A block slipping on a sphere with friction: Exact and perturbative solutions

Tom Prior and E. J. Mele
Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104

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A well studied problem in elementary mechanics is the location of the release point of a particle that slides on the surface of a frictionless sphere when it is released from rest at the top. We generalize this problem to include the effects of sliding friction and solve it by a perturbation expansion in the coefficient of sliding friction and by an exact integration of the equation of motion. A comparison of the two solutions identifies a parameter range where the perturbation series accurately represents the motion of the particle and another range where the perturbative solution fails qualitatively to describe the motion of the particle.

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I. INTRODUCTION

A classic problem in introductory mechanics[1,2] is to determine the release point for a block of mass m that slides without friction on the surface of a sphere or cylinder of radius R starting from rest (that is, with an infinitesimal velocity) at the top (see Fig. 1). The answer is that the release point, the point at which the block first loses contact with the sphere, occurs when it has fallen a vertical distance $h=R/3$. The result is interesting because it is independent of both the particle mass m and the gravitational acceleration g.

It is useful to extend this analysis to include frictional effects, which we model by the frictional force $F_f=\mu F_N$ opposing the motion in the tangent plane of the sphere; F_N is the normal force and μ is the coefficient of (kinetic) friction. This problem is much more challenging because the normal force depends on the particle speed, which in turn depends on the history of the descent of the particle up to its release point.

In this paper we discuss approximate and exact solutions to this problem. We assume that the solution with friction is analytic in the coefficient of friction μ and use this assumption to develop a perturbation expansion in powers of μ for the particle motion up to the release point. It is not generally appreciated that the equation of motion for this problem with friction can be integrated exactly. In the following we develop the perturbation expansion to second order in μ to illustrate its structure.

A comparison of the exact and approximate solutions identifies a parameter regime where the solutions are nearly identical. The comparison also identifies the regime where the solutions are appreciably different and the perturbation solution is no longer meaningful. In particular, the block needs a critical initial impulse to avoid stopping on the surface of the sphere, a situation that is inaccessible by perturbation theory. The discrepancy between the approximate and exact solutions increases as the frictional effects increase, but decreases as more terms are included in the approximate solution.

We used this problem in the context of an honors level introductory course in mechanics to illustrate the application of a perturbation theory to a familiar (though challenging) problem. In Sec. II we present the geometry of the problem, the equation of motion describing the descent of the particle on the surface of the sphere, and summarize its solution for a frictionless sphere using conservation of energy. In Sec. III we formulate the problem in the presence of friction as an integro-differential equation of motion. In Sec. IV we develop a solution by a method of successive approximations in powers of μ. In Sec. V we formulate the original problem as a first-order inhomogeneous ordinary differential equation, which has an exact solution. In Sec. VI we compare the exact and approximate solutions and discuss the features of the motion that fall outside the applicability of the perturbation theory.

II. SOLUTION WITHOUT FRICTION

We first pose a "warm-up" problem before studying the effects of friction. A block of mass m is released at the top of a fixed sphere of radius R with initial velocity v_0 tangent to the surface of the sphere. At what vertical displacement from the top of the sphere does the block lose contact with the sphere?

To solve this problem we use Newton's second law $\boldsymbol{F}_{\text{net}}=m\boldsymbol{a}$, where $\boldsymbol{F}_{\text{net}}$ is the net force on the particle and $\boldsymbol{a}$ is its acceleration. We will also use the work-energy theorem $\int_{r_1}^{r_2}\boldsymbol{F}\,d\boldsymbol{r}=\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2$. We treat the block as a point particle.

It is helpful to introduce a coordinate system whose origin is at the center of the fixed sphere. Let the initial position of the block be $\boldsymbol{r}_0$, the position of the block at some time later be $\boldsymbol{r}$, and the angle between the two be θ. While the block is on the sphere, its distance from the center of the sphere is always R. Because the position of the block is contained in the plane of $\boldsymbol{g}$, and $\boldsymbol{v}_0$, where $\boldsymbol{g}$ is the acceleration due to gravity and $\boldsymbol{v}_0$ is the initial velocity of the block, two scalar variables are enough to completely describe the position of the block. One of these variables, R, is fixed for the section of motion we are considering, so the position of the block can be completely parameterized as a function of θ.

If we resolve forces along the surface normal and define $F_N=|\boldsymbol{F}_N|$, we have

$$mg\cos\theta - F_N = \frac{mv^2}{R}. \tag{1}$$

The application of conservation of energy yields the relation

$$2gR(1-\cos\theta) = v^2 - v_0^2. \tag{2}$$

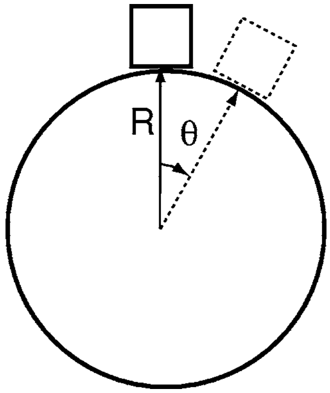

Fig. 1. A block sliding on the surface of a stationary sphere of radius R. the block is approximated as a point particle, and its motion parametrized by the polar angle θ.

To solve for the position at which the block loses contact with the sphere, we combine Eqs. (1) and (2), with the condition $F_N=0$, and solve for $h=R-R\cos\theta$:

$$h=\frac{R}{3}\left(1-\frac{v_0^2}{gR}\right). \tag{3}$$

For the case v_0, as in the original problem posed in Sec. I, $h=R/3$. If $v_0\neq 0$, h decreases with the square of v_0.

III. A MODEL WITH FRICTION

If we include frictional effects, there are three forces acting on the block during the time of interest. These are the force of gravity, the normal force of the sphere on the block, which is always along $\boldsymbol{r}$, and the force of friction on the block, which is always perpendicular to $\boldsymbol{r}$. Because there is no component of the frictional force $\boldsymbol{F}_f$ along $\boldsymbol{r}$, we still have the following relation [after rearranging Eq. (1)]:

$$F_N=mg\cos\theta-\frac{mv^2}{R}. \tag{4}$$

The work-energy theorem yields the following relation, where W is the work done on the block due to friction:

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=mgR(1-\cos\theta)+W. \tag{5}$$

To proceed we need to calculate W. Recall that the force of friction opposes the direction of motion with magnitude μF_N, giving

$$W=\int \boldsymbol{F}_f\, d\boldsymbol{r}=(-\mu R)\int F_N\, d\theta. \tag{6}$$

If we substitute $\boldsymbol{F}_N$ from Eq. (4), we have

$$W=(-\mu R)\int\left(mg\cos\theta-\frac{mv^2}{R}\right)d\theta. \tag{7}$$

We now return to Eq. (5) and substitute Eq. (7) for W and then divide both sides by m. That is, the height at which the block leaves the sphere is independent of its mass.

$$v^2-v_0^2=2gR(1-\cos\theta)+(-2\mu R)\int\left(g\cos\theta-\frac{v^2}{R}\right)d\theta. \tag{8}$$

We proceed by solving Eq. (8) for v. As in the no friction case, we will solve Eq. (8) for $z(\theta)=v^2$, whose positive square root can be taken to give v. Thus we rewrite Eq. (8) in terms of $z(\theta)$:

$$z(\theta)-z_0=2gR(1-\cos\theta)+(-2\mu gR)\times\int_0^\theta\left(\cos\theta-\frac{z(\theta)}{gR}\right)d\theta. \tag{9}$$

IV. APPROXIMATE SOLUTION WITH FRICTION

The solution to Eq. (9) can be approximated by assuming that μ is small and that there exists a series expansion for z in powers of μ about $\mu=0$. We solve for the first few terms of this series expansion:

$$z\approx z|_{\mu=0}+\left.\frac{\partial z}{\partial\mu}\right|_{\mu=0}\mu+\frac{1}{2!}\left.\frac{\partial^2 z}{\partial^2\mu}\right|_{\mu=0}\mu^2+\dots, \tag{10}$$

or

$$z(\theta,\mu)\approx z|_{\mu=0}+a\mu+b\mu^2+\dots, \tag{11}$$

where a and b are functions of θ. The relation obtained after substituting the right-hand side of Eq. (11) for z in Eq. (9) is valid for all μ only if

$$z|_{\mu=0}=2gR(1-\cos\theta)+z_0. \tag{12}$$

We can now solve for the function $a(\theta)$ in the series expansion of Eq. (11), giving

$$a=(-2gR\sin\theta)+2\int_0^\theta(2gR(1-\cos\theta)+z_0)\,d\theta \tag{13a}$$

$$=4gR\theta-6gR\sin\theta+2z_0\theta. \tag{13b}$$

Given $a(\theta)$ we can solve for $b(\theta)$:

$$b=2\int a\,d\theta=4gR(\theta^2+3\cos\theta-3)+2z_0\theta^2. \tag{14}$$

We can continue this process for an arbitrary number of terms, finding each successive term by multiplying the definite integral of the result from the previous iteration by two. For example,

$$c=2\int b\,d\theta,\quad d=2\int c\,d\theta, \tag{15}$$

where the approximate solution for v is given by

$$v\approx\sqrt{z_0+2gR(1-\cos\theta)+\mu(4gR\theta-6gR\sin\theta+2z_0\theta)+\cdots}. \tag{16}$$

V. EXACT SOLUTION WITH FRICTION

Equation (9) can also be solved exactly. This solution is cited in an analysis of the speed of a sled sliding through a valley in Ref. 3. Here we provide some details to allow a quantitative comparison with Eq. (17). We differentiate both sides of Eq. (9) with respect to θ and rearrange terms to obtain

$$\frac{dz}{d\theta} + (-2\mu)z = (2gR \sin\theta - 2\mu gR \cos\theta). \tag{17}$$

Equation (17) is a first-order, linear differential equation of the form

$$\frac{dy}{dx} + p(x)y = q(x), \tag{18}$$

with $y=z$ and $x=\theta$. Equation (18) is solvable by introducing the integrating factor[4] $f(x)=e^{\int p(x)\,dx}$. The following equation holds generally for this choice of f and is solvable by integrating both sides with respect to x:

$$\frac{d}{dx}[f(x)y] = f(x)q(x). \tag{19}$$

This method is now applied to solve Eq. (17). We choose the integrating factor

$$f(\theta) = e^{\int(-2\mu)\,d\theta} = e^{-2\mu\theta}. \tag{20}$$

In analogy to Eq. (19), we have

$$\frac{d}{d\theta}[f(\theta)z] = f(\theta)q(\theta), \tag{21}$$

$$\frac{d}{d\theta}[e^{-2\mu\theta}z] = e^{-2\mu\theta}(2gR\sin\theta - 2\mu gR\cos\theta), \tag{22}$$

$$z = \frac{2gR(2\cos\theta\mu^2 - 3\sin\theta\mu - \cos\theta)}{4\mu^2+1} + Ce^{2\mu\theta}. \tag{23}$$

Equation (23) is the solution for z with C as an integration constant. We can solve for this constant with $z|_{\theta=0}=v_0^2$:

$$C = v_0^2 + gR\left(\frac{3}{4\mu^2+1} - 1\right). \tag{24}$$

If we replace C in Eq. (23) by its solution in Eq. (24), we obtain the following solution for $z=v^2$:

$$z = \frac{2gR(2\cos\theta\mu^2 - 3\sin\theta\mu - \cos\theta)}{4\mu^2+1} + \left(z_0 + gR\left[\frac{3}{4\mu^2+1} - 1\right]\right)e^{2\theta\mu}. \tag{25}$$

Equivalently, z can be replaced by v^2 in Eq. (25), which can then be solved for v:

$$v = \sqrt{\frac{2gR(2\cos\theta\mu^2 - 3\sin\theta\mu - \cos\theta)}{4\mu^2+1} + \left(v_0^2 + gR\left[\frac{3}{4\mu^2+1} - 1\right]\right)e^{2\theta\mu}}. \tag{26}$$

If the argument of the square root is positive, Eq. (26) gives the speed of the particle as a function of its angular displacement θ.

VI. COMPARISON OF PERTURBATIVE AND EXACT SOLUTIONS

By setting $\mu=0$ in Eq. (25) we can verify that z reduces to Eq. (2). The exact solution for z can be expanded in powers of μ to give the same terms that were obtained from the perturbation expansion for z:

$$z \approx z_0 + \left(\frac{\partial z}{\partial\mu}\bigg|_{\mu=0}\right)\mu + \left(\frac{1}{2!}\frac{\partial^2 z}{\partial^2\mu}\bigg|_{\mu=0}\right)\mu^2 + \cdots, \tag{27}$$

$$\frac{\partial z}{\partial\mu} = \frac{1}{(4\mu^2+1)^2}[4e^{2\theta\mu}gR\theta - 6gR\sin\theta + 2e^{2\theta\mu}\theta z_0 + O(\mu)]. \tag{28}$$

If $\partial z/\partial\mu$ is evaluated at $\mu=0$, the resultant expression is the same as that obtained for a in Eq. (13).

Figures 2 and 3 show plots of the speed of the block as a function of the angular displacement from its starting position on the sphere. Figure 2 illustrates the behavior for low friction $\mu=0.5$, and Fig. 3 illustrates the behavior for $\mu=1.0$. The bold curves represent the exact solution, and the

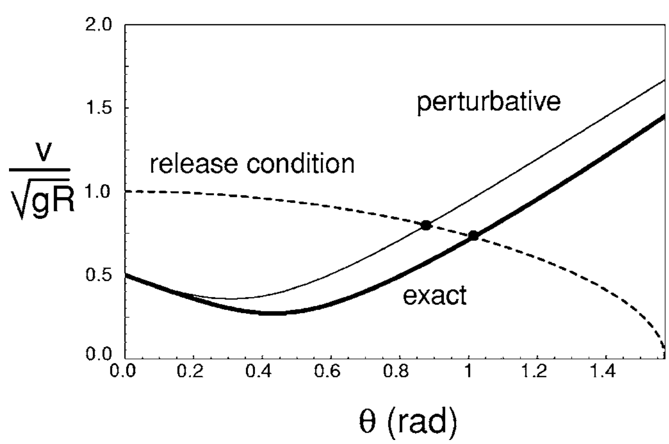

Fig. 2. Solutions for the velocity v of a block as a function of its angular displacement θ from the top of a sphere with free fall acceleration g and the effect of friction with a coefficient of friction $\mu=0.5$. The velocity is plotted in units of $\sqrt{gR}$, the critical velocity for the particle to lose contact with the sphere at $\theta=0$. The dashed curve labeled "release condition" gives the critical velocity for the block to leave the sphere. The thin curve is a perturbative solution to the equation of motion for v^2 to first order in μ [see Eq. (16)]. The bold curve is the exact solution to the equation of motion, Eq. (26). The intersections of these curves with the dashed curve give the predicted release angles. Note that in either solution the block first slows under the influence of friction before speeding up due to the gravitational acceleration.

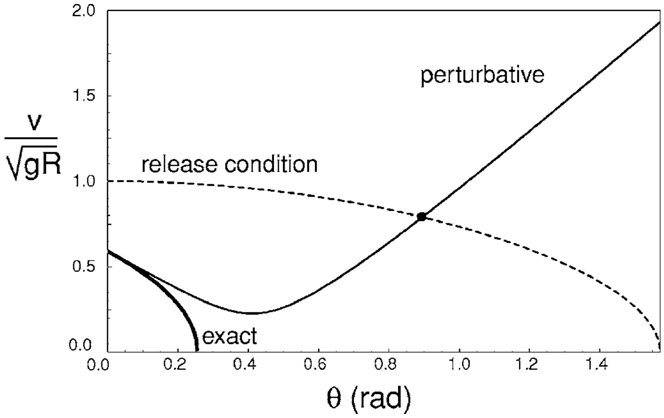

Fig. 3. Same as Fig. 2 with $\mu=1$. Here the first-order perturbative solution predicts that the block slows and then speeds up before reaching the release point given by the intersection with the dashed curve. The exact solution has the block slowing and coming to rest on the surface of the sphere.

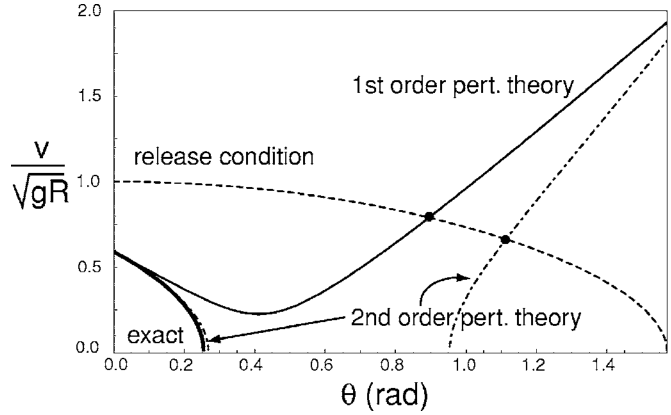

Fig. 4. Same as Fig. 3. The second-order perturbative solution (dashed curve) is in reasonable agreement with the exact solution (bold curve) for small angular displacement, but contains an anomalous branch at larger angles that satisfies the release condition at larger displacement. A final state with the block stationary on the surface of the sphere is inaccessible to any order of perturbation theory in μ.

thin curves represent the solution corresponding to the first-order approximation. The release condition for the particle is indicated as the dashed curve, so that the particle leaves the sphere at the angle at which the dotted line intersects the solution curve. The block is predicted to stop at the angle at which the solution curves intersect the θ axis (at which the velocity of the block is zero).

In both the exact and approximate solutions for low friction the particle initially slows under the influence of friction before speeding up due to the gravitational force. As expected, the speed at a given angle without friction exceeds the speed with friction, and the particle remains in contact with the sphere longer. Because the particle initially slows under the influence of friction, v_0 must exceed a critical value for the particle to avoid stopping on the surface. The exact solution is undefined for angles greater than the angle at which the block has already stopped.

For $\mu \approx 1$ the approximation becomes very poor. In Fig. 3 the approximation predicts that the block will slow down, then speed up, and then leave the sphere at $\theta \approx 0.87$ rad. The exact solution for the velocity of the block predicts that the block will slow down, then stop and remain at $\theta \approx 0.26$ rad.

By extending the perturbation theory to higher order in μ, we can improve the agreement between the approximate and exact solutions over the range where the block is slipping. This improvement is illustrated in Fig. 4, which shows that better agreement for $\mu \approx 1$ is obtained by extending the expansion to second order in μ. However, at this order the perturbation theory still contains an anomalous branch at larger θ where the block satisfies the release condition. Note that when the block comes to rest on the sphere, the coefficient of friction undergoes a nonanalytic change, reverting to the coefficient of static friction. This feature cannot be related to our unperturbed solution at any finite order of the perturbation theory, which treats only the effect of sliding friction.

VII. SUGGESTED PROBLEMS

Problem 1. Perform the iterated series for z in Eq. (15) to obtain $z(\mu)$ through second order in μ. Verify that your result agrees with the quadratic dependence on μ obtained from the exact solution in Eq. (25).

Problem 2. The exact solution can be used to construct a phase diagram in the μ-v_0 plane separating the region where the block sticks to the sphere from the region where the block continues to slip on the sphere and is released. The critical line is the locus of zeros of z in Eq. (25). Write a computer program to evaluate $z(\mu, z_0)$ and search for its zero crossings to construct the phase diagram.

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