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Density Functions for Navigation Function Based Systems

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***Abstract*— In this paper, we present a scheme for constructing density functions for systems that are almost globally asymptotically stable (i.e., systems for which all trajectories converge to an equilibrium except for a set of measure zero) based on Navigation Functions. Although recently-proven converse theorems guarantee the existence of density functions for such systems, results are only existential and the construction of a density function for almost globally asymptotically stable systems remains a challenging task. We show that for a specific class of dynamical systems that are defined based on a navigation function, a density function can be easily derived from the system's underlying navigation function.**

I. INTRODUCTION

For more than a century, Lyapunov's method has been the major tool used in stability analysis of dynamical systems. Recently, however, a new scheme was proposed by Rantzer [7], which can be thought of as a "dual" to Lyapunov's method. Instead of checking for a positive definite "energy-like" function whose directional derivative along the trajectories of the dynamical system is negative definite, in Rantzer's approach, one searches for a positive "density function" such that the divergence of the vector field times the density function is positive almost everywhere. This will guarantee attractivity of the equilibrium for almost all initial conditions. This is of course a weaker result than global asymptotic stability. However, it is a powerful tool for controller synthesis as well as controller composition. This is due to the fact that the synthesis condition for the almost global stability criterion is convex. As a result, (at least in the case of polynomial vector fields) convex optimization can be used to search for density functions *and* the controller simultaneously [6]. Furthermore, the convexity argument allows us to compose different controllers and be able to find a density for the composed system [9].

Since the pioneering work of Rantzer, several authors have been able to prove different results analogous to the ones available for asymptotic stability. For example, Rantzer has shown that given a Lyapunov function which proves global asymptotic stability, one can construct a density function by using the powers of the reciprocal of the Lyapunov function. Also, Monzón [5] and Rantzer [8] have been able to prove converse theorems for almost global stability, similar to converse theorems that guarantee existence of a Lyapunov function for asymptotically stable systems. In [8], Rantzer has proven that existence of density functions is a necessary and sufficient condition for systems that are almost globally stable. Unfortunately, similar to the converse Lyapunov theorems, such results are only existential and can not be used to construct density functions. Some remarks on the structure of density function candidates are discussed in [1] where it is pointed out that the C^1 continuity requirements on the density functions by converse theorems poses strong constraints in the case of systems with negative divergence in the vicinity of their saddle points. The purpose of this paper is to show that in certain special cases, such construction is indeed possible. Specifically, we show that for dynamical systems that are constructed using Rimon Koditschek Navigation Functions [2], one can readily construct a density function using the Navigation Function.

Navigation functions have been proven ex-

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tremely useful for rigorously constructing paths that navigate a kinematic robot in a spherical workspace while avoiding spherical obstacles. The constructive procedure utilizes Morse theory [4] to construct an artificial potential function which is zero at the goal state, and uniformly maximal at the boundary of the workspace and obstacles. Furthermore, all the critical points of this potential are saddle points except for the goal state where the critical point is stable. By constructing a gradient flow based on this potential, it is possible to guarantee that for almost all initial conditions the trajectories converge to the goal state while avoiding obstacles.

One can immediately notice parallel's between the density function and a Navigation Function. This similarity leads us to ask whether it is possible to construct a density function from a Navigation function. We will show that the answer to this question is indeed positive.

The rest of the paper is organized as follows: In Section II we present some preliminary definitions. Section III presents a review of Navigation Functions while section IV reviews some results on Dual Lyapunov Techniques. Our main result is presented in section V. The paper concludes with section VI.

II. PRELIMINARIES

A. Definitions

Let $V : \mathcal{M} \rightarrow R$ be a smooth function and $\mathcal{M} \subset R^n$ a smooth manifold with boundary. A point $p \in \mathcal{M}$ is called a critical point of V if $\nabla V(p) = 0$ where $\nabla V \triangleq \left[\frac{\partial V}{\partial x_1} \ \ldots \ \frac{\partial V}{\partial x_n}\right]^T$ is the gradient of V. The divergence of V is defined as $\mathrm{div(V)} \equiv \nabla \cdot \mathrm{V} \triangleq \frac{\partial \mathrm{V}}{\partial \mathrm{x_1}} + \ldots + \frac{\partial \mathrm{V}}{\partial \mathrm{x_n}}$. A critical point p is called non-degenerate iff the matrix $H_V(p) \triangleq \left[\frac{\partial^2 V}{\partial x^i \partial x^j}\right]$ is non-singular. The matrix $H_V(\cdot)$ is called the Hessian of V where $(x^1, \ldots, x^n)$ is a coordinate system. The matrix $H_V(\cdot)$ is symmetric and the non-degeneracy of p does not depend on the coordinate system [4]. A smooth function V is called a *Morse* function if all its critical points are non-degenerate. Function V is called polar if it has a unique minimum in $\mathcal{M}$ and admissible if it attains the unit value uniformly across the boundary of $\mathcal{M}$, that is $\partial\mathcal{M} = \varphi^{-1}(1)$. The boundary of a $\mathcal{M}$ is denoted by $\partial\mathcal{M}$. Let the function $f(x) = [f_1(x), \ldots f_n(x)]$ denote a vector field. The matrix $J_f(x)$ whose ij'th element is $[J_f(x)]_{ij} = \frac{\partial f_i}{\partial x^j}(x)$ is called the Jacobian of the vector field f at x. Given any $x_0 \in \mathbb{R}^n$, we denote by $\phi_t(x_0)$ for $t \geq 0$ the solution of $\dot{x}(x) = f(x(t))$ with $x(0) = x_0$.

III. NAVIGATION FUNCTIONS

Navigation Functions (NFs) are a special category of Potential Functions. Their negated gradient vector field is attractive towards the goal configuration and repulsive with respect to obstacles. Considering a trivial system described kinematically as $\dot{q} = u$ the basic idea behind navigation functions is to use a control law of the form $u = -\nabla\varphi(q)$ where $\varphi(q)$ is a navigation function, to drive the system to its destination.

It has been shown (Koditschek and Rimon [2]) that strict global navigation (i.e. with a globally attracting equilibrium state) is not possible and a smooth vector field on any sphere world, which has a unique attractor, must have at least as many saddles as obstacles.

Formally a Navigation Function is defined as follows:

Definition 1: [2] Let $\mathcal{F} \subset E^n$ be a compact connected analytic manifold with boundary. A map $\varphi : \mathcal{F} \rightarrow [0, 1]$, is a navigation function if it is:

1) Analytic on $\mathcal{F}$
2) Polar on $\mathcal{F}$, with minimum at $q_d \in \mathring{\mathcal{F}}$
3) Morse on $\mathcal{F}$
4) Admissible on $\mathcal{F}$

For the intuition behind the definition of navigation functions, the interested reader can refer to [2].

IV. DUAL LYAPUNOV TECHNIQUES

The dual Lyapunov criterion for convergence introduced by Rantzer, states that:

Theorem 1: [7] Given the equation $\dot{x} = f(x(t))$, where $f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in \mathrm{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in \mathbb{R}^n : |x| \geq 1\}$ and

$$[\nabla \cdot (f\rho)](x) > 0$$

for almost all x. Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \geq 0$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

The converse result regarding the necessary and sufficient conditions for almost global stability of non-linear systems, is stated below:

Theorem 2: [8] Given $f \in \mathrm{C}^2(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}}))$, suppose that the system $\dot{x} = f(x)$ has a stable equilibrium in $x = 0$ and no solutions with finite escape time. Then, the following two conditions are equivalent:

1) For almost all initial states $x(0)$ the solution $x(t)$ tends to zero as $t \to \infty$.
2) There exists a non-negative $\rho \in \mathrm{C}^1(\mathbb{R}^{\mathrm{n}} \backslash \{0\}, \mathbb{R})$ which is integrable outside a neighborhood of zero and such that $[\nabla \cdot (f\rho)](x) > 0$ for almost all x.

V. Navigation Vector Fields

We call Navigation Vector Field (NVF), a vector field that has navigation like properties. Those properties are captured in the following:

Definition 2: Let $\mathcal{F} \subset E^n$ be a compact connected analytic manifold with boundary. The smooth manifold map $f : \mathcal{F} \to T\mathcal{F}$ is a *navigation vector field* if:

- The system $\dot{x} = f$ is almost GAS
- f is transverse across $\partial\mathcal{F}$

The above definition is motivated by the properties of navigation functions. Clearly the first requirement establishes the almost everywhere convergence of the system $\dot{x} = f$ while the second property establishes that the any trajectory will be safely brought to the origin without collisions. Our next step is to propose a construction of such a vector field which we will call a "canonical" navigation vector field

A. Construction

Let $\lambda_{\min,i}(x_{s,i})$ to be the minimum eigenvalue at the saddle point $x_{s,i}$. The corresponding unit eigenvector is u_i. Let $d_i(x) = \|x - x_{s,i}\|^2$ be the squared metric of the distance of point x from the saddle point i for $i \in \{1 \ldots n_s\}$ where n_s is the number of saddle points. Let I denote the $n \times n$ identity matrix where n is the workspace dimension. We can now define the matrix $U_i = u_i u_i^T + \varepsilon \mathrm{I}$ for $i \in \{1 \ldots n_s\}$ where $0 < \varepsilon \leq 1$. Since the matrix $u_i u_i^T$ is positive semidefinite, the matrix U_i will be positive definite for positive ε. Define $U_{n_s+1} = U_{n_s+2} = \mathrm{I}$. A metric of the distance from the destination configuration can be encoded by using the navigation function, so we can define $d_{n_s+1} = \varphi$ and since the navigation function $\varphi(\partial\mathcal{F}) = 1$ we can encode a metric of the distance from the workspace boundary by denoting $d_{n_s+2} = 1 - \varphi$. Define $\bar{d}_j = \prod_{\substack{i=1 \\ i \neq j}}^{n_s+2} d_i$ Then D_φ is defined as

$$D_\varphi = \mu \sum_{i=1}^{n_s+2} \frac{\bar{d}_i}{\bar{d}_i + d_i} U_i \tag{1}$$

where μ a positive constant. The function $\frac{\bar{d}_i}{\bar{d}_i + d_i}$ is an analytic switch which takes values between zero and 1. The properties of the matrix D_φ are provided in the following:

Lemma 1: The matrix $D_\varphi(x)$ defined in eq. (1) has the following properties

1) a) $D_\varphi(x_{s,i}) = \mu U_{s,i} + \varepsilon\mu \mathrm{I}$
   b) $D_\varphi(\partial\mathcal{F}) = \mu \mathrm{I}$
   c) $D_\varphi(0) = \mu \mathrm{I}$
2) a) $\frac{\partial}{\partial x} D_\varphi(x_{s,i}) = 0$
   b) $\frac{\partial}{\partial x} D_\varphi(0) = 0$
3) $D_\varphi > 0$
4) $q^T D_\varphi q \leq 2(n_s + 2)\mu \|q\|^2, \forall q \in \mathbb{R}^n$

Proof: Property 1: (a) By direct computation we have that at the saddle point i, $d_i(x_{s,i}) = 0, \quad \bar{d}_j = 0$ for $j \neq i$ hence $D_\varphi(x_{s,i}) = \mu U_{s,i} + \varepsilon \mathrm{I}$

(b) At the workspace boundary it holds that $\varphi(\partial\mathcal{F}) = 1$ hence $d_{n_s+2} = 0$ and $d_j = 0$, $j \neq n_s + 2$ and $D_\varphi(\partial\mathcal{F}) = \mu \mathrm{I}$

(c) At the origin $\varphi = 0$ hence $d_{n_s+1} = 0$ and $d_j = 0$, $j \neq n_s + 1$ so $D_\varphi(0) = \mu \mathrm{I}$

Property 2: (a,b) For this property, observe that $d_i'(x_{s,i}) = 0$ for $i \in \{1 \ldots n_s\}$ where $f'(x) = \frac{\partial f(x)}{\partial x}$ and $d'_{n_s+1}(0) = 0$ since $\nabla\varphi(0) = 0$. Also note that $\frac{\partial}{\partial x} \frac{\bar{d}_i}{\bar{d}_i + d_i} = \frac{\bar{d}_i d_i' - \bar{d}_i' d_i}{(\bar{d}_i + d_i)^2}$ so at $x_{s,i}$ and at 0 it will hold that $d_i = d_i' = 0$ and $\bar{d}_j = \bar{d}_j' = 0$ for $j \neq i$ since they will either contain d_i or d_i', so $D_\varphi = 0$ at those locations.

Property 3: Since the matrix $u_i u_i^T$ is a matrix with one eigenvalue equal to unit and the rest eigenvalues zero it follows that the matrix $U_i = u_i u_i^T + \varepsilon \mathrm{I}$ is positive definite for $\varepsilon > 0$. Since the matrix D_φ is the sum of positive definite matrices multiplied by positive scalars, it will still be positive definite.

Property 4: First observe that $0 \leq \frac{\bar{d}_i}{\bar{d}_i+d_i} \leq 1$ Multiplying D_φ left and right with the unit vectors $\hat{q}$ we get: $\hat{q}^T D_\varphi \hat{q} = \mu \sum_{i=1}^{n_s+2} \frac{\bar{d}_i}{\bar{d}_i+d_i} \hat{q}^T U_i \hat{q} \leq \mu \sum_{i=1}^{n_s+2} \hat{q}^T U_i \hat{q} \leq \mu \sum_{i=1}^{n_s+2} (1+\varepsilon) \leq 2(n_s+2)\mu$. Multiplying both sides by $\|q\|^2$ we get the result: $q^T D_\varphi q \leq 2(n_s+2)\mu \|q\|^2, \forall q \in \mathbb{R}^n$ ∎

The main feature of the matrix D_φ is that it allows for local modifications of the vector field in the vicinity of the saddle points. Without loss of generality we assume in the following analysis that the destination configuration of the navigation function is the origin.

B. Main Result

The following is the main result of this paper

Proposition 1: Consider the system

$$\dot{x} = -D_\varphi \nabla \varphi \tag{2}$$

with $D_\varphi(x)$ constructed according to eq. (1). Then there exists an $a_0 > 0$ and an $\varepsilon_0 > 0$ such that the function

$$\rho = \varphi^{-a}$$

is a density function for system (2) as long as $a \geq a_0$ and $0 < \varepsilon \leq \varepsilon_0$.

Proof: Our analysis will be performed for the two dimensional case but the results can be readily extended to higher dimensions. The first observation is that the proposed density function is integrable outside a neighborhood of zero. By construction ρ is positive definite. Setting $f = -D_\varphi \nabla \varphi$ from the divergence criterion, we get: $\nabla \cdot (\rho f) = \nabla \rho f + \rho \nabla \cdot (f)$. We have that $\nabla \rho = -\frac{a}{\varphi^{a+1}} \nabla \varphi$. Hence

$$\nabla \cdot (\rho f) = \frac{1}{\varphi^{a+1}} \left(a \nabla^T \varphi D_\varphi \nabla \varphi - \varphi \nabla \cdot (D_\varphi \nabla \varphi) \right) \tag{3}$$

Expanding the term $\nabla \cdot (D_\varphi \nabla \varphi)$ we get: $\nabla \cdot (D_\varphi \nabla \varphi) = \nabla \cdot \left(\begin{bmatrix} d_{11}\varphi_x + d_{12}\varphi_y \\ d_{21}\varphi_x + d_{22}\varphi_y \end{bmatrix} \right)$ where the notation f_x and f_{xx} denotes the first and second derivatives of f wrt x and d_{ij} is the ij'th element of D_φ. For a navigation function all critical points except the origin are saddle points [2]. At a saddle point $x_{s,i}$ we have that $\nabla \varphi(x_{s,i}) = 0$ hence the terms that contain derivatives of φ are zeroed out. Also $\varphi_{xx} + \varphi_{yy} = \lambda_{\min} + \lambda_{\max}$ since the trace of the Hessian is invariant. Thus we have that: $\nabla \cdot (D_\varphi \nabla \varphi)(x_{s,i}) = u_i^T H_\varphi(x_{s,i}) u_i + \varepsilon\mu(\lambda_{\min,i} + \lambda_{\max,i}) = \mu\lambda_{\min,i} + \varepsilon\mu(\lambda_{\min,i} + \lambda_{\max,i})$

By setting $\varepsilon < \left| \frac{\lambda_{\min,i}}{\lambda_{min,i}+\lambda_{\max,i}} \right| \triangleq \varepsilon_{0,i}$ we get that $\nabla \cdot (D_\varphi \nabla \varphi)(x_{s,i}) < 0$.

Since $x_{s,i}$ is a saddle point, the minimum eigenvalue of the Hessian is necessarily negative (existence of the unstable submanifold).

Hence we have that exactly on the saddle points $\nabla \cdot (\rho f)(x_s) = -\frac{\lambda_{\min}(x_s)}{\varphi^a(x_s)} > 0$

Close to the destination configuration we have that both $\nabla \varphi(0) = 0$ and $\varphi(0) = 0$ hence we need to analyze both terms of eq. 3 to understand its behavior. Noting that (see [2], Proof of Proposition 3.2) $H_\varphi(0) = 2\beta^{-1/k}(0)\mathrm{I}$ and from Lemma 1, property 1, we have that $D_\varphi(0) = \mu\mathrm{I}$ and from property 2 that $\frac{\partial}{\partial x} D_\varphi(0) = 0$, the Taylor expansions of φ and D_φ around the origin are as follows: $\varphi(x) = \beta^{-1/k}(0)\|x\|^2 + O\left(\|x\|^3\right)$, $D_\varphi(x) = \mu\mathrm{I} + \mathrm{O}\left(\|\mathrm{x}\|^2\right)$. For the term $\nabla^T \varphi D_\varphi \nabla \varphi$ we have that $\nabla^T \varphi D_\varphi \nabla \varphi = 4\mu\beta^{-2/k}\|x\|^2 + O(\|x\|^3)$ and for the term $\varphi \nabla \cdot (D_\varphi \nabla \varphi)$ we have that: $\varphi \nabla \cdot (D_\varphi \nabla \varphi) = 4\mu\beta^{-2/k}\|x\|^2 + O(\|x\|^3)$ Hence from eq. (3) we get that: $\varphi^{a+1} \nabla \cdot (\rho f) = (a-1)\, 4\mu\beta^{-2/k}\|x\|^2 + O(\|x\|^3)$ So choosing $a > 1$ will render $\nabla \cdot (\rho f) > 0$ in a neighborhood of zero.

We have until now established the positivity of eq. (3) in the vicinity of critical points. To establish the global positivity of eq. (3), since D_φ is positive definite (property 3 in Lemma 1), we require that: $a > \frac{\max_{x \in \mathcal{F}} \varphi \nabla \cdot (D_\varphi \nabla \varphi)}{\min_{x \in \{\mathcal{F} - \mathcal{B}_\varepsilon(\mathcal{C})\}} \{\nabla^T \varphi D_\varphi \nabla \varphi\|}} \triangleq a_1$, where $\mathcal{C}$ is the set of critical points.

Since the workspace is bounded and the functions D_φ and φ are smooth, the existence of a finite a_1 is guaranteed. Let $\varepsilon_1 = \min_{i \in \{1, \dots n_s\}} \varepsilon_{0,i}$. The positivity of the divergence criterion of Theorem 1 is satisfied by choosing $a_0 = \max\{1, a_1\}$ and $\varepsilon_0 = \min\{1, \varepsilon_1\}$ which completes the proof. ∎

We can now state some properties of the proposed vector field:

Proposition 2: The vector field

$$f = -D_\varphi \nabla \varphi$$

defined in Proposition 1 with $0 < \varepsilon < \varepsilon_0$, where ε_0 is defined in the proof of Proposition 1, is a *navigation vector field*

Proof: By Proposition 1, choosing an $a \geq a_0$ the function $\rho = \varphi^{-a}$ is a density function for (2). Application of the dual Lyapunov criterion (Theorem 1) establishes the almost GAS property of $\dot{x} = f$.

For the transversality property we have that by property 1 of Lemma 1 it holds that $D_\varphi(\partial\mathcal{F}) = \mu\mathrm{I}$. Hence eq. (2) becomes: $\dot{x} = -\mu\nabla\varphi$ since $\mu > 0$ and by property 4 of Definition 1 we have that the vector field on the workspace boundary is transverse ■

Some additional properties of the vector field $-D_\varphi\nabla\varphi$, are provided by the following

Corollary 1: The navigation vector field established in Proposition 2 assuming appropriate choice of parameters, vanishes only at the critical points of φ while its Jacobian is non-degenerate over the critical set of φ

Proof: Since by property 3 of Lemma 1 $D_\varphi > 0$ the vector field vanishes only when $\nabla\varphi = 0$, which is true only at the set of critical points of φ.

We have that $D_\varphi\nabla\varphi = \begin{bmatrix} d_{11}\varphi_x + d_{12}\varphi_y \\ d_{21}\varphi_x + d_{22}\varphi_y \end{bmatrix}$. Taking the Jacobian at a critical point, since $\varphi_x = \varphi_y = 0$ we have that: $\frac{\partial}{\partial x}\left(D_\varphi\nabla\varphi\right) = \begin{bmatrix} d_{11}\varphi_{xx} + d_{12}\varphi_{xy} & d_{11}\varphi_{xy} + d_{12}\varphi_{yy} \\ d_{21}\varphi_{xx} + d_{22}\varphi_{yx} & d_{21}\varphi_{xy} + d_{22}\varphi_{yy} \end{bmatrix} = = D_\varphi H_\varphi$. We know by the Morse property of φ that $\det(\varphi) \neq 0$ at every critical point. By using the relation $\det(AB) = \det(A)\det(B)$ we only need to prove that $\det(D_\varphi) \neq 0$ at the critical points. But from property 3 of Lemma 1 $D_\varphi > 0$ hence the determinant is always positive and the Jacobian is non-degenerate at the critical points. ■

Due to the similarities of $-D_\varphi\nabla\varphi$ with $\nabla\varphi$ we will call the first a "canonical" navigation vector field and the system that this vector field is applied to a "canonical" navigation system.

A comparison of the convergence properties of canonical navigation systems with navigation function based systems is provided by the following result which will allow us to reason about the navigation function based system by examining the canonical system:

Proposition 3: Consider the system

$$\dot{x} = -K\nabla\varphi \tag{4}$$

where K a positive gain. Then there exists a $0 < \mu \leq \mu_0$ such that for almost all the same initial conditions $x_{(4)}(0) = x_{(2)}(0)$ the trajectories of (4) are bounded by the trajectories of (2) as follows: $\varphi\left(x_{(4)}(t)\right) \leq \varphi\left(x_{(2)}(t)\right),\ \forall t \geq 0$ Moreover there exist a spherical neighborhood $\mathcal{B}(0)$ around the origin for which for all $x_{(4)}(0) = x_{(2)}(0) \in \mathcal{B}(0)$ it holds that $\left\|x_{(4)}(t)\right\| \leq \left\|x_{(2)}(t)\right\|,\ \forall t \geq 0$

Proof: See Appendix A ■

VI. Conclusions

We have successfully derived a density function for a navigation function based system. The density function is applicable to a transformed smooth vector field which enjoys the navigation properties of the original navigation function vector field. Under several assumptions, the convergence results derived on the transformed vector field are propagated to the original vector field. This result will enable the exploitation of several features of dual Lyapunov techniques to robotic navigation. Initial results from the application of this approach to robotic navigation are reported in [3].

Further research includes finding density functions that are directly applicable to the primary navigation system.

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APPENDIX

A. Proof of Proposition 3

Proof: Taking the time derivative of φ across the trajectories of system (4) we get:

$$\dot{\varphi}_{(4)} = -K \left\|\nabla\varphi_{(4)}\right\|^2 \tag{5}$$

The time derivative of φ across the trajectories of system (2) is $\dot{\varphi}_{(2)} = -\nabla\varphi^T D_\varphi \nabla\varphi \geq -2(n_s+2)\mu \left\|\nabla\varphi\right\|^2$ by use of the property 4 of Lemma 1. Setting $\mu = \mu_1 \frac{K}{2(n_s+2)}$ with $0 < \mu_1 < 1$ we get

$$\dot{\varphi}_{(2)} > -\mu_1 K \left\|\nabla\varphi_{(2)}\right\|^2 \tag{6}$$

To prove the first part of the Proposition we need to establish that $\dot{\varphi}_{(4)}(x_{(4)}(t)) \leq \dot{\varphi}_{(2)}(x_{(2)}(t))$ for all $t \geq 0$ given that $x_{(4)}(0) = x_{(2)}(0)$. By equations (5) and (6) we have for $t = 0$ that

$$\dot{\varphi}_{(2)}(x_{(2)}(0)) > -\mu_1 \dot{\varphi}_{(4)}(x_{(4)}(0)) \tag{7}$$

By smoothness arguments, there exists a neighborhood of $\mathcal{B}_\varepsilon(x_{(2)}(0))$ around $x_{(2)}(0)$ such that the inequality (7) still holds as long as the initial conditions are not exactly on the saddle point. So in this neighborhood we have that $\varphi\left(x_{(4)}(t)\right) \leq \varphi\left(x_{(2)}(t)\right),\ t \in [0, \delta(\varepsilon)]$ for some increasing function $\delta(\cdot)$. By the selection of μ_1 we have that $\|D_\varphi \nabla\varphi\| \leq K\|\nabla\varphi\|$ hence we can assert that $x_{(4)}$ will exit exit $\mathcal{B}_\varepsilon(x_{(2)}(0))$ first. Let $g_{\max}(a) = \max_{x \in \varphi^{-1}(a)} \|\nabla\varphi(x)\|$ and $g_{\min}(a) = \min_{x \in \{\varphi^{-1}(a) - \mathcal{B}_\varepsilon(\mathcal{S})\}} \|\nabla\varphi(x)\|$ where $\mathcal{S}$ the set of saddle points. Since the reachable set of initial conditions, excluding the set $\mathcal{B}_\varepsilon(x_{(2)}(0))$ is bounded away from saddle points, $g_{\min}$ is non zero. Since the workspace is bounded, and φ is smooth, the maximum value of $\nabla\varphi$ is finite, hence the function $r(a) = \frac{g_{\min}(a)}{g_{\max}(a)}$ is well defined everywhere except at $a = 0$ where the limit exists and is $\lim_{x\to 0} r(x) = \frac{\lambda_{\min}(0)}{\lambda_{\max}(0)}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the Hessian of φ. This can be verified by considering that the origin is a non-degenerate critical point and hence a quadratic one so for appropriate coordinates near the origin $\varphi(x) = \lambda_{\min} x_1^2 + \lambda_{\max} x_2^2$ By setting $\mu_1 \leq \min_{a \in (0,1]} r(a) \triangleq \mu_2$ we have that whenever $\varphi(x_{(4)}(t)) = \varphi(x_{(2)}(t))$ system (4) will have a higher velocity than system (2), hence $\dot{\varphi}(x_{(4)}(t)) < \dot{\varphi}(x_{(2)}(t))$. This means that as long as at some t it is true that $\varphi(x_{(4)}(t)) < \varphi(x_{(2)}(t))$ then it be true for all $t' \geq t$. But since $x_{(4)}$ will exit first $\mathcal{B}_\varepsilon(x_{(4)}(0))$ we have that $\varphi\left(x_{(4)}(t)\right) \leq \varphi\left(x_{(2)}(t)\right),\ \forall t \geq 0$.

Now let $\rho_{\max}$ be the maximum radius of a disk centered at the origin which has no intersections with obstacles. Then this circle contains no saddle points, since saddles occur between the workspace boundary and the obstacles. Alternatively the radius $\rho_{\max}$ can be fixed so that the circle is bounded away from saddle points and obstacles. Moreover we constrain the $\rho_{\max}$ even more such that the Hessian of φ in the disk defined by $\rho_{\max}$ is everywhere positive definite and its minimum eigenvalue is greater than an $\lambda_0 > 0$. Now since the Hessian is positive definite, the level sets of φ inside the circle are convex. Moreover the non-zero minimum eigenvalue establishes that the intersections of the level sets of φ with circles centered at the origin will be performed at obtuse angles, hence the unit vector of the gradient $-\widehat{\nabla\varphi}$ will have a positive projection on the inside pointing unit vector that is perpendicular to the circle's circumference. Denote the value of this projection by $p(x)$. For $\rho \leq \rho_{\max}$ define $g'_{\max}(\rho) = \max_{\|x\|=\rho} \|\nabla\varphi(x)\|$ and $g'_{\min}(\rho) = \min_{\|x\|=\rho} \|p(x)\nabla\varphi(x)\|$. Obviously $g'_{\max}(\rho)$ and $g'_{\min}(\rho)$ are non-zero except at the origin and are bounded due to smoothness and compactness arguments. Hence the function $r'(\rho) = \frac{g'_{\min}(\rho)}{g'_{\max}(\rho)}$ is well defined, finite and nonzero everywhere except at $a = 0$ where the limit exists and is $\lim_{x\to 0} r'(x) = \frac{\lambda_{\min}(0)}{\lambda_{\max}(0)}$. By setting $\mu_1 \leq \min_{\rho \in (0, \rho_{\max}]} r'(\rho) \triangleq \mu_3$ we have that whenever $\left\|x_{(4)}(t)\right\| = \left\|x_{(2)}(t)\right\|$ system (4) will have a velocity whose projection on the perpendicular of the circle's circumference will be higher than the velocity of system (2). This means that as long as at some t it is true that $\left\|x_{(4)}(t))\right\| \leq \left\|(x_{(2)}(t))\right\|$ then it will be true for all $t' \geq t$. But since the initial conditions are the same we have that $\left\|x_{(4)}(t)\right\| \leq \left\|x_{(2)}(t)\right\|,\ \forall t \geq 0$.

Choosing $\mu_0 = \frac{K}{2(n_s+2)} \min\{1, \mu_2, \mu_3\}$ completes the proof. ■