neque substantiam seperantes. Alia est enim persona Patris alia Filii, alia Spiritus Sancti: [. . .] Aeternus Pater, aeternus Filius, aeternus [et] Spiritus Sanctus. Et tamen non tres aeterni, sed unus aeternus. [. . .] Ita Deus Pater, Deus Filius, Deus [et] Spiritus Sanctus. Et tamen non tres dii, sed unus est Deus. [. . .] Pater a nullo est factus: nec creatus, nec genius. Filius a Patre solo est: non factus, nec creatus, sed genius. Spiritus Sanctus a Patre et Filio: non factus, nec creatus, nec genius, sed procedens. Unus ergo Pater, non tres Patres: unus Filius, non tres Filii: unus Spiritus Sanctus, non tres Spiritus Sancti. Et in hac Trinitate nihil prius aut posterius, nihil maius aut minus: Sed totae tres personae coaeternae sibi sunt et coaequales. Ita, ut per omnia, sicut iam supra dictum est, et unitas in Trinitate, et Trinitas in unitate veneranda sit. Qui vult ergo salvus esse, ita de Trinitate sentiat.” Heinrich Denzinger, *Enchiridion symbolorum, definitionum et declarationum de rebus fidei et morum* (Freiburg im Breisgau: Herder, 1911), 17–18 [“But the Catholic faith is this, that we venerate one God in the Trinity, and the Trinity in oneness; neither confounding the persons, nor dividing the substance; for there is one person of the Father, another of
For Scholastic theologians of the thirteenth century, the language most suitable for representing the Holy Trinity is that of Geometry. Following a passage comparing God to light, and the glorified and the damned to refracting and reflecting bodies, Bacon spells out the semiotic demands of representing the Trinity in his *Opus majus*:

It is impossible for the blessed trinity and unity of essence to be represented more aptly by us in an example [. . .] perceptible to the senses than by means of matters pertaining to geometry. For in the triangle alone among all things made is there found a unity of essence with the distinction of three embracing the same essence. *Since each of the three angles occupies the same identical space and the whole of it, as is evident to the sense, and yet they are in fact distinct angles, which is wonderful in a creature, nor is it found elsewhere except in the supreme Trinity. And when on a given line it is required to construct an equilateral triangle, as the first proposition of Euclid states, what can be assumed more properly in order that we may perceive that if God the Father be granted, a trinity of equal persons presents itself?*


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Like Dante’s “tre giri / di tre colori e d’una contenenza,” (Par. 33.116–117) Bacon’s three equal yet distinct angles unify to contain a single space. The appeal for geometric rigor is answered by a simple triangular icon that illustrates the oneness of the three Persons in the Triune God, the scutum fidei. Evolving over several centuries, it draws inspiration from Paul’s call to the faithful that they gird themselves with the arms of faith. Also known as the “Shield of the Trinity,” this popular icon for the doctrine of the Trinity not only enjoys circulation in illustrated manuscripts, it is also extensively applied in church architecture during the Middle Ages from at least the eleventh century and well into the fourteenth century and beyond.

Figure 3. Petrus Picatviensis (1130–1215)

Figure 4. Modern diagram of scutum fidei

30 “propterea accipite armaturam Dei ut possitis resistere in die malo et omnibus perfectis stare state ergo succincti lumbos vestros in veritate et induti loricam iustitiae et calciat pedes in praeparatione evangelii pacis in omnibus sumentes scutum fidei in quo possitis omnia tela nequissimi ignea extinguere et galeam salutis adsumite et gladium Spiritus quod est verbum Dei per omnem orationem et obsecrationem orantes omni tempore in Spiritu et in ipso vigilantes in omni instantia et obsecratione pro omnibus sanctis et pro me ut detur mihi sermo in apertione oris mei cum fiducia notum facere mysterium evangelii pro quo legatione fungor in catena ita ut in ipso audeam prout oportet me loqui.” Ephesians 6. 13–20. [“Therefore take unto you the armour of God, that you may be able to resist in the evil day, and to stand in all things perfect. Stand therefore, having your loins girt about with truth, and having on the breastplate of justice, And your feet shod with the preparation of the gospel of peace In all things taking the shield of faith, wherewith you may be able to extinguish all the fiery darts of the most wicked one. And take unto you the helmet of salvation, and the sword of the Spirit (which is the word of God). By all prayer and supplication praying at all times in the spirit; and in the same watching with all instance and supplication for all the saints: And for me, that speech may be given me, that I may open

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The iconographic conventions of the *scutum fidei* make it readily legible as a symbol of the Athanasian Creed. The three vertices of the triangle are each designated Pater, Spiritus Sanctus and Filius respectively, with the phrase “non est” appearing between each vertex. A fourth node at the center of the triangle represents the boss of the shield (*umbo*) and is designated Deus. Finally, the shared identity of the three Persons with the Godhead is affirmed by connecting the three vertices to the central *umbo* with “est.” Thus it appears—albeit in an isosceles triangle—in a ca. 1208–1216 copy of Peter of Poitier’s *Compendium* with clearly designated nodes, “non est” articulating the perimeter and “est” the bridges to the boss at the center.

Unlike modern versions of the *scutum fidei*, medieval illustrations place the Father and the Holy Spirit on the upper edge of the shield while the Son appears at the bottom along with an image of the crucifix. In the *Chronica Majora* Matthew Paris juxtaposes this version of the *scutum fidei* with a corresponding illustration of Augustine’s *scutum animae*.

Predating Roger Bacon’s concerns about the exigencies of representing the Trinity by several centuries, Augustine had argued in the *De Trinitate* that it is impossible to talk about the Trinity except by analogy to a trinity of the mind, “*memoria et intelligentia et voluntate*”\(^{31}\) (Figure 5):

![Figure 5. Matthew Paris (1200–1259)\(^{32}\)](https://repository.upenn.edu/bibdant/vol1/iss1/3)

This arrangement also appears in John of Wallingford’s *Miscellanea* and William Peraldus’s *Summa contra vitis*:

---


\(^{32}\) Cambridge, Parker Library, MS Corpus Christi College 16II, fol. 49v.
An illustration in the Durham Cathedral manuscript containing Robert Grosseteste’s *Dicta* underscores the *scutum fidei’s* conventional appeal. An avid natural philosopher and renowned scientist, the Bishop of Lincoln’s notes on the semiotics of representing the Trinity provide a clarifying lens through which to appreciate the significance of the *scutum fidei* as an icon with ready popular currency. Grosseteste’s interpretation of the *scutum fidei* as symbol of the Trinity in scientific and metaphysical terms is particularly illuminating. Among the variety of symbolic *scuti* he annotates in his *Dicta*, Grosseteste dedicates *Dictum 95* solely to describing the four nodes of the *scutum fidei* (see Figure 8) in terms of the creative virtues of the Trinity that account for the qualities of intelligibility in the creation of the universe.36

33 London, British Library, MS Cotton Julius D. VII, fol. 3v.
34 London, British Library, MS Harley 3244, fol. 28r. The allegorical illustration in William Peraldus’s *Summa contra vitii* dramatizes of Paul’s admonition to the faithful to take up the Shield of Faith, the Helmet of Salvation and the Sword of the Spirit (*verbum Dei*) for their protection from the fiery arrows of the Enemy. See also Suzanne Lewis, *The Art of Matthew Paris in the Chronica Majora*, (Berkeley: University of California Press in collaboration with Corpus Christi College, Cambridge, 1987), 194 and 494 n.118.
35 Scholastic scientists like Grosseteste, Albertus Magnus, Roger Bacon, Witelo and Theodoric of Freiberg used not only mirrors and quartz prisms, but also spherical beryll crystals and water filled medical flasks to simulate magnified versions of raindrops to observe the effects of light refraction and reflection in nature John Gage, *Color and Meaning: Art, Science, and Symbolism* (Berkeley: University of California Press, 1999), 123–125.
The boss of this shield gleams with inaccessible light. The corner where ‘Father’ is written is figured with the virtue of universal power; the corner where ‘Son’ is written with the image of universal beauty; and the corner where ‘Holy Spirit’ is written with the most suitable bonding of the order of universal things. This shield, engraved in trustfulness operating in love, is the shield of faith.

Applying his observations in experimental optics about the properties and behavior of luminosity, Grosseteste elaborates a
metaphysics of light wherein “God is light,”³⁹ both the point that begets radiating splendor and the spherical volume into which this begotten splendor propagates. In a remarkably cogent synthesis of scientific and theological concern in figuratively representing the Trinity, the most significant concept in this philosophy is the mutual reflectivity of the Three Persons in which:

. . . one expresses from himself a second; but the second reflects himself back into the first and expresses his own reflection out of himself into the first. Or rather, the first is reflected through the second [back] into himself, and this reflection proceeds at once from the first and from the second.⁴⁰

This mutual reflectivity of the Father and the Son from whom the Holy Spirit proceeds is clearly represented by Dante the pilgrim’s vision at the end of Paradiso. In that vision of the Trinity, two Persons of the Triune God, the sources of “l’Amore / che l’uno e l’altro etternalmente spira” (Par. 10.11–12) appear to reflect each other and the Holy Spirit proceeds from the two of them together:

l’un dall’altro come iri da iri
parea riflesso, e ’l terzo parea foco
che quinci e quindi igualmente si spiri. (Par. 33.115–120)

The conceptual parallel in these closely contemporaneous representations of the Trinity indicates that understanding the structural correspondence between the Commedia’s terza rima and the scutum fidei icon will render the Trinitary ontology of the poem’s prosody empirically transparent. This, in turn, is crucial to discerning the structural hermeneutics of the Commedia’s quantitative design as will become apparent shortly.

Like Grosseteste, Dante describes God as light, and he, too, perceives the creator of the universe, who contains all and is contained by none, as both a point and a volume.⁴¹ In the final stage of the pilgrim’s journey, where the poet’s diction is increasingly saturated with the language of geometry, this paradox is ostensibly irreconcilable. However, following Bacon’s lead to consult Euclid’s Elements shows the progressive definitions of fundamental geometric entities that Euclid enumerates can rationally resolve the

⁴⁰ Ibid., 226.
⁴¹ For the punto in which all contingencies are present, see Par. 17.17, 28.16, 28.41, 28.95, 29.19. For the volume that contains all of creation, see Par. 33.85–93.

~ 55 ~
The point, the first fundamental geometric entity defined by Euclid, is without dimension (A in Figure 9). The simple straight line $BC$ extends between two points and has one dimension: length. The plane surface has two dimensions—length and breadth—and its simplest form is the triangle, constructed by connecting a third discrete point $A$ mutually to two other points lying on a discreet line $BC$, resulting in $\triangle ABC$. The last fundamental geometric entity defined in the *Elements* is the volume which has length, breadth, and depth, and is therefore three dimensional. The volume, of which the tetrahedron is the first basic Platonic solid, evolves from the plane by connecting all three vertices of $\triangle ABC$ to a fourth point, $D$, that is located on a separate plane.\(^{42}\)

![Figure 9. Point, line, plane and volume.](image)

Seen through a Euclidian geometric lens, therefore, the “punto” is the dimensionless point of origin, the *alpha* from which all of creation springs, while the “volume” is the *omega*, the complex volume where all of creation, time, and substance is contained. The tetrahedron is thus the ideal three-dimensional model for the unified “trinity of equal persons” in Bacon’s equilateral triangle, each one discreet yet dimensionally indistinguishable within the complex whole, while also comprising an identical, equally situated fourth vertex for the boss of the *scutum fidei*: *Deus*.

Applying this three-dimensional tetrahedral configuration to the Trinity represented by the *scutum fidei* resolves the structural concern of critics who consider the duality of the *rime rilevate* ($A_2–A_3$ and $Z_1–Z_2$) to be a structural flaw of the *terza rima*.\(^{43}\) Rather, these opening and closing rhymes are in fact an integral part of a robust closed structure with no loose ends. In the illustrations below, the *scutum fidei* is presented in a three-dimensional rendering.

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\(^{43}\) Freccero, “*Terza Rima*,” 261.
to effectively make use of the icon’s tetrahedral form in the analysis that follows.

![Figure 10. Scutum fidei as tetrahedron](image)

![Figure 11. Scutum fidei Progressive sequence](image)

In Figure 11, a path at the edges of the tetrahedron adjoining the vertices traces a linear reading of the diagram with vectors from left to right starting with the Pater (P) to the Spiritus Sanctus (S\(^3\)S), then turning to go from the Spiritus Sanctus to the Filius (F), and finally, angling upward from the Filius to the central vertex, where the essential Godhead, Deus (D), is at the boss of the shield. In this tetrahedral construction, beginning with the Pater is by no means arbitrary. Grosseteste’s emphasis on the self-propagating activity of light reflected and reflecting within itself\(^{44}\) underscores the concept that the Pater is the absolutely originative God from whom the second Person is generated.

![Figure 12. Scutum fidei - 2nd cycle](image)

![Figure 13. Terza rima (ABA BCB C)](image)

Once the initial path around the *scutum fidei* has been traced through the four vertices, the essential identity of all Three Persons to the centrally located Deus vertex makes it possible to simply continue forward, this final vertex now becoming a new expression of the Pater that leads to a new iteration of both the Filius and the Spiritus Sanctus as shown in Figure 12.

The sequential pattern thus established, the iteration \((P \rightarrow (S^pS) \rightarrow (F) \rightarrow \) may continue for as long as necessary, with the ever-present possibility of closing with a final arrival at the central vertex, \((D)\), thus generating the pattern of the *terza rima* that cycles through each *canto*.

Figure 13 illustrates how in the resulting iterative expression of the Trinity, each of the three Persons corresponds to one of three axes. Translated into a pattern of versification, each instant represents the rhyme of a verse and all the rhymes in a *canto* are expressed on the axis that begins with the Pater and concludes with the Deus. The Filius, begotten of the Father and mutually reflective with the Father, mirrors the Father, resulting in the same rhyme. The Spiritus Sanctus, which emanates equally from the Father and the Son, yet is also distinct from them, appears in the third axis, its distinct characteristics manifesting as the new rhyme in the second verse of every *terzina*. The essential identity of the three Persons, Deus, includes and expresses the characteristics of the newly introduced rhyme, poised to begin yet another cycle.

As a recursive iteration of the *scutum fidei* throughout the pilgrim’s journey, the *terza rima*’s structural consistency and ubiquity is a signifier for the *Commedia*’s mimetic representation of Divine order and omnipresence in all of Creation. Three quantitative components of this structure are integral to the musico-mathematical architecture of the poem: (#1), the singular point, or verse; (#4), the four-point tetrahedral *scutum fidei*; and (#7), the seven-point iteration of the *scutum fidei* that establishes the pattern of the *terza rima* along the axes of the three Persons of the Trinity. Together with Beatrice’s mathematical identity, these three prosodic units decrypt the rationale for the *canto* lengths of the *Commedia*, revealing a design that has confounded critics for seven hundred years.

**Beatrice as Mathematical Abstraction**

Reflecting on the semiotics of the number nine in Beatrice’s life in the *Vita nova*, the young Dante contemplates its recurring ubiquity in the story of his love for her, invoking the authority of both
Ptolemy and Christian doctrine.\textsuperscript{45} Much like the host of angels concertedly singing Osanna in the poet’s fevered dream of Beatrice’s death,\textsuperscript{46} the nine heavens aligned perfectly at her birth. According to Dante, however, that only approximates the true significance of the number nine.

For in reality, Beatrice is a divine miracle and identifying her with nine (9) is not merely numerological but rather mathematical: \( f(x) = x^2 \), where \( x = 3 \), the Trinity, and the value of \( x^2 \) is 9. Beatrice (9) is a pristine product of her divinely immaculate factor (3).\textsuperscript{47} The entire passage spelling out this mathematical abstraction of Beatrice can be expressed as a series of simple reiterative equations and algebraic statements that come full circle, as illustrated in the following diagnostic diagram:

\[
\begin{align*}
[a] & \quad 3 = \sqrt{9} \\
[b] & \quad \text{for } f(x)=9, \text{ the set of } x = 3 \\
[c] & \quad 3 \times 3 = 9 \\
[d] & \quad \therefore \ 3 \times 3 = 9 \\
[e] & \quad \text{The Trinity } = 3, \text{ i.e. } (1 1 1) \\
[f] & \quad \text{Beatrice } = 9 \\
[g] & \quad \sqrt{9} = 3 = \text{Trinity}
\end{align*}
\]

Because her number corresponds to cosmic order, Beatrice effectively manifests as one of the key organizing principles of Dante’s cosmomimetic poem, determining the pace of the itinerary and the narrative and thematic arrangement of the text, among a variety of other elements. This organizing function is transparent when the thirteen \textit{canto} lengths of the \textit{Commedia} are expressed as multiples

\begin{itemize}
\item \textsuperscript{45} VN 29.22.2.
\item \textsuperscript{46} VN 23.27.7.
\item \textsuperscript{47} The raised letters in brackets in the passage below correspond to discrete algebraic expressions. See the analysis below for the English translation. “. . . più sottilmente pensando, e secondo la infallibile veritate, questo numero fue ella medesima; per similitudine dico, e ciò intendo così. [a]Lo numero del \textit{tre} è la radice del \textit{nove}, però che [b]sanza numero altro alcuno, per se medesimo fa \textit{nove}, si [c]come vedemo manifestamente che \textit{tre} via \textit{tre} fa \textit{nove}. [d]Dunque se lo \textit{tre} è fattore per sè medesimo del \textit{nove}, [e]e lo fattore per sè medesimo de li miracoli è \textit{tre}, cioè Padre e Figlio e Spirito Santo, li quali sono \textit{tre} e \textit{uno}, questa donna fue accompagnata da questo numero del \textit{nove} a dare ad intendere [f]ch’ella era uno \textit{nove}, cioè uno miracolo. [g]la cui radice, cioè del miracolo, è solamente la mirabile Trinitade” (VN 29.23.3).
\end{itemize}
of 9. The results show that each *canto* length can exceed any one of six enneads by +1, +4, or +7.

<table>
<thead>
<tr>
<th>enneads</th>
<th>+1v verses</th>
<th>+4v verses</th>
<th>+7v verses</th>
<th>Musical note (chord)</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>99 + N</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>99+1N = 99+9 = 108</td>
<td></td>
<td>115</td>
<td><em>ut</em></td>
</tr>
<tr>
<td>2</td>
<td>99+2N = 99+18 = 117</td>
<td></td>
<td>124</td>
<td><em>re</em></td>
</tr>
<tr>
<td>3</td>
<td>99+3N = 99+27 = 126</td>
<td></td>
<td>130</td>
<td><em>mi</em></td>
</tr>
<tr>
<td>4</td>
<td>99+4N = 99+36 = 135</td>
<td>136</td>
<td>139</td>
<td><em>fa</em></td>
</tr>
<tr>
<td>5</td>
<td>99+5N = 99+45 = 144</td>
<td>145</td>
<td>148</td>
<td><em>sol</em></td>
</tr>
<tr>
<td>6</td>
<td>99+6N = 99+54 = 153</td>
<td>154</td>
<td>157</td>
<td><em>la</em></td>
</tr>
</tbody>
</table>

Table 1. *Canto* length enneads (N)

Sorted accordingly, the outcomes of this operation fall into three distinct groups in the decidedly Pythagorean proportions of 3:4:6. The first group has three consecutive multiples of 9 plus a single verse (+1v). The second are four consecutive multiples of 9 plus 4 verses (+4v). The third group has six consecutive multiples of 9 plus 7 verses (+7v). Each of the six enneads in turn correlates to a note on the corresponding degree of the Guidonian *scala* and consists of 7 hexachords successively overlapping across a range of 3 octaves:

![Figure 14. The Guidonian scala](image1)

![Figure 15. *Canto* lengths on the Guidonian scala](image2)
The seven Guidonian hexachords all use the same series of mnemonic syllables (\textit{ut–re–mi–fa–sol–la}) to identify the scale of pitches. The three scales that begin on \textit{Γ} (\textit{Gamma}), \textit{G} and \textit{g}, \#1, \#4, and \#7, are designated \textit{hexachorda dura}, while the scales that begin on \textit{C}, \#2 and \#5, are designated \textit{hexachordum natural}, and those that begin on, \textit{F}, \#3 and \#6, are designated \textit{hexachordum molle}. Figure 15 correlates the \textit{canto} length ennead sets \(+1\text{v}, +4\text{v}, \text{and } +7\text{v}\) to the corresponding three \textit{hexachorda dura} \#1, \#4, and \#7.

![Table 2. Canto length frequency of occurrence by hexachordum durum](image)

Table 2 provides a comprehensive look at the number of times each \textit{canto} length is used in the poem. Noteworthy here is the detail that although the number of notes in each \textit{hexachordum} group differs (i.e. 3:4:6), each group is equally represented. Thirty-three \textit{canto} lengths correspond to notes in the lowest register hexachord, \(\#1\) \textit{Γ}–\textit{E}; thirty-four (34) \textit{canto} lengths to notes in the middle register hexachord \(\#4\) \textit{G}–\textit{e}; and another thirty-three (33) \textit{canto} lengths to notes in the highest register hexachord \(\#7\) \textit{g}–\textit{ee}, bringing the total number to one hundred \textit{canti}.

The \textit{canto} lengths in each \textit{hexachordum} group are in turn distributed across the poem, resulting in a proportional cross-correlation with the \textit{cantiche} that offers further empirical evidence of the compositional unity and rational design of the \textit{Commedia}'s quantitative structure. Since this design accounts for the poem’s \textit{canto} lengths and allots a fixed number of verses to each realm, discerning it illuminates the poet’s logistical concern that “pieno son tutte le carte / ordite a questa cantica” (\textit{Purg.} 33.139–140). In this light, the poet’s decision to curtail further details about the winged creatures of the procession in Terrestrial Paradise—“più non spargo / rime, lettor; ch’altra spesa mi strigne” (\textit{Purg.} 29.97–
98)—is not merely a rhetorical conceit, but rather a matter of practical constraint.

Table 3. Canto lengths in each hexachordum group

Collating the musical notes corresponding to each canto length according to the number of times they occur in the Commedia reveals intervallic relationships of a distinctly Pythagorean nature: the octave, the 5th, the 4th, and the 2nd. The notes of the canto lengths that occur only once, namely e, a, and ee (160, 124, and 157), together form intervals of the octave, the 4th and the 5th on the musical scale. The double instance of g (115), the seven instances of E (154), and the nine instances of dd (151) all occur as discrete unisons. The four-fold instances of b and bb (130 and 133) and the sixteen-fold instances of c and cc (139 and 142), form unadorned octaves. The thirteen instances of the C, D, and d (136, 145, and 148) also create an octave, albeit one accompanied by the interval of a 2nd at C–D.

The series of canto length frequencies is also by no means arbitrary. Rather, it closely follows Boethius’s process for generating inequalities from equalities to create geometric series. Although two of the values (7 and 13) in the Commedia’s series of

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48 Boethius, Musica, 1.10–11, 17–19.
49 Boethius, Arithmetica, 6, 1.33. For his enumeration see ibid., 1.32, and ibid., 2.2.
canto length frequencies (1, 2, 4, 7, 9, 13, 16) vary from their counterparts in the Boethian sequence (1, 2, 4, 6, 9, 12, 16), the internal proportions of the geometric series of the sesquialter (2:3 ratio) and the sesquitetius (3:4 ratio) are consistent.

Not only do the binary (1, 2, 4), the ternary (1, 3, 9) and the quadruple (1, 4, 16) multiplex series remain intact, the sesquialter (2:3) or (3:2) and sesquitetius (3:4) or (4:3) proportions of the differences between the squares and the intermediate values also remain constant in the modified series. The mathematical consistency of this new series further underscores the rational unity of the Commedia’s quantitative design.

\[
\begin{align*}
2^2 & \rightarrow (2^2 + 2) \rightarrow (3^2 + 2 + 3) \rightarrow (4^2 + 3 + 4) \\
& = (3^2 - 3) \rightarrow (2^2 + 2 + 3) \rightarrow (4^2 - 4) \\
& = & & = \\
4 & & 9 & & 16
\end{align*}
\]

Table 4. Intermediate intervals of the sesquialter and sesquitetius (2:3:4) ratios.

Boethius explains, as illustrated in Table 4, that in order to generate the square of any whole number, it is enough to know the square of the number immediately preceding or following it. Given consecutive numbers A and B, where A is less than B, the sum of A + B plus the square of A yields the square of B: \(A^2 + [A + B] = B^2\), hence \(2^2 + [2 + 3] = 3^2\). Adding only the smaller number to its square, \(A^2 + [A]\), or subtracting only the larger number from its square \(B^2 - [B]\), yields the intermediate rectangular numbers: \(2^2 + [2] = 6\) and \(3^2 - [3] = 6\).
The intermediate values (7 and 13) in Dante’s series of frequencies are the result of altering a step in Boethius’ equation. Instead of adding the smaller number to its square \(A^2 + [A]\), Dante instead adds the larger number to the square of the smaller number thus: \(A^2 + [B]\). As a result, the relatively stable intermediate rectangular values in Boethius’ multiplex series give way to irregular, unstable prime numbers \(2^2 + [3] = 7\) and \(3^2 + [4] = 13\).

\[
\begin{align*}
2^2 & \rightarrow (2^2 + 3) \rightarrow 3^2 & \rightarrow (3^2 + 4) & \rightarrow 4^2 \\
& = (3^2 - 2) & = (2^2 + 2 + 3) & = (4^2 - 3) & = (3^2 + 3 + 4)
\end{align*}
\]

Table 5. Reformulated intermediate intervals for canto length frequencies

THE MUSIC OF THE COMMEDIA

When the results of this analysis are transcribed following the order of canto lengths as they occur in the Commedia, the outcome is striking. The transcription in Figure 19 presents the sequence of notes in two parallel staves: the grand staff with each note in its native hexachordum register shows a dramatically disjunct contour, while the lower staff in which the same notes have been consolidated by virtue of their solfège identity into a single common register presents a conjunct monophonic contour that is therefore readily available to be sung, heard, and memorized. Where the grand staff annotates the mathematical abstraction of musica universalis, the solfège synthesis translates it into musica instrumentalis.

The notes on the grand staff are color coded to distinguish each of the hexachorda dura by range: the canto lengths in group +1v in red, those in group +4v in green, and those in group +7v in black.

Maintaining the principle of organization of Beatrice’s number, the canto lengths are here transcribed on nine measures. The cantos of Inferno therefore appear in the first three measures, Purgatorio on the next three measures, and Paradiso on the final three. The symmetrical arrangement of canto lengths discussed by
Singleton\textsuperscript{50} and subsequently debated by Pegis\textsuperscript{51} and Logan\textsuperscript{52} are clearly visible in the middle measure of \textit{Inferno} (16, 17, 18) and in the first and second measures of \textit{Purgatorio} (3–9 and 14–20) in the completed transcription.

![Figure 19. Canto length transcription: Guidonian scala and five-line staff](image)

A search of the \textit{Commedia}'s conjunct melodic contour in the Cantus Database (CD) reveals a strong concordance with four liturgical chants particular to Paschal and Marian feasts. This correlation opens a path of inquiry that is currently beyond the scope of this study, although it does invite collaboration with scholars to develop new perspectives on Dante’s poetics. Nevertheless, a brief summary of the search results will serve well to draw the present analysis to a harmonious conclusion.

\textsuperscript{50} Singleton, “Poet’s Number.”
\textsuperscript{51} Pegis, “Numerology and Probability.”
\textsuperscript{52} Logan, “Poet’s Central Numbers.”

\~65~
Of the two melodically concordant antiphons designated for Eastertide, “*Erat autem aspectus ejus sicut*” (CantusID 002647) is specifically designated for Easter Sunday.

![Figure 20. CantusID 002647: Erat autem aspectus ejus sicut](image)

An overview of manuscripts containing this melody during Dante’s time provides a window into its popular currency for Paschal office in the secular cursus. As of this writing, there are 142 concordances documented in Cantus Database records for CantusID 002647 in breviaries and antiphones from the 10th through the early 14th centuries from various libraries and monasteries in Europe, many of them Carolingian. Fifteen of these manuscripts are of Italian provenance, originating in Rome, Pozzeveri, Assisi, Florence, and Vallombrosa in central Italy, as well as Aosta, Chievenna, Milan, Pavia and Venice to the north, Benevento to the south, along with a handful more simply described as “Franciscan breviary from central Italy.”

![Figure 21. CantusID 001333: Alleluia](image)

The “*Alleluia*” (CantusID 001333) antiphon is designated for several Paschal offices including Easter Sunday, Easter Monday, Easter Wednesday, the Octave of Easter, and Paschal feasts in general. It is more widespread with 495 concordances in the Cantus Database to date. In this case, the manuscripts of Italian provenance

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53 Debra Lacoste and Jan Koláček, *Cantus: A Database for Latin Ecclesiastical Chant*, http://cantus.uwaterloo.ca/, Cantus ID 002647. The text of the chant is based on a passage from Matthew 28:3: “1. Vespere autem sabbati, quae lucescit in prima sabbati, venit Maria Magdalene, et altera Maria, videre sepulchrum. 2. Et ecce terraemotus factus est magnus. Angelus enim Domini descendit de caelo : et accedens revolvit lapidem, et sedebat super eum : 3. erat autem aspectus ejus sicut fulgur : et vestimentum ejus sicut nix.” [“1. And in the end of the sabbath, when it began to dawn towards the first day of the week, came Mary Magdalen and the other Mary, to see the sepulchre. 2. And behold there was a great earthquake. For an angel of the Lord descended from heaven, and coming, rolled back the stone, and sat upon it. 3. *And his countenance was as lightning, and his raiment as snow.*” Douay-Rheims. Emphasis added.]

54 *Cantus*, Cantus ID 001333.
in which it is found are both monastic and secular. In addition to
the locations mentioned in the previous summary, these manu-
scripts also originate in Montecassino and Piacenza.

The next concordance is a responsory verse, “Et praeparabili-
tur in misericordia solium” (CantusID 006639b), sung on the sec-
cond Sunday of Advent, the feast anticipating Christ’s birth to the
Virgin Mary.

The manuscripts of Italian provenance containing this chant melody make up a significantly higher percentage of sources than in the previous two examples.

The fourth concordance is “Aque multae non potuerunt
extinguere caritatem” (CantusID 001470), sung on the feasts of the
Immaculate Conception of the Virgin Mary, the birth of the Virgin
Mary, and the Assumption of the Virgin Mary, as well as the feasts
of other virgins including those celebrating Mary’s mother Anna,
Lucia, and Mary Magdalene.

The manuscripts of Italian provenance containing this chant
once again make up a significant proportion in this set of sixty con-
cordances, and they all indicate that, at least in Italy, the chant was
sung for the Common of Virgins celebrating that category of saints
as a whole. An antiphoner from Piacenza assigns this chant to the

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55 Cantus, Cantus ID 006639b. The passage is from Isaiah 16.5: “Et praeparabitur in misericordia solium, et sedebit super illud in veritate in tabernaculo David, judicans et quarens judicium, et velociter reddens quod justum est.” [“And a throne shall be prepared in mercy, and one shall sit upon it in truth in the tabernacle of David, judging and seeking judgment and quickly rendering that which is just.” Douay-Rheims. Emphasis added.]

56 Cantus, Cantus ID 001470. The passage is from Song of Songs 8.7: “Aque multae non potuerunt extinguere caritatem, nec flumina obruent illam. Si dederit homo omnem substantiam domus suae pro dilectione, quasi nihil despiciet eam.” [Many waters cannot quench charity, neither can the floods drown it: if a man should give all the substance of his house for love, he shall despise it as nothing.” Douay-Rheims. Emphasis added]
feast of Mary Magdalene, while another from Sant’Orso in Aosta, along with several German manuscripts and four Carolingian antiphoners, assign the chant and its melody to the feast of the Assumption of the Virgin Mary.

What conclusion might be drawn from the melodic affinity of the *Commedia*’s musically expressed mathematical architecture with Paschal and Marian liturgical plainchant remains to be seen. For now, this much is certain: After 700 years, the mathematical rationale for the *canto* lengths in Dante’s *Commedia* has finally been decrypted through an empirical analysis of the poem’s quantitative structure. The results of this analysis illuminate how the meticulous, systematic care Dante put into elaborating the design of the poem’s quantitative framework accounts for the myriad internal dispositional congruencies that have for decades been separately identified and documented by scholars working in fragmented solitude on questions about the poem’s configuration. The comprehensive manner in which Dante integrates all the *artes liberales*, especially the *quadrivium*, in the compositional framework of the *Commedia* further illuminates the empirical significance of “*fictio rhetorica musicaque poëta*” in his poetics. Even Dante’s choice to write his *magnum opus* in the vernacular, which he describes as the language of the hearth, is mirrored in the orthodoxy and popular currency of both the *scutum fidei* and the liturgical chants, whose melody the musical *incipit* of the poem’s *canto* lengths echoes. The accessibility of the *Commedia*’s mathematical blueprint as *musica instrumentalis*, which appeals to material sensory perception through hearing, endows it with the added benefit of being an efficient memory aid, especially for a poet in exile who spent nearly two decades of his life completing this singular work.

It is difficult to affirm whether the reader of the *Commedia* is meant to discern any of these architectural elements, since by definition, *musica universalis*, the musico-mathematical abstraction, eludes sensory perception. The recurring declarations by scholars and commentators about the ineffable sense of the poem’s comprehensive structural harmony over seven centuries attest to the effects of an exacting rational design that is at once complex, subtle, and eloquently lucid. Apprehending the semiotic coherence of the poem’s mathematical architecture, readers who thereby recognize the authorial transparency of the narrator’s assertions about the quantitative bounds that constrain him to brevity can appreciate the veiled candor and aspirational intimacy with which he addresses his audience. Rather, in tempering and attuning the poem to the

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57 *DVE* 1.1.2.

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Pythagorean harmony of the Ptolemaic cosmos, the *musica universalis* in the textual architecture of the *Commedia* appeals to the reader’s intellectual perception of a unitary order of all things in Dante’s mimetic craft.