Weak Input-to-State Stability Properties for Navigation Function Based Controllers

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***Abstract*— Due to topological constraints, Navigation Functions, are not, except from trivial cases, equivalent to quadratic Lyapunov functions, hence systems based on Navigation Functions cannot directly accept an Input-to-State stability (ISS) characterization. However a relaxed version of Input-to-State stability, namely almost global ISS (aISS), is shown to be applicable. The proposed framework provides compositional capability for navigation function based systems. Cascade as well as feedback interconnections of aISS navigation systems are shown to also possess the aISS property under certain assumptions on the interconnections. Several simulated examples of navigation systems are presented to demonstrate the effectiveness of the proposed scheme.**

I. Introduction

Navigation Functions proposed by Koditschek and Rimon [3] are a valuable tool for robotic navigation due to their closed form feedback structure and their amenability to analysis from a control theoretic perspective. It was shown in [3] that topological constraints prohibit the construction of a globally attracting equilibrium state and smooth vector fields on any sphere world must have at least as many saddles as there are obstacles. For Navigation Functions the saddle points are sets of measure zero and the set of initial conditions that are attracted to them are also a set of measure zero. Hence the asymptotic stability achieved by Navigation Function (NF) based systems is of almost global nature, implying the existence of sets of measure zero that are not attracted to the minimum. This implies that a Lyapunov function candidate for an NF based system is not equivalent to a quadratic Lyapunov function because the level sets of an NF $\varphi(\cdot)$ beyond a certain value $\varphi_{\mathcal{C},\min} = \min_{q \in \mathcal{C} \setminus \{\mathbf{0}\}} \{\varphi(q)|\}$, where $\mathcal{C} \subset R^n$ is the set of critical points, are not homotopically equivalent to the $n-1$ sphere ($\mathbb{S}^{n-1}$).

While navigation function based controllers are attractive for guiding vehicles to their destinations in a known, obstacle-cluttered field, they do not lend themselves to composition with other controllers. For example, it is desirable to be able to compose navigation function based controllers with reactive controllers for avoiding unmodeled obstacles[6]. In systems with human operator oversight, it is often necessary to allow the composition with human operator inputs, so that the operator can locally modify the trajectory of an otherwise autonomous system. In multi-robot systems, it may be necessary to loosely couple multiple robots each navigating independently to its destination.

In this paper, we study the composition of inputs arising from navigation function based controllers with inputs from other sources using the framework of interconnected systems. The central question is if navigation function based systems can be connected to other systems while still exhibiting the desirable property of almost global asymptotic stability. The concept of input-to-state stability introduced by Sontag [11], [12] provides a framework under which such stability like behaviors of non-linear systems can be studied. Since navigation functions are not equivalent to quadratic Lyapunov functions, the notion of ISS cannot be directly applied to them. However an almost global notion of input to state stability introduced by Angeli [1] is applicable to almost globally asymptotically stable systems as is the case of navigation function based systems. The proposed framework in this paper is based on this almost global notion of ISS. While there are established results regarding cascade interconnections of aISS with almost GAS systems [1], results regarding more general interconnections of aISS systems are currently an open research topic. To achieve the propagation of the aISS property of navigation function based systems through general interconnections, we have introduced two input ports to the navigation function based system. The first input port is a general interconnection port and the second port is used in the case of feedback interconnections. We prove that a feedback interconnection of navigation function based aISS systems still possess the aISS property as long as at least one subsystem in the feedback interconnection loop is connected through the second input port. These properties provide compositionality for arbitrary interconnection topologies between aISS systems with the resulting systems possessing the aISS property.

The literature on applications of notions of Input-to-State stability to the motion planning domain is rather restricted, mainly due to the fact that the ISS property is not directly applicable to almost GAS systems, that is oftentimes the case for navigation systems. However the concept of ISS has been successfully applied to the formation control problem [15] where a leader following scheme is considered and an ISS based notion of Leader-to-Formation stability is introduced. Also results characterizing the stability properties of a certain class of non-holonomic systems have been presented in [13]

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where it is shown that those systems enjoy the ISS property.

In this paper we exploit results on Dual Lyapunov methodologies [8] along with recent results regarding the density function of navigation function based systems [4] to construct an asymptotic gain characterization for navigation function based systems. This enables the aISS characterization of those systems as well as the analysis of their interconnections and the propagation of the aISS property.

The rest of the paper is organized as follows: Section II reviews several properties of the navigation functions. Section III reviews some results regarding aISS systems. Section IV presents two classes of navigation function based systems and analyzes their aISS properties. Section V presents simulation results and Section VI concludes the paper.

II. Navigation Function Preliminaries

Navigation Functions (NFs) are real valued maps, realized through cost functions, the negated gradient field of which is attractive towards the goal configuration and repulsive with respect to obstacles. Considering a trivial system described kinematically as

$$\dot{q} = u$$

the basic idea behind navigation functions is to use a control law of the form

$$u = -\nabla\varphi(q)$$

where $\varphi(q)$ is a navigation function, to drive the system to its destination (figure 1). It has been shown (Koditschek and Rimon [3]) that strict global navigation (i.e. with a globally attracting equilibrium state) is not possible and a smooth vector field on any sphere world, which has a unique attractor, must have at least as many saddles as obstacles. It has been shown [3] that navigation properties are invariant under diffeomorphisms hence any world that can be diffeomorphically transformed to a sphere world can accept a navigation function [10], [9], [14]. Recent extensions of Navigation Functions to the multiple disk shaped robots case have been independently proposed by the first author [5] and by [2].

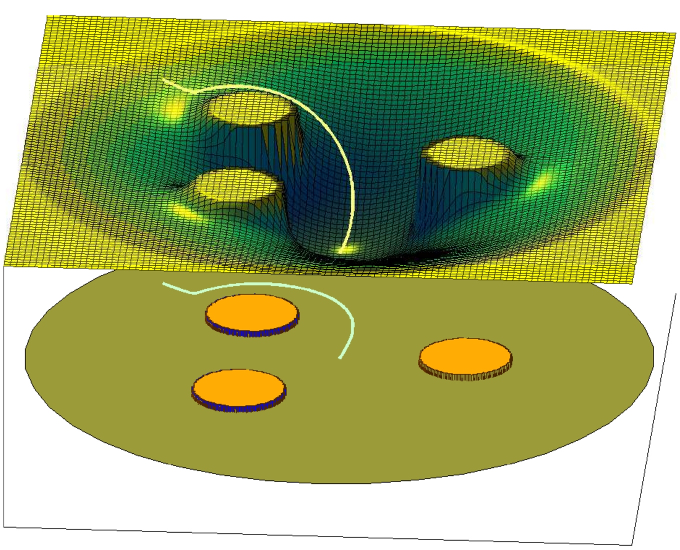

Fig. 1. Navigation Function with three obstacles and the resulting gradient following path

Formally a Navigation Function is defined as follows:

Definition 1: [3] Let $\mathcal{F} \subset E^n$ be a compact connected analytic manifold with boundary. A map $\varphi : \mathcal{F} \to [0, 1]$, is a navigation function if it is:

1) Analytic on $\mathcal{F}$
2) Polar on $\mathcal{F}$, with minimum at $q_d \in \overset{\circ}{\mathcal{F}}$
3) Morse on $\mathcal{F}$
4) Admissible on $\mathcal{F}$

The intuition behind property 1 of Definition 1 is that it is preferable to have an analytic form of the gradient of the vector field to encode actuator commands instead of "patching together" closed form expressions on different portions of space, in order to avoid branching and looping in the control algorithm.

A function φ is called polar if it has a unique minimum on $\mathcal{F}$. By using smooth vector fields one cannot do better than have almost global navigation [3]. By using a polar function on a compact connected manifold with boundary, all initial conditions will either be brought to a saddle point or to the unique minimum: $\mathbf{q}_d$.

A scalar valued function φ is called a Morse function if all its critical points (zero gradient vector field) are non-degenerate, that is its Hessian at the critical points is full rank. The requirement in Definition 1 that a navigation function must be a Morse function, establishes that the initial conditions that bring the system to saddle points are sets of measure zero [7]. In view of this property, all initial conditions away from sets of measure zero are brought to q_d.

The last property of definition 1 requires that the Navigation Function is uniformly maximal across the workspace boundary, that is $\partial\mathcal{F} = \varphi^{-1}(1)$. This property guarantees that the resulting vector field is transverse to the boundary of $\mathcal{F}$. This establishes the safety properties of the Navigation Function, that the system will be collision free.

Some useful properties of Navigation Functions are provided by the following:

Proposition 1: [3] Consider an autonomous system of the form $\dot{x} = -\nabla\varphi(x)$ where φ is a navigation function defined on a compact Riemannian manifold $\mathcal{F}$. Then q_d is asymptotically stable, a.e. (almost everywhere) on $\mathcal{F}$

Remark 1: The almost everywhere condition implies the existence of sets of measure zero of initial conditions that are not attracted to q_d. Those sets are exactly the sets of initial conditions with positive limit set the saddle points.

The following result gives us bounds for the minimum value of the norm of the gradient of a Koditschek-Rimon (K-R) [3] navigation function across the workspace boundary:

Lemma 1: Let $\varphi(q)$ be a navigation function on a sphere world. Assume an K-R construction of $\varphi(q)$. Then it holds that:

$$\|\nabla\varphi(q)\| \geq 2\frac{r_{\min}\left(\left(r_{\min}+d_{o,\min}\right)^2-r_{\min}^2\right)^{n_O-1}}{k\left(r_w+\|q_d\|\right)^k} \triangleq \mathcal{N}_{\min}$$

for all $q \in \partial\mathcal{F}$, where $r_w, r_{\min}, d_{o,\min}, q_d$ and n_O are the workspace radius, the minimum obstacle radius, the minimum distance between obstacles, the destination configuration and the number of obstacles, respectively. k is the tuning parameter used for the K-R construction.

Proof: See Appendix B ■

III. ISS FOR ALMOST GAS SYSTEMS

In this section we will review some results from the ISS literature.

Definition 2: [12] Consider a system of the form $\dot{x} = f(x,u)$ evolving in finite dimensional spaces $\mathbb{R}^n$ with inputs $u \in \mathbb{R}^m$ that are measurable essentially locally bounded. The map $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0) = 0$. The system is input to state stable (ISS) if

$$\|x(t)\| \leq \beta\left(\|x(0)\|, t\right) + \gamma\left(\|u\|_\infty\right) \quad (1)$$

for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and for all $t \geq 0$. Operator $\|u\|_\infty$ denotes the essential supremum of a function $u(\cdot)$.

Unfortunately almost GAS systems cannot be characterized by the above $\beta+\gamma$ type of estimate [1]. An equivalent approach in terms of asymptotic gains is more suitable for such systems and gives rise to an almost global definition of input to state stability:

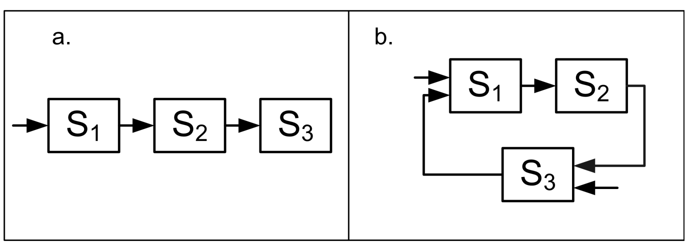

Fig. 2. Examples of a. Cascade interconnections and b. Feedback interconnections

Definition 3: [1] A system $\dot{x} = f(x,u)$ evolving on a smooth manifold M with $f : M \times \mathbb{U} \rightarrow TM$ a locally Lipschitz manifold map satisfying $f(x,u) \in T_xM$ for all $x \in M$ and all $u \in \mathbb{U} \subset \mathbb{R}^m$ is almost ISS (aISS) with respect to an invariant compact set $\mathcal{A} \subset M$, if $\mathcal{A}$ is locally asymptotically stable and

$$\forall u, \forall\ a.\ a.\ \xi \in M,\ \limsup_{t\rightarrow+\infty} |x(t,\xi,u)|_{\mathcal{A}} \leq \gamma\left(\|u\|_\infty\right) \quad (2)$$

where $\gamma \in \mathcal{K}$ and $|\cdot|_{\mathcal{A}}$ denotes the standard point to set distance.

While ISS properties are propagated through cascade interconnections and under small gain conditions for feedback interconnections (Figure 2), propagation of aISS properties through cascades and feedback interconnections is currently an open research topic. A weaker result was established by [1] regarding the interconnection of an almost GAS with an aISS system stating that the resulting system is almost GAS:

Theorem 1: [1] Consider the cascaded system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(y) \end{aligned} \quad (3)$$

with state $z = \left[x^T, y^T\right]^T \in M \times N$ where N a smooth manifold. Assume that f and g satisfy $f(0_M, 0_N) = 0$ and $g(0_N) = 0$ for some points $0_M \in M$ and $0_N \in N$. Let the x-subsystem be almost ISS with respect to the equilibrium 0_M and the input y and the y-subsystem be almost be almost GAS at 0_N. Then the interconnection (3) is almost GAS at $0_{M\times N} := 0_M \times 0_N$

IV. NAVIGATION FUNCTION BASED SYSTEMS

In this section we present some special classes of navigation function based systems, which under certain conditions, enjoy propagation of the aISS property through their interconnections.

We will be primarily concerned with systems that are trivially described by first order kinematic models. Without loss of generality the treatment is performed on the sphere model world where the destination configuration is considered to be the origin.

Consider the system:

$$S: \quad \dot{x} = -K\nabla\varphi(x) + u(t) \quad (4)$$

where $\varphi(x)$ is a navigation function, K a gain and $x \in \mathcal{F}$ where $\mathcal{F} \subset E^n$ a compact Riemannian manifold denoting the system's workspace and the input $u : [0,\infty) \rightarrow \mathcal{U} \subset R^n$.

We will initially study system S since this is the simplest and most frequently encountered NF based system with external input. Based on the properties of navigation functions, we can state the following:

Proposition 2: Let $u_\infty = \sup_{t\geq 0}\{\|u(t)\|\}$. Then any trajectory of S satisfies

$$x(t, x_0, u(t)) \in \mathcal{F}$$

for all $t \geq 0$, $x_0 \in \mathcal{F}$ as long as $u_\infty < K\mathcal{N}_{\min}$ with $\mathcal{N}_{\min}$ as defined in Lemma 1

Proof: See Appendix C ■

Remark 2: By using a scaling function, we can construct the system:

$$S_\sigma: \quad \dot{x} = -K\nabla\varphi(x) + u_\infty\, sat\left(\frac{u(t)}{u_\infty}\right) \quad (5)$$

which satisfies the conditions of Proposition 2 by construction. Function $sat(\cdot)$ is a vector scaling function defined in the Appendix A-1

The following result provides an input to state characterization of the stability of the system S_σ:

Proposition 3: System S_σ is aISS with respect to the origin.

Proof: See Appendix D ■

Let us denote with $S_1 \rightharpoonup S_2$ the cascade interconnection of the output of system S_1 to the input of the system S_2.

We can state the following regarding the interconnections of systems of type S_σ:

Proposition 4: Let $S_{\sigma_1}, S_{\sigma_2} \in S_\sigma$. Then the system formed by the cascade interconnection $S_{\sigma_1} \rightharpoonup S_{\sigma_2}$ is aISS with respect to the origin.

Proof: See Appendix E ■

Remark 3: It can be easily verified that the aISS properties of the interconnection $S_{\sigma_1} \rightharpoonup S_{\sigma_2}$ hold even if we allow external inputs to be added to the second system, i.e. $u_2 = x_1 + v_2$ where $v_2 \in \mathbb{U}$. Also the aISS property is conserved in the more general case where the interconnection is of the form $u_2 = h(x_1) + v_2$ where $\|h(x_1)\|$ is of class $\mathcal{K}$ since composition of class $\mathcal{K}$ functions are still of class $\mathcal{K}$.

Even though the system S_σ maintains the aISS property through cascade interconnections as demonstrated in Proposition 4, this is not the case for feedback interconnections. This is mainly due to the fact that due to topological obstructions, the interconnected system might become trapped away from the destination configuration. To this extend we propose the following construction of a navigation function based vector field with two input ports. Consider the system:

$$S_\pi: \quad \dot{x} = -K\nabla\varphi + (1-\varphi)u_1 + \sigma_\varepsilon\left(-u_2^T\nabla\varphi\right)u_2 \quad (6)$$

where function σ_ε is a smooth function defined in the Appendix A-2 and $u_1, u_2 \in \mathbb{U}$. The first observation is that $S_\pi \supset S_\sigma$[1] since the term $(1-\varphi)u_1$ can take any value away from the workspace boundary. We can state the following:

Proposition 5: The trajectories of system S_π for any measurable u_1, u_2 satisfy:

$$x(t, x_0, u_1(t), u_2(t)) \in \mathcal{F}$$

for all $t \geq 0$, $x_0 \in \mathcal{F}$. Moreover the system S_π with $u_1 \equiv 0$ is almost GAS for all $u_2 \in \mathbb{U}$

Proof: See Appendix F ■

We have the following characterization of the stability of system S_π

Proposition 6: System S_π is aISS with respect to the origin.

Proof: See Appendix G ■

We will use the previously defined symbol to denote cascade interconnections through the input port u_1, i.e. for the systems $S_1, S_2 \in S_\pi$, $S_1 \rightharpoonup S_2$ denotes a cascade interconnection of the output of system S_1 to the input port u_1 of system S_2. We use the notation $S_1 \rightharpoondown S_2$ to denote a cascade interconnection of the output of system S_1 to the input port u_2 of system S_2. In case of feedback interconnections (see figure 3) an arrow edge pointed in the inside of the loop indicates an interconnection to port u_2 e.g. the interconnection between systems S_4 and S_5 in figure 3 is $S_5 \rightharpoondown S_4$. The system shown in figure 3 excluding the connection inside the dotted box is represented by the following representation:

$$(S_1 \rightharpoonup S_2)\,(S_2 \rightharpoonup S_3 \rightharpoonup S_5 \rightharpoondown S_4 \rightharpoonup S_2)\,(S_5 \rightharpoonup S_6)$$

[1]The subset relation between systems implies that the trajectories of system S_1 are included in the trajectories of system $S_2 \supset S_1$

The connection inside the dotted box adds a human input $H \rightharpoonup S_4$. Note that the symbol $\rightarrow$ will be used to denote an arbitrary type of interconnection.

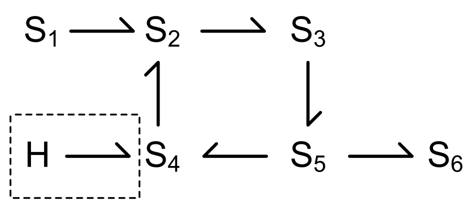

Fig. 3. A complex interconnection including a feedback loop and cascade interconnections.

We can now state the following regarding the interconnections of systems of type S_π:

Proposition 7: Let $S_1, S_2 \in S_\pi$. Then the systems formed by the cascade interconnections $S_1 \rightharpoonup S_2$, $S_1 \rightharpoondown S_2$ and $S_1 \overset{\rightharpoonup}{\rightharpoondown} S_2$ are aISS.

Proof: See Appendix H ■

Regarding feedback interconnections, the aISS property can be propagated through them under some assumptions:

Proposition 8: Let $S_i \in S_\pi$, $i \in \{1 \ldots n\}$. Then the feedback interconnection $S_1 \rightarrow S_2 \rightarrow \ldots \rightarrow S_1$ is aISS if at least one interconnection $\rightarrow$ is of type $\rightharpoondown$.

Proof: See Appendix I ■

V. Simulation Results

To verify the effectiveness of our algorithms we have setup two simulations with 6 S_π type systems in sphere worlds. The system interconnection is the one depicted in figure 3 and is represented by the string:

$$(S_1 \rightharpoonup S_2)\,(S_2 \rightharpoonup S_3 \rightharpoonup S_5 \rightharpoondown S_4 \rightharpoonup S_2)\,(S_5 \rightharpoonup S_6)$$

All systems had three obstacles in their workspace. The workspace radius was set to $1m$. The workspace obstacle characteristics for each system are summarized in Table I:

TABLE I

Obstacle characteristics

S_1, S_3, S_5		S_2, S_4, S_6	
Location	Radius	Location	Radius
(0.4m, 0.2m)	0.2m	(0.2m, 0.4m)	0.2m
(-0.4m, 0.2m)	0.2m	(0.2m, -0.4m)	0.2m
(0.0m, -0.6m)	0.2m	(-0.6m, 0.0m)	0.2m

The initial conditions for each system are shown in Table II: As can be seen from figure 4 the systems are driven safely

TABLE II

Initial Conditions

x_{0_1}	x_{0_2}	x_{0_3}	x_{0_4}	x_{0_5}	x_{0_6}
$0.7m$	$-0.5m$	$-0.7m$	$-0.9m$	$0.01m$	$0.5m$
$0.6m$	$-0.7m$	$0.6m$	$0.01m$	$-0.9m$	$0.7m$

to their destinations without colliding with the workspace obstacles. Figure 5 shows the distance to the destination vs time for each system. We can clearly see the influence of system S_4 on systems S_2 and S_3 and of system S_5 on system S_6. Eventually, in the absence of external inputs all the systems converge to the origin.

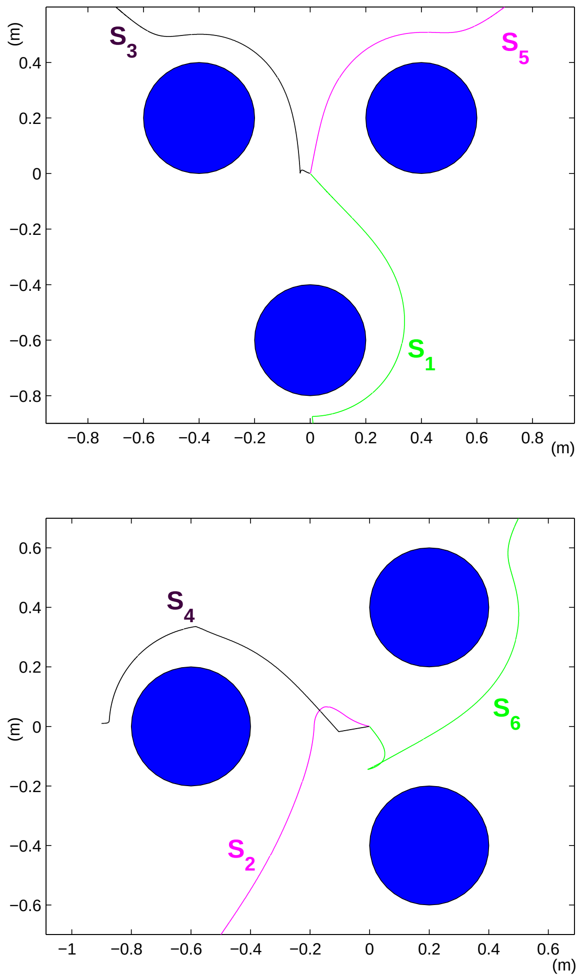

Fig. 4. Trajectories of the interconnected system

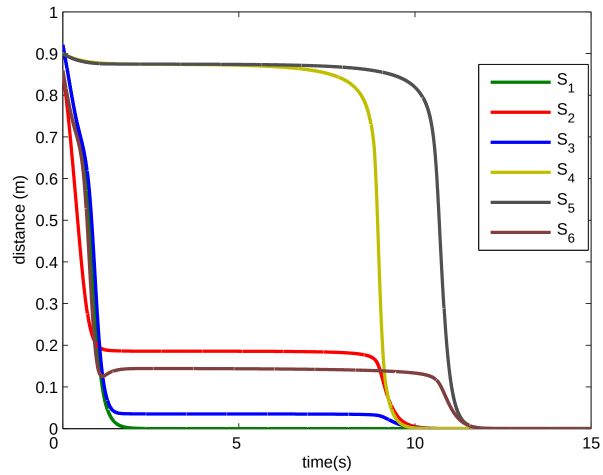

Fig. 5. Distance to the goal vs time

In the second simulation, we consider a situation in which the human operator decides to locally modify the trajectory of the system S_4. Accordingly, an external joystick input is connected to the input port 1 of system S_4. The trajectories of the system are shown in figure 6. As can be seen at a certain time instant the human operator decided that he wanted the system to avoid the nearby obstacle by navigating to the other side of it. As can be seen the external input did not destabilize the interconnected system and the objective of the human operator was achieved. Observe how the external input to system S_4 affected the rest of the systems by comparing the results to the results from the first simulation. As we can see and in this case, the interconnected system was successful in converging safely to its destination.

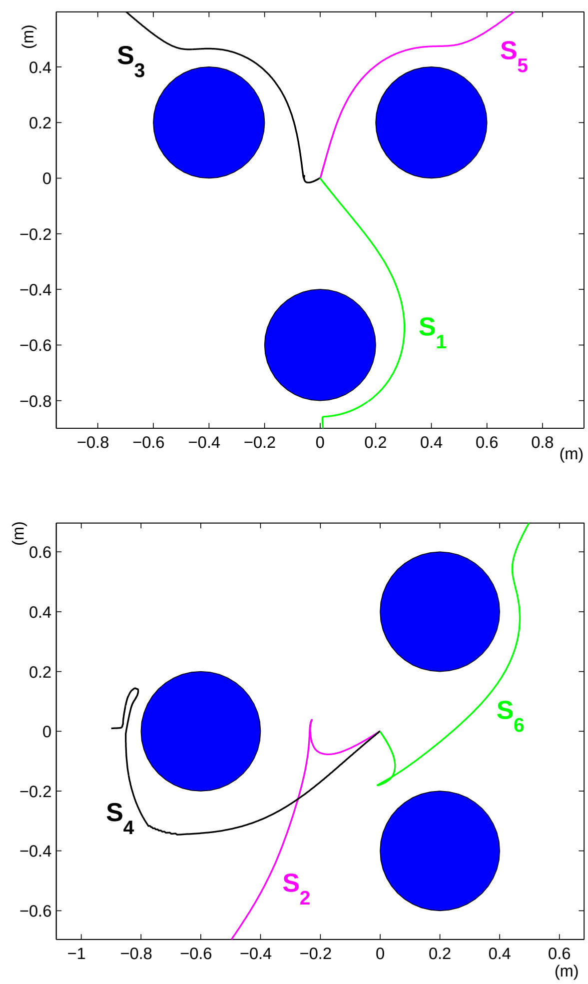

Fig. 6. Trajectories of the system when a joystick input from a human operator is connected to input port 1 of the subsystem S_4

VI. Conclusions

We have presented a methodology to interconnect Navigation Function based systems using both cascade and feedback interconnection architectures. This methodology is shown to have direct applications to coordinating multiple autonomous robots while accommodating human inputs that can be used to locally modify inputs to the system.

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Appendix

A. Definitions

The $sat(\cdot)$ function is defined as:

$$sat(x) \triangleq \begin{cases} x & \|x\| \leq 1 \\ \frac{x}{\|x\|} & \|x\| > 1 \end{cases} \tag{A-1}$$

The $\sigma_\varepsilon(\cdot)$ function is defined as:

$$\sigma_\varepsilon(x) \triangleq \frac{\upsilon(x)}{\upsilon(x) + \upsilon(\varepsilon - x)} \tag{A-2}$$

where $\varepsilon > 0$ and the function $\upsilon(\cdot)$ is defined as: $\upsilon(t) \triangleq \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$

Remark 4: Both $\sigma_\varepsilon(\cdot)$ and $\upsilon(\cdot)$ are smooth functions as can be verified by direct calculation. Function $\sigma_\varepsilon(x)$ is identically zero for $x \leq 0$ and identically 1 for $x \geq \varepsilon$.

B. Proof of Lemma 1

Proof: The gradient of a navigation function is given by:

$$\nabla\varphi(q) = \frac{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}} \nabla\gamma_d - \gamma_d \nabla\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}}{\left(\gamma_d^k + \beta\right)^{\frac{2}{k}}}$$

Over the boundary of $\mathcal{F}$ it holds that $\beta = 0$, hence

$$\nabla\varphi(q)_{|q \in \partial\mathcal{F}} = \frac{\gamma_d \nabla\gamma_d - \gamma_d \nabla\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}}{\gamma_d^2}$$

which after expanding the terms in the gradient and simplifying, becomes:

$$\nabla\varphi(q)_{|q \in \partial\mathcal{F}} = -\frac{\nabla\beta}{k\gamma_d^k}$$

Function β can be expressed as the product $\beta = \beta_i \bar{\beta}_i$ where β_i is the distance to the obstacle function that we are considering and $\bar{\beta}_i$ is the product of the distance functions from the rest of the obstacles. When a robot is over the workspace boundary, it is touching obstacle i, hence $\beta_i = 0$. For φ to be a navigation function in a sphere world, there must be a minimum non-zero distance between obstacles. We denote this minimum distance between obstacles as $d_{o,\min}$. By substituting $\beta = \beta_i \bar{\beta}_i$ we get

$$\nabla\varphi(q)_{|q \in \partial\mathcal{F}} = -\frac{\bar{\beta}_i \nabla\beta_i}{k\gamma_d^k}$$

Taking the norm of both sides we get:

$$\left\|\nabla\varphi(q)_{|q \in \partial\mathcal{F}}\right\| = \frac{\bar{\beta}_i}{k\gamma_d^k} \|\nabla\beta_i\|$$

Substituting $\|\nabla\beta_i\| = 2\|q - q_i\|$ and taking the minimum of the gradient norm across the workspace boundary, we get:

$$\min_{q \in \partial\mathcal{F}} \|\nabla\varphi(q)\| = 2\frac{\min\limits_{q \in \partial\mathcal{F}} \left(\bar{\beta}_i \|q - q_i\|\right)}{\max\limits_{q \in \partial\mathcal{F}} \left(k\gamma_d^k\right)}$$

We have that $\min\|q - q_i\| = r_{\min}$ where $r_{\min}$ is the minimum radius of an obstacle and $\min\bar{\beta}_i = \left(\left(r_{\min} + d_{o,\min}\right)^2 - r_{\min}^2\right)^{n_O - 1}$, where n_O is the number of obstacles in the workspace. Also $\max\limits_{q \in \partial\mathcal{F}} \gamma_d^k = \left(r_w + \|q_d\|\right)^{2k}$, where r_w is the workspace radius and q_d is the destination configuration. Then

$$\min_{q \in \partial\mathcal{F}} \|\nabla\varphi(q)\| = 2\frac{r_{\min}\left(\left(r_{\min} + d_{o,\min}\right)^2 - r_{\min}^2\right)^{n_O - 1}}{k\left(r_w + \|q_d\|\right)^{2k}}$$

■

C. Proof of Proposition 2

Proof: Since $\varphi(\cdot)$ is a navigation function, one of the properties required by its definition [3] is that it is uniformly maximal over the boundary of $\mathcal{F}$. This is sufficient to guarantee that the negated gradient vector field is transverse over the boundary of $\mathcal{F}$. Assume that $\overrightarrow{n}$ is the unit perpendicular vector over $\partial\mathcal{F}$ pointing in the internal of $\mathcal{F}$ then at any $q \in \partial\mathcal{F}$ we have that

$$\begin{aligned}(-K\nabla\varphi(q) + u) \cdot \overrightarrow{n} &= K\|\nabla\varphi(q)\| + u \cdot \overrightarrow{n} \\ &\geq K\|\nabla\varphi(q)\| - u_\infty > 0\end{aligned}$$

The strict inequality shows that the boundary $\partial\mathcal{F}$ is not reachable from initial conditions in the internal $\mathring{\mathcal{F}}$ of $\mathcal{F}$. The existence of a non-zero value of $\mathcal{N}_{\min} = \min_{q\in\partial\mathcal{F}} \|\nabla\varphi(q)\|$ is established by Lemma 1. Hence $\dot{x}$ points inside $\mathcal{F}$ across its boundary and $\mathcal{F}$ is a positive invariant set. ■

D. Proof of Proposition 3

Proof: The proof is based on dual Lyapunov techniques [8]. The concept of combining primal and dual methodologies to derive an asymptotic gain for aISS systems is proposed in [1]. We also make use of a recent result that appears in [4] which states that the navigation function can be used to construct a density function for a "canonical" navigation vector field defined as $f = D_\varphi\nabla\varphi + u$ (see [4] for the definition of D_φ). Results extracted on this vector field can then be transferred to the navigation function based vector field. With φ the navigation function, the density function is given by $\rho = \frac{1}{\varphi^a}$ where a a sufficiently large positive parameter. The dual Lyapunov criterion requires (almost everywhere) positivity of the following:

$$\mathrm{div}\,[\rho f] = -\frac{a}{\varphi^{a+1}}\nabla^T\varphi\,(-KD_\varphi\nabla\varphi + u) - \frac{K}{\varphi^a}\nabla\cdot(D_\varphi\nabla\varphi)$$

The positivity requirement can be satisfied by choosing:

$$\|u\| \leq \frac{aK\nabla^T\varphi D_\varphi\nabla\varphi - K\varphi\nabla\cdot(D_\varphi\nabla\varphi)}{a\|\nabla\varphi\|} \triangleq U(x) \quad \text{(D-1)}$$

The numerator of the above inequality is almost everywhere positive and at the saddle points $\nabla\cdot(D_\varphi\nabla\varphi) < 0$ as is shown in [4]. This implies that there exists a neighborhood $\mathcal{B}_{\delta_i}(x_{s,i})$ of radius δ_i of each saddle point $x_{s,i}$, where $\nabla\cdot(D_\varphi\nabla\varphi) < 0$ due to the smoothness properties of the vector field. Since the origin is a non-degenerate critical point, we can always find a neighborhood of the origin, for which it holds that $\nabla\cdot(D_\varphi\nabla\varphi) > 0$. Let $\mathcal{B}_{\delta_0}(\mathbf{0})$ be the largest spherical neighborhood with radius δ_0 around the origin for which $\nabla\cdot(D_\varphi(x)\nabla\varphi(x)) > 0, \quad \forall x \in \mathcal{B}_{\delta_0}(\mathbf{0})$. Now define the set

$$\mathcal{B}_i = \mathcal{B}_{\delta_0}(\mathbf{0}) \bigcup_{j\neq i} \mathcal{B}_{\delta_j}(x_{s,j})$$

. Since the vector field $\nabla\varphi$ vanishes only at the saddle points and at the destination configuration, for each saddle point i choose δ_i' small enough such that

$$\max_{x\in\partial\mathcal{B}_{\delta_i'}(x_{s,i})} \|\nabla\varphi\| > \min_{x\in\mathcal{F}-\mathcal{B}_i} \|\nabla\varphi\|.$$

Now choose

$$\delta = \min_{i\in\{0,\dots n_s\}} \delta_i$$

where n_s is the number of saddle points. In view of the positivity requirement (D-1) we can see that in each $\mathcal{B}_\delta(x_{s,i})$ region $U(x \in \mathcal{B}_\delta(x_{s,i})) > 0$. Select

$$U_u = \min_{x\in\mathcal{B}_\delta(x_{s,i}),\ \forall i\in\{1,\dots n_s\}} U(x).$$

Now the dual Lyapunov (positivity) requirement will be satisfied near the saddle points, as long as $\|u\| \leq U_u$ there. From the primal Lyapunov analysis we have the requirement: $\nabla^T\varphi\,(-KD_\varphi\nabla\varphi + u) \leq 0$, hence the primal Lyapunov criterion is satisfied by

$$\|u\| \leq \frac{K\nabla^T\varphi D_\varphi\nabla\varphi}{\|\nabla\varphi\|} \triangleq \Phi(x) \quad \text{(D-2)}$$

Now select

$$\Phi_u = \min_{x\in\partial\mathcal{B}_\delta(x_{s,i}),\ \forall i\in\{1,\dots n_s\}} \Phi(x).$$

In view of the above, the primal Lyapunov requirement will be satisfied in the complementary to the saddle proximal region and sufficiently away from the origin as long as $\|u\| \leq \Phi_u$ there. Taking the intersection of the two requirements we have that sufficiently away from the origin the almost everywhere attractivity of the origin will be maintained due to either the primal or the dual Lyapunov criterion, as long as $\|u\| \leq \min\{U_u, \Phi_u\} \triangleq M_u$. Define ν such that $\Phi(x \in \mathcal{B}_\nu'(\mathbf{0})) \leq M_u$ where $\mathcal{B}_\nu'(\mathbf{0})$ is the ball of radius ν around the origin. Construct the function

$$\gamma(x) = \frac{r_w}{\nu^2}x^2$$

where r_w the workspace radius. Then given any measurable input $u(\cdot)$, for almost all initial conditions we have that

$$\forall u\ \forall a.a.\ x_0 \quad \limsup_{t\to\infty} \|x(t, x_0, u)\| \leq \gamma(\|u\|_\infty)$$

due to the compactness of the workspace. As is shown in [4] the rate of convergence of a system under the influence of the canonical vector field $-KD_\varphi\nabla\varphi$ is upper bounded by the rate of convergence of a system using the navigation function vector field $-K\nabla\varphi$, so the same asymptotic gain can be used for that system. Hence all constraints of Definition 3 are satisfied and the proof is complete

■

E. Proof of Proposition 4

Proof: By Proposition 3 we have that S_{σ_1} and S_{σ_2} are aISS and let their asymptotic gains be γ_1 and γ_2 respectively chosen as in Proposition 3. We will use subscripts $i \in \{1, 2\}$ to denote the system S_{σ_i} to which we refer to. By Definition 3 we have that for the first subsystem:

$$\forall u, \forall\ a.\ a.\ \xi \in \mathcal{F}_1,\ \limsup_{t\to+\infty} |x_1(t, \xi, u)| \leq \gamma_1(\|u\|_\infty) \quad \text{(E-1)}$$

and for the second

$$\forall x_1, \forall\ a.\ a.\ \xi \in \mathcal{F}_2,\ \limsup_{t\to+\infty} |x_2(t, \xi, x_1)| \leq \gamma_2(\|x_1\|_\infty) \quad \text{(E-2)}$$

Applying eq. (E-1) to eq. (E-2) we get:

$$\forall u, \forall\ a.\ a.\ \xi \in \mathcal{F}_2,\ \limsup_{t\to+\infty} |x_2(t,\xi,x_1)| \le \gamma_2\left(\gamma_1\left(\|u\|_\infty\right)\right) \quad \text{(E-3)}$$

Combining eq. (E-1) and eq. (E-3) we get:

$$\forall u, \forall\ a.\ a.\ \xi \in \mathcal{F}_1 \times \mathcal{F}_2,\ \limsup_{t\to+\infty} |x_{12}(t,\xi,u)| \le \gamma_{12}\left(\|u\|_\infty\right) \quad \text{(E-4)}$$

where

$$|x_{12}(t,\xi,u)| = |x_1(t,\xi_1,u)| + |x_2(t,\xi_2,x_1)|$$

and

$$\gamma_{12} = \gamma_1\left(\|u\|_\infty\right) + \gamma_2\left(\gamma_1\left(\|u\|_\infty\right)\right)$$

is again of class $\mathcal{K}$. By application of Theorem 1 the interconnection $S_{\sigma_1} \rightharpoonup S_{\sigma_2}$ is almost GAS for zero inputs. Thus in view of the result E-4 all constraints of Definition 3 are satisfied and $S_{\sigma_1} \rightharpoonup S_{\sigma_2}$ is aISS. ∎

F. Proof of Proposition 5

Proof: The navigation function φ is by definition [3] uniformly maximal over the workspace boundary, i.e. $\varphi(q) = 1$ for $q \in \partial\mathcal{F}$. This implies that $-\nabla\varphi$ is transverse across the workspace boundary and points inwards. Taking the inner product of $-\nabla\varphi$ with $\dot{x}$ we have that

$$\dot{x}^T(-\nabla\varphi) = K\|\nabla\varphi\|^2 - (1-\varphi)u_1^T\nabla\varphi - \sigma_\varepsilon\left(-u_2^T\nabla\varphi\right)u_2^T\nabla\varphi$$

At the boundary the second term vanishes and the last term is non-positive since $\sigma_\varepsilon(x)$ is identically zero for $x \le 0$. Hence $\dot{x}^T(-\nabla\varphi) > 0$ across the workspace boundary and the workspace is positive invariant. When $u_1 \equiv 0$ using φ as a Lyapunov function candidate and noting that $\sigma_\varepsilon\left(-u_2^T\nabla\varphi\right)u_2^T\nabla\varphi \le 0$ we have that $\dot{\varphi} \le -K\|\nabla\varphi\|^2 \le 0$ almost everywhere - hence the system S_π with $u_1 \equiv 0$ is almost GAS at 0 ∎

G. Proof of Proposition 6

Proof: The proof follows the same line of thought like the proof of Proposition 3. Denote

$$r_1 = 1 - \varphi$$

and

$$r_2 = \sigma_\varepsilon\left(-u_2^T D_\varphi \nabla\varphi\right)$$

From the dual Lyapunov criterion we get a positivity requirement for the following:

$$\begin{aligned} \operatorname{div}[\rho\dot{x}] \quad = \quad & -\tfrac{a}{\varphi^{a+1}}\nabla^T\varphi\left(-KD_\varphi\nabla\varphi + r_1u_1 + r_2u_2\right) \\ & + \tfrac{1}{\varphi^a}\nabla\cdot\left(-KD_\varphi\nabla\varphi + r_1u_1 + r_2u_2\right) \end{aligned}$$

But from Proposition 5 we know that the system with $u_1 = 0$ is GAS hence by using duality arguments we have that it will hold that almost everywhere:

$$\begin{aligned} & -\tfrac{a}{\varphi^{a+1}}\nabla^T\varphi\left(-\tfrac{K}{2}D_\varphi\nabla\varphi + r_2u_2\right) \\ & + \tfrac{1}{\varphi^a}\nabla\cdot\left(-\tfrac{K}{2}D_\varphi\nabla\varphi + r_2u_2\right) \qquad > 0 \end{aligned}$$

Hence setting $K' = K/2$ we get from the dual Lyapunov criterion, positivity requirement for the following:

$$\begin{aligned} \operatorname{div}[\rho\dot{x}] \quad = \quad & -\tfrac{a}{\varphi^{a+1}}\nabla^T\varphi\left(-K'D_\varphi\nabla\varphi + r_1u_1\right) \\ & + \tfrac{1}{\varphi^a}\nabla\cdot\left(-K'D_\varphi\nabla\varphi + r_1u_1\right) \end{aligned}$$

The positivity of the above can be satisfied by choosing:

$$\|u_1\| \le \frac{aK'\nabla^T\varphi D_\varphi\nabla\varphi - K'\varphi\nabla\cdot\left(D_\varphi\nabla\varphi\right)}{\varphi\|\nabla r_1\| + ar_1\|\nabla\varphi\|} \triangleq U(x) \quad \text{(G-1)}$$

Noting that r_1, $\|\nabla r_1\|$ are upper bounded due to smoothness and compactness arguments as also are φ and $\|\nabla\varphi\|$, the proof continues in the exact same way as the proof of Proposition 3 giving us the asymptotic gain $\gamma(x) = \frac{r_w}{\nu^2}x^2$. Then given any measurable input $u_1(\cdot)$, for almost all initial conditions we have that

$$\forall u_1, u_2\ \forall a.a.\ x_0 \quad \limsup_{t\to\infty}\|x(t,x_0,u_1,u_2)\| \le \gamma\left(\|u_1\|_\infty\right) \quad \text{(G-2)}$$

Using the same arguments as in Proposition 3 we can transfer the same asymptotic gain from the canonical system to the navigation function vector field. Hence all constraints of Definition 3 are satisfied and the proof is complete. ∎

H. Proof of Proposition 7

Proof: As can be seen from the proof of the aISS property of S_π type systems (eq. G-2) the stability characterization does not depend on input u_2. So we will consider the general case $S_1 \overset{\rightharpoonup}{\rightharpoondown} S_2$ which can be reduced to any of the other two.

By Proposition 6 we have that S_1 and S_2 are aISS and let their asymptotic gains be γ_1 and γ_2 respectively chosen as in Proposition 6. We will use subscripts $i \in \{1,2\}$ to denote the system S_i to which we refer to.

By Definition 3 we have that for the first subsystem:

$$\forall u_1, u_2, \forall\ a.\ a.\ \xi \in \mathcal{F}_1, \limsup_{t\to+\infty}|x_1(t,\xi,u_1,u_2)| \le \gamma_1\left(\|u_1\|_\infty\right) \quad \text{(H-1)}$$

and for the second

$$\forall x_1, \tilde{x}_1, \forall\ a.\ a.\ \xi \in \mathcal{F}_2, \limsup_{t\to+\infty}|x_2(t,\xi,x_1,\tilde{x}_1)| \le \gamma_2\left(\|x_1\|_\infty\right) \quad \text{(H-2)}$$

where $\tilde{x}_1 = x_1$ in the case of port 2 interconnections but otherwise is any measurable input. Applying eq. (H-1) to eq. (H-2) we get:

$$\forall u_1, u_2, \forall\ a.\ a.\ \xi \in \mathcal{F}_2, \limsup_{t\to+\infty}|x_2(t,\xi,x_1,\tilde{x}_1)| \le \gamma_2\left(\gamma_1\left(\|u_1\|_\infty\right)\right) \quad \text{(H-3)}$$

Combining eq. (H-1) and eq. (H-3) we get:

$$\forall u_1, u_2, \forall a.a.\ \xi \in \mathcal{F}_1 \times \mathcal{F}_2, \limsup_{t\to+\infty}|x_{12}(t,\xi,u_1,u_2)| \le \gamma_{12}\left(\|u_1\|_\infty\right) \quad \text{(H-4)}$$

where

$$|x_{12}(t,\xi,u_1,u_2)| = |x_1(t,\xi_1,u_1,u_2)| + |x_2(t,\xi_2,x_1,\tilde{x}_1)|$$

and

$$\gamma_{12} = \gamma_1\left(\|u_1\|_\infty\right) + \gamma_2\left(\gamma_1\left(\|u_1\|_\infty\right)\right)$$

is again of class $\mathcal{K}$. By application of Theorem 1 the interconnection $S_1 \overset{\rightharpoonup}{\rightharpoondown} S_2$ is almost GAS for zero inputs. Thus, in view of the result H-4 all constraints of Definition 3 are satisfied and $S_1 \overset{\rightharpoonup}{\rightharpoondown} S_2$ is aISS. ∎

I. Proof of Proposition 8

Proof: The proof is based on the fact that any S_π type system with $\rightharpoondown$ interconnection is almost GAS for zero inputs on port 1. This can be easily seen by performing a Lyapunov type analysis with φ the Lyapunov function candidate:

$$\dot{\varphi} = -K \left\|\nabla\varphi\right\|^2 + \sigma_\varepsilon \left(-u_2^T \nabla\varphi\right) \nabla^T \varphi u_2 \leq -K \left\|\nabla\varphi\right\|^2 \overset{a.e.}{<} 0$$

Hence the feedback interconnection can be broken at the point of $\rightharpoondown$ interconnection and the system can then be treated as a cascaded interconnection. This is demonstrated if we consider the interconnection $S_5 \rightharpoondown S_4$ from figure 3. By the aISS property of system S_4 we have that:

$$\forall u, x_5, \forall\ a.\ a.\ \xi \in \mathcal{F}_4, \limsup_{t\to+\infty} \left|x_4\left(t, \xi, u, x_5\right)\right| \leq \gamma_4 \left(\left\|u_1\right\|_\infty\right) \tag{I-1}$$

which does not depend on the value of x_5. ■