# Dependent Interoperability (Technical Report)

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**Abstract**

In this paper we study the problem of *interoperability*—combining constructs from two separate programming languages within one program—in the case where one of the two languages is dependently typed and the other is simply typed. We present a core calculus called SD, which combines dependently- and simply-typed sub-languages and supports user-defined (dependent) datatypes, among other standard features. SD has "boundary terms" that mediate the interaction between the two sub-languages. The operational semantics of SD demonstrates how the necessary dynamic checks, which must be done when passing a value from the simply-typed world to the dependently typed world, can be extracted from the dependent type constructors themselves, modulo user-defined functions for marshaling values across the boundary. We establish type-safety and other meta-theoretic properties of SD, and contrast this approach to others in the literature.

## 1 Introduction

Dependently-typed languages allow programmers to specify a rich set of properties about their programs that are verifiable during type-checking. This comes at the price of complexity — it is at best extremely time-consuming and at worse infeasible to use dependently-typed languages in large software developments. A natural way to mitigate this weakness is to use a dependently-typed language to provide specifications for critical components while the rest of the system is written in a mainstream programming language. However, care must be taken to ensure that the specifications of the dependently-typed language are respected by "weaker" programming language. In this paper, we study the problem of interoperability between a language with dependent types and a language with simple types, focusing on the key meta-theoretic issues that arise in this setting.

Prior work on interoperability initially focused on the implementation of such interoperability systems. Many languages provide an escape hatch into C, such as Java's JNI [16], or OCaml's [14] and Haskell's [18] FFI. Other work considers how to achieve interoperability by developing a *lingua franca* for languages to talk to each other. Proposals include C [3], the Java virtual machine [17], COM [26], or the .NET framework [30]. More recently, the focus has shifted to understanding the relationship between dynamic and typed languages with contracts [8], blame [33], and the integration of scripting and typed languages [34].

In these systems, dynamic checks ensure that the static guarantees of the typed language are respected by the untyped language. The dynamic check amounts to a simple type tag check, e.g., verifying that $\mathsf{typeof}\ (\lambda x{:}\,S.s)$ is indeed a function. However, the same concerns arise if we consider languages with richer type systems, namely those with dependent types. A simply-typed language will be able to enforce only some of a dependently-typed language's static guarantees during type-checking; the difference must again be made up with dynamic checks. However these dynamic checks must now perform non-trivial computation rather than simply checking type tags.

For example, suppose that your dependently-typed language provides a certified library that you would like to use in your application. For simplicity's sake, let's consider a $\mathsf{List}$ datatype that contains $\mathsf{Int}$s.

$$\begin{aligned}&\mathsf{List} : \mathsf{Int} \Rightarrow *\\&\mathsf{Nil} : (y : \mathsf{Unit}) \to \mathsf{List}\, y\\&\mathsf{Cons} : (y_1 : \mathsf{Int}) \to (y_2 : \mathsf{Int}) \to \mathsf{List}\, y_1 \to \mathsf{List}\, y_1 + 1\end{aligned}$$

$\mathsf{List}$ is indexed by an integer than represents its length, and that invariant is maintained by its two constructors $\mathsf{Nil}$ and $\mathsf{Cons}$. Suppose that our library also has a dependently-typed function $\mathsf{PrettyPrintList5} : \mathsf{List}\, 5 \to \mathsf{Unit}$ that prints out lists of length five in a special way, but instead of giving it a dependently-typed $\mathsf{List}$, we'd like to provide it our standard simply-typed $\mathsf{List}$ instead. Our interoperability layer must not only marshal the $\mathsf{List}$ value between languages, but also ensure that the simply-typed $\mathsf{List}$ has length five.

### 1.1 Contributions and Outline

How do we craft an interoperability layer that can generate such dynamic checks? How does such an interoperability layer affect the meta-theoretic properties of the languages involved? In order to answer these questions, we propose a calculus in the style of Matthews and Findler [20] that combines two languages together — in our case, a simply-typed and dependently-typed language — via boundary terms.

Our work on dependent interoperability contributes the following:

1. A core calculus called SD that combines a simply-typed and dependently-typed lambda calculus extended with user-defined datatypes. While we are aware of previous efforts to combine simply-typed and dependently-typed programming, to our knowledge, this is the first work that looks at the problem from the perspective of language interoperability with the corresponding aim of modifying the languages as little as possible when integrating them.

2. Analysis of the meta-theoretic properties of SD, in particular, a proof of type safety for the language.

3. Exploration of the design space of dependent interoperability, including changes to the design to guarantee termination in the presence of recursive functions and alternatives to directly translating data.

4. A comparison of our system to real world systems such as Coq and Agda that provide limited forms of language interoperability. Such comparisons strengthen our claim that our model faithfully captures dependent interoperability, but also suggests how these real world systems can improve in this area.

We open in Section 2 by expanding on the benefits of dependent interoperability. In Section 3, we describe the syntax and semantics of SD. We discuss the metatheory of SD in Section 4. Next we describe additional interesting properties of SD in Section 5. In Section 6 we compare SD to real world dependently-typed systems that offer interoperability facilities. Finally we discuss related and future work in Section 7 and close in Section 8. In this technical report, we also give a full account of the language in Appendix A and complete proofs of SD's type safety in Appendix B.

## 2 Motivation

Before we discuss SD proper, we first motivate further why dependent interoperability is a useful idea by discussing three use cases in more detail. Along the way we will foreshadow the potential difficulties in creating an interoperability layer that we will solve in Section 3.

1. **Using a simply-typed library in a dependently-typed context.** While our dependently-typed language may be safer to use, it will typically not have all the functionality we would like. For example, we may wish to use a simply-typed library that provides network access, e.g., a function $\mathsf{sendData} : \mathsf{Packet} \to \mathsf{Unit}$, from our dependently-typed program. It is a good bet (although not always true) that our dependent type system is strictly more powerful than the simple type system, so intuition tells us that we shouldn't need any dynamic checks here. Therefore, our interop boundary needs only to

| | $\lambda^{\to}$ | $\lambda^{\cong}$ |
|---|---|---|
| Kinds | | $K$ |
| Types | $S$ | $T$ |
| Terms | $s$ | $t$ |
| Variables | $x$ | $y$ |
| Datatypes | $A$ | $B$ |

Figure 1: Metavariable Conventions for $\lambda^{\to}$ and $\lambda^{\cong}$

marshal the data from the dependently-typed language into the $\mathsf{Packet}$ that the simply-typed function expects to use.

2. **Using a dependently-typed library in a simply-typed context.** The dual of the previous use case is the desire to use dependently-typed code in a simply-typed context. In the introduction, we used the toy example of a $\mathsf{List}\ n$. However, you can imagine wanting to use a verified library for a particular data structure or protocol from a simply-typed context and be assured that the simply-typed data you feed it does not break the properties the verified library enforces. Discovering and enforcing these properties is the primary challenge our interoperability boundary faces.

3. **Verifying properties of simply-typed code**. Finally, because we are working with a dependently-typed language, an interesting question arises. In addition to verifying properties of dependently-typed terms, can we do the same with simply-typed terms? That is, rather than implement a verified library in the dependently-typed language and translating simply-typed data into that library, we would like to verify properties of a simply-typed library directly. Ideally the dependently-typed language would be able to do this all during typechecking, but realistically, complete checking of a term across an interop boundary is impossible. We expect that the result is similar to a hybrid type system [9] where some properties are verified during compilation and the rest are "made up" with dynamic checks.

# 3 Language

Our language SD consists of a *simply-typed* and a *dependently-typed* lambda calculus joined together by boundary terms in the style of Matthews and Findler [20]. Throughout this paper, we use a meta-variable convention to distinguish terms of the simply-typed fragment ($\lambda^{\to}$) and the dependently-typed fragment ($\lambda^{\cong}$) outlined in Figure 1. In addition, there are several judgments that make up SD. In the interest of the brevity, we only present the salient features of each of these judgments. Appendices A and B contain the complete definitions of our system along with proofs.

## 3.1 Syntax

$\lambda^{\to}$ is a standard lambda calculus with simple types as defined in Figure 3. We augment the calculus with pairs $< s_1, s_2 >$, $\mathsf{unit}$, an $\mathsf{error}$ term that will be raised if a boundary check fails, and user-defined data constructors $C$ with corresponding datatypes $A$. Constructors are modeled as taking only a single argument but this is not a limitation since multiple arguments can be combined using pairs. For example, the constructor $\mathsf{Cons}^{\to}$ has type

$$\mathsf{Cons}^{\to} : (\mathsf{List} * \mathsf{Int}) \to \mathsf{List}.$$

In SD we presuppose a signature $\Psi_0$ containing the definitions of these constructors.

The notable addition to $\lambda^{\to}$ is the addition of the typed boundary term $\mathsf{SD}^{S}_{T}t$ which can be read as an interoperability boundary that translates the inner $\lambda^{\cong}$ term $t$ of type $T$ to a $\lambda^{\to}$ term of type $S$. Such boundaries are responsible for *marshaling* data from one side of the boundary to the other and *checking* that this marshaled data is appropriate for the context it will be used in. Our formulation focuses on

| Judgment | Description |
|---|---|
| $\Gamma \vdash s : S$ | $\lambda^{\rightarrow}$ Typing |
| $\Gamma \vdash K$ | $\lambda^{\cong}$ Well-formed Kinds |
| $\Gamma \vdash T : K$ | $\lambda^{\cong}$ Kinding |
| $\Gamma \vdash t : T$ | $\lambda^{\cong}$ Typing |
| $\vdash \Psi$ | Well-formed Signature |
| $\vdash \Gamma$ | Well-formed Context |
| $\mathsf{FO}\,(T)$ | First-order Type |
| $S \Leftrightarrow T$ | Type Translation |
| $\Gamma \vdash K \equiv K'$ | $\lambda^{\cong}$ Kind Equivalence |
| $\Gamma \vdash T \equiv T'$ | $\lambda^{\cong}$ Type Equivalence |
| $\Gamma \vdash t \cong t'$ | $\lambda^{\cong}$ Term Equivalence |
| $s \longrightarrow s'$ | $\lambda^{\rightarrow}$ Evaluation |
| $t \longrightarrow t'$ | $\lambda^{\cong}$ Evaluation |

Figure 2: SD Judgments

$$
\begin{array}{lrcl}
\lambda^{\rightarrow}\text{ Types} & S & ::= & S_1 \rightarrow S_2 \;|\; S_1 * S_2 \;|\; \mathsf{Unit} \;|\; A \\
\lambda^{\rightarrow}\text{ Terms} & s & ::= & x \;|\; \lambda x{:}S.s \;|\; s_1\, s_2 \\
 & & | & <s_1, s_2> \;|\; s.1 \;|\; s.2 \\
 & & | & C\, s \;|\; \mathsf{case}\, s\, \mathsf{of}\, \overline{C_i\, x_i \rightarrow s_i}^{\,i} \\
 & & | & \mathsf{unit} \;|\; \mathsf{error} \;|\; \mathsf{letd}\, y = t\, \mathsf{in}\, s \;|\; \mathsf{SD}^{S}_{T}\, t
\end{array}
$$

$$
\begin{array}{lrcl}
\lambda^{\cong}\text{ Kinds} & K & ::= & * \;|\; T \Rightarrow * \\
\lambda^{\cong}\text{ Types} & T & ::= & (y : T_1) \rightarrow T_2 \;|\; T\, t \\
 & & | & (y : T_1) * T_2 \;|\; \mathsf{Unit} \;|\; B \\
\lambda^{\cong}\text{ Terms} & t & ::= & y \;|\; \lambda y{:}\, T.t \;|\; t_1\, t_2 \\
 & & | & <t_1, t_2> \;|\; t.1 \;|\; t.2 \\
 & & | & C\, t \;|\; \mathsf{case}\, t\, \mathsf{of}\, \overline{C_i\, y_i \rightarrow t_i}^{\,i} \\
 & & | & \mathsf{unit} \;|\; \mathsf{error} \\
 & & | & \mathsf{DS}^{T}_{S}\, s \;|\; t_1 \cong t_2 \rhd t_3
\end{array}
$$

Figure 3: SD Syntax

understanding the latter responsibility: what checks are necessary to ensure type-safety when moving across boundaries?

$\lambda^{\cong}$ is a standard dependently-typed lambda calculus inspired Jia et al's system "Lambda-eek" [13]. The syntax of $\lambda^{\cong}$ as given in Figure 3 mirrors the syntactic forms found in $\lambda^{\rightarrow}$: it has dependent functions and pairs along with unit and error. The types of dependent functions and pairs are written $(y : T_1) \rightarrow T_2$ and $(y : T_1) * T_2$ reflecting the fact that $T_2$ in both cases may contain the bound term variable $y$. A datatype $B$ is now a type-level function that, given a term $t$, produces a type $B\, t$. Consequently, we introduce kinds to classify such type-level functions $T \Rightarrow *$, versus proper kinds $*$.

Constructors in $\lambda^{\cong}$ also take single arguments. Combining multiple arguments using pairs is trickier because of dependent types, but still manageable. For example, the type of dependent $\mathsf{Cons}^{\cong}$ is

$$\mathsf{Cons}^{\cong} : (y_1 : (y_2 : \mathsf{Int}) * (\mathsf{List}\, y_2 * \mathsf{Int})) \rightarrow \mathsf{List}\,(y.1) + 1$$

In effect, we use dependent pairs to introduce additional arguments and then project out the arguments when needed to compute the index of the datatype.

In the interest of simplifying the syntax, the introduction forms for the different constructs are shared between $\lambda^{\rightarrow}$ and $\lambda^{\cong}$. This is not problematic as we can look at a term's sub-terms to determine which syntactic category it belongs to. In particular, the names of constructors $C$ are shared between the two calculi, with the implicit assumption that each constructor has $\lambda^{\rightarrow}$ and $\lambda^{\cong}$ counterparts. This simplifies our reasoning when dealing with translating constructors, as we only need to worry about translating the arguments of the constructor.

We introduce a guard term $t_1 \cong t_2 \rhd t_3$ that is the result of reducing a boundary term $\mathsf{DS}_S^T\, s$. This guard term makes explicit the equivalence check that must occur before we create the marshaled term $t$ from $s$. In our presentation of SD, the only check we need is an equivalence check $t_1 \cong t_2$ that determines whether two $\lambda^{\cong}$ terms are indeed equivalent at runtime.

The attentive reader may notice that guards appear only on the $\lambda^{\cong}$ side of the boundary. Intuitively this is because the types of $\lambda^{\cong}$ make strictly stronger guarantees than $\lambda^{\rightarrow}$. When going from $\lambda^{\cong}$ to $\lambda^{\rightarrow}$, no checks are necessary because the $\lambda^{\cong}$ type system can verify all the properties that the $\lambda^{\rightarrow}$ type system tries to enforce. Conversely, $\lambda^{\rightarrow}$ cannot make such guarantees, so we make up the difference on the $\lambda^{\cong}$ side with dynamic checks in the form of our guards.

In both $\lambda^{\rightarrow}$ and $\lambda^{\cong}$ we introduce $\mathsf{let}$ forms as the standard syntactic sugar over abstraction binding.

$$\mathsf{let}\, x = s_1 \,\mathsf{in}\, s_2 \triangleq (\lambda x{:}\, S_1 . s_2)\, s_1$$
$$\mathsf{let}\, y = t_1 \,\mathsf{in}\, t_2 \triangleq (\lambda y{:}\, T_1 . t_2)\, t_1$$

However, in $\lambda^{\rightarrow}$ we also add the special let binding $\mathsf{letd}\, y = t \,\mathsf{in}\, s$ that crosses from $\lambda^{\rightarrow}$ to $\lambda^{\cong}$ to bind a $\lambda^{\cong}$ term and then returns to evaluate $s$. This form is used in order to avoid duplication of side-effects during evaluation. We discuss $\mathsf{letd}$ in more detail when we talk about the evaluation rules of SD.

## 3.2 Typing and well-formedness

The typing rules for the $\lambda^{\rightarrow}$ fragment are entirely standard, so we do not reproduce them in their entirety here. The only interesting addition is WF_STM_SD, which gives a type to our boundaries $\mathsf{SD}_T^S\, t$. A boundary is well-typed if the contained $\lambda^{\cong}$ term meets the type annotation on the boundary, and if the types on the boundary are compatible, written $S \Leftrightarrow T$. Figure 4 gives these rules.

Our type compatibility relation ensures that we can translate between data of the given types. For compound types such as arrows and pairs, we can translate between them if we can translate between their component types. Translating between $\mathsf{Unit}$ types is trivial. And since datatypes $A$ and $B$ are user-defined, we appeal to user-defined translations between them represented by the meta-function $\mathsf{corr}\,(A, B)$. As a concrete example, it is reasonable to expect that the $\mathsf{List}$ datatypes between the $\lambda^{\rightarrow}$ and $\lambda^{\cong}$ fragments are convertible so that we have $\mathsf{corr}\,(\mathsf{List}^{\rightarrow}, \mathsf{List}^{\cong})$. Note that $S \Leftrightarrow T$ strips away the term-components of a dependent type—it compares types only up to the simply-typed "skeleton". However, compatibility does require that the types of the indices of dependent data are first order, written $\mathsf{FO}\,(T)$. Intuitively, $\mathsf{FO}\,(T)$ means that the type $T$ does not contain any arrows. If we did allow arrows here, then when translating such datatypes we would be forced to compare equality of function values, which is a hard problem. This will become clear in Section 3.3 where we discuss the evaluation rules of SD. Note that the data that we are translating is allowed to contain functions, but the index of that datatype is not.

For $\lambda^{\cong}$ we present several of the kinding and typing rules in Figures 5 and 6 to remind the reader of the intricacies of dependent type systems and foreshadow the technical challenges of translating terms into these types during evaluation.

All programs are typed with respect to some fixed signature $\Psi_0$, which assigns types to constructors $C$ and kinds to datatypes $A$ and $B$. We assume that all the types and kinds in $\Psi_0$ are well-formed in the empty context. Because datatypes are type-level functions, we assign them kinds of the form $T_1 \Rightarrow *$, as shown in WF_DTY_DATA, while the remaining types have kind $*$, e.g., WF_DTY_ARR.

Rules WF_DTM_APP and WF_DTM_PAIR illustrate the dependent nature of abstraction and pairs in $\lambda^{\cong}$. The second component $T_2$ of the types may contain free occurrences of $y$ of type $T_1$, so we must close $T_2$ by substituting for $y$. WF_DTM_CONV is the standard conversion rule that allows us to take advantage of indexed types by establishing equivalences between them (via the type-equivalence judgment $\Gamma \vdash T \equiv T'$

$\boxed{\Gamma \vdash s : S}$

$$\frac{\Gamma \vdash t : T \qquad S \Leftrightarrow T}{\Gamma \vdash \mathsf{SD}_T^S\, t : S}\ \text{WF\_STM\_SD}$$

$\boxed{S \Leftrightarrow T}$

$$\frac{S_1 \Leftrightarrow T_1 \qquad S_2 \Leftrightarrow T_2}{S_1 \to S_2 \Leftrightarrow (y : T_1) \to T_2}\ \text{COMPAT\_ARR} \qquad \frac{S_1 \Leftrightarrow T_1 \qquad S_2 \Leftrightarrow T_2}{S_1 * S_2 \Leftrightarrow (y : T_1) * T_2}\ \text{COMPAT\_PAIR}$$

$$\frac{}{\mathsf{Unit} \Leftrightarrow \mathsf{Unit}}\ \text{COMPAT\_UNIT} \qquad \frac{B{:}T_0 \Rightarrow * \in \Psi_0 \qquad \mathsf{FO}\,(T_0) \qquad \mathsf{corr}\,(A, B)}{A \Leftrightarrow B\ t}\ \text{COMPAT\_DATA}$$

$\boxed{\mathsf{FO}\,(T)}$

$$\frac{\mathsf{FO}\,(T)}{\mathsf{FO}\,(T\ t)}\ \text{FO\_APP} \qquad \frac{}{\mathsf{FO}\,(\mathsf{Unit})}\ \text{FO\_UNIT}$$

$$\frac{\mathsf{FO}\,(T_1) \qquad \mathsf{FO}\,(T_2)}{\mathsf{FO}\,((y : T_1) * T_2)}\ \text{FO\_PAIR} \qquad \frac{\mathsf{constrs}\ B = \overline{C_i}^{\,i} \qquad \overline{C_i{:}(y_i : T_i) \to B\ t_i' \in \Psi_0}^{\,i} \qquad \overline{\mathsf{FO}\,(T_i)}^{\,i}}{\mathsf{FO}\,(B\ t)}\ \text{FO\_DATA}$$

Figure 4: Abridged $\lambda^{\to}$ Typing Rules, Type Compatibility, and First-order Types

$\boxed{\Gamma \vdash K}$

$$\frac{}{\Gamma \vdash *}\textsc{wf\_dkn\_proper} \qquad \frac{\Gamma \vdash T : *}{\Gamma \vdash T \Rightarrow *}\textsc{wf\_dkn\_arr}$$

$\boxed{\Gamma \vdash T : K}$

$$\frac{\begin{array}{c}\Gamma \vdash T_1 : * \\ \Gamma, y{:}T_1 \vdash T_2 : *\end{array}}{\Gamma \vdash (y : T_1) \to T_2 : *}\textsc{wf\_dty\_arr} \qquad \frac{B{:}T \Rightarrow * \in \Psi_0}{\Gamma \vdash B : T \Rightarrow *}\textsc{wf\_dty\_data}$$

$\boxed{\Gamma \vdash t : T}$

$$\frac{\begin{array}{l}\Gamma \vdash t_1 : (y : T_1) \to T_2 \\ \Gamma \vdash t_2 : T_1 \\ \Gamma \vdash [t_2/y]\,T_2 : *\end{array}}{\Gamma \vdash t_1\ t_2 : [t_2/y]\,T_2}\textsc{wf\_dtm\_app} \qquad \frac{\begin{array}{l}\Gamma \vdash t_1 : T_1 \\ \Gamma \vdash t_2 : [t_1/y]\,T_2 \\ \Gamma \vdash (y : T_1) * T_2 : *\end{array}}{\Gamma \vdash <t_1, t_2>: (y : T_1) * T_2}\textsc{wf\_dtm\_pair}$$

Figure 5: Abridged $\lambda^{\cong}$ Typing Rules

as discussed in the next section). With WF_DTM_CTOR, we type a constructor $C$ at some datatype $B\,[t/y]t'$ where we substitute into the term the argument given to $C$. Note that the type of the argument to $C$ does not need to coincide with the type of the index of $B$. Finally when we type cases with WF_DTM_CASE in each branch we remember the refined type $B\ t_i'$ of the branch's associated constructor.

Checking $\mathsf{DS}$ via WF_DTM_DS is analogous to $\mathsf{SD}$ boundaries: the inner term must typecheck and the type annotations must coincide. WF_DTM_GUARD typechecks guards by checking to see if the types involved in the equivalence check are well-typed. In addition, $t$ must be well-typed under the assumption that the check holds. Finally, we require that the types of the guard are first-order with the judgment $\mathsf{FO}\,(T)$. The first-order judgment ensures that the types of guards are never arrows so that we do not have to determine the equivalence of functions.

The judgment $\mathsf{FO}\,(T)$ ensures that the inhabitants of $T$ do not contain function values. In the case of FO_DATA we check that all constructors of $B$ take first-order arguments. We do not need to check that the type of $B$'s index term $t_i$ is first-order, since the index is not part of the values inhabiting $B$.

## 3.3 Evaluation

The evaluation rules of SD are of most interest to us because this is where we do the actual work of checking values and marshaling them across boundaries. Figures 7 and 8 give the syntax of our one-step evaluation contexts which define the standard call-by-value order for our language. In addition, Figures 7 and 8 give also lists the interesting evaluation rules for both languages.

The evaluation of the usual syntactic forms — abstractions, pairs, and constructors — are standard. The interesting rules arise from evaluation of boundary terms. In both languages, the evaluation of boundaries is directed by their type annotations, so there is one rule for each value that might be sent across a boundary.

When we translate lambdas, e.g., a $\lambda^{\to}$ lambda to a $\lambda^{\cong}$ lambda as in EVAL_STM_DS_ABS, the output must be a $\lambda^{\cong}$ lambda. Our translation is similar to Matthews' and Findler's. This new $\lambda^{\cong}$ lambda translates its argument $y$ to $\lambda^{\to}$, supplies that translated argument to the $\lambda^{\to}$ lambda, and translates the $\lambda^{\to}$ result of the application back to $\lambda^{\cong}$.

In the $\mathsf{DS}$ case this is straightforward. However, if we look at the $\mathsf{SD}$ case as presented in EVAL_DTM_SD_ABS, we note that $T_2$ may contain free occurrences of $y$ in the boundary. To fix this problem, we close $T_2$ with the

$\boxed{\Gamma \vdash t : T}$

$$\frac{\Gamma \vdash t : (y\!: T_1) * T_2}{\Gamma \vdash t.1 : T_1}\ \textsc{wf\_dtm\_proj1} \qquad \frac{\begin{array}{l}\Gamma \vdash t : (y\!: T_1) * T_2 \\ \Gamma \vdash [t.1/y]\, T_2 : *\end{array}}{\Gamma \vdash t.2 : [t.1/y]\, T_2}\ \textsc{wf\_dtm\_proj2}$$

$$\frac{\begin{array}{l}C{:}(y\!: T_1) \to B\ t' \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \Gamma \vdash t : T_1 \\ \Gamma \vdash B\, [t/y]t' : *\end{array}}{\Gamma \vdash C\ t : B\, [t/y]t'}\ \textsc{wf\_dtm\_ctor} \qquad \frac{\begin{array}{l}\Gamma \vdash t : T \\ \Gamma \vdash T \equiv T' \\ \Gamma \vdash T' : *\end{array}}{\Gamma \vdash t : T'}\ \textsc{wf\_dtm\_conv}$$

$$\frac{\begin{array}{l}\Gamma \vdash s : S \\ \Gamma \vdash T : * \\ S \Leftrightarrow T\end{array}}{\Gamma \vdash \mathsf{DS}_S^T\, s : T}\ \textsc{wf\_dtm\_ds} \qquad \frac{\begin{array}{l}\Gamma \vdash t_0 : T_0 \\ \Gamma \vdash t_1 : T_0 \\ \mathsf{FO}\,(T_0) \\ \Gamma, t_1 \cong t_0 \vdash t : T\end{array}}{\Gamma \vdash t_1 \cong t_0 \rhd t : T}\ \textsc{wf\_dtm\_guard}$$

$$\frac{\begin{array}{l}\Gamma \vdash t : B\ t' \\ \Gamma \vdash T : * \\ \mathsf{constrs}\ B = \ \overline{C_i}^{\,i} \\ \overline{C_i{:}(y_i\!: T_i) \to B\ t'_i \in \Psi_0}^{\,i} \\ \overline{\Gamma, y_i{:}T_i, t' \cong t'_i, t \cong C_i\ y_i \vdash t_i : T}^{\,i}\end{array}}{\Gamma \vdash \mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \to t_i}^{\,i} : T}\ \textsc{wf\_dtm\_case}$$

Figure 6: Abridged $\lambda^{\cong}$ Typing Rules (cont.)

$$\begin{array}{llcl}
\lambda^{\to}\ \text{Values} & u & ::= & x \ \mid\ \lambda x{:}\,S.x \ \mid <u_1, u_2> \mid\ C\ u \\
\lambda^{\to}\ \text{Contexts} & \mathcal{E}_s & ::= & \Box \ \mid\ \Box\, s \ \mid\ u\,\Box \ \mid < \Box, s > \mid < u, \Box > \\
 & & \mid & \Box.1 \ \mid\ \Box.2 \ \mid\ C\,\Box \ \mid\ \mathsf{letd}\ y = \Box\ \mathsf{in}\ s \\
 & & \mid & \mathsf{case}\,\Box\,\mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i} \ \mid\ \mathsf{SD}_T^S\Box \\
\lambda^{\cong}\ \text{Values} & v & ::= & y \ \mid\ \lambda y{:}\,T.t \ \mid <v_1, v_2> \mid\ C\ v \\
\lambda^{\cong}\ \text{Contexts} & \mathcal{E}_t & ::= & \Box \ \mid\ \Box\, t \ \mid\ v\,\Box \ \mid < \Box, t > \mid < v, \Box > \\
 & & \mid & \Box.1 \ \mid\ \Box.2 \\
 & & \mid & C\,\Box \ \mid\ \mathsf{case}\,\Box\,\mathsf{of}\ \overline{C_i\ y_i \to t_i}^{\,i} \\
 & & \mid & \mathsf{DS}_S^T\Box \ \mid\ \Box \cong t_2 \rhd t \ \mid\ v \cong \Box \rhd t
\end{array}$$

Figure 7: SD Evaluation: Contexts and Rules

$\boxed{s \longrightarrow s'}$

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ \mathsf{argToS}_C\, v = u \end{array}}{\mathsf{SD}^{A}_{(B\ t)}\, C\ v \longrightarrow C\ u}\ \text{EVAL\_STM\_SD\_CONSTR}$$

$$\frac{}{\mathsf{SD}^{(S_1 \to S_2)}_{((y{:}T_1) \to T_2)} \lambda y{:}\, T_1'.t \longrightarrow \lambda x{:}\, S_1.\mathsf{letd}\ y' = \mathsf{DS}^{T_1}_{S_1}\, x\ \mathsf{in}\ \mathsf{SD}^{S_2}_{([y'/y]\, T_2)}((\lambda y{:}\, T_1'.t)\ y')}\ \text{EVAL\_STM\_SD\_ABS}$$

$$\frac{}{\mathsf{SD}^{(S_1 * S_2)}_{((y{:}T_1) * T_2)}\ {<}v_1, v_2{>} \longrightarrow {<}\mathsf{SD}^{S_1}_{T_1}\, v_1, \mathsf{SD}^{S_2}_{([v_1/y]\, T_2)}\, v_2{>}}\ \text{EVAL\_STM\_SD\_PAIR}$$

$\boxed{t \longrightarrow t'}$

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ \mathsf{argToD}_C\, u = v \end{array}}{\mathsf{DS}^{(B\ t)}_{A}(C\ u) \longrightarrow t \cong [v/y]t_1 \rhd (C\ v)}\ \text{EVAL\_DTM\_DS\_CONSTR}$$

$$\frac{}{\mathsf{DS}^{((y{:}T_1) \to T_2)}_{(S_1 \to S_2)} \lambda x{:}\, S_1'.s \longrightarrow \lambda y{:}\, T_1.\mathsf{DS}^{T_2}_{S_2}((\lambda x{:}\, S_1'.s)\, (\mathsf{SD}^{S_1}_{T_1}\, y))}\ \text{EVAL\_DTM\_DS\_ABS}$$

$$\frac{}{\mathsf{DS}^{((y{:}T_1) * T_2)}_{(S_1 * S_2)}\ {<}u_1, u_2{>} \longrightarrow \mathsf{let}\ y' = \mathsf{DS}^{T_1}_{S_1}\, u_1\ \mathsf{in}\ {<}y', \mathsf{DS}^{[y'/y]\, T_2}_{S_2}\, u_2{>}}\ \text{EVAL\_DTM\_DS\_PAIR}$$

$$\frac{}{v \cong v \rhd t \longrightarrow t}\ \text{EVAL\_DTM\_GUARD\_REFL} \qquad \frac{v \neq v'}{v \cong v' \rhd t \longrightarrow \mathsf{error}}\ \text{EVAL\_DTM\_GUARD\_ERROR}$$

Figure 8: SD Evaluation: Contexts and Rules (cont.)

$\lambda^{\rightarrow}$ lambda's translated argument. Thus, boundary type annotations are not simple annotations that can be erased at compile time. They are entities that affect evaluation, so they must have a concrete representation at runtime. Note that the $\mathsf{DS}$ case does not need a substitution due to our choice of creating a $\lambda^{\cong}$ lambda that implicitly captures the free variable found in $T_2$.

This observation that the second type component $T_2$ needs to be closed via a substitution is also applicable when translating pairs. In the EVAL_STM_SD_PAIR case the sub-components are already $\lambda^{\cong}$ terms, so we simply close $T_2$ with $v_1$. In the EVAL_DTM_DS_PAIR case, $u_1$ is a $\lambda^{\rightarrow}$ term, so we need to translate it before substituting into $T_2$. So as a first attempt, we might make the term step to $<\mathsf{DS}^{T_1}_{S_1} u_1, \mathsf{DS}^{[\mathsf{DS}^{T_1}_{S_1} u_1/y]\, T_2}_{S_2} u_2>$. However, that proposal has a different problem: $\mathsf{DS}^{T_1}_{S_1} u_1$ is not a value! In particular, while $u_1$ itself is a value, $T_1$ may contain non-value terms. By duplicating this expression, we potentially duplicate any of its side-effects.

To avoid this, in EVAL_DTM_DS_PAIR we $\mathsf{let}$-bind the first component of the translated pair. This sequences the evaluation at runtime and avoids duplicating side-effects. Similarly, in EVAL_STM_SD_ABS we $\mathsf{let}$-bind the translated argument $x$. However, an interesting technicality arises. The point at which we need to $\mathsf{let}$-bind the argument — which is a $\lambda^{\cong}$ term — lies in $\lambda^{\rightarrow}$! To fix this issue, we use the $\mathsf{letd}$ construct that allows us to bind a value in $\lambda^{\cong}$ and then evaluate a $\lambda^{\rightarrow}$ term. In this context, $\mathsf{letd}$ has a natural interpretation: $\mathsf{letd}$ goes into $\lambda^{\cong}$ to bind a term in the environment, returns back to $\lambda^{\rightarrow}$, and evaluates as normal.

The translation of datatypes is more involved because, in addition to variable capture, we must also check that the translation "respects" the property represented by the datatype's index. For example, in the case of $\mathsf{List}$, a reasonable translation from a $\mathsf{List}^{\rightarrow}$ to $\lambda^{\cong}$ should produce a $\mathsf{List}^{\cong}\, t$ where $t$ is the length of the list. In general, what the translation should do is dependent on the datatypes we are translating.

Thus, in addition to presupposing user-defined constructors $C$ of datatypes $A$ and $B\ t$, we also presuppose user-defined *conversions between arguments* of constructors, with the intent that these conversions preserve the dependent datatype's properties. These conversions come as a pair of functions

$$\begin{gathered}\mathsf{argToS}_C\, v = u\\ \mathsf{argToD}_C\, u = v\end{gathered}$$

responsible for converting constructor arguments from one language to the other. At type-checking time, the arguments $v$ and $u$ could contain free variables making it unclear how to translate them, so we allow $\mathsf{argToS}$ and $\mathsf{argToD}$ to be partial functions. When they are undefined the corresponding boundary term is stuck. To ensure Progress, we require that they are always defined for closed well-typed values. We also require some additional conditions expressing that they are defined "naturally" in the argument that we discuss further in Section 4.3.

$\mathsf{argToS}$ and $\mathsf{argToD}$ can be viewed constructor-indexed user-level functions which, if $C{:}S \rightarrow A \in \Psi_0$, $C{:}(y{:}\,T_1) \rightarrow B\ t \in \Psi_0$, and $B{:}T_2 \Rightarrow * \in \Psi_0$, have the types

$$\begin{gathered}\mathsf{argToS} : T_1 \rightarrow S\\ \mathsf{argToD} : S \rightarrow T_1.\end{gathered}$$

We distinguish them from user-level functions because as we have defined the calculus there is no way to form such mixed types. Also, in addition to their types, we intend that the functions are inverses. That is, the following equations should hold

$$\begin{aligned}&1.\quad (\mathsf{argToS} \circ \mathsf{argToD})(u) = u \text{ with } u : S\\ &2.\quad (\mathsf{argToD} \circ \mathsf{argToS})(v) = v \text{ with } v : T_1.\end{aligned}$$

This makes $\mathsf{argToS}$ and $\mathsf{argToD}$ an isomorphism over the constructor $C$.

In EVAL_STM_SD_CONSTR, we use $\mathsf{argToS}$ to convert the $\lambda^{\cong}$ argument $v$. Intuitively, since we are going from $\lambda^{\cong}$ to $\lambda^{\rightarrow}$, no checks are necessary because the type system of $\lambda^{\cong}$ enforces all the properties that $\lambda^{\rightarrow}$ does and more.

Conversely, in EVAL_DTM_DS_CONSTR, we must verify that the argument converted from $\lambda^{\rightarrow}$ meets the specification demanded by the $\lambda^{\cong}$ datatype. To generate this check, we note that the type of the new

$\boxed{\Gamma \vdash t \cong t'}$

$$\frac{t \cong t' \in \Gamma}{\Gamma \vdash t \cong t'}\text{EQ\_DTM\_ASSUMPTION} \qquad \frac{t \longrightarrow t'}{\Gamma \vdash t \cong t'}\text{EQ\_DTM\_STEP}$$

$$\frac{}{\Gamma \vdash t \cong t}\text{EQ\_DTM\_REFL} \qquad \frac{\Gamma \vdash t' \cong t}{\Gamma \vdash t \cong t'}\text{EQ\_DTM\_SYM}$$

$$\frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ \Gamma \vdash t' \cong t''\end{array}}{\Gamma \vdash t \cong t''}\text{EQ\_DTM\_TRANS} \qquad \frac{\begin{array}{c}\Gamma \vdash t_1 \cong t_1' \\ y \notin \mathbf{dom}(\Gamma)\end{array}}{\Gamma \vdash [t_1/y]t \cong [t_1'/y]t}\text{EQ\_DTM\_SUBST}$$

$$\frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ y \notin \mathbf{dom}(\Gamma)\end{array}}{\Gamma \vdash [v/y]t \cong [v/y]t'}\text{EQ\_DTM\_SUBST\_VAL} \qquad \frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ x \notin \mathbf{dom}(\Gamma)\end{array}}{\Gamma \vdash [u/x]t \cong [u/x]t'}\text{EQ\_DTM\_SSUBST\_VAL}$$

Figure 9: $\lambda^{\cong}$ Term Equivalence

constructor $C\,v$ by WF_DTM_CTOR is $B\,[v/y]t_1$ where $B{:}T_1 \Rightarrow * \in \Psi_0$. The type demanded by the boundary is $B\,t$ and so we must check $t \cong [v/y]t_1$. Note that because of our restriction that $\mathsf{FO}\,(T_1)$, the equality check will never need to compare lambdas, only data of first-order type.

## 3.4 Equivalence

Equivalence checks are the core of a dependently-typed system. Figure 9 outlines the most important of these, equivalence over $\lambda^{\cong}$ terms. We elide $\lambda^{\cong}$ kind equivalence ($\Gamma \vdash K \equiv K'$) and $\lambda^{\cong}$ type equivalence ($\Gamma \vdash T \equiv T'$) as they are standard.

Our term-level equivalence is reflexive, transitive, and symmetric by the EQ_DTM_REFL, EQ_DTM_SYM, and EQ_DTM_TRANS rules. The most interesting of these rules is EQ_DTM_STEP which allows us to use reduction of $t$ in our equivalence relation. This rule is good because we do not need an explicit notion of $\lambda^{\rightarrow}$ equivalence, which would be unnatural. That is, in a real system, the $\lambda^{\cong}$ will only have available to it the ability to evaluate $\lambda^{\rightarrow}$ terms rather than have access to the internals of the entire $\lambda^{\rightarrow}$ program.

One subtlety that sets us apart from dependent languages like Coq and Agda is that our EQ_STM_STEP rule is restricted to *call-by-value* reduction. Pure, strongly normalizing languages have the luxury of allowing arbitrary $\beta$-reductions when comparing types because any order of evaluation gives the same answer. In our language that is not the case because of run-time errors, e.g. $(\lambda y{:}\,\mathsf{Unit}.\mathsf{unit})\,\mathsf{error}$ evaluates to $\mathsf{error}$ under CBV but to $\mathsf{unit}$ under CBN. This problem would get even worse if the language included more interesting side-effects.

For this reason, the type equivalence judgment is defined in terms of the evaluation relation $\longrightarrow$ which is explicitly CBV. Even so, we *do* want to allow reduction of open terms. For example to typecheck the usual *append* function we want $\mathsf{List}\,(0 + y) \equiv \mathsf{List}\,y$. Therefore, our definition of values includes variables. To make that choice work, we are careful to only substitute values for variables. In particular, we need an extra premise in WF_DTM_APP to check that the type $[t_2/y]\,T_2$ is well-kinded. It might not be, since the well-kindedness of $(y{:}\,T_1) \rightarrow T_2$ may depend on $y$ being a value.

### 3.5 Examples

To get a better understanding of how our system works, let's expand on the $\mathsf{List}$ example we've used so far.

The complete set of definitions for our $\mathsf{List}$ datatype are

$$\begin{aligned}
&\mathsf{List} : \mathsf{Int} \Rightarrow * \\
&\mathsf{Nil} : \mathsf{Unit} \to \mathsf{List} \\
&\mathsf{Nil} : (y\!:\!\mathsf{Unit}) \to \mathsf{List}\, 0 \\
&\mathsf{Cons} : (\mathsf{List} * \mathsf{Int}) \to \mathsf{List} \\
&\mathsf{Cons} : (y_1\!:\!(y_2\!:\!\mathsf{Int}) * (\mathsf{List}\, y_2 * \mathsf{Int})) \to \mathsf{List}\,(y_1.1) + 1.
\end{aligned}$$

So the types of our argument conversion functions are

$$\begin{aligned}
&\mathsf{argToS}_{\mathsf{Nil}} : \mathsf{Unit} \to \mathsf{Unit} \\
&\mathsf{argToD}_{\mathsf{Nil}} : \mathsf{Unit} \to \mathsf{Unit} \\
&\mathsf{argToS}_{\mathsf{Cons}} : (y_1 : (y_2 : \mathsf{Int}) * (\mathsf{List}\ y_2 * \mathsf{Int})) \to (\mathsf{List} * \mathsf{Int}) \\
&\mathsf{argToD}_{\mathsf{Cons}} : (\mathsf{List} * \mathsf{Int}) \to (y_1 : (y_2 : \mathsf{Int}) * (\mathsf{List}\ y_2 * \mathsf{Int})).
\end{aligned}$$

Note that the type of the arguments to $\mathsf{Cons}^{\to}$ is a pair whereas $\mathsf{Cons}^{\cong}$ is a triple. This is because the extra $\mathsf{Int}$ carried by $\mathsf{Cons}^{\cong}$ is required to represent the size of the argument $\mathsf{List}$.

Morally, a $\mathsf{List}\, y$ has length $y$ so our conversions needs to respect that property. The conversions of the arguments to $\mathsf{Nil}$ are trivial.

$$\begin{aligned}
&\mathsf{argToS}_{\mathsf{Nil}}\mathsf{unit} = \mathsf{unit} \\
&\mathsf{argToD}_{\mathsf{Nil}}\mathsf{unit} = \mathsf{unit}
\end{aligned}$$

To convert from a $\mathsf{Cons}^{\cong}$ to a $\mathsf{Cons}^{\to}$, we can simply drop the index argument. To convert in the other direction, we must regenerate it by requesting the $\mathsf{List}$'s length.

$$\begin{aligned}
&\mathsf{argToS}_{\mathsf{Cons}}(k, l, v) = (l, v) \\
&\mathsf{argToD}_{\mathsf{Cons}}(l, v) = (\mathsf{length}(l), (l, v))
\end{aligned}$$

This is reminiscent of McBride's work on ornamental types [21] where he also makes the observation that the difference between a simply-typed list and a standard dependently-typed list is the "ornamental" length data.

Matthews and Amhed demonstrate how nested boundaries can enforce specifications over the behavior of the weakly-typed language while being written in a strongly-typed language [19]. In their system, they are only able to express simple type specifications, e.g., that a Scheme function performs at type $\mathsf{Int} \to \mathsf{Int}$. As expected with our dependently-typed language, we are able to express more powerful constraints via this method. For example consider a function $\mathsf{pop}$ over simply-typed $\mathsf{List}$s.

$$\mathsf{pop} : \mathsf{List} \to \mathsf{List}$$

Given this function, we can write a safe variant of $\mathsf{pop}$ in $\lambda^{\cong}$ that simply calls $\mathsf{pop}$ to do the heavy lifting:

$$\begin{aligned}
&\mathsf{safePop} : (n : \mathsf{Int}) \to \mathsf{List}\ n \to \mathsf{List}\ (n-1) \\
&\mathsf{safePop} = \lambda n : \mathsf{Int}.\lambda y : \mathsf{List}\ n.\mathsf{DS}^{\mathsf{List}\ n-1}_{\mathsf{List}}\mathsf{pop}(\mathsf{SD}^{\mathsf{List}}_{\mathsf{List}\ n}y))
\end{aligned}$$

Now, this function will verify via dynamic checks that — provided the length of the subject list $n$ — $\mathsf{pop}$ does the right thing for that list.

Providing this length argument explicitly is annoying, so we can write one more wrapper around this method that is callable directly from $\lambda^{\to}$ and has the signature we want. The difference between this and the original $\mathsf{pop}$ is that now the function will check to see if $\mathsf{pop}$ produces the correct value:

$$\begin{aligned}
\mathsf{verifiedPop}\ &: \mathsf{List} \to \mathsf{List} \\
\mathsf{verifiedPop}\ &= \lambda y : \mathsf{List}. \\
&\quad \mathsf{let}\ l = \mathsf{length}\ y\ \mathsf{in} \\
&\quad \mathsf{SD}^{\mathsf{List}}_{\mathsf{List}\ \mathsf{DS}^{\mathsf{Int}}_{\mathsf{Int}}l-1}( \\
&\qquad \mathsf{safePop}\ (\mathsf{DS}^{\mathsf{Int}}_{\mathsf{Int}}l)\ (\mathsf{DS}^{\mathsf{DS}^{\mathsf{Int}}_{\mathsf{Int}}\mathsf{List}\ l}_{\mathsf{List}}y))
\end{aligned}$$

**Property 1** (Types of $\mathsf{argToD}/\mathsf{argToS}$)**.** *Suppose* $C{:}S \to A \in \Psi_0$ *and* $C{:}(y : T_1) \to B\ t_1 \in \Psi_0$.
*If* $\Gamma \vdash u : S$, *then* $\Gamma \vdash \mathsf{argToD}_C\, u : T_1$ *(if it is defined).*
*If* $\Gamma \vdash v : T_1$, *then* $\Gamma \vdash \mathsf{argToS}_C\, v : S$ *(if it is defined).*

**Property 2** (Correctness of $\mathsf{corr}\,(A, B)$)**.** *If* $\mathsf{corr}\,(A, B)$, *then* $A$ *and* $B$ *have the same constructors* $C_i$.

**Property 3** ($\mathsf{argToD}/\mathsf{argToS}$ respect substitution)**.** *If* $\mathsf{argToD}_C\, u$ *and* $\mathsf{argToS}_C\, v$ *are defined, then*
$\mathsf{argToD}_C([u_1/x_1]u) = [u_1/x_1](\mathsf{argToD}_C\, u)$
$\mathsf{argToD}_C([v_1/y_1]u) = [v_1/y_1](\mathsf{argToD}_C\, u)$
$\mathsf{argToS}_C([u_1/x_1]v) = [u_1/x_1](\mathsf{argToS}_C\, v)$
$\mathsf{argToS}_C([v_1/y_1]v) = [v_1/y_1](\mathsf{argToS}_C\, v)$

**Property 4** ($\mathsf{argToD}/\mathsf{argToS}$ respect $\longrightarrow_{\mathrm{p}}$)**.**
*If* $u \longrightarrow_{\mathrm{p}} u'$, *then* $\mathsf{argToD}_C\, u \longrightarrow_{\mathrm{p}} \mathsf{argToD}_C\, u'$.
*If* $v \longrightarrow_{\mathrm{p}} v'$, *then* $\mathsf{argToS}_C\, v \longrightarrow_{\mathrm{p}} \mathsf{argToS}_C\, v'$.

**Property 5.** $\mathsf{argToD}$ *and* $\mathsf{argToS}$ *are defined for closed values.*

Figure 10: Requirements on the conversion functions

$\mathsf{verifiedPop}$ is a good example of the power of dependent interoperability. We are able to take a simply-typed piece of code and then inject dynamic checks to verify its behavior against a dependently-typed specification.

# 4 Metatheory

Our technical contribution is a proof of type safety for SD: every well-typed term either goes to a value, diverges, or goes to $\mathsf{error}$. We state this result in the usual way, via Preservation and Progress theorems.

The type-safety proof puts some requirements on the user-defined translation-functions $\mathsf{argToD}$, $\mathsf{argToS}$, and $\mathsf{corr}\,(A, B)$. These are stated in figure 10, and we will point out where they are needed. Note that the round-tripping law is not one of the properties needed for type-safety. The term equivalence judgment does not axiomatize this property, so violating it does not lead to type errors. However, we still feel that requiring it rules out bad behavior.

## 4.1 Structural Lemmas

We begin by showing basic structural properties of the type system: Weakening, Substitution, and ignoring redundant assumptions.

Since the different syntactic categories of our language (simple and dependent terms, types and kinds) form a mutually recursive system, the proofs of these lemmas also need to be by mutual induction. The typing judgments call out to the type equivalence judgments, but the equivalence is defined without any reference to types, so the proofs about the equivalence judgments can be done first. For example, Weakening can be proved in two lemmas, each of which is proved using mutual induction.

**Lemma 1** (Weakening for Equivalence)**.**

1. *If* $\Gamma_1, \Gamma_3 \vdash t \cong t'$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash t \cong t'$.
2. *If* $\Gamma_1, \Gamma_3 \vdash T \equiv T'$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash T \equiv T'$.
3. *If* $\Gamma_1, \Gamma_3 \vdash K \equiv K'$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash K \equiv K'$

**Lemma 2** (Weakening)**.**

1. *If* $\Gamma_1, \Gamma_3 \vdash t : T$ *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash t : T$*.*
2. *If* $\Gamma_1, \Gamma_3 \vdash s : S$ *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash s : S$*.*
3. *If* $\Gamma_1, \Gamma_3 \vdash T : *$ *then* $\Gamma_1, \Gamma_3, \Gamma_3 \vdash T : *$*.*
4. *If* $\vdash \Gamma_1, \Gamma_2$ *then* $\vdash \Gamma_1$

The other lemmas are proved by similar mutual inductions. To save space we abbreviate sets of statements like this to $\Gamma \vdash J$, where the $J$ stands for all the judgment forms in the type system (equivalence, typing, and kinding).

For the Preservation proof we need a substitution lemma. Somewhat unusually, it is restricted to substituting values into the judgments, not arbitrary terms. This is because our term equivalence is CBV, so substituting a non-value could block reductions and cause types to no longer be equivalent.

**Lemma 3** (Substitution)**.**

1. *If* $\Gamma, x{:}S_2, \Gamma' \vdash J$ *and* $\Gamma \vdash u_2 : S_2$ *then* $\Gamma, [u_2/x]\Gamma' \vdash [u_2/x]J$*.*
2. *If* $\Gamma, y{:}T_2, \Gamma' \vdash J$ *and* $\Gamma \vdash v_2 : T_2$ *then* $\Gamma, [v_2/y]\Gamma' \vdash [v_2/y]J$*.*

Because we present dependent pattern matching using explicit equality assumptions in the context, we also need a set of structural lemmas stating that we can omit redundant equations and swap equivalent ones. These lemmas are used when proving type preservation of case-expressions and guard expressions: when the scrutinee steps, the corresponding equation changes to a syntactically different but $\beta$-equivalent one.

**Lemma 4** (Cut)**.** *If* $\Gamma \vdash t_1 \cong t_2$ *and* $\Gamma, t_1 \cong t_2, \Gamma' \vdash J$*, then* $\Gamma, \Gamma' \vdash J$*.*

**Lemma 5** (Context Equivalence)**.** *If* $\Gamma \vdash t_1 \cong t_1'$ *and* $\Gamma \vdash t_2 \cong t_2'$ *and* $\Gamma, t_1 \cong t_2, \Gamma' \vdash J$*, then* $\Gamma, t_1' \cong t_2', \Gamma' \vdash J$*.*

Cut is proved like a substitution lemma: each use of the equality assumption is replaced by the explicit derivation of the equation. The Context Equivalence lemma follows as a corollary of Weakening and Cut.

## 4.2 Preservation

We prove preservation by mutual recursion on the simple typing, dependent typing, and kinding judgment.

**Theorem 1** (Preservation)**.**

1. *If* $\Gamma \vdash s : S$ *and* $s \longrightarrow s'$ *then* $\Gamma \vdash s' : S$*.*
2. *If* $\Gamma \vdash [t/y]t_0 : T$ *and* $t \longrightarrow t'$ *then* $\Gamma \vdash [t'/y]t_0 : T$*.*
3. *If* $\Gamma \vdash [t/y]T_0 : K$ *and* $t \longrightarrow t'$ *then* $\Gamma \vdash [t'/y]T_0 : K$*.*

The statement for simple typing is standard but we have generalized the ones for dependent typing and kinding. The reason for this twist is again the CBV-style dependent typesystem: we need to know that the premise $\Gamma \vdash [t_2/y]T_2 : *$ to the WF_DTM_APP rule is preserved when $t_2$ steps. The generalization creates some extra congruence-like cases to deal with, but essentially this is still a standard Preservation proof.

The proof of this theorem informs the typing rules for the interoperability features. We highlight a few interesting cases.

First, the case when a SD-boundary for pairs steps is interesting because we substitute into the type on the SD boundary:

$$\mathsf{SD}^{S_1 * S_2}_{(y{:}T_1) * T_2} <v_1, v_2> \longrightarrow <\mathsf{SD}^{S_1}_{T_1} v_1, \mathsf{SD}^{S_2}_{[v_1/y]T_2} v_2>$$

This is different from prior work on non-dependent interoperability. We might worry that this would interfere with the compatibility check of the type. However, that is not the case, as we have the following lemma, which states that compatibility never looks at the terms embedded inside a type.

**Lemma 6.** *$S \Leftrightarrow T$ iff $S \Leftrightarrow [t/y]\,T$.*

Now, from the derivation of $\mathsf{SD}^{S_1 * S_2}_{(y:T_1)*T_2}\,{<}v_1, v_2{>}$ we get $S_1 * S_2 \Leftrightarrow (y : T_1) * T_2$, so by inversion $S_2 \Leftrightarrow T_2$ and hence $S_2 \Leftrightarrow [v_1/y]\,T_2$, which is the compatibility condition that we need for the term $\mathsf{SD}^{S_2}_{[v_1/y]\,T_2} v_2$ to be well-typed.

Next, consider the case when a $\mathsf{DS}$-boundary for a data constructor steps. This is the case that motivates our handling of dynamic checks:

$$\mathsf{DS}^{(B\ t)}_{A}(C\ u) \longrightarrow t \cong [v/y]t_1 \rhd (C\ v) \text{ where } \mathsf{argToD}_C\, u = v$$

when the signature contains declarations $C : S \to A$ and $C : (y : T_1) \to B\, t_1$. By our requirements on $\mathsf{argToD}$ we know that $\Gamma \vdash v : T_1$, so $\Gamma \vdash C\,v : B\,[v/y]t_1$. By the type conversion rules, that means $\Gamma, t \cong [v/y]t_1 \vdash C\,v : B\,t$. So we wrap the expression in a guard that enforces that equality assumption.

A final interesting case is when a guarded term steps. This motivates the structural lemmas Cut and Context Equivalence. The typing rule looks like

$$\frac{\begin{array}{l}\Gamma \vdash t_0 : T_0 \\ \Gamma \vdash t_1 : T_0 \\ \mathsf{FO}\,(T_0) \\ \Gamma, t_1 \cong t_0 \vdash t : T\end{array}}{\Gamma \vdash t_1 \cong t_0 \rhd t : T}\ \textsc{wf\_dtm\_guard}$$

Consider how the term can step. If $t_1 \longrightarrow t_1'$, then it suffices to show $\Gamma, t_1' \cong t_0 \vdash t : T$. But by the rule $\textsc{eq\_dtm\_step}$, $\Gamma, t_1' \cong t_0$ and $\Gamma, t_1' \cong t_0$ are equivalent contexts. Otherwise, if it steps by $v \cong v \rhd t \longrightarrow t$, then by $\textsc{eq\_dtm\_refl}$ the equation $v \cong v$ was redundant, so by Cut we can show $\Gamma \vdash t : T$ as required. Finally, it may step by $v \cong v' \rhd \mathsf{error}$. Since $\mathsf{error}$ is always well-typed, preservation holds. Although the proof doesn't illustrate it, the $\mathsf{FO}\,(T_0)$ restriction means that we will never go to error unless it is absolutely necessary, when $v$ and $v'$ are unequal first-order values.

## 4.3 Progress

As it turns out, the interoperability features do not add much complication to the Progress part of the proof. However, as is common in languages with dependent pattern matching, we need to do a bit of work to rule out contradictory equalities.

To prove progress we first need to prove a canonical forms lemma.

**Lemma 7** (Canonical Forms)**.**

1. *If $\cdot \vdash v : (y : T_1) \to T_2$ then $v$ is $\lambda y{:}\,T.t$.*
2. *If $\cdot \vdash v : (y : T_1) * T_2$ then $v$ is ${<}v_1, v_2{>}$.*
3. *If $\cdot \vdash v : \mathsf{Unit}$ then $v$ is $\mathsf{unit}$.*
4. *If $\cdot \vdash v : B\,t$ then $v$ is $C\,v'$ and $C{:}(y : T) \to B\,t' \in \Psi_0$.*

This does not follow immediately from inspecting the typing judgment, because of the rule $\textsc{eq\_dty\_incon}$: if we could somehow in the empty context prove $\cdot \vdash C_1\,v_1 \cong C_2\,v_2$ where $C_1 \neq C_2$, then we could assign any term any type. So we need to rule out such an inconsistent equation. However, the way we define the term equivalence judgment $\Gamma \vdash t \cong t'$ makes that difficult. The definition is succinct, but because it has an explicit transitivity rule it doesn't give any leverage for doing induction on it.

Our solution is to define an auxiliary notion of *parallel reduction*, denoted $\longrightarrow_{\mathrm{p}}$, in the style of Takahashi [31]. This relation contains the evaluation relation $\longrightarrow$, but it also allows reducing more than one redex, and reducing inside the body of a lambda expression or other binder. For example, the two parallel reduction rules for applications are:

$$\frac{t_1 \longrightarrow_{\mathrm{p}} t_1' \qquad t_2 \longrightarrow_{\mathrm{p}} t_2'}{t_1\, t_2 \longrightarrow_{\mathrm{p}} t_1'\, t_2'} \qquad \frac{t_1 \longrightarrow_{\mathrm{p}} t_1' \qquad v_2 \longrightarrow_{\mathrm{p}} v_2'}{(\lambda y\colon T.t_1)\, v_2 \longrightarrow_{\mathrm{p}} [v_2'/y]t_1'}$$

As a result, unlike evaluation, parallel reduction is closed under substitution: if $v_1 \longrightarrow_{\mathrm{p}} v_2$ and $t_1 \longrightarrow_{\mathrm{p}} t_2$ then $[v_1/y]t_1 \longrightarrow_{\mathrm{p}} [v_2/y]t_2$ and $[t_1/y]t \longrightarrow_{\mathrm{p}} [t_2/y]t$. We also show that it is confluent. Together, these properties lets us prove a useful characterization of term equivalence.

**Lemma 8** (Parallel reduction contains term equivalence)**.** *If $\cdot \vdash t_1 \cong t_2$, then there exists some $t'$ such that $t_1 \longrightarrow_{\mathrm{p}^*} t'$ and $t_2 \longrightarrow_{\mathrm{p}^*} t'$.*

This lemma rules out the inconsistent equation we were worried about, since reducing a term can never change its constructor. We can then straightforwardly show Canonical Forms and Progress.

**Theorem 2** (Progress)**.**

1. *If $\cdot \vdash t : T$ then either $t \longrightarrow t'$, $t$ is a value, or $t$ is* $\mathsf{error}$.
2. *If $\cdot \vdash s : S$ then either $s \longrightarrow s'$, $s$ is a value, or $s$ is* $\mathsf{error}$.

However, there is a difficulty. In order to prove substitution and confluence of parallel reduction, we need to assume these properties for the $\mathsf{argToD}$ and $\mathsf{argToS}$ functions, because the reduction relation is defined in terms of them. This yields properties 3 and 4 in figure 10.

We expect these requirements to be satisfied by any "natural" definition of $\mathsf{argToD}$ and $\mathsf{argToS}$. For example, one definition that would not respect parallel reduction would be to define

$$\begin{aligned} \mathsf{argToS}_C(\lambda y\colon \mathsf{Unit}.1+1) &= \mathsf{true} \\ \mathsf{argToS}_C(\lambda y\colon \mathsf{Unit}.2) &= \mathsf{false} \end{aligned}$$

But such a function, which examines the body of a $\lambda$-abstraction, could never be written by user code. In practice, we expect the translation functions to do pattern matching and to construct constructor applications and function calls, e.g. $\mathsf{argToD}_{\mathsf{Cons}}$ in section 3.5. Such translation functions automatically satisfy these requirements, because they are just built up from $\lambda^{\rightarrow}$ and $\lambda^{\cong}$ terms.

# 5 Additional Properties

Two important properties of SD that deserve special mention are the soundness of the dependently-typed fragment of the language and decidable typechecking.

## 5.1 Soundness

Soundness of a dependently-typed language is important because a sound language can function as a proof system. Unfortunately, by introducing boundaries that produce errors and defer complete typechecking until runtime, we've removed soundness from $\lambda^{\cong}$.

In the case of $\mathsf{error}$ we can simply consider the empty datatype $\mathsf{false}$ that should have no inhabitants. But due to SD_WF_DTM_ERROR we can ascribe $\mathsf{error}$ that type.

With respect to complete typechecking, consider the term

$$\mathsf{case}\,\mathsf{DS}^{(\mathsf{Foo}\,1)}_{\mathsf{Foo}}\mathsf{mkFoo}\,\mathsf{unit}\,\mathsf{of}\,\mathsf{mkFoo}\,y \rightarrow t$$

Where $\mathsf{Foo} : \mathsf{Int} \Rightarrow *$ and $\mathsf{mkFoo} : (y : \mathsf{Unit}) \rightarrow \mathsf{Foo}\,0$. The boundary typechecks giving $\mathsf{DS}^{(\mathsf{Foo}\,1)}_{\mathsf{Foo}} s$ the type $\mathsf{Foo}\,1$, an uninhabited type. By SD_WF_DTM_CASE, in the only case for $\mathsf{Foo}$ we arrive at the inequality $0 \cong 1 \in \Gamma$ and can thus typecheck the case to $\mathsf{false}$.

$\boxed{s \longrightarrow s'}$ $\boxed{t \longrightarrow t'}$

$$\frac{}{\mathsf{SD}^{S}_{\mathbf{L}}(\mathsf{DS}^{\mathbf{L}}_{S} u) \longrightarrow u}\text{EVAL\_STM\_SD\_LUMP}$$

$$\frac{}{\mathsf{DS}^{T}_{\mathbf{L}}(\mathsf{SD}^{\mathbf{L}}_{T} v) \longrightarrow v}\text{EVAL\_DTM\_DS\_LUMP}$$

Figure 11: Evaluation Rules for Lumps

Note that this is an unavoidable consequence of boundaries. We need to signal errors at runtime and our boundaries necessarily make claims (e.g., above that the boundary expects a $\mathsf{Foo}\,1$ even though it is impossible) that can only be verified at runtime.

However, like Lambda-eek [13], we believe that while an interoperating calculus such as SD is not necessarily suitable as a proof system, it is interesting as a programming language in its own right.

## 5.2 Decidable Typechecking

A related question to the soundness of $\lambda^{\cong}$ is whether the typechecking of SD is decidable in the presence of term evaluation in types. With our current formulation of $\lambda^{\rightarrow}$ and $\lambda^{\cong}$, we believe (but do not prove) that SD is strongly normalizing and thus typechecking of SD is decidable. We believe that this is reasonable given that both $\lambda^{\rightarrow}$ and $\lambda^{\cong}$ appear to be strongly normalizing and the type-directed boundaries that we consider in SD themselves do not contribute any additional computational power to the language.

Irrespective of this, it is clear that we can make SD typechecking undecidable by giving $\lambda^{\rightarrow}$ recursive functions. This is because we determine the equivalence of $t_1 \cong t_2$ by $\beta$-reduction as per the EQ_DTM_STEP rule (Figure 9). With recursive functions in $\lambda^{\rightarrow}$, evaluation of a $\mathsf{DS}$ boundary could end up in an infinite loop.

Because our actual $\lambda^{\rightarrow}$ language will likely be a general-purpose functional language with recursion, how might we recover decidable typechecking in this scenario? One such approach is to introduce a purity check in $\lambda^{\cong}$ that restricts boundaries from being embedded in types. This is a clean way to regain decidable typechecking but at the cost of losing the ability to embed terminating boundary terms in types.

Finally, we may give up the ambition that the typechecker automatically decides term equivalence by evaluating terms, and instead require the programmer to add explicit annotations stating what should be evaluated for how many steps. An example of a language taking this approach is Guru [29].

## 5.3 Lumping and Non-termination

One tempting suggestion to alleviate the problem of decidable typechecking is to limit how we can compute with values across the boundary. Rather than marshaling values, perhaps we can treat data on the other side of the boundary as a opaque *lump* that we can carry around and give back, but otherwise not inspect its contents. We give the evaluation rules in Figure 11. While appealing at first glance, it turns out that this system admits non-termination.

In the lump variant of our rules, we introduce a type $\mathbf{L}$ that represents an opaque lump value contained in a boundary. With lumps, boundaries no longer marshal values between languages or otherwise look at their structure. Instead, boundaries are "canceled out" when they meet each other as per EVAL_STM_SD_LUMP and EVAL_DTM_DS_LUMP. The problem is that it turns out that you can write an infinite loop with these boundaries in a similar manner to type dynamic [1] where you use a pair of functions of type $\mathbf{L} \rightarrow (\mathbf{L} \rightarrow \mathbf{L})$ and $(\mathbf{L} \rightarrow \mathbf{L}) \rightarrow \mathbf{L}$ to encode a term $\Omega$ that loops. The actual terms for these functions and $\Omega$ are the same as Matthews' and Findler's versions for their ML-in-ML calculus [20] but adapted to our boundaries.

Because of this, any interoperability boundary between simply- and dependently-typed languages using a lump style induces undecidable typechecking if boundaries can appear in dependent types and reduce.

# 6 Comparisons

Many real-world dependently-typed languages provide some facilities for interoperability with simply-typed languages. However we know of no language that provides the flexibility suggested by SD. Now that we've established SD and its properties, it is instructive to compare the techniques used by these dependently-typed languages with how SD establishes its interoperability boundaries for two reasons. First, if SD can accurately describe the interoperability features of these languages, then it builds confidence that SD is a good model for dependent interoperability in general. And second, the differences between the two suggests ways that the dependently-typed language can improve its interoperability support, or conversely, why it may be hard to do so.

## 6.1 ATS Data Translation

ATS [6] is built with interoperability with C in mind. Since the two languages share the same data representation, marshaling is relatively trivial. ATS values are typically exposed to C as wrapped structs, e.g., a C int has type `ats_int_type` in ATS. ATS functions can be exposed to C via `extern` declarations and C code can either be inlined into ATS files or referenced as external values or types. In this sense, ATS closely mimics the two-way interoperability boundary of SD.

However, beyond basic type-checking, ATS interoperability makes no attempt at checking to see if dependent type properties are preserved when traveling in and out of C. This is because with arbitrary casts, C code can arbitrarily munge ATS values or otherwise break the type guarantees made by ATS.

## 6.2 Extraction in Coq

The theorem prover Coq [32] provides a mechanism, `Extraction`, that extracts functional programs written in OCaml (or other functional languages such as Haskell) from proofs of specifications [15]. Coq distinguishes between computationally relevant types ($\mathsf{Set}$s) and computationally irrelevant types ($\mathsf{Prop}$s) and uses that information to guide `Extraction`. Datatypes extracted from Coq are translated into comparable datatypes in ML. Alternatively, Coq provides a mechanism for the user to map a Coq datatype and its associated constructors into a ML datatype and its constructors.

For our purposes, `Extraction` is a form of *one-way* interoperability where ML code can use verified Coq code . If we imagine the extracted program as living in $\lambda^{\cong}$ and the ML code living in $\lambda^{\rightarrow}$, then this amounts to only allowing the user to call $\lambda^{\cong}$ code via a $\mathsf{SD}$ boundary.

However, there are several limitations to the one-way interoperability model offered by `Extraction`:

1. **Extracted code does not enforce the properties of datatypes.** By design the extracted code is correct up to the verification done in Coq. However, because of erasure, the extracted code cannot verify that ML data passed to it meets the pre-conditions (if any) to use that code. For example, our $\mathsf{List}\, y$ example datatype would be erased to a simple $\mathsf{List}$ in ML. If the extracted code depends on receiving a non-empty $\mathsf{List}$ then it must trust the user to give it a non-empty $\mathsf{List}$ rather than enforcing that pre-condition itself.

2. **User-defined translation of datatypes is simple macro replacement.** In $\mathsf{SD}$, the user-defined translation function $\mathsf{argToS}$ is any function from the arguments of the $\lambda^{\cong}$ constructor to the $\lambda^{\rightarrow}$ constructor that respects the properties we outlined in the previous sections. In Coq, the analogous `Extract Inductive` command performs a macro-replacement of the occurrences of the datatype and its constructors with the strings specified with the commands. The resulting ML code is not even checked for well-formedness.

### 6.3 Agda Data Translation

Agda [23] provides a foreign-function interface that allows Agda to call into Haskell code. As part of the FFI, the user specifies Haskell functions to call from Agda with the `{-# COMPILED #-}` pragma. The user can also specify translations from Agda datatypes to Haskell datatypes via the `{-# COMPILED_DATA ... #-}` pragma.

Like Coq `Extraction`, the Agda FFI is a *one-way* interoperability layer. The difference is that the FFI allows Agda, the dependently-typed language, to invoke Haskell code, the simply-typed language . Translation occurs when Agda invokes a Haskell function. The arguments are converted to Haskell and the return value converted back to Agda according to the FFI's built in rules to translate Agda types coupled with the declared `COMPILED_DATA` pragmas.

Agda's FFI suffers from problems similar to Coq `Extraction` due to the restrictive nature of Agda's translation function. Agda erases terms in types down to `unit` so the translation has no way of preserving or even checking to see if the properties of dependent types are preserved. Unlike Coq `Extraction`'s macro-based datatype compatibility declarations, Agda's compatibility declarations are type-directed. However, they are still less flexible than SD as you can only map constructors of the same number of arguments and types.

### 6.4 Coq's Program Tactic

Coq's `Program` tactic [28] offers a different flavor of interoperability than `Extraction`. `Program` allows the user to write dependently-typed code in the form of predicate subtyping [27] over terms, but using a simply-typed language instead. This simply-typed language is a relaxed version of Coq's term language, but could very well be OCaml or Haskell instead.

The work flow of `Program` occurs in two steps:

1. The user writes a program in the simply-typed fragment. This includes predicates over types written in the refinement style `{x | P}`. The user does not need to write any proofs during this step.

2. Coq elaborates the program into Coq terms and then generates a series of proof obligations that the user must discharge. The result is a complete Coq term that is the program that meets the specifications outlined via the predicates of the program.

`Program` is an example of a dependently-typed system utilizing the power of a simply typed system to do interesting work. We can view the elaboration step from the simply-typed fragment to Coq as a translation from $\lambda^{\rightarrow}$ to $\lambda^{\cong}$ where we are interested in using $\lambda^{\cong}$ to prove properties of the $\lambda^{\rightarrow}$ program.

## 7 Prior Work

We believe our work is the first to directly address the technical challenges involved with interoperating between a dependently-typed and simply-typed programming language. However, there has been considerable effort in related areas that we highlight here.

**Interoperability Implementation** Since different programming languages typically operate under different runtime environments, much of the early work in interoperability research focuses how to reconcile those environments. Frequently the analysis takes specific pairs of languages, usually C, with other languages such as Java [7], ML [4], and Haskell [5], but sometimes also with other language pairs such as Python to Scheme [25] or SML to Java [22]. Other approaches attempt to develop a *lingua franca* by which two languages can communicate such as C [3], the Java virtual machine, COM [26], or the .NET framework [30].

**Interoperability Semantics** There has been comparatively less work in understanding the semantics of interoperating languages. We extend Matthews's and Findler's original work [20] that showed that even with simple language pairs — untyped and simply-typed lambda calculi — interoperability leads to some surprising results. Their latest work in this area focuses on adding polymorphism to a interoperability setting while preserving parametricity [19].

**Mixing Dependency with Dynamic** A different thread of related research comes from analyses of dependently-typed languages intermixed with type dynamic [1]. Ou et al [24] introduce `simple` and `dependent` constructs in which dynamically-typed and dependently-typed, respectively, exists. They allow for nesting of such constructs (e.g., `simple{dynamic{...}}`) and provide rules for how simple blocks dynamically enforce constraints imposed by dependent blocks. Gronski et al [12] extend this approach to a pure-type system without explicit, separate constructs for dynamic and dependent types. Instead, they include dynamic as a base type and assume the rest of the world is dependent.

**Refinement Types and Contracts** The underlying framework for many of these systems is the theory of refinement types [10] and higher-order contracts [8]. Recently, the study of contracts has gone in many directions, for example assigning blame [33]. Directly relevant to our work is the study of dependent contracts, e.g., the systems studied by Greenberg et al [11].

# 8 Future Work and Conclusion

We tackle the problem of making dependently-typed programming more accessible from the viewpoint of interoperability. Can we author an interoperability boundary between a dependently-typed language and a simply-typed language that preserves the properties enforced by the dependently-typed language? Our solution, the language SD, is able to meet design goals we set forth for such an interoperability layer: using code from one language from within the other language and verifying properties of simply-typed code with the dependently-typed language.

In the future, we would like to apply the ideas in this paper to improve the interop support of real-world languages like Coq and Agda, e.g., adding true "two-way" interoperability. Theoretically, there is also room for more careful analysis: proofs of strong normalization and a theorem characterizing when boundaries can be inserted without changing program behavior in harmful ways.

There are also more design variations for SD worth exploring. In particular, we restrict datatype indices at boundaries to be first-order. While this is not a serious limitation, it would be interesting to adapt ideas from the contracts literature and decompose equality checks of functions into checks at their use sites during type conversion. Finally, we can move beyond the pairing of dependent and simple types are explore other combinations such as dependent and dynamic types and pairings involving linear types.

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# A The Full Language

Figures 12 through 28 gives the full syntax and semantics of SD.

$$
\begin{array}{lllll}
\lambda^{\rightarrow}\ \text{Types} & S & ::= & S_1 \rightarrow S_2 \ \mid\ S_1 * S_2 \ \mid\ \mathsf{Unit} \ \mid\ A \\
\lambda^{\rightarrow}\ \text{Terms} & s & ::= & x \ \mid\ \lambda x{:}S.s \ \mid\ s_1\, s_2 \\
 & & \mid & <s_1, s_2> \mid\ s.1 \ \mid\ s.2 \\
 & & \mid & C\ s \ \mid\ \mathsf{case}\ s\ \mathsf{of}\ \overline{C_i\ x_i \rightarrow s_i}^{\,i} \\
 & & \mid & \mathsf{unit} \ \mid\ \mathsf{error} \ \mid\ \mathsf{letd}\ y = t\ \mathsf{in}\ s \ \mid\ \mathsf{SD}^S_T\, t
\end{array}
$$

$$
\begin{array}{lllll}
\lambda^{\cong}\ \text{Kinds} & K & ::= & * \ \mid\ T \Rightarrow * \\
\lambda^{\cong}\ \text{Types} & T & ::= & (y : T_1) \rightarrow T_2 \ \mid\ T\ t \\
 & & \mid & (y : T_1) * T_2 \ \mid\ \mathsf{Unit} \ \mid\ B \\
\lambda^{\cong}\ \text{Terms} & t & ::= & y \ \mid\ \lambda y{:}\,T.t \ \mid\ t_1\, t_2 \\
 & & \mid & <t_1, t_2> \mid\ t.1 \ \mid\ t.2 \\
 & & \mid & C\ t \ \mid\ \mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \rightarrow t_i}^{\,i} \\
 & & \mid & \mathsf{unit} \ \mid\ \mathsf{error} \\
 & & \mid & \mathsf{DS}^T_S\, s \ \mid\ t_1 \cong t_2 \rhd t_3
\end{array}
$$

Figure 12: SD Syntax

$\boxed{\Gamma \vdash s : S}$

$$\frac{\vdash \Gamma \quad x{:}S \in \Gamma}{\Gamma \vdash x : S}\text{WF\_STM\_VAR} \qquad \frac{\Gamma, x{:}S_1 \vdash s : S_2}{\Gamma \vdash \lambda x{:}S_1.s : S_1 \to S_2}\text{WF\_STM\_ABS}$$

$$\frac{\Gamma \vdash s_1 : S_1 \to S_2 \quad \Gamma \vdash s_2 : S_1}{\Gamma \vdash s_1\, s_2 : S_2}\text{WF\_STM\_APP} \qquad \frac{\Gamma \vdash s_1 : S_1 \quad \Gamma \vdash s_2 : S_2}{\Gamma \vdash <s_1, s_2>: S_1 * S_2}\text{WF\_STM\_PAIR}$$

$$\frac{\Gamma \vdash s : S_1 * S_2}{\Gamma \vdash s.1 : S_1}\text{WF\_STM\_PROJ1} \qquad \frac{\Gamma \vdash s : S_1 * S_2}{\Gamma \vdash s.2 : S_2}\text{WF\_STM\_PROJ2}$$

$$\frac{C{:}S \to A \in \Psi_0 \quad \Gamma \vdash s : S}{\Gamma \vdash C\, s : A}\text{WF\_STM\_CTOR} \qquad \frac{\Gamma \vdash s : A \quad \mathsf{constrs}\ A = \overline{C_i}^{\,i} \quad \overline{C_i{:}S'_i \to A \in \Psi_0}^{\,i} \quad \overline{\Gamma, x_i{:}S'_i \vdash s_i : S}^{\,i}}{\Gamma \vdash \mathsf{case}\ s\ \mathsf{of}\ \overline{C_i\, x_i \to s_i}^{\,i} : S}\text{WF\_STM\_CASE}$$

$$\frac{\vdash \Gamma}{\Gamma \vdash \mathsf{unit} : \mathsf{Unit}}\text{WF\_STM\_UNIT} \qquad \frac{\Gamma \vdash t : T \quad \Gamma, y{:}T \vdash s : S}{\Gamma \vdash \mathsf{letd}\ y = t\ \mathsf{in}\ s : S}\text{WF\_STM\_LETD}$$

$$\frac{\Gamma \vdash t : T \quad S \Leftrightarrow T}{\Gamma \vdash \mathsf{SD}^{S}_{T}\, t : S}\text{WF\_STM\_SD} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \mathsf{error} : S}\text{WF\_STM\_ERROR}$$

Figure 13: $\lambda^{\to}$ Typing

$\boxed{\Gamma \vdash K}$

$$\frac{}{\Gamma \vdash *}\text{WF\_DKN\_PROPER} \qquad \frac{\Gamma \vdash T : *}{\Gamma \vdash T \Rightarrow *}\text{WF\_DKN\_ARR}$$

$\boxed{\Gamma \vdash T : K}$

$$\frac{\begin{array}{c}\Gamma \vdash T_1 : * \\ \Gamma, y{:}T_1 \vdash T_2 : *\end{array}}{\Gamma \vdash (y : T_1) \rightarrow T_2 : *}\text{WF\_DTY\_ARR} \qquad \frac{\begin{array}{l}\Gamma \vdash T : T_1 \Rightarrow * \\ \Gamma \vdash t : T_1\end{array}}{\Gamma \vdash T\, t : *}\text{WF\_DTY\_APP}$$

$$\frac{\begin{array}{c}\Gamma \vdash T_1 : * \\ \Gamma, y{:}T_1 \vdash T_2 : *\end{array}}{\Gamma \vdash (y : T_1) * T_2 : *}\text{WF\_DTY\_PAIR} \qquad \frac{}{\Gamma \vdash \mathsf{Unit} : *}\text{WF\_DTY\_UNIT}$$

$$\frac{B{:}T \Rightarrow * \in \Psi_0}{\Gamma \vdash B : T \Rightarrow *}\text{WF\_DTY\_DATA}$$

Figure 14: $\lambda^{\cong}$ Kinding

$\boxed{\Gamma \vdash t : T}$

$$\frac{\begin{array}{l}\vdash \Gamma \\ y{:}T \in \Gamma\end{array}}{\Gamma \vdash y : T}\ \text{WF\_DTM\_VAR} \qquad \frac{\Gamma, y{:}T_1 \vdash t : T_2}{\Gamma \vdash \lambda y{:}\,T_1.t : (y : T_1) \to T_2}\ \text{WF\_DTM\_ABS}$$

$$\frac{\begin{array}{l}\Gamma \vdash t_1 : (y : T_1) \to T_2 \\ \Gamma \vdash t_2 : T_1 \\ \Gamma \vdash [t_2/y]\,T_2 : *\end{array}}{\Gamma \vdash t_1\ t_2 : [t_2/y]\,T_2}\ \text{WF\_DTM\_APP} \qquad \frac{\begin{array}{l}\Gamma \vdash t_1 : T_1 \\ \Gamma \vdash t_2 : [t_1/y]\,T_2 \\ \Gamma \vdash (y : T_1) * T_2 : *\end{array}}{\Gamma \vdash <t_1, t_2>: (y : T_1) * T_2}\ \text{WF\_DTM\_PAIR}$$

$$\frac{\Gamma \vdash t : (y : T_1) * T_2}{\Gamma \vdash t.1 : T_1}\ \text{WF\_DTM\_PROJ1} \qquad \frac{\begin{array}{l}\Gamma \vdash t : (y : T_1) * T_2 \\ \Gamma \vdash [t.1/y]\,T_2 : *\end{array}}{\Gamma \vdash t.2 : [t.1/y]\,T_2}\ \text{WF\_DTM\_PROJ2}$$

$$\frac{\begin{array}{l}C{:}(y : T_1) \to B\ t' \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \Gamma \vdash t : T_1 \\ \Gamma \vdash B\ [t/y]t' : *\end{array}}{\Gamma \vdash C\ t : B\ [t/y]t'}\ \text{WF\_DTM\_CTOR} \qquad \frac{\begin{array}{l}\Gamma \vdash t : B\ t' \\ \Gamma \vdash T : * \\ \mathsf{constrs}\ B = \overline{C_i}^{\,i} \\ \overline{C_i{:}(y_i : T_i) \to B\ t'_i \in \Psi_0}^{\,i} \\ \overline{\Gamma, y_i{:}T_i, t' \cong t'_i, t \cong C_i\ y_i \vdash t_i : T}^{\,i}\end{array}}{\Gamma \vdash \mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \to t_i}^{\,i} : T}\ \text{WF\_DTM\_CASE}$$

$$\frac{\begin{array}{l}\Gamma \vdash s : S \\ \Gamma \vdash T : * \\ S \Leftrightarrow T\end{array}}{\Gamma \vdash \mathsf{DS}_S^T\, s : T}\ \text{WF\_DTM\_DS} \qquad \frac{\begin{array}{l}\Gamma \vdash t_0 : T_0 \\ \Gamma \vdash t_1 : T_0 \\ \mathsf{FO}\,(T_0) \\ \Gamma, t_1 \cong t_0 \vdash t : T\end{array}}{\Gamma \vdash t_1 \cong t_0 \rhd t : T}\ \text{WF\_DTM\_GUARD}$$

$$\frac{}{\Gamma \vdash \mathsf{unit} : \mathsf{Unit}}\ \text{WF\_DTM\_UNIT} \qquad \frac{\Gamma \vdash T : *}{\Gamma \vdash \mathsf{error} : T}\ \text{WF\_DTM\_ERROR} \qquad \frac{\begin{array}{l}\Gamma \vdash t : T \\ \Gamma \vdash T \equiv T' \\ \Gamma \vdash T' : *\end{array}}{\Gamma \vdash t : T'}\ \text{WF\_DTM\_CONV}$$

Figure 15: $\lambda^{\cong}$ Typing

$\boxed{\vdash \Psi}$

$$\frac{}{\vdash \cdot}\textsc{wf\_sig\_empty} \qquad \frac{\begin{array}{c}\vdash \Psi \\ A \in \Psi\end{array}}{\vdash \Psi, C : S \to A}\textsc{wf\_sig\_sctor} \qquad \frac{\begin{array}{c}\vdash \Psi \\ A \notin \Psi\end{array}}{\vdash \Psi, A}\textsc{wf\_sig\_styctor}$$

$$\frac{\begin{array}{c}\vdash \Psi \\ \cdot \vdash (y : T_2) \to B\ t : *\end{array}}{\vdash \Psi, C : (y : T_2) \to B\ t}\textsc{wf\_sig\_dctor} \qquad \frac{\begin{array}{c}\vdash \Psi \\ \cdot \vdash T : * \\ B \notin \Psi\end{array}}{\vdash \Psi, B : T \Rightarrow *}\textsc{wf\_sig\_dtyctor}$$

$\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \cdot}\textsc{wf\_ctx\_empty} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, x{:}S}\textsc{wf\_ctx\_scons}$$

$$\frac{\begin{array}{c}\vdash \Gamma \\ \Gamma \vdash T : *\end{array}}{\vdash \Gamma, y{:}T}\textsc{wf\_ctx\_dcons} \qquad \frac{\begin{array}{c}\vdash \Gamma \\ \Gamma \vdash t : T \\ \Gamma \vdash t' : T'\end{array}}{\vdash \Gamma, t \cong t'}\textsc{wf\_ctx\_equiv}$$

Figure 16: Auxiliary definitions

$\boxed{\mathsf{FO}\,(T)}$

$$\frac{\mathsf{FO}\,(T)}{\mathsf{FO}\,(T\ t)}\ \textsc{fo\_app} \qquad \frac{\mathsf{FO}\,(T_1) \quad \mathsf{FO}\,(T_2)}{\mathsf{FO}\,((y : T_1) * T_2)}\ \textsc{fo\_pair} \qquad \frac{}{\mathsf{FO}\,(\mathsf{Unit})}\ \textsc{fo\_unit}$$

$$\frac{\mathsf{constrs}\ B = \overline{C_i}^{\,i} \quad \overline{C_i{:}(y_i : T_i) \rightarrow B\ t_i' \in \Psi_0}^{\,i} \quad \overline{\mathsf{FO}\,(T_i)}^{\,i}}{\mathsf{FO}\,(B\ t)}\ \textsc{fo\_data}$$

$\boxed{S \Leftrightarrow T}$

$$\frac{S_1 \Leftrightarrow T_1 \quad S_2 \Leftrightarrow T_2}{S_1 \rightarrow S_2 \Leftrightarrow (y : T_1) \rightarrow T_2}\ \textsc{compat\_arr} \qquad \frac{S_1 \Leftrightarrow T_1 \quad S_2 \Leftrightarrow T_2}{S_1 * S_2 \Leftrightarrow (y : T_1) * T_2}\ \textsc{compat\_pair}$$

$$\frac{}{\mathsf{Unit} \Leftrightarrow \mathsf{Unit}}\ \textsc{compat\_unit} \qquad \frac{B{:}T_0 \Rightarrow * \in \Psi_0 \quad \mathsf{FO}\,(T_0) \quad \mathsf{corr}\,(A, B)}{A \Leftrightarrow B\ t}\ \textsc{compat\_data}$$

Figure 17: Auxiliary definitions (cont.)

$$\boxed{\Gamma \vdash K \equiv K'}$$

$$\frac{}{\Gamma \vdash * \equiv *}\text{EQ\_DKN\_REFL} \qquad \frac{\Gamma \vdash T \equiv T'}{\Gamma \vdash T \Rightarrow * \equiv T' \Rightarrow *}\text{EQ\_DKN\_PI}$$

Figure 18: $\lambda^{\cong}$ Kind Equivalence

$$\boxed{\Gamma \vdash T \equiv T'}$$

$$\frac{B{:}K \in \Psi_0}{\Gamma \vdash B \equiv B}\text{EQ\_DTY\_TREFL} \qquad \frac{\begin{array}{c}\Gamma \vdash T_1 \equiv T_1' \\ \Gamma \vdash T_2 \equiv T_2'\end{array}}{\Gamma \vdash (y : T_1) \to T_2 \equiv (y : T_1') \to T_2'}\text{EQ\_DTY\_PI}$$

$$\frac{\begin{array}{l}\Gamma \vdash C\; v \cong C'\; v' \\ C \neq C'\end{array}}{\Gamma \vdash T \equiv T'}\text{EQ\_DTY\_INCON} \qquad \frac{\begin{array}{c}\Gamma \vdash T_1 \equiv T_1' \\ \Gamma \vdash T_2 \equiv T_2'\end{array}}{\Gamma \vdash (y : T_1) * T_2 \equiv (y : T_1') * T_2'}\text{EQ\_DTY\_SIGMA}$$

$$\frac{}{\Gamma \vdash \mathsf{Unit} \equiv \mathsf{Unit}}\text{EQ\_DTY\_UREFL} \frac{\begin{array}{c}\Gamma \vdash T \equiv T' \\ \Gamma \vdash t \cong t'\end{array}}{\Gamma \vdash T\; t \equiv T'\; t'}\text{EQ\_DTY\_APP}$$

Figure 19: $\lambda^{\cong}$ Type Equivalence

$\boxed{\Gamma \vdash t \cong t'}$

$$\frac{t \cong t' \in \Gamma}{\Gamma \vdash t \cong t'}\text{EQ\_DTM\_ASSUMPTION} \qquad \frac{t \longrightarrow t'}{\Gamma \vdash t \cong t'}\text{EQ\_DTM\_STEP}$$

$$\frac{}{\Gamma \vdash t \cong t}\text{EQ\_DTM\_REFL} \qquad \frac{\Gamma \vdash t' \cong t}{\Gamma \vdash t \cong t'}\text{EQ\_DTM\_SYM} \qquad \frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ \Gamma \vdash t' \cong t''\end{array}}{\Gamma \vdash t \cong t''}\text{EQ\_DTM\_TRANS}$$

$$\frac{\begin{array}{c}\Gamma \vdash t_1 \cong t_1' \\ y \notin \mathbf{dom}(\Gamma)\end{array}}{\Gamma \vdash [t_1/y]t \cong [t_1'/y]t}\text{EQ\_DTM\_SUBST} \qquad \frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ y \notin \mathbf{dom}(\Gamma)\end{array}}{\Gamma \vdash [v/y]t \cong [v/y]t'}\text{EQ\_DTM\_SUBST\_VAL}$$

$$\frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ x \notin \mathbf{dom}(\Gamma)\end{array}}{\Gamma \vdash [u/x]t \cong [u/x]t'}\text{EQ\_DTM\_SSUBST\_VAL}$$

Figure 20: $\lambda^{\cong}$ Term Equivalence

$$\begin{array}{llll}
\lambda^{\rightarrow} \text{ Contexts} & \mathcal{E}_s & ::= & \Box \mid \Box\, s \mid u\,\Box \mid <\Box, s> \mid <u, \Box> \\
& & \mid & \Box.1 \mid \Box.2 \mid C\,\Box \\
& & \mid & \mathsf{case}\,\Box\,\mathsf{of}\,\overline{C_i\,x_i \to s_i}^{\,i} \mid \mathsf{letd}\,y = \Box\,\mathsf{in}\,s \mid \mathsf{SD}_T^S\Box \\
\lambda^{\cong} \text{ Contexts} & \mathcal{E}_t & ::= & \Box \mid \Box\, t \mid v\,\Box \mid <\Box, t> \mid <v, \Box> \\
& & \mid & \Box.1 \mid \Box.2 \mid C\,\Box \\
& & \mid & \mathsf{case}\,\Box\,\mathsf{of}\,\overline{C_i\,y_i \to t_i}^{\,i} \mid \mathsf{DS}_S^T\Box \mid \Box \cong t_1 \rhd t \mid v_0 \cong \Box \rhd t
\end{array}$$

Figure 21: Evaluation contexts

$\boxed{s \longrightarrow s'}$

$$\frac{}{(\lambda x{:}\, S.s_1)\; u_2 \longrightarrow [u_2/x]s_1}\text{EVAL\_STM\_BETA} \qquad \frac{s \longrightarrow s'}{\mathcal{E}_s.s \longrightarrow \mathcal{E}_s.s'}\text{EVAL\_STM\_CTX}$$

$$\frac{}{<u_1, u_2> .1 \longrightarrow u_1}\text{EVAL\_STM\_PROJ1} \qquad \frac{}{<u_1, u_2> .2 \longrightarrow u_2}\text{EVAL\_STM\_PROJ2}$$

$$\frac{}{\mathsf{letd}\; y = v \;\mathsf{in}\; s \longrightarrow [v/y]s}\text{EVAL\_STM\_LETD}$$

$$\frac{}{\mathsf{case}\; C_i\; u \;\mathsf{of}\; \overline{C_i\; x_i \to s_i}^{\,i} \longrightarrow [u/x_i]s_i}\text{EVAL\_STM\_CASE}$$

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\; t_1 \in \Psi_0 \\ \mathsf{argToS}_C\, v = u \end{array}}{\mathsf{SD}^{A}_{(B\; t)}\, C\; v \longrightarrow C\; u}\text{EVAL\_STM\_SD\_CONSTR}$$

$$\frac{}{\mathsf{SD}^{(S_1 \to S_2)}_{((y{:}T_1) \to T_2)} \lambda y{:}\, T_1'.t \longrightarrow \lambda x{:}\, S_1.\mathsf{letd}\; y' = \mathsf{DS}^{T_1}_{S_1}\, x \;\mathsf{in}\; \mathsf{SD}^{S_2}_{([y'/y]\, T_2)}((\lambda y{:}\, T_1'.t)\; y')}\text{EVAL\_STM\_SD\_ABS}$$

$$\frac{}{\mathsf{SD}^{(S_1 * S_2)}_{((y{:}T_1) * T_2)}\; <v_1, v_2> \longrightarrow <\mathsf{SD}^{S_1}_{T_1}\, v_1, \mathsf{SD}^{S_2}_{([v_1/y]\, T_2)}\, v_2>}\text{EVAL\_STM\_SD\_PAIR}$$

Figure 22: $\lambda^{\to}$ Evaluation

$\boxed{t \longrightarrow t'}$

$$\frac{}{(\lambda y{:}\, T.t_1)\; v_2 \longrightarrow [v_2/y]t_1}\text{EVAL\_DTM\_BETA} \qquad \frac{t \longrightarrow t'}{\mathcal{E}_t.t \longrightarrow \mathcal{E}_t.t'}\text{EVAL\_DTM\_CTX}$$

$$\frac{}{<v_1, v_2>\ .1 \longrightarrow v_1}\text{EVAL\_DTM\_PROJ1} \qquad \frac{}{<v_1, v_2>\ .2 \longrightarrow v_2}\text{EVAL\_DTM\_PROJ2}$$

$$\frac{}{\mathsf{case}\ C_i\ v\ \mathsf{of}\ \overline{C_i\ y_i \rightarrow t_i}^{\,i} \longrightarrow [v/y_i]t_i}\text{EVAL\_DTM\_CASE}$$

$$\frac{\begin{array}{l} C{:}S \rightarrow A \in \Psi_0 \\ C{:}(y : T_1) \rightarrow B\ t_1 \in \Psi_0 \\ \mathsf{argToD}_C\, u = v \end{array}}{\mathsf{DS}_A^{(B\ t)}(C\ u) \longrightarrow t \cong [v/y]t_1 \rhd (C\ v)}\text{EVAL\_DTM\_DS\_CONSTR}$$

$$\frac{}{\mathsf{DS}_{(S_1 * S_2)}^{((y{:}T_1)*T_2)}\ <u_1, u_2>\longrightarrow \mathsf{let}\ y' = \mathsf{DS}_{S_1}^{T_1} u_1\ \mathsf{in}\ <y', \mathsf{DS}_{S_2}^{[y'/y]\,T_2} u_2>}\text{EVAL\_DTM\_DS\_PAIR}$$

$$\frac{}{v \cong v \rhd t \longrightarrow t}\text{EVAL\_DTM\_GUARD\_REFL} \qquad \frac{v \neq v'}{v \cong v' \rhd t \longrightarrow \mathsf{error}}\text{EVAL\_DTM\_GUARD\_ERROR}$$

Figure 23: $\lambda^{\cong}$ Evaluation

$\boxed{s \longrightarrow_{\mathrm{p}} s'}$

$$\frac{}{s \longrightarrow_{\mathrm{p}} s}\ \textsc{par\_eval\_stm\_refl} \qquad \frac{}{\mathcal{E}_s.\mathsf{error} \longrightarrow_{\mathrm{p}} \mathsf{error}}\ \textsc{par\_eval\_stm\_error}$$

$$\frac{\begin{array}{c} s_1 \longrightarrow_{\mathrm{p}} s_1' \\ u_2 \longrightarrow_{\mathrm{p}} u_2' \end{array}}{(\lambda x{:}\, S.s_1)\ u_2 \longrightarrow_{\mathrm{p}} [u_2'/x]s_1'}\ \textsc{par\_eval\_stm\_beta} \qquad \frac{\begin{array}{c} s_1 \longrightarrow_{\mathrm{p}} s_1' \\ s_2 \longrightarrow_{\mathrm{p}} s_2' \end{array}}{s_1\ s_2 \longrightarrow_{\mathrm{p}} s_1'\ s_2'}\ \textsc{par\_eval\_stm\_app}$$

$$\frac{\begin{array}{c} S \longrightarrow_{\mathrm{p}} S' \\ s_1 \longrightarrow_{\mathrm{p}} s_1' \end{array}}{\lambda x{:}\, S.s \longrightarrow_{\mathrm{p}} \lambda x{:}\, S'.s'}\ \textsc{par\_eval\_stm\_abs} \qquad \frac{\begin{array}{c} s_1 \longrightarrow_{\mathrm{p}} s_1' \\ s_2 \longrightarrow_{\mathrm{p}} s_2' \end{array}}{<s_1, s_2> \longrightarrow_{\mathrm{p}} <s_1', s_2'>}\ \textsc{par\_eval\_stm\_pair}$$

$$\frac{s \longrightarrow_{\mathrm{p}} s'}{s.1 \longrightarrow_{\mathrm{p}} s'.1}\ \textsc{par\_eval\_stm\_proj}1 \qquad \frac{\begin{array}{c} u_1 \longrightarrow_{\mathrm{p}} u_1' \\ u_2 \longrightarrow_{\mathrm{p}} u_2' \end{array}}{<u_{11}, u_2>\ .1 \longrightarrow_{\mathrm{p}} u_1'}\ \textsc{par\_eval\_stm\_proj}1\textsc{Beta}$$

$$\frac{s \longrightarrow_{\mathrm{p}} s'}{s.2 \longrightarrow_{\mathrm{p}} s'.2}\ \textsc{par\_eval\_stm\_proj}2 \qquad \frac{\begin{array}{c} u_1 \longrightarrow_{\mathrm{p}} u_1' \\ u_2 \longrightarrow_{\mathrm{p}} u_2' \end{array}}{<u_1, u_2>\ .2 \longrightarrow_{\mathrm{p}} u_2'}\ \textsc{par\_eval\_stm\_proj}2\textsc{Beta}$$

Figure 24: Parallel reduction (simple terms)

$$\frac{s \longrightarrow_{\mathrm{p}} s'}{C\ s \longrightarrow_{\mathrm{p}} C\ s'}\ \textsc{par\_eval\_stm\_ctor}$$

$$\frac{\begin{array}{c} s \longrightarrow_{\mathrm{p}} s' \\ \overline{s_i \longrightarrow_{\mathrm{p}} s'_i}^{\,i} \end{array}}{\mathsf{case}\ s\ \mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i} \longrightarrow_{\mathrm{p}} \mathsf{case}\ s'\ \mathsf{of}\ \overline{C_i\ x_i \to s'_i}^{\,i}}\ \textsc{par\_eval\_stm\_case}$$

$$\frac{\begin{array}{c} u \longrightarrow_{\mathrm{p}} u' \\ \overline{s_i \longrightarrow_{\mathrm{p}} s'_i}^{\,i} \end{array}}{\mathsf{case}\ C_i\ u\ \mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i} \longrightarrow_{\mathrm{p}} [u'/x_i]s'_i}\ \textsc{par\_eval\_stm\_caseBeta}$$

$$\frac{\begin{array}{c} S \longrightarrow_{\mathrm{p}} S' \\ T \longrightarrow_{\mathrm{p}} T' \\ t \longrightarrow_{\mathrm{p}} t' \end{array}}{\mathsf{SD}^{S}_{T}\, t \longrightarrow_{\mathrm{p}} \mathsf{SD}^{S'}_{T'}\, t'}\ \textsc{par\_eval\_stm\_sd}$$

$$\frac{\begin{array}{cc} S_1 \longrightarrow_{\mathrm{p}} S'_1 & S_2 \longrightarrow_{\mathrm{p}} S'_2 \\ T_1 \longrightarrow_{\mathrm{p}} T'_1 & T_2 \longrightarrow_{\mathrm{p}} T'_2 \\ T_3 \longrightarrow_{\mathrm{p}} T'_3 & t \longrightarrow_{\mathrm{p}} t' \end{array}}{\mathsf{SD}^{(S_1 \to S_2)}_{((y:T_1) \to T_2)} \lambda y{:}\, T_3.t \longrightarrow_{\mathrm{p}} \lambda x{:}\, S'_1.\mathsf{letd}\ y' = \mathsf{DS}^{T'_1}_{S'_1}\, x\ \mathsf{in}\ \mathsf{SD}^{S'_2}_{([y'/y]\, T'_2)}((\lambda y{:}\, T_3.t')\ y')}\ \textsc{par\_eval\_stm\_sd\_abs}$$

$$\frac{\begin{array}{cc} S_1 \longrightarrow_{\mathrm{p}} S'_1 & S_2 \longrightarrow_{\mathrm{p}} S'_2 \\ T_1 \longrightarrow_{\mathrm{p}} T'_1 & T_2 \longrightarrow_{\mathrm{p}} T'_2 \\ v_1 \longrightarrow_{\mathrm{p}} v'_1 & v_2 \longrightarrow_{\mathrm{p}} v'_2 \end{array}}{\mathsf{SD}^{(S_1 * S_2)}_{((y:T_1) * T_2)}\ {<}v_1, v_2{>} \longrightarrow_{\mathrm{p}} {<}\mathsf{SD}^{S'_1}_{T'_1}\, v'_1, \mathsf{SD}^{S'_2}_{([v'_1/y]\, T'_2)}\, v'_2{>}}\ \textsc{par\_eval\_stm\_sd\_pair}$$

$$\frac{\begin{array}{l} \mathsf{constrs}\ A = \overline{C_i}^{\,i} \\ C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \mathsf{argToS}_C\, v' = u \\ v \longrightarrow_{\mathrm{p}} v' \end{array}}{\mathsf{SD}^{A}_{(B\ t)}\, C\ v \longrightarrow_{\mathrm{p}} C\ u}\ \textsc{par\_eval\_stm\_sd\_constr}$$

$$\frac{}{\mathsf{SD}^{\mathsf{Unit}}_{\mathsf{Unit}}\mathsf{unit} \longrightarrow_{\mathrm{p}} \mathsf{unit}}\ \textsc{par\_eval\_stm\_sd\_unit}$$

Figure 25: Parallel reduction (simple terms, cont.)

$\boxed{t \longrightarrow_{\mathrm{p}} t'}$

$$\frac{}{t \longrightarrow_{\mathrm{p}} t}\ \textsc{par\_eval\_dtm\_refl} \qquad \frac{}{\mathcal{E}_t.\mathsf{error} \longrightarrow_{\mathrm{p}} \mathsf{error}}\ \textsc{par\_eval\_dtm\_error}$$

$$\frac{\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} t_1' \\ t_2 \longrightarrow_{\mathrm{p}} t_2' \end{array}}{t_1\ t_2 \longrightarrow_{\mathrm{p}} t_1'\ t_2'}\ \textsc{par\_eval\_dtm\_app} \qquad \frac{\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} t_1' \\ v_2 \longrightarrow_{\mathrm{p}} v_2' \end{array}}{(\lambda y{:}\, T.t_1)\ v_2 \longrightarrow_{\mathrm{p}} [v_2'/y]t_1'}\ \textsc{par\_eval\_dtm\_appBeta}$$

$$\frac{T \longrightarrow_{\mathrm{p}} T' \qquad t \longrightarrow_{\mathrm{p}} t'}{\lambda y{:}\, T.t \longrightarrow_{\mathrm{p}} \lambda y{:}\, T'.t'}\ \textsc{par\_eval\_dtm\_abs}$$

$$\frac{\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} t_1' \\ t_2 \longrightarrow_{\mathrm{p}} t_2' \end{array}}{<t_1, t_2> \longrightarrow_{\mathrm{p}} <t_1', t_2'>}\ \textsc{par\_eval\_dtm\_pair} \qquad \frac{t \longrightarrow_{\mathrm{p}} t'}{t.1 \longrightarrow_{\mathrm{p}} t'.1}\ \textsc{par\_eval\_dtm\_proj1}$$

$$\frac{\begin{array}{c} v_1 \longrightarrow_{\mathrm{p}} v_1' \\ v_2 \longrightarrow_{\mathrm{p}} v_2' \end{array}}{<v_1, v_2>.1 \longrightarrow_{\mathrm{p}} v_1'}\ \textsc{par\_eval\_dtm\_proj1Beta} \qquad \frac{t \longrightarrow_{\mathrm{p}} t'}{t.2 \longrightarrow_{\mathrm{p}} t'.2}\ \textsc{par\_eval\_dtm\_proj2}$$

$$\frac{\begin{array}{c} v_1 \longrightarrow_{\mathrm{p}} v_1' \\ v_2 \longrightarrow_{\mathrm{p}} v_2' \end{array}}{<v_1, v_2>.2 \longrightarrow_{\mathrm{p}} v_2'}\ \textsc{par\_eval\_dtm\_proj2Beta} \qquad \frac{t \longrightarrow_{\mathrm{p}} t'}{C\ t \longrightarrow_{\mathrm{p}} C\ t'}\ \textsc{par\_eval\_dtm\_ctor}$$

$$\frac{v_1 \neq v_2}{v_1 \cong v_2 \rhd t \longrightarrow_{\mathrm{p}} \mathsf{error}}\ \textsc{par\_eval\_dtm\_guard\_error}$$

Figure 26: Parallel reduction (dependent terms)

$$\frac{\begin{array}{c} t \longrightarrow_{\mathrm{p}} t' \\ \overline{t_i \longrightarrow_{\mathrm{p}} t'_i}^{\,i} \end{array}}{\mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \to t_i}^{\,i} \longrightarrow_{\mathrm{p}} \mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \to t'_i}^{\,i}}\ \textsc{par\_eval\_dtm\_case}$$

$$\frac{\begin{array}{c} v \longrightarrow_{\mathrm{p}} v' \\ \overline{t_i \longrightarrow_{\mathrm{p}} t'_i}^{\,i} \end{array}}{\mathsf{case}\ C_i\ v\ \mathsf{of}\ \overline{C_i\ y_i \to t_i}^{\,i} \longrightarrow_{\mathrm{p}} [v'/y_i]t'_i}\ \textsc{par\_eval\_dtm\_caseBeta}$$

$$\frac{\begin{array}{ll} T \longrightarrow_{\mathrm{p}} T' & S \longrightarrow_{\mathrm{p}} S' \\ s \longrightarrow_{\mathrm{p}} s' & \end{array}}{\mathsf{DS}_{S}^{T}\, s \longrightarrow_{\mathrm{p}} \mathsf{DS}_{S'}^{T'}\, s'}\ \textsc{par\_eval\_dtm\_ds}$$

$$\frac{\begin{array}{ll} T_1 \longrightarrow_{\mathrm{p}} T'_1 & T_2 \longrightarrow_{\mathrm{p}} T'_2 \\ S_1 \longrightarrow_{\mathrm{p}} S'_1 & S_2 \longrightarrow_{\mathrm{p}} S'_2 \\ S_3 \longrightarrow_{\mathrm{p}} S'_3 & s \longrightarrow_{\mathrm{p}} s' \end{array}}{\mathsf{DS}_{(S_1 \to S_2)}^{((y:T_1) \to T_2)} \lambda x{:}\, S_3.s \longrightarrow_{\mathrm{p}} \lambda y{:}\, T'_1.\mathsf{DS}_{S'_2}^{T'_2}((\lambda x{:}\, S'_3.s')\,(\mathsf{SD}_{T'_1}^{S'_1}\, y))}\ \textsc{par\_eval\_dtm\_ds\_abs}$$

$$\frac{\begin{array}{ll} T_1 \longrightarrow_{\mathrm{p}} T'_1 & T_2 \longrightarrow_{\mathrm{p}} T'_2 \\ S_1 \longrightarrow_{\mathrm{p}} S'_1 & S_2 \longrightarrow_{\mathrm{p}} S'_2 \\ u_1 \longrightarrow_{\mathrm{p}} u'_1 & u_2 \longrightarrow_{\mathrm{p}} u'_2 \end{array}}{\mathsf{DS}_{(S_1 * S_2)}^{((y:T_1) * T_2)}\ <u_1, u_2> \longrightarrow_{\mathrm{p}} \mathsf{let}\ y' = \mathsf{DS}_{S'_1}^{T'_1}\, u'_1\ \mathsf{in}\ <y', \mathsf{DS}_{S'_2}^{[y'/y]\,T'_2}\, u'_2>}\ \textsc{par\_eval\_dtm\_ds\_pair}$$

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \mathsf{argToD}_C\, u' = v \\ u \longrightarrow_{\mathrm{p}} u' \\ t \longrightarrow_{\mathrm{p}} t' \end{array}}{\mathsf{DS}_{A}^{(B\ t)}(C\ u) \longrightarrow_{\mathrm{p}} t' \cong [v/y]t_1 \rhd (C\ v)}\ \textsc{par\_eval\_dtm\_ds\_constr}$$

$$\frac{\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} t'_1 \\ t_2 \longrightarrow_{\mathrm{p}} t'_2 \\ t \longrightarrow_{\mathrm{p}} t' \end{array}}{t_1 \cong t_2 \rhd t \longrightarrow_{\mathrm{p}} t'_1 \cong t'_2 \rhd t'}\ \textsc{par\_eval\_dtm\_guard} \qquad \frac{\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} v \\ t_2 \longrightarrow_{\mathrm{p}} v \\ t \longrightarrow_{\mathrm{p}} t' \end{array}}{v \cong v \rhd t \longrightarrow_{\mathrm{p}} t}\ \textsc{par\_eval\_dtm\_guard\_refl}$$

$$\frac{}{\mathsf{DS}_{\mathsf{Unit}}^{\mathsf{Unit}}\mathsf{unit} \longrightarrow_{\mathrm{p}} \mathsf{unit}}\ \textsc{par\_eval\_dtm\_ds\_unit}$$

Figure 27: Parallel reduction (dependent terms, cont.)

$\boxed{S \longrightarrow_{\mathrm{p}} S'}$

$$\frac{}{S \longrightarrow_{\mathrm{p}} S}\text{PAR\_EVAL\_STY\_REFL}$$

$$\frac{\begin{array}{c} S_1 \longrightarrow_{\mathrm{p}} S_1' \\ S_2 \longrightarrow_{\mathrm{p}} S_2' \end{array}}{S_1 \to S_2 \longrightarrow_{\mathrm{p}} S_1' \to S_2'}\text{PAR\_EVAL\_STY\_ARR}$$

$$\frac{\begin{array}{c} S_1 \longrightarrow_{\mathrm{p}} S_1' \\ S_2 \longrightarrow_{\mathrm{p}} S_2' \end{array}}{S_1 * S_2 \longrightarrow_{\mathrm{p}} S_1' * S_2'}\text{PAR\_EVAL\_STY\_PAIR}$$

$\boxed{T \longrightarrow_{\mathrm{p}} T'}$

$$\frac{}{T \longrightarrow_{\mathrm{p}} T}\text{PAR\_EVAL\_DTY\_REFL}$$

$$\frac{\begin{array}{c} T_1 \longrightarrow_{\mathrm{p}} T_1' \\ T_2 \longrightarrow_{\mathrm{p}} T_2' \end{array}}{(y : T_1) \to T_2 \longrightarrow_{\mathrm{p}} (y : T_1') \to T_2'}\text{PAR\_EVAL\_DTY\_ARR}$$

$$\frac{\begin{array}{c} T \longrightarrow_{\mathrm{p}} T' \\ t \longrightarrow_{\mathrm{p}} t' \end{array}}{T\ t \longrightarrow_{\mathrm{p}} T'\ t'}\text{PAR\_EVAL\_DTY\_APP}$$

$$\frac{\begin{array}{c} T_1 \longrightarrow_{\mathrm{p}} T_1' \\ T_2 \longrightarrow_{\mathrm{p}} T_2' \end{array}}{(y : T_1) * T_2 \longrightarrow_{\mathrm{p}} (y : T_1') * T_2'}\text{PAR\_EVAL\_DTY\_PAIR}$$

Figure 28: Parallel reduction (simple and dependent types)

# B Proofs

## B.1 Structural Properties

**Lemma 9.** *Free variables in typing judgments*

1. *If* $\Gamma \vdash t : T$ *then* $\mathbf{fv}(t) \subseteq \mathbf{dom}(\Gamma)$ *and* $\mathbf{fv}(T) \subseteq \mathbf{dom}(\Gamma)$.
2. *If* $\Gamma \vdash T : K$ *then* $\mathbf{fv}(T) \subseteq \mathbf{dom}(\Gamma)$ *and* $\mathbf{fv}(K) \subseteq \mathbf{dom}(\Gamma)$.
3. *If* $\Gamma \vdash K$ *then* $\mathbf{fv}(K) \subseteq \mathbf{dom}(\Gamma)$.
4. *If* $\Gamma \vdash s : S$ *then* $\mathbf{fv}(s) \subseteq \mathbf{dom}(\Gamma)$ *and* $\mathbf{fv}(S) \subseteq \mathbf{dom}(\Gamma)$.

**Lemma 10.** *Weakening for Equivalence*

1. *If* $\Gamma_1, \Gamma_3 \vdash t \cong t'$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash t \cong t'$.
2. *If* $\Gamma_1, \Gamma_3 \vdash T \equiv T'$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash T \equiv T'$.
3. *If* $\Gamma_1, \Gamma_3 \vdash K \equiv K'$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash K \equiv K'$

*Proof.* Proof by mutual induction on the typing derivations. ☐

**Lemma 11.** *Weakening*

1. *If* $\Gamma_1, \Gamma_3 \vdash t : T$ *and* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash t : T$.
2. *If* $\Gamma_1, \Gamma_3 \vdash s : S$ *and* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash s : S$.
3. *If* $\Gamma_1, \Gamma_3 \vdash T : K$ *and* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ *then* $\Gamma_1, \Gamma_3, \Gamma_3 \vdash T : K$.
4. *If* $\Gamma_1, \Gamma_3 \vdash K$ *and* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ *then* $\Gamma_1, \Gamma_3, \Gamma_3 \vdash K$.
5. *If* $\vdash \Gamma_1, \Gamma_2$ *then* $\vdash \Gamma_1$

*Proof.* Proof by mutual induction on the typing derivations. ☐

**Lemma 12** (Values are closed under substitution)**.** *For any values* $v_1$, $v_2$, $u_1$, $u_2$, *the substituted terms* $[v_1/y]v_2$, $[u_1/x]v_2$, $[v_1/y]u_2$, *and* $[u_1/x]u_2$ *are also values.*

*Proof.* Simple induction on the structure of values. ☐

**Lemma 13** (Substitution of $u$s for Equivalence)**.**

1. $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash T \equiv T'$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]T \equiv [u_2/x_2]T'$.
2. $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash K \equiv K'$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]K \equiv [u_2/x_2]K'$.
3. $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash t \cong t'$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]t \cong [u_2/x_2]t'$.

*Proof.* By mutual induction on the three derivations. The cases for $\Gamma, x_2{:}S_2, \Gamma_2 \vdash T \equiv T'$ are:

**Case eq_dty_incon:** The rule looks like

$$\frac{\begin{array}{l}\Gamma \vdash C\ v \cong C'\ v' \\ C \neq C'\end{array}}{\Gamma \vdash T \equiv T'}\ \textsc{eq\_dty\_incon}$$

By the mutual IH we get $\Gamma_2, [u_2/x]\Gamma_2 \vdash [u_2/x](C\ v) \cong [u_2/x](C'\ v')$. Since $[u_2/x](C\ v) = C\,[u_2/x]v$ this is still a contradiction and we can re-apply $\textsc{eq\_dty\_incon}$.

**Case eq_dty_urefl:** The rule looks like

$$\frac{}{\Gamma \vdash \mathsf{Unit} \equiv \mathsf{Unit}}\text{EQ\_DTY\_UREFL}$$

We must show $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x]\mathsf{Unit} \equiv [u_2/x]\mathsf{Unit}$. But $[u_2/x]\mathsf{Unit} = \mathsf{Unit}$, so we can just apply EQ_DTY_UREFL again.

**Case eq_dty_trefl:** Similar to the case for EQ_DTY_UREFL.

**Case eq_dty_pi:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash T_1 \equiv T_1' \\ \Gamma \vdash T_2 \equiv T_2'\end{array}}{\Gamma \vdash (y : T_1) \to T_2 \equiv (y : T_1') \to T_2'}\text{EQ\_DTY\_PI}$$

We must show $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2](y : T_1) \to T_2 \equiv [u_2/x_2](y : T_1') \to T_2$. Since $y$ is a bound variable we can pick it fresh, so this is the same as showing $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash (y : [u_2/x_2]\, T_1) \to [u_2/x_2]\, T_2 \equiv (y : [u_2/x_2]\, T_1') \to [u_2/x_2]\, T_2'$. By the IH we get $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]\, T_1 \equiv [u_2/x_2]\, T_1'$ and $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]\, T_2 \equiv [u_2/x_2]\, T_2'$, then re-apply EQ_DTY_PI.

**Case eq_dty_sigma:** Similar to EQ_DTY_PI.

**Case eq_dty_app:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash T \equiv T' \\ \Gamma \vdash t \cong t'\end{array}}{\Gamma \vdash T\, t \equiv T'\, t'}\text{EQ\_DTY\_APP}.$$

We get $\Gamma_1, [u_2/x]\Gamma_2 \vdash [u_2/x]\, T \equiv [u_2/x]\, T'$ by IH, and $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]t \cong [u_2/x_2]t'$ by the mutual IH. Then we can re-apply EQ_DTY_APP.

The cases for $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash K \equiv K'$ are:

**Case eq_dkn_refl:** Similar to EQ_DTY_UREFL.

**Case eq_dkn_pi:** Similar to EQ_DTY_PI.

The cases for $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash t \cong t'$ are:

**Case eq_dtm_assumption:** We have $t \cong t' \in \Gamma_1, x_2{:}S_2, \Gamma_2$ as a premise to the rule. There are two cases: either $t \cong t' \in \Gamma_1$ or $t \cong t' \in \Gamma_2$. If $t \cong t' \in \Gamma_1$, then we also have $t \cong t' \in \Gamma_1, [u_2/x_2]\Gamma_2$. So the conclusion follows by EQ_DTM_ASSUMPTION followed by EQ_DTM_SSUBST. If $t \cong t' \in \Gamma_2$ then $[u_2/x_2]t \cong [u_2/x_2]t' \in \Gamma_1, [u_2/x_2]\Gamma_2$, so the conclusion follows by just EQ_DTM_ASSUMPTION.

**Case eq_dtm_step:** By EQ_DTM_STEP followed by EQ_DTM_SSUBST.

**Case eq_dtm_refl, eq_dtm_sym, eq_dtm_trans:** These all go directly by IH.

**Case eq_dtm_subst:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash t_1 \cong t_1' \\ y \notin \mathbf{dom}\,(\Gamma)\end{array}}{\Gamma \vdash [t_1/y]t \cong [t_1'/y]t}\text{EQ\_DTM\_SUBST}$$

By the IH (instantiated with $u_2$) we get $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]t_1 \cong [u_2/x_2]t_1'$.

So by EQ_DTM_SUBST, taking $[u_2/x]t$ as the template, we get

$$\Gamma_1, [u_2/x]\Gamma_2 \vdash [[u_2/x_2]t_1/y][u_2/x_2]t \cong [[u_2/x_2]t_1'/y][u_2/x_2]t.$$

Now by the assumption $\Gamma \vdash u_2 : S_2$, lemma 9 and the premise $y \notin \mathbf{dom}\,(\Gamma_1, x_2{:}S_2, \Gamma_2)$ we know that $y$ is not free in $u_2$. So we can commute the substitution to get

$$\Gamma_1, [u_2/x]\Gamma_2 \vdash [u_2/x_2][t_1/y]t \cong [u_2/x_2][t_1'/y]t,$$

which is what we needed to show.

**Case eq_dtm_subst_val:** The typing rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ y \notin \mathbf{dom}\,(\Gamma)\end{array}}{\Gamma \vdash [v/y]t \cong [v/y]t'}\textsc{eq\_dtm\_subst\_val}$$

By the IH we have $\Gamma, [u_2/x]\Gamma_2 \vdash [u_2/x]t \cong [u_2/x]t$. Values are closed under substitution of values (lemma 12), so $[u_2/x]v$ is a value. Applying EQ_DTM_SUBST_VAL we get

$$\Gamma, [u_2/x]\Gamma_2 \vdash [[u_2/x_2]v/y][u_2/x_2]t \cong [[u_2/x_2]v/y][u_2/x_2]t'$$

By the assumption $\Gamma \vdash u_2 : S_2$, lemma 9, and the premise $y \notin \mathbf{dom}\,(\Gamma_1, x_2{:}S_2, \Gamma_2)$ we know that $y$ is not free in $u_2$, so we can commute the substitutions and get

$$\Gamma_1, [u_2/x]\Gamma_2 \vdash [u_2/x_2][v/y]t \cong [u_2/x_2][v/y]t'$$

which is what we needed to show.

**Case eq_dtm_ssubst_val:** Similar to the previous case.

□

**Lemma 14** (Substitution of $v$s for Equivalence)**.**

1. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash T \equiv T'$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]T \equiv [v_2/y_2]T'$.
2. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash K \equiv K'$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]K \equiv [v_2/y_2]K'$.
3. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash t \cong t'$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]t \cong [v_2/y_2]t'$.

*Proof.* Similar to the previous lemma. □

**Lemma 15** (Substitution of $u$s)**.**

1. *If* $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash t_1 : T_1$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]t_1 : [u_2/x_2]T_1$.
2. *If* $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash s_1 : S_1$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]s_1 : [u_2/x_2]S_1$.
3. *If* $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash T : K$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]T : [u_2/x_2]K$.
4. *If* $\Gamma_1, x_2{:}S_2, \Gamma_2 \vdash K$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\Gamma_1, [u_2/x_2]\Gamma_2 \vdash [u_2/x_2]K$.
5. *If* $\vdash \Gamma_1, x_2{:}S_2, \Gamma_2$ *and* $\Gamma_1 \vdash u_2 : S_2$ *then* $\vdash \Gamma_1, [u_2/x_2]\Gamma_2$.

*Proof.* Mutual induction on all the judgments. In the VAR cases we splice in the provided typing derivation. In the WD_DTM_CONV case we appeal to lemma 13. □

**Lemma 16** (Substitution of $v$s)**.**

1. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash t_1 : T_1$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]t_1 : [v_2/y_2]T_1$.
2. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash s_1 : S_1$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]s_1 : [v_2/y_2]S_1$.

3. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash T : K$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]\,T : [v_2/y_2]K$.

4. *If* $\Gamma_1, y_2{:}T_2, \Gamma_2 \vdash K$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\Gamma_1, [v_2/y_2]\Gamma_2 \vdash [v_2/y_2]K$.

5. *If* $\vdash \Gamma_1, y_2{:}T_2, \Gamma_2$ *and* $\Gamma_1 \vdash v_2 : T_2$ *then* $\vdash \Gamma_1, [v_2/y_2]\Gamma_2$.

*Proof.* Similar to the previous lemma. □

**Lemma 17** (Equivalence Cut)**.** *Suppose* $\Gamma \vdash t_1 \cong t_2$. *Then:*

1. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash t \cong t'$, *then* $\Gamma, \Gamma' \vdash t \cong t'$.

2. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash T \equiv T'$, *then* $\Gamma, \Gamma' \vdash T \equiv T'$.

3. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash K \equiv K'$, *then* $\Gamma, \Gamma' \vdash K \equiv K'$.

*Proof.* Mutual induction on the derivations. The only case that doesn't go directly by the IH is EQ_DTM-_ASSUMPTION, where we splice in the provided derivation. □

**Lemma 18** (Typing Cut)**.** *Suppose* $\Gamma \vdash t_1 \cong t_2$. *Then:*

1. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash t : T$, *then* $\Gamma, \Gamma' \vdash t : T$.

2. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash s : S$, *then* $\Gamma, \Gamma' \vdash s : S$.

3. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash T : K$, *then* $\Gamma, \Gamma' \vdash T : K$.

4. *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash K$, *then* $\Gamma, \Gamma' \vdash K$.

5. *If* $\vdash \Gamma, t_1 \cong t_2, \Gamma'$, *then* $\vdash \Gamma, \Gamma'$.

*Proof.* Mutual induction on the derivations. The only case that doesn't go directly by IH is WF_SIG_CONV, where we appeal to lemmma 17. □

**Lemma 19** (Substitution through types)**.** *If* $\Gamma \vdash T_1 : K$ *and* $\Gamma \vdash t_2 \cong t_2'$ *and* $y_2 \notin \mathbf{dom}(\Gamma)$, *then* $\Gamma \vdash [t_2/y_2]\,T_1 \equiv [t_2'/y_2]\,T_1$.

*Proof.* Induction on the derivation of $\Gamma, y{:}T_2 \vdash T_1 : K$. The cases are

**Case wf_dty_arr:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash T_1 : * \\ \Gamma, y{:}T_1 \vdash T_2 : *\end{array}}{\Gamma \vdash (y : T_1) \to T_2 : *}\text{WF\_DTY\_ARR}$$

Since $y$ is a bound variable we can pick it to be different from $y_2$ and push the substitution in, so we must show $\Gamma \vdash (y : [t_2/y_2]\,T_1) \to [t_2/y_2]\,T_2 \equiv (y : [t_2'/y_2]\,T_1) \to [t_2'/y]\,T_2$. By EQ_DTY_PI it suffices to show $\Gamma_1, y : T_2 \vdash [t_2/y_2]\,T_1 \equiv [t_2'/y_2]\,T_1$ and $\Gamma_1 \vdash [t_2/y_2]\,T_2 \equiv [t_2'/y_2]\,T_2$, both of which follow by IH (noting again that $y_2 \neq y$).

**Case wf_dty_pair:** Similar to the previous case.

**Case wf_dty_data:** We must show $\Gamma \vdash [t_2/y_2]B \equiv [t_2'/y_2]B$, that is to say, $\Gamma \vdash B \equiv B$. This follows by EQ_DTY_TREFL.

**Case wf_dty_unit:** Similar to the previous case.

**Case wf_dty_app:** The rule looks like

$$\frac{\begin{array}{l}\Gamma \vdash T : T_1 \Rightarrow * \\ \Gamma \vdash t : T_1\end{array}}{\Gamma \vdash T\,t : *}\ \textsc{wf\_dty\_app}$$

By distributing the substitution and using EQ_DTY_APP, it suffices to show $\Gamma \vdash [t_2/y_2]T \equiv [t_2'/y_2]T$ (which is direct by IH) and $\Gamma \vdash [t_2/y_2]t \cong [t_2'/y_2]t$ (which is by EQ_DTM_SUBST).

□

**Lemma 20** (Context conversion (types))**.** *Suppose* $\Gamma \vdash T_1 \equiv T_2$ *and* $\Gamma \vdash T_2 : *$. *Then:*

1. *If* $\Gamma, y_1{:}T_1, \Gamma' \vdash t : T$, *then* $\Gamma, y_1{:}T_2, \Gamma' \vdash t : T$.
2. *If* $\Gamma, y_1{:}T_1, \Gamma' \vdash s : S$, *then* $\Gamma, y_1{:}T_2, \Gamma' \vdash s : S$.
3. *If* $\Gamma, y_1{:}T_1, \Gamma' \vdash T : K$, *then* $\Gamma, y_1{:}T_2, \Gamma' \vdash T : K$.
4. *If* $\Gamma, y_1{:}T_1, \Gamma' \vdash K$, *then* $\Gamma, y_1{:}T_2, \Gamma' \vdash K$.
5. *If* $\vdash \Gamma, y_1{:}T_1, \Gamma'$, *then* $\vdash \Gamma, y_1{:}T_2, \Gamma'$.

*Proof.* Mutual induction on all the judgments. The only case that doesn't go immediately by IH is WF_DTM_VAR, where we splice in a use of WF_DTM_CONV. □

**Lemma 21** (Context conversion (equations))**.** *If* $\Gamma, t_1 \cong t_2, \Gamma' \vdash J$ *and* $\Gamma \vdash t_1 \cong t_1'$ *and* $\Gamma \vdash t_2 \cong t_2'$, *then* $\Gamma, t_1' \cong t_2', \Gamma' \vdash J$.

*Proof.* The proof can be carried out completely generically for all the judgment forms, using the Weakening and Cut properties.

By weakening (lemmas 10 and 11) on the second two assumptions we have $\Gamma, t_1' \cong t_2' \vdash t_1 \cong t_1'$ and $\Gamma, t_1' \cong t_2' \vdash t_2 \cong t_2'$. So by EQ_DTM_SYM and EQ_DTM_TRANS we know $\Gamma, t_1' \cong t_2' \vdash t_1 \cong t_2$.

By weakening on the first assumption, we have $\Gamma, t_1' \cong t_2', t_1 \cong t_2, \Gamma' \vdash J$. But then by Cut (lemmas 17 and 18) we have $\Gamma, t_1' \cong t_2', \Gamma' \vdash J$ as we claimed. □

## B.2 Preservation

**Lemma 22** (Regularity)**.**

1. *If* $\vdash \Gamma$ *and* $y{:}T \in \Gamma$ *then* $\Gamma \vdash T : *$.
2. *If* $\Gamma \vdash s : S$ *then* $\vdash \Gamma$.
3. *If* $\Gamma \vdash t : T$ *then* $\Gamma \vdash T : *$ *and* $\vdash \Gamma$.

*Proof.* Claim (1) is by induction on $\vdash \Gamma$, using weakening (lemma 11). Claim (2) is an easy induction on $\Gamma \vdash s : S$ (using inversion on $\vdash \Gamma$ in the cases that extend the context.) Claim (3) is by induction on $\Gamma \vdash t : T$. In the WF_DTM_APP, WF_DTM_PROJ2 and WF_DTM_CTOR we have the needed well-formedness directly as a premise to the rule. □

**Lemma 23** (Type equivalence inversion)**.**

1. *If* $\Gamma \vdash (y : T_1) \to T_2 \equiv (y : T_1') \to T_2'$, *then* $\Gamma \vdash T_1 \equiv T_1'$ *and* $\Gamma \vdash T_2 \equiv T_2'$.
2. *If* $\Gamma \vdash (y : T_1) * T_2 \equiv (y : T_1') * T_2'$, *then* $\Gamma \vdash T_1 \equiv T_1'$ *and* $\Gamma \vdash T_2 \equiv T_2'$.
3. *If* $\Gamma \vdash T\,t \equiv T'\,t'$, *then* $\Gamma \vdash T \equiv T'$ *and* $\Gamma \vdash t \cong t'$.

*Proof.* By inversion on the judgment. The three claims are similar, so we only show (1).

For the claim (1), two rules can match the conclusion, namely EQ_DTY_PI and EQ_DTY_INCON. If the derivation ended with EQ_DTY_PI we have $\Gamma \vdash T_1 \equiv T_1'$ and $\Gamma \vdash T_2 \equiv T_2'$ as premises to the rule. If it ended with EQ_DTY_INCON we have an inconsistent premise $\Gamma \vdash C\, v \cong C'\, v'$, so we can derive $\Gamma \vdash T_1 \equiv T_1'$ and $\Gamma \vdash T_2 \equiv T_2'$ by applying EQ_DTY_INCON. □

**Lemma 24** (Type equivalence is an equivalence relation)**.**

1. *For any $T$, we have $\Gamma \vdash T \equiv T$.*
2. *If $\Gamma \vdash T \equiv T'$ then $\Gamma \vdash T' \equiv T$.*
3. *If $\Gamma \vdash T \equiv T'$ and $\Gamma \vdash T' \equiv T''$, then $\Gamma \vdash T \equiv T''$.*

*Proof.* Claim (1) is by induction on the structure of $T$. For each syntactic form (arrow, type application, pair, Unit, and $B$) there is a corresponding equivalence rule (EQ_DTY_PI, EQ_DTY_APP, EQ_DTY_SIGMA, EQ_DTY_UREFL, EQ_DTY_TREFL). In the type application case we use EQ_DTM_REFL.

Claim (2) is an easy induction on $\Gamma \vdash T \equiv T'$.

Claim (3) is by a double induction on $\Gamma \vdash T \equiv T'$ and $\Gamma \vdash T' \equiv T''$. The cases for $\Gamma \vdash T \equiv T'$ and $\Gamma \vdash T' \equiv T''$ are:

**eq_dty_incon and anything, anything and eq_dty_incon:** Here we can directly show $\Gamma \vdash T \equiv T''$ using the inconsistent equality premise.

**Both derivations are eq_dty_urefl** : Then $T = T' = T'' = \mathsf{Unit}$.

**Both derivations are eq_dty_trefl** : Similar.

**Both derivations are eq_dty_pi:** We apply the IH to the sub-derivations.

**Both derivations are eq_dty_sigma:** Similar.

**Both derivations are eq_dty_app:** As premises to the two rules we have

1. $\Gamma \vdash T \equiv T'$
2. $\Gamma \vdash t \cong t'$
3. $\Gamma \vdash T' \equiv T''$
4. $\Gamma \vdash t' \cong t''$.

From the IH applied to (1) and (3) we get $\Gamma \vdash T \equiv T''$. From EQ_DTM_TRANS applied to (2) and (4) we get $\Gamma \vdash t \cong t'$. Then apply EQ_DTY_APP again.

**Other combinations:** Cannot happen, since the top-level structure of the "middle" term would not match up.

□

**Lemma 25** (Kinding inversion)**.**

1. *If $\Gamma \vdash B\, t : *$ and $B{:}T_0 \Rightarrow * \in \Psi_0$, then $\Gamma \vdash t : T_0$.*

*Proof.* Directly by inversion on the judgment we see that the derivation must have been by WF_DTY_APP and WF_DTY_DATA. So we must have $B{:}T \Rightarrow * \in \Psi_0$ and $\Gamma \vdash t : T$. Since there are no duplicate declarations in $\Psi_0$, $T$ must be $T_0$. □

**Lemma 26** (Typing inversion)**.**

1. *If $\Gamma \vdash (\lambda y{:}\, T_1.t) : T'$, then $\Gamma, y{:}T_1 \vdash t : T_2$, and $\Gamma \vdash (y : T_1) \to T_2 \equiv T'$.*

2. *If* $\Gamma \vdash <t_1, t_2>: T'$*, then* $\Gamma \vdash t_1 : T_1$ *and* $\Gamma \vdash t_2 : [t_1/y]\, T_2'$ *and* $\Gamma \vdash (y : T_1) * T_2 \equiv T'$*.*

3. *If* $\Gamma \vdash C\ t : T'$*, then* $C{:}(y : T_1) \to B\ t_1 \in \Psi_0$ *and and* $\Gamma \vdash t : T_1$ *and* $\Gamma \vdash B\,[t/y]t_1 \equiv T'$*.*

*Proof.* Induction on the typing $\Gamma \vdash t : T'$. The cases are:

**Cases wf_dtm_var, app, proj1, proj2, case, ds, guard, error:** In these rules the shape of the term does not match the ones mentioned in the lemma.

**Case wf_dtm_abs:** The typing rule looks like

$$\frac{\Gamma, y{:}T_1 \vdash t : T_2}{\Gamma \vdash \lambda y{:}\,T_1.t : (y : T_1) \to T_2}\text{WF\_DTM\_ABS}$$

So we get $\Gamma, y{:}T_1 \vdash t : T_2$ as an assumption to the rule. By reflexivity (lemma 24) we have $\Gamma \vdash (y : T_1) \to T_2 \equiv (y : T_1) \to T_2$ as required.

**Case wf_dtm_pair, wf_dtm_unit:** Similar to the ABS case.

**Case wf_dtm_ctor:** The typing rule looks like

$$\frac{\begin{array}{l} C{:}(y : T_1) \to B\ t' \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \Gamma \vdash t : T_1 \\ \Gamma \vdash B\,[t/y]t' : * \end{array}}{\Gamma \vdash C\ t : B\,[t/y]t'}\text{WF\_DTM\_CTOR}$$

We have $C{:}(y : T_1) \to B \in \Psi_0$ as a premise, and $\Gamma \vdash [t/y]t' \cong [t/y]t$ by EQ_DTM_REFL.

**Case wf_dtm_conv:** The typing rule looks like

$$\frac{\begin{array}{l} \Gamma \vdash t : T \\ \Gamma \vdash T \equiv T' \\ \Gamma \vdash T' : * \end{array}}{\Gamma \vdash t : T'}\text{WF\_DTM\_CONV}$$

For (1) we get $\Gamma, y{:}T_1 \vdash t : T_2$ and $\Gamma \vdash (y : T_1) \to T_2 \equiv T$ by the IH, so by transitivity (lemma 24) we have $\Gamma \vdash (y : T_1) \to T_2 \equiv T'$ as required. (2) and (3) are similar.

□

**Lemma 27** ($\Leftrightarrow$ ignores terms in types)**.** *$S \Leftrightarrow T$ iff $S \Leftrightarrow [t/y]\,T$.*

*Proof.* We show the left-to-right direction only, as the reverse direction is similar. We proceed by induction on $S \Leftrightarrow T$. The cases are:

**Case compat_arr:** The case looks like

$$\frac{\begin{array}{c} S_1 \Leftrightarrow T_1 \\ S_1 \Leftrightarrow T_1 \end{array}}{S_1 \to S_2 \Leftrightarrow (y' : T_1) \to T_2}\ \text{COMPAT\_ARR}$$

Since $y'$ is a bound variable we can pick it fresh, so that $[t/y]((y' : T_1) \to T_2) = (y : [t/y]\,T_1) \to [t/y]\,T_2$. Then we get $S_1 \Leftrightarrow [t/y]\,T_1$ and $S_2 \Leftrightarrow [t/y]\,T_2$ by IH. So re-applying the rule we get $S_1 \to S_2 \Leftrightarrow (y' : [t/y]\,T_1) \to [t/y]\,T_2$ as required.

**Case compat_pair, compat_unit:** Similar to COMPAT_ARR.

**Case compat_data:** The case looks like

$$\frac{\begin{array}{l} B{:}T_0 \Rightarrow * \in \Psi_0 \\ \mathsf{FO}\,(T_0) \\ \mathsf{corr}\,(A, B) \end{array}}{A \Leftrightarrow B\ t}\ \text{COMPAT\_DATA}$$

The substitution doesn't affect the premises of the rule.

□

**Lemma 28** (Convenient derivable typing rules)**.** *The following rules are derivable:*

$$\frac{\begin{array}{l} \Gamma \vdash t_1 : (y : T_1) \to T_2 \\ \Gamma \vdash v_2 : T_1 \end{array}}{\Gamma \vdash t_1\ v_1 : [v_1/y]\,T_2}\ \text{WF\_DTM\_APPVAL} \qquad \frac{\begin{array}{l} \Gamma, y{:}T_1 \vdash t_2 : T_2 \\ \Gamma \vdash t_1 : T_1 \\ \Gamma \vdash T_2 : * \end{array}}{\Gamma \vdash \mathsf{let}\ y = t_1\ \mathsf{in}\ t_2 : T_2}\ \text{WF\_DTM\_LET}$$

*Proof.* WF_DTM_APPVAL: By regularity (lemma 22) we have $\Gamma \vdash (y : T_1) \to T_2 : *$. By inversion on the kinding relation, that means that $\Gamma, y{:}T_1 \vdash T_2 : *$. So by substitution (lemma 16) we get $\Gamma \vdash [v_2/y]\,T_2 : *$. Then we conclude by WF_DTM_APP.

WF_DTM_LET: Expanding the syntactic sugar, what we need to show is

$$\Gamma \vdash (\lambda y{:}\, T_1.t_2)\ t_1 : T_2$$

By WD_DTM_ABS we know $\Gamma \vdash (\lambda y{:}\, T_1.t_2) : (y : T_1) \to T_2$. From the premise $\Gamma \vdash T_2 : *$ and lemma 9 we know that $y \notin \mathbf{fv}\,(T_2)$, so $[t_2/y]\,T_2 = T_2$. So the same kinding premise also tells us $\Gamma \vdash [t_2/y]\,T_2 : *$. Then the application is well-formed, so $\Gamma \vdash (\lambda y{:}\, T_1.t_2)\ t_1 : [t_1/y]\,T_2$, which is syntactically equal to the type we want. □

**Lemma 29.** *If $B{:}T_0 \Rightarrow * \in \Psi_0$ and $\mathsf{constrs}\,B = \overline{C_i}^{\,i}$ and $C_j{:}(y_j : T_j) \to B\ t_j \in \Psi_0$, then $\cdot, y_j{:}T_j \vdash t_j : T_0$.*

*Proof.* By inversion on the judgment $\vdash \Psi_0$ we get $\cdot \vdash (y_j : T_j) \to B\ t_j : *$. Then use inversion on the kinding judgment. □

**Property 6.** *If $C{:}S \to A \in \Psi_0$ and $C{:}(y : T_1) \to B\ t_1 \in \Psi_0$ and $\Gamma \vdash u : S$, then $\Gamma \vdash \mathsf{argToD}_C\, u : T_1$ (if it is defined).*

**Property 7.** *If $C{:}S \to A \in \Psi_0$ and $C{:}(y : T_1) \to B\ t_1 \in \Psi_0$ and $\Gamma \vdash v : T_1$, then $\Gamma \vdash \mathsf{argToS}_C\, v : S$ (if it is defined).*

**Theorem 3** (Generalized Preservation)**.** *Let $y_0$ be fresh for $\Gamma$, $T$, $S$, $K$. Then*

1. *If $\Gamma \vdash s : S$ and $s \longrightarrow s'$ then $\Gamma \vdash s : S$.*
2. *If $\Gamma \vdash [t_0/y_0]t : T$ and $t_0 \longrightarrow t_0'$ then $\Gamma \vdash [t_0'/y_0]t : T$.*
3. *If $\Gamma \vdash [t_0/y_0]s : S$ and $t_0 \longrightarrow t_0'$ then $\Gamma \vdash [t_0'/y_0]s : S$.*
4. *If $\Gamma \vdash [t_0/y_0]\,T : K$ and $t_0 \longrightarrow t_0'$ , then $\Gamma \vdash [t_0'/y_0]\,T : K$.*

*Proof.* We proceed by mutual induction on the judgments $\Gamma \vdash s : S$, $\Gamma \vdash [t_0/y_0]s : S$, $\Gamma \vdash [t_0/y_0]t : T$, and $\Gamma \vdash [t_0/y_0]\,T : K$

The cases for $\Gamma \vdash s : S$ are mostly routine, but we show the two cases which involve novel language features, namely SD-boundaries and letd-expressions.

**Case sd_stm_sd:** The rule looks like

$$\frac{\begin{array}{l}\Gamma \vdash t : T \\ S \Leftrightarrow T\end{array}}{\Gamma \vdash \mathsf{SD}^{S}_{T} t : S}\textsc{wf\_stm\_sd}$$

The expression reduces either by the context $\mathsf{SD}^{S}_{T}\Box$ or by one of the $\mathsf{DS}$ stepping rules.

- Suppose it was by congruence so $t \longrightarrow t'$. Then by IH $\Gamma \vdash t' : T$ and thus $\Gamma \vdash \mathsf{SD}^{S}_{T} t' : S$.
- Suppose it was by EVAL_STM_SD_ABS. So the step looks like

$$\frac{}{\mathsf{SD}^{(S_1 \to S_2)}_{((y{:}T_1) \to T_2)} \lambda y{:}\, T'_1.t \longrightarrow \lambda x{:}\, S_1.\mathsf{letd}\ y' = \mathsf{DS}^{T_1}_{S_1} x \ \mathsf{in}\ \mathsf{SD}^{S_2}_{([y'/y]\,T_2)} ((\lambda y{:}\, T'_1.t)\ y')}\textsc{eval\_stm\_sd\_abs}$$

  The variable $y'$ is bound, so we rename it to $y$ to reduce clutter. We must show $\Gamma \vdash \lambda x{:}\, S_1.\mathsf{letd}\ y = \mathsf{DS}^{T_1}_{S_1} x \ \mathsf{in}\ \mathsf{SD}^{S_2}_{T_2} ((\lambda y{:}\, T'_1.t)\ y) : S_1 \to S_2$, while we have the fact $\Gamma \vdash (\lambda y{:}\, T'_1.t) : (y : T_1) \to T_2$ available as a premise to the typing rule. By inversion on the judgment $S_1 \to S_2 \Leftrightarrow (y : T_1) \to T_2$ we get $S_1 \Leftrightarrow T_1$ and $S_2 \Leftrightarrow T_2$.
  By Weakening (lemma 11) and WF_DTM_APPVAL (lemma 28) we get $\Gamma, x{:}S_1, y{:}T_1 \vdash (\lambda y{:}\, T'_1.t)\ y : T_2$. So by WF_STM_SD we have

$$\Gamma, x{:}S_1, y{:}T_1 \vdash \mathsf{SD}^{S_2}_{T_2} ((\lambda y{:}\, T'_1.t)\ y) : S_2.$$

  Also, by WF_DTM_DS we immediately get immediately get $\Gamma, x : S_1 \vdash \mathsf{DS}^{T_1}_{S_1} x : T_1$. So by WF_STM_LETD (lemma 28) we have

$$\Gamma, x{:}S_1 \vdash \mathsf{letd}\ y = \mathsf{DS}^{T_1}_{S_1} x \ \mathsf{in}\ \mathsf{SD}^{S_2}_{T_2} ((\lambda y{:}\, T'_1.t)\ y) : S_2.$$

  We conclude by applying WF_STM_ABS.
- Suppose it was by EVAL_STM_SD_PAIR. So the step looks like

$$\frac{}{\mathsf{SD}^{(S_1 * S_2)}_{((y{:}T_1) * T_2)} <v_1, v_2> \longrightarrow <\mathsf{SD}^{S_1}_{T_1} v_1, \mathsf{SD}^{S_2}_{([v_1/y]\,T_2)} v_2>}\textsc{eval\_stm\_sd\_pair}$$

  We must show $\Gamma \vdash <\mathsf{SD}^{S_1}_{T_1} v_1, \mathsf{SD}^{S_2}_{([v_1/y]\,T_2)} v_2> : S_1 * S_2$, while we have the fact $\Gamma \vdash <v_1, v_2> : (y : T_1) * T_2$ available as a premise to the rule. By inversion on the judgment $S_1 * S_2 \Leftrightarrow (y : T_1) * T_2$ we get $S_1 \Leftrightarrow T_1$ and $S_2 \Leftrightarrow T_2$. By inversion (lemma 26) on $\Gamma \vdash <v_1, v_2> : (y : T_1) * T_2$ we find $\Gamma \vdash v_1 : T_1$ and $\Gamma \vdash v_2 : [v_1/y]\,T_2$.
  We wish to apply WFT_STM_PAIR. We directly get the first premise, namely $\Gamma \vdash \mathsf{SD}^{S_1}_{T_1} v_1 : S_1$, by WFT_STM_SD. For the second premise we must show $\Gamma \vdash \mathsf{SD}^{S_2}_{([v_1/y]\,T_2)} v_2 : S_2$. By lemma 27 we have $S_2 \Leftrightarrow [v_1/y]\,T_2$, so this also follows from WFT_STM_SD.
- Suppose it was by EVAL_STM_SD_CONSTR. So the step looks like

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ \mathsf{argToS}_C\, v = u \end{array}}{\mathsf{SD}^{A}_{(B\ t)}\, C\ v \longrightarrow C\ u}\textsc{eval\_stm\_sd\_constr}$$

  We must show $\Gamma \vdash C\ u : A$, while we have the fact $\Gamma \vdash C\ v : B\ t$ available as a premise to the rule. By inversion (lemma 26) we get $\Gamma \vdash v : T_1$. So by property 7 we get that $\Gamma \vdash u : S$, and hence $\Gamma \vdash C\ u : A$ as required.

- Suppose it was by EVAL_STM_SD_UNIT. So the step looks like

$$\frac{}{\mathsf{SD}^{\mathsf{Unit}}_{\mathsf{Unit}}\mathsf{unit} \longrightarrow \mathsf{unit}}\ \text{EVAL\_STM\_SD\_UNIT}$$

We must show $\Gamma \vdash \mathsf{unit} : \mathsf{Unit}$, which is straightforwardly true.

**Case wf_stm_letd:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash t : T \\ \Gamma, y{:}T \vdash s : S\end{array}}{\Gamma \vdash \mathsf{letd}\, y = t \,\mathsf{in}\, s : S}\ \text{WF\_STM\_LETD}$$

We consider the ways the expression $\mathsf{letd}\, y = t \,\mathsf{in}\, s$ may step.

- By EVAL_STM_CTX. So $\mathcal{E}_s$ is $\mathsf{letd}\, y = \Box \,\mathsf{in}\, s$, we have $t \longrightarrow t'$, and the transition looks like $\mathsf{letd}\, y = t \,\mathsf{in}\, s \longrightarrow \mathsf{letd}\, y = t' \,\mathsf{in}\, s$. By mutual IH we know $\Gamma \vdash t' : T$, so we conclude by re-applying WF_STM_LETD.
- By EVAL_STM_LETD. So the transition is $\mathsf{letd}\, y = v \,\mathsf{in}\, s \longrightarrow [v/y]s$. By substitution (part (2) of lemma 16) we get $\Gamma \vdash [v/y]s : S$ as required.

The cases for $\Gamma \vdash [t_0/y_0]s : S$ are all immediate by IH except two, namely

**Case wf_stm_sd.** The situation looks like this:

$$\frac{\begin{array}{c}\Gamma \vdash [t_0/y_0]t : [t_0/y_0]T \\ S \Leftrightarrow [t_0/y_0]T\end{array}}{\Gamma \vdash \mathsf{SD}^{S}_{[t_0/y_0]T}[t_0/y_0]t : S}\ \text{WF\_STM\_SD}$$

(Notice that simple types never contain any term variables, so applying a substitution to $S$ does not do anything). By the mutual IH we get $\Gamma \vdash [t_0'/y_0]t : [t_0/y_0]T$. By applying lemma 27 twice we get $S \Leftrightarrow [t_0'/y_0]T$. So we can re-apply the rule to get $\Gamma \vdash \mathsf{SD}^{S}_{[t_0'/y_0]T}[t_0'/y_0]t : S$ as required.

**Case wf_stm_letd.** We can pick the variable bound by the letd-expression to be different from $y_0$. Then after pushing the substitution in the situation looks like this:

$$\frac{\begin{array}{c}\Gamma \vdash [t_0/y_0]t : T \\ \Gamma, y{:}T \vdash [t_0/y_0]s : S\end{array}}{\Gamma \vdash \mathsf{letd}\, y = [t_0/y_0]t \,\mathsf{in}\, [t_0/y_0]s : S}\ \text{WF\_STM\_LETD}$$

Now by the mutual IH we get $\Gamma \vdash [t_0'/y_0]t : T$, while by IH we get $\Gamma, y{:}T \vdash [t_0'/y]s : S$. So we re-apply the rule to get $\Gamma \vdash \mathsf{letd}\, y = [t_0'/y_0]t \,\mathsf{in}\, [t_0'/y_0]s : S$ as required.

The cases for $\Gamma \vdash [t_0/y_0]t : T$ are:

**Case wf_dtm_var.** Since variables do not step we must have $[t_0/y_0]t = [t_0'/y_0]t$ and the result is trivial.

**Case wf_dtm_abs.** So $[t_0/y_0]t$ is a $\lambda$-abstraction. Considering the possibilities for $t$, this means that either $t$ is $y_0$ or $t$ is a $\lambda$-abstraction. However, the former is impossible because abstractions don't step. We can pick the bound variable in $t$ to be distinct from $y_0$ and push the substitution in. Then the situation looks like this:

$$\frac{\Gamma, y{:}[t_0/y_0]T_1 \vdash [t_0/y_0]t : T_2}{\Gamma \vdash \lambda y{:}[t_0/y_0]T_1.[t_0/y_0]t : (y : [t_0/y_0]T_1) \to T_2}\ \text{WF\_DTM\_ABS}$$

So we need to prove $\Gamma \vdash \lambda y{:}[t_0'/y_0]T_1.[t_0'/y_0]t : (y : [t_0/y_0]T_1) \to T_2$.

By EQ_DTM_STEP we know $\Gamma \vdash t_0 \cong t_0'$, so by lemma 19 we have $\Gamma \vdash [t_0/y_0]T_1 \equiv [t_0'/y_0]T_1$. By the IH we know $\Gamma, y{:}[t_0/y_0]T_1 \vdash [t_0'/y_0]t : T_2$, so by context conversion (lemma 20) we have $\Gamma, y{:}[t_0'/y_0]T_1 \vdash [t_0'/y_0]t : T_2$. Now we re-apply WF_DTM_ABS to get $\Gamma \vdash \lambda y{:}[t_0'/y_0]T_1.[t_0'/y_0]t : (y : [t_0'/y_0]T_1) \to T_2$. Finally by WF_DTM_CONV this gives $\Gamma \vdash \lambda y{:}[t_0'/y_0]T_1.[t_0'/y_0]t : (y : [t_0/y_0]T_1) \to T_2$ as required.

**Case wf_dtm_app** . So $[t_0/y_0]t$ is an application, and $T$ is an arrow type. This can happen in two ways: either $t$ is $y_0$, or $t$ is an application.

If $t$ is an application, we can choose the bound variable in the arrow type to be different from $y_0$ and push the substitution in. Then the situation looks like this:

$$\frac{\begin{array}{l}\Gamma \vdash [t_0/y_0]t_1 : (y\colon T_1) \to T_2 \\ \Gamma \vdash [t_0/y_0]t_2 : T_1 \\ \Gamma \vdash [[t_0/y_0]t_2/y]\, T_2 : *\end{array}}{\Gamma \vdash [t_0/y_0]t_1\, [t_0/y_0]t_2 : [[t_0/y_0]t_2/y]\, T_2}\ \textsc{wf\_dtm\_app}$$

By the IH we get $\Gamma \vdash [t_0'/y_0]t_1 : (y : T_1) \to T_2$ and $\Gamma \vdash [t_0'/y_0]t_2 : T_1$. Since $y_0 \notin \mathbf{fv}\,(T_2)$, we have that $[[t_0/y_0]t_2/y]\, T_2 = [t_0/y_0][t_2/y]\, T_2$ and $[[t_0'/y_0]t_2/y]\, T_2 = [t_0'/y_0][t_2/y]\, T_2$. So by the mutual IH we get $\Gamma \vdash [[t_0'/y_0]t_2/y]\, T_2 : *$. So re-applying WD_DTM_APP we get

$$\Gamma \vdash [t_0'/y_0]t_1\, [t_0'/y_0]t_2 : [[t_0'/y_0]t_2/y]\, T_2.$$

Now by EQ_DTM_STEP we have $\Gamma \vdash t_0 \cong t_0'$, so (again noting that $[[t_0/y_0]t_2/y]\, T_2 = [t_0/y_0][t_2/y]\, T_2$) by lemma 19 we have $\Gamma \vdash [[t_0'/y_0]t_2/y]\, T_2 \equiv [[t_0/y_0]t_2/y]\, T_2$. So by WF_DTM_CONV we get

$$\Gamma \vdash [t_0'/y_0]t_1\, [t_0'/y_0]t_2 : [[t_0/y_0]t_2/y]\, T_2$$

as required.

On the other hand, suppose that the $t$ is $y_0$, that is $t_0$ is an application which steps and we need to show that its type is preserved. The situation looks like

$$\frac{\begin{array}{l}\Gamma \vdash t_1 : (y\colon T_1) \to T_2 \\ \Gamma \vdash t_2 : T_1 \\ \Gamma \vdash [t_2/y]\, T_2 : *\end{array}}{\Gamma \vdash t_1\ t_2 : [t_2/y]\, T_2}\ \textsc{wf\_dtm\_app}$$

We consider the ways $t_1\ t_2$ can step.

- By EVAL_DTM_CTX when $\mathcal{E}_t$ is $\square\, t_2$. So $t_1 \longrightarrow t_1'$. By the IH, $\Gamma \vdash t_1' : (y\colon T_1) \to T_2$ so $\Gamma \vdash t_1'\ t_2 : T_2$.
- By EVAL_DTM_CTX when $\mathcal{E}_t$ is $v\,\square$. So $t_2 \longrightarrow t_2'$. By the IH $\Gamma \vdash t_2' : T_1$, so by WF_DTM_APP we have that $\Gamma \vdash v\ t_2' : [t_2'/y]\, T_2$. So by lemma 19 and one use of WF_DTM_CONV we have that $\Gamma \vdash v\ t_2' : [t_2/y]\, T_2$ (taking advantage of the fact that $\Gamma \vdash t_2 \cong t_2'$ by EQ_DTM_STEP).
- By EVAL_DTM_BETA. So $t_1$ is $\lambda y\colon T_1'.t$ and $t_2$ is some value $v$, and the step is $(\lambda y\colon T_1'.t)\ v \longrightarrow [v/y]t$. By inversion (lemma 26) we know $\Gamma, y{:}T_1' \vdash t : T_2'$ for some $T_2'$ such that $\Gamma \vdash (y : T_1') \to T_2' \equiv (y : T_1) \to T_2$.

  By inversion on the type equality (lemma 23) we know $\Gamma \vdash T_1' \equiv T_1$ and $\Gamma \vdash T_2' \equiv T_2$. So by one application of WF_DTM_CONV we have $\Gamma \vdash v : T_1'$; by substitution (lemma 16) $\Gamma \vdash [v/y]t : T_2'$; and by a second application of WF_DTM_CONV we get $\Gamma \vdash [v/y]t : T_2$.

**Case wf_dtm_pair** So $[t_0/y_0]t$ is a pair. Considering the possibilities for $t$ this means that either $t$ is $y_0$ or $t$ is a pair. However, the former is impossible because pairs don't step. So pushing in the substitution, the situation looks like this:

$$\frac{\begin{array}{l}\Gamma \vdash [t_0/y_0]t_1 : T_1 \\ \Gamma \vdash [t_0/y_0]t_2 : [[t_0/y_0]t_1/y]\, T_2 \\ \Gamma \vdash (y\colon T_1) * T_2 : *\end{array}}{\Gamma \vdash <[t_0/y_0]t_1, [t_0/y_0]t_2> : (y\colon T_1) * T_2}\ \textsc{wf\_dtm\_pair}$$

Now by IH we have $\Gamma \vdash [t_0'/y_0]t_1 : T_1$ and $\Gamma \vdash [t_0'/y_0]t_2 : [[t_0/y_0]t_1/y]\, T_2$. Since $y_0 \notin \mathbf{fv}\,(T_2)$ we know $[[t_0/y_0]t_1/y]\, T_2 = [t_0/y_0][t_1/y]\, T_2$, so by EQ_DTM_STEP and lemma 19 we have $\Gamma \vdash [[t_0'/y_0]t_1/y]\, T_2 \equiv [[t_0/y_0]t_1/y]\, T_2$. So by WF_DTM_CONV (and regularity, lemma 22, to satisfy the kinding premise to conv) we get $\Gamma \vdash [t_0'/y_0]t_2 : [[t_0'/y_0]t_1/y]\, T_2$. Then we can re-apply WD_DTM_PAIR.

**Case wf_dtm_proj1** So $[t_0/y_0]t$ is a projection. This means that either $t$ is a projection, or $t$ is $y_0$.

In the first case, the situation looks like

$$\frac{\Gamma \vdash [t_0/y_0]t : (y\!:T_1) * T_2}{\Gamma \vdash [t_0/y_0]t.1 : T_1}\ \textsc{wf\_dtm\_proj1}$$

By the IH we get that $\Gamma \vdash [t'_0/y_0]t : (y\!:T_1) * T_2$, and we can just re-apply $\textsc{wf\_dtm\_proj1}$.

In the other case, $t_0$ itself is a projection that steps and we need to show preservation for it. We consider the ways it may step:

- By the evaluation context $\square.1$. This is immediate by IH, like the previous case.
- By $\textsc{eval\_dtm\_proj1}$. So the step is $<v_1, v_2> \longrightarrow v_1$. By inversion (lemma 26) and inversion on the equivalence judgment (lemma 23) we have $\Gamma \vdash v_1 : T'_1$ and $\Gamma \vdash T'_1 \equiv T_1$. So by regularity and $\textsc{wf\_dtm\_conv}$ we get $\Gamma \vdash v_1 : T_1$.

**Case wf_dtm_proj2** Similar to the previous case.

**Case wf_dtm_ctor** The typing rule looks like

$$\frac{\begin{array}{l} C{:}(y\!:T_1) \to B\ t' \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \Gamma \vdash t : T_1 \\ \Gamma \vdash B\ [t/y]t' : * \end{array}}{\Gamma \vdash C\ t : B\ [t/y]t'}\ \textsc{wf\_dtm\_ctor}$$

The reasoning is similar to the $\textsc{wf\_dtm\_app}$ case, but there are fewer cases to consider since there is no $\beta$-rule for constructor applications.

**Case wf_dtm_case** So $[t_0/y_0]t$ is a case-expression, which means that either $t$ is a case-expression or $t$ is $y_0$.

In the former case, we pick $y_i$ to be different from $y_0$ and push the substitution in, so the situation looks like

$$\frac{\begin{array}{l} \Gamma \vdash [t_0/y_0]t : B\ t' \\ \Gamma \vdash T : * \\ \mathsf{constrs}\ B = \ \overline{C_i}^{\,i} \\ \overline{C_i{:}(y_i\!:T_i) \to B\ t'_i \in \Psi_0}^{\,i} \\ \overline{\Gamma, y_i{:}T_i, t' \cong t'_i, [t_0/y_0]t \cong C_i\ y_i \vdash [t_0/y_0]t_i : T}^{\,i} \end{array}}{\Gamma \vdash \mathsf{case}\ [t_0/y_0]t\ \mathsf{of}\ \overline{C_i\ y_i \to [t_0/y_0]t_i}^{\,i} : T}\ \textsc{wf\_dtm\_case}$$

By the IH we get $\Gamma \vdash [t'_0/y_0]t : B\ t'$ and $\Gamma, y_i : T_i, t' \cong t'_i, [t_0/y_0]t \cong C_i\ y_i \vdash [t'_0/y_0]t_i : T$. By $\textsc{eq\_dtm\_step}$ and $\textsc{eq\_dtm\_subst}$ we get $\Gamma, y_i{:}T_i, t' \cong t'_i \vdash [t_0/y_0]t \cong [t'_0/y_0]t$. So by context conversion (lemma 21) we get $\Gamma, y_i : T_i, t' \cong t'_i, [t'_0/y_0]t \cong C_i\ y_i \vdash [t'_0/y_0]t_i : T$. We conclude by re-applying $\textsc{wf\_dtm\_case}$.

In the other case, $t_0$ itself is a case-expression which steps, and we must show that its type is preserved. The typing rule looks like

$$\frac{\begin{array}{l} \Gamma \vdash t : B\ t' \\ \Gamma \vdash T : * \\ \mathsf{constrs}\ B = \ \overline{C_i}^{\,i} \\ \overline{C_i{:}(y_i\!:T_i) \to B\ t'_i \in \Psi_0}^{\,i} \\ \overline{\Gamma, y_i{:}T_i, t' \cong t'_i, t \cong C_i\ y_i \vdash t_i : T}^{\,i} \end{array}}{\Gamma \vdash \mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \to t_i}^{\,i} : T}\ \textsc{wf\_dtm\_case}$$

We consider how the expression $\mathsf{case}\, t\, \mathsf{of}\, \overline{C_i\ y_i \to t_i}^{\,i}$ may step:

- By EVAL_DTM_CTX and the evaluation context $\mathsf{case}\,\square\,\mathsf{of}\,\overline{C_i\ y_i \to t_i}^{\,i}$.
  Then $t \longrightarrow t''$ so IH gives us $\Gamma \vdash t'' : T_1$. Additionally, by EQ_DTM_STEP we have $\Gamma, y_i{:}T_i \vdash t \cong t''$, so by context conversion (lemma 21) applied to the premise $\Gamma, y_i{:}T_i, t' \cong t_i', t'' \cong C_i\ y_i \vdash t_i : T$ we get $\Gamma, y_i{:}T_i, t' \cong t_i', t'' \cong C_i\ y_i \vdash t_i : T$. So by WF_DTM_CASE the final result is well-typed.
- By EVAL_DTM_CASE.
  Then $t = C_i\ v$ for some branch $C_i$ of the case expression, and the expression steps to $[v/y_i]t_i$. By inversion (lemma 26) we know $\Gamma \vdash v : T_i$ with $C_i{:}(y_i : T_i) \to B'\ t_i' \in \Psi_0$ and $\Gamma \vdash B'\ [v/y_i]t_i' \equiv B\ t'$. Since the signature cannot contain duplicate declarations, we know that $B' = B$ and that this $T_i$ is the same that was used to typecheck the $i$th branch of the case expression. Then we have (remember that $t$ is $C_i\ v$)
  $$\Gamma, y_i{:}T_i, t' \cong t_i', C_i\ v \cong C_i\ y_i \vdash t_i : T$$
  so by substitution we get
  $$\Gamma, [v/y_i]t' \cong [v/y_i]t_i', C_i\ [v/y_i]v \cong C_i\ v \vdash [v/y_i]t_i : [v/y_i]T$$
  Since $yi$ is a bound variable in the case branch and in the constructor declaration, we can pick it suitably fresh. Then $y_i$ can not occur in $t'$ or in $v$ or in $T$, so we can simplify this to
  $$\Gamma, t' \cong [v/y_i]t_i', C_i\ v \cong C_i\ v \vdash [v/y_i]t_i : T.$$
  Now the two equalities in the context are provable: we get $\Gamma \vdash t' \cong [v/y_i]t_i'$ by inversion (lemma 23) on the derivation $\Gamma \vdash B'\ [v/y_i]t_i' \equiv B\ t'$, while $\Gamma \vdash C_i\ v \cong C_i\ v$ follows by EQ_DTM_REFL. So by Cut (lemma 18) we have
  $$\Gamma \vdash [v/y_i]t_i : T$$
  as required.

**Case wf_dtm_ds** So $[t_0/y_0]t$ is a DS-boundary. Either $t$ is a DS-boundary, or $t$ is $y_0$. In the former case, the situation looks like this:

$$\frac{\begin{array}{c}\Gamma \vdash [t_0/y_0]s : S \\ \Gamma \vdash [t_0/y_0]T : * \\ S \Leftrightarrow [t_0/y_0]T\end{array}}{\Gamma \vdash \mathsf{DS}^{[t_0/y_0]T}_{[t_0/y_0]S}\,[t_0/y_0]s : [t_0/y_0]T}\ \text{WF\_DTM\_DS}$$

By the IH we get $\Gamma \vdash [t_0'/y_0]s : [t_0'/y_0]S$, by the mutual IH we get $\Gamma \vdash [t_0'/y_0]T : *$, and by applying lemma 27 twice we get $S \Leftrightarrow [t_0'/y_0]T$. So re-applying WF_DTM_DS we have $\Gamma \vdash \mathsf{DS}^{[t_0'/y_0]T}_{[t_0'/y_0]S}\,[t_0'/y_0]s : [t_0'/y_0]T$. Then by lemma 19 and WF_DTM_CONV we get $\Gamma \vdash \mathsf{DS}^{[t_0'/y_0]T}_{[t_0'/y_0]S}\,[t_0'/y_0]s : [t_0/y_0]T$ as required.

In the other case, $t_0$ itself is a DS-boundary which steps, and we must show that its type is preserved. We consider the ways the expression $\mathsf{DS}^T_S\, s$ can step:

- By congruence, so $s \longrightarrow s'$. Then by IH $\Gamma \vdash s' : S$ and thus $\Gamma \vdash \mathsf{DS}^T_S\, s' : T$.
- By EVAL_DTM_DS_ABS. So the step looks like
  $$\frac{}{\mathsf{DS}^{((y{:}T_1)\to T_2)}_{(S_1\to S_2)}\,\lambda x{:}\,S_1'.s \longrightarrow \lambda y{:}\,T_1.\mathsf{DS}^{T_2}_{S_2}((\lambda x{:}\,S_1'.s)\,(\mathsf{SD}^{S_1}_{T_1}\,y))}\ \text{EVAL\_DTM\_DS\_ABS}$$
  Then we must show that $\Gamma \vdash \lambda y{:}\,T_1.\mathsf{DS}^{T_2}_{S_2}(\lambda x{:}\,S_1'.s)\,(\mathsf{SD}^{S_1}_{T_1}\,y) : (y : T_1) \to T_2$, while we have the fact $\Gamma \vdash (\lambda x{:}\,S_1'.s) : S_1 \to S_2$ available as a premise to the typing rule.
  Constructing such a typing derivation is straightforward utilizing the fact that since $S_1 \to S_2 \Leftrightarrow (y : T_1) \to T_2$ then $S_1 \Leftrightarrow T_1$ and $S_2 \Leftrightarrow T_2$ (under a context $\Gamma, y{:}T_1$) by inversion of COMPAT_ARR. Eventually we must show that $\Gamma, y{:}T_1 \vdash \lambda x{:}\,S_1.s : S_1 \to S_2$. We are given that $\Gamma \vdash \lambda x{:}\,S_1.s : S_1 \to S_2$ so by weakening (lemma 11) we arrive at the desired result.

- Suppose it was by EVAL_DTM_DS_PAIR. So the step looks like

$$\frac{}{\mathsf{DS}^{((y{:}T_1)*T_2)}_{(S_1*S_2)}\ {<}u_1,u_2{>}\longrightarrow \mathsf{let}\, y' = \mathsf{DS}^{T_1}_{S_1} u_1 \,\mathsf{in}\ {<}y',\mathsf{DS}^{[y'/y]\,T_2}_{S_2} u_2{>}}\text{EVAL\_DTM\_DS\_PAIR}$$

  We have $\Gamma \vdash {<}u_1, u_2{>}: S_1 * S_2$ as a premise to the typing rule, directly by inversion on that typing we get $\Gamma \vdash u_1 : S_1$ and $\Gamma \vdash u_2 : S_2$. By inversion on TRANS_PAIR we get that $S_1 \Leftrightarrow T_1$ and $S_2 \Leftrightarrow T_2$.

  We must construct a typing derivation $\Gamma \vdash \mathsf{let}\, y = \mathsf{DS}^{T_1}_{S_1} u_1 \,\mathsf{in}\ {<}y, \mathsf{DS}^{T_2}_{S_2} u_2{>}: (y : T_1) * T_2$. (Here we rename the $y'$ to $y$ in order to simplify the expression). We will do this by applying the derived rule WF_DTM_LET (lemma 28), which requires showing the two premises $\Gamma \vdash \mathsf{DS}^{T_1}_{S_1} u_1 : T_1$ and $\Gamma, y{:}\,T_1 \vdash {<}y, \mathsf{DS}^{T_2}_{S_2} u_2{>}: (y : T_1) * T_2$. The third (kinding) premise follows immediately by regularity (lemma 22).

  From $\Gamma \vdash u_1 : S_1$ and $S_1 \Leftrightarrow T_1$ we directly get $\Gamma \vdash \mathsf{DS}^{T_1}_{S_1} u_1 : T_1$.

  For the second premise, we want to apply WF_DTM_PAIR. So we must show the three premises of that rule.

  1. $\Gamma, y{:}T_1 \vdash y : T_1$. Immediate by WF_DTM_VAR.
  2. $\Gamma, y{:}T_1 \vdash \mathsf{DS}^{T_2}_{S_2} u_2 : T_2$. Also immediate, using the fact that $S_2 \Leftrightarrow T_2$ as we noted before.
  3. $\Gamma, y{:}T_1 \vdash (y : T_1) * T_2 : *$. By regularity (lemma 22).

- Suppose it was by EVAL_DTM_DS_CONSTR. So the step looks like

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ \mathsf{argToD}_C\, u = v \end{array}}{\mathsf{DS}^{(B\ t)}_{A}(C\ u) \longrightarrow t \cong [v/y]t_1 \rhd (C\ v)}\text{EVAL\_DTM\_DS\_CONSTR}$$

  and we must show $\Gamma \vdash t \cong [v/y]t_1 \rhd C\ v : B\ t$.

  By property 6 we have $\Gamma \vdash v : T_1$. So $\Gamma \vdash C\ v : B\,[v/y]t_1$. By EQ_DTM_ASSUMPTION and EQ_DTY_APP we have $\Gamma, t \cong [v/y]t_1 \vdash B\,[v/y]t_1 \equiv B\,t$, so by WF_DTM_CONV we have $\Gamma, t \cong [v/y]t_1 \vdash C\ v : B\,t$.

  By inversion on the premise $A \Leftrightarrow B\,t$ we get $B : T_0 \Rightarrow * \in \Psi_0$ and $\mathsf{FO}\,(T_0)$, so by lemma 29 and substitution and weakening, we get $\Gamma \vdash [v/y]t_1 : T_0$. Also, from inversion (lemma 25) on the premise $\Gamma \vdash B\ t : *$, we get $\Gamma \vdash t : T_0$. Hence by WF_DTM_GUARD we get $\Gamma \vdash t \cong [v/y]t_1 \rhd C\ v : B\ t$ as required.

- Suppose it was by EVAL_DTM_DS_UNIT. So the step looks like

$$\frac{}{\mathsf{DS}^{\mathsf{Unit}}_{\mathsf{Unit}}\mathsf{unit} \longrightarrow \mathsf{unit}}\text{EVAL\_DTM\_DS\_UNIT}$$

  We must show $\Gamma \vdash \mathsf{unit} : \mathsf{Unit}$, which is straightforwardly true.

**Case wf_dtm_guard** . So $[t_0/y_0]t$ is a guard-expression, which means that either $t$ is a guard-expression or $t$ is $y_0$.

In the former case, we can push the substitution in and the situation looks like this:

$$\frac{\begin{array}{l} \Gamma \vdash [t_0/y_0]t_1 : T_0 \\ \Gamma \vdash [t_0/y_0]t_2 : T_0 \\ \mathsf{FO}\,(T_0) \\ \Gamma, [t_0/y_0]t_1 \cong [t_0/y_0]t_2 \vdash [t_0/y_0]t : T \end{array}}{\Gamma \vdash [t_0/y_0]t_1 \cong [t_0/y_0]t_2 \rhd [t_0/y_0]t : T}\text{WF\_DTM\_GUARD}$$

Directly by the IH we get $\Gamma \vdash [t'_0/y_0]t_1 : T_0$ and $\Gamma \vdash [t'_0/y_0]t_2 : T_0$. The IH also gives us $\Gamma, [t_0/y_0]t_1 \cong [t_0/y_0]t_2 \vdash [t'_0/y_0]t : T$. Now by EQ_DTM_STEP we have $\Gamma \vdash t_0 \cong t'_0$, so by EQ_DTM_SUBST we know $\Gamma \vdash [t_0/y_0]t_1 \cong [t'_0/y_0]t_1$. Similarly for $t_2$. So by Context Conversion (lemma 21) we get $\Gamma, [t'_0/y_0]t_1 \cong [t'_0/y_0]t_2 \vdash [t'_0/y_0]t : T$. We conclude by re-applying WF_DTM_GUARD.

The other possibility is that the entire expression is a guard-expression which steps, and we must prove that its type is preserved. So the typing rule looks like

$$\frac{\begin{array}{l}\Gamma \vdash t_0 : T_0 \\ \Gamma \vdash t_1 : T_0 \\ \mathsf{FO}\,(T_0) \\ \Gamma, t_1 \cong t_0 \vdash t : T\end{array}}{\Gamma \vdash t_1 \cong t_0 \rhd t : T}\ \text{WF\_DTM\_GUARD}$$

We consider the ways the expression $t_1 \cong t_0 \rhd t$ can step:

- By EVAL_DTM_CTX when the context is $\square \cong t_0 \rhd t$ and $t_1 \longrightarrow t'_1$. We must show $\Gamma \vdash t'_1 \cong t_0 \rhd t : T$. By the IH we have $\Gamma \vdash t'_1 : T_0$, so it suffices to show $\Gamma, t'_1 \cong t_0 \vdash t : T$, and then we can re-apply WF_DTM_GUARD. By EQ_DTM_STEP we know that $\Gamma \vdash t_1 \cong t'_1$ and by EQ_DTM_REFL we know $\Gamma \vdash t_0 \cong t_0$. So this follows by context conversion (lemma 21) on the assumption $\Gamma, t_1 \cong t_0 \vdash t : T$.
- By EVAL_DTM_CTX when the context is $v \cong \square \rhd t$. This is similar to the previous case.
- By EVAL_DTM_GUARD_REFL. In this case the expression steps to $t$.
  The premise to the rule says $\Gamma, v \cong v \vdash t : T$, but by EQ_DTM_REFL we have $\Gamma \vdash v \cong v$, so by Cut (lemma 18) we get $\Gamma \vdash t : T$ as required.
- By EVAL_DTM_GUARD_ERROR. In this case the entire expression steps to $\mathsf{error}$, and we can indeed type $\Gamma \vdash \mathsf{error} : T$ as required (using regularity, lemma 22, for the required kinding premise).

**Case wf_dtm_unit** Since $\mathsf{unit}$ doesn't step we must have $[t_0/y_0]t = [t'_0/y_0]t$ and the result is trivial.

**Case wf_dtm_error** Similar to the previous case.

The cases for $\Gamma \vdash [t_0/y_0]\,T : K$ are:

**Case wf_dty_arr.** The type $[t_0/y_0]\,T$ must be some arrow type. Since the variable is bound by the arrow we can pick it to not clash with $y_0$, so the type is in fact of the form $(y\,{:}\,[t_0/y_0]\,T_1) \to [t_0/y_0]\,T_2$, and the situation looks like

$$\frac{\begin{array}{l}\Gamma \vdash [t_0/y_0]\,T_1 : * \\ \Gamma, y{:}[t_0/y_0]\,T_1 \vdash [t_0/y]\,T_2 : *\end{array}}{\Gamma \vdash (y\,{:}\,[t_0/y_0]\,T_1) \to [t_0/y_0]\,T_2 : *}\ \text{WF\_DTY\_ARR}$$

Directly by the IH we get $\Gamma \vdash [t'_0/y_0]\,T_1 : *$. By the IH we get $\Gamma \vdash [t'_0/y_0]\,T_1 : *$ and $\Gamma, y{:}[t_0/y_0]\,T_1 \vdash [t'_0/y_0]\,T_2 : *$. By EQ_DTM_STEP we know $\Gamma \vdash t_0 \cong t'_0$, so by lemma 19 $\Gamma \vdash [t_0/y_0]\,T_1 \equiv [t'_0/y_0]\,T_1$. So by context conversion (lemma 20) we have $\Gamma, y{:}[t'_0/y_0]\,T_1 \vdash [t'_0/y]\,T_2 : *$. Re-apply WF_DTY_ARR to get $\Gamma \vdash (y\,{:}\,[t'_0/y]\,T_1) \to [t'_0/y]\,T_2 : *$ as required.

**Case wf_dty_pair.** Similar to the previous case.

**Case wf_dty_app.** The situation looks like:

$$\frac{\begin{array}{l}\Gamma \vdash [t_0/y_0]\,T : T_1 \Rightarrow * \\ \Gamma \vdash [t_0/y_0]t_1 : T_1\end{array}}{\Gamma \vdash ([t_0/y_0]\,T)\,([t_0/y_0]t_1) : *}\ \text{WF\_DTY\_APP}$$

By the IH we get $\Gamma \vdash [t'_0/y_0]\,T : T_1 \Rightarrow *$, and by the mutual IH we get $\Gamma \vdash [t'_0/y_0]t_1 : T_1$. Then we can re-apply WF_DTY_APP.

**Case wf_dty_data.** Since data constructors $B$ do not step we must have $[t_0/y_0]T = [t_0'/y_0]T$ and the result is trivial.

**Case wf_dtm_conv:** The situation looks like:

$$\frac{\begin{array}{l}\Gamma \vdash [t_0/y_0]t : T \\ \Gamma \vdash T \equiv T' \\ \Gamma \vdash T' : *\end{array}}{\Gamma \vdash [t_0/y]t : T'}\ \text{WF\_DTM\_CONV}$$

By the IH we get $\Gamma \vdash [t_0'/y_0]t : T$. Conclude by re-applying WF_DTM_CONV.

□

## B.3 Progress

**Lemma 30** (Parallel reduction contains evaluation)**.**

1. *If $t \longrightarrow t'$ then $t \longrightarrow_{\mathrm{p}} t'$.*
2. *If $s \longrightarrow s'$ then $s \longrightarrow_{\mathrm{p}} s'$.*

*Proof.* Easy from inspecting the definition of $\longrightarrow_{\mathrm{p}}$. □

**Property 8.** *If* $\mathsf{argToD}_C u$ *is defined, then* $\mathsf{argToD}_C[u_1/x_1]u = [u_1/x_1](\mathsf{argToD}_C u)$ *and* $\mathsf{argToD}_C[v_1/y_1]u = [v_1/y_1](\mathsf{argToD}_C u)$.

**Property 9.** *If* $\mathsf{argToS}_C v$ *is defined, then* $\mathsf{argToS}_C[u_1/x_1]v = [u_1/x_1](\mathsf{argToS}_C v)$ *and* $\mathsf{argToS}_C[v_1/y_1]v = [v_1/y_1](\mathsf{argToS}_C v)$.

**Property 10.** *If $u \longrightarrow_{\mathrm{p}} u'$, then* $\mathsf{argToD}_C u \longrightarrow_{\mathrm{p}} \mathsf{argToD}_C u'$.

**Property 11.** *If $v \longrightarrow_{\mathrm{p}} v'$, then* $\mathsf{argToS}_C v \longrightarrow_{\mathrm{p}} \mathsf{argToS}_C v'$.

**Lemma 31** (Substitution for parallel reduction)**.** *Suppose $u \longrightarrow_{\mathrm{p}} u'$, $s \longrightarrow_{\mathrm{p}} s'$, $v \longrightarrow_{\mathrm{p}} v'$, $t \longrightarrow_{\mathrm{p}} t'$, $S \longrightarrow_{\mathrm{p}} S'$, and $T \longrightarrow_{\mathrm{p}} T'$. Then*

1. $[u/x]s_2 \longrightarrow_{\mathrm{p}} [u'/x]s_2'$
2. $[u/x]t_2 \longrightarrow_{\mathrm{p}} [u'/x]t_2'$
3. $[u/x]S \longrightarrow_{\mathrm{p}} [u'/x]S'$
4. $[u/x]T \longrightarrow_{\mathrm{p}} [u'/x]T'$
5. $[v/y]s_2 \longrightarrow_{\mathrm{p}} [v'/y]s_2'$
6. $[v/y]t_2 \longrightarrow_{\mathrm{p}} [v'/y]t_2'$
7. $[v/y]S \longrightarrow_{\mathrm{p}} [v'/y]S'$
8. $[v/y]T \longrightarrow_{\mathrm{p}} [v'/y]T'$

*Proof.* By mutual induction on $s \longrightarrow_{\mathrm{p}} s'$, $t \longrightarrow_{\mathrm{p}} t'$, $S \longrightarrow_{\mathrm{p}} S'$ and $T \longrightarrow_{\mathrm{p}} T'$. Most of the cases are very similar, we show two representative ones (for substitution of dependent terms $v$ into dependent applications $t_1\ t_2$). We also show the cases involving $\mathsf{DS}/\mathsf{SD}$-boundaries on constructors, since those motivate the substitution properties for $\mathsf{argToD}$ and $\mathsf{argToS}$.

**Case par_eval_dtm_app:** The rule looks like

$$\frac{\begin{array}{l} t_1 \longrightarrow_{\mathrm{p}} t_1' \\ t_2 \longrightarrow_{\mathrm{p}} t_2' \end{array}}{t_1\ t_2 \longrightarrow_{\mathrm{p}} t_1'\ t_2'}\ \textsc{par\_eval\_dtm\_app}$$

By the IH we get $[v/y]t_1 \longrightarrow_{\mathrm{p}} [v'/y]t_1'$ and $[v/y]t_2 \longrightarrow_{\mathrm{p}} [v'/y]t_2'$. So re-applying $\textsc{par\_eval\_dtm\_appBeta}$ we get $[v/y](t_1\ t_2) \longrightarrow_{\mathrm{p}} [v'/y](t_1\ t_2')$ as required.

**Case par_eval_dtm_appBeta:** The rule looks like

$$\frac{t_1 \longrightarrow_{\mathrm{p}} t_1' v_2 \longrightarrow_{\mathrm{p}} v_2'}{(\lambda y_1 \colon T.t_1)\ v_2 \longrightarrow_{\mathrm{p}} [v_2'/y_1]t_1'}\ \textsc{par\_eval\_dtm\_appBeta}$$

Since $y_1$ is a bound variable we can pick so that $y_1 \notin \mathbf{fv}\,(v_2)$. The IH gives us $[v/y]t_1 \longrightarrow_{\mathrm{p}} [v'/y]t_1'$ and $[v/y]v_2 \longrightarrow_{\mathrm{p}} [v'/y]v_2'$. By lemma 12 $[v/y]v_2$ is still a value, so pushing down the substitution and re-applying $\textsc{par\_eval\_dtm\_appBeta}$ we get $[v/y]((\lambda y_1 \colon T.t_1)\ v_2) \longrightarrow_{\mathrm{p}} [[v'/y]v_2'/y_1][v'/y]t_1'$. Noting that $y_1 \notin \mathbf{fv}\,(v_2')$ we have $[[v'/y]v_2'/y_1][v'/y]t_1' = [v'/y][v_2'/y_1]t_1'$, so we have showed $[v/y]((\lambda y_1 \colon T.t_1)\ v_2) \longrightarrow_{\mathrm{p}} [v'/y][v_2'/y_1]t_1'$ as required.

**Case par_eval_dtm_ds_constr:** The rule looks like

$$\frac{\begin{array}{l} C{:}S \to A \in \Psi_0 \\ C{:}(y_1 : T_1) \to B\ t_1 \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \mathsf{argToD}_C\, u' = v_1 \\ u \longrightarrow_{\mathrm{p}} u' \\ t \longrightarrow_{\mathrm{p}} t' \end{array}}{\mathsf{DS}_A^{(B\ t)}(C\ u) \longrightarrow_{\mathrm{p}} t' \cong [v_1/y_1]t_1 \rhd (C\ v_1)}\ \textsc{par\_eval\_dtm\_ds\_constr}$$

By mutual IH we get $[v/y]u \longrightarrow_{\mathrm{p}} [v/y]u'$. So by property 10, $\mathsf{argToD}_C[v/y]u \longrightarrow_{\mathrm{p}} \mathsf{argToD}_C[v'/y]u'$. Also by IH, we have $[v/y]t \longrightarrow_{\mathrm{p}} [v'/y]t$. So re-applying $\textsc{par\_eval\_dtm\_ds\_constr}$, we have that

$$\mathsf{DS}_A^{(B\,[v/y]t)}(C\,[v/y]u) \longrightarrow_{\mathrm{p}} [v'/y]t' \cong [\mathsf{argToD}_C[v'/y]u/y_1]t_1 \rhd (C\,(\mathsf{argToD}_C[v'/y]u)).$$

By property 8 we know that $\mathsf{argToD}_C[v'/y]u = [v'/y](\mathsf{argToD}_C\,u)$, so

$$[\mathsf{argToD}_C[v'/y]u/y_1]t_1 = [[v'/y](\mathsf{argToD}_C\,u)/y_1]t_1.$$

From the assumption that the signature is well-formed ($\vdash \Psi$) we in particular get that $\cdot \vdash (y : T_1) \to B\ t_1 : *$, so $y \notin \mathbf{fv}\,(t_1)$, and $[[v'/y](\mathsf{argToD}_C\,u)/y_1]t_1 = [v'/y][\mathsf{argToD}_C\,u/y_1]t_1$. So pulling the substitutions out, we have in fact shown

$$[v/y]\mathsf{DS}_A^{(B\ t)}(C\ u) \longrightarrow_{\mathrm{p}} [v'/y]t' \cong [v'/y][\mathsf{argToD}_C\,u/y_1]t_1 \rhd (C\,[v'/y](\mathsf{argToD}_C\,u)),$$

as required.

**Case par_eval_stm_sd_constr:** The rule looks like

$$\frac{\begin{array}{l} \mathsf{constrs}\,A = \overline{C_i}^{\,i} \\ C{:}S \to A \in \Psi_0 \\ C{:}(y : T_1) \to B\ t_1 \in \Psi_0 \\ B{:}T_2 \Rightarrow * \in \Psi_0 \\ \mathsf{argToS}_C\, v_1' = u_1 \\ v_1 \longrightarrow_{\mathrm{p}} v_1' \end{array}}{\mathsf{SD}_{(B\ t)}^{A}\, C\ v_1 \longrightarrow_{\mathrm{p}} C\ u_1}\ \textsc{par\_eval\_stm\_sd\_constr}$$

By mutual IH we get $[v/y]v_1 \longrightarrow_{\mathrm{p}} [v'/y]v_1'$. So by re-applying PAR_EVAL_STM_SD_CONSTR we get

$$[v/y](\mathsf{SD}^{A}_{(B\,1)}(C\,v_1)) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C([v'/y]v_1')).$$

By property 9 we can commute the substitution past $\mathsf{argToS}$, so we have in fact shown

$$[v/y](\mathsf{SD}^{A}_{(B\,1)}(C\,v_1)) \longrightarrow_{\mathrm{p}} [v'/y](C\,(\mathsf{argToS}_C\,v_1')),$$

as required.

□

**Lemma 32** (One-step diamond property for parallel reduction)**.**

1. *If $s \longrightarrow_{\mathrm{p}} s_1$ and $s \longrightarrow_{\mathrm{p}} s_2$, then there exists some $s'$ such that $s_1 \longrightarrow_{\mathrm{p}} s'$ and $s_2 \longrightarrow_{\mathrm{p}} s'$.*
2. *If $S \longrightarrow_{\mathrm{p}} S_1$ and $S \longrightarrow_{\mathrm{p}} S_2$, then there exists some $S'$ such that $S_1 \longrightarrow_{\mathrm{p}} S'$ and $S_2 \longrightarrow_{\mathrm{p}} S'$*
3. *If $t \longrightarrow_{\mathrm{p}} t_1$ and $t \longrightarrow_{\mathrm{p}} t_2$, then there exists some $t'$ such that $t_1 \longrightarrow_{\mathrm{p}} t'$ and $t_2 \longrightarrow_{\mathrm{p}} t'$.*
4. *If $T \longrightarrow_{\mathrm{p}} T_1$ and $T \longrightarrow_{\mathrm{p}} T_2$, then there exists some $T'$ such that $T_1 \longrightarrow_{\mathrm{p}} T'$ and $T_2 \longrightarrow_{\mathrm{p}} T'$.*

*Proof.* By induction on the structure of $s$, $S$, $t$ and $T$. In each case we consider the (non-REFL) ways the term/type can step. (If one of the steps is by REFL the result is trivial).

Cases for $s$:

**Case $x$.** Trivial since $x$ doesn't step.

**Case $\mathsf{unit}$, $\mathsf{error}$.** Similar.

**Case $\lambda x{:}S.s$.** The only way this expression can step is by $S$ and $s$ stepping; apply IH.

**Case $s_1\,s_2$.** Consider the pairs of ways that the expression may step:

- Both are PAR_EVAL_STM_APP. In other words, we have $s_1\,s_2 \longrightarrow_{\mathrm{p}} s_{11}\,s_{21}$ and $s_1\,s_2 \longrightarrow_{\mathrm{p}} s_{12}\,s_{22}$. By the IH for $s_1$ we get $s_{11} \longrightarrow_{\mathrm{p}} s_1'$ and $s_{12} \longrightarrow_{\mathrm{p}} s_1'$ for some $s_1'$, and similarly for $s_2$. So then by PAR_EVAL_STM_APP, we get $s_{11}\,s_{12} \longrightarrow_{\mathrm{p}} s_1'\,s_2'$ and $s_{12}\,s_{22} \longrightarrow_{\mathrm{p}} s_1'\,s_2'$ as required.
- One of them is PAR_EVAL_STM_APP and one is PAR_EVAL_STM_BETA. In other words, we have

$$\begin{array}{rl} (\lambda x{:}S.s_0)\,u_2 \longrightarrow_{\mathrm{p}} [u_{21}/x]s_{01} & \quad\text{where } s_0 \longrightarrow_{\mathrm{p}} s_{01} \text{ and } u_2 \longrightarrow_{\mathrm{p}} u_{21} \\ (\lambda x{:}S.s_0)\,u_2 \longrightarrow_{\mathrm{p}} (\lambda x{:}S.s_{02})\,u_{22} & \quad\text{where } s_0 \longrightarrow_{\mathrm{p}} s_{02} \text{ and } u_2 \longrightarrow_{\mathrm{p}} u_{22} \end{array}$$

  By the IH we have $s_{01} \longrightarrow_{\mathrm{p}} s_1'$ and $s_{02} \longrightarrow_{\mathrm{p}} s_0'$ for some $s_0'$, and also $u_{21} \longrightarrow_{\mathrm{p}} u_2'$ and $u_{22} \longrightarrow_{\mathrm{p}} u_2'$ for some $u_2'$. By lemma 31 we have $[u_{21}/x]s_{01} \longrightarrow_{\mathrm{p}} [u_2'/x]s_0'$, and by PAR_EVAL_STM_BETA we have $(\lambda x{:}S.s_{02})\,u_2' \longrightarrow_{\mathrm{p}} [u_2'/x]s_0'$, as required.
- Both of them are PAR_EVAL_STM_BETA. In other words we have

$$\begin{array}{rl} (\lambda x{:}S.s_0)\,u_2 \longrightarrow_{\mathrm{p}} [u_{21}/x]s_{01} & \quad\text{where } s_0 \longrightarrow_{\mathrm{p}} s_{01} \text{ and } u_2 \longrightarrow_{\mathrm{p}} u_{21} \\ (\lambda x{:}S.s_0)\,u_2 \longrightarrow_{\mathrm{p}} [u_{22}/x]s_{02} & \quad\text{where } s_0 \longrightarrow_{\mathrm{p}} s_{02} \text{ and } u_2 \longrightarrow_{\mathrm{p}} u_{22} \end{array}$$

  By the IH we have $s_{01} \longrightarrow_{\mathrm{p}} s_1'$ and $s_{02} \longrightarrow_{\mathrm{p}} s_0'$ for some $s_0'$, and also $u_{21} \longrightarrow_{\mathrm{p}} u_2'$ and $u_{22} \longrightarrow_{\mathrm{p}} u_2'$ for some $u_2'$. Now by lemma 31, $[u_{21}/x]s_{01} \longrightarrow_{\mathrm{p}} [u'/x]s_0'$ and $[u_{22}/x]s_{02} \longrightarrow_{\mathrm{p}} [u'/x]s_0'$ as required.
- One of them is PAR_EVAL_STM_ERROR. By considering the cases for the context $\mathcal{E}_s$, we see that the error transition is either $\mathsf{error}\,s_2 \longrightarrow_{\mathrm{p}} \mathsf{error}$ or $u_1\,\mathsf{error} \longrightarrow_{\mathrm{p}} \mathsf{error}$. Since $\mathsf{error}$ is not a value, the other transition can not be PAR_EVAL_STM_BETA, so it must be either ERROR or APP. If it is ERROR, then both terms step to error and we are done; if it is APP then the term stepped to $\mathsf{error}\,s_2'$ or $u_1'\,\mathsf{error}$, which can again step to $\mathsf{error}$.

**Case** $<s_1, s_2>$**.** Consider the pairs of ways the expression may step:

- By two PAR_EVAL_STM_PAIR transitions. We reason similarly to the case for two PAR_EVAL_STM_APP transitions above.
- By one PAR_EVAL_STM_PAIR and one PAR_EVAL_STM_ERROR transition. Stepping by PAIR will still leave the term in a form where ERROR applies.
- By two PAR_EVAL_STM_ERROR transitions. Then both terms step to $\mathsf{error}$, so we are done.

**Case** $s.1$**,** $s.2$**,** $C\ s$**.** Similar to the previous case.

**Case** $\mathsf{case}\ s\ \mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i}$**.** We consider the ways the expression may step.

- Both transitions are by PAR_EVAL_STM_CASE. We reason as in the above cases.
- One transition was by PAR_EVAL_STM_CASE, and one by PAR_EVAL_STM_CASEBETA. In other words we have

$$\begin{aligned} \mathsf{case}\ C_i\ u\ \mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i} &\longrightarrow_{\mathrm{p}} [u_1/x_i]s_{1\,i} && \text{where } u \longrightarrow_{\mathrm{p}} u_1 \text{ and } s_i \longrightarrow_{\mathrm{p}} s_{1\,i} \\ \mathsf{case}\ C_i\ u\ \mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i} &\longrightarrow_{\mathrm{p}} \mathsf{case}\ C_i\ u_2\ \mathsf{of}\ \overline{C_i\ x_i \to s_{2\,i}}^{\,i} && \text{where } u \longrightarrow_{\mathrm{p}} u_2 \text{ and } s_i \longrightarrow_{\mathrm{p}} s_{2\,i} \end{aligned}$$

  Now by IH we have $u_1 \longrightarrow_{\mathrm{p}} u'$ and $u_2 \longrightarrow_{\mathrm{p}} u'$ for some $u'$, and also $s_{1\,i} \longrightarrow_{\mathrm{p}} s'_i$ and $s_{2\,i} \longrightarrow_{\mathrm{p}} s'_i$ for each $i$. By lemma 31 we get $[u_1/x_i]s_{1\,i} \longrightarrow_{\mathrm{p}} [u'/x_i]s'_i$, and by PAR_EVAL_CASEBETA we get $\mathsf{case}\ C_i\ u_2\ \mathsf{of}\ \overline{C_i\ x_i \to s_{2\,i}}^{\,i} \longrightarrow_{\mathrm{p}} [u'/x_i]s'_i$, as required.
- Both transitions are by PAR_EVAL_STM_CASEBETA. As in the case of two PAR_EVAL_STM_BETA transitions, this follows by IH and lemma 31.
- One transition was by PAR_EVAL_STM_ERROR. So the term must be $\mathsf{case}\ \mathsf{error}\ \mathsf{of}\ \overline{C_i\ x_i \to s_i}^{\,i}$. We see that the other transition must be either CASE or ERROR, and in both cases the resulting terms are joinable at $\mathsf{error}$.

**Case** $\mathsf{letd}\ y = t\ \mathsf{in}\ s$ Consider the pairs of ways the expressions may step:

- Both are PAR_EVAL_STM_LETD. This follows directly by IH, similar to previous congruence cases.
- One transition is PAR_EVAL_STM_ERROR. Considering the possibilities for $\mathcal{E}_s$, we see that the transition must be $\mathsf{letd}\ y = \mathsf{error}\ \mathsf{in}\ s$. Then, since $\mathsf{error}$ is not a value, the only possibilities for the other transition is PAR_EVAL_STM_ERROR (in which case the terms already are joined at $\mathsf{error}$) and PAR_EVAL_STM_LETD (which reduces to a term where PAR_EVAL_STM_ERROR can still fire).
- One transition is PAR_EVAL_STM_LETDBETA, the other is PAR_EVAL_STM_LETD. In other words we have

$$\begin{aligned} \mathsf{letd}\ y = v\ \mathsf{in}\ s &\longrightarrow_{\mathrm{p}} [v_1/y]s_1 && \text{where } v \longrightarrow_{\mathrm{p}} v_1 \text{ and } s \longrightarrow_{\mathrm{p}} s_1 \\ \mathsf{letd}\ y = v\ \mathsf{in}\ s &\longrightarrow_{\mathrm{p}} \mathsf{letd}\ y = v_2\ \mathsf{in}\ s_2 && \text{where } v \longrightarrow_{\mathrm{p}} v_2 \text{ and } s \longrightarrow_{\mathrm{p}} s_2 \end{aligned}$$

  Now by the mutual IH we know $v_1 \longrightarrow_{\mathrm{p}} v'$ and $v_2 \longrightarrow_{\mathrm{p}} v'$ for some $v'$, and by the IH similar for $s$. So by lemma 31 we get $[v_1/y]s_1 \longrightarrow_{\mathrm{p}} [v'/y]s'$, while by PAR_EVAL_LETDBETA we get $\mathsf{letd}\ y = v_2\ \mathsf{in}\ s_2 \longrightarrow_{\mathrm{p}} [v'/y]s'$, as required.
- Both are PAR_EVAL_STM_LETDBETA. In other words we have

$$\begin{aligned} \mathsf{letd}\ y = v\ \mathsf{in}\ s &\longrightarrow_{\mathrm{p}} [v_1/y]s_1 && \text{where } v \longrightarrow_{\mathrm{p}} v_1 \text{ and } s \longrightarrow_{\mathrm{p}} s_1 \\ \mathsf{letd}\ y = v\ \mathsf{in}\ s &\longrightarrow_{\mathrm{p}} [v_2/y]s_1 && \text{where } v \longrightarrow_{\mathrm{p}} v_2 \text{ and } s \longrightarrow_{\mathrm{p}} s_2 \end{aligned}$$

  By mutual IH and IH we get that $v_i \longrightarrow_{\mathrm{p}} v'$ and $s_i \longrightarrow_{\mathrm{p}} s'$, and conclude by lemma 31.

**Case $\mathsf{SD}^{S}_{T}\,t$.** Consider the pairs of ways the expression may step:

- Both are PAR_EVAL_STM_SD. We reason as in previous cases above.
- One transition is by PAR_EVAL_STM_ERROR. Then considering the possible evaluation contexts the term must be $\mathsf{SD}^{S}_{T}\mathsf{error}$, and there is only one possible transition, so the other transition must be PAR_EVAL_STM_ERROR also.
- One transition is by PAR_EVAL_STM_ABS and one is by PAR_EVAL_STM_SD. In other words we have

$$\begin{aligned}&\mathsf{SD}^{(S_1\to S_2)}_{((y:T_1)\to T_2)}\lambda y{:}\,T_3.t \longrightarrow_{\mathrm{p}} \lambda x{:}\,S_{12}.\mathsf{letd}\ y' = \mathsf{DS}^{T_{11}}_{S_{11}}\,x\ \mathsf{in}\ \mathsf{SD}^{S_{21}}_{[y'/y]\,T_{21}}((\lambda y{:}\,T_{31}.t_1)\ y')\\ &\qquad\text{where } T_1 \longrightarrow_{\mathrm{p}} T_{11},\ T_2 \longrightarrow_{\mathrm{p}} T_{21},\ S_1 \longrightarrow_{\mathrm{p}} S_{11},\ S_2 \longrightarrow_{\mathrm{p}} S_{21},\ T_3 \longrightarrow_{\mathrm{p}} T_{31},\ t \longrightarrow_{\mathrm{p}} t_1.\\ &\mathsf{SD}^{(S_1\to S_2)}_{((y:T_1)\to T_2)}\lambda y{:}\,T_3.t \longrightarrow_{\mathrm{p}} \mathsf{SD}^{(S_{12}\to S_{22})}_{((y:T_{12})\to T_{22})}\lambda y{:}\,T_{32}.t_2\\ &\qquad\text{where } T_1 \longrightarrow_{\mathrm{p}} T_{12},\ T_2 \longrightarrow_{\mathrm{p}} T_{22},\ S_1 \longrightarrow_{\mathrm{p}} S_{12},\ S_2 \longrightarrow_{\mathrm{p}} S_{22},\ T_3 \longrightarrow_{\mathrm{p}} T_{32},\ t \longrightarrow_{\mathrm{p}} t_2.\end{aligned}$$

  Now, by the (mutual) IH we get that $T_{11} \longrightarrow_{\mathrm{p}} T_1'$ and $T_{12} \longrightarrow_{\mathrm{p}} T_1'$ for some $T'$, and similarly for the other subterms. Since $y'$ is a value, we know by lemma 31 that $[y'/y]\,T_{21} \longrightarrow_{\mathrm{p}} [y'/y]\,T_2'$. So by a combination of PAR_EVAL_STM_ABS, PAR_EVAL_STM_LETD, PAR_EVAL_DTM_ABS, PAR_EVAL_STM_SD and PAR_EVAL_DTM_DS, we have

$$\begin{aligned}\lambda x{:}\,S_{12}.\mathsf{letd}\ y' = (\mathsf{DS}^{T_{11}}_{S_{11}}\,x)\ \mathsf{in}\ \mathsf{SD}^{S_{21}}_{[y'/y]\,T_{21}}((\lambda y{:}\,T_{31}.t)\ y') &\longrightarrow_{\mathrm{p}}\\ \lambda x{:}\,S_1'.\mathsf{letd}\ y' = \mathsf{DS}^{T_1'}_{S_1'}\,x\ \mathsf{in}\ \mathsf{SD}^{S_2'}_{[y'/y]\,T_2'}((\lambda y{:}\,T_3'.t')\ y')&\end{aligned}$$

  while by PAR_EVAL_STM_DS_ABS we have

$$\mathsf{SD}^{(S_{12}\to S_{22})}_{((y:T_{12})\to T_{22})}\lambda y{:}\,T_{32}.t_2 \longrightarrow_{\mathrm{p}} \lambda x{:}\,S_1'.\mathsf{letd}\ y' = \mathsf{DS}^{T_1'}_{S_1'}\,x\ \mathsf{in}\ \mathsf{SD}^{S_2'}_{[y'/y]\,T_2'}((\lambda y{:}\,T_3'.t')\ y')$$

  as required.
- Both transitions are by PAR_EVAL_STM_SD_ABS. In other words we have

$$\begin{aligned}&\mathsf{SD}^{(S_1\to S_2)}_{((y:T_1)\to T_2)}\lambda y{:}\,T_3.t \longrightarrow_{\mathrm{p}} \lambda x{:}\,S_{12}.\mathsf{letd}\ y' = \mathsf{DS}^{T_{11}}_{S_{11}}\,x\ \mathsf{in}\ \mathsf{SD}^{S_{21}}_{[y'/y]\,T_{21}}((\lambda y{:}\,T_{31}.t_1)\ y')\\ &\qquad\text{where } T_1 \longrightarrow_{\mathrm{p}} T_{11},\ T_2 \longrightarrow_{\mathrm{p}} T_{21},\ S_1 \longrightarrow_{\mathrm{p}} S_{11},\ S_2 \longrightarrow_{\mathrm{p}} S_{21},\ T_3 \longrightarrow_{\mathrm{p}} T_{31},\ t \longrightarrow_{\mathrm{p}} t_1.\\ &\mathsf{SD}^{(S_1\to S_2)}_{((y:T_1)\to T_2)}\lambda y{:}\,T_3.t \longrightarrow_{\mathrm{p}} \lambda x{:}\,S_{12}.\mathsf{letd}\ y' = \mathsf{DS}^{T_{12}}_{S_{12}}\,x\ \mathsf{in}\ \mathsf{SD}^{S_{22}}_{[y'/y]\,T_{22}}((\lambda y{:}\,T_{32}.t_2)\ y')\\ &\qquad\text{where } T_1 \longrightarrow_{\mathrm{p}} T_{11},\ T_2 \longrightarrow_{\mathrm{p}} T_{21},\ S_1 \longrightarrow_{\mathrm{p}} S_{11},\ S_2 \longrightarrow_{\mathrm{p}} S_{21},\ T_3 \longrightarrow_{\mathrm{p}} T_{31},\ t \longrightarrow_{\mathrm{p}} t_1.\end{aligned}$$

  By the (mutual) IH we get that $T_{11} \longrightarrow_{\mathrm{p}} T_1'$ and $T_{12} \longrightarrow_{\mathrm{p}} T_1'$ for some $T'$, and similarly for the other subterms. By reasoning similarly to the previous case we see that both terms step to $\lambda x{:}\,S_1'.\mathsf{letd}\ y' = \mathsf{DS}^{T_1'}_{S_1'}\,x\ \mathsf{in}\ \mathsf{SD}^{S_2'}_{[y'/y]\,T_2'}((\lambda y{:}\,T_3'.t')\ y')$.
- One transition is by PAR_EVAL_STM_SD_PAIR and the other one is by PAR_EVAL_STM_SD. In other words we have (by considering case for how e.g. the expression $S_1 * S_2$ may step)

$$\begin{aligned}\mathsf{SD}^{S_1*S_2}_{(y:T_1)*T_2}<v_1,v_2>\longrightarrow_{\mathrm{p}}<\mathsf{SD}^{S_{11}}_{T_{11}}\,v_{11},\mathsf{SD}^{S_{21}}_{[v_{11}/y]\,T_{21}}\,v_{21}> &\qquad \text{where } S_1 \longrightarrow_{\mathrm{p}} S_{11} \text{ etc}\\ \mathsf{SD}^{S_1*S_2}_{(y:T_1)*T_2}<v_1,v_2>\longrightarrow_{\mathrm{p}} \mathsf{SD}^{S_{12}*S_{22}}_{(y:T_{12})*T_{22}}<v_{12},v_{22}> &\qquad \text{where } S_1 \longrightarrow_{\mathrm{p}} S_{12} \text{ etc}\end{aligned}$$

  By the IH we get $S_{11} \longrightarrow_{\mathrm{p}} S_1'$ and $S_{12} \longrightarrow_{\mathrm{p}} S_1'$ for some $S_1'$, and similarly for the other subterms. By lemma 31 we know $[v_{11}/y]\,T_{21} \longrightarrow_{\mathrm{p}} [v_1'/y]\,T_2'$, so by various congruence rules $<\mathsf{SD}^{S_{11}}_{T_{11}}\,v_{11},\mathsf{SD}^{S_{21}}_{[v_{11}/y]\,T_{21}}\,v_{21}>\longrightarrow_{\mathrm{p}}<\mathsf{SD}^{S_1'}_{T_1'}\,v',\mathsf{SD}^{S_2'}_{[v_1'/y]\,T_2'}\,v_2'>$. Meanwhile by PAR_EVAL_SD_PAIR we have $\mathsf{SD}^{S_{12}*S_{22}}_{(y:T_{12})*T_{22}}<v_{12},v_{22}>\longrightarrow_{\mathrm{p}}<\mathsf{SD}^{S_1'}_{T_1'}\,v',\mathsf{SD}^{S_2'}_{[v_1'/y]\,T_2'}\,v_2'>$ as required.
- Both transitions are by PAR_EVAL_STM_SD_PAIR. Using lemma 31 and congruence rules, we get that the terms are joinable at $<\mathsf{SD}^{S_1'}_{T_1'}\,v',\mathsf{SD}^{S_2'}_{[v_1'/y]\,T_2'}\,v_2'>$.

- One transition is by PAR_EVAL_STM_SD_CONSTR and one is by PAR_EVAL_STM_SD. In other words we have

$$\begin{array}{ll} \mathsf{SD}^{A}_{(B\ t)}(C\ v) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v_1) & \text{where } v \longrightarrow_{\mathrm{p}} v_1 \\ \mathsf{SD}^{A}_{(B\ t)}(C\ v) \longrightarrow_{\mathrm{p}} \mathsf{SD}^{A_2}_{(B\ t_2)}(C\ v_2) & \text{where } A \longrightarrow_{\mathrm{p}} A_2,\ t \longrightarrow_{\mathrm{p}} t_2 \text{ and } v \longrightarrow_{\mathrm{p}} v_2 \end{array}$$

  By the IH, we get that $v_1 \longrightarrow_{\mathrm{p}} v'$ and $v_2 \longrightarrow_{\mathrm{p}} v'$ for some $v'$. By property 11, we have $\mathsf{argToS}_C\, v_1 \longrightarrow_{\mathrm{p}} \mathsf{argToS}_C\, v'$, so $C\,(\mathsf{argToS}_C\, v_1) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v')$. And by PAR_EVAL_STM_CONSTR we have $\mathsf{SD}^{A_2}_{(B\ t_2)}(C\ v_2) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v')$, as required.

- Both transitions are by PAR_EVAL_STM_CONSTR. In other words, we have

$$\begin{array}{ll} \mathsf{SD}^{A}_{(B\ t)}(C\ v) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v_1) & \text{where } v \longrightarrow_{\mathrm{p}} v_1 \\ \mathsf{SD}^{A}_{(B\ t)}(C\ v) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v_2) & \text{where } v \longrightarrow_{\mathrm{p}} v_2 \end{array}$$

  By the mutual IH we know $v_1 \longrightarrow_{\mathrm{p}} v'$ and $v_2 \longrightarrow_{\mathrm{p}} v'$ for some $v'$. By property 11 we know $\mathsf{argToS}_C\, v_1 \longrightarrow_{\mathrm{p}} \mathsf{argToS}_C\, v'$ and $\mathsf{argToS}_C\, v_2 \longrightarrow_{\mathrm{p}} \mathsf{argToS}_C\, v'$. So by congruence we have $C\,(\mathsf{argToS}_C\, v_1) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v')$ and $C\,(\mathsf{argToS}_C\, v_2) \longrightarrow_{\mathrm{p}} C\,(\mathsf{argToS}_C\, v')$ as required.

- One transition is by PAR_EVAL_STM_UNIT. So the term is $\mathsf{SD}^{\mathsf{Unit}}_{\mathsf{Unit}}\mathsf{unit}$, and there is only one possible transition, so the other transition must be PAR_EVAL_STM_UNIT also.

Cases for $S$: These are all trivial, since there are only congruence rules.
Cases for $t$:

**Case** $y$, $\mathsf{unit}$, $\mathsf{error}$**.** These terms do not step.

**Case** $\lambda y{:}\, T.t$**,** $<t_1, t_2>$**,** $t.1$**,** $t.1$**, and** $C\ t$ . These expressions can only step by congruence rules, the proof is similar to some previous cases.

**Case** $t_1\ t_2$ **and** $\mathsf{case}\ t\ \mathsf{of}\ \overline{C_i\ y_i \rightarrow t_i}^{\,i}$ . These are similar to the corresponding cases for simply-typed terms.

**Case** $\mathsf{DS}^{T}_{S}\, s$ We consider the pairs of ways the expression may step.

- Both transitions are by PAR_EVAL_DS. This follows by the IH and reasoning similar to the previous congruence cases.

- One transition is by PAR_EVAL_ERROR. By considering the possible evaluation contexts $\mathcal{E}_t$ we see that the transition must be $\mathsf{DS}^{T}_{S}\mathsf{error} \longrightarrow_{\mathrm{p}} \mathsf{error}$. The only possibilities for the other transition is that it also steps to $\mathsf{error}$ or that it is $\mathsf{DS}^{T}_{S}\mathsf{error} \longrightarrow_{\mathrm{p}} \mathsf{DS}^{T'}_{S'}\mathsf{error}$; in both cases the resulting terms are joinable at $\mathsf{error}$.

- One transition is by PAR_EVAL_DS_ABS, and the other one is by PAR_EVAL_DS. So the term must have the form $\mathsf{DS}^{((y{:}T_1)\rightarrow T_2)}_{(S_1\rightarrow S_2)}\lambda x{:}\, S_3.s$. By inversion on the rules, we know that the only way the subterm $(y{:}\, T_1) \rightarrow T_2$ can step is by congruence if $T_1$ and $T_2$ steps. In other words we have

$$\begin{array}{l} \mathsf{DS}^{(y{:}T_1)\rightarrow T_2}_{S_1\rightarrow S_2}\lambda x{:}\, S_3.s \longrightarrow_{\mathrm{p}} \lambda y{:}\, T_{11}.\mathsf{DS}^{T_{21}}_{S_{21}}((\lambda x{:}\, S_{31}.s_1)\,(\mathsf{SD}^{S_{11}}_{T_{11}}\, y)) \\ \mathsf{DS}^{(y{:}T_1)\rightarrow T_2}_{S_1\rightarrow S_2}\lambda x{:}\, S_3.s \longrightarrow_{\mathrm{p}} \mathsf{DS}^{(y{:}T_{12})\rightarrow T_{22}}_{S_{12}\rightarrow S_{22}}\lambda x{:}\, S_{32}.s_2 \end{array}$$

  where $T_1 \longrightarrow_{\mathrm{p}} T_{11}$ and $T_1 \longrightarrow_{\mathrm{p}} T_{12}$, and similar for the other subterms. By the IH we get $T_{11} \longrightarrow_{\mathrm{p}} T_1'$ and $T_{12} \longrightarrow_{\mathrm{p}} T_1'$ for some $T'$, and similarly for the other subterms. So by a combination of congruence rules, $\lambda y{:}\, T_{11}.\mathsf{DS}^{T_{21}}_{S_{21}}((\lambda x{:}\, S_{31}.s_1)\,(\mathsf{SD}^{S_{11}}_{T_{11}}\, y)) \longrightarrow_{\mathrm{p}} \lambda y{:}\, T_1'.\mathsf{DS}^{T_2'}_{S_2'}((\lambda x{:}\, S_3'.s')\,(\mathsf{SD}^{S_1'}_{T_1'}\, y))$, while by PAR_EVAL_DS_ABS $\mathsf{DS}^{(y{:}T_{12})\rightarrow T_{22}}_{S_{12}\rightarrow S_{22}}\lambda x : S_{32}.s_2 \longrightarrow_{\mathrm{p}} \lambda y : T_1'.\mathsf{DS}^{T_2'}_{S_2'}((\lambda x : S_3'.s')\,(\mathsf{SD}^{S_1'}_{T_1'}\, y))$ as required.

- Both transitions are by PAR_EVAL_DS_ABS. So the transition are

$$\begin{array}{l}\mathsf{DS}^{(y:T_1)\to T_2}_{S_1\to S_2}\,\lambda x{:}\,S_3.s \longrightarrow_{\mathrm{p}} \lambda y{:}\,T_{11}.\mathsf{DS}^{T_{21}}_{S_{21}}((\lambda x{:}\,S_{31}.s_1)\,(\mathsf{SD}^{S_{11}}_{T_{11}}\,y))\\ \mathsf{DS}^{(y:T_1)\to T_2}_{S_1\to S_2}\,\lambda x{:}\,S_3.s \longrightarrow_{\mathrm{p}} \lambda y{:}\,T_{11}.\mathsf{DS}^{T_{22}}_{S_{22}}((\lambda x{:}\,S_{32}.s_2)\,(\mathsf{SD}^{S_{12}}_{T_{12}}\,y))\end{array}$$

  which join at $\lambda y{:}\,T_1'.\mathsf{DS}^{T_2'}_{S_2'}((\lambda x{:}\,S_3'.s')\,(\mathsf{SD}^{S_1'}_{T_1'}\,y))$ by congruence rules.

- One transition is by PAR_EVAL_DS_PAIR, one is by PAR_EVAL_DS. In other words (and considering the possible transitions $(y : T_1) * T_2$ can make), we have

$$\begin{array}{l}\mathsf{DS}^{(y:T_1)*T_2}_{S_1*S_2}\,{<}u_1,u_2{>}\longrightarrow_{\mathrm{p}} \mathsf{let}\,y' = \mathsf{DS}^{T_{11}}_{S_{11}}\,u_{11}\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_{21}}_{S_{21}}\,u_{21}{>}\\ \mathsf{DS}^{(y:T_1)*T_2}_{S_1*S_2}\,{<}u_1,u_2{>}\longrightarrow_{\mathrm{p}} \mathsf{DS}^{(y:T_{12})*T_{22}}_{S_{21}*S_{22}}\,{<}u_{12},u_{22}{>}\end{array}$$

  where $T_1 \longrightarrow_{\mathrm{p}} T_{11}$ and $T_1 \longrightarrow_{\mathrm{p}} T_{12}$, etc. By the IH we get get $T_{11} \longrightarrow_{\mathrm{p}} T_1'$ and $T_{12} \longrightarrow_{\mathrm{p}} T'$, etc. By lemma 31 we know $[y'/y]\,T_{21} \longrightarrow_{\mathrm{p}} [y'/y]\,T_2'$. So by various congruence rules we get $\mathsf{let}\,y' = \mathsf{DS}^{T_{11}}_{S_{11}}\,u_{11}\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_{21}}_{S_{21}}\,u_{21}{>}\longrightarrow_{\mathrm{p}} \mathsf{let}\,y' = \mathsf{DS}^{T_1'}_{S_1'}\,u_1'\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_2'}_{S_2'}\,u_2'{>}$, while by PAR_EVAL_DS_PAIR we have $\mathsf{DS}^{(y:T_{12})*T_{22}}_{S_{21}*S_{22}}\,{<}u_{12},u_{22}{>}\longrightarrow_{\mathrm{p}} \mathsf{let}\,y' = \mathsf{DS}^{T_1'}_{S_1'}\,u_1'\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_2'}_{S_2'}\,u_2'{>}$, as required.

- Both transitions are by PAR_EVAL_DS_PAIR. So the transitions are

$$\begin{array}{l}\mathsf{DS}^{(y:T_1)*T_2}_{S_1*S_2}\,{<}u_1,u_2{>}\longrightarrow_{\mathrm{p}} \mathsf{let}\,y' = \mathsf{DS}^{T_{11}}_{S_{11}}\,u_{11}\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_{21}}_{S_{21}}\,u_{21}{>}\\ \mathsf{DS}^{(y:T_1)*T_2}_{S_1*S_2}\,{<}u_1,u_2{>}\longrightarrow_{\mathrm{p}} \mathsf{let}\,y' = \mathsf{DS}^{T_{12}}_{S_{12}}\,u_{12}\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_{22}}_{S_{22}}\,u_{22}{>}\end{array}$$

  which join at $\mathsf{let}\,y' = \mathsf{DS}^{T_1'}_{S_1'}\,u_1'\,\mathsf{in}\,{<}y',\mathsf{DS}^{[y'/y]\,T_2'}_{S_2'}\,u_2'{>}$ by various congruence rules.

- One transition is by PAR_EVAL_DS_CONSTR and the other one is by PAR_EVAL_DS. In other words we have

$$\begin{array}{rl}\mathsf{DS}^{(B\;t)}_{A}\,C\;u \longrightarrow_{\mathrm{p}} t_1 \cong [\mathsf{argToD}_C\,u_1/y]t \rhd C\,(\mathsf{argToD}_C\,u_1) & \text{where } t \longrightarrow_{\mathrm{p}} t_1 \text{ and } u \longrightarrow_{\mathrm{p}} u_1\\ \mathsf{DS}^{(B\;t)}_{A}\,C\;u \longrightarrow_{\mathrm{p}} \mathsf{DS}^{(B\;t_2)}_{A}\,C\;u_2 & \text{where } t \longrightarrow_{\mathrm{p}} t_2 \text{ and } u \longrightarrow_{\mathrm{p}} u_2\end{array}$$

  By the IH we have $t_1 \longrightarrow_{\mathrm{p}} t'$ and $t_2 \longrightarrow_{\mathrm{p}} t'$ for some $t'$, and $u_1 \longrightarrow_{\mathrm{p}} u'$ and $u_2 \longrightarrow_{\mathrm{p}} u'$ for some $u'$. By property 10, we know $\mathsf{argToD}_C\,u_1 \longrightarrow_{\mathrm{p}} \mathsf{argToD}_C\,u'$ and $\mathsf{argToD}_C\,u_2 \longrightarrow_{\mathrm{p}} \mathsf{argToD}_C\,u'$. So by lemma 31 and congruence rules we have

$$t_1 \cong [\mathsf{argToD}_C\,u_1/y]t \rhd C\,(\mathsf{argToD}_C\,u_1) \longrightarrow_{\mathrm{p}} t' \cong [\mathsf{argToD}_C\,u'/y]t \rhd C\,(\mathsf{argToD}_C\,u')$$

  while by PAR_EVAL_DS_CONSTR (note that the term $t$ comes out of the signature $\Psi_0$ and is therefore always the same) we get

$$\mathsf{DS}^{(B\;t_2)}_{A}\,C\;u_2 \longrightarrow_{\mathrm{p}} t' \cong [\mathsf{argToD}_C\,u'/y]t \rhd C\,(\mathsf{argToD}_C\,u')$$

  as required.

- Both transitions are by PAR_EVAL_DS_CONSTR. In other words we have

$$\begin{array}{rl}\mathsf{DS}^{(B\;t)}_{A}\,C\;u \longrightarrow_{\mathrm{p}} t_1 \cong [\mathsf{argToD}_C\,u_1/y]t \rhd C\,(\mathsf{argToD}_C\,u_1) & \text{where } t \longrightarrow_{\mathrm{p}} t_1 \text{ and } u \longrightarrow_{\mathrm{p}} u_1\\ \mathsf{DS}^{(B\;t)}_{A}\,C\;u \longrightarrow_{\mathrm{p}} t_2 \cong [\mathsf{argToD}_C\,u_2/y]t \rhd C\,(\mathsf{argToD}_C\,u_2) & \text{where } t \longrightarrow_{\mathrm{p}} t_2 \text{ and } u \longrightarrow_{\mathrm{p}} u_2\end{array}$$

  Similarly to the previous case, by IH, property 10, lemma 31, and congruence rules, there terms are joinable at

$$t' \cong [\mathsf{argToD}_C\,u'/y]t \rhd C\,(\mathsf{argToD}_C\,u').$$

- One transition is by PAR_EVAL_DS_UNIT. That is, the transition looks like $\mathsf{DS}^{\mathsf{Unit}}_{\mathsf{Unit}}\mathsf{unit} \longrightarrow_{\mathrm{p}} \mathsf{unit}$. The only possibility is that the other transition is PAR_EVAL_DS_UNIT also.

**Case** $t_1 \cong t_2 \rhd t$ We consider the pairs of ways the expression may step.

- Both transitions are PAR_EVAL_DTM_GUARD.
- One transition is PAR_EVAL_DTM_ERROR. By considering cases for the context $\mathcal{E}_t$, we see that the transition must be one of $\mathsf{error} \cong t_2 \rhd t \longrightarrow_{\mathrm{p}} \mathsf{error}$ or $v_1 \cong \mathsf{error} \rhd t \longrightarrow_{\mathrm{p}} \mathsf{error}$. Since $\mathsf{error}$ is not a value, neither of PAR_EVAL_DTM_GUARD_REFL or PAR_EVAL_DTM_GUARD_ERROR applies, so we know the other transition must be by either PAR_EVAL_DTM_GUARD, in which case the reduct can step to $\mathsf{error}$ also, or PAR_EVAL_DTM_ERROR, in which case we are immediately done.
- One transition is PAR_EVAL_DTM_GUARD_REFL. So the term must be of the form $v \cong v \rhd t$. The only other transitions that can match is congruence or REFL. If the other transition is also REFL we can conclude directly by IH. If the other transition is PAR_EVAL_DTM_GUARD, then we have

$$\begin{array}{rl} v \cong v \rhd t \longrightarrow_{\mathrm{p}} t_1 & \quad \text{where } t \longrightarrow_{\mathrm{p}} t_1 \\ v \cong v \rhd t \longrightarrow_{\mathrm{p}} v_{21} \cong v_{22} \rhd t_2 & \quad \text{where } v \longrightarrow_{\mathrm{p}} v_{12},\ v \longrightarrow_{\mathrm{p}} v_{22} \text{ and } t \longrightarrow_{\mathrm{p}} t_2. \end{array}$$

  By the IH we have that $t_1 \longrightarrow_{\mathrm{p}} t'$ and $t_2 \longrightarrow_{\mathrm{p}} t'$ for some $t'$, and by PAR_EVAL_DTM_GUARD_REFL, we have $v_{21} \cong v_{22} \rhd t_2 \longrightarrow_{\mathrm{p}} t'$ as required.
- One transition is PAR_EVAL_DTM_GUARD_ERROR. The reasoning is similar to the previous case.

Cases for $T$: These are all trivial, since there are only congruence rules. ☐

**Lemma 33** (Confluence of parallel reduction)**.** *If $t \longrightarrow_{\mathrm{p}^*} t_1$ and $t \longrightarrow_{\mathrm{p}^*} t_2$, then there exists some $t'$ such that $t_1 \longrightarrow_{\mathrm{p}^*} t'$ and $t_2 \longrightarrow_{\mathrm{p}^*} t'$.*

*Proof.* This is a simple corollary of the 1-step version (lemma 32), by "diagram chasing to fill in the rectangle" (see e.g. [2], lemma 3.2.2). ☐

**Lemma 34** (Compatibility of parallel reduction)**.** *Suppose $t \longrightarrow_{\mathrm{p}} t'$ and $s \longrightarrow_{\mathrm{p}} s'$. Then*

- $[t/y]t_1 \longrightarrow_{\mathrm{p}} [t'/y]t_1$
- $[t/y]s_1 \longrightarrow_{\mathrm{p}} [t'/y]s_1$
- $[t/y]T_1 \longrightarrow_{\mathrm{p}} [t'/y]T_1$
- $[t/y]S_1 \longrightarrow_{\mathrm{p}} [t'/y]S_1$
- $[s/x]t_1 \longrightarrow_{\mathrm{p}} [s'/x]t_1$
- $[s/x]s_1 \longrightarrow_{\mathrm{p}} [s'/x]s_1$
- $[s/x]T_1 \longrightarrow_{\mathrm{p}} [s'/x]T_1$
- $[s/x]S_1 \longrightarrow_{\mathrm{p}} [s'/x]S_1$

*Proof.* Mutual induction on the structure of $t_1$, $s_1$, $T_1$ and $S_1$. These are all similar, so we show only cases for $t_1$.

**Case** $y_1$ There are two cases depending on whether $y_1 = y$ or not. If it is, we must show $t \longrightarrow_{\mathrm{p}} t'$, which we have by assumption. If not we must show $y_1 \longrightarrow_{\mathrm{p}} y_1$, which is true by PAR_EVAL_DTM_REFL.

**Case** $t_1\ t_2$ . We note that $[t/y](t_1\ t_2) = [t/y]t_1\ [t/y]t_2$, so we can then apply the IH.

**Case** $<t_1, t_2>$ **,** $t_1.1$ **,** $t_1.2$**,** $C\ t_1$**,** $\mathsf{DS}^{T_1}_{S_1} s_1$**, and** $t_{11} \cong t_{12} \rhd t_1$**.** Similar to the previous case, except we also appeal to the mutual IH for $T_1$ and $S_1$.

**Case** $\lambda y_1 \colon T_1 . t_1$ . Since $y_1$ is a bound variable we can pick it so that $y_1 \neq y$. Then $[t/y](\lambda y_1 \colon T_1 . t_1) = \lambda y_1 \colon [t/y]\, T_1 . [t/y] t_1$, and we conclude by (direct and mutual) IH.

**Case** $\mathsf{case}\; t_1 \;\mathsf{of}\; \overline{C_i\; y_i \rightarrow t_i}^{\,i}$ . Similar to the previous case.

**Case** $\mathsf{unit}$**.** Trivial since $[t/y]\mathsf{unit} = [t'/y]\mathsf{unit} = \mathsf{unit}$.

**Case** $\mathsf{error}$**.** Similar to the previous case.

□

**Lemma 35** (Parallel reduction contains term equivalence)**.** *If* $\cdot \vdash t_1 \cong t_2$*, then there exists some* $t'$ *such that* $t_1 \longrightarrow_{\mathrm{p}^*} t'$ *and* $t_2 \longrightarrow_{\mathrm{p}^*} t'$*.*

*Proof.* We proceed by induction on $\cdot \vdash t_1 \cong t_2$. The cases are

**Case eq_dtm_assumption:** This cannot happen because the context is empty.

**Case eq_dtm_step:** We are given $t_1 \longrightarrow t_2$ as a premise to the rule. By lemma 30 we have $t_1 \longrightarrow_{\mathrm{p}} t_2$, so take $t' = t_2$.

**Case eq_dtm_refl:** Take $t' = t$.

**Case eq_dtm_sym:** Immediate from the IH.

**Case eq_dtm_trans:** The IH gives us that there exists $t_1$ and $t_2$ such that $t \longrightarrow_{\mathrm{p}^*} t_1$, $t' \longrightarrow_{\mathrm{p}^*} t_1$, $t' \longrightarrow_{\mathrm{p}^*} t_2$ and $t'' \longrightarrow_{\mathrm{p}^*} t_2$. So by confluence (lemma 33) applied to $t'$ we know there exists $t_3$ such that $t_1 \longrightarrow_{\mathrm{p}^*} t_3$ and $t_1 \longrightarrow_{\mathrm{p}^*} t_3$, which is what we needed.

**Case eq_dtm_subst:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash t_1 \cong t_1' \\ y \notin \mathbf{dom}\,(\Gamma)\end{array}}{\Gamma \vdash [t_1/y]t \cong [t_1'/y]t}\;\text{EQ\_DTM\_SUBST}$$

The IH gives us two chains of reductions:

$$\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} t_{12} \longrightarrow_{\mathrm{p}} t_{13} \ldots \longrightarrow_{\mathrm{p}} t_n \\ t_1' \longrightarrow_{\mathrm{p}} t_{12}' \longrightarrow_{\mathrm{p}} t_{13}' \ldots \longrightarrow_{\mathrm{p}} t_n. \end{array}$$

By lemma 34 we can lift these to chains

$$\begin{array}{c} [t_1/y]t \longrightarrow_{\mathrm{p}} [t_{12}/y]t \longrightarrow_{\mathrm{p}} [t_{13}/y]t \ldots \longrightarrow_{\mathrm{p}} [t_n/y]t \\ [t_1'/y]t \longrightarrow_{\mathrm{p}} [t_{12}'/y]t \longrightarrow_{\mathrm{p}} [t_{13}'/y]t \ldots \longrightarrow_{\mathrm{p}} [t_n/y]t. \end{array}$$

which is what we need.

**Case eq_dtm_subst_val:** The rule looks like

$$\frac{\begin{array}{c}\Gamma \vdash t \cong t' \\ y \notin \mathbf{dom}\,(\Gamma)\end{array}}{\Gamma \vdash [v/y]t \cong [v/y]t'}\;\text{EQ\_DTM\_SUBST\_VAL}$$

The IH gives us two chains of reductions

$$\begin{array}{c} t_1 \longrightarrow_{\mathrm{p}} t_{12} \longrightarrow_{\mathrm{p}} t_{13} \ldots \longrightarrow_{\mathrm{p}} t_n \\ t_1' \longrightarrow_{\mathrm{p}} t_{12}' \longrightarrow_{\mathrm{p}} t_{13}' \ldots \longrightarrow_{\mathrm{p}} t_n. \end{array}$$

Use part (6) of lemma 31 to get

$$\begin{array}{c} [v/y]t_1 \longrightarrow_{\mathrm{p}} [v/y]t_{12} \longrightarrow_{\mathrm{p}} [v/y]t_{13} \ldots \longrightarrow_{\mathrm{p}} [v/y]t_n \\ [v/y]t_1' \longrightarrow_{\mathrm{p}} [v/y]t_{12}' \longrightarrow_{\mathrm{p}} [v/y]t_{13}' \ldots \longrightarrow_{\mathrm{p}} [v/y]t_n, \end{array}$$

which is what we need.

**Case eq_dtm_ssubst_val:** Similar to the previous case.

□

**Lemma 36.** *If $C_1 \neq C_2$, then we never have $\cdot \vdash C_1\, v_1 \cong C_2\, v_2$.*

*Proof.* By the previous lemma there must be some term $t'$ such that $C_1\, v_1 \longrightarrow_{p^*} t'$ and $C_2\, v_2 \longrightarrow_{p^*} t'$. But that is impossible: by looking at the rules for $\longrightarrow_{p}$ we see that they can never change the outermost constructor of a term. □

**Lemma 37** (Canonical forms for $s$s)**.**

1. *If $\cdot \vdash u : S_1 \to S_2$ then $u$ is $\lambda x{:}\, S.s$.*
2. *If $\cdot \vdash u : S_1 * S_2$ then $u$ is $<u_1, u_2>$.*
3. *If $\cdot \vdash u : \mathsf{Unit}$ then $u$ is $\mathsf{unit}$.*
4. *If $\cdot \vdash u : A$ then $u$ is $C\, u'$ and $C{:}S \to A \in \Psi_0$.*

*Proof.* By induction on the typing judgment $\cdot \vdash s : S$. The cases are:

**Case wf_stm_var:** This is impossible since the context is empty.

**Case wf_stm_abs:** The type is an arrow type, so only the first case applies. $s$ is indeed a $\lambda$-expression.

**Case wf_stm_pair,wf_stm_ctor, wf_stm_unit:** Similar to the previous case, $s$ does indeed have the right form.

**Case wf_stm_app, proj1, proj2, case, letd, sd, error:** In these rules, the subject of the typing is not a value.

□

**Lemma 38** (Canonical forms for $t$s)**.**

1. *If $\cdot \vdash v : (y : T_1) \to T_2$ then $v$ is $\lambda y{:}\, T.t$.*
2. *If $\cdot \vdash v : (y : T_1) * T_2$ then $v$ is $<v_1, v_2>$.*
3. *If $\cdot \vdash v : \mathsf{Unit}$ then $v$ is $\mathsf{unit}$.*
4. *If $\cdot \vdash v : B\, t$ then $v$ is $C\, v'$ and $C{:}(y : T) \to B\, t' \in \Psi_0$.*

*Proof.* By induction on the typing judgment $\cdot \vdash t : T$. The cases are:

**Case wf_dtm_var:** This is impossible since the context is empty.

**Case wf_dtm_abs:** The type is an arrow type, so only the first item applies. $t$ is indeed a $\lambda$-expression.

**Case wf_dtm_pair, wf_dtm_unit, wf_dtm_ctor:** Similar to the ABS case.

**Case wf_dtm_app, proj1, proj2, case, ds, guard, error:** In these rules the subject of the typing is not a value.

**Case wf_dtm_conv:** The typing rule looks like

$$\frac{\begin{array}{l}\Gamma \vdash t : T \\ \Gamma \vdash T \equiv T' \\ \Gamma \vdash T' : *\end{array}}{\Gamma \vdash t : T'}\text{WF\_DTM\_CONV}$$

We have as an assumption that the top-level shape of $T'$ is $\rightarrow$, $*$, $\mathsf{Unit}$ or $B\,t$, and we want to invoke the IH on the premise $\cdot \vdash t : T$. So we need to establish that $T$ has the same top-level shape as $T'$. This follows by a case analysis on the judgment $\cdot \vdash T \equiv T'$. We see that all the type-equivalence rules preserve the top-level shape of the type except EQ_DTY_INCON and inconsistency is ruled out by lemma 36.

□

**Property 12.**

1. *If* $C{:}S \rightarrow A \in \Psi_0$ *and* $\mathsf{corr}\,(A, B)$*, then* $C{:}(y : T_1) \rightarrow B\,t_1 \in \Psi_0$ *for some* $T_1$ *and* $t_1$*.*
2. *If* $C{:}(y : T) \rightarrow B\,t' \in \Psi_0$ *and* $\mathsf{corr}\,(A, B)$*, then* $C{:}S \rightarrow A \in \Psi_0$ *for some* $S$*.*

**Lemma 39** (Case coverage)**.** *If* $\Gamma \vdash \mathsf{case}\; C\;v\;\mathsf{of}\;\overline{C_i\;y_i \rightarrow t_i}^{\,i} : T$ *and* $\vdash \Psi_0$*, then* $C$ *is one of the constructors* $C_i$*.*

**Property 13.**

1. *If* $C{:}S \rightarrow A \in \Psi_0$ *and* $C{:}(y : T_1) \rightarrow B\,t_1 \in \Psi_0$ *and* $\cdot \vdash u : S$*, then* $\mathsf{argToD}_C\,u$ *is defined.*
2. *If* $C{:}S \rightarrow A \in \Psi_0$ *and* $C{:}(y : T_1) \rightarrow B\,t_1 \in \Psi_0$ *and* $\cdot \vdash v : T_1$*, then* $\mathsf{argToS}_C\,v$ *is defined.*

**Theorem 4** (Progress)**.**

1. *If* $\cdot \vdash t : T$ *then either* $t \longrightarrow t'$*,* $t$ *is a value, or* $t$ *is* $\mathsf{error}$*.*
2. *If* $\cdot \vdash s : S$ *then either* $s \longrightarrow s'$*,* $s$ *is a value, or* $s$ *is* $\mathsf{error}$*.*

*Proof.* Proof by mutual induction on the judgments $\cdot \vdash s : S$ and $\cdot \vdash t : T$.

The cases for $\cdot \vdash t : T$ are:

**Case wf_dtm_var:** Impossible since the context is empty.

**Case wf_dtm_abs:** $t$ is already a value.

**Case wf_dtm_app:** The case looks like

$$\frac{\begin{array}{l}\Gamma \vdash t_1 : (y : T_1) \rightarrow T_2 \\ \Gamma \vdash t_2 : T_1 \\ \Gamma \vdash [t_2/y]\,T_2 : *\end{array}}{\Gamma \vdash t_1\;t_2 : [t_2/y]\,T_2}\text{WF\_DTM\_APP}$$

By the IH we get that $t_1$ and $t_2$ either step, are values, or are $\mathsf{error}$.

If either of them are $\mathsf{error}$, then the term steps by EVAL_DTM_ERROR. Otherwise, if $t_1$ steps, then the entire term steps by EVAL_DTM_APP. Similarly if $t_1$ is a value and $t_2$ steps.

Finally, suppose both $t_1$ and $t_2$ are values. Then, by lemma 38 $t_1$ must be a $\lambda$-abstraction, so the application steps by EVAL_DTM_BETA.

**Case wf_dtm_proj1,wf_dtm_proj2:** similar to APP.

**Case wf_dtm_pair:** By the IH the components of the pair are $\mathsf{error}$, one steps (in which case the entire expression steps), or they are values (in which case the entire expression is a value).

**Case wf_dtm_ctor:** Similar to PAIR.

**Case wf_dtm_case:** The case looks like

$$\dfrac{\begin{array}{l}\Gamma \vdash t : B\, t' \\ \Gamma \vdash T : * \\ \mathsf{constrs}\, B = \overline{C_i}^{\,i} \\ \overline{C_i{:}(y_i : T_i) \to B\, t'_i \in \Psi_0}^{\,i} \\ \overline{\Gamma, y_i{:}T_i, t' \cong t'_i, t \cong C_i\, y_i \vdash t_i : T}^{\,i}\end{array}}{\Gamma \vdash \mathsf{case}\, t\, \mathsf{of}\, \overline{C_i\, y_i \to t_i}^{\,i} : T}\text{WF\_DTM\_CASE}$$

By the IH, $t$ either steps, is $\mathsf{error}$, or is a value. In the first two cases the entire expression steps. If $t$ is a value, then by canonical forms (lemma 38), we know that $t$ is $C\, v$ and $C{:}(y : T) \to B\, t' \in \Psi_0$. Then by lemma 39 we know $C$ is one of the branches of the case expression, so it can step by WF_DTM_CASE.

**Case wf_dtm_ds:** The case looks like

$$\dfrac{\begin{array}{l}\Gamma \vdash s : S \\ \Gamma \vdash T : * \\ S \Leftrightarrow T\end{array}}{\Gamma \vdash \mathsf{DS}_S^T\, s : T}\text{WF\_DTM\_DS}$$

By the mutual IH for $s$, $s$ steps, is a value, or is $\mathsf{error}$. If it steps or is an $\mathsf{error}$, the entire expression steps. So suppose s is a value.

By inversion on the judgment $S \Leftrightarrow T$ there are four possibilities:

- $S$ is $S_1 \to S_2$ and $T$ is $(y : T_1) \to T_2$. By canonical forms (lemma 37), $s$ must be a $\lambda$-abstraction, so the entire expression steps by EVAL_DTM_DS_ABS.
- $S$ is $S_1 * S_2$ and $T$ is $(y : T_1) * T_2$. By canonical forms (lemma 37), $s$ must be a pair, so the entire expression steps by EVAL_DTM_DS_PAIR.
- $S$ and $T$ are unit. By canonical forms (lemma 37) $s$ must be $\mathsf{unit}$, so the entire expression steps by EVAL_DTM_DS_ABS.
- $S$ is $A$, $T$ is $B\, t$, and $\mathsf{corr}\,(A, B)$. By canonical forms (lemma 37, $s$ is $C\, u'$ and $C{:}S \to A \in \Psi_0$. So by property 12, $C : (y : T_1) \to B\, t_1 \in \Psi_0$ for some $T_1$ and $t_1$. We have $\cdot \vdash u : S$ by assumption, so $\mathsf{argToD}_C\, u$ is defined by property 13. Then the entire expression steps to $t \cong [v/y]t_1 \rhd (C\, (\mathsf{argToD}_C\, u))$ by EVAL_DTM_SD_CONSTR.

**Case wf_dtm_guard:** The case looks like

$$\dfrac{\begin{array}{l}\Gamma \vdash t_0 : T_0 \\ \Gamma \vdash t_1 : T_0 \\ \mathsf{FO}\,(T_0) \\ \Gamma, t_1 \cong t_0 \vdash t : T\end{array}}{\Gamma \vdash t_1 \cong t_0 \rhd t : T}\text{WF\_DTM\_GUARD}$$

By the IH, $t_1$ and $t_0$ and each is a value, $\mathsf{error}$, or steps. If they step or are error the entire expression steps by EVAL_DTM_CTX or EVAL_DTM_ERROR. Finally, if $t_1$ and $t_2$ are both values, then the expression steps by EVAL_DTM_GUARD_REFL or EVAL_DTM_GUARD_ERROR.

**Case wf_dtm_error:** Here $t$ is $\mathsf{error}$ as required.

**Case wf_dtm_conv:** The case looks like

$$\frac{\begin{array}{l}\Gamma \vdash t : T \\ \Gamma \vdash T \equiv T' \\ \Gamma \vdash T' : *\end{array}}{\Gamma \vdash t : T'}\ \text{WF\_DTM\_CONV}$$

so the conclusion follows directly by the IH for $t$.

The cases for $\cdot \vdash s : S$ are mostly routine, but we show the one involving an SD-boundary:

**Case wf_stm_sd:** The case looks like

$$\frac{\begin{array}{c}\Gamma \vdash t : T \\ S \Leftrightarrow T\end{array}}{\Gamma \vdash \mathsf{SD}^{S}_{T} t : S}\ \text{WF\_STM\_SD}$$

By the mutual IH, $t$ either, steps, is $\mathsf{error}$, or is a value. If it steps or is $\mathsf{error}$ the entire expression steps, so assume it is a value.

By inversion on the judgment $S \Leftrightarrow T$ there are four possibilities:

- $S$ is $S_1 \to S_2$ and $T$ is $(y : T_1) \to T_2$. By canonical forms (lemma 38) we know $t$ is a lambda. So the expression steps by EVAL_STM_SD_ABS.
- $S$ is $S_1 * S_2$ and $T$ is $(y : T_1) * T_2$. By canonical forms (lemma 38) we know $t$ is a pair of values. So the expression steps by EVAL_STM_SD_PAIR.
- $S$ is $\mathsf{Unit}$ and $T$ is $\mathsf{Unit}$. By canonical forms (lemma 38) we know $t$ is $\mathsf{unit}$. So the expression steps by EVAL_STM_SD_UNIT.
- $S$ is $A$ and $T$ is $B\, t$ and $\mathsf{corr}\,(A, B)$. By canonical forms (lemma 38) we know $t$ is $C\, v'$ and $C{:}(y : T) \to B\, t' \in \Psi_0$. So by property 12 we have $C{:}S \to S \in \Psi_0$ for some $S$. By property 13 we know $\mathsf{argToS}_C\, v$ is defined. Then the expression steps by EVAL_STM_SD_CONSTR to $C\, (\mathsf{argToS}_C\, v)$.

$\square$