BANKRUPTCY AND TRANSACTION COSTS IN GENERAL FINANCIAL MODELS

Matthew Hoelle

A DISSERTATION

in

Economics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2010

Supervisor of Dissertation
Signature___________________________
Felix Kubler
Professor, University of Zurich and Swiss Finance Institute

Graduate Group Chairperson
Signature___________________________
Dirk Krueger, Professor, Economics

Dissertation Committee
Felix Kubler, Professor, University of Zurich and Swiss Finance Institute
Dirk Krueger, Professor, Economics
Andrew Postlewaite, Professor, Economics

Acknowledgement

I would like to acknowledge the wonderful opportunity that I have been granted over my past 5 years at the University of Pennsylvania, including one year as a visiting scholar at the University of Zurich. This opportunity has allowed me to advance my education under the tutelage of some of the finest economists in the world. I will thank these individuals in turn, but first and foremost, I would never have had this opportunity if not for the loving support of my parents, Tom and Lydia. For their phone calls and interest in my research, I am grateful. This dissertation is dedicated to them.

I would like to thank the late Dave Cass. His role as the 'General' and instructor of the most idiosyncratic first year GE course imaginable sparked my interest in the field. His insightful discussions and critiques helped me develop as a researcher.

I would like to thank Felix Kubler. After serving as his teaching assistant for two semesters, I recognized that if I wanted to produce research in GE, he would be an invaluable person to have on my team. Following his move from the University of Pennsylvania to the University of Zurich, Felix was gracious enough to fund my one year visit to his new department, the Swiss Banking Institute. Working alongside Felix for one year, I was able to develop the papers that are contained in this dissertation.

I would like to thank Dirk Krueger. He has certainly proved to be a helpful advisor for the macroeconomic applications of my research, but more importantly, Dirk was the Director of Graduate Studies (DGS) and approved my leave to the University of Zurich. His support for all graduate students regardless of their research endeavors certainly goes above and beyond his duties as DGS.

I would like to thank Andrew Postlewaite for agreeing to serve on my committee and for many helpful discussions on bankruptcy and asymmetric information.

I would like to thank Kelly Quinn at the University of Pennsylvania and Ruth Hafliger at the University of Zurich for their essential roles in providing administrative assistance, without which I would never have been able to navigate the daily life of a graduate student.

I would like to thank my colleagues, especially my first-year study group of Cristiano Costa, Huanlei Ni, and Klenio Barbosa. It is often said that nourishment in the early years of one's life has a larger impact than at any other time. As that first year of graduate school was when I began my academic life, the countless discussions and laughs shared with this group are, and will ever be, greatly impactful.

Finally, I acknowledge financial support from the Fontaine Fellowship at the University of Pennsylvania and financial support from the Swiss Banking Institute at the University of Zurich.

Abstract

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Matthew Hoelle

Supervisor: Felix Kubler

General financial models have become workhorse models in the fields of macroeconomics and finance. These models have been developed and extensively studied by general equilibrium theorists. What makes them so applicable for macroeconomics and finance is the well accepted fact that models with a representative agent and without financial frictions yield equilibrium outcomes that are inconsistent with the empirical realities of financial markets. The general financial models are characterized by two main features: heterogeneous agents and financial frictions. The ability of these models to be applied in the fields of macroeconomics and finance in the future depends upon the frontier research in general equilibrium today. Over the past 20 years, research in general equilibrium has predominantly focused on a single friction: incomplete financial markets. The papers contained in this dissertation will analyze the equilibrium effects, both positive and normative, of two seldom researched frictions: bankruptcy and transaction costs. It is the hope that by studying financial frictions in isolation, we may learn which frictions have the greatest effect on welfare, which frictions are most able to be controlled by the government, and how to satisfactorily analyze the equilibrium of a process that is concurrently being restrained by several frictions.

Contents

Chapter 1

Introduction

The models studied in the field of general equilibrium capture a trading environment in its purest form. Trades are both competitive and anonymous: competitive in that households can trade any amount of good or asset at the posted market price and anonymous in that households only adjust their trading decisions as the price changes and not based on who they trade with. It is these non-strategic interactions that characterize markets in which a sufficiently large number of households trade.

While economists continue to search for the perfect model that is consistent with the empirical realities of the latest trendy topic, whether that topic is money, fiscal stimulus, or banking regulation, the benchmark has remained the Walras-Arrow-Debreu model (Walras, 1874; Arrow and Debreu, 1954) in a static setting and the GEI model in a dynamic setting. These models represent the frictionless version of economic trade and are the progenitor of entire fields of economic thought, including real business cycle macroeconomics and finance.

The incentives that the households face, namely the market prices, are determined endogenously. While this may make it difficult to prove analytical results in these models, it is only with such generality that an economist can quantify the normative and positive effects of a policy change. For the theory to be well-defined, households must have perfect foresight for the entire future history of market prices so that they can make optimal contingency plans for all possible future realizations.

I will now discuss several modeling decisions in general equilibrium as they relate to the field in general and specifically my three research papers.

Time horizon

The trading environment can either occur over a finite number of periods or over an infinite number of periods. The households have perfect foresight and know the length of time that they will be participating in the trading environment. A finite time horizon is the natural first choice due to the conceptual problems of infinite time.

For many applications, a finite time horizon permits a robust analysis. Suppose that a proposed model has the following two conditions: (i) households are alive for the entire length of the economy and (ii) any choices by households are immediately resolved in the following period and have no impact on future consumption. Then a finite time horizon is appropriate. In this case, the choices of households only have consequences for the following period. Concerning the choices made in the terminal period, adjustments can be made to the consumption bundle so that the decisions by households at this terminal period have similar consequences as prior decisions.

With an infinite time horizon, there are two possibilities: either households can be infinite-lived or they can be finite-lived. When the households are finite-lived, households that 'die out' each period are replaced by a new generation of households. This is the overlapping generations model (Samuelson, 1958). While this model is certainly the most plausible as it mirrors a realistic demographic structure, more research has been devoted to the case of infinite-lived households.

The first conceptual difficulty with infinite-lived households is how they compare consumption bundles at different points in time. Households that are indeed immortal would never consider consumption today to be any more valuable than consumption tomorrow, because at any point in time the horizon of all future time periods remains unchanged. However, to prove the existence of equilibria in this setting, the households must discount future consumption. Specifically, the preference orderings must be continuous in the Mackey topology. Discounting is natural for finite-lived households or households with an uncertain length of life, but appears to be unnatural for infinite-lived households.

The second conceptual difficulty is that in the infinite horizon, the preferences are represented by expected utility, despite a modeler's reluctance to claim that the expected utility axioms are satisfied or that probabilistic beliefs are required for household choice. The expected utility form is certainly tractable and satisfies the desirable property of intertemporal consistency, but leaves much to be desired on other fronts.

Chapter 2 utilizes a finite time horizon, specifically two time periods, to model bankruptcy. Chapter 3 considers the dynamic aspects of bankruptcy, aspects that can only be analyzed by extending to an infinite time horizon. Chapter 4 uses a finite time horizon, specifically two time periods, to model transaction costs.

Households

In general equilibrium, the number of households needs to either be countably infinite or uncountable to justify the competitive trade assumption that households are price takers. There are then two possibilities. The first is that households are idiosyncratic, that is, their preferences and endowments in all time periods are independent of those of other households. Typically, these idiosyncrasies are generated by a random process that is common to all households. Otherwise, the modeler would have to provide details as to what the preferences and endowments are for an infinite number of households.

The second possibility is that each households belongs to one of a finite number of types. The type of a household does not change throughout time. In this formulation, the preferences and endowments of a household are entirely dependent on what the preferences and endowments are for any household of the same type.

In reality, what determines a household's preferences and endowments is likely to be a combination of factors that are unique to the household and factors that are shared by all households of a particular type (e.g., occupation). A majority of general equilibrium models are written with a finite number of types. In this way, it is possible to analytically compute equilibria and analyze welfare under the Pareto criterion.

Chapters 2 and 4 are both models with a finite number of types of households. Chapter 3 is a model with a continuum of idiosyncratic households that belongs to the class of Bewley (1986) models. In these models, a household's idiosyncratic endowment is determined by a random Markov process. Reasons for using such a model will be discussed in section 1.2.

Preferences and endowments

The preferences of households will be formulated over the entire consumption bundle, which includes consumption both in the current period and in all possible realizations in future time periods. In order to have well-defined and objective preferences, the households are assumed to have perfect foresight. That is, households can correctly anticipate the prices in all realizations and how their decisions in the current period will affect the chosen consumption bundle in all realizations. Perfect foresight is a necessary condition of another desirable property of preferences, intertemporal consistency, and is a standard assumption in nearly all economic models of competitive markets.

The preferences are uniquely represented (up to a monotonic transformation) by a utility function. The economy will then be specified in terms of the utility functions of households rather than the preference relations. While the length of the time horizon will determine whether a restricted utility form, such as expected utility, or a more general form will be used, the utility functions must satisfy four standard assumptions. In order to obtain regularity results and compute equilibria, it is assumed that the utility function for each household is C^2, differentiably strictly increasing, differentiably strictly quasi-concave, and satisfies the boundary condition. These will be referred to as the "smooth assumptions". The boundary condition states that the closure of the upper contour set with respect to any strictly positive consumption must lie in the strictly positive orthant.

The endowments to households can be both the endowment of physical commodities and the endowment (potentially) of any financial contracts, though in the following chapters only the endowment of physical commodities is considered. The specification of the endowments of physical commodities for all households is crucial as the endowment summed over all households equals the aggregate resources in a closed economy. In Chapters 2 and 3, the economies are closed. In Chapter 4, the transaction costs create an environment in which asset trade on the markets reduces the aggregate resources.

The assignment of endowments ultimately determines how resources are allocated, but "generically" does not affect the efficiency of an equilibrium allocation according to the Pareto criterion. A feasible allocation is one in which the households consume all, and no more, than the aggregate resources. A feasible allocation is efficient under the Pareto criterion (henceforth, Pareto efficient) if there does not exist any other feasible allocation such that the utility is strictly higher for some and weakly higher for all households. A "generic" subset $A \subseteq B \subseteq \mathbb{R}^n$ of a set B is one in which A is open in B and A has full measure in B (meaning that the complement A_B^C with respect to B is a measure zero subset of B).

Suppose that for a particular assignment of endowments, the resulting equilibrium allocation is Pareto inefficient. Then over a generic subset of endowments, the resulting equilibrium allocations are Pareto inefficient. As a partial converse, suppose that for a robust subset of endowments (a subset that is not measure zero), the resulting equilibrium allocations are Pareto efficient. Then, for any possible choice of endowments, the resulting equilibrium allocation must be Pareto efficient.

The intratemporal dynamics of trade

There are several different interpretations of how the abstract Walras-Arrow-Debreu markets can be translated into the actual trade of commodities and assets by households. In the standard general equilibrium models with full commitment, we do not have to wed ourselves to one interpretation as the outcomes are equivalent. I will discuss two alternatives:

1. Market facilitator interpretation: The facilitator knows the preferences and endowments of all households. Given this, the facilitator can post prices for commodities and assets that satisfy the aggregate consistency conditions. These conditions are the standard market clearing conditions stating that total purchases must equal total sales. Households approach the market facilitator, sell their endowments, and choose how much of each commodity to buy and how much of each asset to trade *taking as given the listed prices.* The facilitator can be thought of as either an omnibenevolent individual or as a zero-profit firm with the technology to transfer commodities and assets to the unit of account and vice-versa.

2. Random matching interpretation: Each period, two households are randomly matched. The prices of a potential transaction are posted, so the households have only to decide how much they are willing to trade (if any) at the posted prices. Within the same time period, two new households are randomly matched. The process continues until no matching of households generates trade between the pair. All of the matching occurs simultaneously and at zero cost. The prices are known by all households and are the ones that satisfy the aggregate consistency conditions.

In the model of transaction costs in Chapter 4, the size of the transaction costs is assumed to be a parameter of the model. The transaction costs are only present for the trading of assets, not commodities. The model can fit either interpretation. Under the "market facilitator interpretation", the facilitator would be a financial broker who is responsible for purchasing assets from sellers and hoarding these assets for eventual sale to purchasers. The storage technology of this broker will be the transaction costs. As these storage costs are lost to the economy, the transaction costs will be real costs implying a reduction in the aggregate resources. In this way, the transaction costs can be endogenized (Martins-da-Rocha and Vailakis, 2010).

Under the "random matching interpretation", suppose that a search cost must be paid whenever two households are matched. Due to the limited trading capacity of each matched pair, it is clear that for a larger asset sale or purchase, more matchings will be required and hence higher search costs must be paid. In this way, the eventual assumption that the transaction costs are a convex function of the asset size (either purchase or sale) can be justified.

Financial contracts

Finally, as the models of interest are not static models without a financial sector, but dynamic models with a financial sector (called general financial models or GEI models), I must discuss the nature of the financial contracts. In each time period, there are a finite number of different aggregate states that can be realized. These aggregate states are observed by all households. As such, financial contracts can be written on the history of aggregate state realizations.

The contracts will be resolved in just one time period. The contracts traded in time t will represent a claim to the numeraire commodity in time $t+1$. The value of the claim, what I will later call the asset payout, will be conditional on the aggregate states in time t and $t+1$. For example, if a contract promises the claim to a constant amount of the numeraire commodity regardless of the realization of the aggregate state, then this will be called a "risk-free bond".

Given the claims of financial contracts, market prices will be endogenously determined so that the aggregate consistency conditions are satisfied. In all the following chapters, the financial contracts are in zero net supply, so the aggregate consistency conditions simply require that the total number of contracts sold equals the total number purchased. If the financial contracts were to be claims to the production bundle of a public firm (a "stock"), then the financial contracts would be in unit net supply. In this case, the economy is not closed as the aggregate resources are equal to the sum of the production bundles and the total household endowments.

A financial contract traded in time t will be bought or sold in terms of the unit of account in time t (as dictated by the market price) and the claims will be made in time $t+1$ in terms of the numeraire commodity. Such a "numeraire financial contract" is a special case of a "real financial contract" in which the claim is made over the vector of commodities in time $t+1$. The other type of financial contract is a "nominal financial contract" in which the claim is made over the unit of account in time $t+1$.

I will now argue for the use of "numeraire financial contracts" against those who argue that the stocks and bonds that are traded in the actual financial markets have payouts in terms of the unit of account. Suppose that the numeraire commodity is "money" loosely defined as currency and "liquid" assets, where "liquid" corresponds to whatever specification of M1, M2, M3, or M4 you are most familiar. Consumption of this commodity can certainly provide utility. Then the numeraire financial contracts will have claims in terms of money. This differs from the nominal financial contracts whose claims are in terms of the unit of account. The price of money does not enter into these latter claims. The price of money is endogenously determined by the monetary policy and the extent to which households use it to facilitate the exchange of goods. It is directly related to the inflation rate, which is the rate by which the price for a representative bundle of goods has increased from one period to the next.

So with nominal financial contracts, there is a disconnection between the real side and the financial side as the claims do not account for the changing price level. The numeraire financial contracts, on the other hand, have claims in terms of the purchasing value of the unit of account. This contract is adjusted for inflation, so a risk-free bond would actually be an inflation-indexed bond. Now that the argument has been reduced to a choice between offering inflation-indexed bonds and fixed-income bonds, the choice of the former is not indefensible.

There are a finite number of financial contracts and their claims are linear functions of the asset position and are linearly independent. This simply means that upon removing a contract, trading in a convex combination of the remaining contracts cannot replicate the claims of that missing contract. The task is not to find an 'optimal' structure on the claims of contracts as in Demange and Laroque (1995). Even in their stylized model without any of the frictions that I wish to consider, the task proved challenging. Rather, the claims of the contracts, what I will later call the asset structure, will be considered a fixed parameter in all three models in the following chapters.

Conditional on time t being reached, suppose there are $S < \infty$ possible realizations of the aggregate state in time $t + 1$. Further, suppose that the households do not receive idiosyncratic realizations each period, but rather belong to a finite number of types. Thus the preferences and endowments of all households of a particular type are only determined by the aggregate state realizations. Let the number of financial contracts traded at t be equal to J. These contracts were assumed to be independent, so if $J = S$, then the contracts span the entire space of uncertainty. The case $J = S$ will be denoted the "complete markets case" and the complimentary case $J < S$ will be denoted the "incomplete markets case".

Incomplete markets is a friction unto itself and provides one final reason why only numeraire financial contracts are considered rather than nominal financial contracts. With nominal financial contracts and incomplete markets, for a generic subset of endowments, the resulting equilibrium allocation is indeterminate. This means that the set of possible equilibria is a continuum and any comparative statics exercises become meaningless.

For the frictions that I consider, namely bankruptcy and transaction costs, the distinction between complete and incomplete markets is less important. Under both of my frictions, for a generic subset of endowments, the resulting equilibrium allocation is Pareto inefficient. Thus, the connection between the static trading environment of the Walras-Arrow-Debreu model and the dynamic environment of the GEI model is broken and all analyses must be undertaken in the dynamic environment. Any results that I prove will not be particular to the complete markets case or the incomplete markets case and this is especially noteworthy for the normative result of Chapter 4.

All financial contracts can be reduced to the market price at which the contract trades and the fixed promises that each unit of the contract has a claim to. This is in part due to the implied assumption that the financial contracts are not secured with collateral and their trade is entirely competitive (linear pricing). The only thing that is variable in the model will be the price of the contract. In keeping with the traditions of general equilibrium, I will call the financial contract simply an "asset". The asset is traded at an asset price in time t and the prescribed claims will be called "asset payouts" in time $t+1$. The asset payouts of all assets for all realizations of the aggregate states will be collectively called the "asset structure". The asset structure will be fixed throughout and no discussion will be given of its origins.

I will now introduce the models that will be presented in Chapters 2-4. Chaper 2 introduces a 2-period model of bankruptcy. Chapter 3 considers the dynamic consequences of bankruptcy in an infinite time horizon model by making use of the "credit score". Chapter 4 introduces a model with transaction costs on asset trades and analyzes the welfare implications of a planner adjustment of these costs.

1.1 Bankruptcy in a 2-Period Model

The model is a 2-period model with uncertainty in which a finite number of aggregate states can be realized in the second period. Denote the first time period as $t = 0$ and the second time period as $t = 1$. The model is a general financial model (GEI) as assets are available for trade to allow households to transfer wealth across time and states of uncertainty. The costs and benefits of an asset trade are separated across time in the favor of the seller. The seller collects the benefits of an asset sale at $t = 0$ and then has a negative claim on the asset payouts at $t = 1$ (indebted by the amount of the payout). As no mechanism exists to perfectly and costlessly enforce the commitment that the seller made at $t = 0$, the option for the seller to renege is available. We have now entered the class of default and bankruptcy models.

The household that bought the asset at $t = 0$ lies on the other side of this transaction. The buyer has paid the cost of the asset already and the benefit of the asset at $t = 1$ is equal to the positive claim on the asset payout. However, if the seller does not repay what it owes, then the buyer will not reap any benefit from the asset. Thus, for financial trade to even take place, there must exist clearly specified penalties for a seller that chooses to renege. These penalties serve two purposes: first as a deterrent to the seller, and second as a way to transfer wealth from the seller to the buyer if the seller chooses to renege at $t = 1$.

The penalties incurred by the seller who chooses to renege depend upon whether the asset was secured or unsecured. If the asset is secured, then choosing to renege means the seller must forfeit some pre-contracted amount of collateral. For an unsecured asset, that is not the case.

Default is a term that encompasses many mechanisms. Throughout these chapters, default will be the decision by a seller of a particular secured asset to renege. Default is thus a decision made on an asset-by-asset basis. The penalties for default are independent of the other assets that are held.

Bankruptcy, as it is defined in the legal code and implemented in my model, is the process by which a household discharges its unsecured debt. Bankruptcy is thus a decision made over the entire portfolio of assets. The penalties of bankruptcy include the loss of all nonexempt asset purchases in a household's portfolio. From the definitions of default and bankruptcy, it should be clear that when there is but a single asset available for trade, bankruptcy is simply a special case of default where the collateral is defined as the amount of perishable commodity at $t = 1$ with value equal to the cost of bankruptcy. So the mechanism of bankruptcy cannot be studied independent of that for default unless there is more than one asset.

I only consider unsecured asset markets for three reasons. First, I wish to isolate the effects of the bankruptcy policy. Second, complementary research research by Geanakoplos and Zame (2002) on secured assets has already been developed. Third, even with only secured assets, if the collection of payouts allows for "recourse", then unsecured asset markets may still play a role. Without recourse, when a seller of a secured asset defaults, it forfeits the collateral to the buyer and the process ends. The buyer does not have the legal right to seek further repayment. However, with recourse, if the value of the collateral does not cover the amount of debt owed, then the outstanding amount becomes unsecured debt. With this unsecured debt, the seller then has to decide, given its other asset holdings, whether to repay the amount in whole or declare bankruptcy.

Just as in the case with secured assets (Geanakoplos and Zame, 2002) in which no households would be willing to buy an asset that wasn't backed by collateral, there must be nontrivial costs of bankruptcy. While the purpose of the bankruptcy legal code is to ultimately grant a discharge of debt on a household's asset sales, the costs of bankruptcy are actually only a function of the household's asset purchases. First, the bankrupt household must forfeit its nonexempt asset purchases. These purchases are confiscated by the bankruptcy court and used to pay back creditors. Second, in order to maintain possession of its exempt asset purchases, the bankrupt household must pay the administrative cost of submitting a detailed record of its asset purchases to the court. The cost is assumed to be strictly increasing in the value of the asset purchases. To drive home the point, if a household has only asset sales, then it can discharge its entire debt without cost.

The bankruptcy code allows eligible households to renege on their obligations and declare bankruptcy, but does not permit them to renege on paying these costs of bankruptcy. Eligible households, in keeping with the 2005 Bankruptcy Abuse Prevention and Consumer Protection Act, will be those households whose average income over the prior 6 months is above the median level for their state of residence.

Whether a household is eligible for bankruptcy or not, there is a limited liability condition on the total value that it can owe. In this 2-period model without the possibility for future borrowing, the maximum amount that a household can owe, in either bankruptcy costs or financial obligations, is equal to the value of its endowments of physical commodities. In this situation, the household will have zero consumption and whatever value is still owed beyond this maximum amount will be discharged debt and will lower the payouts for creditors. As the utility function satisfies the boundary condition, a condition of equilibrium is that households will never make financing decisions such that they owe this maximum amount in any state at $t = 1$.

Consider a particular asset j and define the sellers of this asset as "debtors for asset j" and the buyers of this asset as "creditors for asset j". As is common with large financial institutions, it is assumed that the funds collected from debtors, both solvent and bankrupt, are pooled and then used to pay back the creditors. The creditors are anonymous so they are reimbursed in proportion to their asset purchase. The decision by a single debtor to declare bankruptcy and return to the pool less than what it originally owed will then lower the asset payout for all creditors. This reduction in payouts will affect the market clearing prices for assets.

These asset prices will be independent of the size of the loan. To rationalize this outcome, I must interpret the intratemporal dynamics of trade to be occurring via the random matching of a debtor and a creditor. In this way, a single creditor cannot know what the total size of a debtor's asset sale will be and the debtor has no incentive to reveal this information. The asset price cannot be conditioned on the asset size. This is in contrast to the polar case espoused by Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007). In their model, zero profit financial intermediaries coordinate the asset trades and each debtor would request their entire asset sale from an intermediary. A larger asset sale, as it entails a higher chance of less-than-full repayment to creditors, must then lead to a strictly lower asset price. Otherwise, a financial intermediary could make strictly positive profit by cherry-picking the smaller asset sales.

In my model, the asset price is constant across asset size whereas in Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), the asset price is strictly decreasing in asset size. Neither outcome is per se 'correct' as the pricing function for financial contracts is in reality a step function. Both outcomes have their own advantages, but ultimately result from the chosen interpretation of the intratemporal dynamics of trade: "market facilitator interpretation" for their outcome and "random matching interpretation" for mine.

As households will typically hold a diverse portfolio composed of both sales and purchases, a natural question arises. If a household must use the payouts from its purchases in order to repay the debt it owes on its sales, can bankruptcy by other households lower these payouts enough to force the household into bankruptcy? Such a mechanism will be called a "chain reaction of bankruptcy". This possibility for a chain reaction was proposed by Dubey, Geanakoplos, and Shubik (2005) and the equivalent jargon "contagion" has been studied frequently in finance.

In all cases, a household that would otherwise have remained solvent had it received its full payout is forced to declare bankruptcy when its purchases return a lower value. In the context of bankruptcy, a further mechanism exists that can exacerbate the problem. The funds that are collected from bankrupt households do not have a fixed nominal value, but are directly related to the value of the household's asset purchases. If bankruptcy occurs in the economy, the court will then have less valuable asset purchases that it can use to pay back creditors. The described chain reaction occurs instantaneously and does not introduce any dynamic elements into the equilibrium.

It appears that chain reactions are an inescapable aspect of a bankruptcy equilibrium that exacerbates the inefficient allocation of resources. However, this may not be the case. In Chapter 2, I will prove in a simple model that chain reactions are not possible and will provide rationale for why they may be unlikely in more general models.

The answer has to do with the dynamics of the model. The asset prices must be such that markets clear. These asset prices determine the size of a household's asset trade and are ultimately determined by creditors' expectations about their future asset payouts. In this way, the equilibrium can self-correct away from chain reaction. Bankruptcy may induce lower asset payouts for creditors, but this will also lower the asset price, and creditors will be able to increase their asset purchases. In the end, both the value of the asset purchases paid at $t = 0$ and the value of the asset purchases received at $t = 1$ may not change for the creditors. The value of a household's portfolio remains nearly unchanged, so if the household did not declare bankruptcy at the original asset payouts and prices, then it will not do so even when other households' bankruptcy declarations drive down the payouts.

1.2 Bankruptcy in an Infinite Horizon Model

The 2-period model is limited because the decision to declare bankruptcy is always a static one. After declaring bankruptcy and paying the appropriate penalties, there are no dynamic costs or reputation costs to pay. This is not an adequate representation of actual financial markets. At any given point in time, the terms at which a household can borrow depend on its past financial decisions. Also, the decision to declare bankruptcy in this period will affect its access to credit in the future.

The model by Sabarwal (2003) included a finite number of periods (≥ 2) where the investment constraints for households could be conditioned on a household's repayment history. In this way, a household's current access to credit can be based on its past financial decisions. But what about the other direction? A current bankruptcy declaration must impose consequences for household borrowing multiple periods into the future. With a finite time horizon, it is not possible to adjust the payouts in the terminal period so that the bankruptcy decision in that terminal period, with all its imputed dynamic costs, is identical to the bankruptcy decision faced in previous periods. An infinite time horizon solves this problem and allows for a cleaner equilibrium analysis as the possibility exists for stationary equilibrium.

Modern financial institutions use the credit score as an inexpensive summary of a household's past financial decisions. While there are other means by which lenders can forecast a borrower's repayment likelihood in order to set the terms of borrowing, unsecured loans (typically smaller than secured ones) require a measure that is inexpensive and commonly used. The FICO credit score is used by at least 75% of financial institutions and is inexpensive compared to the cost of acquiring a household's entire credit history. Bankrupty is the single worst thing that a household can do to its credit score, so it is natural for this dynamic model of bankruptcy to incorporate an adaptation of the FICO credit score.

The credit score serves two purposes in actual financial markets. First, it allows the financial markets, in an environment where borrowers have private information about their repayment likelihood, to form beliefs about these likelihoods and thus set appropriate terms of borrowing. Second, for a household deciding whether or not to declare bankruptcy, the credit score captures the reputation costs that must be paid. The household has perfect foresight and knows how its decision today will affect its credit score in the future. As the credit score changes, the subsequent interest rates for borrowing will change. The exact level of interest rates are determined endogenously, but the 'likely' outcome is that a bankruptcy declaration will lead to higher interest rates for borrowing in the future.

With this formulation, I have moved away from the past literature's ad hoc representation of dynamic bankruptcy costs. In the works of Kehoe and Levine (1993) and Alvarez and Jermann (2000), if a household decided to declare bankruptcy, it was permanently excluded from the market and forced to subsist on endowments alone. In their formulation, individual rationality constraints are added so that no one declares bankruptcy in equilibrium.

In Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), households are forbidden from borrowing on the asset market for the length of time that a bankruptcy declaration remains on the credit report. This is a representative view of the macroeconomics literature. This is an improvement, but still an exogenous market participation restriction is used to simplify the model.

In my setup, the asset prices dictate whether or not households are lent to and at what terms. The asset prices are conditioned on whether or not the household has a bankruptcy declaration on its credit report. Implicitly, this information is available and able to be accessed by the markets without cost.

The FICO credit score contains 5 components: payment history (35%), amount owed and amount of available credit (30%), length of credit history (15%), mix of credit (10%), and new credit (10%). In the model, the credit score will only be payment history, specifically bankruptcy history. There are three reasons why only bankruptcy history makes up the credit score in the model. First, the final three components of the FICO credit score are equal for all households in the model. Second, in a model without investment constraints, the second component of the FICO credit score is not appropriate. Finally, declaring bankruptcy is the single worst thing a household can do to lower its credit score. Without transaction costs, a household would never carry debt across time periods and would only have incomplete repayment when declaring bankruptcy.

Thus, the "credit score" in the model will be the number of periods since a prior bankruptcy declaration. I use quotation marks with "credit score" as the version in the model does not have all the features of the actual credit score that most people are familiar with. According to the legal code, a notice of a past bankruptcy declaration (called a "bankruptcy flag") must be removed after 10 years. Thus, a household whose last bankruptcy occurred more than 10 years ago has the same "credit score" as a household who has never declared bankruptcy. The "credit score" will take integer values in the set $0-10$, where the most recent declarers have a "credit score" of 0 and those without a bankruptcy flag have a "credit score" of 10.

The "credit score" in my model serves the two purposes of actual credit scores as previously introduced. First, the markets can use this variable to form beliefs about a borrower's repayment likelihood and appropriate asset prices can be set. Second, households with perfect foresight know how a bankruptcy declaration will determine their "credit score" over the foreseeable future (it will reset the "credit score" to 0) and what this will imply for the endogenous asset prices.

Past literature, notably Chatterjee, Corbae, and Ríos-Rull (2008) and Elul and Gottardi (2008), model credit scores as only satisfying the first purpose. In both of their models, if a household declares bankruptcy in some period, then in that period, and in every period thereafter until the bankruptcy flag is removed, there is a probability $p > 0$ that the bankruptcy flag is removed. The parameter p is constant across time and is chosen so that on average a bankruptcy flag remains on a credit report for 10 years. So, the financial markets receive either a "Yes, bankruptcy flag" or "No, bankruptcy flag" signal. This still allows them to form beliefs about repayment likelihood. However, these random signals do not fully capture the reputation costs of bankruptcy. Rather than perfect foresight and penalties for up to 10 years, the household's reputation costs of bankruptcy are determined as an expectation over a random, artificial construct of the model.

Additionally, with my formulation of the "credit score", I will show in Chapter 3 what assumptions on the parameters are required so that equilibrium properties of the "credit score" match the empirical properties of the actual credit score. Such empirical facts that will be verified are:

i. a household's access to credit increases significantly as it moves from "credit score" 9 to "credit score" 10 (Musto, 2004).

ii. interest rates for borrowers are strictly decreasing functions of the "credit score".

1.3 Transaction Costs and Planner Intervention

Transaction costs can be direct (examples include taxes, fees for market facilitators, and search costs), implicit (adverse selection), or even hidden (when a scarce resource such as collateral or money is required to support a transaction that can then not be used for any other concurrent transactions). As mentioned in the previous section, without transaction costs in the model (and intermittent closing of markets can be thought of as imposing infinitely large transaction costs on trade), households would never choose to carry debt across time periods. In fact, there are many economic phenomena that cannot result in models when transaction costs are not present.

The rationale to justify the absence of transaction costs in most economic models is that economic analysis requires simple models to be able to get answers to the specific questions at hand. So even though all agree that transaction costs are present in actual financial markets, they will only be explicitly modeled when the question of interest involves transaction costs. Chapter 4 will look at transactions costs imposed on the trade of assets and how adjustments in these transaction costs by an omniscient social planner can lead to a welfare gain.

When modeling transaction costs, the following questions must be answered: (i) what form should the transaction costs take?, (ii) who or what determines the transaction costs?, and (iii) where do the costs go after being collected from the households? There are of course different answers to these modeling questions, but I make choices that are the most plausible and the most widely adopted in the previous literature.

The transaction costs will be determined by a function whose domain is the set of possible asset positions. This function will be heterogeneous across households and is assumed to be C^3 and differentiably strictly convex. While this rules out such common transaction costs as a linear tax scheme, the assumptions are required both to maintain the convexity of the households' budget sets and to allow standard results in differential topology to be applied.

The codomain of the function is the set of asset positions. Thus, the transaction costs are real transaction costs meaning that assets are the economic object collected from each transaction. As such, in the presence of transaction costs, the number of assets still available for trade will decrease with implications for the market clearing asset prices. As with all other chapters, the assets will be numeraire assets meaning that each asset represents a claim to an amount of the numeraire commodity. As the numeraire assets are removed, the aggregate supply of the numeraire commodity is decreased with implications for the market clearing commodity prices.

I assume that the transaction costs in the economy are fixed by outside forces. These outside forces include the unmodeled technology of a market facilitator to transfer trade offers into actual transactions and the legal and market institutions in place. Having some control over the legal institutions will be a social planner. Suppose the social planner can adjust the tax rate, but cannot make any other changes. So the social planner is clearly not capable of completely removing all the transaction costs. The justification for planner intervention is the fact that for a generic subset of endowments, the resulting equilibrium allocation will be Pareto inefficient. The exact problem of the planner will not be addressed for it suffices to mentions only its motivation. Given the original equilibrium in which allocations are inefficient, the planner adjusts the transaction costs in such a way so as to effect a Pareto improvement. That is, all households will have strictly higher utility after the planner adjustment as compared to the original equilibrium.

As I do not model the planner as a government whose expenditures are publicly financed by some part of transaction costs, it is assumed that the value of the collected transaction costs is simply discarded. An equivalent assumption is that the value is spent in a way that does not affect the utility of any of the households. This assumption may be an extreme view of public finance, but even if the collected transaction costs are used for some worthwhile expenditure, the allocation will remain inefficient. The government may still seek to adjust the transaction costs to achieve a welfare gain without sacrificing the worthwhile expenditure. If the adjustments of the transaction costs are budget neutral and lead to a welfare gain when the transaction costs are discarded (my result in Chapter 4), then a welfare gain would also be achieved by a government who can adjust the transaction costs while still providing the same level of worthwhile expenditure.

Constrained suboptimality

When dealing with a friction, the question to ask is if the resulting equilibrium allocation satisfies some optimal welfare properties. My result is in the class of "generic constrained suboptimality" results. As such, I will briefly describe the progression of thought in this line of literature.

The literature began with the recognition that while the complete markets case of GEI, as envisioned by Arrow (1953), Debreu (1959), and Radner (1972), wherein households transfer wealth across time and states of uncertainty using a complete set of assets, guarantees that the equilibrium allocations are Pareto efficient, this optimality result relies crucially on the assumption that markets are complete. As recognized by Hart (1975) among others, if the set of assets is incomplete, then for a generic subset of endowments, the resulting equilibrium allocations are Pareto suboptimal.

When speaking about suboptimality, often an omniscient planner is introduced as the welfare properties concern non-market allocation of resources. The planner would know the preferences and endowments of all households. Pareto suboptimality then means that the planner can make household-specific transfers such the utility will be increased for all households. If the planner makes transfers that depend upon a household's preferences or endowments, then these transfers are not "anonymous". If the planner makes transfer that are independent across households, then these are anonymous. My result in Chapter 4 considers anonymous intervention.

The above comparison between what the planner can do and what the households can do using the available markets seems unfair. The households can only use the available set of incomplete asset markets while the planner can make transfers as if a complete set of asset markets existed. So, for a sensible suboptimality result, the planner cannot be free to make any transfers, but must make transfers that respect the fixed asset structure.

This is the notion of "constrained suboptimality". An equilibrium allocation is constrained suboptimal if the planner can choose the asset holdings for all households (that satisfy aggregate consistency) and make wealth transfers in the initial period such that the households will then choose the consumption bundle in each state and their resulting utility is strictly higher. The households' choice of their consumption bundle in each state is a static choice as the wealth of households in each state is fixed by the planner's choice of their assets and the asset structure. This intervention by the planner is not anonymous.

The following constrained suboptimality result, the original constrained suboptimality result, was first stated by Stiglitz (1982) and was then later formally proven by Geanakoplos and Polemarchakis (1986). When the asset structure is incomplete and there are at least two physical commodities in each state, then for a generic subset of endowment and utilities, the equilibrium allocations are constrained suboptimal.

The historical timeline above indicates that most of the work in the field considered incomplete markets as the friction. I will shortly discuss the recent work, including mine, on frictions other than incomplete markets. Before that, I will need to clarify the statement of the "generic constrained suboptimality" theorem as a technical result used in the proof of the theorem will also be used in my result. Specifically, I will discuss what it means for a result to hold over a generic subset of the set of utility functions when this set by definition is not a subset of a finite-dimensional Euclidean space.

Quadratic perturbations

This technique is used in both Geanakoplos and Polemarchakis (1986) and Citanna, Kajii, and Villanacci (1998). I follow the exposition of Citanna, Kajii, and

Villanacci (1998) beginning on pg. 505. The set of utility functions belongs to the infinite-dimensional set of functions from the consumption set $\mathbb{R}_+^G$ to $\mathbb{R}$ where G is the total number of goods in the economy. The following result extends Debreu (1970, 1972, 1976). For a generic subset of endowments, the equilibrium allocations for any asset structure containing numeraire assets are finite and locally unique.

Choose any endowment within this generic subset and choose any utility function $\overline{u}^h$ satisfying the smooth assumptions described in the opening monologue. The resulting equilibrium allocations for household h, denoted as $\overline{x}_i^h \in \mathbb{R}_+^G$ for $i = 1, 2, ..,$ are finite in number. These allocations are further locally unique. Consider any allocation x^h within an open set around $\overline{x}_i^h$ and write the utility function for x^h in terms of the $G \times G$ symmetric perturbation matrix A^h :

$$u^h\left(x^h; A^h\right) = \overline{u}^h\left(x^h\right) + \frac{1}{2}\left[\left(x^h - \overline{x}_i^h\right)^T A^h \left(x^h - \overline{x}_i^h\right)\right].$$

If the norm of A^h is small enough, then the function $u^h\left(\cdot; A^h\right)$ is a subset of the set of utility functions satisfying the smooth assumptions. The symmetric matrix A^h belongs to the $G(G+1)/2-$dimensional Euclidean space. Moreover, if a sequence of perturbation matrices converge $A^n \rightarrow A$, then a sequence of utility functions converge $u^h(\cdot; A^n) \rightarrow u^h(\cdot; A)$ in the C^3 uniform converge topology (this is the topology chosen for the set of utility functions). This can be verified by taking derivatives of $u^h\left(x^h; A^h\right)$.

Any utility function $u^h(\cdot)$ that is in an $\epsilon-$neighborhood around the function $\overline{u}^h(\cdot)$ can be represented by $u^h\left(\cdot; A^h\right)$ with a symmetric matrix A^h in an $\epsilon'-$neighborhood around $\overline{A}^h = 0$. Thus, for any utility function $\overline{u}^h(\cdot)$, I can state results for generic subsets of $\mathbb{R}^{G(G+1)/2}$, the Euclidean space containing all possible perturbation matrices.

With this technique, I am able to state results holding over a generic subset of utility functions. I will also consider generic subsets of the transaction costs functions. Recall that these functions specify the amount of transaction cost that must be paid for any possible asset positions.

"Constrained" suboptimality

There are certainly more frictions than just incomplete markets that can restrict market performance. Even if only considering incomplete markets, there are a variety of different restrictions that you can impose on the planner's intervention. For instance, I previously defined that an allocation is constrained suboptimal if a planner can adjust the assets of households and make wealth transfers in the initial period such that the utility of all households strictly increases. What if the planner was prevented from making wealth transfers in the initial period so that a household's initial wealth is the sum of the value of the endowments and the value of the assets chosen by the planner? In that case, a generic "constrained" suboptimality result holds, but stronger assumptions are required. Notably, an upper bound on the number of households is needed.

So for any set of "constraints" facing a planner, the question of interest is whether planner intervention can lead to a Pareto improvement. A general system of equations that restrict the intervention of the planner will be labeled the "constraints". If, over a generic subset of parameters, the planner is able to intervene and effect a Pareto improvement while adhering to the "constraints", then the original equilibrium allocation is generically "constrained" suboptimal. When the "constraints" are those from Geanakoplos and Polemarchakis (1986) as described twice already, then these are simply constraints and the welfare result is titled generic constrained suboptimality.

The paper of Citanna, Kajii, and Villanacci (1998) set a framework so that for any set of "constraints", an equilibrium allocation is generically "constrained" suboptimal if a rank condition (ND) and a dimensionality condition are met. Several works, including Cass and Citanna (1998) on the impact of financial innovation, have since utilized this framework. I will focus attention on those papers dealing with taxation as this is a special case of transaction costs.

Pareto-improving tax schemes

The papers by Citanna, Polemarchakis, and Tirelli (2006) and del Mercato and Villanacii (2006) both consider the normative effects of a planner introducing a tax/subsidy scheme. The asset structure for both must be incomplete and there are no other frictions affecting the original equilibrium. The intervention is anonymous in Citanna, Polemarchakis, and Tirelli (2006) and subsequently requires an upper bound on the number of households whereas del Mercato and Villanacci (2006) have a non-anonymous intervention but do not require an upper bound on the number of households. The intervention in Citanna, Polemarchakis, and Tirelli (2006) is taxes on asset purchases and the collected taxes are then redistributed back to the households lump sum using a fixed distribution scheme. The intervention of del Mercato and Villanacci (2006) is the taxation of outside money (basically a risk-free bond) leading to an adjustment of each household's portfolio. In either case, the result is that over a generic subset of endowments and utilities, there exists a tax scheme that the planner can employ such that the resulting allocation provides higher utility for all households.

In these papers, taxation is seen as a partial fix of the ineffiencies caused by incomplete markets. This suggest that the primary purpose of taxation is for the reallocation of resources for normative gains. Well before resources can be reallocated, a tax system must be imposed that provides funds for the creation and enforcement of the legal institutions (to protect property rights) and for the provision of public goods. So it would seem prudent to analyze an economy in which the only friction is from a given tax system and then determine if there exists a way in which the planner could adjust the tax system in order to effect a Pareto improvement. This is the contribution I make in Chapter 4.

In Chapter 4, I consider an economy in which transaction costs are already present. The planner then makes anonymous adjustments to the transaction costs that satisfy budget balance. Budget balance is satisfied when the value collected from all the transaction costs does not change. The asset structure can be either complete or incomplete. The only friction under consideration is transaction costs and that friction is also the tool in the hands of the planner. I prove that given an upper bound on the number of households, for a generic subset of endowments, utilities, and transaction costs, there exists a planner adjustment of the transaction costs such that resulting allocation provides strictly higher utility to all households.

The result clearly has normative implications for tax policy. The limitation of the result is that it assumes that the planner is omniscient, namely that it knows the possible parameters (tuples of preferences, endowments, and transaction costs) that the finite number of households can have. Given that a planner may not be omniscient, the following question is naturally of interest. For any parameters of the household (lying in a generic subset), does there exist a 'universal' adjustment by the planner such that all households receive strictly higher utility? If so, then an uninformed planner can choose this 'universal' adjustment and still effect the desired Pareto improvement. My result remains silent on such a question, but my hunch from working with these models is that no such 'universal' adjustment exists.

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Chapter 2

Bankruptcy in a 2-Period Model

2.1 Introduction

In the development and study of general financial models, two fundamental features have been commitment and perfect foresight. With these two foundations, households are modeled as making all economic decisions for the entire finite length of the model at the initial date-event. At this node, households not only directly trade in the commodity and asset markets currently open, but also make contingent transactions in all future commodity and asset markets. While this simple framework has been shown to be an invaluable tool for analyzing financial markets, the reality is that no mechanism exists to freely and perfectly enforce commitment.

In this work, I relax the assumption that households must fulfill their commitments in the financial markets. Simply put, households are permitted to sell assets in the initial date-event, reap the benefit (increase in wealth) at that time, and then decide not to repay the debt owed at a future date-event. A similar idea was developed by Dubey et al. (2005) and termed "default". In their paper, a household chooses to hold a portfolio of assets. At the time when the debt from a past sale of an asset comes due, the household can choose what fraction (if any) of this debt to repay. The cost of less than complete repayment is utility loss, with greater loss for a greater amount of debt left outstanding.

My model of bankruptcy differs noticeably from Dubey et al. (2005). As above, a household chooses to hold a portfolio of assets. At any future date-event, the household considers the value of the entire portfolio. If the value is sufficiently negative, the household may choose to default on the entire portfolio. The idea of defaulting on an entire portfolio and not making good on any commitments is what I term "bankruptcy". The household only has a binary choice: bankruptcy or solvency. Declaring bankruptcy is not costless, but unlike Dubey et al. (2005), the cost is a financial one affecting the budget set, rather than changing the households' preferences.

Concerning bankruptcy and not default, the two theoretical contributions are by Sabarwal (2003) and by Araujo and Pascoa (2002). In both of these papers, exemption levels are exogenously specified for all physical commodities and these levels can vary across households. A bankruptcy declaration would force the bankrupt household to forfeit its endowments of commodities above the exemption level. In my work, I suggest that the bankruptcy exemptions written in the legal code are actually written on asset purchases and not on physical commodities. Under my formulation, a bankruptcy equilibrium with exemptions is guaranteed to exist so long as the size of the exemptions is bounded. The exemptions are only introduced to better represent the U.S. legal code; they play no role in guaranteeing existence.

There are two types of credit that can be obtained by debtors: secured credit and unsecured credit. With secured credit, the market sets both an asset price and a collateral requirement for an asset sale. Thus, a household declaring bankruptcy would be forced to forfeit this collateral (and possible additional resources if the creditor is granted 'recourse'). The creditors are treated equally and their asset payouts depend upon the pooled repayments of all debtors. Secured credit has already been analyzed in the model by Geanakoplos and Zame (2002).

This model considers only unsecured credit, the second type of credit. Unsecured credit is exactly as its name indicates, an asset for which there are no collateral requirements when selling. A household declaring bankruptcy on unsecured credit still repays their creditors, but this repayment is not specified by the financial contract. Unlike secured credit, this repayment may be zero. In the model, bankrupt households must forfeit their nonexempt asset purchases and must pay a bankruptcy cost that is strictly increasing in the value of the purchases. This cost is the cost of filing detailed records of a bankrupt household's asset purchases with the court in order to obtain bankruptcy exemptions. It is these funds that are used by the perfectly efficient bankruptcy court (which operates without any loss of resources) to pay back a bankrupt household's creditors. As with secured credit, the creditors are treated equally and their asset payouts depend upon the pooled repayments of all debtors.

Chapter 7 bankruptcy is the most common form of household bankruptcy (accounting for over 70% of the individual bankruptcies[1]) and is also the most interesting to model. Briefly, chapter 7 allows a bankrupt household to completely discharge its financial debt at the cost of forfeiting its nonexempt assets.[2] The cost of declaring chapter 7 bankruptcy is not tied at all to the total amount of debt accrued. In contrast, chapter 13 (which composes all but the remaining 0.1% of individual bankruptcies[3]) involves a 5-year repayment plan between the debtor and its creditors.[4] Such a dynamic cannot be modeled in a 2-period model and even if it could, a house-

[1] http://en.wikipedia.org/wiki/Chapter_7_bankruptcy
[2] 11 U.S.C. §7
[3] http://en.wikipedia.org/wiki/Chapter_7_bankruptcy
[4] 11 U.S.C. §1322(d)

hold is still responsible (eventually) for all or most of its debt. Chapter 7 bankruptcy and the unsecured credit markets together present the greatest potential for abuse of bankruptcy privileges. For this reason, a household holding debt in the unsecured markets will opt for chapter 7 bankruptcy.

The obvious abuse of bankruptcy privileges is that a household would borrow as much as possible knowing that its debt would be erased in the following period. A recent 2005 clarification of the bankruptcy law sought to reduce the number of abuses. It states that any household applying for chapter 7 bankruptcy must pass a "means test". Simply put, if a household's average income over the previous 6 months is above the state median, then (unless a large number of statutorily allowed expenses can be deducted) the household's chapter 7 claims are transferred to chapter 13.[5] In this instance, the household would not be able to discharge the entire debt and would be subject to the 5-year repayment plan as regulated by chapter 13. Therefore, if the household will fail the "means test" in some state next period, then the household will not run up debt beyond its means to repay in that state. Chatterjee et al. (2007) show that this new "means test" policy is welfare improving in a dynamic macroeconomic model of bankruptcy.

While investment constraints may be observed in reality, a bankruptcy model must be written in a manner so that assets are endogenously constrained. Otherwise, the households would only choose to sell assets at the arbitrary constraint and nowhere else. Araujo and Pascoa (2002) have two models in their paper: one with investment constraints (model 1) and one without (model 2). Model 2 of Araujo and Pascoa (2002) was the first work in the literature to remove the constraints and bound the assets endogenously. I do neither of the following, but the two most straightforward

[5] 11 U.S.C. §707(b)(1) and the Bankruptcy Abuse Prevention and Consumer Protection Act of 2005

methods to bound assets without investment constraints are either (i) to assume that the cost of bankruptcy strictly increases with the size of the debt or (ii) to assume that the bankruptcy cost is in terms of wage garnishments that approach complete as the size of the debt becomes unbounded. Araujo and Pascoa took the second approach. Such an assumption requires that the cost of bankruptcy depends on a household's debt at the time of bankruptcy. This is contrary to the U.S. legal framework as summarized above.

Further looking at model 2 of Araujo and Pascoa (2002), existence is only guaranteed when the asset payouts for creditors are strictly concave functions of the assets ("nonproportional reimbursement"). Such an assumption is required in their setup as the assets may be linearly dependent for bankrupt households. With dependent payouts, the asset choices of households may not be bounded, so it is clear that some additional assumption is required. The problem with the assumption of "nonproportional reimbursement" is that it contradicts the U.S. bankruptcy code. The code states that creditors with a claim in a bankruptcy case are divided into 6 classes.[6] Repayment for a lower class will only occur once the classes above have been repaid in full. Thus, throughout this work, I will assume that all creditors belong to the same class and are repaid equally, that is, the model is constructed using the rule of "proportional reimbursement".

After the existence of the general model is proven, I will analyze an economy with two households and two assets. With only a single asset available for trade, the model of bankruptcy is identical to the Dubey et al. (2005) model of default, so at least two assets must be considered. In both the aforementioned Dubey et al. (2005) work and in the finance literature, there exists the possibility for chain reaction of default (or contagion). Briefly, a chain reaction of default occurs when default by one household

[6] 11 U.S.C. §726

lowers the asset payouts by just enough so that other households will now default, other households that would have remained solvent had they received their entire asset payouts. I prove for the simple economy with two households that a chain reaction of bankruptcy cannot be an equilibrium phenomenon. Thus, households who declare bankruptcy given the prices of the bankruptcy equilibrium would have made the same decision given the prices of the full commitment equilibrium.

Suppose that bankruptcy imposes real costs on the economy. Such costs would be the resources lost in the process whereby the bankruptcy court liquidates bankrupt households' assets and distributes these funds to the creditors. I dedicate a brief section to explaining how the real costs are incorporated and how the proof of existence generalizes. As it turns out, the only significant difference, and an interesting implication, is that the asset payouts for creditors are smaller when bankruptcy imposes real costs on the economy.

This paper is organized into four remaining sections. Section 2.2 introduces the model and proves the existence of a bankruptcy equilibrium, the fundamental concept in this paper. Section 2.3 considers the possibility for a chain reaction of bankruptcy in a simple economy with two households. Section 2.4 generalizes the model to include real costs of bankruptcy. Section 2.5 contains the existence proof.

2.2 The Model

I consider a 2-period general financial model with uncertainty. Let there be S states of uncertainty in the second time period and denote the first time period as state $s = 0$, so that the states belong to the finite set $s \in \mathcal{S} = \{0, ..., S\}$. In each state, there are L physical commodities and the commodity $l = L$ will be the numeraire (meaning that all other commodities in that state are priced relative to $l = L$). I will denote commodities with a subscript and states in parentheses.

Let the set of households be denoted by $h \in \mathcal{H}$ where $\mathcal{H}\tilde{}[0,1]$ and $\int_{h\in\mathcal{H}} dh = 1$. I will denote household variables and parameters with a superscript. Define the Borel-measurable function $\mu(A) : \mathcal{H} \rightarrow [0,1]$ as

$$\mu(A) = \int_{h\in A\subseteq\mathcal{H}} dh.$$

There are a finite number of distinct types of households denoted by $f \in \mathcal{F} = \{1, ..., F\}$. Define $\mathcal{H}_f = \{h \in \mathcal{H} : h \text{ is of type } f\}$. I explicity assume, though the assumption is without loss of generality, that $\mu(\mathcal{H}_f) = \frac{1}{F}$ $\forall f$. Further, to be of the same type f, I require that $\forall h, h' \in \mathcal{H}_f$, $X^h = X^{h'} = X^f$ (consumption set), $u^h(\cdot) = u^{h'}(\cdot) = u^f(\cdot)$ (utility function), and $e^h = e^{h'} = e^f$ (endowments). These household primitives will be introduced shortly. Throughout this paper, I will refer to both households and types of households as simply households. No confusion should arise, but households will be linked with equilibrium variables (since households of the same type may make different optimizing decisions), while types will be linked with parameters (as all households of the same type have identical parameters).

Define $G = L(S+1)$ and denote household consumption as $x^h \in \mathbb{R}^G_+$. Concerning notation, $x^h(s) \in \mathbb{R}^L_+$ is the vector of consumption by household h of all commodities in state s and $x^h_l(s) \in \mathbb{R}_+$ is the scalar denoting the consumption by household h of good (s,l), or the l^{th} physical commodity in state s. The primitives for the households are the consumption set X^f, the utility function $u^f : X^f \rightarrow \mathbb{R}$, and the endowment of physical commodities $e^f \in \mathbb{R}^G_+$ in all states. To characterize and compute equilibria, I assume that the model satisfies standard smooth assumptions:

A.1 $X^f = \mathbb{R}^G_+$.

A.2 u^f is C^2, differentiably strictly increasing (i.e., $\forall x^f \in X^f$, $Du^f(x^f) >> 0$), differentiably strictly quasi-concave (i.e., $\forall x^f \in intX^f$ s.t. $Du^f(x^f)\Delta_f = 0$ and $\Delta_f \neq 0$, then $\Delta_f^T D^2 u^f(x^f)\Delta_f < 0$), and satisfies the boundary condition (i.e., $\forall x^f \in intX^f$, $clU^f(x^f) \subset intX^f$ where $U^f(x^f) = \{x' \in X^f : u(x') \geq u(x^f)\}$).

A.3 $e^f >> 0$.

To transfer wealth between states of uncertainty, the households have access to numeraire assets in zero net supply. Let there by $J \leq S$ assets that are traded at state $s = 0$ and return strictly positive payouts $r_j(s) > 0$ in states $s > 0$. I collect the asset payouts into the $S \times J$ yields matrix

$$Y = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}$$

with the payouts in terms of the numeraire commodity $l = L$. The assumptions placed on the parameter Y are:

A.4 Y is a strictly positive matrix with full column rank and is in general position.[7]

Denote the portfolio of household assets as $z^h \in \mathbb{R}^J$ where $z_j^h \in \mathbb{R}$ is the scalar denoting the amount of asset j held by household h. For each asset j, I will call a household a creditor (on asset j) if $z_j^h \geq 0$ and a debtor (on asset j) if $z_j^h < 0$.

Let the equilibrium commodity prices be denoted by $p \in \mathbb{R}^G \backslash \{0\}$. Under assumption $(A.2)$, the prices are strictly positive $p >> 0$. As commodity $l = L$ is the numeraire, I normalize $p_L(s) = 1$ $\forall s \geq 0$. The assets pay out in this numeraire

[7] By general position, I mean that any J rows of Y will be linearly independent.

commodity, so Y has real units. To consider the nominal value of the asset payouts, I will make the identity transformation

$$Y = \begin{bmatrix} p_L(1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & p_L(S) \end{bmatrix} \cdot \begin{bmatrix} r_1(1) & ... & r_J(1) \\ \vdots & & \vdots \\ r_1(S) & ... & r_J(S) \end{bmatrix}.$$

Thus, I view the yields matrix Y as a matrix with payouts in terms of the unit of account. Let the equilibrium asset prices be denoted by $q \in \mathbb{R}^J$. The asset prices can be thought of as the nominal returns of the assets in state $s = 0$. Define the $(S + 1) \times J$ returns matrix as

$$R = \begin{pmatrix} -q \\ Y \end{pmatrix},$$

where R (as with Y) pays out in the unit of account.

The model introduced thus far is standard with one exception: the asset payouts are assumed to be strictly positive. The importance of this assumption will be made clear shortly. In all lack of commitment models, there are two technical issues to consider when proving existence: convexity and boundedness.

2.2.1 Convexity

The budget set written in terms of $z^h \in \mathbb{R}^J$ is not convex. For some given state $s > 0$, the asset payouts are a concave function of assets for bankrupt households and the asset payouts are a concave function of assets for solvent households, but over the entire domain of asset holdings (and allowing a household to choose to be either bankrupt or solvent), the asset payouts are not a concave function of assets.

This is akin to the problem contained in the Dubey et al. (2005) model of default.

In their model, the threshold between being a debtor and a creditor was exactly set at zero asset holdings. By defining $\theta^h \in \mathbb{R}_+^J$ as asset purchases and $\phi^h \in \mathbb{R}_+^J$ as asset sales, it is unambiguous (and state independent) that a debtor for asset j is a household with $\phi_j^h > 0$. By defining twice as many asset variables $\left(\theta^h, \phi^h\right) \in \mathbb{R}_+^{2J}$ as compared to $z^h \in \mathbb{R}^J$, Dubey et al. (2005) are able to maintain the convexity of the household's optimization problem.[8] In my model, the threshold between being bankrupt and solvent depends on all the assets in the portfolio and the state-specific payouts of each. Therefore, in order to employ the Dubey et al. (2005) scheme to convexify the budget set, I would need to not only split the assets into purchases and sales, but also introduce an additional S variables that would denote the different values that a portfolio could take in each state $s > 0$. For obvious reasons, I choose not to do this.

Moreover, the bankruptcy problem in reality is a nonconvex one, either a household is or is not bankrupt. In order to obtain existence, I will therefore need a continuum of each of the finite types of households $f \in \mathcal{F}$.[9] With the continuum, the integrated budget set across all households of type $f \in \mathcal{F}$ will be convex. Further, while the demand correspondence for each individual household is guaranteed to be upper hemicontinuous, but not convex-valued, the demand correspondence for each type $f \in \mathcal{F}$ will be convex-valued.

[8] Dubey et al. (2005) have artificially doubled the number of independent assets. Consider the statement from Zame (1993): "by separating purchases from sales, I have allowed for the possibility that agents go long and short in (i.e., buy and sell) the same security. I have implicitly contemplated this possibility in the [GEI] model, but when default is not possible, such an action is irrelevant. However, when default is possible, such an action may not be irrelevant; it may benefit a trader to go long and short in the same security *if he does not intend to meet all his obligations* [sic]".

[9] This was recognized independently by both Sabarwal (2003) and Araujo and Pascoa (2002).

2.2.2 Boundedness, bankrupt households

When households declare bankruptcy, the asset payouts for creditors will be determined by the pooled repayments of all debtors. The creditors will receive proportional reimbursement, so their asset payout is $r_j^h(s) = \rho_j(s) r_j(s)$. The endogenous variable $\rho_j(s)$ is the overall repayment rate. The overall repayment rate is equal to the weighted sum of the individual repayment rates of all debtors. A formal definition will be provided shortly. The asset payouts for a particular household will then differ depending upon if the household is a creditor or a debtor:

$$r_j^h(s) = \rho_j(s) r_j(s) \quad \text{if } z_j^h \geq 0 \ \text{ and } \ r_j^h(s) = r_j(s) \quad \text{if } z_j^h < 0 \quad \forall j.$$

Solvent households are required to fulfill their financial commitments entirely. The financial payout for a solvent household in state $s > 0$ is given by:

$$r^h(s) \cdot z^h = \sum\nolimits_j r_j^h(s) z_j^h.$$

A bankruptcy discharge may not always be permitted. In this model, households with income above the median level are not eligible for bankruptcy and must fulfill their financial commitments. I will denote the $l = 1$ physical commodity as the income of the household from the labor it provides. I assume that for some state $s > 0$, the household will receive an income above the median. Define $e_{med}(s)$ to be the median income in state $s > 0$.[10] The "means test" assumption can be state formally.

A.5 $\forall f \in \mathcal{F}$, $\exists s > 0$ s.t. $e_1^f(s) > e_{med}(s)$.

[10] $e_{med}(s)$ is defined as $e : \mu(A') = \mu(A'') = 0.5$ where $A' = \{f \in \mathcal{F} : e^f(s) \leq e\}$ and $A'' = \{f \in \mathcal{F} : e^f(s) \geq e\}$.

Define $\mathcal{S}_f^* = \{s > 0 : e_1^f(s) \leq e_{med}(s)\}$ as the set of states at which households of type f are "eligible" to declare bankruptcy.[11]

Let $\chi_j \in [0, \overline{\chi_j}]$ be the exemption level (in terms of the unit of account) for asset j. The presence of exemptions is not required for the proof of existence, so I allow for no exemptions, $\chi_j = 0$. To guarantee that the asset choices are bounded, I must assume that the exemption level is bounded above, $\chi_j \leq \overline{\chi_j}$ with $\overline{\chi_j} < \infty$. Upon declaring bankruptcy, a household discharges its debt on all its asset sales, forfeits its asset purchases with value above the exemption level, and pays the cost of bankruptcy.

The process of receiving bankruptcy exemptions and that of paying the cost of bankruptcy are closely related. To receive the bankruptcy exemptions, a bankrupt household must submit detailed records of all its asset purchases to the bankruptcy court. This process typically requires the household to hire legal counsel. The cost of submitting the records (or of hiring the legal counsel) will be strictly increasing in the value of the household's asset purchases.

Define the variables $(x)^+ = \max\{x, 0\}$ and $(x)^- = \min\{x, 0\}$. Define $\mathcal{H}_s'$ as the set of bankrupt households in state $s > 0$. The financial payout for a bankrupt household $h \in \mathcal{H}_s'$ is given by:

$$\sum_j \min\left\{r_j^h(s)\left(z_j^h\right)^+, \chi_j\right\} - \sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ .$$[12]

The first term signifies that a bankrupt household may keep up to χ_j of the value of its purchases $r_j^h(s)\left(z_j^h\right)^+$ for each asset j. The second term is the cost of bankruptcy. The assumption that the bankruptcy cost is strictly increasing in the value of the

[11] If I were to introduce spot prices into the definition of $\mathcal{S}_f^*$, then a small price change may switch $s \in \mathcal{S}_f^*$ to $s \notin \mathcal{S}_f^*$ or vice-versa. As a result, the budget correspondence would not be continuous.

[12] As with the financial payout for solvent households, there is an assumption that the payout cannot be smaller than the negative value of a household's endowments, that is, households are not permitted to hold negative wealth. In equilibrium, this lower bound would never be reached due to assumption $(A.2)$.

asset purchases is given simply as:

A.6 $\alpha >> 0$.

The following intuition motivates claim 1 that assumptions $(A.5) - (A.6)$ suffice to prove the boundedness of assets for bankrupt households. If Arrow securities were permitted, it would be optimal for any household to sell an arbitrarily large amount of the Arrow security paying out in state s. Such a sale would result in an arbitrage profit, higher payout in the initial state without reducing payout in any other state.

Arrow securities are not permitted in this model (assumption $(A.4)$). However, a household may be able to replicate an Arrow security with some combination of the available assets. Such a combination of assets must contain both purchases and sales. To obtain an arbitrage profit with this combination of assets, the household must transact an arbitrarily large amount of each asset. This will force the household to pay an arbitrarily large cost of bankruptcy on its asset purchases. Thus, the assumption $(A.6)$ with $\alpha >> 0$ prevents the most basic form of arbitrage imaginable for a bankruptcy model.

Claim 2.1 *For bankrupt households $h \in \cup_{s>0}\mathcal{H}'_s$, the assets z^h are bounded.*

Proof. Consider a household $h \in \cup_{s>0}\mathcal{H}'_s$ of type f. If h declares bankruptcy in some state, then its wealth is equal to

$$p(s)e^f(s) + \sum_j \min\left\{r_j^h(s)\left(z_j^h\right)^+, \chi_j\right\} - \sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ \qquad (2.1)$$

in those states and at least one such state exists since $h \in \cup_{s>0}\mathcal{H}'_s$. If h remains solvent in some state, either by choice or by force $[s \notin \mathcal{S}_f^*]$, then its wealth is equal to

$$p(s)e^f(s) + \sum_j r_j^h(s)z_j^h \qquad (2.2)$$

in those states and at least one such state exists by $(A.5)$. Suppose that there exists an optimal sequence of asset choices $\left(z_j^\nu\right)_{\forall\nu\in\mathbb{N}}$ such that $z_j^\nu \to \infty$ as $\nu \to \infty$ for some j. From (2.1) and with $\alpha >> 0$ and $\chi_j \leq \overline{\chi_j}$ bounded above, h would be consuming outside the consumption set $X^f = \mathbb{R}_+^G$ for some $\nu < \infty$ (contradiction). With z^h bounded above, suppose there exists an optimal sequence of asset choices $(z_k^\nu)_{\forall\nu\in\mathbb{N}}$ such that $z_k^\nu \to -\infty$ as $\nu \to \infty$ for some k. Then from (2.2) and with $r_j^h(s) = r_j(s) > 0$, h would be consuming outside of the consumption set $X^f = \mathbb{R}_+^G$ for some $\nu < \infty$ (contradiction). ■

2.2.3 Repayment rates

As this model only contains 2 periods, a household h will declare bankruptcy in some state $s > 0$ if (i) the household is eligible, $s \in \mathcal{S}_f^*$ with $h \in \mathcal{H}_f$ and (ii) the choice is optimal:

$$\sum_j \min\left\{r_j^h(s)\left(z_j^h\right)^+, \chi_j\right\} - \sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ > \sum_j r_j^h(s) z_j^h. \tag{2.3}$$

For these bankrupt households, the nonexempt asset purchases and the bankruptcy cost are turned over to the bankruptcy court. The total collected value is:

$$\sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+. \tag{2.4}$$

The court then liquidates the assets and returns the value to those creditors that are owed funds by the bankrupt household. A bankrupt household has total debt $\sum_j r_j(s)\left(z_j^h\right)^-$. I make the following distribution assumption.

A.7 If $h \in \mathcal{H}_s'$, then $\forall j$, the fraction $\frac{r_j(s)\left(z_j^h\right)^-}{\sum_j r_j(s)\left(z_j^h\right)^-}$ of (2.4) is returned to the asset j creditors.

Define the individual repayment rate $\delta^h(s) \in [0,1]$ as the repayment rate of any debtor h in state $s > 0$. If h is a debtor on multiple assets, assumption $(A.7)$ implies that the individual repayment rate is identical for all those assets in state s. Define $\delta^h(s) = 1$ for solvent households. From assumption $(A.7)$, a bankrupt household originally owes $r_j(s)\left(z_j^h\right)^-$ on asset j in state s, but only pays back $\delta^h(s) r_j(s)\left(z_j^h\right)^-$ of this value where:

$$\delta^h(s) = -\frac{\sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+}{\sum_j r_j(s)\left(z_j^h\right)^-}. \tag{2.5}$$

The bankruptcy inequality (2.3) holds iff $\delta^h(s) < 1$. Further, the financial payouts can now be written in an identical manner for both solvent and bankrupt households using $\delta^h(s)$:

$$\sum\nolimits_j \rho_j(s) r_j(s)\left(z_j^h\right)^+ + \sum\nolimits_j \delta^h(s) r_j(s)\left(z_j^h\right)^- . \tag{2.6}$$

The overall repayment rate $\rho_j(s)$ is determined in equilibrium such that the repayment rate expected by creditors is equal to the weighted sum of the individual repayment rates across all debtors:

$$\rho_j(s)\int_{h\in\mathcal{H}} \left(z_j^h\right)^+ + \int_{h\in\mathcal{H}} \delta^h(s)\left(z_j^h\right)^- = 0 \quad \forall j, \forall s > 0. \tag{2.7}$$

Equation (2.7) implies that

$$\int_{h\in\mathcal{H}} \left(\sum\nolimits_j \rho_j(s) r_j(s)\left(z_j^h\right)^+ + \sum\nolimits_j \delta^h(s) r_j(s)\left(z_j^h\right)^-\right) = 0 \quad \forall s > 0.$$

2.2.4 Definition of bankruptcy equilibrium

I will now write down the budget set for a household $h \in \mathcal{H}_f$:

$$B^h(p,q,\rho) = \left\{ \begin{array}{l} (x,z) \in X^h \times \mathbb{R}^J : \\ p(0)(e^f(0) - x(0)) - qz \geq 0, \\ \forall s \in \mathcal{S}_f^* : \quad p(s)\left(e^f(s) - x(s)\right) + \\ \max\left\{ r^h(s) z^h, \sum_j \left[\min\left\{ r_j^h(s)\left(z_j^h\right)^+, \chi_j \right\} - \alpha_j r_j^h(s)\left(z_j^h\right)^+ \right] \right\} \geq 0, \\ \forall s \notin \mathcal{S}_f^* : \quad p(s)(e^f(s) - x(s)) + r^h(s) \cdot z^h \geq 0. \end{array} \right\}.$$

I define a *bankruptcy equilibrium* as $\left((x^h, z^h)_{h\in\mathcal{H}}, p, q, \rho\right)$ s.t.

$$\forall h \in \mathcal{H}, \text{ given } (p,q,\rho) \qquad (x^h, z^h) \in \underset{subj\ to\ (x,z)\in B^h(p,q,\rho)}{\arg\max}\, u^h(x). \tag{H}$$

$$\begin{aligned} \int_{h\in\mathcal{H}} z_j^h &= 0 \quad \forall j. \\ \frac{1}{F}\sum\nolimits_{f\in\mathcal{F}} e_l^f(s) &= \int_{h\in\mathcal{H}} x_l^h(s) \quad \forall (l,s). \end{aligned} \tag{M}$$

$$\rho_j(s)\int_{h\in\mathcal{H}} \left(z_j^h\right)^+ + \int_{h\in\mathcal{H}} \delta^h(s)\left(z_j^h\right)^- = 0 \quad \forall j, \forall s > 0. \tag{2.7}$$

As with the Dubey et al. (2005) model of default, a no-trade outcome is always an equilibrium. By no-trade, I mean that the creditors expect the overall repayment rate $\rho_j(s) = 0 \;\; \forall s > 0$ and will not purchase asset j. The equilibrium asset price will be $q_j = 0$, so debtors will not sell asset j. This no-trade outcome trivially satisfies equation (2.7). Thus, pessimistic beliefs by the creditors can be self-fulfilling in equilibrium.

The no-trade equilibrium is not rational, though certainly an equilibrium. The reason that the beliefs by creditors are irrational is that assumption $(A.5)$ forces all debtors to repay their debt in some state $s > 0$. Thus, creditors should rationally expect a strictly positive repayment rate in some state and will be willing to purchase a strictly positive amount of asset j. Adhering to the idea suggested by Dubey et al. (2005), I need only introduce some tremble that prevents creditors from forming beliefs that $\rho_j(s) = 0 \;\; \forall s > 0$. The existence of an equilibrium with this tremble is actually what is shown in section 2.5.

2.2.5 Boundedness, solvent households

If $\int_{h\in\mathcal{H}} \left(z_j^h\right)^+ > 0$, then equation (2.7) can be rearranged (using the market clearing condition $\int_{h\in\mathcal{H}} z_j^h = 0$) as:

$$\rho_j(s) = 1 + \frac{\int_{h\in\mathcal{H}} \left(1 - \delta^h(s)\right) \left(z_j^h\right)^-}{\int_{h\in\mathcal{H}} \left(z_j^h\right)^+}. \tag{2.8}$$

When $\int_{h\in\mathcal{H}} \left(z_j^h\right)^+ > 0$, the overall repayment rate $\rho_j(s) \in [0,1]$. Recalling that $\delta^h(s) < 1$ if $h \in \mathcal{H}'_s$, the following intuitive results hold:

1. If $z_j^h \geq 0 \;\; \forall h \in \mathcal{H}'_s$ (or $\mathcal{H}'_s = \emptyset$), then $\rho_j(s) = 1$.

2. If $z_j^h < 0$ for some $h \in \mathcal{H}'_s$, then $\rho_j(s) < 1$.

Claim 2.1 above showed how the bankruptcy setup is able to bound the assets z^h for bankrupt households. Claim 2.2 will bound the assets z^h for entirely solvent households, that is, those households that do not declare bankruptcy in any state $s > 0$, the households $h \notin \cup_{s>0}\mathcal{H}'_s$. Collect the household-specific asset payouts

$\left(r_j^h(s)\right)_{\forall j, \forall s>0}$ into the household specific yields matrix Y^h. For the following claim, define the set of entirely solvent households of type f as $\mathcal{H}_f^* = \{h \in \mathcal{H}_f : h \notin \cup_{s>0} \mathcal{H}'_s\}$.

Claim 2.2 *For any $f \in \mathcal{F}$, the set $\int_{h \in \mathcal{H}_f^*} z^h$ is bounded.*

Proof. Suppose that $\int_{h \in \mathcal{H}_f^*} z^h$ is not bounded for some $f \in \mathcal{F}$. Divide the assets into subsets s.t. for sequences of assets $(z_1^\nu, .., z_J^\nu)_{\forall \nu \in \mathbb{N}}$:

1. If $\int_{h \in \mathcal{H}_f^*} z_j^\nu \to +\infty$ as $\nu \to \infty$, then $j \in \mathcal{J}^+$.

2. If $\int_{h \in \mathcal{H}_f^*} z_k^\nu \to -\infty$ as $\nu \to \infty$, , then $k \in \mathcal{J}^-$.

Under the assumption that $\int_{h \in \mathcal{H}_f^*} z^h$ is not bounded, then $\mathcal{J}^+ \cup \mathcal{J}^- \neq \emptyset$. Since $Y^h z^h$ is bounded, then $\mathcal{J}^+ \neq \emptyset$, $\mathcal{J}^- \neq \emptyset$, and Y^h does not have full rank in the limit. Denote the $S \times (\#\mathcal{J}^+ + \#\mathcal{J}^-)$ matrix Y^ν as the endogenous payout matrix with only assets $j \in \mathcal{J}^+ \cup \mathcal{J}^-$. For all submatrices of Y^ν with the number of rows equal to $(\#\mathcal{J}^+ + \#\mathcal{J}^-)$, call these submatrices$Y_n^\nu$ for $n = 1, ..., N$ where $N = \begin{pmatrix} S \\ \#\mathcal{J}^+ + \#\mathcal{J}^- \end{pmatrix}$:

$$\det Y_n^\nu \to 0 \ \ \forall n \text{ as } \nu \to \infty.$$

Y^ν has terms $\left(r_j^h(s), r_k^h(s)\right)_{j \in \mathcal{J}^+, k \in \mathcal{J}^-}$. By the definition of the asset payouts:

$$\begin{aligned} r_j^h(s) &= \rho_j^\nu(s) r_j(s) \ \ \forall \nu, \ \forall j \in \mathcal{J}^+. \\ r_k^h(s) &= r_j(s) \ \ \forall \nu, \ \forall k \in \mathcal{J}^-. \end{aligned}$$

For $j \in \mathcal{J}^+$, recall the definition of $\rho_j^\nu(s) \ \ \forall s \in \mathcal{S}_1$:

$$\rho_j^\nu(s) = 1 + \frac{\displaystyle\int_{h\in\mathcal{H}} \left(1-\delta^h(s)\right)\left(z_j^\nu\right)^-}{\displaystyle\int_{h\in\mathcal{H}} \left(z_j^\nu\right)^+}. \tag{2.8}$$

Since $\displaystyle\int_{h\in\mathcal{H}_f^*} z_j^\nu \to +\infty$, then $\displaystyle\int_{h\in\mathcal{H}} \left(z_j^\nu\right)^+ \to \infty$ as $\nu \to \infty$. The term $\displaystyle\int_{h\in\mathcal{H}} \left(1-\delta^h(s)\right)\left(z_j^\nu\right)^-$ is bounded because $\displaystyle\int_{h\in\mathcal{H}'_s} \left(z_j^\nu\right)^-$ is bounded by claim 1 and $\delta^h(s) = 1$ for $h \notin \mathcal{H}'_s$. Thus, from (2.8), $\rho_j^\nu(s) \to 1 \ \ \forall s > 0$ as $\nu \to \infty$.

Therefore, $Y_n^\nu \to Y_n \ \ \forall n$ as $\nu \to \infty$, where Y_n has terms $\left(r_j(s), r_k(s)\right)_{j\in\mathcal{J}^+, k\in\mathcal{J}^-}$. As $\det Y_n^\nu \to 0$, then $\det Y_n = 0 \ \ \forall n$.

From $(A.4)$, Y has full column rank and is in general position. Thus, any square submatrix Y_n with J rows will have full rank. Also, for any number of columns $J' \le J$, $\exists$ a square submatrix with J' rows and full rank. This contradicts that $\det Y_n = 0 \ \ \forall n$. ■

Theorem 2.1 states the general existence of a *bankruptcy equilibrium.* The proof of this theorem is contained in section 2.5.

Theorem 2.1 *Given assumptions* $(\mathbf{A.1})$–$(\mathbf{A.7})$, *a bankruptcy equilibrium* $\left((x^h, z^h)_{h\in\mathcal{H}}, p, q, \rho\right)$ *exists.*

2.3 Absence of Chain Reactions

I will analyze an economy with two types of households $F = 2$, two assets $J = 2$, and two states of uncertainty $S = 2$.[13]

[13] Recall that a model with only a single asset is identical to the model of default by Dubey et al. (2005).

To compute bankruptcy equilibria, I must specify the system of equations that characterize the equilibria. The first order conditions with respect to assets are given by:

$$\lambda^h \hat{R}^h = 0.$$

The $1 \times (S+1)$ vector λ^h is the vector of Lagrange multipliers for each of the $(S+1)$ budget constraints in the household's problem (H). The returns matrix $\hat{R}^h$ is the household-specific $(S+1) \times J$ matrix

$$\hat{R}^h = \begin{pmatrix} -q \\ \hat{Y}^h \end{pmatrix}.$$

The yields matrix $\hat{Y}^h$ is the household-specific $S \times J$ matrix with terms $\hat{r}_j^h(s)$ defined as:

$$\hat{r}_j^h(s) = \rho_j(s) r_j(s) \quad \text{if } z_j^h \geq 0 \text{ and } \quad \hat{r}_j^h(s) = \delta^h(s) r_j(s) \quad \text{if } z_j^h < 0 \quad \forall j.$$

I will compare the GEI equilibrium with the bankruptcy equilibrium. I will say that a "chain reaction of bankruptcy" has occurred if there exists a household h who chooses to remain solvent in some state $s > 0$ given the GEI equilibrium prices and chooses to declare bankruptcy in that state s given the bankruptcy equilibrium prices. If such a phenomenon occurs, the declaration of bankruptcy by other households has forced an otherwise solvent household into a bankruptcy position. This "chain reaction" is instantaneous, but can be examined by comparing the GEI equilibrium prices with the bankruptcy equilibrium prices.

A natural conjecture would be that a household with a mixture of both asset purchases and sales, upon receiving a lower payout for its purchases due to bankruptcy declarations by other households, will then choose to declare bankruptcy. I prove that with two households, this conjecture is false, that is, a chain reaction of bankruptcy cannot occur. While the result is only valid for two households, the result suggests that chain reactions of bankruptcy would be unlikely even with three or more households.

The costs of bankruptcy $(\alpha_1, \alpha_2) >> 0$ and the exemption levels $(\chi_1, \chi_2) \in [0, \overline{\chi_1}] \times [0, \overline{\chi_2}]$ can take any values. Suppose that the household endowments belong to a generic subset:

$$e^1(1) \neq e^2(1) \quad \text{and} \quad e^1(2) \neq e^2(2).$$

The household with the higher endowment in each state will have endowment above the median level e_{med}. So by assumption $(A.5)$, one type of household can declare bankruptcy in each state and no type is eligible to declare in both states.

Let the yields matrix be the full rank 2×2 matrix Y. Take any state $s \in \{1, 2\}$. Let household h be a household of type f with $s \in \mathcal{S}_f^*$. Let $k \neq h$ be a household of the other type, $k \notin \mathcal{H}_f$. If household h does not declare bankruptcy in s, the asset payouts for both households are:

$$r^h(s) = (r_1(s), r_2(s)) = r^k(s).$$

If household h does declare bankruptcy, the following result holds:

Claim 2.3 $\left(z_1^h\right) \cdot \left(z_2^h\right) < 0$ *for bankrupt household* $h \in \mathcal{H}_s'$.

Proof. If $z_1^h \geq 0$ and $z_2^h \geq 0$, then

$$\textstyle\sum_j r_j^h(s) z_j^h \geq \sum_j \min\left\{r_j^h(s)\left(z_j^h\right)^+, \chi_j\right\} - \sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+,$$

so h would not choose to declare bankruptcy (contradicting $h \in \mathcal{H}'_s$).

If $z_1^h < 0$ and $z_2^h < 0$, then

$$\delta^h(s) = -\frac{\sum_j \alpha_j r_j^h(s) \left(z_j^h\right)^+ + \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+}{\sum_j r_j(s) \left(z_j^h\right)^-} = 0.$$

The payout matrix for h now contains the row $\hat{r}^h(s) = \delta^h(s) \left(r_1(s), r_2(s)\right) = (0, 0)$. The equilibrium prices q can never be such that assets 1 and 2 are dependent for household h.[14] With $\alpha = \overrightarrow{0}$, household h could make an arbitrage profit by purchasing arbitrarily large amounts of one asset and selling arbitrarily large amounts of the other asset. With $\alpha >> \overrightarrow{0}$, an arbitrage profit is ruled out, but household h will still optimize by purchasing one asset and selling the other. Thus $\delta^h(s) > 0$ and there exists only one asset sale $z_j^h < 0$. ■

Suppose without loss of generality that $z_1^h > 0$ and $z_2^h < 0$. This holds for all households of type f with $s \in \mathcal{S}_f^*$. Then the individual repayment rate is equal to:

$$\delta^h(s) = -\frac{\alpha_1 r_1(s) z_1^h + \left(r_1^h(s) z_1^h - \chi_1\right)^+}{r_2(s) z_2^h}.$$

With household h declaring bankruptcy, the overall repayment rate is reduced. For asset $j = 1$, the overall repayment rate $\rho_1(s) = 1$ since the debtors are of type $\hat{f}$ with $s \notin \mathcal{S}_{\hat{f}}^*$. For asset 2 from equation (2.8),

$$\rho_2(s) = 1 + \frac{\left(1 - \delta^h(s)\right) z_2^h}{z_2^k}$$

[14] Consider when household h of type f declares bankruptcy in state $s = 1 \in \mathcal{S}_f^*$ with $\delta^h(1) = 0$. The first order conditions with respect to assets for h are given by: $\lambda^h \begin{pmatrix} -q_1 & -q_2 \\ 0 & 0 \\ r_1^h(2) & r_2^h(2) \end{pmatrix} = 0$. To satisfy no arbitrage, $\frac{q_1}{q_2} = \frac{r_1^h(2)}{r_2^h(2)}$. Even in the unlikely case that the asset prices satisfy this equality, the choice of assets is arbitrary as the assets are dependent. Thus, some of the possible choices include assets that will still satisfy the statement of this claim: $\left(z_1^h\right) \cdot \left(z_2^h\right) < 0$.

where z_2^k is the asset purchase of all households of type $\hat{f}$ with $s \notin \mathcal{S}^*_{\hat{f}}$. By market clearing, the asset choice is given by $z_2^k = -z_2^h$. Thus,

$$\rho_2(s) = 1 + \frac{\left(1-\delta^h(s)\right) z_2^h}{-z_2^h} = \frac{-z_2^h + \left(1-\delta^h(s)\right) z_2^h}{-z_2^h} = \frac{-\delta^h(s) z_2^h}{-z_2^h} = \delta^h(s).$$

Therefore, the asset payouts in state s are given by:

$$\begin{aligned} \hat{r}^h(s) &= \left(r_1(s), \delta^h(s) r_2(s)\right) \\ \hat{r}^k(s) &= \left(r_1(s), \rho_2(s) r_2(s)\right) \end{aligned}$$

where h are households of type f with $s \in \mathcal{S}^*_f$ and k are households of type $\hat{f}$ with $s \notin \mathcal{S}^*_{\hat{f}}$. In equilibrium, $\hat{r}^h(s) = \hat{r}^k(s)$.

The same holds $\forall s > 0$, so $\hat{Y}^h = \hat{Y}$ for all households h and both types of households face the same returns matrix $\hat{R} = \begin{pmatrix} -q \\ \hat{Y} \end{pmatrix}$.

The GEI model is the canonical model with full commitment. Without bankruptcy, the returns matrix for the GEI model is $R = \begin{pmatrix} -q \\ Y \end{pmatrix}$.

Claim 2.4 *The set of bankruptcy equilibrium allocations is identical to the set of GEI equilibrium allocations.*

Proof. I will show that the equilibrium allocations are equal by showing that they are both identical to the set of Arrow-Debreu equilibrium allocations. This is standard for the GEI equilibria with complete markets. The state prices for the GEI equilibria are found as $(\pi(0), \pi(1), \pi(2)) : \pi(0) = 1$ and

$$(\pi(1), \pi(2)) = \left(\frac{\lambda^h(1)}{\lambda^h(0)}, \frac{\lambda^h(2)}{\lambda^h(0)}\right) = (q_1, q_2) \cdot (Y)^{-1}.^{15} \tag{2.9}$$

From these state prices, the Arrow-Debreu prices can be defined.

The bankruptcy model has complete markets as well. Let the asset prices for bankruptcy equilibria be specified as $(\hat{q}_1, \hat{q}_2)$. Then the state prices for the bankruptcy equilibria are exactly equal to the state prices of the GEI equilibria:

$$(\pi(1), \pi(2)) = \left(\frac{\lambda^h(1)}{\lambda^h(0)}, \frac{\lambda^h(2)}{\lambda^h(0)}\right) = (\hat{q}_1, \hat{q}_2) \cdot \left(\hat{Y}\right)^{-1}. \tag{2.10}$$

Thus, the Arrow-Debreu prices are the same and the GEI equilibrium allocations and bankruptcy equilibrium allocations are equivalent to the Arrow-Debreu equilibrium allocations (Pareto optimal allocations). ■

Recall that I defined a "chain reaction of bankruptcy" as that event whereby a household h chooses to declare bankruptcy in some state $s > 0$ at the bankruptcy equilibrium prices and yet chooses to remain solvent in s at the GEI equilibrium prices.

Theorem 2.2 *In the simple economy specified with two household types $F = 2$, two assets $J = 2$, and two states of uncertainty $S = 2$, it is impossible to have a chain reaction of bankruptcy.*

Proof. Specify the GEI equilibrium prices as $\left(p, q, \overrightarrow{1}\right)$ and the bankruptcy equilibrium prices as $(\hat{p}, \hat{q}, \hat{\rho})$. If $\left(p, q, \overrightarrow{1}\right) = (\hat{p}, \hat{q}, \hat{\rho})$, the proof is trivial, so suppose $\left(p, q, \overrightarrow{1}\right) \neq (\hat{p}, \hat{q}, \hat{\rho})$. Consider any household $h \in \mathcal{H}_f$ and let $\hat{z}^h$ be the asset choices

[15] With complete markets, $\left(\frac{\lambda^h(1)}{\lambda^h(0)}, \frac{\lambda^h(2)}{\lambda^h(0)}\right) = \left(\frac{\lambda^k(1)}{\lambda^k(0)}, \frac{\lambda^k(2)}{\lambda^k(0)}\right)$ where $h \in \mathcal{H}_f$ and $k \notin \mathcal{H}_f$.

under $(\hat{p}, \hat{q}, \hat{\rho})$. In the GEI equilibrium, without the opportunity for bankuptcy, denote z^h as the asset choices given prices $\left(p, q, \overrightarrow{1}\right)$.

Suppose that h declares bankruptcy in state $s \in \mathcal{S}_f^*$ given the prices $(\hat{p}, \hat{q}, \hat{\rho})$. As $\hat{\delta}^h(s) < 1$ for some h, s, then $\rho < \overrightarrow{1}$ and $\hat{Y} < Y$.[16] From $(2.9) - (2.10)$, then $q > \hat{q}$. From the previous claim, $p = \hat{p}$ and the equilibrium consumption choices are identical for both sets of prices $\left(p, q, \overrightarrow{1}\right)$ and $(\hat{p}, \hat{q}, \hat{\rho})$. Thus the asset payouts are identical:

$$Yz^h = \hat{Y}\hat{z}^h = \begin{pmatrix} p(1)\left(x^h(1) - e^f(1)\right) \\ p(2)\left(x^h(2) - e^f(2)\right) \end{pmatrix}.$$

Premultiplying by $(\pi(1), \pi(2)) >> 0$, then $(2.9) - (2.10)$ imply $qz^h = \hat{q}\hat{z}^h$.

In the bankruptcy equilibrium, the individual repayment rate is

$$\hat{\delta}^h(s) = -\frac{\sum_j \alpha_j r_j(s)\left(\hat{z}_j^h\right)^+ + \sum_j \left(r_j(s)\hat{z}_j^h - \chi_j\right)^+}{\sum_j r_j(s)\left(\hat{z}_j^h\right)^-} < 1.$$

Notice that $\hat{r}_j(s) = r_j(s)$ for the asset j such that $\hat{z}_j^h \geq 0$. The debtors on this asset can only be households $k \in \mathcal{H}_{\hat{f}}$ where $s \notin \mathcal{S}_{\hat{f}}^*$ implies $\hat{\delta}^k(s) = 1 \ \ \forall k \in \mathcal{H}_{\hat{f}}$.

Under the prices $\left(p, q, \overrightarrow{1}\right)$, h chooses the assets z^h given that the economy *does not permit bankruptcy*. If given the opportunity to declare bankruptcy, h may change the assets to y^h. This is obvious given that the payout matrix directly depends upon if bankrupty is declared or not. It is feasible for the household under the prices $(p, q, 1)$ to choose $y^h = \hat{z}^h$, the assets chosen under the prices $(\hat{p}, \hat{q}, \hat{\rho})$ when permitted to declare bankruptcy. If $y^h = \hat{z}^h$, then the individual repayment rate under $\left(p, q, \overrightarrow{1}\right)$

[16] When writing inequalities with matrices, the matrices are considered as a vector containing all their elements.

is:

$$\delta^h(s) = -\frac{\sum_j \alpha_j r_j(s)\left(y_j^h\right)^+ + \sum_j \left(r_j(s) y_j^h - \chi_j\right)^+}{\sum_j r_j(s)\left(y_j^h\right)^-} = \hat{\delta}^h(s) < 1.$$

The financial payout in state s is $Y(s)z^h = \sum_j r_j(s)\left(z_j^h\right)^+ + \sum_j \delta^h(s) r_j(s)\left(z_j^h\right)^-$. The assets y^h provide the same payout as z^h in state s (since $\hat{Y}(s)y^h = \hat{Y}(s)\hat{z}^h = Y(s)z^h$) and strictly higher payout in state $s' \notin \mathcal{S}_f^*$ (since $\hat{\rho}(s') \leq \overrightarrow{1}$ implies $Y(s')y^h > \hat{Y}(s')\hat{z}^h = Y(s')z^h$). In state $s = 0$, to compare qy^h against qz^h, the equalities $qz^h = \hat{q}\hat{z}^h = \hat{q}y^h$ allow me to compare qy^h against $\hat{q}y^h$. From above, the asset prices satisfy $q > \hat{q}$. Thus, $\sum_j q_j\left(y_j^h\right)^- < \sum_j \hat{q}_j\left(y_j^h\right)^-$. Even though, $\sum_j q_j\left(y_j^h\right)^+ \geq \sum_j \hat{q}_j\left(y_j^h\right)^+$, this additional cost of purchases is exactly canceled by strictly higher payouts $\rho(s') = \overrightarrow{1} \geq \hat{\rho}(s')$.

Therefore, y^h dominates z^h given that the prices are $\left(p, q, \overrightarrow{1}\right)$ and that the economy permits bankruptcy. With the asset choice y^h, since $\delta^h(s) = \hat{\delta}^h(s) < 1$, then h declares bankruptcy. The optimal asset choice must then satisfy $\delta^h(s) \leq \hat{\delta}^h(s)$ for h to pick the optimal choice over y^h. So h declares bankruptcy in state s given prices $\left(p, q, \overrightarrow{1}\right)$. This completes the proof. ■

2.4 Real Costs of Bankruptcy

Now suppose that bankruptcy imposes real costs on the economy. Define $\rho_{j,RC}(s)$ as the overall repayment rate of debtors in such an economy where RC stands for 'real costs'. The amount turned over by bankrupt households to the courts have value $\sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+$. Assume that the bankruptcy courts are not completely efficient in liquidating the assets of bankrupt households and distributing their value to the creditors. That is, suppose that only $\beta \sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \beta \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+$ with $\beta \in [0, 1)$ is made available to the creditors. The difference $(1-\beta)\sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + (1-\beta)\sum_j \left(r_j^h(s)z_j^h - \chi_j\right)^+$ is irrevocably lost to the economy (a real loss of the numeraire commodity). As a result, the market clearing conditions are appropriately adjusted to account for this loss:

$$\begin{aligned} \int_{h\in\mathcal{H}} z_j^h &= 0 \ \ \forall j. && \text{(M)} \\ \frac{1}{F}\sum\nolimits_{f\in\mathcal{F}} e_l^f(s) &= \int_{h\in\mathcal{H}} x_l^h(s) \quad \forall (l,s) \notin \{(L,1),...,(L,S)\}. \end{aligned}$$

$$\tfrac{1}{F}\sum\nolimits_{f\in\mathcal{F}} e_L^f(s) - \int_{h\in\mathcal{H}'_s} (1-\beta)\left[\sum\nolimits_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \sum\nolimits_j \left(r_j^h(s)z_j^h - \chi_j\right)^+\right] = \int_{h\in\mathcal{H}} x_L^h(s)$$
$$\forall s > 0.$$

The individual repayment rate $\delta_{RC}^h(s)$ is defined as $\delta_{RC}^h(s) = 1$ for solvent households $h \notin \mathcal{H}'_s$ and

$$\delta_{RC}^h(s) = -\frac{\beta\left[\sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \sum_j \left(r_j^h(s)z_j^h - \chi_j\right)^+\right]}{\sum_j r_j(s)\left(z_j^h\right)^-} \tag{2.11}$$

for bankrupt households $h \in \mathcal{H}'_s$. The equation for the financial payouts remains the same in terms of $\delta^h(s)$ as was defined in (2.5) :

$$\sum\nolimits_j \rho_j(s) r_j(s)\left(z_j^h\right)^+ + \sum\nolimits_j \delta^h(s) r_j(s)\left(z_j^h\right)^- . \tag{2.12}$$

The resulting equilibrium conditions are similar to (2.7) :

$$\rho_{j,RC}(s)\int_{h\in\mathcal{H}} \left(z_j^h\right)^+ + \int_{h\in\mathcal{H}} \delta_{RC}^h(s)\left(z_j^h\right)^- = 0 \ \ \forall j, s > 0. \tag{2.13}$$

Claim 2.5 *Equation* (2.13) *implies that*

$$\begin{aligned} &\int_{h\in\mathcal{H}} \left(\sum_j \rho_{j,RC}(s) r_j(s) \left(z_j^h\right)^+ + \sum_j \delta^h(s) r_j(s) \left(z_j^h\right)^-\right) = \\ &-\int_{h\in\mathcal{H}'_s} (1-\beta)\left[\sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ + \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+\right] \quad \forall s>0. \end{aligned} \tag{2.14}$$

Proof. Equation (2.12) can be written equivalently as:

$$\begin{array}{ll} \sum_j \rho_{j,RC}(s) r_j(s) \left(z_j^h\right)^+ + \sum_j r_j(s) \left(z_j^h\right)^- & \text{for } h \notin \mathcal{H}'_s \\ \sum_j \rho_{j,RC}(s) r_j(s) \left(z_j^h\right)^+ - \sum_j \alpha_j r_j^h(s)\left(z_j^h\right)^+ - \sum_j \left(r_j^h(s) z_j^h - \chi_j\right)^+ & \text{for } h \in \mathcal{H}'_s \end{array}.$$

Using these expressions, then (2.14) simplifies to:

$$\begin{aligned} &\int_{h\in\mathcal{H}} \left(\sum_j \rho_{j,RC}(s) r_j(s) \left(z_j^h\right)^+\right) + \int_{h\notin\mathcal{H}'_s} \left(\sum_j r_j(s) \left(z_j^h\right)^-\right) \\ &-\beta \int_{h\in\mathcal{H}'_s} \left[\sum_j \alpha_j r_j(s) \left(z_j^h\right)^+ + \sum_j \left(r_j^h(s) \left(z_j^h\right)^+ - \chi_j\right)^+\right] = 0. \end{aligned} \tag{2.15}$$

Using the definition of $\delta_{RC}^h(s)$ from (2.11), then (2.15) simplifies to:

$$\int\limits_{h\in\mathcal{H}} \left(\sum_j \rho_{j,RC}(s) r_j(s) \left(z_j^h\right)^+\right) + \int\limits_{h\notin\mathcal{H}'_s} \left(\sum_j r_j(s) \left(z_j^h\right)^-\right) + \int\limits_{h\in\mathcal{H}'_s} \delta_{RC}^h(s) \left(\sum_j r_j(s) \left(z_j^h\right)^-\right) = 0.$$

By definition, $\delta_{RC}^h(s) = 1$ for $h \notin \mathcal{H}'_s$, so using equation (2.13) finishes the argument.

■

With this claim, it is straightforward to write a proof for the existence of a bankruptcy equilibrium with real costs using the results contained in sections 2.2 and 2.5.

If $\int_{h\in\mathcal{H}} \left(z_j^h\right)^+ > 0$, then I can rearrange equation (2.13) :

$$\rho_{j,RC}(s) = 1 + \frac{\int_{h\in\mathcal{H}} \left(1-\delta_{RC}^h(s)\right)\left(z_j^h\right)^-}{\int_{h\in\mathcal{H}} \left(z_j^h\right)^+}. \tag{2.16}$$

The equation (2.16) yields the following three results. The first two are the same as in section 2.2, while the third one compares the model without real costs to the model with real costs. The subscript RC will be used to distinguish the overall repayment rate $\rho_{j,RC}(s)$ in the model with real costs from the overall repayment rate $\rho_j(s)$ in the model without real costs.

1. If $z_j^h \geq 0 \ \ \forall h \in \mathcal{H}_s'$ (or $\mathcal{H}_s' = \emptyset$), then $\rho_{j,RC}(s) = 1$.

2. If $z_j^h < 0$ for some $h \in \mathcal{H}_s'$, then $\rho_{j,RC}(s) < 1$.

3. $\rho_{j,RC}(s) \leq \rho_j(s)$ with strict inequality if $z_j^h \cdot \left(\sum_j \left(z_j^h\right)^+\right) < 0$ for some $h \in \mathcal{H}_s'$.

Proof. (Statement 3)

For the third statement, the equality $\rho_{j,RC}(s) = \rho_j(s)$ is obvious if $z_j^h \cdot \left(\sum_j \left(z_j^h\right)^+\right) \geq 0 \ \ \forall h \in \mathcal{H}_s'$. This is because $\delta_{RC}^h(s) = \delta^h(s) = 0 \ \forall h \in \mathcal{H}_s'$ if $\sum_j \left(z_j^h\right)^+ = 0$ and $\rho_{j,RC}(s) = \rho_j(s) = 1$ if $z_j^h \geq 0 \ \forall h \in \mathcal{H}_s'$.

i. To show that $\rho_{j,RC}(s) < \rho_j(s)$ under the condition $z_j^h \cdot \left(\sum_j \left(z_j^h\right)^+\right) < 0$, first consider the ceterus paribus analysis in which the households' asset choices z^h are held fixed. With $\beta < 1$, then $\delta_{RC}^h(s) < \delta^h(s)$ and from (2.16), $\rho_{j,RC}(s) < \rho_j(s)$.

ii. Now allow the households to equilibrate under the new repayment rates $\rho_{j,RC}(s)$. There exists values of β arbitrarily close to 0 so that $\rho_{j,RC}(s) < \rho_j(s)$. Suppose that as β increases outside this open neighborhood, the equilibrium overall repayment rates converge $\rho_{j,RC}(s) \to \rho_j(s)$ from below. Then the equilibrium asset choices in the model with real costs approach the asset choices in the model without real costs. If the choices are identical, then $\rho_{j,RC}(s) < \rho_j(s)$. Thus, it can never be that $\rho_{j,RC}(s) \geq \rho_j(s)$ as this contradicts part (i) of the proof.

■

Consider the normative impact of an increase in the real costs, a decrease in the parameter β. When $\rho_{j,RC}(s) < \rho_j(s)$ for some j, this change will be Pareto inferior (utility will not increase for any households and will strictly decrease for some).[17] From a normative standpoint, it is obvious that the planner would want to reduce the real costs parameterized by β, but it is not clear how the planner could use a tax scheme to finance such a reduction, a tax scheme that would need to be approved by some/most/all households. Without a definitive normative result, the main benefit of this extension is that under a more realistic bankruptcy setup, one with real costs from the inefficiency of the bankruptcy process, existence is guaranteed and the effects of this inefficiency on prices and allocation can be quantified.

[17] There are three types of households to consider: solvent ones that remain so after the increase, bankrupt ones that remain so after the increase, and solvent household that now declare bankruptcy as a result of the lower overall repayment rate ρ. Solvent creditors are strictly worse off as their asset payouts are lower. Bankrupt households are indifferent. For the households that change their bankruptcy decision, they could have chosen a portfolio to put them in a bankrupt position previously (and taken advantage of the potential spanning ability of bankruptcy/default as in Zame, 1993), but they chose not to. Thus, these households prefer to remain solvent. However, the prices are now such that the households will declare bankruptcy, a decision that makes them at least as worse off.

2.5 Proof of Theorem 2.1

Define the upper bound $\bar{x} \in \mathbb{R}^G_+$ as

$$\bar{x}_l(s) = 2\sum_{f\in\mathcal{F}} e^f_l(s) \;\; \forall l, s.$$

Define the bounded budget set as

$$\bar{B}^h(p,q,\rho) = \left\{ \begin{array}{l} (x,z) \in X^h \times \mathbb{R}^J : \; x \leq \bar{x} \\ p(0)(e^f(0) - x(0)) - qz \geq 0, \\ \forall s \in \mathcal{S}^*_f : p(s)(e^f(s) - x(s)) + \\ \max\left\{ r^h(s)\cdot z, \sum_j \left[\min\left\{r^h_j(s)(z_j)^+, \chi_j\right\} - \alpha_j r^h_j(s)(z_j)^+\right]\right\} \geq 0, \\ \forall s \notin \mathcal{S}^*_f : p(s)(e^f(s) - x(s)) + r^h(s)\cdot z \geq 0 \end{array} \right\}.$$

In equilibrium, the constraints $x \leq \bar{x}$ will never bind. Since u^h is continuous and quasi-concave, adding the nonbinding constraints $x \leq \bar{x}$ is innocuous and does not change the household decision.

Lemma 2.1 $\displaystyle\int_{h\in\mathcal{H}_f} \bar{B}^h(p,q,\rho)$ *is a compact set for each type* $f \in \mathcal{F}$.

Proof. The sets are trivially closed. For boundedness, see the proof of claims 2.1 and 2.2 in section 2.2. ■

I will define a correspondence with the same name as the budget set $\bar{B}^h(p,q,\rho)$ such that the value of the correspondence is equal to this set. The correspondence $\bar{B}^h : \Delta^*(\omega) \rightrightarrows \bar{B}^h$ defined as $(p,q,\rho) \longmapsto \bar{B}^h(p,q,\rho)$ has domain $\Delta^*(\omega)$, the price space to be defined shortly. The price space restricts $p(s) > 0 \;\; \forall s \geq 0$. The correspondence $\bar{B}^h$ is well-defined and upper hemicontinuous (uhc).

Lemma 2.2 $\bar{B}^h$ *is lower hemicontinuous (lhc).*

Proof. Consider a sequence $(p^\nu, q^\nu, \rho^\nu) \to (p, q, \rho)$ and $(x, z) \in \bar{B}^h(p, q, \rho)$. To verify lhc, I will construct a sequence (x^ν, z^ν) s.t. (i) $\exists N^*$ s.t. $(x^\nu, z^\nu) \in \bar{B}^h(p^\nu, q^\nu, \rho^\nu)$ $\forall \nu \geq N^*$ and (ii) $(x^\nu, z^\nu) \to (x, z)$.

The proof method will be to determine some scaling fraction $\theta^\nu(s) \in [0,1]$ $\forall s \geq 0$. This scaling fraction must converge $\theta^\nu(s) \to 1$ $\forall s \geq 0$. Further, by defining

$$\begin{aligned} x^\nu(s) &= \left(\min_{s\geq 0} \theta^\nu(s)\right) \cdot x(s) \quad \forall s \geq 0 \\ z^\nu &= \left(\min_{s\geq 0} \theta^\nu(s)\right) \cdot z, \end{aligned}$$

the vector (x^ν, z^ν) will now be affordable. I define $\theta^\nu = \left(\min_{s\geq 0} \theta^\nu(s)\right)$ for simplicity.

In order for a household to declare bankruptcy, (i) it must be eligible to do so and (ii) it must find it optimal to do so. By definition, the set $\mathcal{S}_f^*$ is independent of prices. Thus, I only focus attention on the static bankruptcy maximization:

$$\max \left\{ r^h(s) \cdot z, \sum_j \left[\min \left\{ r_j^h(s)(z_j)^+, \chi_j \right\} - \alpha_j r_j^h(s)(z_j)^+ \right] \right\}.$$

I will need to show that $(\theta^\nu(s))_{s\geq 0} \to \overrightarrow{1}$ and $\left((x^\nu(s))_{\forall s\geq 0}, z^\nu\right)$ is affordable given prices (p^ν, q^ν, ρ^ν) $\forall \nu \geq N^*$.

Let $s = 0.$

If $p(0)(e^f(0) - x(0)) - qz > 0$, then $\exists M_0$ s.t. $\forall \nu \geq M_0$,

$$p^\nu(0)(e^f(0) - x(0)) - q^\nu z > 0.$$

For these $\nu \geq M_0$, set $\theta^\nu(0) = 1$.

If $p(0)(e^f(0) - x(0)) - qz = 0$, then $p(0)x(0) + qz > 0$ and $\exists N_0$ s.t. $\forall \nu \geq N_0$,

$p^\nu(0)x(0) + q^\nu z > 0$. Define $\theta^\nu(0)$ as:

$$\theta^\nu(0) = \left\{ \begin{array}{ll} \frac{p^\nu(0)e^f(0)}{p^\nu(0)x(0)+q^\nu z} & \text{if } p^\nu(0)e^f(0) < p^\nu(0)x(0) + q^\nu z \\ & \text{and } p^\nu(0)x(0) + q^\nu z \neq 0 \\ 1 & \text{otherwise} \end{array} \right\}.$$

In the limit, $p^\nu(0)x(0) + q^\nu z \to p^\nu(0)e^f(0)$ and $\forall \nu \geq N_0$, $p^\nu(0)x(0) + q^\nu z > 0$. Thus $\theta^\nu(0) \to 1$. To verify feasibility,

$$p^\nu(0)x^\nu(0) + q^\nu z^\nu = \theta^\nu \left(p^\nu(0)x(0) + q^\nu z\right).$$

Since $\theta^\nu \leq \theta^\nu(0)$ and $p^\nu(0)x(0) + q^\nu z > 0$, then

$$p^\nu(0)x^\nu(0) + q^\nu z^\nu \leq p^\nu(0)e^f(0).$$

Thus $(x^\nu(0), z^\nu)$ satisfies the budget constraint for $s = 0$ given $(p^\nu(0), q^\nu)$.

Let $s > 0$.

Define the financial payout as

$$w(z, \rho; s) = \max\left\{r^h(s) \cdot z, \sum_j \left[\min\left\{r_j^h(s)(z_j)^+, \chi_j\right\} - \alpha_j r_j^h(s)(z_j)^+\right]\right\}.$$

The following two claims will allow me to streamline the argument.

Claim 2.6 $w(\theta z, \rho^\nu; s) \geq \theta w(z, \rho^\nu; s) \quad \forall \theta \in [0, 1]$.

Claim 2.7 $\min\left\{\theta r_j^h(s)(z_j)^+, \chi_j\right\} \geq \theta \min\left\{r_j^h(s)(z_j)^+, \chi_j\right\} \quad \forall \theta \in [0, 1]$.

Proof of Claim 2.7

Case 1: If $\min\left\{r_j^h(s)(z_j)^+, \chi_j\right\} = r_j^h(s)(z_j)^+$, then $\min\left\{\theta r_j^h(s)(z_j)^+, \chi_j\right\} = \theta r_j^h(s)(z_j)^+$. Obviously, $\theta r_j^h(s)(z_j)^+ \geq \theta\left(r_j^h(s)(z_j)^+\right)$. Case 2: Suppose that $\min\left\{r_j^h(s)(z_j)^+, \chi_j\right\} = \chi_j$. If $\min\left\{\theta r_j^h(s)(z_j)^+, \chi_j\right\} = \chi_j$, then obviously $\chi_j \geq \theta\chi_j$. If $\min\left\{\theta r_j^h(s)(z_j)^+, \chi_j\right\} = \theta r_j^h(s)(z_j)^+$, then $\theta r_j^h(s)(z_j)^+ \geq \theta\chi_j$ as we are considering case 2.■

Proof of Claim 2.6

Case 1: If $w(z, \rho^\nu; s) = r^h(s)\cdot z$, then $w(\theta z, \rho^\nu; s) \geq \theta\left(r^h(s)\cdot z\right)$ by definition. Obviously, $\theta r^h(s)\cdot z \geq \theta\left(r^h(s)\cdot z\right)$, so $w(\theta z, \rho^\nu; s) \geq \theta w(z, \rho^\nu; s)$ holds. Case 2: Now consider $w(z, \rho^\nu; s) = \sum_j \min\left\{r_j^h(s)(z_j)^+, \chi_j\right\} - \sum_j \alpha_j r_j^h(s)(z_j)^+$. By definition, $w(\theta z, \rho^\nu; s) \geq \sum_j \min\left\{\theta r_j^h(s)(z_j)^+, \chi_j\right\} - \theta\sum_j \alpha_j r_j^h(s)(z_j)^+$. Using claim 2.7 for all assets j, then

$$\begin{gathered}\sum_j \min\left\{\theta r_j^h(s)(z_j)^+, \chi_j\right\} - \theta\sum_j \alpha_j r_j^h(s)(z_j)^+ \geq \\ \theta\left(\sum_j \min\left\{r_j^h(s)(z_j)^+, \chi_j\right\} - \sum_j \alpha_j r_j^h(s)(z_j)^+\right).\end{gathered}$$

In sum, $w(\theta z, \rho^\nu; s) \geq \theta w(z, \rho^\nu; s)$, completing the argument.■

Returning to the proof of *Lemma 2.2,* if $p(s)(e^f(s) - x(s)) + w(z, \rho; s) > 0$, then $\exists M_s$ s.t. $\forall\nu \geq M_s$,

$$p^\nu(s)(e^f(s) - x(s)) + w(z, \rho^\nu; s) > 0.$$

For these $\nu \geq M_s$, set $\theta^\nu(s) = 1$.

If $p(s)(e^f(s) - x(s)) + w(z, \rho; s) = 0$, then $p(s)x(s) - w(z, \rho; s) > 0$ and $\exists N_s$ s.t. $\forall\nu \geq N_s$, $p^\nu(s)x(s) - w(z, \rho^\nu; s) > 0$. Define $\theta^\nu(s)$ as:

$$\theta^\nu(s) = \left\{\begin{array}{cl} \frac{p^\nu(s)e^f(s)}{p^\nu(s)x(s) - w(z,\rho^\nu;s)} & \begin{array}{l}\text{if } p^\nu(s)e^f(s) < p^\nu(s)x(s) - w(z, \rho^\nu; s) \\ \text{and } p^\nu(s)x(s) - w(z, \rho^\nu; s) \neq 0\end{array} \\ 1 & \text{otherwise} \end{array}\right\}.$$

In the limit, $p^{\nu}(s)x(s)-w(z,\rho^{\nu};s) \to p^{\nu}(s)e^{f}(s)$ and $\forall \nu \geq N_s$, $p^{\nu}(s)x(s)-w(z,\rho^{\nu};s) > 0$. Thus $\theta^{\nu}(s) \to 1$. To verify feasibility,

$$p^{\nu}(s)x^{\nu}(s) - w(z^{\nu},\rho^{\nu};s) = \theta^{\nu}p^{\nu}(s)x(s) - w(\theta^{\nu}z,\rho^{\nu};s)$$

for $\theta^{\nu} \in [0,1]$. From claim 2.6,

$$\theta^{\nu}p^{\nu}(s)x(s) - w(\theta^{\nu}z,\rho^{\nu};s) \leq \theta^{\nu}\left(p^{\nu}(s)x(s) - w(z,\rho^{\nu};s)\right).$$

Since $\theta^{\nu} \leq \theta^{\nu}(s)$ and $p^{\nu}(s)x(s) - w(z,\rho^{\nu};s) > 0$, then

$$\theta^{\nu}\left(p^{\nu}(s)x(s) - w(z,\rho^{\nu};s)\right) \leq p^{\nu}(s)e^{f}(s).$$

In summary,

$$p^{\nu}(s)x^{\nu}(s) - w(z^{\nu},\rho^{\nu};s) \leq p^{\nu}(s)e^{f}(s).$$

Thus $(x^{\nu}(s), z^{\nu})$ satisfies the budget constraint for $s > 0$ given $(p^{\nu}(s), \rho^{\nu}(s))$.

Setting $N^* = \max\{M_0, N_0, ..., M_S, N_S\}$, then (i) $(x^{\nu}, z^{\nu}) \in \bar{B}^h(p^{\nu}, q^{\nu}, \rho^{\nu})$ $\forall \nu \geq N^*$ and (ii) $(x^{\nu}, z^{\nu}) \to (x,z)$. This completes the proof of the lemma.[18] ■

Define the demand correspondence $\Upsilon^h = \left\{(x,z) \in \arg\max_{(x,z)\in\bar{B}^h(p,q,\rho)} u^h(x)\right\}$. This correspondence is well-defined (using continuity of u^h and compactness of $\bar{B}^h$) and uhc (using the maximum principle and lemma 2.2).

Define the values of the correspondence Υ^h as the set $\Upsilon^h(p,q,\rho)$. This set is a

[18] This proof of lhc of the budget correspondence is the key step to ensuring that the demand correspondence is uhc. The demand correspondence is certainly not continuous as it will contain jumps that correspond to the jump from solvency to bankruptcy in all states $s > 0$, but these jumps are still uhc (household is indifferent between the allocation before and after the jump).

set-valued function that is Borel-measurable with respect to $\mathcal{H}$. Thus, I define

$$\Upsilon^f(p,q,\rho) = \int_{h\in\mathcal{H}_f} \Upsilon^h(p,q,\rho).$$

From Aumann (1966), the set $\Upsilon^f(p,q,\rho)$ is convex. The correspondence Υ^f (appropriately defined as the mapping from the price space $\Delta^*(\omega)$ to the set Υ^f) is convex-valued. As Υ^h is well-defined and uhc $\forall h \in \mathcal{H}$, then Υ^f is well-defined and uhc $\forall f \in \mathcal{F} = \{1, .., F\}$.

Define the bounded budget set $\bar{B}^f(p,q,\rho) = \int_{h\in\mathcal{H}_f} \bar{B}^h(p,q,\rho)$ for each household type $f \in \mathcal{F}$. From lemma 1, this set is compact. From Aumann (1966), this set $\bar{B}^f(p,q,\rho)$ is convex. It is trivially nonempty.

Define the price space as (using a normalization other than $p_L(s) = 1 \ \forall s \geq 0$):

$$\Delta^*(\omega) = \left\{ \begin{array}{l} (p,q,\rho) \in \mathbb{R}^G \times \mathbb{R}^J \times \mathbb{R}^{SJ} : \\ p(s) \in \Delta^{L-1} \quad \forall s \geq 0, \\ p_l(s) \geq \omega \quad \forall l, s \\ 0 \leq q_j \leq \frac{1}{\omega} \quad \forall j \\ 0 \leq \rho_j(s) \leq \frac{1}{\omega} \quad \forall j, \forall s > 0 \end{array} \right\}.$$

$\omega > 0$ is small. By assumption $(A.2)$, $p(s) >> 0$, so the restriction $p_l(s) \geq \omega \ \ \forall (l,s)$ is innocuous for ω small. With $\omega > 0$, the price space $\Delta^*(\omega)$ is nonempty, convex, and compact.

Define the price correspondence by $\Psi : \underset{f\in\mathcal{F}}{\times} \bar{B}^f \rightrightarrows \Delta^*(\omega)$ where

$$\Psi(x,z) = \left\{ \begin{array}{l} (p,q,\rho) \in \Delta^*(\omega) : \rho \text{ satisfies } (2.7) \\ (p,q) \in \\ \arg\max \end{array} \left[\begin{array}{l} p(0)\left(\int\limits_{h\in\mathcal{H}} x^h(0) - \frac{1}{F}\sum_f e^f(0)\right) + q \int\limits_{h\in\mathcal{H}} z^h + \\ \sum_{s>0}\left(p(s)\left(\int\limits_{h\in\mathcal{H}} x^h(s) - \frac{1}{F}\sum_f e^f(s)\right)\right) \end{array} \right] \right\}.$$

The equilibrium condition (2.7) in particular implies

$$\int\limits_{h\in\mathcal{H}} \sum_j \rho_j(s) r_j(s) \left(z_j^h\right)^+ + \int\limits_{h\in\mathcal{H}} \sum_j \delta^h(s) r_j(s) \left(z_j^h\right)^- = 0 \ \forall s > 0.$$

This correspondence Ψ is trivially well-defined and uhc. Since $\Delta^*(\omega)$ is convex and the objective function is linear in (x,z), the correspondence Ψ is convex-valued.

Define the overall correspondence $\Gamma : \Delta^*(\omega) \times \underset{f\in\mathcal{F}}{\times} \bar{B}^f \rightrightarrows \Delta^*(\omega) \times \underset{f\in\mathcal{F}}{\times} \bar{B}^f$ as the Cartesian product of the correspondences $\underset{f\in\mathcal{F}}{\times} \Upsilon^f$ and Ψ. This correspondence Γ is a well-defined, convex-valued, and upper hemicontinuous correspondence that is also a self-map from a nonempty, convex, and compact set into itself. From Kakutani, there exists a fixed point. From the definition of the demand correspondence, the fixed point satisfies the household optimization problem (H), which is the first part of the definition of a *bankruptcy equilibrium*.

Lemma 2.3 *The fixed point from Kakutani satisfies the market clearing conditions on both commodities and assets.*

Proof. From Walras' law and equilibrium condition (2.7) :

$$\begin{array}{l} p(0)\left(\frac{1}{F}\sum\limits_{f\in\mathcal{F}} e^f(0) - \int\limits_{h\in\mathcal{H}} x^h(0)\right) - q\int\limits_{h\in\mathcal{H}} z^h = 0 \\ p(s)\left(\frac{1}{F}\sum\limits_{f\in\mathcal{F}} e^f(s) - \int\limits_{h\in\mathcal{H}} x^h(s)\right) = 0 \quad \forall s > 0 \end{array}.$$

By assumption $(A.2)$, the commodity prices $p(s) >> 0 \;\; \forall s \geq 0$. By assumption $(A.4)$ and with $\rho > 0$ by $(A.5)$, then $q >> 0$. From the definition of the price correspondence Ψ and taking the limit of the price space $\Delta^*(\omega)$ as $\omega \rightarrow 0$, the above equations imply that markets clear:

$$\frac{1}{F}\sum_{f\in\mathcal{F}} e_l^f(s) - \int_{h\in\mathcal{H}} x_l^h(s) = 0 \quad \forall l, s.$$
$$\int_{h\in\mathcal{H}} z_j^h = 0 \quad \forall j.$$

The proof technique has been well documented and is hence omitted (see for instance the proof of theorem 1 in Dubey et al. (2005)). ∎

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Chapter 3

Bankruptcy in an Infinite Horizon Model

3.1 Introduction

The convergence of lack of commitment models from the macroeconomics literature (represented by Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007)) and default models from the general equilibrium literature (as pioneered by Dubey, Geanakoplos, and Shubik (2005)) offers the potential of models that are both competitive with incentives endogenously determined and tractable for quantitative analysis. This work is another step toward that potential. The model will capture the key features of chapter 7 bankruptcy by individuals holding debt in the unsecured credit markets. This model is not one of default (as in Dubey et al. (2005)) in which a household chooses how much of its debt to repay on an asset-by-asset basis. Rather this work considers bankruptcy, or the decision by a household not to repay debt over its entire portfolio of assets. The bankruptcy decision is inherently a binary decision: either a household is or is not bankrupt. To avoid the nonconvexity problems that would otherwise result, I must assume that there are a continuum of households.

Several papers in the general equilibrium literature (Araujo and Pascoa (2002), Sabarwal (2003), and Hoelle (2010)) have addressed the question of bankruptcy in a 2-period general financial model. Sabarwal even extended the analysis to a longer, finite time horizon and suggested that investment constraints be set based on a household's repayment history. This only captures the "backward looking" effects of bankruptcy: how a prior bankruptcy declaration affects a household's access to credit in the current period. Faced with the bankruptcy decision in the current period, a successful model must also incorporate the "forward looking" effects of bankruptcy: how a current bankruptcy declaration will affect access to credit in the future. With a finite time horizon, these "forward looking" effects would unravel from the final period. The model to study the dynamic effects of bankruptcy must have an infinite time horizon. Already working with a continuum of households, the natural model to develop will be an adaptation from the class of Bewley models (Bewley, 1986).[1]

The goal of this paper is to model the modern financial markets for unsecured credit. The most common example to keep in mind will be the credit card markets. In the model, the markets will be competitive and anonymous, competitive in that the asset prices will be the same for all loan sizes and anonymous in that the asset prices will not depend on a household's parameters. In order to protect creditors, whose payouts are reduced when debtors declare bankruptcy, the financial markets have asset prices that are conditioned on a debtor's current income and credit score. This is a primitive of the markets. Implicitly, I am assuming that the markets cannot have the asset prices depend on a more complete credit history, because it is prohibitively costly to verify any additional information.

[1] Bewley (1986) contains frequent references to his earlier works in this extended line of his research.

The FICO credit score is used by more than 75% of lending institutions and contains 5 components: payment history (35%), amount owed and amount of available credit (30%), length of credit history (15%), mix of credit (10%), and new credit (10%).[2] In the model, the credit score will only be payment history, specifically bankruptcy history. There are three reasons why only bankruptcy history makes up the credit score in the model. First, the final three components of the FICO credit score are equal for all households in the model. Second, in a model without investment constraints, the second component of the FICO credit score is not appropriate. Finally, declaring bankruptcy is the single worst thing a household can do to lower its credit score. Without transaction costs, a household would never carry debt across time periods and would only have incomplete repayment when declaring bankruptcy. Thus, the credit score in the model will be the number of periods since a prior bankruptcy declaration.

The credit score will be an imperfect, yet informative, signal about a household's private information. The private information in the model will be the random Markov process that governs a household's income realizations. The earliest work to analyze the problem of private information in general equilibrium was Prescott and Townsend (1984a, 1984b). The private information in these papers is a household's utility function. Subsequent research by Bisin and Gottardi (1999) considers the private information to be the income realizations. They provide a solution to the asymmetric information problem by showing how a price system can be set up, specifically a system that includes a bid-ask spread, so that competitive equilibria are guaranteed to exist.

I propose the following model for analyzing bankruptcy in which private infor-

[2] FICO stands for Fair Isaac and Company and the information was lifted from the website: www.myfico.com.

mation surfaces naturally. There are a finite number of possible aggregate states, the realizations of which are commonly observed. Households will trade financial contracts in order to transfer income across time and between aggregate states. A household's income is determined by a Markov process. The Markov process is private information of each household. By assumption, a household's current income provides no information about which Markov process governs its income realizations. The debtors will be partitioned into "pools". These pools are equivalence classes over income and credit score. Debtors can only sell assets that have been priced for their pool. This asset price will be proportional to the weighted repayment rates of all debtors within that pool. Equilibrium asset prices must be set such that the asset markets clear and no household has an arbitrage opportunity.

One purpose of the credit score is to inform the market of the probability that a certain debtor will repay its debt next period. For this reason, most papers studying credit scores (notably Chatterjee et al. (2008) and Elul and Gottardi (2008)) model them as follows. If a household declares bankruptcy in some period, then in that period, and in every period thereafter until the bankruptcy flag is removed, there is a probability $p > 0$ that the bankruptcy flag is removed. The parameter p is constant across time and is chosen so that on average a bankruptcy flag remains on a credit report for 10 years. This setup has the advantage that it is simple and asset prices can be conditioned (partly) on repayment likelihood. However, it misses the key point that a bankruptcy declaration by a debtor carries with it costs that last for 10 years. This is the second purpose of the credit score. The debtor has perfect foresight of the future asset prices for each of the pools to which it will belong, because it knows with certainty what its credit score will be in the future. The debtor can make its asset choices based on the actual dynamics of the bankruptcy process and not simply as the expectation over some random, artificial construct of the model.

Private information is necessary for dynamic bankruptcy decisions. The financial markets are such that asset prices depend on a debtor's income and credit score, but not on its Markov income process. This is because that information cannot be verified at any cost; it is private. To prove that private information is necessary, suppose that the asset prices depend not only on a debtor's income and credit score, but also on its Markov income process. I prove that the pool of borrowers will be independent across credit scores in equilibrium. Thus, there are not any dynamic costs of bankruptcy and all households eligible for bankruptcy will simply declare when their immediate payout from bankruptcy exceeds their immediate payout from solvency. Such an outcome is unsatisfactory given the care taken to use an infinite time horizon because the bankruptcy decision is entirely static and could be satisfactorily modeled using only 2 time periods (e.g., Araujo and Pascoa (2002), Sabarwal (2003), and Hoelle (2010)). Further the equilibrium outcome completely contradicts the empirical facts about bankruptcy. These can be summarized succinctly by Musto (1999) (among others): a bankruptcy flag on a household's credit report significantly impairs its ability to borrow on the financial markets.

In a simple economy with private information, I prove analytical results stating that a borrower faces strictly higher asset prices (lower interest rates) as its credit score increases. The household that is the bad credit risk is the one with the following private information: given a low income realization, the probability is high that low income will be realized in the following period. Call this probability the 'persistence of low income'. The assumption needed to prove that asset prices strictly increase with credit score is that the persistence of low income for the bad credit risk households is larger than that for the good credit risk households by at least 'the appropriate amount'. Further sufficient conditions and a discussion of 'the appropriate amount' are presented.

This paper makes three contributions. First, the general model is introduced and the existence of a bankruptcy equilibrium is proven (section 3.2). Second, I show that the bankruptcy decision simplifies to a static decision when either all households have identical private information or when the market allows asset prices to be conditioned on this information (section 3.3). Finally, I use theoretical results for a simple economy to show that the asset prices will be strictly increasing with the credit score (section 3.4). Section 3.5 concludes and discusses the next steps in this line of research. Section 3.6 collects all the proofs that are too long to be included in the main sections.

3.2 The Model

Let the length of the model be described by an infinite-dimension, discrete time process $t \in \{0, 1, .., T\}$ where $T \to \infty$. Suppose there is a continuum (with unit measure) of households $h \in \mathcal{H}\tilde{}[0, 1]$.

There is one physical commodity in each time period, which can be thought of as money or a composite bundle of multiple commodities. Households are infinite-lived and have ex-ante identical utility. Let the utility function $U : \ell_\infty^+ \to \mathbb{R}$ be given as:

$$U(c^h) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c^h(t)\right).$$

I assume that $\beta \in (0, 1)$ and the Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is C^2, differentiably strictly increasing, differentiably strictly concave, and satisfies the Inada condition (that is, $u'\left(c^h(t)\right) \to \infty$ as $c^h(t) \to 0$).

3.2.1 Primitives

Aggregate States

Let there be a finite number of possible states $s \in \mathcal{S} = \{1, .., S\}$ with $S < \infty$. Let the aggregate state that occurs in time period t be denoted by $s(t)$. The notation s^τ will be the entire sequence of states from $t = 0$ to $t = \tau$, $s^\tau = (s(0), ..., s(\tau))$. At time period τ, the current state $s(\tau)$ and all prior states $s^{\tau-1}$ are known by all households. The transition between states is common knowledge and given by the Markov transition matrix $\Omega \in \mathbb{R}^{S,S}$. Elements of Ω are denoted by $\omega(s'|s)$, that is, the probability that state s' occurs tomorrow given that the state today is s.

Income

Households are endowed each period with some strictly positive amount of the commodity, which for simplicity will be called income. The realized income for household h in time period t is denoted by $e^h(t)$. The set of possible income realizations is a finite set $\mathcal{E} = \{e_1, ..., e_I\}$ with $I < \infty$ and $e_i > 0 \;\; \forall i$. Without loss of generality, $e_1 < e_2 < ... < e_I$. The income realizations will be randomly determined by a Markov process that will depend both on the aggregate state and the household type.

Markov income process

The Markov income process will be private information. The set of possible Markov processes is a finite set $k = \{1, ..., K\}$ with $K < \infty$. A household with Markov process k is said to be of "type k" and this is unchanged over time. Let $\Pi_s^k \in \mathbb{R}^{I,I}$ be the transition matrix for a household of type k from period t to period $t + 1$ given that the aggregate state in time period t is $s(t) = s$. Denote the elements of the matrix Π_s^k as $\pi_s^k(i, j)$. The term $\pi_s^k(i, j)$ is the probability that a household of type

k with income $e^h(t) = e_i$ and state $s(t) = s$ realizes the income $e^h(t+1) = e_j$. For simplicity, when $e = e_i$ and $e' = e_j$, define $\pi_s^k(e'|e) = \pi_s^k(i,j)$.

I would like to be able to use the law of large numbers to state that $\pi_s^k(i,j)$ is also the fraction of all households of type k with income $e^h(t) = e_i$ that transition to income $e^h(t+1) = e_j$ given that the aggregate state is $s(t) = s$. As described by Judd (1985), when a continuum of agents draw realizations from an iid random process, the set of sample realizations that satisfy the law of large numbers is not measurable. There are several possible solutions to this problem. First, I could incorporate a small amount of dependence into the income process as suggested by Feldman and Gilles (1985). Second, I could specify the set of households not as a continuum, but rather as a process for which the law of large numbers is applicable and which has the same cardinality as the continuum. Such a process is the hyperfinite process as analyzed by Sun (1998).[3] I will choose the latter solution.

For each aggregate state s and each household type k, define the distribution over the set of possible incomes $\mathcal{E}$ as $\mu_s^k \in \Delta^{I-1}$. Each term of the vector μ_s^k is denoted $\mu_s^k(i)$ and this is the fraction of all households of type k that have income $e^h(t+1) = e_i$ given $s(t) = s$. Mathematically, μ_s^k is the unique eigenvector of Π_s^k satisfying the constraint $\sum_{i=1}^I \mu_s^k(i) = 1$. I assume that the transition matrices are such that the income distributions will be identical across household types and aggregate states: $\mu_s^k = \mu_{s'}^{k'}$ $\forall k, k'$ and $\forall s, s'$. That is, there is no aggregate risk.[4]

[3]There are two reasons to model the set of households as a hyperfinite process. First, the asymptotic properties of finite processes are embedded in the limit setting. In particular, the exact law of large numbers applies. Second, the external cardinality of the index sets of a hyperfinite process is the same as the cardinality of the continuum.

[4]At a minimum, I must require that $\mu_s^k = \mu_s^{k'}$ $\forall k, k'$ and $\forall s$ for if the distribution differs across types, the income of a household carries information about its type. The equilibrium asset prices, as determined in this framework, depend upon the repayment rates of all debtors, regardless of their type. Thus, I could certainly allow the income to be informative of a household's type and the bankruptcy equilibrium would still be well-defined. I choose not to and instead keep the household's type as entirely private information.

I claim that $\mu_s^k = \mu_s^{k'}$ $\forall k, k'$ and $\forall s$ implies $\mu_s^k = \mu_{s'}^{k'}$ $\forall k, k'$ and $\forall s, s'$. Suppose otherwise, that is, Define $\overline{e} = \sum_{i=1}^{I} \mu_s^k(i) \cdot e_i$ for any (s, k) as the aggregate income (identical across states). Likewise, define the median income (identical across states):

$e_{med} = e^*$ where $\sum_{i=1}^{I} \mu_s^k(i) \cdot 1\{e_i \leq e^*\} = \sum_{i=1}^{I} \mu_s^k(i) \cdot 1\{e_i \geq e^*\} = 0.5$ for any (s, k).

"Credit score"

I will use the quotation marks around "credit score" to indicate that the "credit score" in the model is equivalent to a household's bankruptcy history. It does not contain all the elements that we associate with the actual FICO credit score.

I will model bankruptcy to satisfy the following three legal facts:

1. A household cannot declare bankruptcy twice in any 6-year period.
2. A bankruptcy declaration remains on a household's credit report for 10 years.
3. According to recent legislation[5], households that fail the "means test" cannot declare bankruptcy. A household fails this test if its income is above the median.

Let the length of a time period be equal to 5 years. Define the "credit score" $b \in \mathcal{B} = \{0, 1, 2\}$ as the number of periods since the prior bankruptcy declaration (if one exists). If a household h declares bankruptcy in time period t, then $b^h(t) = 0$. From Fact 1, the household cannot then declare bankruptcy in time period $t + 1$, so $b^h(t+1) = 1$. If a household decides to declare bankruptcy in time period $t+2$, then $b^h(t + 2) = 0$. Otherwise, $b^h(t + 2) = 2$.

suppose $\mu_s^k = \mu_s^{k'} \neq \mu_{s'}^k = \mu_{s'}^{k'}$ for some s, s', k, k'. It is not true that given the stationary distribution $\mu_s^k = \mu_s^{k'}$ for state $s(t) = s$ that the distribution for state $s(t+1) = s' \neq s$ will be equal across types k, k'. This is simply because the transition matrix will not (generically) become stationary in one period. In summary, for the income to not carry information about a household's type, I must assume that $\mu_s^k = \mu_{s'}^{k'}$ $\forall k, k'$ and $\forall s, s'$.

[5] U.S.C. §707(b)(1) and the Bankruptcy Abuse Prevention and Consumer Protection Act of 2005.

Finally, for any time period in which $b^h(\tau) = 2$, if the household decides to remain solvent next period, then $b^h(\tau + 1) = 2$. In accordance with Fact 2, a household whose last bankruptcy occurred two periods ago has the same "credit score" as a household that has never declared bankruptcy.

Fact 3 implies that if at some time period t the income realization $e^h(t) > e_{med}$, then the household cannot declare bankruptcy. Thus, $b^h(t) = 0$ only if $e^h(t) \leq e_{med}$ and $b^h(t-1) > 0$. I will say that a household is "eligible" for bankruptcy in time t if $e^h(t) \leq e_{med}$ and $b^h(t-1) > 0$.

3.2.2 Financial markets

The aggregate states are observable by all households. As such, financial contracts can be written on the realizations of these states. The financial contracts will specify the terms of trade for one-period numeraire assets in zero net supply. The number of assets and the payouts of each will be exogenously fixed.

The asset price is the market value of the claims to the asset payouts in the following time period. A creditor has a positive claim and a debtor has a negative claim. As the benefits of the trade are separated across time (the debtor gains wealth in the current time period and owes funds in the following period) and no mechanism exists to perfectly and costlessly enforce commitment, the debtor may choose to renege on its financial commitment. The financial contract must then clearly specify what the consequences are for such an action. As with the asset payouts, the consequences of bankruptcy will be exogenously specified.

Maintaining the assumptions that the markets are both competitive and anonymous, the asset prices will be linear. Without investment constraints or collateral requirements, the markets do not have the option of offering a menu of financial contracts that attempt to endogenously separate debtors by their private information. The only free variable is the asset price. Debtors will only trade the financial contracts that offer the highest asset price.

Define $J \leq S$ to be the number of different asset payouts available for trade in all time periods. The payouts for assets traded in time t will depend on both the aggregate states $s(t)$ and $s(t+1)$. These payouts are in terms of the single physical commodity and are given by the $S \times J$ payout matrix $R(s)$:

$$R(s) = \begin{bmatrix} r_1(1|s) & .. & r_J(1|s) \\ : & .. & : \\ r_1(S|s) & .. & r_J(S|s) \end{bmatrix}_{S\times J} .$$

As is standard in the literature, the matrix $R(s)$ is nonnegative, full rank, and in general position $\forall s \in \mathcal{S}$. By general position, I mean that any J rows of $R(s)$ will have full rank.

The total number of assets available for trade in any time period is given by $3 \cdot I \cdot J$. For each asset payout $j \in \{1, .., J\}$, for each "credit score" $b \in \{0, 1, 2\}$, and for each income $e \in \{e_1, ..., e_I\}$, an asset is defined. The asset (j, e, b) traded in time period t by household h will be denoted $z^h_{j,e,b}(t)$ with associated asset price $q_{j,e,b}(t)$.

A household will be a debtor on asset (j, e, b) if $z^h_{j,e,b}(t) < 0$ and a creditor otherwise. The following is an exogenous constraint of the financial markets. A household h can only be a debtor on asset (j, e, b) in time period t if $e^h(t) = e$ and $b^h(t) = b$. A household can be a creditor for any of the assets.

I will define a debtor "pool" as an equivalence class over "credit score" and current income. That is, a pool contains all debtors with the same "credit score" and current income. If the "credit score" is b and the current income is e, then I will speak of "pool (e, b)". The asset price $q_{j,e,b}(t)$ will be proportional to the weighted repayment rate of all debtors within the pool (e, b).

3.2.3 Bankruptcy assumptions

The first of three assumptions needed to prevent households from having unbounded asset sales and an arbitrage opportunity is:

Assumption A: $\forall k$, $\forall s$, and $\forall e$, $\exists e'$ such that $e' > e_{med}$ and $\pi_s^k(e'|e) > 0$.

The payout matrix $R(s)$ satisfies the following assumption:

Assumption B: $R(s)$ has strictly positive terms $\forall s \in \mathcal{S}$.

Before getting to assumption C, I need to specify what the asset payouts will be for creditors when bankruptcy is declared by other households.

Bankruptcy declarations by some debtors implies that the payouts for creditors must be diluted. The asset payouts will be considered on an asset-by-asset basis. The funds paid on asset (j, e, b) are collected from all debtors on this asset. These collected funds are then used as the payouts to the asset (j, e, b) creditors. Creditors are proportionately reimbursed. If a creditor was originally supposed to receive the payout $r_j\left(s'|s\right) \cdot z_{j,e,b}^h(t) \geq 0$ for holding asset (j, e, b), then in the presence of bankruptcy, the creditor will only receive $\rho_{j,e,b}(t+1) \cdot r_j\left(s'|s\right) \cdot z_{j,e,b}^h(t) \geq 0$ where $\rho_{j,e,b}(t+1) \in [0, 1]$ is the overall repayment rate for asset (j, e, b). The overall repayment rate $\rho_{j,e,b}(t+1)$ is endogenously determined and will be defined shortly.

A household who chooses to remain solvent and not declare bankruptcy in time period $t + 1$ will have financial payout:

$$r^h\left(s'|s, t+1\right) \cdot z^h(t) = \sum_{j,e,b} r_{j,e,b}^h\left(s'|s, t+1\right) \cdot z_{j,e,b}^h(t)$$

where the household-specific asset payouts $r^h\left(s'|s,t+1\right)$ are defined as:

$$r^h_{j,e,b}\left(s'|s,t+1\right) = \left\{ \begin{array}{ll} r_j\left(s'|s\right) & \text{if } z^h_{j,e,b}(t) < 0 \\ \rho_{j,e,b}(t+1)\cdot r_j\left(s'|s\right) & \text{if } z^h_{j,e,b}(t) \geq 0 \end{array} \right\}. \tag{3.1}$$

The obvious benefit of a household declaring bankruptcy is that it can escape its debt obligations. This benefit may be the most important consideration for the household, but what drives the existence result is what happens to those assets (if any) that the household has purchased.

The exemption level for each set of asset purchases $(j,e,b)_{\forall e,b}$ will be given by $\chi_j \in \mathbb{R}_+$ $(\chi_j < \infty)$. If a household declares bankruptcy, it can keep up to χ_j of the value of asset purchases $(j,e,b)_{\forall e,b}$, but must forfeit any value greater than χ_j. In order to obtain the exemptions, a bankrupt household must submit a detailed record of each asset purchase to the bankruptcy court. Assuming that a bankrupt household has asset purchases with value:

$$r^h\left(s'|s,t+1\right)\cdot\left(z^h(t)\right)^+ = \sum_{j,e,b} r^h_{j,e,b}\left(s'|s,t+1\right)\cdot\left(z^h_{j,e,b}(t)\right)^+,$$

then the nominal cost of submitting the detailed record to the bankruptcy court will be given by:

$$\alpha\cdot r^h\left(s'|s,t+1\right)\cdot\left(z^h(t)\right)^+ = \sum_{j,e,b}\alpha_j\cdot r^h_{j,e,b}\left(s'|s,t+1\right)\cdot\left(z^h_{j,e,b}(t)\right)^+.$$

Throughout, the notation will be that $(y)^+ = \max\{y,0\}$ and $(y)^- = \min\{y,0\}$.

The next assumption is the third and final assumption needed to prevent the households from having unbounded asset sales:

Assumption C: $\alpha_j > 0 \;\; \forall j.$

The total financial payout for a household that declares bankruptcy in time period $t+1$ is the value of asset purchases below the exemption level minus the cost of bankruptcy:

$$\sum_j \left[\min \left\{ \sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right) \cdot \left(z^h_{j,e,b}(t)\right)^+, \chi_j \right\} - \alpha_j \sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right) \cdot \left(z^h_{j,e,b}(t)\right)^+ \right].$$

At time period $t+1$, a household eligible for bankruptcy faces a binary decision: it can either choose to remain solvent, with payout denoted by $S^h(t+1)$, or it can choose to declare bankruptcy, with payout denoted by $B^h(t+1)$, where:

$$\begin{aligned} S^h(t+1) &= \sum_{j,e,b} r^h_{j,e,b}\left(s'|s,t+1\right) \cdot z^h_{j,e,b}(t) \qquad (3.2) \\ B^h(t+1) &= \sum_j \min \left\{ \sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right) \cdot \left(z^h_{j,e,b}(t)\right)^+, \chi_j \right\} \\ &\quad - \sum_j \alpha_j \sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right) \cdot \left(z^h_{j,e,b}(t)\right)^+ \end{aligned}$$

3.2.4 Bankruptcy equilibrium

The household problem

The household problem for each household $h \in \mathcal{H}$ will be the sequence of consumption $c^h = \left(c^h(t)\right)_{\forall t}$, assets $z^h = \left(z^h(t)\right)_{\forall t} = \left(\left(z^h_{j,e,b}(t)\right)_{\forall j,e,b}\right)_{\forall t}$, and "credit score" $b^h = \left(b^h(t)\right)_{\forall t}$ that maximizes $U(c^h)$ and are feasible given the sequence of asset prices $q = (q(t))_{\forall t} = \left((q_{j,e,b}(t))_{\forall j,e,b}\right)_{\forall t}$ and overall repayment rates $\rho = (\rho(t+1))_{\forall t} = \left(\left(\rho_{j,e,b}(t+1)\right)_{\forall j,e,b}\right)_{\forall t}$. The feasibility requirement means that the vector of sequences $\left(c^h, z^h, b^h\right)$ lies in the budget set $B^h(q,\rho)$.

Given (q,ρ),

$$\left(c^h, z^h, b^h\right) \in \underset{subj\ to\ \left(c^h,z^h,b^h\right) \in B^h(q,\rho)}{\arg\max} U(c^h)$$

$$B^h(q,\rho) = \left\{ \begin{array}{l} \left(c^h, z^h, b^h\right) = \left(c^h(t), z^h(t), b^h(t)\right)_{\forall t} : \\ c^h(t) \geq 0, \quad z^h(t) \in \mathbb{R}^{3IJ}, \quad b^h(t) \in \{0,1,2\}, \\ c^h(0) + \sum_{j,e,b} q_{j,e,b}(0) z^h_{j,e,b}(0) \leq e^h(0), \\ c^h(t) + \sum_{j,e,b} q_{j,e,b}(t) z^h_{j,e,b}(t) \leq e^h(t) + \left\{ \begin{array}{cc} S^h(t) & \text{if } b^h(t) > 0 \\ \max\left\{B^h(t), S^h(t)\right\} & \text{if } b^h(t) = 0 \end{array} \right\}, \\ \left(z^h_{j,e,b}(t)\right)^- \cdot \left(1 - 1\left\{e^h(t) = e\right\} \cdot 1\left\{b^h(t) = b\right\}\right) = 0 \end{array} \right\}. \tag{3.3}$$

A few notes are in order. First, the payout of the bankrupt household is defined such that the household can never be forced to repay more than it initially owed.[6] Second, the final equation concerning $\left(z^h_{j,e,b}(t)\right)^-$ expresses the exogenous constraint of the financial markets, which states that a household h can only be a debtor on asset (j,e,b) in time period t if both $e^h(t) = e$ and $b^h(t) = b$.

Repayment rates

This section will detail how the individual repayment rates for each household are endogenously determined and how the individual rates ultimately determine the overall repayment rates. Let the aggregate states be $s(t) = s$ in time period t and $s(t+1) = s'$ in time period $t+1$.

The individual repayment rate $\delta^h(t+1) \in [0,1]$ for household h in time $t+1$ will be the same for each asset. That is, if h owes $r_j\left(s'|s\right) \cdot z^h_{j,e,b}(t) < 0$ on asset (j,e,b) and only repays $\delta^h(t+1) \cdot r_j\left(s'|s\right) \cdot z^h_{j,e,b}(t)$ of it, then when h owes $r_k\left(s'|s\right) \cdot z^h_{k,e,b}(t) < 0$ on asset k, it will only repay $\delta^h(t+1) \cdot r_k\left(s'|s\right) \cdot z^h_{k,e,b}(t)$ of it. This will follow by definition.

[6] A household would choose to declare bankruptcy even though the payout from bankruptcy is less than the payout from solvency only if the equilibrium prices dictate that the asset prices are higher for pools (e,b) with $b=0$ or $b=1$ compared to pool $(e,\hat{b})$ with $\hat{b}=2$. I do not rule out this counter-intuitive scenario in the specification of the model.

If a household remains solvent in time period $t+1$ or if the household declares bankruptcy with $\max\left\{B^h(t+1), S^h(t+1)\right\} = S^h(t+1)$, the individual repayment rate is unity, $\delta^h(t+1) = 1$.

When a household declares bankruptcy in time $t+1$ with $\max\left\{B^h(t+1), S^h(t+1)\right\} = B^h(t+1)$, the total funds that are confiscated by the bankruptcy court are the bankruptcy cost and the value of any nonexempt asset purchases. These collected funds have value:

$$num = \sum_j \left[\alpha_j \sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right)\left(z^h_{j,e,b}(t)\right)^+ + \left(\sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right)\cdot\left(z^h_{j,e,b}(t)\right)^+ - \chi_j\right)^+\right]. \tag{3.4}$$

The bankruptcy court decides how this value will be divided among the different creditors. Intuitively, if a household was a debtor only on the single asset (j,e,b), then the entire value should be used to pay back the asset (j,e,b) creditors. Define the total debt of a bankrupt household as the payouts of its asset sales:

$$den = \sum_{j,e,b} r_j\left(s'|s\right)\cdot\left(z^h_{j,e,b}(t)\right)^-. \tag{3.5}$$

The fraction $\frac{r_j(s'|s)\cdot\left(z^h_{j,e,b}(t)\right)^-}{\sum_{j,e,b} r_j(s'|s)\cdot\left(z^h_{j,e,b}(t)\right)^-} \in [0,1]$ of the total value num will be returned to the asset (j,e,b) creditors for each asset. This household h originally owed $-r_j\left(s'|s\right)\cdot\left(z^h_{j,e,b}(t)\right)^-$, but is only paying $r_j\left(s'|s\right)\cdot\left(z^h_{j,e,b}(t)\right)^-\cdot\frac{num}{den}$. Thus, the individual repayment rate for a bankrupt household is defined as:

$$\delta^h(t+1) = -\frac{num}{den} \in [0,1].^7 \tag{3.6}$$

[7] Since $B^h(t+1) \geq S^h(t+1)$, then $\sum_j \min\left\{\sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right)\cdot\left(z^h_{j,e,b}(t)\right)^+, \chi_j\right\} - \alpha\cdot r^h\left(s'|s,t+1\right)\cdot\left(z^h(t)\right)^+ \geq r^h\left(s'|s,t+1\right)\cdot z^h(t)$. This implies $-r\left(s'|s\right)\cdot\left(z^h(t)\right)^- \geq \alpha\cdot r^h\left(s'|s,t+1\right)\cdot$

In equilibrium, the expectations held by creditors about the overall repayment rates $\rho_{j,e,b}(t+1)$ must be equal to the weighted individual repayment rates across all debtors:

$$\rho_{j,e,b}(t+1)\int_{h\in\mathcal{H}}\left(z^h_{j,e,b}(t)\right)^+ = -\int_{h\in\mathcal{H}}\delta^h(t+1)\left(z^h_{j,e,b}(t)\right)^- . \tag{3.7}$$

The overall repayment rate $\rho_{j,e,b}(t+1)\geq 0$. If $\int_{h\in\mathcal{H}}\left(z^h_{j,e,b}(t)\right)^- = 0$ (no trade in this asset), the value for $\rho_{j,e,b}(t+1)$ is not pinned down by (3.7). This keeps open the possibility that undue pessimism by the creditors about their payouts can be self-fulfilling in equilibrium (as in Dubey et al. (2005)).[8] However, if $\int_{h\in\mathcal{H}}\left(z^h_{j,e,b}(t)\right)^- < 0$, the overall repayment rate $\rho_{j,e,b}(t+1)\in(0,1]$. This is best seen by using the market clearing condition $\int_{h\in\mathcal{H}} z^h_{j,e,b}(t) = 0$ (to be introduced shortly) to rewrite equation (3.7) as:

$$\rho_{j,e,b}(t+1) = 1 + \frac{\int_{h\in\mathcal{H}}\left(1-\delta^h(t+1)\right)\left(z^h_{j,e,b}(t)\right)^-}{\int_{h\in\mathcal{H}}\left(z^h_{j,e,b}(t)\right)^+}. \tag{3.8}$$

Definition of a bankruptcy equilibrium

I will now define a bankruptcy equilibrium. As the bankruptcy equilibrium is defined in terms of infinite sequences of choices $(c,z,b) = \left(c^h(t), \left(z^h_{j,e,b}(t)\right)_{\forall j,e,b}, b^h(t)\right)_{\forall t, h\in\mathcal{H}}$ and prices $(q,\rho) = \left(\left(q_{j,e,b}(t)\right)_{\forall j,e,b}, \left(\rho_{j,e,b}(t+1)\right)_{\forall j,e,b}\right)_{\forall t}$, the definition is often referred to as the "sequential" definition in the literature (e.g., Miao, 2006).

Definition 3.1 *A bankruptcy equilibrium is a sequence of household choices* (c,z,b)

$\left(z^h(t)\right)^+ + \sum_j\left(\sum_{e,b} r^h_{j,e,b}\left(s'|s,t+1\right)\cdot\left(z^h_{j,e,b}(t)\right)^+ - \chi_j\right)^+$. By definition, $-den \geq num$ or $-\frac{num}{den}\leq 1$.

[8] This undue pessimism is in fact irrational, though still possible as an equilibrium. With the iid Markov processes and assumption 1, there will always exist some realization of the income process in the next time period such that a subset of households cannot declare bankruptcy. This subset will have unity repayment rates and keep the overall repayment rate ρ bounded above 0.

and prices (q, ρ) *such that*

(i) the household problem (3.3) *is satisfied,*

(ii) markets clear, $\int_{h\in\mathcal{H}} c^h(t) = \bar{e} \ \ \forall t$ *and* $\int_{h\in\mathcal{H}} z^h(t) = \overrightarrow{0} \ \ \forall t,$

(iii) repayment rates ρ *are determined according to equation* (3.7) :

$$\rho_{j,e,b}(t+1)\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^+ = -\int_{h\in\mathcal{H}} \delta^h(t+1)\left(z^h_{j,e,b}(t)\right)^-, \ \textit{and}$$

(iv) $q_{j,e,b}(t) > 0 \ \forall j, e, b, t,$ *and*

(v) If $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^- = 0$ *for* $(e,b) \neq (e_1, 0)$, *then* $\rho_{j,e,b}(t+1) = \frac{q_{j,e,b}(t)}{q_{j,e_1,0}(t)}.$

A few notes are in order. First, if creditors believe that the overall repayment rate $\rho_{j,e,b}(t+1) = 0$, then this belief is self-fulfilling as $q_{j,e,b}(t) = 0$ and no trade occurs. I am not interested in the existence of such trivial equilibria. Thus, the definition of a bankruptcy equilibrium requires that all asset prices are strictly positive. Second, the possibility exists that even given strictly positive asset prices, households may choose not to trade certain assets. For these closed asset markets, the overall repayment rate is not pinned down by (3.7). The creditors need to form beliefs, off the equilibrium path, that inform them about the repayment rate *were they to* decide to purchase those assets. Further explanation of condition (v) is provided in the subsection 'asset pricing'.

Existence of a bankruptcy equilibrium

Let the time horizon be discrete with $t \in \{0, ..., \bar{T}\}$ and $\bar{T} < \infty$. I will show in theorem 3.1 that a bankruptcy equilibrium exists for this finite time horizon. The proof is contained in section 3.6.

Theorem 3.1 *Under assumptions* ***A-C,*** *the truncated bankruptcy equilibrium exists for all parameters* $\mathcal{E}$, $\left(\Pi^1_s, ..\Pi^K_s\right)_{\forall s}$, β, $u(\cdot)$, χ, α, *and* $\left(r_j(s'|s)\right)_{\forall j,s,s'}$.

Theorem 3.2 states that an equilibrium with the discrete, infinite-length time horizon exists and that this infinite-length equilibrium is actually the limit of the appropriately defined truncated equilibrium as $\bar{T} \to \infty$. The proof of theorem 3.2 is contained in section 3.6.

Theorem 3.2 *If a truncated bankruptcy equilibrium exists for any finite length* $\bar{T}$*, then the limit as* $\bar{T} \to \infty$ *is well defined (all equilibrium variables are uniformly bounded) and the equilibrium at the limit is the desired bankruptcy equilibrium for the infinite-time horizon.*

The existence that will be proven will be the existence of the stated "sequential" bankruptcy equilibrium. It remains an open question as to whether a recursive bankruptcy equilibrium exists. Miao (2006) proves the existence of both a sequential equilibrium and a recursive equilibrium and shows that they are payoff equivalent in a model without bankruptcy. The key assumption in his work is an exogenous bound on asset holdings.

Two nonexistence results in full commitment models without an exogenous bound on asset holdings suggest that a recursive equilibrium cannot be proven to exist in my bankruptcy model. First, Krebs (2004) proved that a recursive equilibrium with a compact state space cannot exist unless there are binding investment constraints on assets. Second, Kubler and Schmedders (2002) proved that a recursive equilibrium may not exist even when the state space contains the entire vector of contemporaneous variables.

I refuse to include investment constraints in my model not only for the stated reason of Krebs (2004) that investment constraints introduce an additional friction, but mostly because this additional friction will completely dictate the resulting equilibrium. In a model with bankruptcy, if asset sales are not endogenously curtailed in some manner, then all debtors will sell the most assets permitted by the investment constraint and no fewer.

Asset pricing

Consider the set of assets $(j, e, b)_{\forall e,b}$. Let the asset price $q_{j,e_1,0}(t)$ be the base price that all other prices $(q_{j,e,b}(t))_{\forall(e,b)\neq(e_1,0)}$ will be determined relative to. The base price $q_{j,e_1,0}(t)$ will be determined from one market clearing condition.

A household that is a creditor for asset (j, e, b) can also be a creditor for asset (j, e', b') for all $(e', b') \neq (e, b)$. The payouts from either remaining solvent or declaring bankruptcy can be rewritten (using equations (3.4) – (3.6)) as:

$$S^h(t+1) = B^h(t+1) = \sum_{j,e,b} \rho_{j,e,b}(t+1) r_j\left(s'|s\right)\left(z^h_{j,e,b}(t)\right)^+ + \delta^h(t+1)\sum_{j,e,b} r_j\left(s'|s\right)\left(z^h_{j,e,b}(t)\right)^- \tag{3.9}$$

where the only difference between $S^h(t+1)$ and $B^h(t+1)$ is that $\delta^h(t+1) = 1$ for solvent households and $\delta^h(t+1) \leq 1$ is endogenously determined by (3.6) for bankrupt households. Thus any household that is a creditor in both assets (j, e, b) and (j, e', b') will have payouts $\rho_{j,e,b}(t+1)r_j\left(s'|s\right)$ and $\rho_{j,e',b'}(t+1)r_j\left(s'|s\right)$, respectively.

The definition of a bankruptcy equilibrium specifies that the following "equal returns conditions" are equilibrium conditions:

$$\frac{\rho_{j,e,b}(t+1)}{q_{j,e,b}(t)} = \frac{\rho_{j,e_1,0}(t+1)}{q_{j,e_1,0}(t)} \quad \forall t. \tag{3.10}$$

Recall that $\rho_{j,e_1,0}(t+1) = 1$ by definition if the asset $(j, e_1, 0)$ is traded in equilibrium.

The discussion of condition (3.10) will be in three parts. For the first part, I justify that $\int_{h\in\mathcal{H}} \left(z^h_{j,e_1,0}(t)\right)^- < 0 \;\; \forall j$ in equilibrium. Recall that the pool $(e_1, 0)$ has the lowest current income and are required to repay their debt in $t + 1$. In the full commitment model, I can show that over a generic subset of endowments, there will exist an equilibrium price $q_j > 0$ such that some households will sell the asset z_j and others will buy the asset. Thus, in the bankruptcy model in which the pool $(e_1, 0)$ is a full commitment pool, at any strictly positive prices $q_{j,e_1,0}(t) > 0$, the households do not optimally choose assets $z^h_{j,e_1,0}(t) = 0$ as this choice is dominated by their choice specified in the full commitment model (the choice being generically different from 0). Thus, asset trades occur and some price $q_{j,e_1,0}(t) > 0$ can be found to clear markets.

The second part of the discussion considers the closed asset market $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^- = 0$ for some $(e, b) \neq (e_1, 0)$. From the previous paragraph, $\int_{h\in\mathcal{H}} \left(z^h_{j,e_1,0}(t)\right)^- < 0$ and so $\rho_{j,e_1,0}(t+1) = 1$ by definition. The value for $\rho_{j,e,b}(t+1)$ is not pinned down by (3.7), but the creditors must have the beliefs specifying what the repayment rate would be if they were to trade in asset (j, e, b). These beliefs off the equilibrium path are given by condition (3.10), rearranged:

$$\rho_{j,e,b}(t+1) = \frac{q_{j,e,b}(t)}{q_{j,e_1,0}(t)}. \tag{3.11}$$

If $q_{j,e,b}(t) \in (0, q_{j,e_1,0}(t)]$, then $\rho_{j,e,b}(t+1)$ satisfies the sensible belief requirement $\rho_{j,e,b}(t+1) \in (0, 1]$ when it is set using (3.11). The strict inequality $q_{j,e,b}(t) > q_{j,e_1,0}(t)$ cannot hold or else an arbitrage opportunity exists. This opportunity is for a household h with $e^h(t) = e$ and $b^h(t) = b$ to remain solvent while holding an equal number of sales of (j, e, b) and purchases of $(j, e_1, 0)$. The payouts of these two assets are equal, so letting the number of asset trades become unbounded leads to an arbitrage profit.

The third part of the discussion considers the open asset market $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^- < 0$ for some $(e, b) \neq (e_1, 0)$. Suppose, without loss of generality, that $\frac{\rho_{j,e,b}(t+1)}{q_{j,e,b}(t)} < \frac{1}{q_{j,e_1,0}(t)}$. Then, with the financial payouts expressed as in (3.9), it is clear that no creditor would be willing to purchase asset (j,e,b). The return, defined as payout divided by price, is relatively lower. Thus, it cannot be that $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^- < 0$ as this violates market clearing $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^- + \int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t)\right)^+ = 0$. So, no arbitrage requires the equality $\frac{\rho_{j,e,b}(t+1)}{q_{j,e,b}(t)} = \frac{1}{q_{j,e_1,0}(t)}$ whenever the asset market (j,e,b) is open.

The equation (3.10) pins down the asset prices for $J\,(3I-1)$ assets relative to the base prices $\left(q_{j,e_1,0}(t)\right)_{\forall j}$.

I will utilize the following equilibrium condition throughout sections 3.3 and 3.4. I state the condition as a claim.

Claim 3.1 *A household h with $e^h(t)=e$ and $b^h(t)=b$ will only be a debtor on asset (j,e,b) in time t if either:*

(i) $\delta^h(t+1) < \rho_{j,e,b}(t+1)$ for some realization $(e^h(t+1), s(t+1))$ or

(ii) $\sum_{e,b} \left(z^h_{j,e,b}(t)\right)^+ = 0$.

Proof. Consider a household with $e^h(t)=e$ and $b^h(t)=b$. Suppose that the sum $\sum_{e,b} \left(z^h_{j,e,b}(t)\right)^+ > 0$. The household is receiving the price $q_{j,e,b}(t)$ for its sales of asset (j,e,b) and paying the price of $q_{j,e',b'}(t)$ for its purchases of some asset (j,e',b') with $(e',b') \neq (e,b)$. Given solvency, the expected return on the asset sale (j,e,b) is $\frac{1}{q_{j,e,b}(t)}$ and the expected return on the asset purchase (j,e',b') is $\frac{\rho_{j,e',b'}(t+1)}{q_{j,e',b'}(t)}$. From (3.10), the return on the sale (what is owed) exceeds the return on the purchase: $\frac{1}{q_{j,e,b}(t)} \geq \frac{\rho_{j,e',b'}(t+1)}{q_{j,e',b'}(t)}$. It is only optimal for household h to sell asset (j,e,b) if bankruptcy is declared for some realization $(e^h(t+1), s(t+1))$ and at that realization, the return on the sale is less than the return on the purchase: $\frac{\delta^h(t+1)}{q_{j,e,b}(t)} < \frac{\rho_{j,e',b'}(t+1)}{q_{j,e',b'}(t)}$. Using (3.10), then $\delta^h(t+1) < \rho_{j,e,b}(t+1)$. ■

3.3 Necessity of Private Information

The theorem in this section will state that private information is a necessary condition for dynamic bankruptcy choice to occur in all bankruptcy equilibrium. A dynamic bankruptcy choice is simply defined as the complement of a static bankruptcy choice. A bankruptcy choice is static if an eligible household declares bankruptcy in any time period t whenever $B^h(t+1) > S^h(t+1)$.

To prove necessity, suppose that the Markov income processes are no longer private information of the households. In this section, it will be taken as given that the financial markets allow the asset prices to be conditioned on income, "credit score", and the household's type k. There are now $3 \cdot I \cdot K \cdot J$ assets available for trade in each time period. Let the asset holdings be denoted by $z^h_{j,e,b,k}(t)$ and the asset prices by $q_{j,e,b,k}(t)$. The exogenous restriction is that a household h of type k can only be a debtor on asset (j,e,b,k) in time period t if $b^h(t) = b$ and $e^h(t) = e$.

The parameters $(\chi \in \mathbb{R}^J_+, \alpha \in \mathbb{R}^J_{++})$, by definition of their domains, impose an endogenous bound on the assets (j,e,b,k). Suppose that the values for these parameters are such that the chosen portfolios $\left(z^h_{j,e,b,k}\right)_{\forall e,b,k}$ satisfy the following two conditions:

- $\sum_j \min\left\{\sum_{e,b,k} r^h_{j,e,b,k}(s'|s,t+1) \cdot \left(z^h_{j,e,b,k}(t)\right)^+, \chi_j\right\} = \sum_{j,e,b,k} r^h_{j,e,b,k}(s'|s,t+1) \cdot \left(z^h_{j,e,b,k}(t)\right)^+$
- $\sum_j \alpha_j \sum_{e,b,k} r^h_{j,e,b,k}(s'|s,t+1) \cdot \left(z^h_{j,e,b,k}(t)\right)^+ \simeq 0$ (negligible).

The parameters (χ, α) that satisfy the above conditions will be called "large values" of χ and "small values" of α.

Theorem 3.3 *Suppose that χ takes on "large values" and α takes on "small values". Then, if the asset prices depend on a household's type k (no private information),* *there exists a bankruptcy equilibrium in which all eligible households h will declare bankruptcy in any time period t in which $B^h(t) > S^h(t)$.*

Proof. Suppose that $q_{j,e,1,k}(t) = q_{j,e,2,k}(t) \ \ \forall j,e,k,t$ whenever $\int_{h\in\mathcal{H}} \left(z^h_{j,e,1,k}(t)\right)^- < 0$ and $\int_{h\in\mathcal{H}} \left(z^h_{j,e,2,k}(t)\right)^- < 0$. The asset prices are ordered as $q_{j,e,0,k}(t) \geq q_{j,e,1,k}(t) = q_{j,e,2,k}(t) \ \ \forall j,e,k,t$. A bankruptcy declaration does not have any negative effects on future asset prices. As a result, an eligible household will declare bankruptcy in time period $t+1$, for any possible realization $\left(e^h(t+1), s(t+1)\right)$, whenever $B^h(t+1) > S^h(t+1)$.[9]

Consider the statement of claim 3.1: a household h of type k with $e^h(t) = e$ and $b^h(t) = b$ will only be a debtor on asset (j,e,b,k) in time t if (i) $\delta^h(t{+}1) < \rho_{j,e,b,k}(t{+}1)$ for some realization $(e^h(t+1), s(t+1))$ or (ii) $\sum_{e,b,k} \left(z^h_{j,e,b,k}(t)\right)^+ = 0$. A debtor in case (ii) will find it optimal to declare bankruptcy if $\delta^h(t+1) < 1$ holds.

Consider both cases simultaneously. Recall that upon declaring bankruptcy, the individual repayment rate is defined as $\delta^h(t+1) = -\frac{num}{den}$ from (3.6) with expressions for num and den in (3.4) – (3.5). Under the assumptions that χ has "large values" and α has "small values", $num \simeq 0$. As a debtor on asset (j,e,b,k), $den < 0$. Therefore, if a household h declares bankruptcy at some realization $(e^h(t{+}1), s(t{+}1))$, $\delta^h(t{+}1) \simeq 0$, which implies $B^h(t{+}1) > S^h(t{+}1)$. In fact, no matter which realizations $(e^h(t+1), s(t+1))$ a household h decides to declare bankruptcy at, $\delta^h(t+1) \simeq 0$, which implies $B^h(t{+}1) > S^h(t{+}1)$. Thus, any debtor on asset (j,e,b,k) will declare bankruptcy whenever it is eligible to do so. For $b \in \{1,2\}$, define $\%_{j,e,b,k}(t+1)$ as

[9] A household may declare bankruptcy if $S^h(t{+}1) \geq B^h(t{+}1)$, but $\delta^h(t{+}1) = 1$ in this case. From claim 3.1, the equality $\delta^h(t{+}1) = 1$ is satisfied iff household h is not a debtor on any asset (j,e,b,k). The implication of claim 3.1 is obvious under case (i). For case (ii) of claim 3.1, the implication also holds as $\sum_{e,b,k} \left(z^h_{j,e,b,k}(t)\right)^+ = 0$ for a debtor on asset (j,e,b,k) implies $S^h(t+1) < B^h(t+1)$ using the assumptions that χ has "large values" and α has "small values" (see the paragraph in the body after this footnote).

the probability that a debtor on asset (j, e, b, k) will remain solvent in $t+1$, that is, the probability that the debtor will have income $e(t+1) > e_{med}$:

$$\%_{j,e,b,k}(t+1) = \sum_{e':e'>e_{med}} \pi^k_{s(t)}(e'|e) \quad \text{for } b \in \{1,2\}.$$

From the law of large numbers, $\%_{j,e,b,k}(t+1)$ is equivalently the percentage of debtors on asset (j, e, b, k) that will remain solvent in $t+1$.

The decision to declare bankruptcy is independent of the size of the sale $z^h_{j,e,b,k}(t) < 0$. With χ "large" and α "small", the individual repayment rates for any household h are $\delta^h(t+1) \simeq 0$ whenever bankruptcy is declared. As defined in (3.7), the overall repayment rate $\rho_{j,e,b,k}(t+1)$ is then given by:

$$\rho_{j,e,b,k}(t+1) = \%_{j,e,b,k}(t+1) = \sum_{e':e'>e_{med}} \pi^k_{s(t)}(e'|e) \quad \text{for } b \in \{1,2\}.$$

The overall repayment rates are equal across "credit scores": $\rho_{j,e,1,k}(t+1) = \rho_{j,e,2,k}(t+1) \ \ \forall j, e, k$. From (3.10), the asset prices are equal across "credit scores": $q_{j,e,1,k}(t) = q_{j,e,2,k}(t) \ \ \forall j, e, k$. This completes the argument. ■

3.4 Asset Prices Strictly Increasing in "Credit Score"

The "credit score" $b(t)$ in equilibrium should contain nontrivial information about debtors' repayment rates. In this way, the pools of debtors will be heterogeneous and different "credit scores" (in particular, higher) will lead to different asset prices (in particular, higher).

I will continue to employ the assumption that χ takes on "large values" and α takes on "small values". Additionally, I remove all aggregate uncertainty and consider only 2 household types and 2 incomes, that is, $J = S = 1$ and $K = I = 2$. With $J = 1$, the subscript j is dropped when denoting assets. The single asset is a risk-free bond with a payout of 1 for both income realizations, given full commitment.

The transition matrices will be such that $\mu^k = (0.5, 0.5)$ $\forall k$, that is, $\Pi^k = \begin{bmatrix} \pi^k & 1-\pi^k \\ 1-\pi^k & \pi^k \end{bmatrix}$ with $\pi^1 > \pi^2$. In this economy, $b^h(t) = 0$ only if $e^h(t) = e_1$.

Let θ be the fraction of households of type $k = 1$. Define the endogenous fraction $\tilde{\theta}_{e,b}(t)$ to be the fraction of all households of type $k = 1$ in time t with income $e(t) = e$ and "credit score" $b(t) = b$.

The theorem of this section will state that all bankruptcy equilibria in all time periods have asset prices satisfying:

$$q_{e_1,1}(t) < q_{e_1,2}(t). \tag{3.12}$$

For the theorem to be true, I must rule out the equilibrium from section 3 in which $q_{e,1}(t) = q_{e,2}(t)$ $\forall e, t$ and all eligible households subsequently declare bankruptcy whenever $B(t) > S(t)$.

Lemma 3.1 *Suppose that χ takes on "large values" and α takes on "small values". Suppose that $J = S = 1$ and $K = I = 2$. Suppose that $\mu^k = (0.5, 0.5)$ $\forall k$ and $\pi^1 > \pi^2$. Given the initial conditions: (i) no households with $b(0) = 0$ and (ii) $\tilde{\theta}_{e_1,1}(0) = \tilde{\theta}_{e_2,1}(0) > \tilde{\theta}_{e_1,2}(0) = \tilde{\theta}_{e_2,2}(0)$, then $\tilde{\theta}_{e_1,1}(t) > \tilde{\theta}_{e_1,2}(t)$ $\forall t$.*

The proof of lemma 3.1 is contained in section 3.6.

Suppose (for contradiction) that $q_{e,1}(t) = q_{e,2}(t)$ $\forall e, t$ and all eligible households subsequently declare bankruptcy whenever $B(t) > S(t)$. Then from lemma 1, the probability that a debtor in pool (e_1, b) will remain solvent in $t+1$ is given by:

$$\%_{e_1,b}(t+1) = \tilde{\theta}_{e_1,b}(t)\left(1-\pi^1\right) + \left(1-\tilde{\theta}_{e_1,b}(t)\right)\left(1-\pi^2\right) \text{ for } b \in \{1,2\}.$$

As $\tilde{\theta}_{e_1,1}(t) > \tilde{\theta}_{e_1,2}(t) \quad \forall t$, then $\%_{e_1,1}(t+1) < \%_{e_1,2}(t+1)$. With χ "large" and α "small", then $\rho_{e_1,b}(t+1) = \%_{e_1,b}(t+1)$ for $b \in \{1,2\}$, so $\rho_{e_1,1}(t+1) < \rho_{e_1,2}(t+1)$. This contradicts that $q_{e,1}(t) = q_{e,2}(t)$.

The main theorem of this section is stated as follows.

Theorem 3.4 *Suppose that χ takes on "large values" and α takes on "small values". Suppose that $J = S = 1$ and $K = I = 2$. Suppose that $\mu^k = (0.5, 0.5)$ $\forall k$. Use the initial conditions: (i) no households with $b(0) = 0$ and (ii) $\tilde{\theta}_{e_1,1}(0) = \tilde{\theta}_{e_2,1}(0) > \tilde{\theta}_{e_1,2}(0) = \tilde{\theta}_{e_2,2}(0)$. Then, for every economy $((e_1, e_2), \beta, u(\cdot), \theta)$ satisfying these assumptions, $\exists \Delta > 0$ s.t. when $\pi^1 \geq \pi^2 + \Delta$, $q_{e_1,1}(t) < q_{e_1,2}(t) \quad \forall t$.*

The proof of theorem 3.4 is contained in section 3.6. Short of a characterization of Δ, it is instructive to see what sufficient conditions, albeit endogenous conditions, for the inequality $q_{e_1,1}(t) < q_{e_1,2}(t)$ $\forall t$ look like. Lemma 3.2 provides such sufficient conditions. The proof of lemma 3.2 is contained in section 3.6.

Lemma 3.2 *Suppose that χ takes on "large values" and α takes on "small values". Suppose that $J = S = 1$ and $K = I = 2$. Suppose that $\mu^k = (0.5, 0.5)$ $\forall k$. Use the initial conditions: (i) no households with $b(0) = 0$ and (ii) $\tilde{\theta}_{e_1,1}(0) = \tilde{\theta}_{e_2,1}(0) > \tilde{\theta}_{e_1,2}(0) = \tilde{\theta}_{e_2,2}(0)$. Suppose that in equilibrium, type $k = 1$ eligible households in pool $(e_1, 1)$ at t will declare bankruptcy at $t+1$. Suppose further that type $k = 2$ eligible households in pool $(e_1, 2)$ at t will not declare bankruptcy at $t+1$ unless they were also in pool $(e_1, 2)$ at $t-1$. Then, for every economy $((e_1, e_2), \beta, u(\cdot), (\pi^1, \pi^2), \theta)$ satisfying these assumptions, $q_{e_1,1}(t) < q_{e_1,2}(t) \quad \forall t$.*

3.5 Conclusion

This work has contributed a framework for the analysis of bankruptcy in the class of Bewley models. There is a natural information gap between the debtors and the market, namely that debtors know their repayment likelihood. It has been taken as given in the model that the asset prices can depend on a debtor's current income and "credit score". These two measures are verifiable by the market at minimal cost and lead to more efficient trade than if the asset prices were identical for all debtors. I prove the existence of a bankruptcy equilibrium, show that private information is a necessary condition for a dynamic bankruptcy decision, and also provide conditions on the parameters such that the asset prices are strictly increasing functions of the "credit score".

The next step in this line of research is to extend the asset structure to allow for assets traded on the secured credit markets following the example set by Geanakoplos and Zame (2002). For the secured assets, the households are given the option to default with the entire cost of default being the loss of the value of the contracted collateral. A specific case of default will be foreclosure when the asset under consideration is a home mortgage. In my proposed model, a household defaulting on an asset will still be responsible for the outstanding debt owed after the confiscation of the collateral (creditors are said to have "recourse"). This outstanding debt is namely the difference between debt owed and the value of the collateral. If this outstanding debt is positive, it will become unsecured debt. The household may pay for it either out of endowments and new asset purchases or the household can decide to declare bankruptcy over its entire portfolio. Such a model allows for the explicit analysis of two interesting mechanisms: (i) how a default on a single asset may force a household (eventually) into bankruptcy and (ii) how a menu of asset prices and collateral requirements can be determined in the markets for secured assets. In this manner, the collateral levels can be endogenized without the insurance/collateral intertemporal trade-off of Araujo, Orrillo, and Pascoa (2000).

3.6 Proofs

Proof of Theorem 3.1

The proof will be carried out in four steps. In **Step 1**, I will define the budget set and the price space and prove that they are nonempty, convex, and compact. In **Step 2**, I will define the demand correspondence and prove that it is well-defined, convex-valued, and upper hemicontinuous. In **Step 3**, I will define the price correspondence and prove that it is well-defined, convex-valued, and upper hemicontinuous. In **Step 4**, I will apply Kakutani's theorem to find a fixed point of the Cartesian product of the demand correspondences and the price correspondence. I will show that this fixed point satisfies the definition of a bankruptcy equilibrium from section 3.2.

Step 1: Budget set and price space

Recall that the payouts for households depend upon if they remain solvent, $S(t)$, or declare bankruptcy, $B(t)$, and were defined in equation (3.2) :

$$\begin{aligned} S^h(t) &= p(t)\sum_{j,e,b} r^h_{j,e,b}\left(s'|s,t\right)\cdot z^h_{j,e,b}(t-1) \\ B^h(t) &= p(t)\sum_{j}\min\left\{\sum_{e,b} r^h_{j,e,b}\left(s'|s,t\right)\cdot\left(z^h_{j,e,b}(t-1)\right)^+,\chi_j\right\} \\ &\quad -p(t)\sum_{j}\alpha_j\sum_{e,b} r^h_{j,e,b}\left(s'|s,t\right)\cdot\left(z^h_{j,e,b}(t-1)\right)^+ \end{aligned} \tag{3.2}$$

The commodity price in each time period will be given by the sequence $p =$

$(p(t))_{\forall t} \in \mathbb{R}^{\overline{T}+1}$. In the definition of equilibrium, I made the normalization $p(t) = 1$ $\forall t$. I do not make that same normalization in the proof.

If assets are traded in time periods $t = 0, ..., \overline{T}-1$, the overall repayment rates ρ are defined in time period $t = 1, ..., \overline{T}-1$. I assume with the finite horizon that households are not permitted to declare bankruptcy in the final time period $\overline{T}$. This restriction is innocuous as the actual bankruptcy equilibrium has an infinite time horizon without a final period $\overline{T}$. Then $\rho = (\rho(t+1))_{\forall t} = \left(\left(\rho_{j,e,b}(t+1)\right)_{\forall j,e,b}\right)_{\forall t} \in \mathbb{R}_{+}^{3IJ\left(\overline{T}-1\right)}$ are the overall repayment rates. The asset prices are $q = (q(t))_{\forall t} = \left((q_{j,e,b}(t))_{\forall j,e,b}\right)_{\forall t} \in \mathbb{R}^{3IJ\overline{T}}$.

Define the sequence of household assets as $z^h = \left(z^h(t)\right)_{\forall t} = \left(\left(z^h_{j,e,b}(t)\right)_{\forall j,e,b}\right)_{\forall t} \in \mathbb{R}^{3IJ\overline{T}}$. Define the sequence of household consumption as $c^h = \left(c^h(t)\right)_{\forall t} \in \mathbb{R}_{+}^{\overline{T}+1}$. Define the upper bound on consumption $\bar{c} = 2 \cdot e_I$.

Define the bounded budget set for each household $h \in \mathcal{H}$ as:

$$\bar{B}^h(p,q,\rho) = \left\{ \begin{array}{l} \left(c^h, z^h, b^h\right) \in \mathbb{R}_{+}^{\bar{T}+1} \times \mathbb{R}^{3IJ\overline{T}} \times \mathcal{B}^{\bar{T}-1} : \\ c^h(t) \leq \bar{c} \;\; \forall t, \\ p(0)\left(e^h(0) - c^h(0)\right) - \sum\limits_{j,e,b} q_{j,e,b}(0) z^h_{j,e,b}(0) \geq 0, \\ p(t)\left(e^h(t) - c^h(t)\right) - \sum\limits_{j,e,b} q_{j,e,b}(t) z^h_{j,e,b}(t) + \left\{ \begin{array}{ll} S^h(t) & \text{if } b^h(t) > 0 \\ B^h(t) & \text{if } b^h(t) = 0 \end{array} \right\} \geq 0, \\ p(\bar{T})\left(e^h(\bar{T}) - c^h(\bar{T})\right) + p(\bar{T}) \sum\limits_{j,e,b} r_j\left(s(\bar{T})|s(\bar{T}-1)\right) z^h_{j,e,b}(\bar{T}-1) \geq 0. \end{array} \right\}. \tag{3.13}$$

In equilibrium, the constraints $c^h(t) \leq \bar{c}$ will be nonbinding. Since the utility function is quasi-concave and continuous, it is innocuous to add the constraints to the budget set as the optimal solutions to the household problem will not be affected. $\bar{B}^h(p,q,\rho)$ is nonempty.

Lemma 3.3 $\int \bar{B}^h(p,q,\rho)$ *is convex.*

Proof. $\bar{B}^h(p,q,\rho)$ is a set-valued function (terminology of Aumann, 1966) or a correspondence. From Aumann (1966), $\int \bar{B}^h(p,q,\rho)d\Phi$ is convex provided that $\bar{B}^h(p,q,\rho)$ is a set-valued function defined on the set of households $\mathcal{H}\tilde{}\ [0,1]$ [10] and the values of $\bar{B}^h(p,q,\rho)$ are subsets of $\mathbb{R}_+^{\bar{T}+1} \times \mathbb{R}^{3IJ\bar{T}} \times \mathcal{B}^{\bar{T}-1}$. ■

The proof of lemma 3.4 is found at the completion of the *proof of theorem 3.1.*

Lemma 3.4 $\bar{B}^h(p,q,\rho)$ *is compact.*

In terms of $\epsilon > 0$ small, define the price space as:

$$\Delta_\epsilon^* = \left\{ (p,q,\rho) : \begin{array}{c} p(t) \geq \epsilon \ \ \forall t \\ \epsilon \leq q_{j,e,b}(t) \leq \frac{1}{\epsilon} \ \ \forall t, \forall j, e, b \\ p(t) + \sum_j q_{j,e_1,0}(t) = 1 \ \ \forall t \\ \epsilon \leq \rho_{j,e,b}(t+1) \leq 1 \ \ \forall t, \forall j, e, b \end{array} \right\}. \tag{3.14}$$

By definition, when $\int_{h\in\mathcal{H}} \left(z_{j,e,b}^h(t)\right)^+ > 0$ (asset market open), the overall repayment rate $\rho_{j,e,b}(t+1) \in (0,1]$. In fact, $\rho_{j,e,b}(t+1)$ is bounded below by $\min_{k,s} \sum_{e'>e_{med}} \pi_s^k(e'|e) > 0$ (assumption 1). So set $\epsilon > 0$ small below this bound. If $\int_{h\in\mathcal{H}} \left(z_{j,e,b}^h(t)\right)^+ = 0$ (asset market closed), then $\rho_{j,e,b}(t+1) \in (0,1]$ is fixed according to the "equal returns condition" (3.10). The price space Δ_ϵ^* is nonempty, convex, and compact.

Step 2: Demand correspondence

I will write down the household's truncated optimization problem and define the

[10] Recall that the set of households is actually a hyperfinite process in order to be able to apply the law of large numbers.

household demand. The household problem (H) is given by

$$\max_{c^h, z^h, b^h} E_0 \sum_{t=0}^{\bar{T}} \beta^t u\left(c^h(t)\right) \tag{H}$$

$$subj\ to\ \left(c^h, z^h, b^h\right) \in \bar{B}^h(p, q, \rho).$$

I will define the household demand correspondence as

$$\Upsilon^h : \Delta_\epsilon^* \rightrightarrows \bar{B}^h$$

such that given $(p, q, \rho) \in \Delta_\epsilon^*$, $\left(\tilde{c}^h, \tilde{z}^h, \tilde{b}^h\right) \in \Upsilon^h(p, q, \rho)$ iff $\left(\tilde{c}^h, \tilde{z}^h, \tilde{b}^h\right)$ solves (H). Υ^h is well-defined and Υ, defined such that $\Upsilon(p, q, \rho) = \int \Upsilon^h(p, q, \rho) \quad \forall(p, q, \rho) \in \Delta_\epsilon^*$, is convex-valued. I will show that the correspondence Υ^h is upper hemicontinuous (uhc). The proof of lemma 3.5 is contained after the *proof of theorem 3.1.*

Lemma 3.5 Υ^h *is a uhc correspondence.*

Step 3: Price correspondence

I will now write down the price correspondence

$$\Psi : \int \bar{B}^h \rightrightarrows \Delta_\epsilon^*.$$

Given $\left(c^h, z^h, b^h\right)_{h\in\mathcal{H}}$, by definition $(p, q, \rho) \in \Psi\left(\left(c^h, z^h, b^h\right)_{h\in\mathcal{H}}\right)$ iff the following conditions hold: (i) $(p, q, \rho) \in \Delta_\epsilon^*$, (ii) the overall repayment rates ρ satisfy (3.7), (iii) (q, ρ) satisfy the "equal returns condition" (3.10), and (iv) (p, q) satisfy the following one-period maximization problems ($\forall t : 0 \le t \le \bar{T} - 1$):

$$(p(t), q(t)) \in \arg\max\left\{p(t)\left(\int c^h(t)\, d\Phi - \bar{e}\right) + \sum_{j,e,b} q_{j,e,b}(t) \int z^h_{j,e,b}(t) d\Phi\right\}. \tag{3.15}$$

The correspondence Ψ is well-defined, convex-valued, and uhc. This statement is trivial given that the objective function in (3.15) is linear in (p, q) and the price space Δ_ϵ^* is compact.

Step 4: Market clearing

Define the overall equilibrium correspondence as the Cartesian product $\Upsilon \times \Psi$. The overall correspondence is well-defined, convex-valued, and uhc. It maps from the Cartesian product $\int \bar{B}^h \times \Delta_\epsilon^*$ into itself. The set $\int \bar{B}^h \times \Delta_\epsilon^*$ is nonempty, convex, and compact. Applying Kakutani's fixed point theorem yields a fixed point of this overall equilibrium correspondence. By definition, the fixed point is such that the household choice vector $\left(c^h, z^h, b^h\right)$ satisfies the household optimization problem (H) $\forall h \in \mathcal{H}$.

Walras' Law yields the following equations:

$$\begin{array}{l} p(0)\left(\int c^h(0) - \bar{e}\right) + \sum\limits_{j,e,b} q_{j,e,b}(0) \int z_{j,e,b}^h(0) = 0 \\ p(t)\left(\int c^h(t) - \bar{e}\right) + \sum\limits_{j,e,b} q_{j,e,b}(t) \int z_{j,e,b}^h(t) - \int\limits_{h:\delta^h(t)=1} S^h(t) - \int\limits_{h:\delta^h(t)<1} B^h(t) = 0 \\ p(\bar{T})\left(\int c^h(\bar{T}) - \bar{e}\right) - p(\bar{T}) \sum\limits_{j,e,b} r_j\left(s(\bar{T})|s(\bar{T}-1)\right) \int z_{j,e,b}^h(\bar{T}-1) = 0 \end{array} . \tag{3.16}$$

The sum $\int\limits_{h:\delta^h(t)=1} S^h(t) + \int\limits_{h:\delta^h(t)<1} B^h(t)$ is given by:

$$p(t) \sum_{j,e,b} \rho_{j,e,b}(t) r_j\left(s'|s\right) \int\limits_{h\in\mathcal{H}} \left(z_{j,e,b}^h(t-1)\right)^+ + p(t) \sum_{j,e,b} r_j\left(s'|s\right) \int\limits_{h\in\mathcal{H}} \delta^h(t)\left(z_{j,e,b}^h(t-1)\right)^-$$

after using equation (3.9) where $\delta^h(t) = 1$ for households with payout $S^h(t)$ and $\delta^h(t) < 1$ is endogenously determined from equations (3.4) – (3.6) for the households with payout $B^h(t)$.

The equilibrium equation (3.7)

$$\rho_{j,e,b}(t)\int_{h\in\mathcal{H}}\left(z^h_{j,e,b}(t-1)\right)^+ = -\int_{h\in\mathcal{H}}\delta^h(t)\left(z^h_{j,e,b}(t-1)\right)^-. \tag{3.7}$$

then yields that the sum $\displaystyle\int\limits_{h:\delta^h(t)=1} S^h(t) + \int\limits_{h:\delta^h(t)<1} B^h(t) = 0.$

The equations from Walras' Law in (3.16) can be reduced to:

$$\begin{array}{l} p(t)\left(\int c^h(t)-\bar{e}\right)+\sum\limits_{j,e,b} q_{j,e,b}(t)\int z^h_{j,e,b}(t)=0 \\ p(\bar{T})\left(\int c^h(\bar{T})-\bar{e}\right)-p(\bar{T})\sum\limits_{j,e,b} r_j\left(s(\bar{T})|s(\bar{T}-1)\right)\int z^h_{j,e,b}(\bar{T}-1)=0 \end{array}. \tag{3.17}$$

The proof of lemma 6 is found at the completion of the *proof of theorem 3.1.*

Lemma 3.6 *Given equations* (3.17), *the definition of the price correspondence dictates that the markets for both commodities and assets clear, that is:*

$$\begin{aligned} \int c^h(t)-\bar{e} &= 0 \ \ \forall t \\ \int z^h_{j,e,b}(t) &= 0 \ \forall t,\ \ \forall j,e,b. \end{aligned}$$

This completes the *proof of theorem 3.1.*

Proof of Lemma 3.4

Consider any sequence of state realizations $s^{\bar{T}}$. To prove this result, I recognize that consumption is bounded by definition. Then, beginning in time period $\bar{T}$, I will show that the assets are bounded by backward induction.[11]

Final period: $t = \bar{T}$

The term $p(\bar{T}) \sum_{j,e,b} r_j \left(s(\bar{T})|s(\bar{T}-1)\right) z^h_{j,e,b}(\bar{T}-1)$ is bounded according to the budget constraint at time period $t = \bar{T}$. The payout matrix $R\left(s(\bar{T}-1)\right)$ has full column rank since I do not allow bankruptcy in the final period $\bar{T}$. Consider the summed asset $z^h_j(\bar{T}-1) = \sum_{e,b} z^h_{j,e,b}(\bar{T}-1)$. There are J summed assets and as $R\left(s(\bar{T}-1)\right)$ has full column rank equal to J, the summed assets $\left(z^h_j(\bar{T}-1)\right)_{\forall j}$ are bounded. Thus, and most importantly, the value of the assets $\sum_{j,e,b} q_{j,e,b}(\bar{T}-1) z^h_{j,e,b}(\bar{T}-1) = \sum_j q_{j,e_1,0}(\bar{T}-1) z^h_j(\bar{T}-1)$ is bounded. Without bankruptcy, no arbitrage requires, in agreement with the "equal returns condition", that $q_{j,e_1,0}(\bar{T}-1) = q_{j,e,b}(\bar{T}-1) \;\; \forall e, b$.

Backward induction: $t < \bar{T}$

Let $s = s(t-1)$ and $s' = s(t)$. With $\sum_{j,e,b} q_{j,e,b}(t) z^h_{j,e,b}(t)$ bounded (this is the inductive hypothesis), the budget constraint at time period $t < \bar{T}$ dictates that a household's financial payout, either $S^h(t)$ or $B^h(t)$, is bounded. I will first consider those households with $E_{t-1}\delta^h(t) < 1$ and then those households with $E_{t-1}\delta^h(t) = 1$.

Part I: Households with $E_{t-1}\delta^h(t) < 1$

A household will choose a vector of assets $\left(z^h_{j,e,b}(t-1)\right)_{\forall j,e,b}$ in time period $t-1$. From assumption A, there exists a possible realization $e^h(t) = e'$ such that $e' > e_{med}$ and the household cannot declare bankruptcy at t. Given that the household has financial payout $B^h(t) > S^h(t)$ for at least one realization by definition, then the asset choices $\left(z^h_{j,e,b}(t-1)\right)_{\forall j,e,b}$ are made with the expectation of bankruptcy given some realizations at t and solvency given others.

[11] There is a natural indeterminacy in a household's asset choice as all the assets $(j,e,b)_{\forall e,b}$ provide "equal returns" to creditors. For solvent households, it is certainly possible that some assets $z^h_{j,e,b}(t)$ are not bounded, but the summed asset $z^h_j(t) = \sum_{e,b} z^h_{j,e,b}(t)$ will be proven to be bounded and most importantly the value $\sum_{e,b} q_{j,e,b}(t) z^h_{j,e,b}(t)$ will be proven to be bounded.

Suppose for contradiction that there exists a sequence of asset choices $\left(z^{\nu}_{j,e,b}(t-1)\right)$ s.t. $z^{\nu}_{j,e,b}(t-1) \to +\infty$ as $\nu \to \infty$. For the realizations at which $B^h(t) > S^h(t)$, assumptions B and C together with $\chi_j < \infty$ imply that the term $B^h(t)$ is unbounded below. This is a contradiction.

Knowing that the asset choices are bounded above, suppose for contradiction that there exists a sequence of asset choices $\left(z^{\nu}_{j,e,b}(t-1)\right)$ s.t. $z^{\nu}_{j,e,b}(t-1) \to -\infty$ as $\nu \to \infty$. Then, for the realizations at which the household does not declare bankruptcy (financial payout is $S^h(t)$), assumption B and the prior result dictate that $S^h(t)$ is unbounded below. This is a contradiction

The value $\sum_{j,e,b} q_{j,e,b}(t-1) z^h_{j,e,b}(t-1)$ is bounded.

Part II: Households with $E_{t-1}\delta^h(t) = 1$

A household will choose a vector of assets $\left(z^h_{j,e,b}(t-1)\right)_{\forall j,e,b}$ in time period $t-1$. From part I, I already know that the assets $\left(z^h_{j,e,b}(t-1)\right)_{\forall j,e,b}$ are bounded for all households with $E_{t-1}\delta^h(t) < 1$.

Impose an artificial bound on the asset choices of households with $E_{t-1}\delta^h(t) = 1:$

$$-D_j \leq \sum_{e,b} z^h_{j,e,b}(t-1) \leq D_j \quad \forall j.$$

Define the vector $D = (D_j)_{\forall j} \in \mathbb{R}^J_+$. Then $\bar{B}^h(p,q,\rho)$ is compact, markets clear, and the equation (3.8) holds for all assets:

$$\rho_{j,e,b}(t) = 1 + \frac{\int_{h\in\mathcal{H}} \left(1-\delta^h(t)\right)\left(z^h_{j,e,b}(t-1)\right)^-}{\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t-1)\right)^+}. \tag{3.8}$$

Consider what happens as $D \to \infty$. If the constraints cease to bind, the assets are bounded and the proof is finished. I will assume that some of the constraints continue to bind as $D \to \infty$, that is, the set $\bar{B}^h(p, q, \rho)$ becomes unbounded, and show that this leads to a contradiction.

For the set of assets for which the constraints continue to bind as $D \to \infty$, define J^* as the number of different j. Then the matrix of endogenous payouts of these assets must have rank $< J^*$. This must be the case, otherwise the bounded matrix product $S^h(t) = R^h(s) \cdot z^h(t-1)$ would imply bounded asset choices. Define $\mathcal{A}^*$ as the set of assets (j, e, b) that become unbounded $\int_{h\in\mathcal{H}} \left|z^h_{j,e,b}(t-1)\right| \to \infty$ as $D \to \infty$.

For the assets $(j, e, b) \in \mathcal{A}^*$, as the markets must clear, then both $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t-1)\right)^+$ and $\int_{h\in\mathcal{H}} \left(z^h_{j,e,b}(t-1)\right)^-$ become unbounded. I have already shown that $\left(z^h_{j,e,b}(t-1)\right)_{\forall j,e,b}$ are bounded for all households with $E_{t-1}\delta^h(t) < 1$. Thus, in equation (3.8), the numerator term $\int_{h\in\mathcal{H}} (1 - \delta^h(t)) \left(z^h_{j,e,b}(t-1)\right)^-$ remains bounded. In the limit as $D \to \infty$, the overall repayment rates $\rho_{j,e,b}(t) \to 1$ for all assets $(j, e, b) \in \mathcal{A}^*$.

The asset payouts $r^h_j(s'|s,t) \to r_j(s'|s)$ as $\int_{h\in\mathcal{H}} \bar{B}^h(p, q, \rho)$ becomes unbounded for all assets $(j, e, b) \in \mathcal{A}^*$. The payout matrix $R(s(t-1))$ is in general position. Thus, all the assets $(j, e, b) \in \mathcal{A}^*$ will have an endogenous payout matrix of full rank J^*. This contradiction implies that all the constraints $-D_j \leq \sum_{e,b} z^h_{j,e,b}(t-1) \leq D_j$ cease to bind as $D \to \infty$. The value $\sum_{j,e,b} q_{j,e,b}(t-1) z^h_{j,e,b}(t-1)$ is bounded.

Proof of Lemma 3.5

I will define the budget correspondence $\bar{B}^h : \Delta^*_\epsilon \rightrightarrows \bar{B}^h$ such that given $(p, q, \rho) \in \Delta^*_\epsilon$, the values of the correspondence $\bar{B}^h$ will be the entire budget set $\bar{B}^h(p, q, \rho)$. This correspondence is trivially uhc. The following proof will show that $\bar{B}^h$ is also lhc. The utility function in the programming problem (H) is continuous, so applying the theorem of the maximum, Υ^h is a uhc correspondence.

Claim 3.2 *$\bar{B}^h$ is an lhc correspondence.*

Proof. Consider a sequence $(p^\nu, q^\nu, \rho^\nu) \to (p, q, \rho)$ with $\left(c^h, z^h, b^h\right) \in \bar{B}^h(p, q, \rho)$ for some h. I will drop the household superscript for all household variables and parameters. I will find some scaling factor $(\theta^\nu(t))_{0 \le t \le \bar{T}}$ such that when the variables (c, z, b) are appropriately scaled:

$$\begin{aligned}
c^\nu(t) &= \left(\min_{0 \le t \le \bar{T}} \theta^\nu(t)\right) c(t) \ \ \forall t \\
z^\nu_{j,e,b}(t) &= \left(\min_{0 \le t \le \bar{T}} \theta^\nu(t)\right) z_{j,e,b}(t) \ \ \forall t, \ \forall j, e, b \\
b^\nu(t) &= b(t),
\end{aligned}$$

$\exists N$ s.t. $\forall \nu \ge N$, $(c^\nu, z^\nu, b^\nu) \in \bar{B}(p^\nu, q^\nu, \rho^\nu)$ and $(c^\nu, z^\nu, b^\nu) \to (c, z, b)$. For simplicity, define $\theta^\nu = \left(\min_{0 \le t \le \bar{T}} \theta^\nu(t)\right)$.

The budget set $\bar{B}(p, q, \rho)$ has the so-called scaling propety, so called by Dubey et al. (2005), meaning that it is fairly straightforward to define the sequence of scaling fractions $\theta^\nu(t) \in [0, 1]$ for $0 \le t \le \bar{T}$.

Consider any time period t. I define the financial payout in time period t as $w_t(p, \rho, z)$. If a household declares bankruptcy in time period t with $B(t) > S(t)$, then $w_t(p, \rho, z) = B(t)$. Otherwise, $w_t(p, \rho, z) = S(t)$. The initial condition is obviously $w_0(p, \rho, z) = 0$.

Consider any time period t and if $t > 0$, let the aggregate states be $s = s(t-1)$ and $s' = s(t)$. If $e(t) - c(t) - \sum_{j,e,b} q_{j,e,b}(t) z_{j,e,b}(t) + w_t(p, \rho, z) > 0$, then $\exists M_t$ s.t. $e(t) - c(t) - \sum_{j,e,b} q^\nu_{j,e,b}(t) z_{j,e,b}(t) + w_t(p^\nu, \rho^\nu, z) > 0$ holds $\forall \nu \ge M_t$. Define $\theta^\nu(t) = 1$ for this case.

Otherwise, $e(t) - c(t) - \sum_{j,e,b} q_{j,e,b}(t) z_{j,e,b}(t) + w_t(p, \rho, z) = 0$ and $c(t) + \sum_{j,e,b} q_{j,e,b}(t) z_{j,e,b}(t) - w_t(p, \rho, z) = e(t) > 0$. For simplicity, define $q(t) \cdot z(t) = \sum_{j,e,b} q_{j,e,b}(t) z_{j,e,b}(t)$. Define

$\theta^\nu(t) \in [0,1]$ as:

$$\theta^\nu(t) = \left\{ \begin{array}{ccc} \frac{e(t)}{c(t)+q^\nu(t)\cdot z(t)-w_t(p^\nu,\rho^\nu,z)} & \text{if} & \begin{array}{c} e(t) < c(t)+q^\nu(t)\cdot z(t) - w_t(p^\nu,\rho^\nu,z) \text{ and} \\ c(t)+q^\nu(t)\cdot z(t) - w_t(p^\nu,\rho^\nu,z) \neq 0 \end{array} \\ 1 & \text{otherwise} & \end{array} \right\} \forall \nu. \tag{3.18}$$

With $c(t)+q^\nu(t)\cdot z(t) - w_t(p^\nu,\rho^\nu,z) \to e(t)$ and the fact that $\exists N_t$ such that the term $c(t)+q^\nu(t)\cdot z(t) - w_t(p^\nu,\rho^\nu,z) > 0 \ \ \forall \nu \geq N_t$, then the fraction $\theta^\nu(t) \to 1$ as $\nu \to \infty$. Define $\theta^\nu = \min_{0\leq t\leq T} \theta^\nu(t)$ as the fraction by which both consumption and assets are scaled down:

$$\begin{aligned} c^\nu(t) &= \theta^\nu \cdot c(t) \ \ \forall t \\ z^\nu_{j,e,b}(t) &= \theta^\nu \cdot z_{j,e,b}(t) \ \ \forall t, \ \forall j,e,b. \end{aligned}$$

The "credit score" $b(t)$ is held fixed in the sequence. If the financial payout in time period t is given by $w_t(p,\rho,z) = B(t)$, then in the sequence $w_t(p^\nu,\rho^\nu,z) = B^\nu(t)$. I have left to show that the scaled consumption and asset choices (c^ν, z^ν) are such that the budget constraints are satisfied in each time period.

Consider any time period t. As a first step for the following argument, I will prove that the following inequality holds:

$$\theta^\nu w_t(p^\nu,\rho^\nu,z) \leq w_t(p^\nu,\rho^\nu,z^\nu) \tag{3.19}$$

where by definition $z^\nu_{j,e,b}(t-1) = \theta^\nu \cdot z_{j,e,b}(t-1) \ \ \forall j,e,b$. There are two possible cases to consider and I will attack them each in order:

Case 1:

Suppose that a household decides to remain solvent in time period t, that is,

$$w_t(p^\nu, \rho^\nu, z) = S(t) = p^\nu(t) \sum_{j,e,b} \left[\rho^\nu_{j,e,b}(t)\, r_j(s'|s)\, (z_{j,e,b}(t-1))^+ + r_j(s'|s)\,(z_{j,e,b}(t-1))^-\right].$$

As this term is linear in $z_{j,e,b}(t-1)$, then obviously

$$w_t(p^\nu, \rho^\nu, \theta^\nu z) = \theta^\nu w_t(p^\nu, \rho^\nu, z)$$

and so the inequality (3.19) is satisfied.

Case 2:

Suppose that a household decides to declare bankruptcy in time period t with $B(t) > S(t)$, that is, $w_t(p^\nu, \rho^\nu, z) = B(t)$ where

$$\begin{aligned} B(t) &= p^\nu(t) \sum_j \min\left\{\sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\,(z_{j,e,b}(t-1))^+, \chi_j\right\} \\ &\quad - p^\nu(t) \sum_j \alpha_j \sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\,(z_{j,e,b}(t-1))^+. \end{aligned}$$

The second term is linear in $z_{j,e,b}(t-1)$, so I have only left to show that $(\forall j)$:

$$\begin{aligned} &\theta^\nu \min\left\{\sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\cdot(z_{j,e,b}(t-1))^+, \chi_j\right\} \\ \le\; &\min\left\{\sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\cdot\left(z^\nu_{j,e,b}(t-1)\right)^+, \chi_j\right\}. \end{aligned} \tag{3.20}$$

Subcase (a): In this first subcase, the entire value of the asset purchases is exempt, that is, $\sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\cdot(z_{j,e,b}(t-1))^+ \le \chi_j$. Then

$$\sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\cdot\left(z^\nu_{j,e,b}(t-1)\right)^+ = \theta^\nu \sum_{e,b} \rho^\nu_{j,e,b}(t)\, r_j(s'|s)\cdot(z_{j,e,b}(t-1))^+ \le \chi_j.$$

The terms are then linear in $z_{j,e,b}(t-1)$, so inequality (3.20) is satisfied.

Subcase (b): Consider the second subcase where not all of the asset purchases are exempt, that is, $\sum_{e,b} \rho^{\nu}_{j,e,b}(t)\, r_j(s'|s) \cdot (z_{j,e,b}(t-1))^{+} > \chi_j$. If $\theta^{\nu} \sum_{e,b} \rho^{\nu}_{j,e,b}(t)\, r_j(s'|s) \cdot (z_{j,e,b}(t-1))^{+} > \chi_j$, then trivially $\theta^{\nu}\chi_j \leq \chi_j$ and inequality (3.20) is met. If $\theta^{\nu} \sum_{e,b} \rho^{\nu}_{j,e,b}(t)\, r_j(s'|s) \cdot (z_{j,e,b}(t-1))^{+} \leq \chi_j$, then the inequality from (3.20) becomes

$$\theta^{\nu}\chi_j \leq \theta^{\nu} \sum_{e,b} \rho^{\nu}_{j,e,b}(t)\, r_j(s'|s) \cdot (z_{j,e,b}(t-1))^{+},$$

which is satisfied under this subcase (b).

I will now show that the budget constraint is satisfied in time period t. Using inequality (3.19) :

$$c^{\nu}(t) + q^{\nu}(t) \cdot z^{\nu}(t) - w_t(p^{\nu}, \rho^{\nu}, z^{\nu}) \leq \theta^{\nu}\left(c(t) + q^{\nu}(t) \cdot z(t) - w_t(p^{\nu}, \rho^{\nu}, z)\right)$$

By definition $\theta^{\nu} \leq \theta^{\nu}(t)$ and $c(t) + q^{\nu}(t) \cdot z(t) - w_t(p^{\nu}, \rho^{\nu}, z) > 0 \quad \forall \nu \geq N_t$ which implies:

$$c^{\nu}(t) + q^{\nu}(t) \cdot z^{\nu}(t) - w_t(p^{\nu}, \rho^{\nu}, z^{\nu}) \leq \theta^{\nu}(t)\left(c(t) + q^{\nu}(t) \cdot z(t) - w_t(p^{\nu}, \rho^{\nu}, z)\right).$$

From the definition of $\theta^{\nu}(t)$ in equation (3.18), then

$$c^{\nu}(t) + q^{\nu}(t) \cdot z^{\nu}(t) - w_t(p^{\nu}, \rho^{\nu}, z^{\nu}) \leq e(t).$$

Thus, $\exists N = \max\{.., M_t, N_t, ..\}$ s.t. $\forall \nu \geq N$, $(c^{\nu}, z^{\nu}, b^{\nu}) \in \bar{B}^h(p^{\nu}, q^{\nu}, \rho^{\nu})$ and also $(c^{\nu}, z^{\nu}, b^{\nu}) \to (c, z, b)$. This completes the proof. ■

Proof of Lemma 3.6

At our disposal are the simplified equations from Walras' Law as found previously in equations (3.17) :

$$\begin{array}{l} p(t)\left(\int c^h\,(t)-\overline{e}\right)+\sum_{j,e,b} q_{j,e,b}(t)\int z^h_{j,e,b}(t)=0 \\ p(\bar{T})\left(\int c^h\,(\bar{T})-\overline{e}\right)-p(\bar{T})\sum_{j,e,b} r_j\left(s(\bar{T})|s(\bar{T}-1)\right)\int z^h_{j,e,b}(\bar{T}-1)=0 \end{array}. \tag{3.17}$$

The proof methodology will be to take the equation for time period $t : 0 \leq t \leq \bar{T}-1$ from (3.17) and use it to show that

$$\begin{aligned} \int c^h\,(t)-\overline{e} &= 0 \\ \int z^h_{j,e,b}(t) &= 0\ \ \forall j. \end{aligned}$$

I will rewrite the first equation in (3.17) only in terms of the base price $q_{j,e_1,0}(t)$ using the "equal returns condition" (3.10). I will use the fact that $\rho_{j,e_1,0}(t+1)=1$.

$$p(t)\left(\int c^h\,(t)-\overline{e}\right)+\sum_{j} q_{j,e_1,0}(t)\sum_{e,b}\rho_{j,e,b}(t+1)\int z^h_{j,e,b}(t)=0. \tag{3.21}$$

The normalization will be adjusted for this lemma so that $p(t)+\sum_j q_{j,e_1,0}(t)=1$.

Suppose $\displaystyle\int c^h\,(t)-\overline{e}>0$. The definition of the price correspondence Ψ requires $p(t)=1-J\epsilon$ and $q_{j,e_1,0}(t)=\epsilon\ \ \forall j$. The equation for time period t from (3.21) yields:

$$(1-J\epsilon)\left(\int c^h\,(t)-\overline{e}\right)+\epsilon\sum_{j,e,b}\rho_{j,e,b}(t+1)\int z^h_{j,e,b}(t)=0.$$

There is then an upper bound given by:

$$\left(\int c^h\,(t)-\overline{e}\right)\leq\frac{-\epsilon\displaystyle\sum_{j,e,b}\rho_{j,e,b}(t+1)\int z^h_{j,e,b}(t)}{1-J\epsilon}. \tag{3.22}$$

Suppose $\sum_{e,b} \rho_{j,e,b}(t+1) \int z^h_{j,e,b}(t) > 0$. The definition of the price correspondence Ψ requires $p(t) = \epsilon$, $q_{j,e_1,0}(t) = 1 - J\epsilon$, and $q_{k,e_1,0}(t) = \epsilon \ \ \forall k \neq j$. The equation for time period t from (3.21) yields:

$$(1 - J\epsilon) \sum_{e,b} \rho_{j,e,b}(t+1) \int z^h_{j,e,b}(t) = -\epsilon \sum_{k,e,b:k \neq j} \rho_{k,e,b}(t+1) \int z^h_{k,e,b}(t) - \epsilon \left(\int c^h(t) - \overline{e} \right)$$

As $\int c^h(t) \geq 0$, then an upper bound is given by:

$$\sum_{e,b} \rho_{j,e,b}(t+1) \int z^h_{j,e,b}(t) \leq \frac{-\epsilon \sum_{k,e,b:k \neq j} \rho_{k,e,b}(t+1) \int z^h_{k,e,b}(t) + \epsilon \overline{e}}{(1 - J\epsilon)} \tag{3.23}$$

From the proof of lemma 3.4, $\sum_{e,b} z^h_{j,e,b}(t)$ is bounded below $\forall h$ and thus so is $\sum_{e,b} \rho_{j,e,b}(t+1) \int z^h_{j,e,b}(t)$. Take a sequence $\epsilon \to 0$ such that defining Δ^*_ϵ with this ϵ, the upper bounds (3.22) and (3.23) imply:

$$\begin{aligned} \int c^h(t) - \overline{e} &\leq 0 \\ \sum_{e,b} \rho_{j,e,b}(t+1) \int z^h_{j,e,b}(t) &\leq 0 \ \ \forall j. \end{aligned}$$

As $p(t) > 0$ and $q_{j,e_1,0}(t) > 0 \ \ \forall j$, then (3.21) implies ($\forall t : 0 \leq t \leq \bar{T} - 1$):

$$\begin{aligned} \int c^h(t) - \overline{e} &= 0 \\ \sum_{e,b} \rho_{j,e,b}(t+1) \int z^h_{j,e,b}(t) &= 0 \ \ \forall j. \end{aligned} \tag{3.24}$$

Define the adjusted asset holdings for each household as:

$$\theta^h_{j,e,b}(t) = \rho_{j,e,b}(t+1)z^h_{j,e,b}(t) \ \ \forall j, e, b.$$

The price for all the adjusted assets $(j, e, b)_{\forall e,b}$ will be $q_{j,e_1,0}(t)$[12]. Then, I have the condition:

$$\sum_{e,b} \int \theta^h_{j,e,b}(t) = 0 \ \ \forall j.$$

Consider any j. There is a natural indeterminacy in the assets in that both adjusted assets (j, e, b) and $(j, e_1, 0)$ provide "equal returns" to the creditors. The adjusted asset $\theta^h_{j,e,b}(t)$ can be thought of as a full commitment asset with price $q_{j,e_1,0}(t)$. Thus, if $\sum_{e,b} \int \theta^h_{j,e,b}(t) = 0$, then an appropriate division of the adjusted asset purchases $\left(\theta^h_{j,e,b}(t)\right)_{\forall h,e,b}$ will yield $\int \theta^h_{j,e,b}(t) = 0 \ \ \forall e, b$.

If $\int \theta^h_{j,e,b}(t) = \int \rho_{j,e,b}(t+1)z^h_{j,e,b}(t) = 0 \ \ \forall j, e, b$, then the desired market clearing condition holds: $\int z^h_{j,e,b}(t) = 0 \ \ \forall j, e, b$.

Finally, consider time period $\bar{T}$. What remains of the last equation from (3.17) is $p(\bar{T}) \left(\int c^h \left(\bar{T}\right) d\Phi - \bar{e} \right) = 0$. As $p(\bar{T}) > 0$, then $\int c^h \left(\bar{T}\right) d\Phi - \bar{e} = 0$.

Proof of Theorem 3.2

[12] A creditor is indifferent between spending $q_{j,e,b}(t)z^h_{j,e,b}(t)$ and $q_{j,e_1,0}(t)\theta^h_{j,e,b}(t) = q_{j,e,b}(t)z^h_{j,e,b}(t)$ on its purchase and also indifferent between the payout $\rho_{j,e,b}(t+1)z^h_{j,e,b}(t)$ and $1 \cdot \theta^h_{j,e,b}(t) = \rho_{j,e,b}(t+1)z^h_{j,e,b}(t)$.

A debtor is indifferent between spending $q_{j,e,b}(t)z^h_{j,e,b}(t)$ and $q_{j,e_1,0}(t)\theta^h_{j,e,b}(t) = q_{j,e,b}(t)z^h_{j,e,b}(t)$ on its sale. From the second equation of (3.24), $\sum_{e,b} \rho_{j,e,b}(t+1) \int \left(z^h_{j,e,b}(t)\right)^+ = -\sum_{e,b} \rho_{j,e,b}(t+1) \int \left(z^h_{j,e,b}(t)\right)^-$. Using the definition of Ψ, namely that equation (3.7) holds, then $\sum_{e,b} \rho_{j,e,b}(t+1) \int \left(z^h_{j,e,b}(t)\right)^- = \sum_{e,b} \int \delta^h(t+1) \left(z^h_{j,e,b}(t)\right)^-$. Thus, for any $z^h_{j,e,b}(t) < 0$, I can equivalently define the adjusted asset holdings as $\theta^h_{j,e,b}(t) = \delta^h(t+1)z^h_{j,e,b}(t)$. Using the payouts as written in (3.9), a debtor is indifferent between the payouts $\delta^h(t+1)z^h_{j,e,b}(t)$ and $1 \cdot \theta^h_{j,e,b}(t) = \delta^h(t+1)z^h_{j,e,b}(t)$.

Equation (3.25) is the tranvsersality condition:

$$\lim_{t\to\infty} \sum\nolimits_{s^{t-1},e^{t-1}} \sum\nolimits_{s(t),e(t)} \lambda^h\left(s^t,e^t\right) \sum_{j,e,b} q_{j,e,b}(t) z^h_{j,e,b}(t) = 0. \tag{3.25}$$

The term $e^t = (e(0), ..., e(t))$ is the income realizations up to time period t. Given the joint sequence of realizations $\left(s^t, e^t\right)$, the Lagrange multiplier in time period t can be written as $\lambda^h\left(s^t, e^t\right)$. This Lagrange multiplier is associated with the budget constraint at t that must be satisfied for household h given the joint realization of aggregate states and incomes.

The transversality condition is a necessary condition of the household's optimization problem (3.3) and is natural in theoretical work. Basically, the condition does not allow the optimal portfolio to "leave value" at infinity. Taking all the contingent budget constraints, multiplying them by their respective Lagrange multipliers, and summing over a truncated time horizon yields the single budget constraint:

$$\sum_{t=0}^{\bar{T}} \sum_{s^t,e^t} \lambda^h\left(s^t,e^t\right)\left(c^h\left(s^t,e^t\right) - e^h(t)\right) + \sum_{s^{\bar{T}},e^{\bar{T}}} \lambda^h\left(s^{\bar{T}},e^{\bar{T}}\right)\left(\sum_{j,e,b} q_{j,e,b}(\bar{T}) z^h_{j,e,b}(\bar{T})\right) = 0.$$

With a finite time horizon $(\bar{T} < \infty)$, the assets $z^h(\bar{T})$ are not available for trade. The single budget constraint above is then equivalent to the Arrow-Debreu budget constraint. Without a final time period $(\bar{T} \to \infty)$, it must be that the second term $\sum_{s^{\bar{T}},e^{\bar{T}}} \lambda^h\left(s^{\bar{T}},e^{\bar{T}}\right)\left(\sum_{j,e,b} q_{j,e,b}(\bar{T}) z^h_{j,e,b}(\bar{T})\right)$ vanishes in the limit.

I will use the transversality condition to rule out Ponzi schemes. A natural thought to have concerning Ponzi schemes in a bankruptcy model is whether it is possible for a household to run up an unboundedly large debt using a Ponzi scheme and then simply expunge the debt by declaring bankruptcy. In this model, such a situation is not possible since the transversality condition must hold for *all* realizations of the aggregate state and income. Consider the decision to initiate a Ponzi scheme at time period τ. It is possible that the realizations (s^t, e^t) will provide incomes for the household in each period such that $e^h(t) > e_{med}$ $\forall t \geq \tau$. With these realizations, the household can never have the debt from the Ponzi scheme erased. Therefore, it is not possible for the household to begin a Ponzi scheme in any period τ as such a scheme would not satisfy the requisite transversality condition (3.25).

The variables $\left(\left(c^h, z^h\right)_{h\in\mathcal{H}}, q\right) = \left(\left(c^h(t), \left(z^h_{j,e,b}(t)\right)_{\forall j,e,b}\right)_{h\in\mathcal{H}}, \left(q_{j,e,b}(t)\right)_{\forall j,e,b}\right)_{\forall t: 0\leq t\leq \bar{T}}$ have been shown within the *proof of theorem 3.1* to be bounded. These bounds on the variables do not depend on the time period nor on the size of $\bar{T}$. Thus, the variables are uniformly bounded. This completes the proof.

Proof of Lemma 3.1

This proof will make use of the following lemma (lemma 3.7). Define γ^* to be the probability that a household will declare bankruptcy next period. The proof of lemma 3.7 follows the completion of the *proof of lemma 3.1.*

Lemma 3.7 *When $\gamma^* > 0$, the asset sale of household h with $e^h(t) = e_1$ is strictly decreasing in γ^*.*

Consider the initial conditions: (i) no households with $b(0) = 0$ and (ii) $\tilde{\theta}_{e_1,1}(0) = \tilde{\theta}_{e_2,1}(0) > \tilde{\theta}_{e_1,2}(0) = \tilde{\theta}_{e_2,2}(0)$. Naturally, all households begin with zero initial wealth. From lemma 3.7, type $k = 1$ households borrow more and declare more often. Thus, $\tilde{\theta}_{e_1,0}(1) > \theta$. There are no households with $b(1) = 1$. As $E\left(\tilde{\theta}_{e_1,b}(1); b\right) = \theta$, then $\tilde{\theta}_{e_1,2}(1) < \theta$. I will prove the lemma by considering three cases.

Case I: If $\tilde{\theta}_{e_1,0}(t) \geq \theta$, then $\tilde{\theta}_{e_1,0}(t+1) \geq \theta$ $\forall t$. Consider any $t \geq 2$. Since $\pi^1 > \pi^2$, then more $k = 1$ survive from $(e_1, 0)$ at t to $(e_1, 1)$ at $t + 1$. Thus, $\tilde{\theta}_{e_1,1}(t + 1) >$

$\tilde{\theta}_{e_1,0}(t) \geq \theta$. Under the condition for case I, $\tilde{\theta}_{e_1,0}(t+1) \geq \theta$. As $E\left(\tilde{\theta}_{e_1,b}(t+1); b\right) = \theta$, then $\tilde{\theta}_{e_1,2}(t+1) < \theta$. Therefore, $\tilde{\theta}_{e_1,1}(t+1) > \tilde{\theta}_{e_1,2}(t+1)$.

For cases II and III, consider any time period t with $\tilde{\theta}_{e_1,0}(t) \geq \theta$ and $\tilde{\theta}_{e_1,0}(t+1) < \theta$. As $E\left(\tilde{\theta}_{e_1,b}(t); b\right) = \theta$, then $E\left(\tilde{\theta}_{e_1,b}(t); b \in \{1,2\}\right) \leq \theta$ and $E\left(\tilde{\theta}_{e_1,b}(t+1); b \in \{1,2\}\right) > \theta$. From lemma 3.7, which states that $k = 1$ declares bankruptcy more than $k = 2$, $\tilde{\theta}_{e_1,0}(t+2) \geq E\left(\tilde{\theta}_{e_1,b}(t+1); b \in \{1,2\}\right) > \theta$.

Case II: Let $\tilde{\theta}_{e_1,0}(t) \geq \theta$. As $\pi^1 > \pi^2$, then $\tilde{\theta}_{e_1,1}(t+1) > \tilde{\theta}_{e_1,0}(t) \geq \theta$. If $\tilde{\theta}_{e_1,2}(t+1) \geq \tilde{\theta}_{e_1,1}(t+1)$, then from lemma 3.7 (more bankruptcy from $k = 1$) $\tilde{\theta}_{e_1,0}(t+2) > \tilde{\theta}_{e_1,1}(t+1)$. Again, as $\pi^1 > \pi^2$, then $\tilde{\theta}_{e_1,1}(t+3) > \tilde{\theta}_{e_1,0}(t+2)$. Continuing, for any τ, the sequence holds:

$$\tilde{\theta}_{e_1,2}(t+2\tau+3) \geq \tilde{\theta}_{e_1,1}(t+2\tau+3) > \tilde{\theta}_{e_1,0}(t+2\tau+2) > \tilde{\theta}_{e_1,1}(t+2\tau+1).$$

In the limit, $E\left(\tilde{\theta}_{e_1,b}(t+2\tau+3); b \in \{1,2\}\right) \rightarrow 1$, a contradiction.

Case III: Let $\tilde{\theta}_{e_1,0}(t+1) < \theta$. As $\pi^1 > \pi^2$, then $\tilde{\theta}_{e_1,1}(t+2) > \tilde{\theta}_{e_1,0}(t+1)$. $\tilde{\theta}_{e_1,0}(t+2) > \theta$ implies $E\left(\tilde{\theta}_{e_1,b}(t+2); b \in \{1,2\}\right) < \theta$. If $\tilde{\theta}_{e_1,2}(t+2) \geq \tilde{\theta}_{e_1,1}(t+2)$, then $\theta > \tilde{\theta}_{e_1,1}(t+2) > \tilde{\theta}_{e_1,0}(t+1)$. From lemma 3.7 (more bankruptcy from $k = 1$) $\tilde{\theta}_{e_1,0}(t+3) > \tilde{\theta}_{e_1,1}(t+2) > \tilde{\theta}_{e_1,0}(t+1)$. Continuing, for any τ, the sequence holds:

$$\theta > \tilde{\theta}_{e_1,0}(t+2\tau+3) > \tilde{\theta}_{e_1,1}(t+2\tau+2) > \tilde{\theta}_{e_1,0}(t+2\tau+1).$$

In the limit, $\tilde{\theta}_{e_1,0}(t+2\tau+1) \rightarrow \theta$ and $\tilde{\theta}_{e_1,1}(t+2\tau+2) > \tilde{\theta}_{e_1,0}(t+2\tau+1)$. This contradicts $\theta > \tilde{\theta}_{e_1,1}(t+2\tau+2)$.

Proof of Lemma 3.7

The probability $\gamma^* > 0$ implies that in time period $t+1$, consumption $c^h(t+1) = e_1 + B^h(t+1) - q(t+1) \cdot z^h(t+1)$. The bankruptcy payout $B^h(t+1)$ does not depend on the asset sale $\left(z^h_{e,b}(t)\right)^-$.

There is the realization in time period $t+1$, with the remaining probability $1-\gamma^*$, such that $c^h(t+1) = e_2 + S^h(t+1) - q(t+1) \cdot z^h(t+1)$. Solvency is required as $e_2 > e_{med}$ and $S^h(t+1)$ contains the term $\left(z^h_{e,b}(t)\right)^-$. This latter consumption $c^h(t+1) = e_2 + S^h(t+1) - q(t+1) \cdot z^h(t+1)$ is greater than the former, $c^h(t+1) = e_1 + S^h(t+1) - q(t+1) \cdot z^h(t+1)$. Define the first as $c^h_{high}(t+1)$ and the second as $c^h_{low}(t+1)$.

For derivatives with respect to $\left(z^h_{e,b}(t)\right)^-$, the Euler equation from the household problem (3.3) is given by:

$$u'\left(c^h(t)\right) = \beta(1-\gamma^*)u'\left(c^h_{high}(t+1)\right).$$

Define the function $F(\gamma^*, \left(z^h_{e,b}(t)\right)^-) = u'\left(c^h(t)\right) - \beta(1-\gamma^*)u'\left(c^h_{high}(t+1)\right) = 0$. Then from the implicit function theorem:

$$\frac{\partial \left(z^h_{e,b}(t)\right)^-}{\partial \gamma^*} = -\frac{\frac{\partial F}{\partial \gamma^*}}{\frac{\partial F}{\partial \left(z^h_{e,b}(t)\right)^-}} = -\frac{\beta u'\left(c^h_{high}(t+1)\right)}{-q_{e,b}(t)u''\left(c^h(t)\right) - \beta(1-\gamma^*)u''\left(c^h_{high}(t+1)\right)}.$$

This equation is obtained as $\frac{\partial c^h(t)}{\partial \left(z^h_{e,b}(t)\right)^-} = -q_{e,b}(t)$ and $\frac{\partial c^h_{high}(t+1)}{\partial \left(z^h_{e,b}(t)\right)^-} = 1$. Let's consider the signs of the terms in $\frac{\partial \left(z^h_{e,b}(t)\right)^-}{\partial \gamma^*}$:

- $u'\left(c^h_{high}(t+1)\right) > 0$ as $u(\cdot)$ is strictly increasing.
- $-q_{e,b}(t)u''\left(c^h(t)\right) > 0$ as $u(\cdot)$ is strictly concave.
- $-\beta(1-\gamma^*)u''\left(c^h_{high}(t+1)\right) > 0$ as $u(\cdot)$ is strictly concave.

This completes the argument that $\frac{\partial\left(z^h_{e,b}(t)\right)^-}{\partial\gamma^*} < 0$.

Proof of Theorem 3.4

To prove the result $q_{e_1,1}(t) < q_{e_1,2}(t)\ \forall t$, I must prove that the overall repayment rates satisfy $\rho_{e_1,1}(t{+}1) < \rho_{e_1,2}(t{+}1)\ \forall t$. Define $\sigma^k_{e,b}(t{+}1)$ as the conditional repayment rate in $t+1$ of debtors of type k in pool (e,b) given that the household is eligible to declare. Define $\bar{z}^k_{e,b}(t)$ as the mean asset choice $z^h_{e,b}(t) < 0$ of all type k debtors in pool (e,b). By definition, the overall repayment rate for each pool (e,b) is:

$$\rho_{e,b}(t{+}1) = 1 - \frac{\tilde{\theta}_{e,b}(t)\left(\pi^1\left(1-\sigma^1_{e,b}(t+1)\right)\right)\bar{z}^1_{e,b}(t) + \left(1-\tilde{\theta}_{e,b}(t)\right)\left(\pi^2\left(1-\sigma^2_{e,b}(t+1)\right)\right)\bar{z}^2_{e,b}(t)}{\tilde{\theta}_{e,b}(t)\bar{z}^1_{e,b}(t) + \left(1-\tilde{\theta}_{e,b}(t)\right)\bar{z}^2_{e,b}(t)}. \tag{3.26}$$

Case I: $\pi^2 \to 0$ and $\pi^1 \in (0,1)$

From section 3.3, hypothetically if $\pi^1 = \pi^2$, then the bankruptcy equilibrium is such that all eligible debtors on asset $(e_1,1)$ declare bankruptcy. Thus, in this case with $\pi^1 > \pi^2$, all $k=1$ eligible households will declare from pool $(e_1,1)$. The conditional repayment rates are $\left(\sigma^1_{e_1,1}(t+1), \sigma^2_{e_1,1}(t+1)\right) = (0,1)$ and $\left(\sigma^1_{e_1,2}(t+1), \sigma^2_{e_1,2}(t+1)\right) = (0,1)$. Using equation (3.26) yields:

$$\begin{aligned}
\rho_{e_1,1}(t+1) &= 1 - \frac{\pi^1\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^2_{e_1,1}(t)} \\
\rho_{e_1,2}(t+1) &= 1 - \frac{\pi^1\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}.
\end{aligned}$$

As $\tilde{\theta}_{e_1,1}(t) \to 1$ and $\tilde{\theta}_{e_1,2}(t)$ is uniformly bounded below 1, then $\rho_{e_1,1}(t+1) \to 1-\pi^1$ and $\rho_{e_1,2}(t+1) > 1-\pi^1$.

Case II: $\pi^1 \to 1$ and $\pi^2 \leq \theta$

The conditional repayment rates are $\sigma^1_{e_1,1}(t+1) = 0$, $\sigma^2_{e_1,1}(t+1) \leq 1$, $\sigma^1_{e_1,2}(t+1) = 0$, and $\sigma^2_{e_1,2}(t+1) \geq 0$. Using equation (3.26) yields:

$$\begin{aligned}
\rho_{e_1,1}(t+1) &\leq 1 - \frac{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^2_{e_1,1}(t)} \\
\rho_{e_1,2}(t+1) &\geq 1 - \frac{\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \pi^2\left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}
\end{aligned}$$

Under $\pi^1 \to 1$, then $\tilde{\theta}_{e_1,2}(t) \to 0$. Thus,

$$\rho_{e_1,2}(t+1) \geq 1 - \pi^2.$$

Using lemma 3.7, $\left|\bar{z}^1_{e_1,1}(t)\right| > \left|\bar{z}^2_{e_1,1}(t)\right|$, so then

$$\frac{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^2_{e_1,1}(t)} > \frac{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^1_{e_1,1}(t)} = \tilde{\theta}_{e_1,1}(t).$$

With $\pi^1 \to 1$, then $\tilde{\theta}_{e_1,1}(t) > \theta$. All told,

$$\rho_{e_1,1}(t+1) < 1 - \theta.$$

The assumption $\pi^2 \leq \theta$ finishes the case.

Using the continuity of the household's asset choices as a function of (π^1, π^2), then $\exists \Delta > 0$ so that as long as $\pi^1 \geq \pi^2 + \Delta$, the overall repayment rates are ordered $\rho_{e_1,1}(t+1) < \rho_{e_1,2}(t+1)$ and this holds for every time period $t+1$.

Proof of Lemma 3.2

I will prove that these are sufficient conditions by showing that they imply the inequality $\rho_{e_1,1}(t+1) < \rho_{e_1,2}(t+1)$ for every time period $t+1$. Recall the conditions are:

1. Suppose that in equilibrium, type $k=1$ eligible households in pool $(e_1,1)$ at t will declare bankruptcy at $t+1$.

2. Suppose further that type $k=2$ eligible households in pool $(e_1,2)$ at t will not declare bankruptcy at $t+1$ unless they were also in pool $(e_1,2)$ at $t-1$.

The conditional repayment rates are $\left(\sigma^1_{e_1,1}(t+1), \sigma^2_{e_1,1}(t+1)\right) = (0,1)$ for pool $(e_1,1)$. Out of all the $k=2$ households in pool $(e_1,2)$, at least $\frac{1}{1+\pi^2}$ of them will not declare. These are the households who were *not* in the pool $(e_1,2)$ at $t-1$. Thus, the conditional repayment rates are $\left(\sigma^1_{e_1,1}(t+1) = 0, \sigma^2_{e_1,1}(t+1) \geq \frac{1}{1+\pi^2}\right)$.

From equation (3.26) :

$$\begin{aligned}
\rho_{e_1,1}(t+1) &= 1 - \frac{\pi^1\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^2_{e_1,1}(t)} \\
\rho_{e_1,2}(t+1) &\geq 1 - \frac{\pi^1\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}.
\end{aligned}$$

Thus, lemma 3.2 is complete upon showing the following inequality:

$$\frac{\pi^1\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^2_{e_1,1}(t)} > \frac{\pi^1\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}. \tag{3.27}$$

The left-hand side of inequality (3.27) is equal to the product:

$$\pi^1\tilde{\theta}_{e_1,1}(t) \cdot \frac{\bar{z}^1_{e_1,1}(t)}{\tilde{\theta}_{e_1,1}(t)\bar{z}^1_{e_1,1}(t) + \left(1-\tilde{\theta}_{e_1,1}(t)\right)\bar{z}^2_{e_1,1}(t)}. \tag{3.28}$$

Define the convex combination $\eta\bar{z}^1_{e_1,1}(t) + (1-\eta)\bar{z}^2_{e_1,1}(t)$ as $\bar{z}_{e_1,1}(t;\eta)$ for any value of $\eta \in [0,1]$. Then (3.28) is given by:

$$\pi^1\tilde{\theta}_{e_1,1}(t) \cdot \frac{\bar{z}_{e_1,1}(t;1)}{\bar{z}_{e_1,1}(t;\tilde{\theta}_{e_1,1}(t))}.$$

Define the two terms separately as:

$$\begin{aligned} A &= \pi^1\tilde{\theta}_{e_1,1}(t) \geq 0. \\ B &= \frac{\bar{z}_{e_1,1}(t;1)}{\bar{z}_{e_1,1}(t;\tilde{\theta}_{e_1,1}(t))} \geq 0. \end{aligned}$$

The right-hand side of inequality (3.27) is equal to the product:

$$\left[\pi^1\tilde{\theta}_{e_1,2}(t) + \pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)\right] \cdot \frac{\bar{z}^*_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)} \tag{3.29}$$

where $\bar{z}^*_{e_1,2}(t)$ is defined by:

$$\bar{z}^*_{e_1,2}(t) = \frac{\pi^1\tilde{\theta}_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)}{\pi^1\tilde{\theta}_{e_1,2}(t) + \pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)}.$$

This can be rewritten as a convex combination

$$\bar{z}^*_{e_1,2}(t) = \eta_{e_1,2}(t)\bar{z}^1_{e_1,2}(t) + \left(1-\eta_{e_1,2}(t)\right)\bar{z}^2_{e_1,2}(t)$$

where $\eta_{e_1,2}(t) = \frac{\pi^1\tilde{\theta}_{e_1,2}(t)}{\pi^1\tilde{\theta}_{e_1,2}(t)+\pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)}$. Using the notation for convex combination, then (3.29) is given by:

$$\left[\pi^1\tilde{\theta}_{e_1,2}(t) + \pi^2\frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)\right] \cdot \frac{\bar{z}_{e_1,1}(t;\eta_{e_1,2}(t))}{\bar{z}_{e_1,1}(t;\tilde{\theta}_{e_1,2}(t))}.$$

Define the two terms separately as:

$$\begin{aligned} C &= \pi^1 \tilde{\theta}_{e_1,2}(t) + \pi^2 \frac{\pi^2}{1+\pi^2}\left(1 - \tilde{\theta}_{e_1,2}(t)\right) \geq 0. \\ D &= \frac{\bar{z}_{e_1,1}(t; \eta_{e_1,2}(t))}{\bar{z}_{e_1,1}(t; \tilde{\theta}_{e_1,2}(t))} \geq 0. \end{aligned}$$

The proof is complete upon showing that $AB > CD$. Households $k = 1$ declare bankruptcy from pool $(e_1, 1)$ and $k = 2$ do not. From lemma 3.7, then $\frac{\bar{z}^1_{e_1,2}(t)}{\bar{z}^2_{e_1,2}(t)} < \frac{\bar{z}^1_{e_1,1}(t)}{\bar{z}^2_{e_1,1}(t)}$ and $\left|\bar{z}^1_{e_1,1}(t)\right| > \left|\bar{z}^2_{e_1,1}(t)\right|$. Define two additional terms:

$$\begin{aligned} B' &= \frac{1}{\tilde{\theta}_{e_1,1}(t)}. \\ D' &= \frac{\eta_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)}. \end{aligned}$$

With the two facts from lemma 3.7, then $B' \geq \kappa D' \geq 0$ for some $\kappa \in \mathbb{R}_+$ implies $B > \kappa D \geq 0$. This implication is an immediate corollary to the footnote.[13] With $A \geq 0$ and $C \geq 0$, in order to prove $AB > CD$, it suffices to show $AB' \geq CD'$.

The first term, AB', is given by:

$$AB' = \pi^1.$$

The second term, CD', is given by:

$$CD' = \left[\pi^1 \tilde{\theta}_{e_1,2}(t) + \pi^2 \frac{\pi^2}{1+\pi^2}\left(1 - \tilde{\theta}_{e_1,2}(t)\right)\right] \frac{\eta_{e_1,2}(t)}{\tilde{\theta}_{e_1,2}(t)}.$$

[13] I prove that $\frac{e}{f} > \kappa \frac{x}{y} > 0$, $e > f > 0$, and $\frac{1}{\theta_1} \geq \frac{1}{\theta_2} > 0$ implies the inequality $(*) : \frac{e}{\theta_1 e + (1-\theta_1) f} > \kappa \frac{x}{\theta_2 x + (1-\theta_2) y} > 0$. Define $\hat{x} = \kappa x$. Dividing the left-hand side by e and the right-hand side by $\hat{x}$ yields that $(*)$ holds iff $\theta_1 + (1-\theta_1)\frac{f}{e} < \theta_2 + (1-\theta_2)\frac{y}{\hat{x}}$. As $\frac{y}{\hat{x}} > \frac{f}{e}$, it suffices to show $\theta_1 + (1-\theta_1)\frac{f}{e} \leq \theta_2 + (1-\theta_2)\frac{f}{e}$. Both sides are convex combinations of 1 and $\frac{f}{e} < 1$. The inequality holds as $\theta_1 \leq \theta_2$.

Using the definition of $\eta_{e_1,2}(t) = \frac{\pi^1 \tilde{\theta}_{e_1,2}(t)}{\pi^1 \tilde{\theta}_{e_1,2}(t) + \pi^2 \frac{\pi^2}{1+\pi^2}\left(1-\tilde{\theta}_{e_1,2}(t)\right)}$, then

$$CD' = \left[\pi^1 \tilde{\theta}_{e_1,2}(t) + \pi^2 \frac{\pi^2}{1+\pi^2}\left(1 - \tilde{\theta}_{e_1,2}(t)\right)\right] \left[\frac{\pi^1}{\pi^1 \tilde{\theta}_{e_1,2}(t) + \pi^2 \frac{\pi^2}{1+\pi^2}\left(1 - \tilde{\theta}_{e_1,2}(t)\right)}\right] = \pi^1.$$

Therefore, the inequality $AB' \geq CD'$ is trivially satisfied. This finished the *proof of lemma 3.2*. Notice that the proof is valid for any values of $\theta \in [0,1]$ and $\pi^1 > \pi^2$.

3.7 References

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Chapter 4

Transaction Costs and Planner Intervention

4.1 Introduction

Transaction costs are pervasive in financial markets, both those in the real world and those studied in economic models. Some transaction costs are measurable and apparent such as a tax imposed by the government on the trade of an asset. Other transaction costs are unmeasurable, but are the accepted explanation for why beneficial trade does not occur. The models of derivative asset pricing rely on transaction costs to justify the pricing of an otherwise redundant asset. In other financial models, the holding of an asset is divided into purchases and sales. Without transaction costs, these two variables would be indeterminate.

Recently, interest in explaining the emergence of transaction costs has arisen. One explanation for transaction costs is that they emerge because a financial intermediary is required to facilitate asset trade. This intermediary must be compensated a market wage for the labor required to produce such a service. The most recent work that corrects many of the shortfalls of the previous literature is Martins-da-Rocha and Vailakis (2010). My work does not attempt to explain why transaction costs emerge, rather it studies conditions under which an adjustment of these transaction costs can improve market welfare. The welfare criterion that will be used in this paper will be the Pareto criterion.

To illustrate the normative ramifications of transaction costs, suppose that enough assets exist to span all states of uncertainty. If transaction costs were removed from the model, then all households would perfectly insure against future risk by trading financial assets. As a result, the equilibrium allocation would be Pareto optimal, meaning that a planner cannot intervene and make some households better off without making others worse off. However, with transaction costs, the equilibrium allocation is inefficient and there is justification for planner intervention.

The planner will intervene by scaling the transaction costs either up or down. The underlying shape of the transaction costs functions will remain the same. The intervention must satisfy fiscal balance, meaning that the value of all transactions costs will be identical before and after the intervention. If the transaction costs are taxes, this statement says that the planner's tax reform must be revenue neutral, that is, the tax revenue collected cannot change. The main result states that over a generic subset of parameters and subject to an upper bound on the number of households, there exists an open set of planner interventions that lead to a Pareto superior allocation. As an immediate corollary, consider an intervention within this open set of planner interventions. The planner could then increase all transaction costs locally, still make all households better off, and keep a small profit for itself.

Recent papers by Citanna et al. (2006) and del Mercato and Villanacci (2006) analyze the normative impact of a government tax policy. Both papers, although each in a different setup, arrive at the same conclusion. That conclusion is that with an incomplete markets setup, for a generic subset of endowments and utilities, the introduction of a tax can be Pareto improving. My result differs from both works in two key aspects. First, I focus entirely on one friction (transaction costs) and do not require an incomplete markets setting. Second, these two papers prove that for an economy without tax frictions, introducing taxes to redistribute wealth will lead to a Pareto improvement. However, tax frictions must certainly be present in any economy before a government can redistribute wealth. I prove the regularity of a transaction costs equilibrium, an equilibrium in which tax frictions are already present, and then prove my generic planner intervention result given that equilibrium.

My paper is a descendant of the works by Cass and Citanna (1998) and Elul (1995) that questioned whether financial innovation is always welfare improving. In a setting of incomplete financial markets, both of the above papers prove that there is an open set of payoffs for the new assets (under additional dimensional restrictions) such that the introduction of this new asset actually makes all households worse off. As governments are not in the business of creating new assets, I claim that it is more interesting to study the planner adjustments of transaction costs (taxes), a frequent action performed by governments. The framework used to prove the Cass and Citanna (1998) result, the Citanna et al. (2006) result, the del Mercato and Villanacci (2006) result, and the result presented in this paper was developed by Citanna, Kajii, and Villanacci (1998).

This paper is organized into three remaining sections. In section 4.2, I introduce the general equilibrium model with transaction costs in the financial markets and define an equilibrium. In section 4.3, I state and prove the main result of this paper. In section 4.4, I provide the proofs of two lemmas stated in section 4.2.

4.2 The Model

Consider a 2 period general equilibrium model with S states of uncertainty in the second time period. Denoting the first period as the $s = 0$ state, I will number the states as $s \in \mathcal{S} = \{0, ..., S\}$. At each state, $H \geq 2$ households trade and consume $L \geq 2$ physical commodities. There are a finite number of both households and physical commodities with $h \in \mathcal{H} = \{1, .., H\}$. The commodities are denoted by the variable x. Define the total number of goods as $G = L(S + 1)$ and then the consumption set is the entire nonnegative orthant, $x^h \in \mathbb{R}^G_+ \ \ \forall h \in \mathcal{H}$. Concerning notation, the vector $x \in \mathbb{R}^{HG}_+$ contains the consumptions for all households, the vector $x^h(s) \in \mathbb{R}^L_+$ contains the consumption by household h in state s (of all commodities), and the scalar $x^h_l(s) \in \mathbb{R}_+$ is the consumption by household h of the good (s, l) or the l^{th} physical commodity in state s.

Households are endowed with commodities in all states. These endowments are denoted by e. I assume that all households have strictly positive endowments:

Assumption 1 $\quad e^h >> 0 \ \ \forall h \in \mathcal{H}$.[1]

In addition to endowments, the household primitives include the utility functions $u^h : \mathbb{R}^G_+ \rightarrow \mathbb{R}$ subject to the following assumptions:

Assumption 2 $\quad u^h$ is C^3, differentiably strictly increasing (i.e., $Du^h(x^h) >> 0$ $\forall x^h \in \mathbb{R}^G_+$), differentiably strictly concave (i.e., $D^2u^h(x^h)$ is negative definite $\forall x^h \in \mathbb{R}^G_{++}$), and satisfies the boundary condition ($clU^h(x^h) \subset \mathbb{R}^G_{++}$ where $U^h(x^h) = \{x' \in \mathbb{R}^G_{++} : u^h(x') \geq u^h(x^h)\}$) $\forall h \in \mathcal{H}$.

Define the commodity prices as $p \in \mathbb{R}^G \backslash \{0\}$. Under assumption 2, the prices satisfy $p >> 0$. Of all the physical commodities in each state, the final one $(l = L)$

[1] The notation $e^h >> 0$ means that $e^h_l(s) > 0 \ \ \forall (l, s)$.

is called the numeraire commodity, meaning that all other commodities are priced relative to this one. For simplicity, I normalize the price of the numeraire commodity $p_L(s) = 1$ in every state $s \in \mathcal{S}$.

The commodities are perishable, so the households require financial markets to transfer wealth between states. I assume that there are J assets ($J \leq S$). These assets are numeraire assets meaning that the payout of each asset is in terms of the numeraire commodity $l = L$. The payouts are assumed to be nonnegative and are collected in the $S \times J$ yields matrix Y :

$$Y = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & ... & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}.$$

To get the payoff in terms of the unit of account, I make the preserving transformation

$$Y = \begin{bmatrix} p_L(1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & p_L(S) \end{bmatrix} \cdot \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & ... & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}.$$

Summarizing what I said so far concerning the parameter Y :

Assumption 3 Y is a nonnegative and full rank $S \times J$ yields matrix.

The assets are in zero net supply and are denoted by the variable θ. As with consumption, $\theta^h \in \mathbb{R}^J$ is the portfolio held by an individual household h, $\theta \in \mathbb{R}^{HJ}$ are the portfolios of all households, and $\theta_j^h \in \mathbb{R}$ is the amount of asset j held by household h.

For each asset j, there exists an asset price $q_j \in \mathbb{R}$, which can be viewed as the payoff of the asset (in terms of the unit of account) in state $s = 0$. Combining the endogenous asset prices with the exogenous payouts, I will represent the overall returns of the financial markets in the $(S+1)\times J$ returns matrix R :

$$R=\begin{bmatrix} -q \\ Y \end{bmatrix}.$$

I will model the transaction costs as costs imposed on the trade of financial assets. Initially, all households have zero asset holdings. Upon trading assets, households must pay the real transaction costs. The transaction costs for the entire portfolio (paid in the unit of account) will be determined by the mapping

$$\begin{aligned} F^h &: \mathbb{R}^J \to \mathbb{R}_+ \\ F^h\left(\theta^h\right) &= \sum\nolimits_j q_j \cdot f_j^h\left(\theta^h\right). \end{aligned}$$

For any portfolio θ^h, the value $F^h\left(\theta^h\right)$ is the value that must be paid as transaction costs. The transaction costs are nonnegative. The transaction costs depend (linearly) upon the asset price level. This is natural since the transaction costs will represent a loss of some physical amount of assets. The transaction costs are heterogeneous across households.

The mapping f_j^h is the actual physical amount of asset j that must be paid as transaction costs. These costs (in terms of asset j) will depend not only on the position of asset j, θ_j^h, but also on the position of the other assets θ_{-j}^h. The transaction costs mappings (parameters of the model) are given by the vector-valued function $f^h=\left(f_1^h,..,f_J^h\right)$ where

$$f_j^h:\mathbb{R}^J\to\mathbb{R}_+\ \ \forall j.$$

I impose the following assumptions on f_j^h.

Assumption 4 f_j^h is C^3, differentiably strictly convex in θ_j^h, and satisfies $f_j^h(\theta) = 0$ for any $\theta \in \mathbb{R}^J$ with $\theta_j = 0\ \forall h$ and $\forall j$. By strict convexity in θ_j^h, I mean that $a^T D^2 f_j^h\left(\theta^h\right) a \geq 0\ \ \forall a$ with strict inequality if $a_j \neq 0$ and $\theta_j^h \neq 0\ \ \forall j$.

Claim 4.1 *Given $q >> 0$, F^h is C^3, differentiably strictly convex, and satisfies $F^h(0) = 0$.*

Proof. The first and the last are obvious. For the second, note that the $J \times J$ Hessian for F^h can be equivalently written as:

$$D^2 F^h\left(\theta^h\right) = \sum\nolimits_j q_j \cdot D^2 f_j^h\left(\theta^h\right)$$

where q_j is a scalar multiplier of the Hessians $D^2 f_j^h\left(\theta^h\right)$. With $q >> 0$, since $a^T D^2 f_j^h\left(\theta^h\right) a \geq 0\ \ \forall a$, then $a^T D^2 F^h\left(\theta^h\right) a \geq 0\ \ \forall a$. For strict inequality, if $\theta_j^h \neq 0\ \ \forall j$ and since $a^T D^2 f_j^h\left(\theta^h\right) a > 0\ \ \forall a$ with $a_j \neq 0$ and this holds $\forall j$, then $a^T D^2 F^h\left(\theta^h\right) a > 0\ \ \forall a \neq 0$. ■

Though this paper does not offer an explanation for why the transaction costs are strictly convex, the recent work by Martins-da-Rocha and Vailakis (2009) may shed some light on the question. Their work models transaction costs as an endogenous result of the labor that must be input to produce financial intermediation. The labor to intermediate a financial transaction can be supplied by any of the households in the economy (pure competition). The production set for intermediation needs to be convex. Further, households receive a convex disutility from labor. As a result, the equilibrium transaction costs for a portfolio $\theta^h \in \mathbb{R}^J$ are a function of the utility loss from providing the labor necessary to intermediate θ^h. Martins-da-Rocha and

Vailakis implement a linear transaction costs structure (a constant commission paid by households on all asset trades). With a linear transaction costs structure, there is only one variable to endogenize: the slope. However, with a convex production set and convex disutility, the intuitive idea (though harder to implement) is that the per-unit transaction costs will strictly increase with the size of the trade. This would endogenously generate the strict convexity of transaction costs that I assume in my model.

Define the **canonical representation** for the transaction costs mappings as that specification in which the transaction costs are independent across assets. In this case, f_j^h is only a function of θ_j^h and $D^2F^h\left(\theta^h\right)$ is a positive definite, diagonal matrix.

The transaction costs are paid in terms of the numeraire assets and can be likened to a sieve which collects a certain percentage of the total asset trade. Since the assets are numeraire and pay out in the real physical commodity $l = L$, the transaction costs have a real effect in that the sieve is removing the commodities $l = L$ from the total resources of the economy. A *transaction costs equilibrium* is thus defined as follows.

Definition 4.1 $\left(\left(x^h, \theta^h\right)_{h\in\mathcal{H}}, p, q\right)$ *is a transaction costs equilibrium if*

1. *$\forall h \in \mathcal{H}$, given (p, q),*

   $\left(x^h, \theta^h\right)$ is an optimal solution to the household's maximization problem

$$\begin{array}{ll} \max\limits_{x\geq 0,\theta} & u^h(x) \\ subj\ to & p(0)(e^h(0)-x(0)) - q\theta - \sum_j q_j \cdot f_j^h(\theta) \geq 0 \\ \forall s>0 & p(s)(e^h(s)-x(s)) + \sum_j r_j(s)\theta_j \geq 0 \end{array} \qquad \text{(HP)}$$

2. Markets Clear

$$\begin{aligned}
&\textstyle\sum_h x_l^h(s) = \sum_h e_l^h(s) \quad \forall (l,s) \notin \{(L,1),..,(L,S)\}. \\
&\textstyle\sum_h x_L^h(s) = \sum_h e_L^h(s) + \sum_h \sum_j r_j(s) \cdot \theta_j^h \quad \forall s > 0. \\
&\textstyle\sum_h \theta_j^h + \sum_h f_j^h\left(\theta^h\right) = 0 \quad \forall j.
\end{aligned}$$

The existence of such an equilibrium is well-known and hence its proof is omitted.

The total financial payout in $s = 0$ for some asset j including both the asset price and the transactions costs is given by $-q_j \cdot \tilde{f}_j^h\left(\theta^h\right)$ where

$$\begin{aligned}
\tilde{f}_j^h &: \Theta_E \rightarrow \tilde{f}_j^h\left(\Theta_E\right) \\
\tilde{f}_j^h\left(\theta^h\right) &= \theta_j^h + f_j^h\left(\theta^h\right).
\end{aligned}$$

Θ_E is the set containing all potential equilibrium portfolios (E for equilibrium), that is, assets that satisfy household optimization and market clearing. So far nothing I have said indicates that $\Theta_E \neq \mathbb{R}^J$, but claims 4.2 and 4.3 will do just that. By construction $\tilde{f}_j^h$ satisfies the conditions of assumption 4. Let $\tilde{f}^h = \left(\tilde{f}_1^h, .., \tilde{f}_J^h\right)$ be the Cartesian product of $\left(\tilde{f}_j^h\right)_{\forall j}$ with $\tilde{f}^h : \Theta_E \rightarrow \tilde{f}^h\left(\Theta_E\right)$.

Claim 4.2 *In equilibrium,* $q \cdot D\tilde{f}^h\left(\theta^h\right) >> 0 \ \ \forall \theta^h \in \Theta_E$.

Proof. The following are the first order conditions of the household's problem (HP)

with respect to θ^h where λ^h are the Lagrange multipliers:

$$\lambda^h \begin{pmatrix} -\left(q_1,..,q_J\right)\cdot D\tilde{f}^h\left(\theta^h\right) \\ r(1) \\ : \\ r(S) \end{pmatrix} = 0_{1\times J}. \tag{4.1}$$

This is best seen as the (j,k) element of $D\tilde{f}^h\left(\theta^h\right)$ is $\frac{\partial \tilde{f}^h_j\left(\theta^h\right)}{\partial \theta_k}$ and the first order condition for any one asset θ^h_k is given as:

$$\lambda^h \begin{pmatrix} -\sum_j q_j \frac{\partial \tilde{f}^h_j\left(\theta^h\right)}{\partial \theta_k} \\ r_k(1) \\ : \\ r_k(S) \end{pmatrix} = 0. \tag{4.2}$$

From (4.2) with $\sum_{s>0} \lambda^h(s) r_k(s) > 0$ and $\lambda^h(0) > 0$, then $-\sum_j q_j \frac{\partial \tilde{f}^h_j\left(\theta^h\right)}{\partial \theta_k} < 0 \ \ \forall k$. This finishes the proof. ■

Take the canonical representation for the transaction costs in which f^h_j is only a function of θ^h_j. Then $\tilde{f}^h_j$ is only a function of θ^h_j and $D\tilde{f}^h\left(\theta^h\right)$ is a diagonal matrix.

Claim 4.3 *Under the canonical representation, equilibrium conditions imply $q >> 0$ and $\tilde{f}^h : \Theta_E \to \tilde{f}^h\left(\Theta_E\right)$ is an invertible function.*

Proof. From the previous claim, $q \cdot D\tilde{f}^h\left(\theta^h\right) >> 0 \ \ \forall \theta^h \in \Theta_E$. Under the canonical representation, $D\tilde{f}^h\left(\theta^h\right)$ is a diagonal matrix. Thus, if I can show that $q >> 0$, then $q \cdot D\tilde{f}^h\left(\theta^h\right) >> 0$ implies that $D\tilde{f}^h\left(\theta^h\right)$ has strictly positive diagonal elements for all $\theta^h \in \Theta_E$. Applying the Inverse Function Theorem would yield that $\tilde{f}^h$ is an invertible function.

Consider any asset j and suppose for contradiction that $q_j \leq 0$. Then $q_j < 0$ and $\frac{\partial \tilde{f}_j^h(\theta^h)}{\partial \theta_j} < 0 \;\; \forall h$ from (4.2). By the definition of

$$\frac{\partial \tilde{f}_j^h(\theta^h)}{\partial \theta_j} = 1 + Df_j^h\left(\theta_j^h\right) < 0,$$

then $Df_j^h\left(\theta_j^h\right) < -1$. Since $f_j^h : \mathbb{R} \to \mathbb{R}_+$ has the global minimum at $\theta_j^h = 0$, then $\theta_j^h < 0$. From the market clearing condition:

$$\textstyle\sum_h \tilde{f}_j^h\left(\theta_j^h\right) = 0,$$

there exists some households such that $\tilde{f}_j^h\left(\theta_j^h\right) \leq 0$. For these households, the financial payout in state $s = 0$ is given by

$$-q_j \cdot \tilde{f}_j^h\left(\theta_j^h\right) \leq 0$$

and the payout in states $s > 0$ is given by

$$\begin{pmatrix} \vdots \\ r_j(s)\theta_j^h \\ \vdots \end{pmatrix} < 0.$$

As a result, these households are not optimizing as $\theta_j^h = 0$ is affordable and strictly preferred. Thus $q_j < 0$ and $\frac{\partial \tilde{f}_j^h(\theta^h)}{\partial \theta_j} < 0$ cannot be an equilibrium outcome for any household h. ■

To proceed, I will need to use the inverse function of $\tilde{f}^h : \Theta_E \to \tilde{f}^h\left(\Theta_E\right)$. Under the canonical representation, $\tilde{f}^h$ is invertible. Without the canonical representation, $\tilde{f}^h$ may not be invertible. I will return to this point in lemma 4.2. For now, I state the results conditional on $\tilde{f}^h$ being an invertible function.

Claim 4.4 *If $\tilde{f}^h : \Theta_E \to \tilde{f}^h(\Theta_E)$ is an invertible mapping and $Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ is a nonnegative matrix for all equilibrium θ^h, then $q >> 0$.*[2]

Proof. Since $\tilde{f}^h$ is invertible, the matrix $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ has full rank. Thus $Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ is a full rank matrix. From the first order conditions given in (4.1) :

$$qD\tilde{f}^h\left(\theta^h\right) = \frac{\left(\lambda^h(1), .., \lambda^h(S)\right)}{\lambda^h(0)} \cdot Y.$$

Thus, the asset prices q are given by:

$$q = \frac{\left(\lambda^h(1), .., \lambda^h(S)\right)}{\lambda^h(0)} \cdot Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}.$$

Since $Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ is a nonnegative, full rank matrix, there exists at least one strictly positive element in each column. As $\lambda^h >> 0$, then $q >> 0$. ■

I will define the new asset variable $\eta^h \in \mathbb{R}^J$ such that

$$\eta^h = \tilde{f}^h\left(\theta^h\right) \quad \text{or } \eta_j^h = \tilde{f}_j^h\left(\theta^h\right) \quad \forall j.$$

If $\tilde{f}^h : \Theta_E \to \tilde{f}^h(\Theta_E)$ is invertible, then $\exists g^h : \tilde{f}^h(\Theta_E) \to \Theta_E$ such that

$$\begin{aligned} g^h &= \left(\tilde{f}^h\right)^{-1} \\ g^h\left(\eta^h\right) &= \theta^h. \end{aligned}$$

[2]Since Y is nonnegative by assumption 3 and under the canonical representation $D\tilde{f}^h\left(\theta^h\right)$ is a strictly positive, diagonal matrix, then for an open set of matrices around the canonical representation, $Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ is nonnegative. For this open set, the nonnegativity assumption in claim 4.4 need not be stated.

The vector $g^h = \left(g_1^h, .., g_J^h\right)$ is such that $g_j^h : \tilde{f}^h\left(\Theta_E\right) \to \mathbb{R}$ is C^3 $\forall j$. Further, if $\eta_j^h \neq 0$ $\forall j$, then $\theta_j^h \neq 0$ $\forall j$ since $\eta_j^h = \tilde{f}_j^h\left(\theta^h\right) = \theta_j^h + f_j^h\left(\theta^h\right)$.

With this alternative asset, I will redefine a transaction costs equilibrium.

Definition 4.2 $\left(\left(x^h, \eta^h\right)_{h\in\mathcal{H}}, p, q\right)$ *is a* $\beta-$*transaction costs equilibrium if*

1. $\forall h \in \mathcal{H}$, *given* (p, q),

   $\left(x^h, \eta^h\right)$ *is an optimal solution to the household's maximization problem*

$$\begin{array}{ccc} \max\limits_{x\geq 0,\eta} & u^h(x) & \\ subj\ to & p(0)(e^h(0)-x(0)) - q\eta \geq 0 & . \\ \forall s>0 & p(s)(e^h(s)-x(s)) + \sum_j r_j(s) g_j^h(\eta) \geq 0 & \end{array} \qquad \text{(HP2)}$$

2. *Markets Clear*

$$\begin{array}{l} \sum_h x_l^h(s) = \sum_h e_l^h(s) \quad \forall (l,s) \notin \{(L,1), .., (L,S)\}. \\ \sum_h x_L^h(s) = \sum_h e_L^h(s) + \sum_h \sum_j r_j(s) g_j^h\left(\eta^h\right) \quad \forall s>0. \\ \sum_h \eta_j^h = 0 \quad \forall j. \end{array}$$

Define the total financial payout in each state $s > 0$ as the function

$$\begin{aligned} G_s^h &: \mathbb{R}^J \to \mathbb{R} \\ G_s^h\left(\eta^h\right) &= \sum\nolimits_j r_j(s) \cdot g_j^h\left(\eta^h\right). \end{aligned}$$

Then $G^h : \mathbb{R}^J \to \mathbb{R}^S$ defined as the Cartesian product $G^h = \left(G_1^h, ..., G_S^h\right)$ is given equivalently by:

$$G^h\left(\eta^h\right) = Y \cdot \begin{pmatrix} g_1^h(\eta^h) \\ \vdots \\ g_J^h(\eta^h) \end{pmatrix}$$

where $\begin{pmatrix} g_1^h(\eta^h) \\ \vdots \\ g_J^h(\eta^h) \end{pmatrix} = g^h\left(\eta^h\right)$. Thus, the derivative of $G^h\left(\eta^h\right)$ (an $S \times J$ matrix) is given by:

$$DG^h\left(\eta^h\right) = Y \cdot Dg^h\left(\eta^h\right).$$

$Dg^h\left(\eta^h\right)$ has full rank and so $DG^h\left(\eta^h\right)$ has full column rank.

Define the $(S+1) \times G$ price matrix

$$P = \begin{bmatrix} p(0) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & p(S) \end{bmatrix}.$$

I can characterize the $\beta-$*transaction costs equilibria* with a system of equations Φ. Define $n = H(G+J+S+1)+J+G-(S+1)$ as the number of variables. Given parameters $\sigma = \left(e^h, u^h, f^h\right)_{h\in\ \mathcal{H}}$, the variables $\xi = \left(\left(x^h, \lambda^h, \eta^h\right)_{h\in\mathcal{H}}, p, q\right)$ constitute a $\beta-$*transaction costs equilibrium* iff $\Phi(\xi, \sigma) = 0 \in \mathbb{R}^n$ where

$$
\Phi(\xi,\sigma) =
\begin{array}{ll}
 & \vdots \\
(FOCx) & Du^h(x^h) - \lambda^h P \\
 & \vdots \\
 & \vdots \\
(BC) & \begin{array}{l} p(0)(e^h(0) - x^h(0)) - q\eta^h \\ p(s)(e^h(s) - x^h(s)) + \sum_j r_j(s) g_j^h\left(\eta^h\right) \ \ \forall s > 0 \end{array} \\
 & \vdots \\
 & \vdots \\
(FOC\eta) & \lambda^h \begin{pmatrix} -q \\ Y \cdot Dg^h(\eta^h) \end{pmatrix} \\
 & \vdots \\
(MCx) & \sum_{h\in\mathcal{H}}(e_l^h(s) - x_l^h(s)) \ \ \forall l \neq L, \forall s \geq 0 \\
(MC\eta) & \sum_{h\in\mathcal{H}} \eta^h
\end{array}
$$

Claim 4.5 *If $\tilde{f}^h : \Theta_E \to \tilde{f}^h\left(\Theta_E\right)$ is an invertible mapping and $D^2F^h\left(\theta^h\right) \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is a positive semidefinite matrix for all equilibrium θ^h, then $\sum_{s>0} \lambda^h(s) \cdot D^2G_s^h\left(\eta^h\right)$ is a negative semidefinite matrix.*[3]

Proof. I will employ the Einstein summation convention in this proof for notational simplicity. A good reference is Lee (2006).

[3]Since $D^2F^h\left(\theta^h\right)$ is positive semidefinite from claim 1 and under the canonical representation $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is a strictly positive, diagonal matrix, then for an open set of matrices around the canonical representation, $D^2F^h\left(\theta^h\right) \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is positive semidefinite. For this open set, the semidefinite assumption in claim 4.5 need not be stated.

As $g^h = \left(\tilde{f}^h\right)^{-1}$, then for any θ^h :

$$\begin{aligned} g^h \circ \tilde{f}^h\left(\theta^h\right) &= \theta^h \\ Dg^h\left(\tilde{f}^h\left(\theta^h\right)\right) \cdot D\tilde{f}^h\left(\theta^h\right) &= I_J. \end{aligned}$$

Define $\eta^h = \tilde{f}^h\left(\theta^h\right)$ and $\phi^j = \sum_{s>0} \lambda^h(s) r^j(s) > 0$. Then premultiply the above equation by $\left(\phi^1, .., \phi^J\right)$ to obtain:

$$\left(\phi^1, .., \phi^J\right) Dg^h\left(\tilde{f}^h\left(\theta^h\right)\right) \cdot D\tilde{f}^h\left(\theta^h\right) = \left(\phi^1, .., \phi^J\right). \tag{4.3}$$

Equation (4.3) is equivalent to (using the Einstein summation convention):

$$\phi^j Dg_j^h\left(\tilde{f}^h\left(\theta^h\right)\right) \cdot D\tilde{f}^h\left(\theta^h\right) = \left(\phi^1, .., \phi^J\right).$$

Taking a second derivative yields:

$$\phi^j D^2 g_j^h\left(\eta^h\right) \cdot \left(D\tilde{f}^h\left(\theta^h\right)\right)^2 + \phi^j D_k g_j^h\left(\eta^h\right) D^2 \tilde{f}_k^h\left(\theta^h\right) = 0. \tag{4.4}$$

Define $\left(\psi^1, .., \psi^J\right)$ such that $\psi^k = \phi^j D_k g_j^h\left(\eta^h\right)$. Then (4.4) can be written as:

$$\phi^j D^2 g_j^h\left(\eta^h\right) \cdot \left(D\tilde{f}^h\left(\theta^h\right)\right)^2 + \psi^k D^2 \tilde{f}_k^h\left(\theta^h\right) = 0. \tag{4.5}$$

From the first order conditions with respect to η_k^h of the problem $(HP2)$:

$$\lambda^h \begin{pmatrix} -q_k \\ r^j(1) D_k g_j^h\left(\eta^h\right) \\ \vdots \\ r^j(S) D_k g_j^h\left(\eta^h\right) \end{pmatrix} = 0.$$

By the definition of ϕ^j and ψ^k, the terms $\psi^k = \lambda^h(0) q^k \ \ \forall k$. Thus, (4.5) reduces to

$$\phi^j D^2 g_j^h\left(\eta^h\right) \cdot \left(D\tilde{f}^h\left(\theta^h\right)\right)^2 + \lambda^h(0) q^k D^2 \tilde{f}_k^h\left(\theta^h\right) = 0. \tag{4.6}$$

By definition, $F^h\left(\theta^h\right) = q^k f_k^h\left(\theta^h\right)$. Since $\tilde{f}^h\left(\theta^h\right) = \theta^h + f^h\left(\theta^h\right)$, then

$$D^2 F^h\left(\theta^h\right) = q^k D^2 \tilde{f}_k^h\left(\theta^h\right).$$

By definition, $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) = \phi^j D^2 g_j^h\left(\eta^h\right)$.

Thus, inserting these definitions into (4.6) and rearranging terms yields the final equation:

$$\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) = -\lambda^h(0) D^2 F^h\left(\theta^h\right) \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}. \tag{4.7}$$

If $D^2 F^h\left(\theta^h\right) \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is a positive semidefinite matrix, then $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is a negative semidefinite matrix. This completes the proof. ∎

As a well-known regularity result extended to this model, I state without proof the following lemma with associated well-known corollary.

Lemma 4.1 *The matrix $D_\xi \Phi_{|\Phi(\xi,\sigma)=0}$ has full row rank on a generic subset of endowments $\mathcal{E} = \{(e^h)_{h \in \mathcal{H}} : e^h >> 0\}$.*

Corollary 4.1 *Over a generic subset of endowments* $\mathcal{E} = \{(e^h)_{h\in \mathcal{H}} : e^h >> 0\}$, *then (i)* $\exists l < L$ *(without loss of generality,* $l = 1$*) such that* $(e_1^h(s) - x_1^h(s)) \neq 0 \ \ \forall s > 0$, $\forall h$ *and (ii)* $\eta_j^h \neq 0 \ \ \forall j, \forall h$.

Critical in defining the $\beta-$transaction costs equilibrium is that the mapping $\tilde{f}^h : \Theta_E \rightarrow \tilde{f}^h(\Theta_E)$ is invertible. Up until now, the results only hold conditional on the mapping $\tilde{f}^h$ being invertible. Under the canonical representation, the mapping $\tilde{f}^h$ is invertible. No known conditions exist to guarantee that $\tilde{f}^h$ is always invertible for the general representation.[4] Lemma 4.2 will find an open set in which all mappings $\tilde{f}^h$ will be invertible and this is possible as the set of invertible matrices is an open set. The proof of lemma 4.2 is contained in section 4.4.

Lemma 4.2 *There exists an open set of transaction costs mappings* $\left(f^h\right)_{h\in \mathcal{H}}$ *such that for mappings in this set and endowments* $\left(e^h\right)_{h\in \mathcal{H}}$ *in a generic subset of* $\mathcal{E}$, *the mapping* $\tilde{f}^h : \Theta_E \rightarrow \tilde{f}^h(\Theta_E)$ *is invertible* $\forall h \in \mathcal{H}$.

The next lemma will be useful in the proof of the main theorem. The result from the lemma is sufficient to prove that, over a generic subset of endowments, all equilibrium allocations are Pareto inefficient. The proof of lemma 4.3 is contained in section 4.4.

Lemma 4.3 *With* $H \leq S$, *the matrix*

$$\begin{pmatrix} \lambda^1(1)(e_1^1(1) - x_1^1(1)) & .... & \lambda^H(1)(e_1^H(1) - x_1^H(1)) \\ \vdots & & \vdots \\ \lambda^1(S)(e_1^1(S) - x_1^1(S)) & .... & \lambda^H(S)(e_1^H(S) - x_1^H(S)) \end{pmatrix}$$

[4] It is possible to use investment constraints to restrict the asset trade to a subset such that $\left(\tilde{f}^h\right)_{h\in\mathcal{H}}$ are invertible over the restricted domains. This, however, adds an unwanted additional friction to this transaction costs model.

has full column rank on a generic subset of endowments $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$.

The fact that g_j^h is strictly concave leads to the inefficiency of the equilibrium allocation. If g_j^h was a linear function of η_j^h only ($\forall h$ and $\forall j$), then the equilibrium would exactly equal the GEI equilibrium. With complete markets $J = S$, the equilibrium allocation would be Pareto optimal.

This inefficiency in the equilibrium allocation, a generic result given lemma 4.3, justifies planner intervention. The planner will scale the transaction costs, either up or down, while satisfying fiscal balance. For asset j, $q_j \sum_{h\in\mathcal{H}} f_j^h\left(\theta^h\right)$ is the total value of the asset lost due to the transaction costs. The planner will intervene by setting the new transaction costs at $\left(1+\gamma_j\right)\cdot f_j^h\left(.\right).$ The planner tool γ_j can either be positive (an increase in transaction costs) or negative (a decrease in transaction costs).[5]

The planner has tools given by the vector $\gamma = \left(..., \gamma_j, ..\right) \in \mathbb{R}^J$. I will call any equilibrium that results following planner intervention the planner updated equilibrium. This is in contrast to the original $\beta-$transaction costs equilibrium. If $\gamma = \overrightarrow{0}$, the planner is taking no action and the planner updated equilibrium is identical to the original $\beta-$transaction costs equilibrium.

As a result of the planner intervention, the households are likely to make different optimizing decisions. Define the asset choices of the planner updated equilibrium as $\left(\hat{\theta}^h\right)_{h\in\mathcal{H}}.$ Define the asset prices of the planner updated equilibrium as $(\hat{q}_j)_{\forall j}.$ Fiscal balance requires that the value of transaction costs is identical both before and after the planner intervention:

$$\sum_j q_j \sum_{h\in\mathcal{H}} f_j^h\left(\theta^h\right) = \sum_j \hat{q}_j\left(1+\gamma_j\right)\sum_{h\in\mathcal{H}} f_j^h\left(\hat{\theta}^h\right).^6 \qquad \text{(BB)}$$

[5] The analysis is local, so clearly $1+\gamma_j > 0$.

[6] Thanks to Antonio Penta for suggesting the rough idea that evolved into the current (BB).

I will call this the budget balance (BB) equation. Planner inaction ($\gamma = \overrightarrow{0}$) trivially satisfies the budget balance equation.

The planner tool γ is of dimension J. However due to the budget balance equation (BB), the planner only has $J-1$ degrees of freedom in choosing γ. To obtain the result that the planner can use the vector γ to generically effect a Pareto improvement, there must be as many free tools as households. Thus, throughout this work, the assumption $H \leq J-1$ is essential.[7]

4.3 The Result

Theorem 4.1 *Under assumptions 1-4 with both $2 \leq H \leq J-1$ and $L \geq 2$ and for parameters $\sigma = \left(e^h, u^h, f^h\right)_{h \in \mathcal{H}}$ belonging to a generic subset of $\Sigma = \mathcal{E} \times \mathcal{U} \times \mathcal{F}$ where $\mathcal{E} = \{(e^h)_{h \in \mathcal{H}} : e^h >> 0\}$, $\mathcal{U}$ is the set of utility functions satisfying assumption 2, and $\mathcal{F}$ is the set of transaction costs mappings satisfying assumption 4 and belonging to the open set given in lemma 4.2, then given the original $\beta-$transaction costs equilibrium allocation, there exists a planner policy satisfying (BB) such that the planner updated allocation is Pareto superior and the new equilibrium is regular.*

Proof The implication of the theorem is that an open set of γ exists (call it A) such that if $\gamma \in A$, then all households are strictly better off in the planner updated equilibrium, provided that $H \leq J-1$. As an immediate corollary, take any $\gamma \in A$. Then, $\exists \epsilon > 0$ s.t. for the new planner intervention $\gamma' = \gamma + (\epsilon, ..., \epsilon)$, all households remain strictly better off and the planner receives profit equal to $\epsilon \sum_j \hat{q}_j \sum_{h \in \mathcal{H}} f_j^h \left(\hat{\theta}^h\right) > 0$.

[7] If the assumption $H \leq J-1$ appears restrictive, using the idea from Cass and Citanna (1998), the parameter H can be viewed as the number of different types of households. All households of the same type will have parameters (endowments, utilities, and transaction costs mappings) that lie in an open set around the specified parameters for $h \in \mathcal{H}$.

The proof of this theorem will follow the framework of Citanna, Kajii, and Villanacci (1998), henceforth simply CKV. The principal task will be to show that the vector of household utilities $U(\xi, \gamma) = \left(u^1(x^1), .., u^H(x^H)\right)$ is a submersion.

Picking a vector of parameters $\bar{\sigma} = \left(\bar{e}^h, \bar{u}^h, \bar{f}^h\right)_{h \in \mathcal{H}}$ such that $\left(\bar{e}^h\right)_{h \in \mathcal{H}}$ belongs to a generic subset of $\mathcal{E}$, then all resulting $\beta-$*transaction costs equilibria* are regular values of Φ. In particular, this means that there exists an open set Σ' around $\bar{\sigma}$ such that for any parameters $\sigma \in \Sigma'$, the resulting equilibria satisfy the rank condition of lemma 4.1. The set of $\left(x^h\right)_{h \in \mathcal{H}}$ such that $\left(u^1(x^1), .., u^H(x^H)\right) >> \left(u^1(\bar{x}^1), .., u^H(\bar{x}^H)\right)$ is an open set where $\left(\bar{x}^h\right)_{h \in \mathcal{H}}$ is the equilibrium allocation resulting from the original parameters $\bar{\sigma}$. As such, if for some planner tool γ^*, the resulting planner updated allocation is Pareto superior, then all planner updated equilibrium allocations given γ in an open neighborhood around γ^* are Pareto superior as well.

With the planner intervention, the function $\tilde{f}_j^h\left(\theta^h; \gamma\right) = \theta_j^h + \left(1 + \gamma_j\right) f_j^h\left(\theta^h\right)$. As γ is in a local neighborhood around $\overrightarrow{0}$, then $\tilde{f}^h(.; \gamma) = \left(.., \tilde{f}_j^h(.; \gamma), ..\right)$ remains invertible. Define $\eta^h = \tilde{f}^h\left(\theta^h; \gamma\right)$ and $g^h\left(\eta^h; \gamma\right) = \theta^h$. Thus, the asset payouts for all households will be a function of the planner tool γ.

Take as given the original $\beta-$transaction costs equilibrium $\left(\left(x^h, \lambda^h, \eta^h\right)_{h \in \mathcal{H}}, p, q\right)$. Given parameters $\sigma = \left(e^h, u^h, f^h\right)_{h \in \mathcal{H}}$, the variables $\hat{\xi} = \left(\left(\hat{x}^h, \hat{\eta}^h\right)_{h \in \mathcal{H}}, \hat{p}, \hat{q}\right)$ and policy parameters γ constitute a planner updated equilibrium iff $\Gamma(\hat{\xi}, \gamma, \sigma) = 0$. Γ has one more equation than the system Φ used to define a $\beta-$*transaction costs equilibrium* and is defined as:

$$
\begin{array}{lll}
\Gamma(\hat{\xi},\gamma,\sigma) & = & \\
 & & \vdots \\
(FOCx) & & Du^h(\hat{x}^h) - \hat{\lambda}^h \hat{P} \\
 & & \vdots \\
 & & \vdots \\
(BC) & & \begin{array}{l} \hat{p}(0)(e^h(0) - \hat{x}^h(0)) - \hat{q}\hat{\eta}^h \\ \hat{p}(s)(e^h(s) - \hat{x}^h(s)) + \sum_j r_j(s) g_j^h\left(\hat{\eta}^h;\gamma\right) \ \ \forall s > 0 \end{array} \\
 & & \vdots \\
 & & \vdots \\
(FOC\eta) & & \hat{\lambda}^h \begin{pmatrix} -\hat{q} \\ Y \cdot Dg^h(\hat{\eta}^h;\gamma) \end{pmatrix} \\
 & & \vdots \\
(MCx) & & \sum_{h\in\mathcal{H}}(e_l^h(s) - \hat{x}_l^h(s)) \ \ \forall l \neq L, \forall s \geq 0 \\
(MC\eta) & & \sum_{h\in\mathcal{H}} \hat{\eta}^h \\
(BB) & & \sum_j \hat{q}_j\left(1+\gamma_j\right) \sum_{h\in\mathcal{H}} \left(\hat{\eta}_j^h - g_j^h(\hat{\eta}^h;\gamma)\right) - \sum_j q_j \sum_{h\in\mathcal{H}}\left(\eta_j^h - g_j^h(\eta^h;\gamma)\right)
\end{array}
$$

By definition, if $\Gamma(\hat{\xi}, \overrightarrow{0}, \sigma) = 0$ and $\Phi(\xi,\sigma) = 0$, then $\hat{\xi} = \xi$.

Define the $(H + n + 1) \times (n + J)$ matrix Ψ_0 :

$$
\Psi_0 = \begin{pmatrix} D_\xi U(\hat{\xi},\gamma) & 0 \\ D_\xi \Gamma(\hat{\xi},\gamma,\sigma) & D_\gamma \Gamma(\hat{\xi},\gamma,\sigma) \end{pmatrix}.
$$

From CKV, if Ψ_0 has full row rank, $\exists \hat{\xi} \neq \xi$ s.t. $\hat{\xi}$ satisfies $\Gamma = 0$ (for some γ) and $U(\hat{\xi}) > U(\xi)$. The matrix Ψ_0 is square if $H + 1 = J$, but if $H + 1 < J$, then there are more columns than rows and I must remove some columns (it does not matter which) in order to obtain a square matrix Ψ. This matrix Ψ does not have full rank iff $\exists \nu \in \mathbb{R}^{H+n+1}$ s.t. $\Phi'(\hat{\xi}, \gamma, \nu, \sigma) = 0$ where

$$\Phi'(\hat{\xi}, \gamma, \nu, \sigma) = \begin{pmatrix} \Psi^T \nu \\ \nu^T \nu / 2 - 1 \end{pmatrix}.$$

I will have proven the theorem if I can show that for a generic choice of $\sigma \in \Sigma$, there does not exist (ξ, ν) s.t.

$$\begin{aligned} \Phi(\xi, \sigma) &= 0 \\ \Phi'(\xi, \overrightarrow{0}, \nu, \sigma) &= 0. \end{aligned} \tag{Φ, Φ'}$$

Counting equations and unknowns, (Φ, Φ') has n equations in Φ, n variables ξ, $H + n + 2$ equations in Φ', and only $H + n + 1$ variables ν. I must show that over a generic subset of parameters (exactly which generic subset will be discussed next), the system of equations (Φ, Φ') (more equations than variables) has full rank. To show full rank of (Φ, Φ'), I will reference the (ND) condition of CKV, which is a sufficient condition for the full rank of (Φ, Φ'). The condition states that for $\gamma = \overrightarrow{0}$ and $\hat{\xi} = \xi$, the matrix

$$\begin{pmatrix} \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} & D_\sigma \Phi' \end{pmatrix} \text{ has full row rank} \tag{ND}$$

where σ are the parameters on which the genericity statement is made.

When $\hat{\xi} = \xi$, the budget balance equation simplifies to:

$$\sum\nolimits_j q_j \gamma_j \sum\nolimits_{h \in \mathcal{H}} \left(\eta_j^h - g_j^h(\eta^h; \gamma)\right) = 0. \tag{BB}$$

It is this equation that will be used in the set of equations $\Gamma(\hat{\xi}, \gamma, \sigma)$. Additionally, as $g_j^h(\eta^h; \overrightarrow{0})$ is equal to the original function $g_j^h(\eta^h)$ as defined in section 4.2, I will hide the dependence of $\left(g^h(\eta^h), Dg^h(\eta^h), D^2G_s^h\left(\eta^h\right)\right)$ on γ up until the point when I need to consider the first order conditions of $g^h(\eta^h; \gamma)$ with respect to γ.

For simplicity, I divide the vector ν^T into subvectors that each represent a certain equation in Ψ. Define $\nu^T = \left(\Delta u^T, \Delta x^T, \Delta\lambda^T, \Delta\eta^T, \Delta p^T, \Delta q^T, \Delta b\right) \in \mathbb{R}^{H+n+1}$ where each subvector corresponds sensibly to an equation (row) in Ψ as follows:

$$
\begin{aligned}
\Delta u^T &\Longleftrightarrow U(\xi, \hat{\xi}) \\
\Delta x^T &\Longleftrightarrow FOCx \\
\Delta\lambda^T &\Longleftrightarrow BC \\
\Delta\eta^T &\Longleftrightarrow FOC\eta \\
\Delta p^T &\Longleftrightarrow MCx \\
\Delta q^T &\Longleftrightarrow MC\eta \\
\Delta b &\Longleftrightarrow BB.
\end{aligned}
$$

With $\gamma = \overrightarrow{0}$, the variables $\left(\left(\hat{x}^h, \hat{\lambda}^h, \hat{\eta}^h\right)_{h\in\mathcal{H}}, \hat{p}, \hat{q}\right) = \left(\left(x^h, \lambda^h, \eta^h\right)_{h\in\mathcal{H}}, p, q\right)$. A subset of the equations $\nu^T\Psi = 0$ are given by (corresponding to derivatives with respect to $\left((x^h, \lambda^h, \eta^h)_{h\in\mathcal{H}}\right)$ in that order):

$$\begin{array}{c} \vdots \\ \Delta u_h Du^h(x^h) + \Delta x_h^T D^2 u^h(x^h) - \Delta\lambda_h^T P - \Delta p^T \Lambda = 0. \quad (4.8.a) \\ \vdots \end{array} \qquad (4.8)$$

$$\begin{array}{c} \vdots \\ -\Delta x_h^T P^T + \Delta\eta_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}^T = 0. \quad (4.8.b) \\ \vdots \end{array}$$

$$\begin{array}{c} \vdots \\ \Delta\lambda_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix} + \Delta\eta_h^T \sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) + \\ \Delta q^T + \Delta b\left(\left(.., q_j\gamma_j, ..\right)\left(I_J - Dg^h(\eta^h)\right)\right) = 0. \\ \vdots \end{array} \quad (4.8.c)$$

where Λ is the $(G - S - 1) \times G$ matrix

$$\Lambda = \begin{bmatrix} \begin{pmatrix} I_{L-1} & 0 \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} I_{L-1} & 0 \end{pmatrix} \end{bmatrix}$$

and $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is the $J \times J$ negative semidefinite matrix defined in section 4.2.[8]

For simplicity, I break up the analysis into two cases. They are Case I: $\Delta x_h^T \neq 0$

[8] The term $\left(.., q_j\gamma_j, ..\right)\left(I_J - Dg^h(\eta^h)\right)$ is the $1 \times J$ derivative matrix of (BB) with respect to η^h. The proof of the result requires me to prove the (ND) condition for $\overrightarrow{\gamma} = 0$. Thus, this term has value 0 and will be ignored in future analysis.

$\forall h \in \mathcal{H}$ and Case II: $\Delta x_h^T = 0$ for some $h \in \mathcal{H}$. In Case I, I show that (ND) holds over a generic subset of parameters. In Case II, I show that the system of equations (Φ, Φ') will generically not have any solution.

Case I: $\Delta x_h^T \neq 0 \ \ \forall h \in \mathcal{H}$

Claim 4.6 $\left(\Delta u_h, \Delta p^T, \Delta q^T\right) \neq 0 \ \ \forall h \in \mathcal{H}$.

Proof. Suppose that $\left(\Delta u_h, \Delta p^T, \Delta q^T\right) = 0$ for some h. Then $(4.8.a)$ reads

$$\Delta x_h^T D^2 u^h(x^h) - \Delta \lambda_h^T P = 0.$$

Postmultiplying by Δx_h and using $(4.8.b)$, I obtain

$$\Delta x_h^T D^2 u^h(x^h) \Delta x_h = \Delta \lambda_h^T \begin{pmatrix} -q \\ Y D g^h(\eta^h) \end{pmatrix} \Delta \eta_h$$

and using $(4.8.c)$ with $\Delta q^T = 0$, I finally reach

$$\Delta x_h^T D^2 u^h(x^h) \Delta x_h = -\Delta \eta_h^T \left(\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right) \right) \Delta \eta_h. \qquad (4.9)$$

As $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)$ is a negative semidefinite matrix, then the right-hand side of (4.9) is nonnegative. Meanwhile, from assumption 2, the left-hand side is strictly negative. The contradiction finishes the claim. ■

Claim 4.7 *For* $\gamma = \overrightarrow{0}$, *then* $D_u \Phi' = \begin{pmatrix} A^* \\ \overrightarrow{0} \end{pmatrix}$ *where* A^* *has full row rank and corresponds to the rows for derivatives with respect to* $\left(x^h\right)_{h \in \mathcal{H}}$.

Proof. Consider the space of utility functions $\left(u^h\right)_{h \in \mathcal{H}} \in \mathcal{U}$, where u^h satisfies assumption 2. The space $\mathcal{U}$ is infinite-dimensional and is endowed with the C^3 uniform convergence topology on compact sets. This means that a sequence of functions $\{u^\nu\}$ converges uniformly to u iff $\{Du^\nu\}$, $\{D^2u^\nu\}$, and $\{D^3u^\nu\}$ uniformly converge to Du, D^2u, and D^3u, respectively. Additionally, any subspace of $\mathcal{U}$ is endowed with the subspace topology of the topology of $\mathcal{U}$. I will use the regularity result from lemma 1 to define utility functions as locally belonging to the finite-dimensional subset $\mathcal{A} \subseteq \mathcal{U}$.

Using lemma 4.1, pick a regular value $\bar{\sigma}$. For that $\bar{\sigma}$, there exist finitely many equilibria $\left(\bar{\xi}_i, \bar{\sigma}\right)$ $i = 1, .., I$. Further, there exist open sets Σ' and $A_i'^h$ s.t. $\bar{x}_i^h \in A_i'^h$, the sets $A_i'^h$ are disjoint across i, and $\forall \sigma \in \Sigma'$, $\exists!$ equilibrium $x_i^h \in A_i'^h$. Choose $A_i'^h$ such that the closure $\bar{A}_i'^h$ is compact and there exist disjoint open sets $\tilde{A}_i'^h$ s.t. $A_i'^h \subset \bar{A}_i'^h \subset \tilde{A}_i'^h$.

For each household, define a bump function $\delta^h : \mathbb{R}_+^G \to [0,1]$ with I bumps as $\delta^h = 1$ on $A_i'^h$ and $\delta^h = 0$ on $(\tilde{A}_i'^h)^c$. Now, I define u^h in terms of a $G \times G$ symmetric matrix A^h as:

$$u^h(x^h; A^h) = \bar{u}(x^h) + \frac{1}{2}\delta^h(x^h) \sum\nolimits_i \left[(x^h - \bar{x}_i^h)^T A^h (x^h - \bar{x}_i^h)\right]. \tag{4.10}$$

Thus, the space of symmetric matrices $A^h \in \mathcal{A}$ is a finite dimensional subspace of $\mathcal{U}$. Since $\mathcal{A}$ has the subspace topology of $\mathcal{U}$, then $u^h(\cdot; A^\nu) \to u^h(\cdot; A)$ iff $A^\nu \to A$. This can be seen by taking derivatives and noting that the function $\bar{u}$ stays fixed at the regular value.

Taking derivatives with respect to $x^h \in A_i'^h$ yields:

$$\begin{aligned} D_x u^h(x^h; A^h) &= D\bar{u}(x^h) + A^h(x^h - \bar{x}_i^h) \\ D_{xx}^2 u^h(x^h; A^h) &= D^2\bar{u}(x^h) + A^h. \end{aligned}$$

$\mathcal{A}$ is a $G(G+1)/2$ dimensional space, so write A^h as the vector

$$\left((A^h_{i,i})_{i=1,..,G}, (A^h_{i,j})_{i<j,i=1,..,G-1}\right).$$

Postmultiply D^2_{xx} by Δx_h :

$$D^2_{xx}u^h(x^h;A^h)\Delta x_h = D^2\bar{u}(x^h)\Delta x_h + A^h\Delta x_h.$$

Taking derivatives with respect to the parameter u^h is equivalent to taking derivatives with respect to A^h :

$$D_u\left(D^2_{xx}u^h(x^h;A^h)\Delta x_h\right) = D_A\left(A^h\Delta x_h\right)$$

$$= \begin{pmatrix} \Delta x_h^1 & 0 & 0 & & & \\ 0 & ... & 0 & \Sigma(1) & ... & \Sigma(G-1) \\ 0 & 0 & \Delta x_h^G & & & \end{pmatrix} \in \mathbb{R}^{G,G(G+1)/2}$$

where the submatrix $\Sigma(i)$ is defined as

$$\Sigma(i) = \begin{pmatrix} 0 \in \mathbb{R}^{i-1,G-i} \\ \begin{pmatrix} \Delta x_h^{i+1} & ... & \Delta x_h^G \\ \Delta x_h^i & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \Delta x_h^i \end{pmatrix} \end{pmatrix} \in \mathbb{R}^{G,G-i}.$$

Thus, since $\Delta x_h \neq 0$ (without loss of generality $\Delta x_h^1 \neq 0$), then

$$rank D_A\left(D^2_{xx}u^h(x^h;A^h)\Delta x_h\right) = G. \tag{4.11}$$

Out of all the rows Ψ^T, the utility function u^h only appears in the row for derivatives with respect to x^h. This row in Ψ^T for one household h is given by (as in (4.8.a)):

$$\begin{array}{cccccc} U(\xi,\hat{\xi}) & FOCx & BC & FOC\eta & MCx & MC\eta \\ \left(Du^h(x^h)\right)^T & D^2u^h(x^h) & -P^T & 0 & -\Lambda^T & 0 \end{array}.$$

Thus, taking the derivative $D_{A^h}\Phi' = D_{A^h}\Psi^T\nu$, the only nonzero element is

$$D_{A^h}\left(\left(Du^h(x^h)\right)^T\Delta u_h + D^2u^h(x^h)\Delta x_h - P^T\Delta\lambda_h - \Lambda^T\Delta p\right)$$

$$= D_{A^h}\left(\left(Du^h(x^h;A^h)\right)^T\Delta u_h\right) + D_{A^h}\left(D^2u^h(x^h;A^h)\Delta x_h\right).$$

From the construction of A^h, $D_xu^h(x^h;A^h) = D\bar{u}(x^h) + A^h(x^h - \bar{x}_i^h) = D\bar{u}(x^h)$ for $\gamma = \overrightarrow{0}$ (since $x^h = \bar{x}_i^h$). Thus $D_{A^h}(D_xu^h(x^h;A^h)\Delta u_h) = 0$. Using (4.11), then $D_{A^h}\left(D^2_{xx}u^h(x^h;A^h)\Delta x_h\right)$ is a full rank matrix of size $G\times(G(G+1)/2)$. Thus $A^* = \begin{bmatrix} \ldots & 0 & 0 \\ 0 & D_{A^h}\left(D^2_{xx}u^h(x^h;A^h)\Delta x_h\right) & 0 \\ 0 & 0 & \ldots \end{bmatrix}$ has full row rank. ■

Consider the space of transaction costs mappings for all households $\left(F^h\right)_{h\in\mathcal{H}}$. By definition, $F^h\left(\theta^h\right) = \sum_j q_j\cdot f_j^h\left(\theta^h\right)$ depends on the endogenous asset prices. In equilibrium, $q >> 0$ and all the results follow by letting the asset prices $(q_j)_{\forall j}$ be fixed at some strictly positive values.

Claim 4.8 *For* $\gamma = \overrightarrow{0}$ *and if* $\Delta\eta_h \neq 0\ \ \forall h\in\mathcal{H}$, *then* $D_F\Phi' = \begin{pmatrix} \overrightarrow{0} \\ B^* \\ \overrightarrow{0} \end{pmatrix}$ *where* B^* *has full row rank and corresponds to the rows for derivatives with respect to* $\left(\eta^h\right)_{h\in\mathcal{H}}$.

Proof. Consider the space of functions $\left(F^h\right)_{h\in\mathcal{H}}\in\mathcal{F}$, where $F^h\left(\theta^h\right) = \sum_j q_j\cdot f_j^h\left(\theta^h\right)$ as in section 4.2 and f_j^h satisfies assumption 4. The space $\mathcal{F}$ is infinite-dimensional and is endowed with the C^3 uniform convergence topology on compact sets (same as $\mathcal{U}$). I will use the regularity result from lemma 4.1 to define transaction costs mappings as locally belonging to the finite-dimensional subset $\mathcal{B} \subseteq \mathcal{F}$.

Exactly as with utility functions, I define F^h in terms of a $J \times J$ symmetric matrix B^h as:

$$F^h(\theta^h; B^h) = \bar{F}(\theta^h) + \frac{1}{2}\delta^h(\theta^h) \sum\nolimits_i \left[(\theta^h - \bar{\theta}_i^h)^T B^h (\theta^h - \bar{\theta}_i^h) \right]. \tag{4.12}$$

Thus, the space of symmetric matrices $B^h \in \mathcal{B}$ is a finite dimensional subspace of $\mathcal{B}$.

Taking derivatives with respect to θ^h yields:

$$\begin{aligned} D_\theta F^h(\theta^h; B^h) &= D\bar{F}(\theta^h) + B^h(\theta^h - \bar{\theta}_i^h) \\ D_\theta^2 F^h(\theta^h; B^h) &= D^2\bar{F}(\theta^h) + B^h. \end{aligned}$$

Recall the analysis in the proof of claim 4.5, namely equation (4.7) :

$$\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right) = -\lambda^h(0) D^2 F^h \left(\theta^h\right) \left[D\tilde{f}^h \left(\theta^h\right) \right]^{-2}. \tag{4.7}$$

Thus, I replace $D^2 F^h \left(\theta^h\right)$ by $D^2\bar{F}(\theta^h) + B^h$ and post-multiply both sides of (4.7) by $\Delta\eta_h$ to yield:

$$\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right) \Delta\eta_h = -\lambda^h(0) \left(D^2\bar{F}(\theta^h) + B^h \right) \left[D\tilde{f}^h \left(\theta^h\right) \right]^{-2} \Delta\eta_h.$$

Taking derivatives with respect to the parameter F^h is equivalent to taking derivatives with respect to B^h :

$$D_B \left(-\lambda^h(0) \left(D^2\bar{F}(\theta^h) + B^h \right) \left[D\tilde{f}^h \left(\theta^h\right) \right]^{-2} \Delta\eta_h \right) =$$

$$= -\lambda^h(0) \left(\left[D\tilde{f}^h \left(\theta^h \right) \right]^{-2} \begin{pmatrix} \Delta\eta_h^1 & 0 & 0 & & & \\ 0 & ... & 0 & \Sigma(1) & ... & \Sigma(J-1) \\ 0 & 0 & \Delta\eta_h^J & & & \end{pmatrix} \right) \in \mathbb{R}^{J,J(J+1)/2}$$

where the submatrix $\Sigma(i)$ is defined as

$$\Sigma(i) = \begin{pmatrix} 0 \in \mathbb{R}^{i-1,J-i} \\ \begin{pmatrix} \Delta\eta_h^{i+1} & ... & \Delta\eta_h^J \\ \Delta\eta_h^i & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \Delta\eta_h^i \end{pmatrix} \end{pmatrix} \in \mathbb{R}^{J,J-i}.$$

With $\lambda^h(0) > 0$ and $\left[D\tilde{f}^h \left(\theta^h \right) \right]^{-2}$ a full rank matrix, I only need to verify that $\begin{pmatrix} \Delta\eta_h^1 & 0 & 0 & & & \\ 0 & ... & 0 & \Sigma(1) & ... & \Sigma(J-1) \\ 0 & 0 & \Delta\eta_h^J & & & \end{pmatrix}$ has full rank. If $\Delta\eta_h \neq 0$ (without loss of generality $\Delta\eta_h^1 \neq 0$), then

$$rank D_{B^h} \left(\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h \right) \Delta\eta_h \right) = J. \tag{4.13}$$

As the above development with utilities u^h reveals, although the function g^h appears in both rows for derivatives with respect to λ^h and η^h (see equations (4.8.b) and (4.8.c)), the only nonzero derivatives $D_{B^h}\left(\Psi^T \nu\right)$ are those due to the second derivative $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)$ in (4.8.c). Using (4.13) and if $\Delta\eta_h \neq 0$, then the $J \times (J(J+1)/2)$ matrix $D_{B^h} \left(\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right) \Delta\eta_h \right)$ has full rank. Thus, if

$\Delta\eta_h \neq 0\ \forall h \in \mathcal{H}$, then $B^* = \begin{bmatrix} \ldots & 0 & 0 \\ 0 & D_{B^h}\left(\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)\Delta\eta_h\right) & 0 \\ 0 & 0 & \ldots \end{bmatrix}$ has full row rank. ■

The matrix $\left(\begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \quad D_B\Phi' \right)$ is given below (where the rows correspond to the equilibrium variables $\left((x^h, \lambda^h, \eta^h)_{h\in\mathcal{H}}, p, q\right)$, policy variables (γ), and vector v^T in that order). To conserve on space, I will employ the following conventions:

$$c\left(A^h\right) = \begin{pmatrix} A^1 \\ : \\ A^H \end{pmatrix}, \quad r\left(A^h\right) = \begin{pmatrix} A^1 & \ldots & A^H \end{pmatrix}, \quad d\left(A^h\right) = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & A^H \end{pmatrix}$$

where (c, r, d) stand for column, row, and diagonal, respectively. Further, define $\Omega^h = \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}$, $\bar{D}^2 g^h = \sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$, $\dot{A}^h = D_{A^h}\left(D^2 u^h(x^h)\Delta x_h\right)$, and $\dot{B}^h = D_{B^h}\left(\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)\Delta\eta_h\right)$. The matrix $\left(\begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \quad D_B\Phi' \right)$

is given by:

$$\begin{pmatrix} r\left(Du^h(x^h)^T\right) & d\left(D^2u^h\right) & d(-P^T) & 0 & c(-\Lambda^T) & 0 & 0 & d\left(\dot{A}^h\right) & 0 \\ 0 & d\left(-P\right) & 0 & d(\Omega^h) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d\left((\Omega^h)^T\right) & d(\bar{D}^2 g^h) & 0 & c(I_J) & 0 & 0 & d\left(\dot{B}^h\right) \\ 0 & r(-\Lambda_2) & ** & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r(-\eta^h|\overrightarrow{0}) & r(-\lambda^h(0)I_J) & 0 & 0 & \Upsilon_2 & 0 & 0 \\ 0 & 0 & \Upsilon_1 & ** & 0 & 0 & \Upsilon_3 & 0 & 0 \\ r(\Delta u_h) & r(\Delta x_h^T) & r(\Delta\lambda_h^T) & r(\Delta\eta_h^T) & \Delta p^T & \Delta q^T & \Delta b & 0 & 0 \end{pmatrix}$$

where Λ_2 is the $(G-S-1)\times G$ matrix

$$\Lambda_2 = \begin{bmatrix} \begin{pmatrix} \lambda^h(0)I_{L-1} & 0 \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} \lambda^h(S)I_{L-1} & 0 \end{pmatrix} \end{bmatrix}$$

and the submatrices $(**)$ will not be considered in the analysis. The submatrix Υ_1 is the $J \times H(S+1)$ matrix defined as the transpose of the derivative of the budget constraints with respect to γ. The submatrix Υ_2 is the $J \times 1$ matrix defined as the transpose of the derivative of (BB) with respect to q. The submatrix Υ_3 is the $J \times 1$ matrix defined as the transpose of the derivative of the budget balance equation (BB)

with respect to γ. The last two submatrices are defined as:

$$\Upsilon_2 = \begin{pmatrix} \vdots \\ \gamma_j \sum_{h\in\mathcal{H}} \left(\eta_j^h - g_j^h(\eta^h;\gamma)\right) \\ \vdots \end{pmatrix}$$

$$\Upsilon_3 = \begin{pmatrix} \vdots \\ q_j \sum_{h\in\mathcal{H}} \left(\eta_j^h - g_j^h(\eta^h;\gamma)\right) \\ \vdots \end{pmatrix} - \sum_{h\in\mathcal{H}} D_\gamma g^h\left(\eta^h;\gamma\right) \begin{pmatrix} \vdots \\ q_j\gamma_j \\ \vdots \end{pmatrix}.$$

The derivatives are evaluated when $\gamma = \overrightarrow{0}$, so Υ_2 is the zero vector and $\Upsilon_3 = \begin{pmatrix} \vdots \\ q_j \sum_{h\in\mathcal{H}} \left(\eta_j^h - g_j^h(\eta^h;\gamma)\right) \\ \vdots \end{pmatrix}$.

Claim 4.9 *The submatrix Υ_1 is given by $\Upsilon_1 = r\left(\overrightarrow{0}|T^h \cdot Y^T\right)$ where T^h are full-rank diagonal matrices $\forall h \in \mathcal{H}$.*

Proof. The budget constraints are given by:

$$\begin{matrix} \vdots \\ p(0)(e^h(0) - x^h(0)) - q\eta^h \\ p(s)(e^h(s) - x^h(s)) + \sum_j r_j(s) g_j^h\left(\eta^h;\gamma\right) \;\; \forall s > 0 \\ \vdots \end{matrix}.$$

The matrix T^h is the $J \times J$ matrix equal to the transpose of $D_\gamma g^h\left(\eta^h;\gamma\right)$, the derivative with respect to γ. From the implicit function theorem, $D_\gamma g^h\left(\eta^h;\gamma\right) = \left[D_\gamma \tilde{f}^h\left(\theta^h;\gamma\right)\right]^{-1}$. The derivative matrix $D_\gamma \tilde{f}^h\left(\theta^h;\gamma\right) = \begin{pmatrix} f_1^h\left(\theta^h\right) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & f_J^h\left(\theta^h\right) \end{pmatrix}$.

From corollary 4.1, the diagonal terms are all strictly positive. Thus, $D_\gamma \tilde{f}^h\left(\theta^h;\gamma\right)$ is invertible and $D_\gamma g^h\left(\eta^h;\gamma\right)$ is a full-rank diagonal matrix. ■

I will consider two subcases:

Subcase A: $\Delta\eta_h^T \neq 0\ \forall h \in \mathcal{H}$

I want to show that the matrix $\left(\begin{pmatrix}\Psi^T \\ \nu^T\end{pmatrix} \quad D_A\Phi' \quad D_B\Phi'\right)$ has full rank. From claims 4.7 and 4.8 and since $\left(\Delta u_h, \Delta p^T, \Delta q^T\right) \neq 0\ \ \forall h \in \mathcal{H}$ (claim 4.6), then the first, second, and last row blocks are linearly independent from the others. Thus, the matrix $\left(\begin{pmatrix}\Psi^T \\ \nu^T\end{pmatrix} \quad D_A\Phi' \quad D_B\Phi'\right)$ has full row rank iff the submatrix

$$\begin{pmatrix} -P & 0 & 0 & & \Omega^1 & 0 & 0 \\ 0 & .. & 0 & 0 & 0 & .. & 0 \\ 0 & 0 & -P & & 0 & 0 & \Omega^H \\ ... & -\Lambda_2 & ... & ** & & 0 & \\ & 0 & & r(-\eta^h|\overrightarrow{0}) & & r\left(-\lambda^h(0)I_J\right) & \\ & 0 & & \Upsilon_1 & & ** & \end{pmatrix}$$

has full row rank. By the definition of Λ_2, the $[H(S+1)+G-(S+1)] \times HG$ submatrix $\begin{pmatrix} -P & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -P \\ ... & -\Lambda_2 & ... \end{pmatrix}$ is a full rank matrix. I have left to show that the matrix $\begin{pmatrix} r(-\eta^h|\overrightarrow{0}) & r\left(-\lambda^h(0)I_J\right) \\ \Upsilon_1 & ** \end{pmatrix}$ has full rank. Since $\Upsilon_1 = r\left(\overrightarrow{0}|T^h \cdot Y^T\right)$ (claim 4.9) for a full rank matrix T^h and Y^T has full row rank, then the final row is linearly independent. The matrix $r\left(-\lambda^h(0)I_J\right)$ has full rank, so the submatrix $\begin{pmatrix} r(-\eta^h|\overrightarrow{0}) & r\left(-\lambda^h(0)I_J\right) \\ \Upsilon_1 & ** \end{pmatrix}$ is a full rank matrix. This concludes the proof under subcase A.

Subcase B: $\Delta\eta_h^T = 0$ for some $h \in \mathcal{H}$

Recall the system of equations (a subset of the equations $\nu^T\Psi = 0$):

$$\vdots$$

$$\Delta u_h Du^h(x^h) + \Delta x_h^T D^2u^h(x^h) - \Delta\lambda_h^T P - \Delta p^T \Lambda = 0. \quad (4.8.a) \qquad (4.8)$$

$$\vdots$$

$$\vdots$$

$$-\Delta x_h^T P^T + \Delta\eta_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}^T = 0. \quad (4.8.b)$$

$$\vdots$$

Suppose $\exists h' \in \mathcal{H}$ such that $\Delta\eta_{h'}^T = 0$. Postmultiply (4.8.a) by $\Delta x_{h'}$ and use (4.8.b) and Φ to obtain:

$$\Delta x_{h'}^T D^2u^{h'}(x^{h'})\Delta x_{h'} - \Delta p^T \Lambda \Delta x_{h'} = 0.$$

The left term is strictly negative (by assumption 2). Thus $\Delta p^T \neq 0$.

The matrix $\left(\begin{pmatrix}\Psi^T \\ \nu^T\end{pmatrix} \quad D_A\Phi' \quad D_B\Phi'\right)$ is given by:

$$\begin{pmatrix}
r\left(Du^h(x^h)^T\right) & d\left(D^2u^h\right) & d(-P^T) & 0 & c(-\Lambda^T) & 0 & 0 & d\left(\dot{A}^h\right) & 0 \\
0 & d\left(-P\right) & 0 & d(\Omega^h) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d\left((\Omega^h)^T\right) & d(\bar{D}^2g^h) & 0 & c(I_J) & 0 & 0 & d\left(\dot{B}^h\right) \\
0 & r(-\Lambda_2) & ** & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & r(-\eta^h|\overrightarrow{0}) & r(-\lambda^h(0)I_J) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Upsilon_1 & ** & 0 & 0 & \Upsilon_3 & 0 & 0 \\
r(\Delta u_h) & r(\Delta x_h^T) & r(\Delta\lambda_h^T) & r(\Delta\eta_h^T) & \Delta p^T & \Delta q^T & \Delta b & 0 & 0
\end{pmatrix}$$

From claim 4.7 and since $\Delta p^T \neq 0$, then the first and last row blocks are linearly independent from the others. As in subcase A, it is known that the submatrix $\begin{pmatrix} -P & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -P \\ ... & -\Lambda_2 & ... \end{pmatrix}$ is a full rank matrix. Thus, the matrix $\left(\begin{pmatrix}\Psi^T \\ \nu^T\end{pmatrix} \quad D_A\Phi' \quad D_B\Phi'\right)$ has full rank iff the submatrix

$$\begin{pmatrix}
(\Omega^1)^T & 0 & 0 & \bar{D}^2g^1 & 0 & 0 & I_J \\
0 & ... & 0 & 0 & ... & 0 & : \\
0 & 0 & (\Omega^H)^T & 0 & 0 & \bar{D}^2g^H & I_J \\
(-\eta^1|\overrightarrow{0}) & ... & (-\eta^H|\overrightarrow{0}) & -\lambda^1(0)I_J & ... & -\lambda^H(0)I_J & 0 \\
\left(\overrightarrow{0}|T^1\cdot Y^T\right) & ... & \left(\overrightarrow{0}|T^H\cdot Y^T\right) & ** & ... & ** & 0
\end{pmatrix} \tag{4.14}$$

has full row rank where Υ_1 has been replaced using claim 4.9. By definition, $(\Omega^h)^T =$

$\begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}^T$. If the submatrix

$$M = \begin{pmatrix} (YDg^1(\eta^1))^T & 0 & 0 & I_J \\ 0 & ... & 0 & : \\ 0 & 0 & (YDg^H(\eta^H))^T & I_J \\ T^1 \cdot Y^T & ... & T^H \cdot Y^T & 0 \end{pmatrix} \tag{4.15}$$

has full row rank, then since $\begin{pmatrix} -\lambda^1(0)I_J & ... & -\lambda^H(0)I_J \end{pmatrix}$ has full rank, the submatrix (4.14) would have full row rank.

Claim 4.10 *The matrix M as defined in* (4.15) *has full row rank.*

Proof. To verify full row rank of M, I will pre-multiply M by $\omega^T = \left(\left(\omega^T_{\eta^h}\right)_{h\in\mathcal{H}}, \omega^T_\gamma\right)$ and verify that $\omega^T M = 0$ implies $\omega^T = 0$.

Take any household $h \in \mathcal{H}$. The equations of $\omega^T M = 0$ are given by:

$$\begin{array}{c} \vdots \\ \omega^T_{\eta^h}(YDg^h(\eta^h))^T + \omega^T_\gamma T^h \cdot Y^T = 0. \quad (4.16.a) \\ \vdots \\ \sum_{h\in\mathcal{H}} \omega^T_{\eta^h} = 0. \quad (4.16.b) \end{array} \tag{4.16}$$

Since $(YDg^h(\eta^h))^T = \left(Dg^h(\eta^h)\right)^T Y^T$, then equation (4.16.$a$) becomes:

$$\left(\omega^T_{\eta^h}\left(Dg^h(\eta^h)\right)^T + \omega^T_\gamma T^h\right) Y^T = 0.$$

With Y^T full row rank, then

$$\omega_{\eta^h}^T \left(Dg^h(\eta^h)\right)^T + \omega_\gamma^T T^h = 0$$

$$\omega_{\eta^h} = -\left(Dg^h(\eta^h)\right)^{-1} T^h \omega_\gamma \tag{4.17}$$

where equation (4.17) follows by taking transposes and noting that $Dg^h(\eta^h)$ is invertible and T^h is diagonal. This equation (4.17) holds $\forall h \in \mathcal{H}$. From $(4.16.b)$:

$$\sum_{h\in\mathcal{H}} \left(Dg^h(\eta^h)\right)^{-1} T^h \omega_\gamma = 0. \tag{4.18}$$

By definition, $\left(Dg^h(\eta^h)\right)^{-1} = D\tilde{f}^h(\theta^h)$ and T^h is a diagonal matrix with diagonal terms that are all nonzero. If f^h is given by the canonical representation, then $D\tilde{f}^h(\theta^h)$ is a diagonal matrix with strictly positive diagonal terms. Thus, $D\tilde{f}^h(\theta^h)\cdot T^h$ is a diagonal matrix and the diagonal terms are all nonzero. This holds $\forall h \in \mathcal{H}$. Adding up over all households, the matrix $\sum_{h\in\mathcal{H}} D\tilde{f}^h(\theta^h) \cdot T^h$ is diagonal and the diagonal terms are all nonzero.

The matrix $\sum_{h\in\mathcal{H}} D\tilde{f}^h(\theta^h) \cdot T^h$ has full rank under the canonical representation for $\left(f^h\right)_{h\in\mathcal{H}}$. The transaction costs mappings that are used in the statement of the theorem are those defined in lemma 4.2 as belonging to an open set around the canonical representation. In this open set, $\sum_{h\in\mathcal{H}} D\tilde{f}^h(\theta^h) \cdot T^h$ has full rank.

Therefore, $\sum_{h\in\mathcal{H}} \left(Dg^h(\eta^h)\right)^{-1} \cdot T^h$ has full rank and so (4.18) implies that $\omega_\gamma = 0$. From (4.17), $\omega_{\eta^h} = 0 \; \forall h \in \mathcal{H}$. As $\omega^T = 0$, the matrix M has full row rank. ■

This concludes the proof under case I (both subcases).

Case II: $\Delta x_h^T = 0$ for some $h \in \mathcal{H}$

I will show that over a generic subset of $\mathcal{E} = \{(e^h)_{h\in\mathcal{H}} : e^h >> 0\}$, the system of equations (Φ, Φ') has no solution. Recall the system of equations (a subset of the equations $\nu^T \Psi = 0$):

$$\begin{array}{c} \vdots \\ \Delta u_h Du^h(x^h) + \Delta x_h^T D^2 u^h(x^h) - \Delta \lambda_h^T P - \Delta p^T \Lambda = 0. \quad (4.8.a) \\ \vdots \\ \\ \vdots \\ -\Delta x_h^T P^T + \Delta \eta_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}^T = 0. \quad (4.8.b) \\ \vdots \\ \\ \vdots \\ \Delta \lambda_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix} + \Delta \eta_h^T \sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) + \Delta q^T = 0. \quad (4.8.c) \\ \vdots \end{array} \tag{4.8}$$

Suppose $\exists h' \in \mathcal{H}$ such that $\Delta x_{h'}^T = 0$. From (4.8.a) and Φ, I obtain

$$\begin{aligned} \Delta u_{h'} Du^{h'}(x^{h\prime}) - \Delta \lambda_{h'}^T P - \Delta p^T \Lambda &= 0 \\ Du^{h'}(x^{h'}) - \lambda^{h\prime} P &= 0 \end{aligned}$$

which together imply that $\Delta p^T = 0$ and $\Delta \lambda_{h'}^T = \Delta u_{h'} \lambda^{h\prime}$. From (4.8.$b$), since $YDg^h(\eta^h)$ has full column rank, then $\Delta \eta_{h'}^T = 0$. From (4.8.c) and Φ, after plugging in $\Delta \lambda_{h'}^T = \Delta u_{h'} \lambda^{h\prime}$ and $\Delta \eta_{h'}^T = 0$ to (4.8.c), then $\Delta q^T = 0$.

For all other $h \neq h'$, postmultiply $\Delta u_h Du^h(x^h)$ by Δx_h and use both first order conditions in Φ and (4.8.b) to get $\Delta u_h Du^h(x^h) \Delta x_h = 0$. Next, postmultiply (4.8.a)

by Δx_h and use (4.8.b) and (4.8.c) (as in the proof of claim 4.6) to arrive at

$$\Delta x_h^T D^2 u^h(x^h)\Delta x_h = -\Delta\eta_h^T \left(\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)\right)\Delta\eta_h. \tag{4.9}$$

In (4.9), the left hand side is strictly negative if $\Delta x_h \neq 0$ and the right hand side is nonnegative. Thus $\Delta x_h^T = 0\ \forall h \in \mathcal{H}$. From (4.8.$b$), since $YDg^h(\eta^h)$ has full column rank, then $\Delta\eta_h^T = 0\ \ \forall h \in \mathcal{H}$.

Thus $\forall h \in \mathcal{H}$, $\Delta\lambda_h^T = \Delta u_h \lambda^h$ and $\left(\left(\Delta u_h, \Delta\lambda_h^T\right)_{h\in\mathcal{H}}, \Delta b\right)$ are the only nonzero elements of ν. As such, the following is the equation from $\nu^T\Psi = 0$ corresponding to derivatives with respect to p :

$$\sum\nolimits_{h\in\mathcal{H}} \Delta\lambda_h^s \left(e_{\setminus L}^h(s) - x_{\setminus L}^h(s)\right)^T = 0 \quad \forall s \geq 0 \tag{4.19}$$

where $\left(e_{\setminus L}^h(s) - x_{\setminus L}^h(s)\right)$ is the $(L-1)-$dimensional vector of household negative excess demand with the numeraire commodity excluded. For the analysis to hold at this point, I must use the assumption that $L \geq 2$. Plugging in $\Delta\lambda_h^T = \Delta u_h \lambda^h$ into equation (4.19) and only considering the first physical commodity $l = 1$ and the final $s > 0$ states, I have

$$\sum\nolimits_{h\in\mathcal{H}} \Delta u_h \lambda^h(s)\left(e_1^h(s) - x_1^h(s)\right)^T = 0 \quad \forall s > 0$$

or in matrix notation

$$\begin{pmatrix} \lambda^1(1)(e_1^1(1) - x_1^1(1)) & .... & \lambda^H(1)(e_1^H(1) - x_1^H(1)) \\ \vdots & & \vdots \\ \lambda^1(S)(e_1^1(S) - x_1^1(S)) & .... & \lambda^H(S)(e_1^H(S) - x_1^H(S)) \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_H \end{pmatrix} = 0.$$

From lemma 4.3, generically on $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$, $\Delta u_h = 0\ \ \forall h \in \mathcal{H}$. Thus $\Delta\lambda_h^T = 0\ \ \forall h \in \mathcal{H}$.

The following is the equation from $\nu^T\Psi = 0$ corresponding to derivatives with respect to γ :

$$\sum\nolimits_{h\in\mathcal{H}} \Delta\lambda_h^T \left(\Upsilon_1\right)^T + \Delta b \left(\Upsilon_3\right)^T = 0.$$

Since $\Delta\lambda_h^T = 0\ \forall h \in \mathcal{H}$ and $\left(\Upsilon_3\right)^T = \left(.., q_j \sum_{h\in\mathcal{H}} \left(\eta_j^h - g_j^h(\eta^h;\gamma)\right), ..\right)$ has generic full row rank by corollary 4.1, then $\Delta b = 0$. The entire vector $\nu^T = 0$, which cannot be since Φ' guarantees that $\nu^T\nu/2 = 1$. I conclude that generically case II is not possible. This completes the proof of the theorem. ■

4.4 Proofs of Lemmas

Proof of Lemma 4.2

Proof. From lemma 4.1, take any endowment $\left(e^h\right)_{h\in\mathcal{H}}$ from the generic subset of $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$. Then, given the canonical representation for f^h, the resulting equilibrium variables will be regular values of Φ. Define the parameters as $\bar{\sigma} = \left(\bar{e}^h, \bar{u}^h, \bar{f}^h\right)_{h\in\mathcal{H}}$ where $\bar{f}^h$ has the canonical representation. For that $\bar{\sigma}$, there exist finitely many equilibria $\left(\bar{\xi}_i, \bar{\sigma}\right)$ $i = 1, .., I$ where $\bar{\xi}_i = \left(\left(\bar{x}_i^h, \bar{\eta}_i^h\right)_{h\in\mathcal{H}}, \bar{p}_i, \bar{q}_i\right)$. Implicit in lemma 1 is the result that there exists an open set Σ' for all regular values $\bar{\sigma}$ and open sets Ξ_i' such that $\bar{\xi}_i \in \Xi_i'$ $\forall i = 1, .., I$. Further, the sets Ξ_i' are disjoint across i and $\forall\sigma \in \Sigma'$, $\exists!$ equilibrium $\xi_i \in \Xi_i'$.

The parameters $\sigma = \left(e^h, u^h, f^h\right)_{h\in\mathcal{H}}$ in the open set Σ' will be composed of transaction costs mappings $\left(f^h\right)_{h\in\mathcal{H}}$ in an open set around the canonical representation. At $\bar{\sigma}$, then $\tilde{f}^h(\bar{\theta}^h) = \left(\bar{\theta}_1^h, ..., \bar{\theta}_J^h\right) + \left(\bar{f}_1^h(\bar{\theta}^h), ..., \bar{f}_J^h(\bar{\theta}^h)\right)$ and equilibrium conditions imply that the mapping $\tilde{f}^h$ is invertible (claim 4.3). The set of invertible matrices is open. Thus, for any parameters $\sigma \in \Sigma'$, the mapping $\tilde{f}^h$ defined as $\tilde{f}^h(\theta^h) = \left(\theta_1^h, ..., \theta_J^h\right) + \left(f_1^h(\theta^h), ..., f_J^h(\theta^h)\right)$ will also be invertible. ■

Proof of Lemma 4.3

Proof To prove this, first define

$$Z = \begin{pmatrix} \lambda^1(1)(e_1^1(1) - x_1^1(1)) & .... & \lambda^H(1)(e_1^H(1) - x_1^H(1)) \\ \vdots & & \vdots \\ \lambda^1(S)(e_1^1(S) - x_1^1(S)) & .... & \lambda^H(S)(e_1^H(S) - x_1^H(S)) \end{pmatrix}.$$

I will show that generically on $\mathcal{E}$, the matrix

$$M' = D_{\xi,\omega} \begin{pmatrix} \Phi \\ Z\omega \\ \omega^T\omega/2 - 1 \end{pmatrix} = \begin{pmatrix} D_\xi \Phi_{|\Phi(\xi,\sigma)=0} \\ D_{\xi,\omega} Z\omega \\ \begin{pmatrix} 0 & | & \omega \end{pmatrix} \end{pmatrix}$$

has full row rank. Since M' has more rows than columns, if M' has full row rank, then the equations $\begin{pmatrix} \Phi \\ Z\omega \\ \omega^T\omega/2 - 1 \end{pmatrix} = 0$ will generically not hold. Thus, Z will generically have full column rank. To show that generically on $\mathcal{E}$, the matrix M' has full row rank, I have to show that the extended matrix

$$M = \begin{pmatrix} M' & | & \begin{pmatrix} D_e \Phi_{|\Phi(\xi,\sigma)=0} \\ D_e Z\omega \\ 0 \end{pmatrix} \end{pmatrix}$$

has full row rank.

Since this proof is independent from the proof in the body, notation will be repeated. To show that M has full row rank, premultiply by the row vector $u^T = \left(\Delta x^T, \Delta\lambda^T, \Delta\eta^T, \Delta p^T, \Delta q^T, \Delta z^T, \Delta\omega\right)$. The lemma is proved upon showing that $u^T = 0$. For convenience, the vector u^T is divided into the indicated subvectors which correspond sensibly with the following equations of $\begin{pmatrix} \Phi \\ Z\omega \\ \omega^T\omega/2 - 1 \end{pmatrix}$:

$$\begin{aligned}
\Delta x^T &\iff FOCx \\
\Delta\lambda^T &\iff BC \\
\Delta\eta^T &\iff FOC\eta \\
\Delta p^T &\iff MCx \\
\Delta q^T &\iff MC\eta \\
\Delta z^T &\iff Z\omega \\
\Delta\omega &\iff \omega^T\omega/2 - 1.
\end{aligned}$$

I shall list the equations of $u^T M = 0$ in the order that is most convenient to obtain $u^T = 0$. At my disposal are $\Phi(\xi, \sigma) = 0$ and $\omega \neq 0$.

First, for the columns corresponding to derivatives with respect to x^h and e^h for any $h \in \mathcal{H}$:

$$\begin{aligned}
\Delta x_h^T D^2 u^h(x^h) - \Delta\lambda_h^T P - \Delta p^T \Lambda - \Delta z^T \Lambda_3^h &= 0 \\
\Delta\lambda_h^T P + \Delta p^T \Lambda + \Delta z^T \Lambda_3^h &= 0
\end{aligned}$$

where the matrices P and Λ are as defined previously and Λ_3^h is the $S \times G$ matrix

$$\Lambda_3^h = \left[0 \;\; \middle| \;\; \begin{pmatrix} \begin{pmatrix} \lambda^h(1)\omega^h & \vec{0} \end{pmatrix} & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & \begin{pmatrix} \lambda^h(S)\omega^h & \vec{0} \end{pmatrix} \end{pmatrix} \right].$$

By assumption 2, $\left(\Delta x_h^T, \Delta \lambda_h^T\right) = 0\ \forall h \in \mathcal{H}$, $\Delta p_l(s) = 0 \quad \forall (l,s) \notin \{(1,1),..,(1,S)\}$, and

$$\Delta p_1(s) + \Delta z_s \lambda^h(s)\omega^h = 0 \quad \forall s > 0 \text{ and } \forall h \in \mathcal{H}. \tag{4.20}$$

Second, for the columns corresponding to derivatives with respect to η^h for any $h \in \mathcal{H}$ and q :

$$\begin{aligned} \Delta\eta_h^T \sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) + \Delta q^T &= 0 \qquad (4.21) \\ \textstyle\sum_{h\in\mathcal{H}} \Delta\eta_h^T(-\lambda^h(0)) &= 0. \end{aligned}$$

From corollary 4.1, for a generic subset of $\mathcal{E}$, $\eta_j^h \neq 0\ \forall j$, $\forall h$. By the definition of $\eta_j^h = \tilde{f}_j^h\left(\theta^h\right)$, this implies that $\theta_j^h \neq 0\ \forall j$, $\forall h$. For any $h \in \mathcal{H}$, from claim 4.5, the matrix $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is negative semidefinite. Moreover, from equation (4.7) (recall the equation is given by

$$\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) = -\lambda^h(0) D^2 F^h\left(\theta^h\right)\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}), \tag{4.7}$$

if $D^2F^h\left(\theta^h\right)\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is positive definite, then $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is negative definite. By definition, $D^2F^h\left(\theta^h\right)$ is positive definite so long as $\theta_j^h \neq 0$ $\forall j$. Multiplication by $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ preserves the positive definiteness (for open sets of transaction costs mappings $\tilde{f}^h$ around the canonical representation). Thus,

$\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)$ is negative definite.

Postmultiply the first equation of (4.21) by $\Delta\eta_h \lambda^h(0)$. The first term

$$\lambda^h(0)\Delta\eta_h^T \left(\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)\right) \Delta\eta_h \leq 0 \quad (< 0 \text{ if } \Delta\eta_h \neq 0) \quad \forall h \in \mathcal{H}.$$

This is because the matrix $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)$ is negative definite and $\lambda^h(0) > 0$. The second term $\Delta q^T \Delta\eta_h \lambda^h(0)$ will be equal to 0 when summed over all households. The only way that

$$\sum_{h\in\mathcal{H}} \lambda^h(0)\Delta\eta_h^T \left(\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)\right) \Delta\eta_h + \sum_{h\in\mathcal{H}} \Delta q^T \Delta\eta_h \lambda^h(0) = 0$$

is if $\Delta\eta_h^T = 0 \; \forall h \in \mathcal{H}$.[9] From (4.21), $\Delta q^T = 0$.

Finally, for the columns corresponding to derivatives with respect to λ^h for any $h \in \mathcal{H}$ and ω :

$$\begin{aligned} \Delta z^T \Lambda_4^h &= 0 \\ \Delta z^T Z + \Delta\omega\,(\omega) &= 0 \end{aligned} \tag{4.22}$$

where Λ_4^h is the $S \times (S+1)$ matrix

$$\Lambda_4^h = \left(\begin{array}{c|ccc} & (e_1^h(1) - x_1^h(1))\omega^h & 0 & 0 \\ 0 & 0 & ... & 0 \\ & 0 & 0 & (e_1^h(S) - x_1^h(S))\omega^h \end{array} \right).$$

[9] The key realization with this model is that the nonlinearity in the asset payouts (in this case, the inclusion of the negative definite second derivative matrix $\left(\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)\right)$ in (4.21)) leads to the inefficiency in equilibrium allocation, whereas the inefficiency is absent with linear asset payouts.

From (4.22), I obtain that

$$\Delta z_s(e_1^h(s) - x_1^h(s))\omega^h = 0 \quad \forall h \in \mathcal{H} \text{ and } \forall s > 0.$$

Generically (corollary 4.1), $(e_1^h(s) - x_1^h(s)) \neq 0 \;\; \forall s, \forall h$ and since $\omega \neq 0$, then for some h, $(e_1^h(s) - x_1^h(s))\omega^h \neq 0 \;\; \forall s > 0$. Thus $\Delta z_s = 0 \;\; \forall s > 0$. From (4.20), the remaining terms of Δp^T are equal to 0, namely $\Delta p_1(s) = 0$ for $s > 0$. With $\omega \neq 0$, the scalar $\Delta\omega = 0$ (from (4.22)). Thus $u^T = 0$ and the proof of lemma 4.3 is complete.■

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