Network Synthesis for Dynamical System Stabilization

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***Abstract*—We present our recent results in the area of distributed control over wireless networks. In our previous work, we introduced the concept of a Wireless Control Network (WCN), where the network acts as a decentralized structured controller. In this case, the network is not used only as a communication medium (as in traditional control paradigms), but instead as a fully distributed computational substrate. We show that the dynamics of the plant dictate the types of network topologies that can be used to stabilize the system. Finally, we describe how to obtain a stabilizing configuration for the WCN if the topological conditions are satisfied.**

I. Introduction

With recent advancements in wireless technology, multi-hop wireless networks have emerged as a cost-effective way to monitor the performance of large-scale industrial control systems. Wireless sensor networks are used to route sensor measurements to gateways and data centers to enable efficient plant management. In recent years, Wireless Networked Control Systems (WNCSs) – that employ wireless networks to close the loop – have started to find their place in industrial automation. These systems primarily use multi-hop wireless networks as a communication medium. In this case the nodes simply route information to and from a dedicated controller (as shown in Fig. 1(a)).

In [1] we introduced the concept of the Wireless Control Network (WCN), where each node in the network implements a simple distributed algorithm (based on a linear iterative procedure). This causes the entire network to behave as a linear dynamical system, with sparsity constraints imposed by the network topology (Fig. 1(b)). In this paper we present an overview of our recent results from [1], [2], [3]. We show that the linear iterative strategy allows us to incorporate dynamics into the network, which enables us to simultaneously analyze the interaction between the network and the plant. In this case, the plant and the network can be considered as a *structured linear system*, thus allowing us to reason about the effects that the plant's structure and the network topology have on the ability to stabilize the system. Unlike standard procedures for WNCS synthesis, which are usually only focused on minimization of network induced delays, the WCN synthesis procedure explicitly takes into account the plant's dynamics (see Fig. 2).

This paper is organized as follows: In Section II we describe an extended WCN scheme that allows us to frame the network synthesis problem as a decentralized feedback control problem. In Section III we present sufficient topological conditions that guarantee stabilizability of the plant with a WCN. Finally, in Section IV we show the procedure that can be used to obtain link weights for which the closed-loop system is stable.

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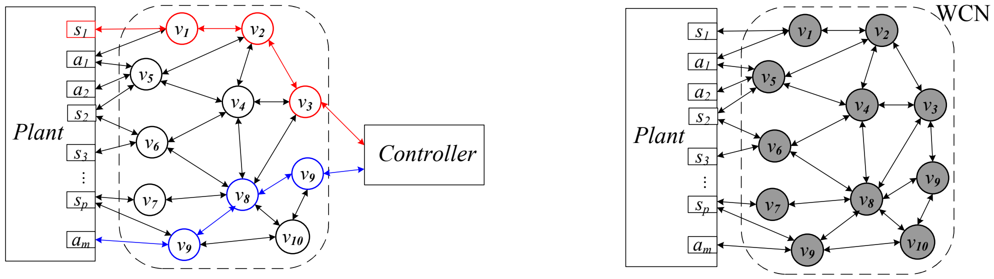
A. Notation and Terminology

We use $\mathbf{e}_i$ to denote the column vector (of appropriate size) with a 1 in its i-th position and 0's elsewhere. The symbol $\mathbf{I}$ denotes the identity matrix of the appropriate dimensions, and $\mathbf{A}'$ indicates the transpose of matrix $\mathbf{A}$.

1) Graph Theory: A graph is an ordered pair $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is a set of vertices (or nodes), and $\mathcal{E}$ is a set of ordered pairs of vertices, called directed edges. The vertices in the set $\mathcal{N}_{v_i} = \{v_j | (v_j, v_i) \in \mathcal{E}\}$ are the neighbors of vertex v_i. A *path* P from vertex v_{i_0} to vertex v_{i_t} is a sequence of vertices $v_{i_0} v_{i_1} \cdots v_{i_t}$ such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}$ for $0 \leq j \leq t-1$. The nonnegative integer t is the *length* of the path. A path is called a *cycle* if its start vertex and end vertex are the same, and no other vertex appears more than



(a) Wireless Networked Control System

(b) Wireless Control Network

Figure 1. (a) Standard architectures for Wireless Networked Control Systems; Red links/nodes - routing data from the plant's sensors to the controller; Blue links/nodes - routing data from the controller to the plant's actuators; (b) A multi-hop Wireless Control Network, where the network acts as a distributed controller.

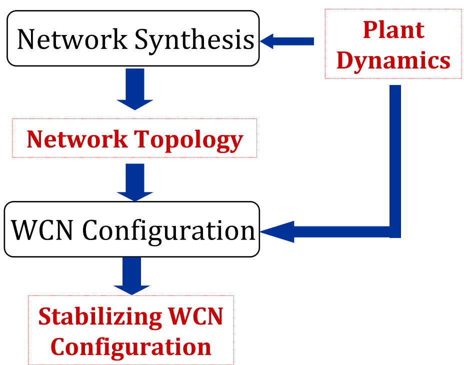


Figure 2. Diagram of the WCN synthesis procedure. The procedure takes into account dynamics of the plant while designing the network.

once in the path. We will call a graph *disconnected* if there exists at least one pair of vertices $v_i, v_j \in \mathcal{V}$ such that there is no path from v_j to v_i. A graph is said to be *strongly connected* if there is a path from every vertex to every other vertex.

II. Wireless Control Network

Consider the system presented in Fig. 1(b), where a wireless network is used to control a discrete-time system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ (i.e., a plant)[1] of the form:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k], \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ denotes the state vector, and $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$. The sensors from the set $\mathcal{S} = \{s_1, s_2, \ldots, s_p\}$ provide the measurements of the plant's output vector $\mathbf{y}[k] = \begin{bmatrix} y_1[k] & y_2[k] & \ldots & y_p[k] \end{bmatrix}'$, while the input vector $\mathbf{u}[k] = \begin{bmatrix} u_1[k] & u_2[k] & \ldots & u_m[k] \end{bmatrix}'$ contains the input signals applied to the plant by the actuators from the set $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$.

The radio connectivity in the network is described by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is the set of N nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the communication topology (i.e., edge $(v_j, v_i) \in \mathcal{E}$ if node v_i can receive information directly from node v_j).[2] In addition, we define $\mathcal{V}_S \subseteq \mathcal{V}$ as the set of nodes that can receive information directly from at least one sensor, and $\mathcal{V}_A \subseteq \mathcal{V}$ as the set of nodes whose transmissions can be heard by at least one actuator. In this paper we utilize the 'extended' graph $\bar{\mathcal{G}} = \{\mathcal{V} \cup \mathcal{S} \cup \mathcal{A}, \mathcal{E} \cup \mathcal{E}_{in} \cup \mathcal{E}_{out}\}$ that includes the initial graph $\mathcal{G}$, the plant's sensors and actuators, and the edge sets:

$$\mathcal{E}_{out} = \left\{ (s_l, v_i) \;\middle|\; \begin{array}{c} s_l \in \mathcal{S}, v_i \in \mathcal{V}_S, \\ v_i \text{ can receive values from sensor } s_l \end{array} \right\}, \tag{2}$$

$$\mathcal{E}_{in} = \left\{ (v_i, a_l) \;\middle|\; \begin{array}{c} a_l \in \mathcal{A}, v_i \in \mathcal{V}_A, \\ \text{actuator } a_l \text{ can receive values from } v_i \end{array} \right\}. \tag{3}$$

The basic WCN scheme is proposed in [1], [2], where each node in the network maintains a (possibly vector) state. In addition, each node implements a simple linear iterative procedure, where at every time step (i.e., once every communication frame) the node updates its state to be a linear combination of its previous state and the states of its neighbors. Furthermore, each node from the set $\mathcal{V}_S$ (i.e., a neighbor of one or more plant sensors) includes in its update procedure a linear combination of measurements (i.e., plant outputs) provided by the sensors in its neighborhood. Denoting node v_i's state at time step k by $z_i[k]$, the update procedure is given by:

$$z_i[k+1] = w_{ii} z_i[k] + \sum_{v_j \in \mathcal{N}_{v_i}} w_{ij} z_j[k] + \sum_{s_j \in \mathcal{N}_{v_i}} h_{ij} y_j[k]. \tag{4}$$

Note that in the above equation, the neighborhood $\mathcal{N}_{v_i}$ of vertex v_i is with respect to the graph $\bar{\mathcal{G}}$.

In this work we employ a slightly modified approach from [2], where we allow each actuator $a_i, (1 \leq i \leq m)$, to maintain a (possibly vector) state z_{a_i} and to use the same linear iterative procedure to update its state:[3]

$$z_{a_i}[k+1] = w_{a_i} z_{a_i}[k] + \sum_{v_j \in \mathcal{N}_{a_i}} g_{ij} z_j[k] \tag{5}$$

The proposed scheme is reminiscent to the consensus reaching algorithms (e.g., [4], [5]) and algorithms used in linear network coding [6]. With linear network coding, to increase the rate of information transmission through a network, nodes

[1]The use of WCNs for control of continuous-time linear time-invariant plants is described in [3].

[2]We assume that each node can communicate with itself, i.e., for all v_i, $(v_i, v_i) \in \mathcal{E}$.

[3]For simplicity, in this work we present the case when all nodes and actuators maintain a scalar state. A more general formulation, where each node and actuator maintains a vector state is described in [1], [2].

in the network mathematically combine incoming packets before transmitting them. On the other hand, the focus of consensus-reaching algorithms that use linear iterative updates is to determine whether all nodes (i.e., agents) in the network can agree (i.e., converge) on a specific value. The main difference in our approach is that the WCN is used as a feedback mechanism in order to *stabilize a dynamical plant*. Thus, to achieve this, each actuator a_i applies input u_i, which at step k is derived as:

$$u_i[k] = t_{a_i} z_{a_i}[k] + \sum_{v_j \in \mathcal{N}_{a_i}} k_{ij} z_j[k], \tag{6}$$

where t_{a_i} and the k_{ij}'s are scalars to be chosen.

To specify the evolution of the states of all nodes and actuators in the network, we define the node state vector $\mathbf{z}[k] = \begin{bmatrix} z_1[k] & z_2[k] \dots & z_N[k] \end{bmatrix}'$ and the actuator state vector $\mathbf{z}_a[k] = \begin{bmatrix} z_{a_1}[k] & z_{a_2}[k] \dots & z_{a_m}[k] \end{bmatrix}'$. Thus, these states evolve as:

$$\mathbf{z}[k+1] = \mathbf{W}\mathbf{z}[k] + \mathbf{H}\mathbf{y}[k] \ , \tag{7}$$

$$\mathbf{z}_a[k+1] = \mathbf{W}_a \mathbf{z}_a[k] + \mathbf{G}\mathbf{z}[k]. \tag{8}$$

In the above equations, the matrix $\mathbf{W}_a \in \mathbb{R}^{m \times m}$ is a diagonal matrix, and the matrices $\mathbf{W} \in \mathbb{R}^{N \times N}$, $\mathbf{H} \in \mathbb{R}^{N \times p}$ and $\mathbf{G} \in \mathbb{R}^{m \times N}$ have sparsity constraints imposed by the underlying WCN topology – the connections between the nodes in the network (for matrix $\mathbf{W}$), from the sensors to the nodes (for $\mathbf{H}$), and from the nodes to the actuators (for $\mathbf{G}$). For example, for all $i \in \{1, \dots, N\}$, $w_{ij} = 0$ if $v_j \notin \mathcal{N}_{v_i} \cup \{v_i\}$, $h_{ij} = 0$ if $s_j \notin \mathcal{N}_{v_i}$, and $g_{ij} = 0$ if $v_j \notin \mathcal{N}_{a_i}$.

Aggregating the node and actuator states into the network state vector $\hat{\mathbf{z}} = \begin{bmatrix} \mathbf{z}[k]' & \mathbf{z}_a[k]' \end{bmatrix}'$, the behavior of the network can be described as:

$$\begin{aligned} \hat{\mathbf{z}}[k+1] &= \underbrace{\begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{G} & \mathbf{W}_a \end{bmatrix}}_{\mathbf{W}_d} \hat{\mathbf{z}}[k] + \underbrace{\begin{bmatrix} \mathbf{H} \\ \mathbf{0} \end{bmatrix}}_{\mathbf{H}_d} \mathbf{y}[k] \\ \mathbf{u}[k] &= \mathbf{T}_a \mathbf{z}_a[k] + \mathbf{K}\mathbf{z}[k] = \underbrace{\begin{bmatrix} \mathbf{K} & \mathbf{T}_a \end{bmatrix}}_{\mathbf{G}_d} \hat{\mathbf{z}}[k], \end{aligned} \tag{9}$$

where $\mathbf{T}_a \in \mathbb{R}^{m \times m}$ is a diagonal matrix, and $\mathbf{K} \in \mathbb{R}^{m \times N}$ is a structured matrix with sparsity constraints imposed by the links from nodes in the network to the actuators. From (9) we observe that the linear iterative strategy employed by all nodes and actuators causes the entire network to behave as a *structured dynamical compensator*.

To be able to describe the closed-loop system we denote with $\hat{\mathbf{x}}[k] = \begin{bmatrix} \mathbf{x}[k]' & \mathbf{z}[k]' & \mathbf{z}_a[k]' \end{bmatrix}'$ the overall system state that contains the state of the plant and states of the nodes and actuators. Using this notation, the overall closed-loop system can be described as:

$$\hat{\mathbf{x}}[k+1] = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{G}_d \\ \mathbf{H}_d\mathbf{C} & \mathbf{W}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \hat{\mathbf{z}}[k] \end{bmatrix} \triangleq \hat{\mathbf{A}}\hat{\mathbf{x}}[k]. \tag{10}$$

with the matrices $\mathbf{W}_d, \mathbf{H}_d, \mathbf{G}_d$ defined in (9).

One of the goals of our work has been to determine conditions on the network topology for which there exist a set of link weights (i.e., the structured matrices $\mathbf{W}_d, \mathbf{H}_d, \mathbf{G}_d$) that results in a stable closed-loop system. We consider WCNs where each wireless nodes maintains a scalar state, while actuators are allowed to maintain vector states. This is inspired by the fact that actuators are usually not power constrained, which allows them to employ more powerful CPUs. On the other hand, wireless nodes are usually battery-operated, and thus utilize low-power resource constrained microcontrollers.

III. Topological Conditions for System Stabilization

In this section we present topological conditions on the network topology for which there exists a set of links weights that guarantees system stability. To extract these conditions we use results from the structured system theory, which allows us to employ graph-theoretic tools to analyze linear systems. Thus, we start with an overview of structured system theory, before showing that the WCN scheme allows us to exploit the structural graph of the closed-loop system. Finally, we present sufficient topological conditions that ensure the existence of a stabilizing WCN configuration.

A. Structured Linear Systems

Consider a system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ of the form:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k], \end{aligned} \tag{11}$$

where $\mathbf{x}[k] \in \mathbb{R}^n, \mathbf{u}[k] \in \mathbb{R}^m, \mathbf{y}[k] \in \mathbb{R}^p$ and the matrices are of the appropriate dimensions.

A linear system of the form $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ is said to be *structured* if each element of the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is either a fixed zero or an independent free parameter [7]. Therefore, two systems are structurally equivalent if they have the same number of states, inputs and outputs, and their system matrices (i.e., $\mathbf{A}, \mathbf{B}, \mathbf{C}$) have fixed zeros in the same locations. A structured system Σ can be represented via a directed graph $\mathcal{G}_\Sigma = \{\mathcal{V}_\Sigma, \mathcal{E}_\Sigma\}$, which is sometimes referred to as a *structural graph*. The vertex set $\mathcal{V}_\Sigma = \{\mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}\}$ consists of the set of state vertices $\mathcal{X} = \{x_1, ..., x_n\}$, and sets of input vertices $\mathcal{U} = \{u_1, ..., u_m\}$ and output vertices $\mathcal{Y} = \{y_1, ..., y_p\}$. Furthermore, the edge set is defined as $\mathcal{E}_\Sigma = \mathcal{E}_\mathbf{A} \cup \mathcal{E}_\mathbf{B} \cup \mathcal{E}_\mathbf{C}$, where $\mathcal{E}_\mathbf{A} = \{(x_i, x_j) | a_{ji} \neq 0\}$, $\mathcal{E}_\mathbf{B} = \{(u_i, x_j) | b_{ji} \neq 0\}$, $\mathcal{E}_\mathbf{C} = \{(x_i, y_j) | c_{ji} \neq 0\}$.

To illustrate this consider the dynamical plant specified by:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ \lambda_3 & \lambda_4 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda_6 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \lambda_7 & 0 \\ 0 & 0 \\ 0 & \lambda_8 \\ 0 & 0 \end{bmatrix}, \\ \mathbf{C} &= \begin{bmatrix} \lambda_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{10} \\ 0 & \lambda_{11} & 0 & 0 \end{bmatrix}. \end{aligned} \tag{12}$$

The structural graph for the plant is presented in Fig. 3.

The main focus of the structured system theory is on the properties of a structured system that can be inferred purely from the zero/nonzero structure of the system matrices. These properties are *generic*, meaning that they hold *almost everywhere* – for almost any choice of free parameters (i.e., the set of parameters for which the property does not hold has Lebesgue measure zero [7]).

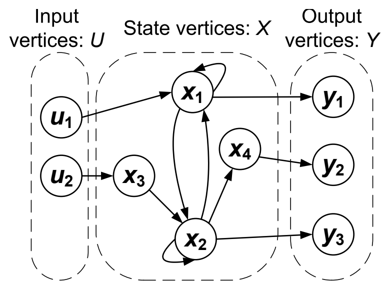


Figure 3. A graph associated with the structured plant specified in (12).

B. Structural Graph of the Closed-loop System

As described in [2], to obtain a structural graph of the closed-loop system we view the plant $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ and the WCN as a linear system $\tilde{\Sigma}$. If we initially disregard the effects of the actuators on the plant, the system $\tilde{\Sigma}$ can be described as:

$$\tilde{\mathbf{x}}[k+1] = \begin{bmatrix} \mathbf{x}[k+1] \\ \mathbf{z}[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{HC} & \mathbf{W} \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} \mathbf{x}[k] \\ \mathbf{z}[k] \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\tilde{\mathbf{B}}} \mathbf{u}[k],$$

$$\tilde{\mathbf{y}}[k] = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{E}_{\mathcal{V}_A} \end{bmatrix}}_{\tilde{\mathbf{C}}} \begin{bmatrix} \mathbf{x}[k] \\ \mathbf{z}[k] \end{bmatrix}, \tag{13}$$

where $\mathbf{E}_{\mathcal{V}_A} = \begin{bmatrix} \mathbf{e}'_{i_1} & \mathbf{e}'_{i_2} & \ldots & \mathbf{e}'_{i_t} \end{bmatrix}$ selects the states from the nodes in the neighborhood of the actuators (i.e., the nodes from the set $\mathcal{V}_A = \{v_{i_1}, v_{i_2}, ..., v_{i_t}\}$, with $t = |\mathcal{V}_A|$). With this representation of the interaction between the plant and the WCN, the states of the nodes from the set $\mathcal{V}_A$ are specified as the output of the system $\tilde{\Sigma}$, and the system is described as $\tilde{\Sigma} = (\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$.

For the system $\tilde{\Sigma}$ we obtain the structural graph $\mathcal{G}_{\tilde{\Sigma}} = \{\mathcal{V}_{\tilde{\Sigma}}, \mathcal{E}_{\tilde{\Sigma}}\}$ from the structural graph of the initial plant Σ and the network graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

$$\mathcal{V}_{\tilde{\Sigma}} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{V}, \quad \mathcal{E}_{\tilde{\Sigma}} = \mathcal{E}_{\mathbf{A}} \cup \mathcal{E} \cup \mathcal{E}_{\tilde{\mathcal{O}}}$$

where (for $\mathcal{E}_{out}$ defined in (2))

$$\mathcal{E}_{\tilde{\mathcal{O}}} = \{(x_i, v_j) \in \mathcal{X} \times \mathcal{V}_S | \exists y_k, (x_i, y_k) \in \mathcal{E}_{\mathbf{C}}, (y_k, v_j) \in \mathcal{E}_{out}\}.$$

is the edge set between the state vertices connected to a plant output and all network nodes in the neighborhood of the corresponding plant sensor.[4] It is worth noting that although $\mathcal{G}$ denotes the graph of the 'physical' network, when each of the nodes maintains a scalar state we can also use $\mathcal{G}$ as the structural graph of the WCN.

C. Topological Conditions

From the structural graph description of the plant and the WCN, the problem of system stabilization using the WCN can be framed as a decentralized control problem with feedback constraints [8]. Using the concept of structural fixed modes [9], [10], [11], from Theorem 3 from [2] we can state the following theorem.

[4]More details about this mapping, including some limitations of this approach, can be obtained from [2].

Theorem 1: Almost any system structurally equivalent to system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ can be stabilized with a WCN if for each plant state vertex $x_i \in \mathcal{X}$ in the structural graph $\mathcal{G}_{\tilde{\Sigma}}$ there exists a cycle that contains the state vertex $x_i \in \mathcal{X}$ and any WCN vertex from $\mathcal{V}$. □

The following corollary introduces a straightforward sufficient condition on the network topology.

Corollary 1: Almost every structurally controllable and detectable system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ can be stabilized with a strongly connected WCN if all sensors and all actuators are connected to the network. □

However, the above results do not guarantee that any plant with the specified structure can be stabilized using a WCN that satisfies the sufficient conditions. In the general case, for any given plant we can use the sufficient topological conditions derived in [2].

Theorem 2 ([2]): Consider the detectable and stabilizable system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, along with a WCN. Let d denote the largest geometric multiplicity of any unstable eigenvalue of $\mathbf{A}$. Suppose the connectivity of the network is at least d, and each actuator has at least d WCN nodes in its neighborhood. Then, for almost any choice of parameters in $\mathbf{W}$ and $\mathbf{H}$ such that $\mathbf{W}$ is stable, the system $\tilde{\Sigma}$ can be stabilized via a dynamic compensator at each actuator. □

An interesting byproduct of the above result is that the network diameter, and thus delays in the network, does not affect stabilizability of the system. Thus, with appropriate compensators at the actuators, if a network satisfies the topological conditions the system can be stabilized with the WCN despite the path lengths in the network.

IV. Extracting a Stabilizing Configuration

If the topological conditions from the previous section are satisfied, a stabilizing configuration for the WCN with dynamical compensators at actuators can be found using a simple modification of the numerical procedure for the basic WCN described in [1].

The closed-loop system described by (10) is stable if the matrix $\hat{\mathbf{A}} = \hat{\mathbf{A}}(\mathbf{W_d}, \mathbf{H_d}, \mathbf{G_d})$ has all eigenvalues inside the unit circle. Since matrices $\mathbf{W_d}, \mathbf{H_d}, \mathbf{G_d}$ are structured, finding a stabilizing configuration for the system described in (10) is a problem equivalent to finding a stabilizing configuration for the basic WCN. Therefore, a stabilizing configuration can be obtained using the numerical procedure specified in Algorithm 1, which is a simple extension of the algorithm used for the basic WCN [1], [12]. In addition, a procedure similar to the one from from [1] can be used to extract a stabilizing configuration for the closed-loop system with unreliable communication links. If the links can be modeled as independent Bernoulli processes, the stabilizing configuration guarantees mean square stability of the system.

For example, consider the system presented in Fig. 4 where the plant is specified by:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2.1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix}. \tag{14}$$

If each node maintains a scalar state and the actuator (acting as a dynamical compensator) maintains a state from $\mathbb{R}^2$,

Algorithm 1 Stabilizing closed-loop system with the WCN

1. Find feasible points $\mathbf{X_0}, \mathbf{Y_0}, \mathbf{W_0}$, $\mathbf{H_0}$, $\mathbf{G_0}$ that satisfy the constraints

$$\begin{bmatrix} \mathbf{X}_0 & \hat{\mathbf{A}}_0^T \\ \hat{\mathbf{A}}_0 & \mathbf{Y}_0 \end{bmatrix} \succ 0, \quad \begin{bmatrix} \mathbf{X_0} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y_0} \end{bmatrix} \succeq 0,$$

$$\hat{\mathbf{A}}_0 = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{G}_0 \\ \mathbf{H}_0\mathbf{C} & \mathbf{W_0} \end{bmatrix},$$

$$(\mathbf{W_0}, \mathbf{H}_0, \mathbf{G}_0) \in \Psi, \;\; \mathbf{X}_0, \mathbf{Y}_0 \in \mathbb{S}_{++}^{n+N}.$$

If a feasible point does not exist, then it is not possible to stabilize the system with this network topology.

2. At iteration k $(k \geq 0)$, from $\mathbf{X_k}, \mathbf{Y_k}$ obtain the matrices $\mathbf{X_{k+1}}, \mathbf{Y_{k+1}}, \mathbf{W_{k+1}}, \mathbf{H_{k+1}}, \mathbf{G_{k+1}}$ by solving the following LMI problem

$$\min tr(\mathbf{Y_k}\mathbf{X_{k+1}} + \mathbf{X_k}\mathbf{Y_{k+1}})$$

$$\begin{bmatrix} \mathbf{X}_{k+1} & \hat{\mathbf{A}}_{k+1}^T \\ \hat{\mathbf{A}}_{k+1} & \mathbf{Y}_{k+1} \end{bmatrix} \succ 0, \quad \begin{bmatrix} \mathbf{X_{k+1}} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y_{k+1}} \end{bmatrix} \succeq 0,$$

$$\hat{\mathbf{A}}_{k+1} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{G}_{k+1} \\ \mathbf{H}_{k+1}\mathbf{C} & \mathbf{W_{k+1}} \end{bmatrix},$$

$$(\mathbf{W_{k+1}}, \mathbf{H}_{k+1}, \mathbf{G}_{k+1}) \in \Psi, \;\; \mathbf{X}_{k+1}, \mathbf{Y}_{k+1} \in \mathbb{S}_{++}^{n+N}.$$

3. If the matrix

$$\hat{\mathbf{A}}_{k+1} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{G}_{k+1} \\ \mathbf{H}_{k+1}\mathbf{C} & \mathbf{W}_{k+1} \end{bmatrix}$$

is Schur, stop the algorithm. Otherwise, set $k = k+1$ and go to the step **2**.

using the aforementioned algorithm we obtained the following stabilizing configuration:

$$\mathbf{W} = \begin{bmatrix} 1.94 & -0.26 \\ 37.67 & -3.35 \end{bmatrix}, \;\; \mathbf{W_a} = \begin{bmatrix} -2.28 & 5.12 \\ -1.83 & 3.75 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \;\; \mathbf{G} = \begin{bmatrix} 0 & 1.0 \\ 0 & 0.54 \end{bmatrix}, \;\; \mathbf{K} = \begin{bmatrix} 0 \\ 0.12 \end{bmatrix}', \;\; \mathbf{T_a} = \begin{bmatrix} -0.39 \\ 0.61 \end{bmatrix}'.$$

V. Conclusion

In this paper we have presented a high-level overview of the concept of a Wireless Control Network, where the network itself acts as a distributed structured controller. We have described topological conditions for a WCN that ensure that both structured and numerically specified plants can be stabilized with the WCN, along with a procedure that can be used to obtain a stabilizing set of link weights. However, in our work we did not investigate topological conditions for which we can guarantee the existence of a robust WCN configuration that maintains stability even in presence of node and link failures. This will be an avenue for future work.

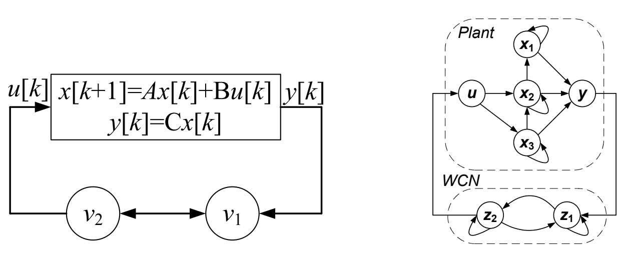


Figure 4. (a) An example of a WCN where the plant is specified in (14); (b) Graph description of the system.

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