CONTINUOUS LOGIC AND PROOF THEORY FOR FIRST-ORDER ŁUKASIEWICZ LOGIC

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"Kant already taught -and indeed it is part of parcel of his doctrine- that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone."

-David Hilbert, On the Infinity.

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May your bowl always be filled with snacks.

May your bowl always be filled with snacks and free of Coconut.

ABSTRACT

CONTINUOUS LOGIC AND PROOF THEORY FOR FIRST-ORDER ŁUKASIEWICZ LOGIC

Jin Wei

Henry Towsner

Continuous model theory traces back to Chang and Keisler (1966) and was recently redeveloped by Yaacov et al. (2008) to study complete metric structures with applications in operator algebras and many other fields. The underlying logic is called continuous logic, which is a natural generalization of first-order classical logic by extending the truth values to the continuous set $[0, 1]$. From a syntactic point of view, continuous logic is an extension of (first-order) Łukasiewicz logic, one of the famous fuzzy logics.

The motivations for continuous logic were mostly semantic, while Yaacov and Pedersen (2010) provided a Hilbert-style proof system with proofs of its soundness and (approximate) completeness. In this thesis, we will discuss further about syntactic properties of continuous logic and proof theory for first-order Łukasiewicz logic. We will present a hypersequent calculus GŁ$\forall$ for first-order Łukasiewicz logic, first discovered by Baaz and Metcalfe (2010). The main result will be our proof of approximate completeness of GŁ$\forall$ with respect to the $[0, 1]$-semantics for arbitrary first-order formulas, while the original proof only applies to prenex formulas as pointed out in Gerasimov (2020). We will also explore some of the consequences and potential applications to continuous model theory, particularly in term of proof mining and extracting quantitative information from proofs using continuous ultraproducts.

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CHAPTER 1

INTRODUCTION

1.1. Motivations

Model theory is a study of mathematical structures using first-order logic. It often applies to algebraic structures, such as linear orders, graphs, algebraic closed fields, and etc (See examples in Marker (2002)). Model theory is generally considered harder to apply to analytic structures, such as Banach spaces and C^*-algebras, mainly due to the fact that metric completeness cannot be expressed in first-order logic as first-order logic does not allow quantifying over subsets.

Another related phenomenon is the use of ultraproducts. Ultraproduct is a very effective tool in model theory by the following theorem of Łos.

Theorem 1.1.1. *(Łos) Given a language $\mathcal{L}$, a collection of $\mathcal{L}$-structure $(\mathfrak{M}_i)_{i\in I}$, and an ultrafilter $\mathcal{U}$ on the index I, the ultraproduct $\prod_{i\in I}\mathfrak{M}_i/\mathcal{U}$ is a $\mathcal{L}$-structure. Furthermore, for each $\mathcal{L}$-formula φ we have that*

$$\prod_{i\in I}\mathfrak{M}_i/\mathcal{U} \models \varphi \iff \{i \in I \mid \mathfrak{M}_i \models \varphi\} \in \mathcal{U}$$

In particular, ultraproduct gives a criterion to check if a class of mathematical structures is an elementary class/axiomatizable.

Theorem 1.1.2. *(Keisler) Given a language $\mathcal{L}$, a class of $\mathcal{L}$-structures $\mathcal{C}$ is axiomatizable if and only if it satisfies the following conditions:*

- *$\mathcal{C}$ is closed under isomorphisms.*
- *$\mathcal{C}$ is closed under ultraproducts.*
- *The complement of $\mathcal{C}$ is closed under ultrapowers.*

The class of complete metric spaces can not be axiomatizable since it is not closed under ultra-

products: the reals $\mathbb{R}$ is metric complete while the hyperreals, viewed as an ultrapower of $\mathbb{R}$, is not complete by noticing the subset of infinitesimals does not have a least upper bound. It is consistent with the above observation that metric completeness is not a first-order property.

Introduced first by Chang and Keisler (1966) and recently redeveloped by Yaacov et al. (2008), continuous model theory is a natural generalization of classical model theory, specializing in dealing with complete metric spaces. Rather than classical structures $(M, =)$ with an equality predicate

$$=: M \times M \to \{0, 1\}$$

we now consider continuous structures, which are complete and bounded metric spaces (M, d) with a metric predicate

$$d : M \times M \to [0, 1]$$

One can draw the analogy by either viewing d as a generalized equality, namely $d(x, y)$ measures how close x, y are being equal, or viewing the equality predicate as a discrete metric.

This viewpoint requires the underlying logic to admit arbitrary truth values in $[0, 1]$, hence the name continuous logic. Furthermore, we generally axiomatize the equality predicate as an equivalence relation together with the following congruence rules for function and relation symbols:

$$\frac{x = y}{f(x; \bar{z}) = f(y; \bar{z})} \qquad \frac{x = y}{\varphi(x; \bar{z}) \iff \varphi(y; \bar{z})}$$

In continuous logic, the focus is less about the exact equality, but more about approximations. Hence, we instead would expect for any $\epsilon \in (0, 1]$ there exists $\delta \in (0, 1]$ such that:

$$\frac{d(x, y) < \delta}{d(f(x; \bar{z}), f(y; \bar{z})) < \epsilon} \qquad \frac{d(x, y) < \delta}{|\varphi(x; \bar{z}) - \varphi(y; \bar{z})| < \epsilon}$$

which forces the interpretation of function and relation symbols in continuous logic to be uniformly continuous functions for each argument.

This new formulation of model theory using continuous logic immediately inherits many nice model-theoretic properties, such as the Compactness Theorem, a notion of ultraproduct and corresponding

Łos Theorem, Löwenheim-Skolem, the Omitting Type Theorem, etc. See Yaacov et al. (2008) for further discussions on the model-theoretic side of continuous logic. Importantly, complete metric spaces now become an elementary class in continuous logic, which brings many applications. For instance, continuous model theory was used in Goldbring (2022) to interpret a recent breakthrough of the Connes Embedding problem, a classical open problem in the theory of tracial von Neumann algebras. Boutonnet et al. (2017) is another example where the continuous version of Keisler-Shelah Theorem was used to find uncountably many separable II_1 factors with non-isomorphic ultrapowers.

1.2. Connections to Łukasiewicz Logic

From a syntactic point of view, continuous logic is an extension of first-order Łukasiewicz logic. First introduced by Jan Łukasiewicz in 1917 as a three-valued logic, Łukasiewicz logic is the strongest among the three fuzzy propositional logics where truth values are taken as any possible value between 0 and 1. The following key theorem explains the use of Łukasiewicz logic in continuous logic where formulas are valid in $[0, 1]$ if it is always evaluated as 0 in all possible interpretations:

Theorem 1.2.1. *A proposition φ is valid in $[0, 1]$ if and only if φ is provable from Łukasiewicz logic.*

Continuous logic extends first-order Łukasiewicz logic by adding an “equality" predicate d and corresponding “equivalence" and “congruence" axioms. Simiar to classical first-order logic with equality, continuous logic can be constructed through three stages and the following charts demonstrates the analogy:

	Discrete Syntax	Discrete Semantics
Propositional	Classical Logic	$\{0,1\}$/Boolean Algebras
Predicate	First-Order Logic	First-order Structures
Predicate with Equality	First-Order Logic with Equality	First-Order Structures with Equality
	Continuous Syntax	Continuous Semantics
Propositional	Łukasiewicz Logic	$[0,1]$/MV-Algebras
Predicate	First-Order Łukasiewicz Logic	Standard $[0,1]$-Semantics
Predicate with Equality	Continuous Logic	Continuous (Pre)-Structures

1.3. Proof-Theoretic Investigations

A proof system for continuous logic was introduced by Yaacov and Pedersen (2010) based on a Hilbert-style formulation of Łukasiewicz logic. They provided a proof of soundness and approximate completeness, in the sense that any formula valid in all continuous (pre)-structures can be proved in continuous logic to be arbitrarily close to the truth 0. However, one cannot expect to have a full completeness and there are counterexamples where a formula is valid and does not have a proof in continuous logic, only its approximations.

It is related to the fact that valid formulas in first-order $[0,1]$-semantics are not recursively enumerable (in fact Π^0_2-complete, see Hájek (1998)). Therefore, there is no reasonable set of axioms or rules that is complete with respect to the $[0,1]$-semantics. The approximate form of completeness is optimal in that way and it is consistent with the philosophy of continuous model theory that **exact values of truth are replaced by approximations of truth**.

Proof theory of (first-order) Łukasiewicz logic has been studied extensively in the past. In particular, a cut-free Gentzen-style proof system for propositional Łukasiewicz logic GŁ was first introduced by Metcalfe et al. (2005) in the form of hypersequent calculus, containing proofs of completeness and cut admissibility. A syntactic cut elimination proof was also provided in Ciabattoni and Metcalfe (2003).

Remark 1. *The author of this thesis and his colleague, Eben Blaisdell, were searching for a Gentzen-*

style proof system for continuous logic and arrived independently at the system GŁ, both unaware of the existing literature. A cut elimination procedure and a completeness proof are also independently discovered. We include our work in this thesis to offer alternative proofs and expository details, without the intention to make any claim for credit.

Baaz and Metcalfe (2010) extended the system GŁ to the first-order system GŁ∀. They proved its soundness and approximate completeness in the same paper. The proof takes any first-order formula to its equivalent prenex formula. Then it utilizes Skolemization and an approximate form of Herbrand's Theorem to convert any given valid first-order formula φ to a valid quantifier-free formula φ' that roughly approximates φ. The propositional completeness of GŁ∀ guarantees a derivation of φ' and we re-introduce quantifiers from it to derive arbitrary approximations of φ.

Gerasimov (2020) remarked that their completeness result only applies to prenex formulas, since GŁ∀ cannot prove equivalence between first-order formulas to its prenex form without the *cut* rule. It thus remains open whether GŁ∀ is complete with respect to $[0, 1]$-semantics for arbitrary first-order formulas. In Wei (2024), we discovered a correct proof of approximate completeness of GŁ∀ by generalizing the idea in Baaz and Metcalfe (2010). In fact, our result is even stronger since it also applies to arbitrary hypersequents, not just formulas. We will present our proof as the main result of this thesis.

Lastly, we will update our progress in finding a theory of proof mining in continuous logic. Proof mining is a subfield in proof theory that focuses on extracting quantitative information lost in mathematical proofs due to uses of non-effective principles. It often relies on interpreting proofs with restricted means, typically some reasonable extension of first-order arithmetic, and uses tools such as the double-negation translation and Gödel's functional interpretation to extract computational content. This approach has been proven remarkably successful across various mathematical disciplines. See Kohlenbach (2008) for more details.

In classical model theory, ultraproducts often obscure constructive information in proofs. Using proof mining, Towsner (2024) gave a systematic approach to eliminate uses of ultraproducts in proofs

and to yield quantitative information. There could be an analogy of this approach in continuous model theory. In fact, proof mining and logical metatheorems for continuous model theory were already investigated in Avigad and Iovino (2013) and Günzel and Kohlenbach (2016) in the form of positive bounded logic where continuous predicates are treated as analytic judgments on bounded metric structures. There were also individual applications, such as in Goldbring et al. (2018) where explicit sentences distinguishing non-isomorphic McDuff's II_1 factors were extracted using proof mining.

In the end, we hope to generalize existing proof mining methodologies to continuous model theory by exploring proof-theoretic properties of continuous logic. Different from the earlier approaches that represent continuous logic in analysis and first-order logic, we try to build up a native theory of functional interpretation and proof mining whose transformations of proofs directly take place within continuous logic. This new approach could enable direct applications of existing proof-theoretic results in first-order logic to the continuous setting. This new direction requires developments of Gentzen-style proof systems, cut elimination, an intuitionistic fragment, realizability, and proof interpretations in the continuous setting.

CHAPTER 2

CONTINUOUS LOGIC

W present the formal definitions for continuous structures. The definitions are mainly due to Yaacov et al. (2008) and Yaacov and Pedersen (2010), while we also made some significant changes as in Remark 2, 3, 5.

2.1. Continuous Structures

Definition 2.1.1. *A* ***continuous signature*** $\mathcal{L}$ *(with a metric) is a collection of the following objects:*

- *A binary relation symbol* d*.*
- *Function symbols* f *together with an arity* $n_f \in \mathbb{N}$ *and* $\Delta_f : (0,1] \to (0,1]$*.*
- *Relation symbols* r *together with an arity* $n_r \in \mathbb{N}$ *and* $\Delta_r : (0,1] \to (0,1]$*.*

In addition, we say f *is a* ***constant symbol*** *when* $n_f = 0$ *and* r *is a* ***truth constant*** *when* $n_r = 0$*.*

Remark 2. *Yaacov and Pedersen (2010) distinguish between continuous signatures and continuous signatures with a metric. Since all structures of interest contain a metric, we assume unless state otherwise all continuous signatures contain a distinguished metric symbol.*

Definition 2.1.2. *Given a continuous signature* $\mathcal{L}$*, a* ***continuous*** $\mathcal{L}$***-pre-structure*** $\mathcal{M}$ *is a nonempty set* M *together with the following interpretation of* $\mathcal{L}$*-symbols:*

- $d^{\mathfrak{M}} : M \times M \to [0,1]$ *is a* ***pseudo-metric****.*
- $f^{\mathfrak{M}} : M^{n_f} \to M$ *satisfies that for each* $i < n_f$ *and* $\epsilon \in (0,1]$*:*

$$(\forall x, y, \bar{z})(d^{\mathfrak{M}}(x,y) < \Delta_f(\epsilon) \implies d^{\mathfrak{M}}(f^{\mathfrak{M}}(\bar{x};\bar{z}), f^{\mathfrak{M}}(\bar{y};\bar{z})) \leqslant \epsilon)$$

The function Δ_f *is thus referred as a* ***modulus of uniform continuity for*** f*.*

- $r^{\mathfrak{M}} : M^{n_r} \to [0,1]$ *satisfies that for each* $i < n_r$ *and* $\epsilon \in (0,1]$*:*

$$(\forall x, y, \bar{z})(d^{\mathfrak{M}}(x,y) < \Delta_f(\epsilon) \implies |r^{\mathfrak{M}}(x;\bar{z}) - r^{\mathfrak{M}}(y;\bar{z})| \leqslant \epsilon)$$

The function Δ_r *is called a* ***modulus of uniform continuity for*** r*.*

A ***continuous*** $\mathcal{L}$***-structure*** $\mathcal{M}$ *is a a continuous* $\mathcal{L}$*-pre-structure whose interpretation of* d *happens to be a* ***complete metric****.*

One can verify that classical first-order structures can indeed be recovered by taking a discrete metric d.

Remark 3. *The function* Δ_f *(*Δ_r*) are external moduli witnessing uniform continuity of* f *(*r*) with respect to the metric in any interpretation and they are not syntactic objects in continuous logic (see the later Definition 2.2.9). One could also assign individual moduli of uniform continuity* $\delta_{f,i}$ *(*$\delta_{r,i}$*) for each argument* $i < n_f$ *(*$i < n_i$*). Both were discussed in Yaacov and Pedersen (2010).*

The definition of $\mathcal{L}$-terms and $\mathcal{L}$-formulas is very similar to classical first-order logic (see Avigad (2022) for instance), except that the logic connectives are $\{\bot, \to, \frac{1}{2}, \forall\}$.

Definition 2.1.3. *The collection of* $\mathcal{L}$***-terms*** *is defined inductively as:*

- *A countable collection of variables* $V = \{v_i \mid i \in \mathbb{N}\}$ *as* $\mathcal{L}$*-terms.*
- *If* f *is a function symbol and* $\bar{t}$ *is a tuple of* $\mathcal{L}$*-term with* $|\bar{t}| = n_f$*, then* $f(\bar{t})$ *is also an* $\mathcal{L}$*-term.*

The collection of $\mathcal{L}$***-formulas*** *is defined inductively as:*

- $\bot$ *is an* $\mathcal{L}$*-formula.*
- *For any* $\mathcal{L}$*-term* t_1, t_2*, we have that* $d(t_1, t_2)$ *is an* $\mathcal{L}$*-formula.*
- *If* r *is a relation symbol and* $\bar{t}$ *is a tuple of* $\mathcal{L}$*-term with* $|\bar{t}| = n_r$*, then* $r(\bar{t})$ *is an* $\mathcal{L}$*-formula.*
- *If* φ *and* ψ *are* $\mathcal{L}$*-formulas, then* $\varphi \to \psi$ *and* $\frac{1}{2}\varphi$ *are also* $\mathcal{L}$*-formulas.*

- *If φ is an $\mathcal{L}$-formula and x is a variable, then $(\forall x)\varphi$ is also an $\mathcal{L}$-formula.*

Definition 2.1.4. *Given a continuous $\mathcal{L}$-pre-structure $\mathfrak{M}$, an $\mathfrak{M}$-**assignment** is a mapping $\sigma : V \to M$. Fixing a $\mathfrak{M}$-assignment σ, we define **the interpretation** of $\mathcal{L}$-terms, denoted as $t^{\mathfrak{M},\sigma}$, as the following:*

- *If $v \in V$ is a variable, then $v^{\mathfrak{M},\sigma} := \sigma(v)$.*
- *$f(\bar{t})^{\mathfrak{M},\sigma} := f^{\mathfrak{M}}(\bar{t}^{\mathfrak{M},\sigma})$ where $\bar{t}^{\mathfrak{M},\sigma} := (t_1^{\mathfrak{M},\sigma}, \ldots, t_k^{\mathfrak{M},\sigma})$.*

*We define **the interpretation/value** of $\mathcal{L}$-formulas, denoted as $\varphi^{\mathfrak{M},\sigma}$, in $[0,1]$ as the following:*

- $\bot^{\mathfrak{M},\sigma} := 1$.
- $d(t_1,t_2)^{\mathfrak{M},\sigma} := d^{\mathfrak{M}}(t_1^{\mathfrak{M},\sigma}, t_2^{\mathfrak{M},\sigma})$.
- $r(\bar{t})^{\mathfrak{M},\sigma} := r^{\mathfrak{M}}(\bar{t}^{\mathfrak{M},\sigma})$.
- $(\varphi \to \psi)^{\mathfrak{M},\sigma} := \psi^{\mathfrak{M},\sigma} \dot{-} \varphi^{\mathfrak{M},\sigma} = \begin{cases} \psi^{\mathfrak{M},\sigma} - \varphi^{\mathfrak{M},\sigma} & \psi^{\mathfrak{M},\sigma} > \varphi^{\mathfrak{M},\sigma} \\ 0 & \psi^{\mathfrak{M},\sigma} \leqslant \varphi^{\mathfrak{M},\sigma} \end{cases}$,

  and $(\frac{1}{2}\varphi)^{\mathfrak{M},\sigma} := \frac{1}{2}\varphi^{\mathfrak{M},\sigma}$.
- $((\forall x)\varphi)^{\mathfrak{M},\sigma} := \sup_{m\in M}(\varphi[m/x])^{\mathfrak{M},\sigma}$ *where $\varphi[m/x]$ denotes the substitution of x in φ by m.*

*When $\varphi^{\mathfrak{M},\sigma} = 0$ for all assignments σ, we say $\mathfrak{M}$ **is a model** of φ and denote it as $\mathfrak{M} \models_Q \varphi$. A $\mathcal{L}$-formula φ is called **valid**, denoted as $\models_Q \varphi$, if $\mathfrak{M} \models_Q \varphi$ for all continuous $\mathfrak{L}$-pre-structure $\mathfrak{M}$.*

*For an **$\mathcal{L}$-theory** Γ, namely a set of $\mathcal{L}$-formulas, we say $\mathfrak{M}$ **is a model** of Γ or $\mathfrak{M} \models_Q \Gamma$, if $\mathfrak{M} \models_Q \varphi$ for all $\varphi \in \Gamma$. We say Γ **entails** φ or $\Gamma \models_Q \varphi$ if whenever $\mathfrak{M} \models_Q \Gamma$, we have that $\mathfrak{M} \models_Q \varphi$. A $\mathcal{L}$-theory Γ is called **satisfiable** if there exists a $\mathcal{L}$-pre-structure $\mathfrak{M}$ such that $\mathfrak{M} \models_Q \Gamma$.*

*If $\mathcal{L}$-pre-structures in the above definitions are replaced by $\mathcal{L}$-structures, we have a notion of **complete model** $\mathfrak{M} \models_{QC} \varphi$, **complete entailment** $\Gamma \models_{QC} \varphi$, **complete validity**, and **complete satisfiability** respectively.*

Remark 4. *Different from the traditions, valid formulas in continuous logic have truth value* 0 *while truth value* 1 *is considered a total falsity. It is consistent with the interpretation of* d *as a metric: the predicate* $d(x, y)$ *takes value at* 0 *when* x, y *are indeed equal to each other.*

The interpretation of $\varphi \to \psi$ *is defined such that* $\varphi \to \psi$ *is valid if and only if* $\varphi^{\mathfrak{M},\sigma} \geqslant \psi^{\mathfrak{M},\sigma}$ *for all possible assignments and pre-structures.*

Using $\bot, \to, \forall$, we can define more familiar logic connectives.

Definition 2.1.5. *We have the following definitions of logic connectives:*

- $\neg\varphi := \varphi \to \bot$.
- $\top := \bot \to \bot$.
- $\varphi \vee \psi := (\varphi \to \psi) \to \psi$ *and* $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$.
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.
- $\varphi \oplus \psi := \neg\varphi \to \psi$ *and* $\varphi \odot \psi := \neg(\neg\varphi \oplus \neg\psi)$.
- $(\exists x)\varphi := \neg((\forall x)\neg\varphi)$

Connectives $\vee$ *and* $\wedge$ *are called* ***weak disjunction*** *and* ***weak conjunction*** *and they can be viewed as the additive fragment based on the connection to affine logic as discussed later in Section 3.1;* $\oplus$ *and* $\odot$ *are called* ***strong disjunction*** *and* ***strong conjunction*** *as the multiplicative fragment.*

Through some calculations, we can derive the following equalities, which also explain their meanings:

- $(\neg\varphi)^{\mathfrak{M},\sigma} = 1 - \varphi^{\mathfrak{M},\sigma}$.
- $\top^{\mathfrak{M},\sigma} = 0$.
- $(\varphi \vee \psi)^{\mathfrak{M},\sigma} = \min(\varphi^{\mathfrak{M},\sigma}, \psi^{\mathfrak{M},\sigma})$ and $(\varphi \wedge \psi)^{\mathfrak{M},\sigma} = \max(\varphi^{\mathfrak{M},\sigma}, \psi^{\mathfrak{M},\sigma})$.
- $(\varphi \leftrightarrow \psi)^{\mathfrak{M},\sigma} = |\varphi^{\mathfrak{M},\sigma} - \psi^{\mathfrak{M},\sigma}|$.

- $(\varphi \oplus \psi)^{\mathfrak{M},\sigma} = \max(\varphi^{\mathfrak{M},\sigma} + \psi^{\mathfrak{M},\sigma} - 1, 0)$ and $(\varphi \odot \psi)^{\mathfrak{M},\sigma} = \min(\varphi^{\mathfrak{M},\sigma} + \psi^{\mathfrak{M},\sigma}, 1)$.
- $((\exists x)\varphi)^{\mathfrak{M},\sigma} = \inf_{m \in M}(\varphi[m/x])^{\mathfrak{M},\sigma}$.

Remark 5. *Yaacov and Pedersen (2010) define $\varphi \wedge \psi := (\varphi \to \psi) \to \psi$ to accompany the exchange of truth value 0 and 1. It brings some undesired consequences, since $\vee$ would then behave like a conjunction while $\wedge$ would behave like a disjunction. For instance, they define $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \vee (\psi \to \varphi)$.*

We decide to keep the syntactic definition of $\vee$ and $\wedge$ unchanged, since the exchange of truth value 0 and 1 only happens to the semantic level. The advantages will be more clear once we introduce the Gentzen-style proof system later in Definition 3.2.4 where $\vee$ will have the same admissible introduction rules as $\vee$ in classical logic.

Definition 2.1.6. *The dyadic truth values $\mathbb{D} := \{\frac{k}{2^n} \mid k \leqslant 2^n\}$ can also be defined by using $\frac{1}{2}$:*

- $\mathbf{1} := \bot$ *and* $\mathbf{0} := \bot \to \bot$.
- $\frac{\mathbf{1}}{\mathbf{2^{k+1}}} := \frac{1}{2}(\frac{\mathbf{1}}{\mathbf{2^k}})$.
- $\frac{\mathbf{k}}{\mathbf{2^n}} := \overbrace{\frac{\mathbf{1}}{\mathbf{2^n}} \odot \cdots \odot \frac{\mathbf{1}}{\mathbf{2^n}}}^{k}$ *for* $k \leqslant 2^n$.

One can verify that the interpretation of boldface formulas are exactly the underlying truth value.

2.2. Continuous Logic

We now present the syntax of continuous (pre-)structures, namely a Hilbert-style proof system for continuous logic. First, we clarify some definitions.

Definition 2.2.1. *Given a (Hilbert-style)* ***logic*** *L, namely a set of axioms, we say a formula φ is* ***provable*** *in L, denoted as $\vdash_L \varphi$ if it can be obtained through the following two rules:*

$$\frac{}{\vdash_L \varphi}\ \varphi \text{ is an axiom in } L \qquad\qquad \frac{\vdash_L \varphi \qquad \vdash_L \varphi \to \psi}{\vdash_L \psi}\ \textit{modus ponens}$$

We extend the provability relation to $\Gamma \vdash_L \varphi$ or Γ ***proves*** *φ* ***in*** *L to express that $\vdash_{L \cup \Gamma} \varphi$.*

We say Γ *is* ***consistent*** *if* $\Gamma \nvdash_L \bot$.

Definition 2.2.2. *(Łukasiewicz Logic) Given an infinite collection of propositional variables and connectives* $\{\bot, \to\}$*, Łukasiewicz logic Ł consists of the following axioms:*

A1 $\vdash \varphi \to (\psi \to \varphi)$.

A2 $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$.

A3 $\vdash ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

A4 $\vdash (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.

We also present the standard $[0,1]$-semantics for Łukasiewicz logic:

Definition 2.2.3. *(Standard* $[0,1]$*-Assignment) A* $[0,1]$***-truth assignment*** *is a mapping* v *from the set of propositional variables to* $[0,1]$*. We extend* v *to all propositions by the followings:*

- $v(\bot) := 1$.
- $v(\varphi \to \psi) := v(\psi) \dot{-} v(\varphi)$.

We say a proposition φ *is* ***valid*** *if* $v(\varphi) = 0$ *for all* $[0,1]$*-truth assignments. We also extend the notion of entailment relation* $\models_{[0,1]}$ *accordingly.*

Theorem 2.2.4. *(Soundness and Completeness of Ł) For any proposition* φ*, we have that*

$$\vdash_Ł \varphi \iff \models_{[0,1]} \varphi.$$

Proof. Soundness essentially reduces to verify that the corresponding inequalities hold in arbitrary $[0,1]$-assignments. For instance, A3 is equivalent to the inequality $\min(v(\varphi), v(\psi))) \leqslant \min(v(\psi), v(\varphi))$.

Completeness is more involved. The usual proof makes use of completeness of Ł over MV-algebras and the fact that valid formulas in $[0,1]$ coincide with valid formulas in all MV-algebras. See Hájek (1998) for details. □

Remark 6. *Notice that we only include soundness and completeness of Łukasiewicz logic in terms of validities. Strong completeness in terms of provability* $\Gamma \vdash_{Ł} \varphi$ *does not hold for Łukasiewicz logic, unless* Γ *is finite. See further discussions in Montagna (2007).*

Now we move to the first-order setting:

Definition 2.2.5. *(First-Order Łukasiewicz Logic) Given a classical signature* $\mathcal{L}$*, first-order Łukasiewicz logic Ł∀ consists generalizations of all axioms in Ł, where propositional variables are replaced by* $\mathcal{L}$*-formulas, together with generalizations of the following:*

A5 $\vdash \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$.

A6 $\vdash \forall x\varphi \to \varphi[t/x]$.

A7 $\vdash \varphi \to \forall x\varphi$ *where* x *is not free in* φ.

Definition 2.2.6. *(Standard* $[0,1]$*-Structure) An* $[0,1]$***-structure*** *is similar to a first-order* $\mathcal{L}$*-pre-structure except that the interpretation of function symbol* f *is an* ***arbitrary*** *function* $M^{n_f} \to M$ *and the interpretation of relation symbol* r *is an* ***arbitrary*** *function* $M^{n_r} \to [0,1]$*. Entailment relation, validity, and satisfiability are defined accordingly. Without ambiguity, we also denote the entailment relation as* $\models_{[0,1]}$.

Theorem 2.2.7. *(Soundness of Ł∀) For any first-order formula* φ *in a language* $\mathcal{L}$*,*

$$\vdash_{Ł\forall} \varphi \Longrightarrow \models_{[0,1]} \varphi.$$

Proof. It reduces to simple computations to show that the inequalities defined by A5-A7 are satisfied by the interpretation of $\forall x$ by $\sup_m$ in all $[0,1]$-structures. □

As discussed earlier, we can only have an approximate form of completeness for first-order Łukasiewicz

logic. The following observation defines a notion of approximation:

$$\models_{[0,1]} A \oplus \overbrace{(\varphi \odot \cdots \odot \varphi)}^{n} \text{ if and only if } A^{\mathfrak{M},\sigma} \leqslant \frac{1}{n+1} \text{ for all } [0,1]\text{-interpretation } \mathfrak{M}, \sigma.$$

Theorem 2.2.8. *(Approximate Completeness of Ł∀) For any first-order formula φ,*

$$\models_{[0,1]} \varphi \Longrightarrow \vdash_{Ł\forall} \varphi \oplus \overbrace{(\varphi \odot \cdots \odot \varphi)}^{n} \text{ for all } n \in \mathbb{N}.$$

Proof. See the proof in Hájek (1998). □

To reach continuous logic, we add axioms related to the metric predicate d. The connective $\frac{1}{2}$ is needed to express uniform continuity.

Definition 2.2.9. *Continuous logic CŁ is first-order Łukasiewicz logic together with operator $\frac{1}{2}$ and metric d and generalizations of the following axioms:*

Defining axioms for $\frac{1}{2}$:

A8 $\vdash (\frac{1}{2}\varphi \to \varphi) \to \frac{1}{2}\varphi$.

A9 $\vdash \frac{1}{2}\varphi \to (\frac{1}{2}\varphi \to \varphi)$.

Equivalence axioms establishing that d is a pseudo-metric:

A10 $\vdash d(x,x)$, *where we use $d(x,y)$ to denote dxy.*

A11 $\vdash d(x,y) \to d(y,x)$.

A12 $\vdash d(x,y) \to (d(y,z) \to d(x,z))$.

Congruence axioms expressing uniform continuity for function and relation symbols:

A13 $\vdash (d(x,y) \to q) \vee (p \to d(f(x;\bar{z}), f(y;\bar{z})))$.

for each $f \in \mathcal{F}$, $\epsilon \in (0,1]$, *and* $p, q \in \mathbb{D}$ *with* $p > \epsilon$ *and* $q < \Delta_f(\epsilon)$.

A14 $\vdash (d(x,y) \to q) \vee (p \to (r(x;\bar{z}) \leftrightarrow r(y;\bar{z})))$.

for each $r \in \mathcal{R}$, $\epsilon \in (0,1]$, *and* $p, q \in \mathbb{D}$ *with* $p > \epsilon$ *and* $q < \Delta_r(\epsilon)$.

Remark 7. *In A13 and A14, we made use of equivalence in classical logic between* $\varphi \to \psi$ *and* $\neg\varphi \vee \psi$ *to translate the uniform continuity condition. We cannot directly translate the uniform continuity condition in the following way*

$$(q \to d(x,y)) \to (p \to d(f(x;\bar{z}), f(y;\bar{z})))$$

because $\to$ *has different semantic interpretations for classical logic and continuous logic, which are not equivalent.*

2.3. Completeness of Continuous Logic

Notice that we have two notions of semantics for continuous logic, continuous $\mathcal{L}$-pre-structures and continuous $\mathcal{L}$-structures. It is not hard to predict completeness of continuous logic over $\mathcal{L}$-pre-structures since the syntax precisely defines a pseudo-metric space along with uniform continuity of function and relation symbols. To express conditions for a complete metric in the syntax level is impossible similarly to first-order logic.

Fortunately, we will see that $\mathcal{L}$-pre-structures and $\mathcal{L}$-structures are indistinguishable in continuous logic by the following key theorem due to Yaacov and Pedersen (2010):

Theorem 2.3.1. *[Existence of Metric Completion] Let* $\mathfrak{M}$ *be a continuous* $\mathcal{L}$*-pre-structure. Then there exists a continuous* $\mathcal{L}$*-structure* $\widehat{\mathfrak{M}}$ *such that* $\mathfrak{M}$ *is elementarily equivalent to* $\widehat{\mathfrak{M}}$. *In other words, for any* $\mathcal{L}$*-formula* φ

$$\mathfrak{M} \models_Q \varphi \iff \widehat{\mathfrak{M}} \models_{QC} \varphi.$$

Proof. The proof is essentially using the fact that metric completion of a pseudo-metric space respects $\mathcal{L}$-pre-structure evaluations. See more details in Yaacov and Pedersen (2010). □

For a start, we have a strong soundness result.

Theorem 2.3.2. *(Strong Soundness of Continuous Logic) Fix a continuous language $\mathcal{L}$, an $\mathcal{L}$-theory Γ, and an $\mathcal{L}$-formula φ. If $\Gamma \vdash_{C\text{Ł}} \varphi$ then $\Gamma \models_{Q} \varphi$ and $\Gamma \models_{QC} \varphi$.*

A weak form of completeness does hold:

Theorem 2.3.3. *[Weak Soundness and Completeness of Continuous Logic] Given a continuous language $\mathcal{L}$ and an $\mathcal{L}$-theory Γ, the followings are equivalent:*

- Γ *is consistent.*
- Γ *is satisfiable.*
- Γ *is completely satisfiable.*

Proof. Given a consistent theory Γ, we extends Γ to a theory Δ that is maximally consistent and Henkin complete in the sense of continuous logic. One can build a pseudo-metric space using Δ, namely a term $\mathcal{L}$-model for Γ. Theorem 2.3.1 is used to obtain a complete $\mathcal{L}$-model for Γ. □

In first-order logic, weak completeness and strong completeness are equivalent. It is unfortunately not the case for continuous logic and Łukasiewicz logic due to the failure of the deduction theorem. We will discuss further in Section 3.1.

Theorem 2.3.4. *(Approximate Strong Soundness and Completeness of Continuous Logic) For a continuous language $\mathcal{L}$, an $\mathcal{L}$-theory Γ, and an $\mathcal{L}$-formula φ, the following are equivalent:*

- $\Gamma \models_{Q} \varphi$.
- $\Gamma \models_{QC} \varphi$.
- $\Gamma \vdash_{C\text{Ł}} \varphi \oplus \overbrace{(\varphi \odot \cdots \odot \varphi)}^{n}$ *for all* $n \in \mathbb{N}$.
- $\Gamma \vdash_{C\text{Ł}} \frac{\mathbf{1}}{\mathbf{2^n}} \to \varphi$ *for all* $n \in \mathbb{N}$ *for all* $n \in \mathbb{N}$.

Proof. We make use of Theorem 2.3.3 and a weak form of the Deduction Theorem in continuous logic. See Yaacov and Pedersen (2010). □

2.4. Connective $\frac{1}{2}$ and Definability of Rationals

The only logic connective in continuous logic that does not have a counterpart in first-order logic is the $\frac{1}{2}$ operator. There are two motivations for adding this operator in continuous logic:

- We need a collection of truth values dense in $[0, 1]$ to express uniform continuity axioms A13 and A14.
- Furthermore, we expect our choice of connectives to be complete, in the sense that all possible operations on truth values are expressible in some way.

Indeed, we cannot explicitly express any truth value other than $0, 1$ in Łukasiewicz logic by the following result of McNaughton (1951).

Theorem 2.4.1. *(McNaughton) A function $\Phi : [0, 1]^n \to [0, 1]$ is explicitly definable by composing connectives $\{\bot, \to\}$ iff Φ is uniformly continuous and piecewise linear (McNaughton functions). In other words, Φ is uniformly continuous and there are finitely many $\Phi_i(\bar{A}) = \sum_{j=1}^{n}(a_{ij}A_j + b_i)$ with integer a_{ij}, b_i such that for all $\bar{A} \in [0, 1]^n$ there exists an i such that $\Phi(\bar{A}) = \Phi_i(\bar{A})$.*

Corollary 2.4.2. *The only truth values in $[0, 1]$ explicitly definable as functions $\{*\} \to [0, 1]$ in Łukasiewicz logic are 0 and 1.*

However, we can implicitly define all rational truth values in Łukasiewicz logic, namely as unique solutions to certain propositional equations. This establishes that Łukasiewicz logic does not have the Beth definability property. See more discussions in Haniková (2019).

Theorem 2.4.3. *A truth value t in $[0, 1]$ is implicitly definable in Łukasiewicz logic if and only if t is rational.*

Proof. For the forward definition, if t is implicitly definable, then t is a solution to a linear polynomial

with rational coefficients by applying Theorem 2.4.1. Hence, t has to be rational.

For the backward definition, we provide the following construction: for value $\frac{1}{n+1}$, consider the formula

$$\Phi_{\frac{1}{n+1}}(\varphi) := (\neg\varphi) \leftrightarrow \overbrace{(\varphi \odot \cdots \odot \varphi)}^{n}$$

Notice that $v \models_{[0,1]} \Phi_{\frac{1}{n+1}}$ if and only if $v(\varphi) = \frac{1}{n+1}$ for any assignment v.

For value $\frac{m}{n+1}$ with $m \leqslant n+1$, consider the formula

$$\Phi_{\frac{m}{n+1}}(\varphi, \psi) := \Phi_{\frac{1}{n+1}}(\varphi) \wedge (\psi \leftrightarrow \overbrace{\varphi \odot \cdots \odot \varphi}^{m})$$

We have that $v \models_{[0,1]} \Phi_{\frac{m}{n+1}}(\varphi, \psi)$ if and only if $v(\varphi) = \frac{1}{n+1}$ and $v(\psi) = \frac{m}{n+1}$ for any assignment. □

The above result can extend to continuous logic by noticing $\frac{1}{3}$ is implicitly definable but not explicitly definable in the normal sense. In fact, Yaacov et al. (2008) extended the notion of explicit definability so that the Beth definability property still holds for continuous logic.

Definition 2.4.4. *A predicate $P : M^n \to [0,1]$ is **definable** in $\mathfrak{M}$ if and only if there exists a sequence of formulas $(\varphi_i(\bar{x}) \mid i \in \mathbb{N})$ such that $\varphi_i^{\mathfrak{M}}(\bar{x})$ converges to $P(x)$ uniformly.*

We can utilizes Theorem 2.4.3 to reformulate A13-14 without using $\frac{1}{2}$, provided that there are infinitely many unused relation symbols in the language:

A13´ $\vdash \Phi_p(r_1, r_2) \wedge \Phi_q(r_3, r_4) \wedge ((d(x,y) \to r_4) \vee (r_2 \to d(f(x;\bar{z}), f(y;\bar{z}))))$.

for each $f \in \mathcal{F}$, $\epsilon \in (0,1]$, and $p, q \in \mathbb{Q}$ with $p > \epsilon$ and $q < \Delta_f(\epsilon)$.

A14´ $\vdash \Phi_p(s_1, s_2) \wedge \Phi_q(s_3, s_4) \wedge ((d(z,w) \to s_4) \vee (s_2 \to (r(x;\bar{z}) \leftrightarrow r(y;\bar{z}))))$.

for each $r \in \mathcal{R}$, $\epsilon \in (0,1]$, and $p, q \in \mathbb{Q}$ with $p > \epsilon$ and $q < \Delta_r(\epsilon)$.

where r_i, s_j are fresh new relation symbols and Φ_p denotes the defining equation for rational p as in Theorem 2.4.3.

The second motivation for using $\frac{1}{2}$ is related to expressibility of arbitrary logic connectives. In classical logic, $\{\bot, \to\}$ is a complete set of connectives in the sense that any n-ary function $\{0,1\}^n \to \{0,1\}$ is expressible by composing $\{\bot, \to\}$.

In the continuous setting, we expand logic connectives to all uniformly continuous functions $[0,1]^n \to [0,1]$ and there are continuum many such. It is thus impossible to give a countable list of basic connectives since logic compositions have to be finite (see Baratella (2015) for discussions of infinitary continous logic). Instead, we choose $\{\bot, \frac{1}{2}, \to\}$, whose compositions form a **dense subset** of all possible logic connectives and we refer it as a **full** set of connectives.

By combining Theorem 2.4.1 and Theorem 2.4.3, we can require all continuous signatures to include an infinite set of relation symbols and obtain a full set of connectives using rational linear functions *à la* Stone–Weierstrass.

The benefit of replacing $\frac{1}{2}$ with unnamed relations symbols is that there are less axioms and rules to include when exploring different proof systems of continuous logic. Furthermore, the analogy between first-order logic and continuous logic now becomes stricter. Alternatively, one can include all rational values as basic logic connectives to obtain a conservative extension of (first-order) Łukasiewicz logic, which is referenced as Rational Pavelka logic. See Hájek et al. (2000) for further discussions.

CHAPTER 3

PROOF THEORY FOR PROPOSITIONAL ŁUKASIEWICZ LOGIC

3.1. Preliminary

The Deduction theorem is an important property of classical first-order logic and makes Gentzen-style proof systems possible. See Avigad (2022) for general details.

Definition 3.1.1. *(Deduction Theorem) A logic L with a Hilbert-style proof system has the* ***Deduction Theorem*** *if the followings are equivalent:*

- $\Gamma, \varphi \vdash_L \psi$
- $\Gamma \vdash_L \varphi \to \psi$

As discussed earlier, the Deduction Theorem fails in Łukasiewicz logic and continuous logic by the following example.

Example 1. *Consider $\varphi, \varphi \to (\varphi \to \psi) \vdash \psi$. We have a derivation in Łukasiewicz logic with a simply proof tree iterating the mp rule:*

$$\dfrac{\vdash_Ł \varphi \qquad \dfrac{\vdash_Ł \varphi \qquad \vdash_Ł \varphi \to (\varphi \to \psi)}{\vdash_Ł \varphi \to \psi}}{\vdash_Ł \psi}$$

On the other hand, $\varphi \to (\varphi \to \psi) \vdash_Ł \varphi \to \psi$ is not derivable since $\varphi \to (\varphi \to \psi) \nvDash_{[0,1]} \varphi \to \psi$. For instance we can pick a $[0,1]$-assignment v where $v(\varphi) = \frac{1}{2}$ and $v(\psi) = 1$. We have that $v \models_{[0,1]} \varphi \to (\varphi \to \psi))$ but $v \nvDash_{[0,1]} \varphi \to \psi$.

A weak form of the Deduction Theorem for Łukasiewicz logic and continuous logic does hold.

Theorem 3.1.2. *(Weak Deduction Theorem) Given a theory Γ and formulas φ, ψ, the followings are equivalent for both Łukasiewicz logic and continuous logic:*

1. $\Gamma, \varphi \vdash \psi$.

2. $\Gamma \vdash \overbrace{(\varphi \to (\varphi \to \cdots (\varphi}^{n} \to \psi) \cdots)$ *for some* n.

Proof. See Yaacov and Pedersen (2010). □

While $\vdash$ (external implication) and $\to$ (internal implication) behave the same in the presence of the Deduction Theorem, they depart from each other in Łukasiewicz logic: $\vdash$ allows uses of the same premise any number of times as seen in Example 1 while $\to$ requires a record of times a premise being used. This difference is also suggested by their interpretation:

- $\varphi \models_{[0,1]} \psi$ means if φ is evaluated at 0 then ψ is also evaluated at 0.
- $\models_{[0,1]} \varphi \to \psi$ means $v(\psi) \dot{-} v(\varphi) = 0$ for all valutations v.

In order to preserve the Deduction Theorem for the sake of a Gentzen-style proof system, we will alter the provability relation $\vdash_{Ł}$ and consequentially the entail relation $\models_{[0,1]}$ so that they also keep track of the number of times premises being used. It is equivalent to give up the following structural principle:

Definition 3.1.3 (Contraction)**.** *A proof system has* ***contraction*** *if the followings are equivalent:*

- $\Gamma, \varphi \vdash \psi$
- $\Gamma, \varphi, \varphi \vdash \psi$

Both classical logic and Łukasiewicz logic have contraction by how the provablity relation is defined. Classical logic without contraction is called **affine logic**, one of the substructural logics. The underlying difference between classical logic and affine logic is that a theory Γ is viewed as a **set** of formulas in classical logic while it is viewed as a **multiset** in affine logic.

Example 1 in the affine setting will have a derivation of $\varphi, \varphi, \varphi \to (\varphi \to \psi) \vdash \psi$ while $\varphi, \varphi \to (\varphi \to \psi) \vdash \psi$ will not be provable due to lack of contraction. The example will no longer counter the Deduction Theorem and we actually obtain a natural Gentzen-style proof system in this setting.

3.2. Hypersequent Calculus for Propositional Łukasiewicz Logic

Towards a Gentzen-style proof system of Łukasiewicz logic, we first define the notation of sequents and hypersequents in affine setting. To avoid confusion, we tend to use Greek letters $\varphi, \psi, \cdots$ in the case of Hilbert system with contraction and Latin letters $A, B, \cdots$ in the case of Gentzen system in affine setting.

Definition 3.2.1. *(Affine Sequent and Hypersequent) A* ***sequent*** *is of the form* $\Gamma \Rightarrow \Delta$ *where* Γ *and* Δ *are* ***finite multisets*** *of formulas. A* ***hypersequent*** *is a* ***finite multiset*** *of sequents, denoted as* $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, while a sequent can be viewed as a singleton hypersequent.*

For simplicity, we also adopt the following notations:

- nA *denotes the multiset* $\{\overbrace{A, \ldots, A}^{n}\}$ *where* A *is a formula.*
- $n\Gamma$ *denotes the multiset* $\overbrace{\Gamma \cup \cdots \cup \Gamma}^{n}$ *where* Γ *is a multiset of formulas.*
- *A formula is* ***atomic*** *if it is either a propositional variable or* $\bot$*.*

We first add affine features to the standard $[0,1]$-semantics.

Definition 3.2.2. *(Affine* $[0,1]$*-Semantics) Similar to the standard* $[0,1]$*-semantics, we define an assignment* v *mapping from propositional variables to* $[0,1]$ *and extend* v *to include all propositions in the same way. Furthermore, we extend* v *to include all sequents and hypersequents whose values are in* $\mathbb{R}$*.*

- $v(\bot) := 1$.
- $v(A \to B) := v(B) \dot{-} v(A)$.
- $v(\Gamma \Rightarrow \Delta) := \sum_{A \in \Delta} v(A) - \sum_{B \in \Gamma} v(B)$.
- $v(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) := \min_{1 \leqslant i \leqslant n} v(\Gamma_i \Rightarrow \Delta_i)$

*We say a hypersequent $\mathcal{H}$ is **valid** if $v(\mathcal{H}) \leqslant 0$ for any assignment v and we denote as $\models_{[0,1]} \mathcal{H}$. Informally, $\mathcal{H}$ is valid if and only if some of its component sequent represents an inequality valid in all possible assignments.*

Theorem 3.2.3. *We can connect the affine $[0,1]$-semantics and the standard $[0,1]$-semantics in the following way:*

1. *If $\Gamma \models_{[0,1]} A$ with set Γ, then there exists a mutliset Δ such that $\models_{[0,1]} \Delta \Rightarrow A$ and the support (removing duplications) of Δ is exactly Γ.*

2. *If $\models_{[0,1]} \Delta \Rightarrow A$ with multiset Δ, then $\Gamma \models_{[0,1]} A$ where Γ is the support of Δ.*

Proof. The forward direction is an application of the Weak Deduction Theorem in the semantics. The backward direction can be simply inferred by noticing that there exists a unique correspondence between a standard $[0,1]$-assignment and an affine $[0,1]$-assignment. □

Now we are ready to present the hypersequent calculus GŁ for propositional Łukasiewicz logic, due to Metcalfe et al. (2005).

Definition 3.2.4. *(GŁ) A hypersequent $\mathcal{H}$ is provable from GŁ, denoted as $\vdash_{G\text{Ł}} \mathcal{H}$, if it is derivable using a proof tree constructed from the following rules.*

Initial sequents:

$$\frac{}{A \Rightarrow A}\, id \qquad \frac{}{\Rightarrow} \Rightarrow \qquad \frac{}{\bot \Rightarrow A} \bot \Rightarrow$$

Structural rules:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\, ec \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\, ew$$

$$\frac{\mathcal{G} \mid \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}{\mathcal{G} \mid \Gamma_0 \Rightarrow \Delta_0 \mid \Gamma_1 \Rightarrow \Delta_1}\, split \qquad \frac{\mathcal{G} \mid \Gamma_0 \Rightarrow \Delta_0 \qquad \mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1}{\mathcal{G} \mid \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}\, mix$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}\, wl$$

Logical rules:

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta} \to\Rightarrow \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta \qquad \mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta} \Rightarrow\to$$

Remark 8. *GŁ is a two-sided (hyper)-sequent calculus. Different from classical two-sided sequent calculus LK where the left side is interpreted conjunctively and the right side disjunctively, both sides of $\Rightarrow$ in GŁ are interpreted conjunctively, as suggested by its affine $[0,1]$-semantics.*

Remark 9. *The disjunctive feature of GŁ appears in the hypersequent level where each component sequent is interpreted (additively) disjunctively. For instance, $A \Rightarrow B \mid B \Rightarrow A$ is derivable, in comparison to $\vdash_{LK} A, \neg A$ in classical logic. It indicates GŁ is a non-constructive logic. Combined with the later Lemma 3.3.3, GŁ allows proofs by case analysis without knowing precisely which case actually holds.*

Theorem 3.2.5. *(Strong Soundness of GŁ) For any hypersequent $\mathcal{H}$, if $\vdash_{G\text{Ł}} \mathcal{H}$ then $\models_{[0,1]} \mathcal{H}$.*

The *cut* rule in GŁ takes a simpler form, which is equivalent to the familiar *cut* rule.

Definition 3.2.6 (Cut)**.** *We have the following rule additional to GŁ.*

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\, cut$$

The following theorem establishes that the notion of validity is preserved among the standard and affine proof systems and semantics.

Theorem 3.2.7. *(Completeness of GŁ) For any formula A, the followings are equivalent:*

1. $\vdash_{\text{Ł}} A$.
2. $\models_{[0,1]} A$.
3. $\vdash_{G\text{Ł}+cut} \Rightarrow A$.

4. $\models_{[0,1]} \Rightarrow A$.

5. $\vdash_{G\text{Ł}} \Rightarrow A$.

Proof. The equivalence between 1 and 2 is exactly Theorem 2.2.4. The equivalence between 2 and 4 is a consequence of the above Theorem 3.2.3 between the affine $[0,1]$-semantics and the standard $[0,1]$-semantics.

$3 \implies 4$ and $5 \implies 4$ are exactly the Soundness Theorem 3.2.5 plus soundness of the *cut* rule. We will present the proof $1 \implies 3$ by showing all axioms and rules in Ł are derivable in GŁ:

A1

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}}{B, A \Rightarrow A}}{\Rightarrow A \to (B \to A)}$$

A2

$$\dfrac{\dfrac{\dfrac{}{A, B, C \Rightarrow A, B, C}}{A, B \to C, A \to B \Rightarrow C}}{\Rightarrow (A \to B) \to ((B \to C) \to (A \to C))}$$

A3

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}}{B \to A, A \Rightarrow A}}{A \Rightarrow (B \to A) \to A} \quad \dfrac{}{B \Rightarrow B} \quad \dfrac{\dfrac{\dfrac{}{A, B \Rightarrow A, B}}{B, B \to A \Rightarrow A}}{B \Rightarrow (B \to A) \to A}}{B, A \Rightarrow B, (B \to A) \to A}}{B \Rightarrow A \to B, (B \to A) \to A}}{(A \to B) \to B \Rightarrow (B \to A) \to A}}{\Rightarrow ((A \to B) \to B) \to ((B \to A) \to A)}$$

A4

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, \bot \Rightarrow A, \bot}}{A, A \to \bot \Rightarrow \bot} \quad \dfrac{}{B \Rightarrow B}}{A, A \to \bot, B \Rightarrow \bot, B} \quad \dfrac{\dfrac{\dfrac{}{\bot \Rightarrow B} \quad \dfrac{}{A \Rightarrow A}}{A, \bot \Rightarrow A, B}}{A, A \to \bot \Rightarrow B}}{A, A \to \bot \Rightarrow B \to \bot, B}}{A, (B \to \bot) \to (A \to \bot) \Rightarrow B}}{\Rightarrow ((B \to \bot) \to (A \to \bot)) \to (A \to B)}$$

Modus Ponens

$$\dfrac{\Rightarrow A \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A, B \Rightarrow A, B}}{A, A \to B \Rightarrow B}}{A, A \to B \Rightarrow A, B}}{A \to B \Rightarrow B} \quad \Rightarrow A \to B}{\dfrac{A \to B \Rightarrow A \to B, B}{\Rightarrow B}\, cut}$$

It remains to show 3 $\implies$ 5, namely cut elimination for GŁ. Since it is rather involved, we will provide its formal proof in the next Section 3.3. □

Remark 10. *Notice that hypersequents and related rules in GŁ are never needed if only validity problem is considered. However, hypersequents, especially the ew and ec rules, are essential to show cut elimination and strong completeness of GŁ. Metcalfe et al. (2005) also describes a pure sequent calculus GŁ$_s$ for Łukasiewicz logic, which is an interesting topic on its own.*

3.3. Cut Elimination

We will show proofs in GŁ with the *cut* rule can be transformed into proofs in GŁ without *cut*. Similar proof can also be found in Ciabattoni and Metcalfe (2003).

First, we will show that both $\Rightarrow\!\to$ and $\to\!\Rightarrow$ are invertible rules.

Lemma 3.3.1. *(Inversion Lemma for $\Rightarrow\!\to$) The following rules are admissible in GŁ and GŁ+cut.*

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta} \qquad\qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

Furthermore, no new cut is introduced in the derivation.

Proof. A standard proof by induction on the proof tree of $\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta$. □

Different from classical logic, we also have an inversion lemma for $\rightarrow\Rightarrow$ by utilizing hypersequent rules and the following lemmas.

Lemma 3.3.2. *(Weakening) Suppose that we can derive $\mathcal{H}$ from assumption hypersequents $\{\mathcal{G}_i\}$ in GŁ. Then for any hypersequent $\mathcal{L}$, we can derive $\mathcal{L} \mid \mathcal{H}$ from assumptions $\{\mathcal{L} \mid \mathcal{G}_i\}$.*

Proof. Proof by induction where the *ew* rule by $\mathcal{L}$ is used in the main cases. □

Lemma 3.3.3. *(Proof by Case) Assume we have the following derivations in GŁ:*

$$\frac{\mathcal{G} \mid \mathcal{L}_0 \qquad \mathcal{G} \mid \mathcal{R}}{\mathcal{G} \mid \mathcal{H}} \qquad\qquad \frac{\mathcal{G} \mid \mathcal{L}_1 \qquad \mathcal{G} \mid \mathcal{R}}{\mathcal{G} \mid \mathcal{H}}$$

Then we have:

$$\frac{\mathcal{G} \mid \mathcal{L}_0 \mid \mathcal{L}_1 \qquad \mathcal{G} \mid \mathcal{R}}{\mathcal{G} \mid \mathcal{H}}$$

Proof. It is a consequence of Lemma 3.3.2 and external contraction.

$$\dfrac{\dfrac{\mathcal{G} \mid \mathcal{L}_0 \mid \mathcal{L}_1 \qquad \dfrac{\mathcal{G} \mid \mathcal{R}}{\mathcal{G} \mid \mathcal{L}_0 \mid \mathcal{R}}}{\mathcal{G} \mid \mathcal{L}_0 \mid \mathcal{H}} \qquad \dfrac{\mathcal{G} \mid \mathcal{R}}{\mathcal{G} \mid \mathcal{R} \mid \mathcal{H}}}{\dfrac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}}$$

□

Remark 11. *Lemma 3.3.3 can be viewed as a **disjunction elimination** rule.*

Equipped with the above lemma, we can obtain the invertibility of $\rightarrow$ on the left.

Lemma 3.3.4. *(Inversion Lemma for Left $\rightarrow$) The following rule is admissible in GŁ and GŁ+cut.*

$$\frac{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta}$$

Furthermore, no new cut is introduced in the derivation.

Proof. A proof by induction. The only difficult case is when the immediate rule is a right $\to$ introduction:

$$\frac{\mathcal{G} \mid \Gamma, A \to B, C \Rightarrow D, \Delta \qquad \mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow C \to D, \Delta}$$

By inductive hypothesis, we have derivations for two hypersequents $\mathcal{G} \mid \Gamma, C, B \Rightarrow A, D, \Delta \mid \Gamma, C \Rightarrow D, \Delta$ and $\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta$.

Applying Lemma 3.3.3 twice, it suffices to derive $\mathcal{G} \mid \Gamma, B \Rightarrow C \to D, A, \Delta \mid \Gamma \Rightarrow C \to D, \Delta$ for the following four cases:

1. $\mathcal{G} \mid \Gamma, C, B \Rightarrow A, D, \Delta$ and $\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta$.
2. $\mathcal{G} \mid \Gamma, C, B \Rightarrow A, D, \Delta$ and $\mathcal{G} \mid \Gamma \Rightarrow \Delta$.
3. $\mathcal{G} \mid \Gamma, C \Rightarrow D, \Delta$ and $\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta$.
4. $\mathcal{G} \mid \Gamma, C \Rightarrow D, \Delta$ and $\mathcal{G} \mid \Gamma \Rightarrow \Delta$.

Case 1 and case 4 can be easily resolved by applying $\Rightarrow\!\to$ and ew to the given assumptions.

In case 2, we combine the two assumptions to obtain the following:

$$\cfrac{\cfrac{\mathcal{G} \mid \Gamma, B, C \Rightarrow D, A, \Delta \qquad \mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Gamma, B, C \Rightarrow D, A, \Delta, \Delta}}{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma, C \Rightarrow D, \Delta}$$

Then by Lemma 3.3.3 again, it suffices to derive $\mathcal{G} \mid \Gamma, B \Rightarrow C \to D, A, \Delta \mid \Gamma \Rightarrow C \to D, \Delta$ for the following two cases:

1. $\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta$. We combine with the assumption $\mathcal{G} \mid \Gamma, C, B \Rightarrow A, D, \Delta$ to derive:

$$\cfrac{\cfrac{\mathcal{G} \mid \Gamma, B, C \Rightarrow D, A, \Delta \qquad \mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, B \Rightarrow C \to D, A, \Delta}}{\mathcal{G} \mid \Gamma, B \Rightarrow C \to D, A, \Delta \mid \Gamma \Rightarrow C \to D, \Delta}$$

2. $\mathcal{G} \mid \Gamma, C \Rightarrow D, \Delta$. We combine with the assumption $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ to derive:

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma, C \Rightarrow D, \Delta \qquad \mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow C \to D, \Delta}}{\mathcal{G} \mid \Gamma, B \Rightarrow C \to D, A, \Delta \mid \Gamma \Rightarrow C \to D, \Delta}$$

Case 3 can be resolved by an argument similar to case 2. □

Theorem 3.3.5. *(Cut Elimination for GŁ) If hypersequent $\mathcal{H}$ is deriable in GŁ with cut, then $\mathcal{H}$ is also derivable in GŁ without cut.*

Proof. Without loss of generality, we may assume that the *cut* rule appears as the last inference rule and the cut formula is of the form $A \to B$:

$$\dfrac{\mathcal{G} \mid \Gamma, A \to B \Rightarrow A \to B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

Apply the left inversion Lemma 3.3.4, we obtain a derivation of $\mathcal{G} \mid \Gamma, B \Rightarrow A, A \to B, \Delta \mid \Gamma \Rightarrow A \to B, \Delta$. Applying the right inversion Lemma 3.3.1 twice, we obtain the derivable hypersequent

$$\mathcal{G} \mid \Gamma, B, A \Rightarrow B, A, \Delta \mid \Gamma \Rightarrow \Delta$$

By performing two *cut* on formulas A and B with lower rank, we have

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \Gamma, B, A \Rightarrow B, A, \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, B \Rightarrow B, \Delta \mid \Gamma \Rightarrow \Delta}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

Now it suffices to consider the case $\mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta$ when the cut formula A is atomic. Through some case analysis, we can reduce to the following three scenarios:

- The formula A on the right side of $\Rightarrow$ is introduced by the *id* rule. We obtain the resulting hypersequent $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ by not mixing with $A \Rightarrow A$ from the top of the proof tree.

- The formula A on the right side of $\Rightarrow$ is introduced by the $\bot \Rightarrow$ rule and A on the left is introduced by wk. We obtain $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ by not mixing with $\bot \Rightarrow A$ and an internal weakening with $\bot$ instead of A.

- The hypersequent $\mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta$ is obtained from $\mathcal{G}$ by ew. We obtain $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ by an external weakening of $\mathcal{G}$ with $\Gamma \Rightarrow \Delta$.

□

3.4. Strong Completeness of GŁ

Here we present our proof of strong completeness of GŁ with respect to the affine $[0, 1]$-semantics. Similar ideas were already hinted in Colacito et al. (2021) and Metcalfe et al. (2005).

Lemma 3.4.1. *If $\Gamma \Rightarrow \Delta$ is a valid sequent consisting only of atomic formulas, then we have a deduction of $\Gamma \Rightarrow \Delta$ from GŁ.*

Proof. If the sequent is of the form $\Gamma, A \Rightarrow A, \Delta$, then $\Gamma \Rightarrow \Delta$ remains valid. Combing it with the following derivation:

$$\dfrac{\Gamma \Rightarrow \Delta \qquad \dfrac{}{A \Rightarrow A}\, id}{\Gamma, A \Rightarrow A, \Delta}\, mix$$

we may assume without loss of generality that $\Gamma \cap \Delta = \varnothing$.

If Δ is empty, we are done either by the $\Rightarrow$ rule if Γ is empty or iterating wl if Γ is not empty. Otherwise, the sequent is now the form $\Gamma \Rightarrow n_1 A_1, \ldots, n_k A_k$ (see Definition 3.2.1 for notations). Suppose there are $n < n_1 + \cdots + n_k$-many $\bot$ in Γ, we define an interpretation v where $v(A_i) = 1$ for each i and $v(B) = 0$ otherwise. Then $v(\Gamma \Rightarrow n_1 A_1, \ldots, n_k A_k) = n_1 + \cdots + n_k - n > 0$, which contradicts with validity of $\Gamma \Rightarrow n_1 A_1, \ldots, n_k A_k$. Therefore, we must have $n \geqslant n_1 + \cdots + n_k$ and we now construct a deduction of $\Gamma \Rightarrow n_1 A_1, \ldots, n_k A_k$ by $n_1 + \cdots + n_k$-many $\bot \Rightarrow$ rules together with wl and mix. □

We will prove completeness of GŁ by a bottom-up proof search. We need the following variant of Farkas' lemma from linear programming to construct the leaf hypersequents. The proof of this variant can be found in Ball (2023).

Lemma 3.4.2. *(Farkas Lemma) Given matrices $M \in \mathbb{R}^{m_1 \times n}$ and $N \in \mathbb{R}^{m_2 \times n}$ and vectors $a \in \mathbb{R}^{m_1}$ and $b \in \mathbb{R}^{m_2}$. Exactly one of the following holds.*

- *There exists $x \in \mathbb{R}^n$ satisfying*

$$\begin{aligned} Mx &\leqslant a \\ Nx &\ll b \\ x &\geqslant 0 \end{aligned}$$

- *There exist $\lambda \in \mathbb{R}^{m_1}$ and $\mu \in \mathbb{R}^{m_2}$ satisfying*

$$\begin{aligned} \lambda &\geqslant 0 \\ \mu &\geqslant 0 \\ \lambda^T M + \mu^T N &\geqslant 0 \\ \lambda^T a + \mu^T b &\leqslant 0 \\ \lambda^T a + \mu^T b < 0 \text{ or } \mu &> 0 \end{aligned}$$

where $v \leqslant w$ means $v_i \leqslant w_i$ for all i and $v \ll w$ means $v_i < w_i$ for all i.

Theorem 3.4.3. *(Strong Completeness of GŁ) If $\mathcal{H}$ is a valid hypersequent, then $\mathcal{H}$ is derivable from GŁ.*

Proof. We construct a search tree for a deduction of valid hypersequent $\mathcal{H}$.

- $\mathcal{H}$ is the base node.

- At node $\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B$, search for deductions of $\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta$ and $\mathcal{G} \mid \Gamma \Rightarrow \Delta$.
- At node $\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta$, search for deduction of $\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta$.
- At node $\mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta$ where A is either $\bot$ or atomic, search for a deduction of $\mathcal{G} \mid \Gamma \Rightarrow \Delta$.

By applying the $\to$ introduction rules and mix with identity sequents, it suffices to deal with valid hypersequents consisting of only atomic formulas. Let $A_1, \ldots, A_m$ enumerate all propositional variables appearing in $\mathcal{H}$ and say $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$. Since $\mathcal{H}$ is a valid hypersequent, the following system of inequalities of $m \times m$-matrix M and $k \times m$-matrix N

$$\begin{aligned} v(A_1) &\leqslant 1 \\ &\vdots \\ v(A_m) &\leqslant 1 \\ -a_{11}v(A_1) - \ldots - a_{1m}v(A_m) &< -c_0 \\ &\vdots \\ -a_{k1}v(A_1) - \ldots - a_{km}v(A_m) &< -c_k \end{aligned}$$

has no nonnegative solution for any interpretation v where a_{ij} represents the multiplicity of A_j in $\Gamma_i \Rightarrow \Delta_i$ with negative a_{ij} if $x_j \in \Gamma_i$, positive if $x_j \in \Delta_i$, and zero otherwise; c_i represents the multiplicity of $\bot$ in $\Gamma_i \Rightarrow \Delta_i$ with the same parity definition as a_{ij}.

Applying the Farkas lemma 3.4.2 to the above system of inequalities, we obtain some $\lambda \geqslant 0$ and $\mu \geqslant 0$. We may further assume λ and μ are vectors of integers since all coefficients of the system are integers. We obtain the following inequalities:

$$\begin{gathered} \lambda_j - \sum_{i=1}^{k} \mu_i a_{ij} \geqslant 0 \text{ for each } i \\ -\sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{k} \mu_i c_i \geqslant 0 \end{gathered}$$

Hence the following holds for all interpertation v:

$$(\sum_{j=1}^{m} \lambda_j - \sum_{i=1}^{k} \mu_i c_i) \cdot 1 + (-\lambda_1 + \sum_{i=1}^{k} \mu_i a_{i1}) \cdot v(A_1) + \cdots + (-\lambda_m + \sum_{i=1}^{k} \mu_i a_{im}) \cdot v(A_m) \leqslant 0$$

Adding λ_i-many inequalities $v(A_i) \leqslant 1$ for each i, we simplify it into the following inequality holding for any interpretation v:

$$(\sum_{i=1}^{k} \mu_i c_i) \cdot 1 + (\sum_{i=1}^{k} \mu_i a_{i1}) \cdot v(A_1) + \cdots + (\sum_{i=1}^{k} \mu_i a_{im}) \cdot v(A_m) \leqslant 0$$

Now we construct a sequent $\Gamma \Rightarrow \Delta$ where we add $\mu_i|a_{ij}|$-many A_j in Γ if a_{ij} is positive and to Δ if a_{ij} is negative; similarly, we add $\mu_i|ci|$-many $\bot$ to Γ if c_i is positive and to Δ if c_i is negative. By definition, $\Gamma \Rightarrow \Delta$ is a valid sequent consisting only of atomic formulas since the above inequality holds for all interpretation v. We then have a deduction of $\Gamma \Rightarrow \Delta$ by Lemma 3.4.1. Since $\Gamma \Rightarrow \Delta$ essentially contains μ_i-many copies of $\Gamma_i \Rightarrow \Delta_i$ by design, we can recover the original hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_k \Rightarrow \Delta_k$ by the following derivation where the last steps are ew if $\mu_i = 0$ and ec if $\mu_i \neq 0$.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta}{\underbrace{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_1 \Rightarrow \Delta_1}_{\mu_1} \mid \cdots \mid \underbrace{\Gamma_k \Rightarrow \Delta_k \mid \cdots \mid \Gamma_k \Rightarrow \Delta_k}_{\mu_k}}\, split}{\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_k \Rightarrow \Delta_k}\, ec/ew$$

□

CHAPTER 4

PROOF THEORY FOR FIRST-ORDER ŁUKASIEWICZ LOGIC

4.1. Preliminary

In this chapter, we will extend the hypersequent calculus to first-order Łukasiewicz logic and present our approximate completeness result.

Remark 12. *For simplicity of the Approximate Herbrand's Theorem 4.4.5 later on, we choose* $\exists$ *as the primitive quantifier and define*

$$\forall x A := \neg(\exists x(\neg A))$$

despite the inconsistency with the name GŁ$\forall$*.*

4.2. Hypersequent Calculus for First-Order Łukasiewicz Logic

Definition 4.2.1. *GŁ*$\forall$ *is obtained from GŁ by adding the following two rules and replacing propositions with first-order formulas:*

$$\frac{\mathcal{G} \mid \Gamma, A(c) \vdash \Delta}{\mathcal{G} \mid \Gamma, (\exists x)A[x/c] \Rightarrow \Delta}\,\exists \Rightarrow \qquad\qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A[t/x], \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)A, \Delta} \Rightarrow \exists$$

where c *does not appear in the conclusion* $\mathcal{G} \mid \Gamma, (\exists x)A[x/c] \Rightarrow \Delta$*.*

Now we also introduce the affine semantics for first-order Łukasiewicz logic accordingly.

Definition 4.2.2. *(Affine* $[0,1]$*-Structure) An* $[0,1]$*-structure* $\mathfrak{M}$ *is a set* M *together with its interpretation for function and relation symbols.*

- *If* f *is an* n*-ary function symbol, then* $f^{\mathfrak{M}}$ *is a function* $M^n \to M$*.*
- *If* r *is an* n*-ary relation symbol, then* $r^{\mathfrak{M}}$ *is a function* $M^n \to [0,1]$*.*
- $\bot^{\mathfrak{M}} = 1$*.*
- $(A \to B)^{\mathfrak{M}} = B^{\mathfrak{M}} \dot{-} A^{\mathfrak{M}}$*.*

- $((\exists x)A)^{\mathfrak{M}} = \inf_{m \in M}(A[m/x])^{\mathfrak{M}}$.
- $(\Gamma \Rightarrow \Delta)^{\mathfrak{M}} := \sum_{A \in \Delta} A^{\mathfrak{M}} - \sum_{B \in \Gamma} B^{\mathfrak{M}}$.
- $(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n)^{\mathfrak{M}} := \min_{1 \leqslant i \leqslant n}(\Gamma_i \Rightarrow \Delta_i)^{\mathfrak{M}}$

*We say $\mathcal{H}$ is **valid** if $\mathcal{H}$ contains no free variable and $\mathcal{H}^{\mathfrak{M}} \leqslant 0$ for all $[0,1]$-structures $\mathfrak{M}$. We denote it as $\models_{[0,1]} \mathcal{H}$ without causing any ambiguity.*

Theorem 4.2.3. *(Strong Soundness of GŁ∀) For any hypersequent $\mathcal{H}$, if $\vdash_{G\text{Ł}\forall} \mathcal{H}$ then $\models_{[0,1]} \mathcal{H}$.*

The benefit of the affine setting is that approximations of formulas can now be expressed very naturally and effectively.

Definition 4.2.4. *We adopt the following notations for any given formula A, multisets Γ, Δ, and natural number n:*

- $\Gamma \Rightarrow_{\frac{1}{n}} \Delta$ *denotes the sequent*

$$\bot, n\Gamma \Rightarrow n\Delta$$

  whose intended meaning is that $\Gamma \Rightarrow \Delta$ is valid up to an $\frac{1}{n}$ error in all interpretations, since $\bot$ has truth value 1.

- *We generalize the above notation to hypersequents $\mathcal{H}$. Say $\mathcal{H}$ is of the form $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_k \Rightarrow \Delta_k$. Then $\mathcal{H}_{\frac{1}{n}}$ is the hypersequent*

$$\Gamma_1 \Rightarrow_{\frac{1}{n}} \Delta_1 \mid \cdots \mid \Gamma_k \Rightarrow_{\frac{1}{n}} \Delta_k$$

- *$A[\bar{t}/\bar{x}], \Gamma[\bar{t}/\bar{x}], \mathcal{H}[\bar{t}/\bar{x}]$ represent substitutions of free variables $\bar{x}$ by terms $\bar{t}$ on formula, sequent, and hypersequent levels.*

4.3. A Motivating Example

The proof of the Approximate Completeness of GŁ∀ is rather complicated. The following serves as a motivating example from which the actual proof generalizes.

Example 2. *(Drinker Paradox) Let* $\varphi := \exists x(A(x) \to \forall y A(y))$ *where* A *is a unary predicate symbol.*

With some computation, we have that $\models_{[0,1]} \varphi$ and $\vdash_{\mathrm{GŁ}\forall + cut} \Rightarrow \varphi$. However, φ is not provable in GŁ∀ as indicated in Baaz and Metcalfe (2010). Instead, we have derivations of arbitrary approximation of the form $\Rightarrow_{\frac{1}{n}} \varphi$ for each n. To search for such derivations, we perform Skolemization and invert the connective $\to$ from bottom up. We implicitly assume the admissible introduction rules for $\forall$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(x) \Rightarrow A(f(x))}{A(x), \bot \Rightarrow A(f(y)), \bot}}{A(x), A(f(y)) \to \bot \Rightarrow \bot}}{A(x), \exists y(A(y) \to \bot) \Rightarrow \bot}}{A(x) \Rightarrow \exists y(A(y) \to \bot) \to \bot}}{A(x) \Rightarrow \forall y A(y)}}{\Rightarrow A(x) \to \forall y A(y)}}{\Rightarrow \exists x(A(x) \to \forall y A(y))}$$

Unfortunately, the top sequent $A(x) \Rightarrow A(f(x))$ may not have any term witnesses. Instead, for each n, there are approximate Herbrand terms $x, f(x), f(f(x)) \ldots, f^{n-1}(x)$ such that the following is valid

$$A(x) \Rightarrow_{\frac{1}{n}} A(f(x)) \mid A(f(x)) \Rightarrow_{\frac{1}{n}} A(f^2(x)) \mid \cdots \mid A(f^{n-1}(x)) \Rightarrow_{\frac{1}{n}} A(f^n(x))$$

Since the above hypersequent is quantifier-free, we guarantee to have a derivation of the above valid hypersequent by the Strong Completeness Theorem 3.4.3 of GŁ, which is presented below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{n\bot \Rightarrow nA(f^n(x))}}{n\bot, nA(x) \Rightarrow nA(f^n(x))}}{n\bot, nA(x), nA(f(x)), \ldots, nA(f^{n-1}(x)) \Rightarrow nA(f(x)), \ldots, nA(f^n(x))}}{\bot, nA(x) \Rightarrow nA(f(x)) \mid \bot, nA(f(x)) \Rightarrow nA(f^2(x)) \mid \cdots \mid \bot, nA(f^{n-1}(x)) \Rightarrow nA(f^n(x))}}{A(x) \Rightarrow_{\frac{1}{n}} A(f(x)) \mid A(f(x)) \Rightarrow_{\frac{1}{n}} A(f^2(x)) \mid \cdots \mid A(f^{n-1}(x)) \Rightarrow_{\frac{1}{n}} A(f^n(x))}$$

Then we will follow the initial search tree down from top to recover $\Rightarrow_{\frac{1}{n}} \exists x(A(x) \to \forall y A(y)$ by re-introducing connectives in the reverse order for all the n component sequents. We also need to admit a $\frac{1}{n}$-version of connective introduction rules for doing that.

Ultimately, we have the following derivation:

$$\dfrac{\dfrac{\dfrac{}{n\bot \Rightarrow nA(f^n(x))}}{\vdots}}{\dfrac{A(x) \Rightarrow_{\frac{1}{n}} A(f(x)) \mid A(f(x)) \Rightarrow_{\frac{1}{n}} A(f^2(x)) \mid \cdots \mid A(f^{n-1}(x)) \Rightarrow_{\frac{1}{n}} A(f^n(x))}{\vdots}}{\dfrac{\Rightarrow_{\frac{1}{n}} \exists x(A(x) \to \forall y A(y) \mid \cdots \mid \Rightarrow_{\frac{1}{n}} \exists x(A(x) \to \forall y A(y)}{\Rightarrow_{\frac{1}{n}} \exists x(A(x) \to \forall y A(y)} ec}$$

Hence, we obtain derivations of the arbitrary approximations $\vdash_{\text{GŁ}\forall} \Rightarrow_{\frac{1}{n}} \varphi$ for the given valid formula φ, which is the expected result from the Approximate Completeness Theorem for GŁ$\forall$.

The following section generalizes the above proof-search argument to arbitrary valid hypersequents.

4.4. Approximate Strong Completeness of GŁ$\forall$

Given a valid hypersequent $\mathcal{H}$ in first-order Łukasiewicz logic without free variables (universalizing if necessary), we will construct a search tree for a derivation of $\vdash_{\frac{1}{n}} \mathcal{H}$.

Definition 4.4.1. *(Skolemization Tree) Given a hypersequent $\mathcal{H}$, we build a binary tree $T_{\mathcal{H}}$ of hypersequents with free variables $\bar{x}$ to remove quantifiers and $\to$. The order can be taken arbitrarily if multiple conditions of the followings are satisfied.*

Base case: *$\mathcal{H}$ is the base node.*

Right $\to$ case: *If the **left-most** leaf $\mathcal{G}$ has the form*

$$\mathcal{L} \mid \Gamma \Rightarrow A \to B, \Delta$$

*we attach to $\mathcal{G}$ a **left** node $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ and a **right** node $\mathcal{L} \mid \Gamma \Rightarrow \Delta$. We **associate** sequent $\Gamma, A \Rightarrow B, \Delta$ in the left $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ **with** sequent $\Gamma \Rightarrow \Delta$ in the right $\mathcal{L} \mid \Gamma \Rightarrow \Delta$ and*

also ***associate*** *the remaining component sequents in* $\mathcal{L}$ *between* $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ *and* $\mathcal{L} \mid \Gamma \Rightarrow \Delta$ *accordingly.*

For any other leaf $\mathcal{R}$ *in the tree, there are two cases:*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^* \Rightarrow A \to B, \Delta^*$ *where sequent* $\Gamma^* \Rightarrow A \to B, \Delta^*$ *in* $\mathcal{R}$ *is previously associated with sequent* $\Gamma \Rightarrow A \to B, \Delta$ *in* $\mathcal{G}$*. We attach to* $\mathcal{R}$ *two nodes* $\mathcal{L}^* \mid \Gamma^*, A \Rightarrow B, \Delta^*$ *and* $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$*. Both new nodes are associated with the new left-most leaf* $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ *accordingly with both* $\Gamma^*, A \Rightarrow B, \Delta^*$ *and* $\Gamma^* \Rightarrow \Delta^*$ *associated with* $\Gamma, A \Rightarrow B, \Delta$*.*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ *(with* $A \to B$ *absent on the right of* $\Rightarrow$*) where* $\Gamma^* \Rightarrow \Delta^*$ *is previously associated with* $\Gamma \Rightarrow A \to B, \Delta$ *in* $\mathcal{G}$*. We attach* $\mathcal{R}$ *itself in this node and associate it with* $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ *accordingly such that* $\Gamma^* \Rightarrow \Delta^*$ *is associated with* $\Gamma, A \Rightarrow B, \Delta$*.*

Left $\to$ ***case:*** *If the left-most leaf* $\mathcal{G}$ *has the form*

$$\mathcal{L} \mid \Gamma, A \to B \Rightarrow \Delta$$

we attach to $\mathcal{G}$ *the node* $\mathcal{L} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta$*.*

For any other leaf $\mathcal{R}$*,*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^*, A \to B \Rightarrow \Delta^*$ *where* $\Gamma^*, A \to B \Rightarrow \Delta^*$ *is previously associated with* $\Gamma, A \to B \Rightarrow \Delta$ *in* $\mathcal{G}$*. We attach to* $\mathcal{R}$ *the node* $\mathcal{L}^* \mid \Gamma^*, B \Rightarrow A, \Delta^* \mid \Gamma^* \Rightarrow \Delta^*$*. We carry over the associations with the new left-most node* $\mathcal{L} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta$ *such that* $\Gamma^*, B \Rightarrow A, \Delta^*$ *is associated with* $\Gamma, B \Rightarrow A, \Delta$ *and* $\Gamma^* \Rightarrow \Delta^*$ *is associated with* $\Gamma \Rightarrow \Delta$*.*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ *(with* $A \to B$ *absent on the left of* $\Rightarrow$*) where* $\Gamma^* \Rightarrow \Delta^*$ *is previously associated with* $\Gamma, A \to B \Rightarrow \Delta$ *in* $\mathcal{G}$*. We attach* $\mathcal{R}$ *itself in this node and associate it with* $\mathcal{L} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta$ *accordingly such that* $\Gamma^* \Rightarrow \Delta^*$ *is associated with* $\Gamma, A \Rightarrow B, \Delta$*.*

Right $\exists$ ***case:*** *If the left-most leaf* $\mathcal{G}$ *takes the form*

$$\mathcal{L} \mid \Gamma \Rightarrow (\exists y)A, \Delta$$

we attach to $\mathcal{G}$ *the node* $\mathcal{L} \mid \Gamma \Rightarrow A[x/y], \Delta$ *where* x *is a new variable symbol.*

For any other leaf $\mathcal{R}$*,*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^* \Rightarrow (\exists y)A, \Delta^*$ *where* $\Gamma^* \Rightarrow (\exists y)A, \Delta^*$ *is previously associated with* $\Gamma \Rightarrow (\exists y)A, \Delta$ *in* $\mathcal{G}$*. We attach to* $\mathcal{R}$ *the node* $\mathcal{L}^* \mid \Gamma^* \Rightarrow A[x/y], \Delta^*$ *with the same variable* x*. We carry over the associations with the new left-most node* $\mathcal{L}^* \mid \Gamma^* \Rightarrow A[x/y], \Delta^*$ *such that* $\Gamma^* \Rightarrow A[x/y], \Delta^*$ *is associated with* $\Gamma \Rightarrow A[x/y], \Delta$*.*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ *(with* $(\exists y)A$ *absent on the right of* $\Rightarrow$*) where* $\Gamma^* \Rightarrow \Delta^*$ *is previously associated with* $\Gamma \Rightarrow (\exists y)A, \Delta$ *in* $\mathcal{G}$*. We attach* $\mathcal{R}$ *itself in this node and associate it with* $\mathcal{L}^* \mid \Gamma^* \Rightarrow A[x/y], \Delta^*$ *such that* $\Gamma^* \Rightarrow \Delta^*$ *is associated with* $\Gamma \Rightarrow A[x/y], \Delta$*.*

Left $\exists$ ***case:*** *If the left-most leaf* $\mathcal{R}$ *takes the form*

$$\mathcal{L} \mid \Gamma, (\exists y)A \Rightarrow \Delta$$

we attach to $\mathcal{G}$ *the node* $\mathcal{G} \mid \Gamma, A[f(\bar{x})/y] \Rightarrow \Delta$ *where* f *is a new function symbol and* $\bar{x}$ *enumerates all free variables appearing at the current stage of the tree.*

For all other leaf $\mathcal{R}$*,*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^*, (\exists y)A \Rightarrow \Delta^*$ *where* $\Gamma^*, (\exists y)A \Rightarrow \Delta^*$ *is previously associated with* $\Gamma, (\exists y)A \Rightarrow \Delta$ *in* $\mathcal{G}$*. We attach to* $\mathcal{R}$ *the node* $\mathcal{G}^* \mid \Gamma^*, A[f(\bar{x})/y] \Rightarrow \Delta^*$ *with the same* f *and* $\bar{x}$*. We carry over the associations with the new left-most node* $\mathcal{L}^* \mid \Gamma^*, A[f(\bar{x})/y] \Rightarrow \Delta^*$ *such that* $\Gamma^*, A[f(\bar{x})/y] \Rightarrow \Delta^*$ *is associated with* $\Gamma, A[f(\bar{x})/y] \Rightarrow \Delta$*.*

- $\mathcal{R}$ *has the form* $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ *(with* $(\exists y)A$ *absent on the left of* $\Rightarrow$*) where* $\Gamma^* \Rightarrow \Delta^*$ *is*

previously associated with $\Gamma, (\exists y)A \Rightarrow \Delta$ *in* $\mathcal{G}$*. We attach* $\mathcal{R}$ *itself in this node and associate it with* $\mathcal{L}^* \mid \Gamma^*, A[f(\bar{x})/y] \Rightarrow \Delta^*$ *such that* $\Gamma^* \Rightarrow \Delta^*$ *is associated with* $\Gamma, A[f(\bar{x})/y] \Rightarrow \Delta$*.*

Remark 13. *The idea of the tree construction is rather simple while it is painful to give a precise definition: we have branching when inverting connectives because of the* $\Rightarrow\!\!\rightarrow$ *rule combines two assumptions. We work backward using the left-most branch with the requirement that the other branches synchronously copy the exact reversing steps from the left-most branch and skipping if the principle formula is missing. Notice that hypersequent in right nodes are "sub-hypersequents" of the left-most one.*

Let h denote the max length of paths in $T_{\mathcal{H}}$. We define level sets collecting all nodes sharing the same depth.

Definition 4.4.2 (Level Sets)**.** *For each* $i \leqslant h$*, define*

$$X_i := \{\sigma(i) : \sigma \text{ is a path in } T_{\mathcal{H}}\}$$

The following lemma justifies Remark 13.

Lemma 4.4.3. *Given hypersequent* $\mathcal{H}$ *and its tree* $T_{\mathcal{H}}$ *with max depth* h*, we have the following facts:*

- *The left-most path ends with a hypersequent with only atomic formulas.*
- *All paths in* $T_{\mathcal{H}}$ *has length* h*.*
- *For each* $i \leqslant h$*, say* $\mathcal{G}$ *and* $\mathcal{R}$ *are in* X_i *where* $\mathcal{G}$ *is in the left-most path. Then any formula appearing in* $\mathcal{R}$ *also appears in* $\mathcal{G}$*.*

Proof. We prove by an induction on i. The tree construction terminates exactly when the left-most leaf consists no $\rightarrow$ and quantifiers. Tree construction on other branches always follow the construction on the left-most branch, except missing certain formulas due to the Right $\rightarrow$ case. $\square$

Given a valid hypersequent $\mathcal{H}$, the level set X_i of $T_{\mathcal{H}}$ keeps an approximate form of "satisfiability", namely the following meta-judgment for all n:

$$\text{“}\exists \bar{x}(\bigwedge_{\mathcal{G}\in X_i} \mathcal{G}_{\frac{1}{n}}(\bar{x}))\text{"}$$

Lemma 4.4.4. *Say $\mathcal{H}$ is a valid hypersequent and $i \leqslant h$. Then for any structure $\mathfrak{M}$ and any natural number n, the following inequality holds*

$$\inf_{\bar{m}\in M} \max_{\mathcal{G}\in X_i} (\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

where $\bar{x}$ are the free variables appearing in X_i.

Proof. We proceed by induction on i.

In the base case for $i = 1$, we have that $X_1 = \{\mathcal{H}\}$. Since $\mathcal{H}$ is valid and contains no free variable, we have for each $\mathfrak{M}$ and n that:

$$\inf_{\bar{m}\in M} (\mathcal{H}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} = (\mathcal{H}_{\frac{1}{n}})^{\mathfrak{M}} = n(\mathcal{H}^{\mathfrak{M}} - \frac{1}{n}) < n \cdot \mathcal{H}^{\mathfrak{M}} \leqslant 0$$

In the inductive case for $i+1$, there are four cases based on the left-most node in X_{i+1}. Let $\mathfrak{M}$ and n be arbitary.

Right $\rightarrow$ case: When the left-most node in X_{i+1} takes the form $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ with parent node $\mathcal{L} \mid \Gamma \Rightarrow A \rightarrow B, \Delta$ in X_i, the induction hypothesis give

$$\inf_{\bar{m}} \max_{G\in X_i} (\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

It implies for each $\epsilon > 0$ there are $\bar{m} \in M$ such that

$$\max_{G\in X_i} (\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

In particular, we have that

$$(\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A \to B, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$$

Since $(A \to B)^{\mathfrak{M}} = B^{\mathfrak{M}} \dot{-} A^{\mathfrak{M}} = \max(B^{\mathfrak{M}} - A^{\mathfrak{M}}, 0)$, we have

$$\max((\mathcal{L}_{\frac{1}{n}} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}}, (\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}}) < \epsilon$$

and thus obtain two inequalities

$$(\mathcal{L}_{\frac{1}{n}} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$$
$$(\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$$

The first inequality covers $\mathcal{L} \mid \Gamma, A \Rightarrow B, \Delta$ in X_{i+1} and the second inequality covers the element $\mathcal{L} \mid \Gamma \Rightarrow \Delta$ in X_{i+1}.

For the rest of elements in X_{i+1}, notice that there are two possibilities due to the definition of $T_{\mathcal{H}}$.

1. Nodes $\mathcal{L}^* \mid \Gamma^*, A \Rightarrow B, \Delta^*$ and $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ in X_{i+1} have their parent node $\mathcal{L}^* \mid \Gamma^* \Rightarrow A \to B, \Delta^*$ in X_i. We then also have

   $$(\mathcal{L}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} A \to B, \Delta^*)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

   In a similar proof, we derive the two desired inequalities $(\mathcal{L}^*_{\frac{1}{n}} \mid \Gamma^*, A \Rightarrow_{\frac{1}{n}} B, \Delta^*)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$ and $(\mathcal{L}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$.

2. Node $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ in X_{i+1} has its parent node $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ in X_i. We simply obtain

   $$(\mathcal{L}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

Together, we have that $(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$ holds for any $\mathcal{G} \in X_{i+1}$. Therefore, for any $\epsilon > 0$ there are

$\bar{m}$ such that

$$\max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

It implies the promised inequality

$$\inf_{\bar{m} \in M} \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

Left $\rightarrow$ case: When the left-most node in X_{i+1} takes the form $\mathcal{L} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta$ with parent node $\mathcal{L} \mid \Gamma, A \rightarrow B \Rightarrow \Delta$ in X_i, the induction hypothesis give

$$\inf_{\bar{m}} \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

Pick $\bar{m}$ for each $\epsilon > 0$ such that

$$(\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A \rightarrow B, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} \leqslant \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

Since $(A \rightarrow B)^{\mathfrak{M}}$ on the left of $\Rightarrow$ and the hyper-connective $\mid$ are both interpreted as minimum, we compute

$$\begin{aligned}
& (\mathcal{L}_{\frac{1}{n}} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} \\
= \quad & \min((\mathcal{L}_{\frac{1}{n}} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}}, (\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}}) \\
= \quad & (\mathcal{L}_{\frac{1}{n}} \mid \Gamma, A \rightarrow B \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} \\
< \quad & \epsilon
\end{aligned}$$

For other element $\mathcal{R}$ in X_{i+1}, we have $(\mathcal{R}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$ by a similar argument as before. Thus, we have for each ϵ there exists $\bar{m}$ such that

$$\max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

and consequently

$$\inf_{\bar{m}\in M}\max_{G\in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

Right $\exists$ case: When the left-most node in X_{i+1} takes the form $\mathcal{L} \mid \Gamma \Rightarrow A[x/y], \Delta$ with parent node $\mathcal{L} \mid \Gamma \Rightarrow (\exists y)A, \Delta$ in X_i, pick $\bar{m}$ for each $\epsilon > 0$ by induction hypothesis such that

$$(\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} (\exists y)A, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} \leqslant \max_{G\in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \frac{\epsilon}{2}$$

Since $(\exists y)A$ is interpreted as an infinimum, there exists some $m \in M$ satisfying

$$A[m/y, \bar{m}/\bar{x}]^{\mathfrak{M}} < (\exists y)A[\bar{m}/\bar{x}]^{\mathfrak{M}} + \frac{\epsilon}{2}$$

We then compute

$$\begin{aligned}(\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A, \Delta)[m/y, \bar{m}/\bar{x}]^{\mathfrak{M}} &< (\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} (\exists y)A, \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

For other element $\mathcal{R}$ in X_{i+1}, there are two possibilities.

1. Node $\mathcal{L}^* \mid \Gamma^* \Rightarrow A[x/y], \Delta^*$ in X_{i+1} with its parent node $\mathcal{L}^* \mid \Gamma^* \Rightarrow (\exists y)A, \Delta^*$ in X_i. We have by induction hypothesis that

   $$(\mathcal{L}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} (\exists y)A, \Delta^*)[\bar{m}/\bar{x}]^{\mathfrak{M}} \leqslant \max_{G\in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \frac{\epsilon}{2}$$

   By using the same m, we similarly yield the desired inequality $(\mathcal{L}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A[x/y]\Delta)[m/x, \bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$.

2. Node $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ in X_{i+1} has its parent node $\mathcal{L}^* \mid \Gamma^* \Rightarrow \Delta^*$ in X_i. We simply obtain

$$(\mathcal{L}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)[\bar{m}/\bar{x}]^{\mathfrak{M}} \leqslant \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

Together, we conclude that $(\mathcal{G}_{\frac{1}{n}}[m/x, \bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$ holds for each $\mathcal{G} \in X_{i+1}$. Therefore, for any $\epsilon > 0$ there are tuples $(\bar{m}, m)$ such that

$$\max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[m/x, \bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

It implies the promised inequality

$$\inf_{m \in M} \inf_{\bar{m} \in M} \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

Left $\exists$ case: When the left-most node in X_{i+1} takes the form $\mathcal{L} \mid \Gamma, A[f(\bar{x})/y] \Rightarrow \Delta$ with parent node $\mathcal{L} \mid \Gamma, (\exists y)A \Rightarrow \Delta$ in X_i, the induction hypothesis give

$$\inf_{\bar{m}} \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

Pick $\bar{m}$ for each $\epsilon > 0$ such that

$$(\mathcal{L}_{\frac{1}{n}} \mid \Gamma, (\exists y)A \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} \leqslant \max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

Since $(\exists y)A$ is interpreted as a supremum on the left of $\Rightarrow$, we have that

$$(\mathcal{G}_{\frac{1}{n}} \mid \Gamma, A[f(\bar{x})/y] \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} \leqslant (\mathcal{G}_{\frac{1}{n}} \mid \Gamma, (\exists y)A \Rightarrow_{\frac{1}{n}} \Delta)[\bar{m}/\bar{x}]^{\mathfrak{M}} < \epsilon$$

For other element $\mathcal{R}$ in X_{i+1}, we have $(\mathcal{R}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$ by a similar argument. Thus, we have for each ϵ there exists $\bar{m}$ such that

$$\max_{G \in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} < \epsilon$$

and consequently

$$\inf_{\bar{m}\in M}\max_{G\in X_i}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

□

Notice that X_h consists of hypersequents with no quantifiers. For any given n, the above Lemma 4.4.4 applying to X_h yields

$$\sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X_h}(\mathcal{G}_{\frac{1}{n}}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$$

It can be viewed as a meta-Σ_1^0-judgment stating

$$\text{“}(\exists\bar{x})(\bigwedge_{\mathcal{G}\in X_h}\mathcal{G}_{\frac{1}{n}}(\bar{x}))\text{”}$$

where $\bigwedge_{\mathcal{G}\in X_h}\mathcal{G}_{\frac{1}{n}}(\bar{x})$ is quantifier-free. We will adopt an approximate version of Herbrand's Theorem to find Herbrand witnesses.

Theorem 4.4.5 (Approximate Herbrand Theorem)**.** *Suppose X is a finite set of quantifier-free hypersequents with shared free variables $\bar{x}$ and n is a natural number. Then there are terms $\bar{t}_1,\ldots,\bar{t}_k$ such that*

$$\sup_{\mathfrak{M}}\min_{1\leqslant i\leqslant k}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}) < \sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}]^{\mathfrak{M}}) + \frac{1}{n}$$

If we additionally have that

$$\sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}]^{\mathfrak{M}}) \leqslant 0$$

then the hypersequent

$$\mathcal{G}^1_{\frac{1}{n}}[\bar{t}_1/\bar{x}] \mid \cdots \mid \mathcal{G}^k_{\frac{1}{n}}[\bar{t}_k/\bar{x}]$$

is valid for each $\mathcal{G}^1,\ldots,\mathcal{G}^k \in X$.

Proof. We generalize the proof appearing in Baaz and Metcalfe (2010). Let P collect all formulas of

the form $r(\bar{t})$ where r is a relation symbol and $\bar{t}$ are terms in the language. Notice that a structure $\mathfrak{M}$ is uniquely associated with a map $v_{\mathfrak{M}} \in [0,1]^P$. For simplicity, let

$$d := \sup_{\mathfrak{M}} \inf_{\bar{m} \in M} \max_{\mathcal{G} \in X} (\mathcal{G}[\bar{m}/\bar{x}])^{\mathfrak{M}}$$

Given arbitrary term $\bar{t}$, we define

$$\begin{aligned} S(\bar{t}) &:= \{v_{\mathfrak{M}} \in [0,1]^P : \max_{\mathcal{G} \in X} (\mathcal{G}[\bar{t}/\bar{x}]^{\mathfrak{M}}) \geqslant d + \frac{1}{n}\} \\ &= \bigcup_{\mathcal{G} \in X} \{v_{\mathfrak{M}} \in [0,1]^P : (\mathcal{G}[\bar{t}/\bar{x}]^{\mathfrak{M}}) \geqslant d + \frac{1}{n}\} \end{aligned}$$

All $S(\bar{t})$ are closed subsets of $[0,1]^P$ because $\rightarrow$, $\Rightarrow$, and $|$ are interpreted as continuous functions on $\mathbb{R}$ with absence of quantifiers in $\mathcal{G}$ and finite unions of closed sets are still closed. Define

$$\mathcal{S} := \{S(\vec{t}) \mid \bar{t} \text{ is a term}\} \subseteq \mathcal{P}([0,1]^P)$$

Suppose towards a contradiction that there are no finite list of terms $\bar{t}_1, \ldots, \bar{t}_k$ satisfying

$$\sup_{\mathfrak{M}} \min_{i} \max_{\mathcal{G} \in X} (\mathcal{G}[\bar{t}_i/\bar{x}])^{\mathfrak{M}} < d + \frac{1}{n}$$

Then for any given list of terms $\bar{t}_1, \ldots, \bar{t}_k$, there exists a structure $\mathfrak{M}$ such that

$$\max_{\mathcal{G} \in X} (\mathcal{G}[\bar{t}_i/\bar{x}])^{\mathfrak{M}} \geqslant d + \frac{1}{n}$$

for all $1 \leqslant i \leqslant k$. Hence, $v_{\mathfrak{M}} \in \bigcap_i S(\bar{t}_i)$ and we have shown that $\mathcal{S}$ has the finite intersection property.

Since $\mathcal{S}$ is a collection of closed subsets of $[0,1]^P$ with finite intersection property and $[0,1]^P$ is a compact space by Tychonoff's Theorem, $\bigcap \mathcal{S}$ is nonempty and thus there exists a structure $\mathfrak{M}$ such

that $\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{t}/\bar{x}]^{\mathfrak{M}}) \geqslant d + \frac{1}{n}$ for all terms $\bar{t}$. It would particularly imply that

$$\begin{aligned}
\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}]^{\mathfrak{M}}) &\geqslant d + \frac{1}{n} \\
&= \sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}])^{\mathfrak{M}} + \frac{1}{n} \\
&> \sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}])^{\mathfrak{M}}
\end{aligned}$$

which itself is a contradiction. Therefore, there must exist a list of Herbrand terms $\bar{t}_1, \ldots, \bar{t}_k$ as promised.

If furthermore we have $\sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}])^{\mathfrak{M}} \leqslant 0$, then for any given $\mathcal{G}^1, \ldots, \mathcal{G}^k$ in X we have that

$$(\mathcal{G}^i_{\frac{1}{n}}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}) \leqslant \max_{\mathcal{G}\in X}(\mathcal{G}_{\frac{1}{n}}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}$$

Therefore,

$$\begin{aligned}
\sup_{\mathfrak{M}}\min_{1\leqslant i\leqslant k}(\mathcal{G}^i_{\frac{1}{n}}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}) &\leqslant \sup_{\mathfrak{M}}\min_{i}\max_{\mathcal{G}\in X}(\mathcal{G}_{\frac{1}{n}}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}) \\
&= \sup_{\mathfrak{M}}\min_{i}\max_{\mathcal{G}\in X}(n\cdot(\mathcal{G}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}) - 1) \\
&= n\cdot\sup_{\mathfrak{M}}\min_{i}\max_{\mathcal{G}\in X}((\mathcal{G}[\bar{t}_i/\bar{x}]^{\mathfrak{M}}) - \frac{1}{n}) \\
&< n\cdot\sup_{\mathfrak{M}}\inf_{\bar{m}\in M}\max_{\mathcal{G}\in X}(\mathcal{G}[\bar{m}/\bar{x}])^{\mathfrak{M}} \\
&\leqslant 0
\end{aligned}$$

which exactly means that $\mathcal{G}^1_{\frac{1}{n}}[\bar{t}_1/\bar{x}] \mid \cdots \mid \mathcal{G}^k_{\frac{1}{n}}[\bar{t}_k/\bar{x}]$ is a valid hypersequent. □

Applying the above Theorem 4.4.5 to X_h, we arrive at some Herbrand witnesses $\bar{t}_1, \ldots, \bar{t}_k$ such that the following hypersequent consisting of only atomic formulas

$$\mathcal{G}^1_{\frac{1}{n}}[\bar{t}_1/\bar{x}] \mid \cdots \mid \mathcal{G}^k_{\frac{1}{n}}[\bar{t}_k/\bar{x}]$$

is valid for each $\mathcal{G}^1, \ldots, \mathcal{G}^k$ in X_h. From now on, we omit $\bar{x}$ and denote it as

$$\mathcal{G}^1_{\frac{1}{n}}(\bar{t}_1) \mid \cdots \mid \mathcal{G}^k_{\frac{1}{n}}(\bar{t}_k)$$

to simplify the notation.

The hypersequents $\mathcal{G}^1_{\frac{1}{n}}(\bar{t}_1) \mid \cdots \mid \mathcal{G}^k_{\frac{1}{n}}(\bar{t}_k)$ are derivable by Strong Completeness Theorem 3.4.3 for GŁ. We will recover $\mathcal{H}_{\frac{1}{n}}$ by re-introducing connectives following the reverse order of the Skolemization tree $T_{\mathcal{H}}$. For that purpose, we need the following admissible rules that handle introduction rules for connectives in the sense of $\Rightarrow_{\frac{1}{n}}$.

Lemma 4.4.6. *The followings are derivable rules in GŁ and GŁ∀ for any n.*

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid n\Gamma \Rightarrow n\Delta}\, mul \qquad\qquad \frac{\mathcal{G} \mid n\Gamma \Rightarrow n\Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\, div$$

Proof. The *mul* rule is simply an application of *mix* of n-copies of $\mathcal{G} \mid \Gamma \Rightarrow \Delta$. The *div* is the result of applying *split* and *ec*. □

Lemma 4.4.7. *We have left and right introduction rules for $\rightarrow$ in the sense of $\Rightarrow_{\frac{1}{n}}$ for any n.*

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow_{\frac{1}{n}} A, \Delta \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow_{\frac{1}{n}} \Delta} \rightarrow\!\Rightarrow_{\frac{1}{n}} \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta \qquad \mathcal{G} \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta}{\mathcal{G} \mid \Gamma \Rightarrow_{\frac{1}{n}} A \rightarrow B, \Delta} \Rightarrow_{\frac{1}{n}}\!\rightarrow$$

Proof. The $\rightarrow\!\Rightarrow_{\frac{1}{n}}$ rule can be derived by iterating the following derivable rules and apply *ec*.

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta}\, ew}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta}\, \rightarrow\!\Rightarrow$$

and

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta}\, wl$$

For the $\Rightarrow_{\frac{1}{n}}\!\rightarrow$ rule, we first notice that $\mathcal{G} \mid \bot, n\Gamma, iA \Rightarrow iB, n\Delta$ is derivable for any $0 < i < n$.

$$\dfrac{\dfrac{\mathcal{G} \mid \bot, n\Gamma, nA \Rightarrow nB, n\Delta}{\mathcal{G} \mid i\bot, ni\Gamma, niA \Rightarrow niB, nk\Delta}\, mul \qquad \dfrac{\mathcal{G} \mid \bot, n\Gamma \Rightarrow n\Delta}{\mathcal{G} \mid (n-i)\bot, n(n-i)\Gamma \Rightarrow n(n-i)\Delta}\, mul}{\dfrac{\mathcal{G} \mid n\bot, n^2\Gamma, niA \Rightarrow niB, n^2\Delta}{\mathcal{G} \mid \bot, n\Gamma, iA \Rightarrow iB, n\Delta}\, div}\, mix$$

For each i and j, we have

$$\dfrac{\mathcal{G} \mid \bot, n\Gamma, (i+1)A \Rightarrow (i+1)B, j(A \to B), n\Delta \qquad \mathcal{G} \mid \bot, n\Gamma, iA \Rightarrow iB, j(A \to B), n\Delta}{\mathcal{G} \mid \bot, n\Gamma, iA \Rightarrow iB, (j+1)(A \to B), n\Delta}$$

We thus have a derivation of $\mathcal{G} \mid \bot, n\Gamma \Rightarrow n(A \to B), n\Delta$ by a recursion on j. □

Lemma 4.4.8. *We have left and right introduction rules for $\exists$ in the sense of $\Rightarrow_{\frac{1}{n}}$ for any n. For the left rule, we require that c is not free in Γ, Δ, and $\mathcal{G}$.*

$$\dfrac{\mathcal{G} \mid \Gamma, A(c) \Rightarrow_{\frac{1}{n}} \Delta}{\mathcal{G} \mid \Gamma, (\exists x)A[x/c] \Rightarrow_{\frac{1}{n}} \Delta}\, \exists \Rightarrow_{\frac{1}{n}} \qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow_{\frac{1}{n}} A(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow_{\frac{1}{n}} (\exists x)A[x/t], \Delta}\, \Rightarrow_{\frac{1}{n}} \exists$$

(This lemma and a similar proof also appeared in Gerasimov (2020)).

Proof. In the case of $\exists \Rightarrow_{\frac{1}{n}}$, take fresh new variables $x_1, \ldots, x_n$. We have deductions $\mathcal{G} \mid \bot, n\Gamma, nA(x_i) \Rightarrow n\Delta$ for each i by substituting x with x_i. We thus have the following derivation for $\mathcal{G} \mid \bot, n\Gamma, n(\exists x)A \Rightarrow n\Delta$.

$$\dfrac{\dfrac{\dfrac{\overbrace{\mathcal{G} \mid \bot, n\Gamma, nA(x_1) \Rightarrow n\Delta \qquad \cdots \qquad \mathcal{G} \mid \bot, n\Gamma, nA(x_n) \Rightarrow n\Delta}^{n}}{\mathcal{G} \mid n\bot, n^2\Gamma, n^2A(x_1), \ldots, n^2A(x_n) \Rightarrow n^2\Delta}\, mix}{\mathcal{G} \mid \overbrace{\bot, n\Gamma, A(x_1), \ldots, A(x_n) \Rightarrow n\Delta \mid \cdots \mid \bot, n\Gamma, A(x_1), \ldots, A(x_n) \Rightarrow n\Delta}^{n}}\, split}{\dfrac{\mathcal{G} \mid \bot, n\Gamma, A(x_1), \ldots, A(x_n) \Rightarrow n\Delta}{\mathcal{G} \mid \bot, n\Gamma, n(\exists x)A \Rightarrow n\Delta}\, \exists \Rightarrow}\, ec$$

The $\Rightarrow_{\frac{1}{n}} \exists$ rule can be derived by simply iterating $\Rightarrow \exists$, since there is no eigen-conditions. □

Now we are ready to present the main result. We need a few definitions to label hypersequents to keep track of the order of connective introductions.

Definition 4.4.9. *Let a valid hypersequent $\mathcal{H}$ with no free variables, its tree $T_{\mathcal{H}}$ and leafs X_h defined as above, Herbrand witnesses $\bar{t}_1, \ldots, \bar{t}_k$ for $\bar{x}$ from Lemma 4.4.5, and arbitrary natural number n be given.*

*A **labeled hypersequent** is of the form $\mathcal{G}^\sigma$ where $\mathcal{G}$ is a hypersequent and σ is a path in $T_{\mathcal{H}}$ such that the following holds*

- *$\mathcal{G} = \mathcal{L}_{\frac{1}{n}}(\bar{t}_i)$ where $\mathcal{L}$ is the last node of σ and $\bar{t}_i$ is some Herbrand witness.*

*A **multi-labeled hypersequent** $\mathfrak{G}$ is a multiset of labeled hypersequents of the form*

$$\mathcal{G}_1^{\sigma_1} \mid \cdots \mid \mathcal{G}_l^{\sigma_l}$$

*where $0 \leqslant l \leqslant k$. We also refer $\mathcal{G}_i^{\sigma_i}$ as a **component** of $\mathfrak{G}$.*

*We say a labeled hypersequent $\mathcal{G}^\sigma$ is **derivable** if $\mathcal{G}$ is derivable; a multi-labeled hypersequent $\mathcal{G}_1^{\sigma_1} \mid \cdots \mid \mathcal{G}_k^{\sigma_k}$ is **derivable** if $\mathcal{G}_1 \mid \cdots \mid \mathcal{G}_k$ is derivable.*

*The **stage** of a multi-labeled hypersequent $\mathcal{G}_1^{\sigma_1} \mid \cdots \mid \mathcal{G}_k^{\sigma_k}$ is the tuple $(|\sigma_1|, \ldots, |\sigma_k|)$ where $|\sigma_i|$ is the length of path σ_i in tree $T_{\mathcal{H}}$. The **size** of $\mathcal{G}_1^{\sigma_1} \mid \cdots \mid \mathcal{G}_k^{\sigma_k}$ is $\sum_{1 \leqslant i \leqslant k} |\sigma_i|$.*

Definition 4.4.10. *Given a finite set $\mathbb{X}$ of multi-labeled hypersequents, we say $\mathbb{X}$ is **well-formed** if the following conditions hold:*

1. *There exists some element $\mathfrak{G}$ such that each label in $\mathfrak{G}$ is an initial segment of the left-most path in $T_{\mathcal{H}}$. We call $\mathfrak{G}$ the **left-most element**.*

2. *For any element $\mathfrak{G} \mid \mathcal{L}^\sigma \mid \mathfrak{L}$ in $\mathbb{X}$ where σ extends some path $\eta \frown (\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta) \frown (\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta)$ in $T_{\mathcal{H}}$, there exists some element $\mathfrak{G} \mid \mathcal{R}^\tau \mid \mathfrak{L}$ in $\mathbb{X}$ such that τ extends $\eta \frown (\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta) \frown (\mathcal{G} \mid \Gamma \Rightarrow \Delta)$.*

3. *Every element of $\mathbb{X}$ has the same number of components and the same stage.*

4. *Every element of $\mathbb{X}$ is derivable in GŁ∀.*

*We define $\mu(\mathbb{X})$ the **size** of $\mathbb{X}$ to be the size of its left-most element.*

The prototype of a well-formed set is the following:

Definition 4.4.11. *Define the following set of multi-labeled hypersequents*

$$\mathbb{X}_0 := \{(\mathcal{G}^1_{\frac{1}{n}}(\bar{t}_1))^{\sigma_1} \mid \cdots \mid (\mathcal{G}^k_{\frac{1}{n}}(\bar{t}_k))^{\sigma_k} : \mathcal{G}^i \in X_h \text{ and } \sigma_i \text{ is the maximal path ending with } \mathcal{G}^i\}$$

and a left-most element in $\mathbb{X}_0$

$$\mathfrak{G}_0 := (\mathcal{G}_{\frac{1}{n}}(\bar{t}_1))^{\sigma} \mid \cdots \mid (\mathcal{G}_{\frac{1}{n}}(\bar{t}_k))^{\sigma}$$

where σ is the left-most maximal path in $T_{\mathcal{H}}$ ending with $\mathcal{G}$.

Lemma 4.4.12. *The set $\mathbb{X}_0$ is well-formed.*

Proof. We verify each condition in Definition 4.4.10.

1. $\mathfrak{G}_0$ is by definition a left-most element.
2. $\mathbb{X}_0$ enumerates all paths in $T_{\mathcal{H}}$ with any combination of witnesses. Hence, for any $\mathfrak{G} \mid \mathcal{G}^{\sigma}$ in $\mathbb{X}_0$ where σ extends some left branch at a split in $T_{\mathcal{H}}$, say τ is a path in $T_{\mathcal{H}}$ extending the right branch and ending with $\mathcal{R}$. Then $\mathfrak{G} \mid \mathfrak{L}^{\tau}$ is also in $\mathbb{X}_0$ by design.
3. Elements of $\mathbb{X}_0$ all have the same stage $(h, \ldots, h)$ by Lemma 4.4.3 and k-many components.
4. Since $\mathbb{X}_0$ consists of valid quantifier-free hypersequents by Lemma 4.4.5, its elements are derivable in GŁ by Theorem 3.4.3 and thus also derivable in GŁ$\forall$.

□

Starting from $\mathbb{X}_0$, we will iteratively perform a reduction of labels in $\mathbb{X}_0$ until all labels are of the shortest form $(\mathcal{H})$.

Definition 4.4.13. *We define a function F on well-formed sets of multi-labeled hypersequents $\mathbb{X}$ with left-most element $\mathfrak{G}$ by the following. The order can be arbitrarily chosen if multiple conditions are satisfied.*

(1): *If $\mathfrak{G}$ contains a component of the form $\mathcal{L}^\sigma$ where*

$$\sigma = \eta \frown (\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta) \frown (\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta \mid \Gamma \Rightarrow \Delta)$$
$$\mathcal{L} = (\mathcal{G}_{\frac{1}{n}} \mid \Gamma, B \Rightarrow_{\frac{1}{n}} A, \Delta \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)(\bar{t}_i)$$

For simplicity, we write $\mathfrak{G}$ as $\mathfrak{L} \mid \mathcal{L}^\sigma$ since multi-labeled hypersequents are multisets of labeled hypersequents.

By well-formedness of $\mathbb{X}$, $\mathfrak{L} \mid \mathcal{L}^\sigma$ is derivable. Now consider the following proof using Lemma 4.4.7 (without labels)

$$\frac{\mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma, B \Rightarrow_{\frac{1}{n}} A, \Delta \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)(\bar{t}_i)}{\mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma, A \to B \Rightarrow_{\frac{1}{n}} \Delta)(\bar{t}_i)} \to\Rightarrow_{\frac{1}{n}}$$

We first replace $\mathfrak{G}$ in $\mathbb{X}$ with the derived hypersequent with a shorter label

$$\mathfrak{L} \mid ((\mathcal{G}_{\frac{1}{n}} \mid \Gamma, A \to B \Rightarrow_{\frac{1}{n}} \Delta)(\bar{t}_i))^{\eta \frown (\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta)}$$

For other elements $\mathfrak{R}$ in $\mathbb{X}$ with component $(\mathcal{L}^*)^{\sigma^*}$ associated with $\mathcal{L}^\sigma$ in $\mathfrak{G}$, we denote $\mathfrak{R}$ as $\mathfrak{L}^* \mid (\mathcal{L}^*)^{\sigma^*}$. There are two cases due to the construction in $T_{\mathcal{H}}$ in Definition 4.4.1.

- *When $(\mathcal{L}^*)^{\sigma^*}$ takes the following form*

$$\sigma^* = \eta^* \frown (\mathcal{G}^* \mid \Gamma^*, A \to B \Rightarrow \Delta^*) \frown (\mathcal{G}^* \mid \Gamma^*, B \Rightarrow A, \Delta^* \mid \Gamma^* \Rightarrow \Delta^*)$$
$$\mathcal{L}^* = (\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^*, B \Rightarrow_{\frac{1}{n}} A, \Delta^* \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i)$$

We derive the following from $\mathfrak{L}^* \mid \mathcal{L}^*$ *similarly to the case for* $\mathfrak{G}$

$$\mathfrak{L}^* \mid ((\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^*, A \to B \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i))^{\eta^* \frown (\mathcal{G}^* \mid \Gamma^*, A \to B \Rightarrow \Delta^*)}$$

and replace $\mathfrak{R}$ *in* $\mathbb{X}$ *with it.*

- *When* $(\mathcal{L}^*)^{\sigma^*}$ *takes the following form* $\mathfrak{L}^* \mid (\mathcal{L}^*)^{\sigma^*}$ *where*

$$\sigma^* = \eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*) \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*)$$
$$\mathcal{L}^* = (\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i)$$

We replace $\mathcal{R}$ *with*

$$\mathfrak{L}^* \mid ((\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i))^{\eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*)}$$

Now we define $F(\mathbb{X})$ *to be the result of substituting* $\mathfrak{G}$ *and other elements in* $\mathbb{X}$*. We check* $F(\mathbb{X})$ *is still well-formed.*

- *Property (1) is clear as the replacement of* $\mathfrak{G}$ *is still a left-most element.*
- *The only way Property (2) fails is when we remove labels on the right branch of a split while keeping the left branch. Since we do not remove any label of a split at this stage, Property (2) holds for* $F(\mathbb{X})$ *given that it holds for* $\mathbb{X}$*.*
- *Elements in* $F(\mathbb{X})$ *are kept with the same number of components and the same stage.*
- *Elements in* $F(\mathbb{X})$ *are all provided with a derivation.*

(2): *If the condition in* (1) *fails and* $\mathfrak{G}$ *is of the form* $\mathfrak{L} \mid \mathcal{L}^\sigma$ *where*

$$\sigma = \eta \frown (\mathcal{G} \mid \Gamma \Rightarrow \exists y A, \Delta) \frown (\mathcal{G} \mid \Gamma \Rightarrow A[x/y], \Delta)$$
$$\mathcal{L} = (\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A[x/y], \Delta)(\bar{t}_i)$$

By well-formedness of $\mathbb{X}$, $\mathfrak{L} \mid \mathcal{L}^\sigma$ *is derivable. Consider the following proof using Lemma 4.4.8*

$$\frac{\mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A[x/y], \Delta)(\bar{t_i})}{\mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \exists y A, \Delta)(\bar{t_i})} \Rightarrow_{\frac{1}{n}} \exists$$

We then replace $\mathfrak{G}$ *in* $\mathbb{X}$ *with the derived hypersequent with a shorter label*

$$\mathfrak{L} \mid ((\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \exists y A, \Delta)(\bar{t_i}))^{\eta \frown (\mathcal{G} \mid \Gamma \Rightarrow \exists y A, \Delta)}$$

For other elements $\mathfrak{R}$ *in* $\mathbb{X}$*, we similarly replace it with either*

$$\mathfrak{L}^* \mid ((\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \exists y A, \Delta^*)(\bar{t_i}))^{\eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \exists y A, \Delta^*)}$$

or

$$\mathfrak{L}^* \mid ((\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t_i}))^{\eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*)}$$

according to its cases.

Define $F(\mathbb{X})$ *to be the result of substituting* $\mathfrak{G}$ *and other elements in* $\mathbb{X}$*. By a similar argument as in* (1)*,* $F(\mathbb{X})$ *remains to be well-formed.*

(3): *If conditions in* (1) *and* (2) *fail and* $\mathfrak{G}$ *is of the form* $\mathfrak{L} \mid \mathcal{L}^\sigma$ *where*

$$\sigma = \eta \frown (\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta) \frown (\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta)$$
$$\mathcal{L} = (\mathcal{G}_{\frac{1}{n}} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta)(\bar{t_i})$$

Notice that σ *cannot be the right branch of a split due to the choice of* $\mathfrak{G}$ *being the left-most element. Since* $\mathbb{X}$ *is well-formed,* $\mathfrak{L} \mid \mathcal{L}^\sigma$ *is derivable. Furthermore, the multi-labeled hypersequent* $\mathfrak{L} \mid \mathcal{R}^\tau$ *where*

$$\tau = \eta \frown (\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta) \frown (\mathcal{G} \mid \Gamma \Rightarrow \Delta)$$
$$\mathcal{R} = (\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)(\bar{t_i})$$

exists in $\mathbb{X}$ *by condition (2) and (3) of* $\mathbb{X}$ *being well-formed. Therefore,* $\mathfrak{L} \mid \mathcal{R}^\tau$ *is also derivable. Now consider the following proof using Lemma 4.4.7*

$$\frac{\mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma, A \Rightarrow_{\frac{1}{n}} B, \Delta)(\bar{t}_i) \qquad \mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} \Delta)(\bar{t}_i)}{\mathfrak{L} \mid (\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A \to B, \Delta)(\bar{t}_i)} \Rightarrow_{\frac{1}{n}} \to$$

We then replace both $\mathfrak{L} \mid \mathcal{L}^\sigma$ *and* $\mathfrak{L} \mid \mathcal{R}^\tau$ *in* $\mathbb{X}$ *with*

$$\mathfrak{L} \mid ((\mathcal{G}_{\frac{1}{n}} \mid \Gamma \Rightarrow_{\frac{1}{n}} A \to B, \Delta)(\bar{t}_i))^{\eta \frown (\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta)}$$

For other elements $\mathfrak{R}$ *in* $\mathbb{X}$ *with* $(\mathcal{L}^*)^{\sigma^*}$ *associated with* $\mathcal{L}^\sigma$*, there are two cases.*

- $\mathfrak{R}$ *is of the form* $\mathfrak{L}^* \mid (\mathcal{L}^*)^{\sigma^*}$ *where*

$$\sigma^* = \eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow A \to B, \Delta^*) \frown (\mathcal{G}^* \mid \Gamma^*, A \Rightarrow B, \Delta^*)$$
$$\mathcal{L}^* = (\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^*, A \Rightarrow_{\frac{1}{n}} B, \Delta^* \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i)$$

  By well-formedness again, $\mathbb{X}$ *contains some unique element* $\mathfrak{L}^* \mid (\mathcal{R}^*)^{\tau^*}$ *such that*

$$\tau^* = \eta^* \frown (\mathcal{G}^* \mid \Gamma^*, A \to B \Rightarrow \Delta^*) \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*)$$
$$\mathcal{R}^* = (\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i)$$

  We similarly derive the following

$$\mathfrak{L}^* \mid ((\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} A \to B, \Delta^*)(\bar{t}_i))^{\eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow A \to B, \Delta^*)}$$

  and replace both $\mathfrak{L}^* \mid (\mathcal{L}^*)^{\sigma^*}$ *and* $\mathfrak{L}^* \mid (\mathcal{R}^*)^{\tau^*}$ *in* $\mathbb{X}$ *with it.*

- $\mathfrak{R}$ *is of the form* $\mathfrak{L}^* \mid (\mathcal{L}^*)^{\sigma^*}$ *where*

$$\sigma^* = \eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*) \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*)$$
$$\mathcal{L}^* = (\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i)$$

We replace $\mathcal{R}$ with the following

$$\mathfrak{L}^* \mid ((\mathcal{G}^*_{\frac{1}{n}} \mid \Gamma^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i))^{\eta^* \frown (\mathcal{G}^* \mid \Gamma^* \Rightarrow \Delta^*)}$$

Define $F(\mathbb{X})$ to be the result of substituting $\mathfrak{G}$ and other elements in $\mathbb{X}$. Now we check well-formedness of $F(\mathbb{X})$:

- *Property (1) is clear as the replacement of $\mathfrak{G}$ is still a left-most element.*
- *The only way Property (2) fails is when we remove labels on the right branch at a split while keeping the left branch. Since we simultaneously remove both branching labels, Property (2) for $F(\mathbb{X})$ is preserved.*
- *Elements in $F(\mathbb{X})$ are kept with the same number of components and the same stage.*
- *Elements in $F(\mathbb{X})$ are all provided with a derivation.*

(4): *If conditions in* (1), (2), *and* (3) *fail, then components of $\mathfrak{G}$ must have labels either $(\mathcal{H})$ or of the form*

$$\eta \frown (\mathcal{G} \mid \Gamma, \exists y A \Rightarrow \Delta) \frown (\mathcal{G} \mid \Gamma, A[f(\bar{x})/y] \Rightarrow \Delta)$$

Assume not all labels are $(\mathcal{H})$ and say $\mathfrak{G}$ is of the form

$$\mathcal{L}_1^{\sigma_1} \mid \cdots \mid \mathcal{L}_k^{\sigma_k}$$

and let $f_i(\bar{x}_i)$ be the corresponding Skolem function (add a dummy function in the case of $(\mathcal{H})$ label). Then each $\mathcal{L}_i$ is either $\mathcal{H}_{\frac{1}{n}}$ or of the form

$$((\mathcal{G}_i)_{\frac{1}{n}} \mid \Gamma_i, A_i[f_i(\bar{x}_i)/y_i] \Rightarrow_{\frac{1}{n}} \Delta_i)[\bar{t}_i/\bar{x}]$$

By construction, $\bar{x}_i$ enumerates all free variables in $\mathcal{L}_i$ before substitution and thus $\bar{t}_i$ enumerates

all unbounded terms in $\mathcal{L}_i$ except $f_i(\bar{x}_i)[\bar{t}_i/\bar{x}]$. Hence, $f_i(\bar{x}_i)[\bar{t}_i/\bar{x}]$ does not appear elsewhere in $\mathcal{L}_i$ otherwise $f_i(\bar{x}_i)[\bar{t}_i/\bar{x}]$ would be a subterm of $\bar{t}_i$.

We now take m such that $f_m(\bar{x}_m)[\bar{t}_m/\bar{x}]$ is not a proper subterm of any $f_i(\bar{x}_i)[\bar{t}_i/\bar{x}]$. If there exists $j \neq m$ such that

$$f_m(\bar{x}_m)[\bar{t}_m/\bar{x}] = f_j(\bar{x}_j)[\bar{t}_j/\bar{x}]$$

then we must have $\sigma_m = \sigma_j$ and thus $\mathcal{L}_m = \mathcal{L}_j$ since the witness $\bar{t}_m$ and $\bar{t}_j$ are identified at the label σ_m. Let $j_1, \ldots, j_l$ enumerate all such $j \neq m$ and write

$$\mathfrak{G} = \mathfrak{L} \mid \mathcal{L}_m^{\sigma_m} \mid \mathcal{L}_{j_1}^{\sigma_{j_1}} \mid \cdots \mid \mathcal{L}_{j_l}^{\sigma_{j_l}}$$

We remove all such identical labeled hypersequents from $\mathfrak{G}$ by using ec since $\mathcal{L}_m = \mathcal{L}_{j_1} = \cdots = \mathcal{L}_{j_l}$.

$$\frac{\mathfrak{L} \mid \mathcal{L}_m \mid \mathcal{L}_{j_1} \mid \cdots \mid \mathcal{L}_{j_l}}{\mathfrak{L} \mid \mathcal{L}_m} \, ec$$

Now we have that $f_m(\bar{x}_m)[\bar{t}_m/\bar{x}]$ does not appear elsewhere in the entire $\mathfrak{L} \mid \mathcal{L}_m$. We can introduce $\exists y_m$ by using Lemma 4.4.8 and add back $\mathcal{L}_{j_1}, \ldots, \mathcal{L}_{j_l}$ by ew.

$$\frac{\dfrac{\mathfrak{L} \mid ((\mathcal{G}_m)_{\frac{1}{n}} \mid \Gamma_m, A_m[f(\bar{x}_m)/y_m], \Rightarrow_{\frac{1}{n}} \Delta_m)(\bar{t}_m)}{\mathfrak{L} \mid ((\mathcal{G}_m)_{\frac{1}{n}} \mid \Gamma_m, \exists y_m A_m, \Rightarrow_{\frac{1}{n}} \Delta_m)(\bar{t}_m)} \, \exists \Rightarrow_{\frac{1}{n}}}{\mathfrak{L} \mid ((\mathcal{G}_m)_{\frac{1}{n}} \mid \Gamma_m, \exists y_m A_m, \Rightarrow_{\frac{1}{n}} \Delta_m)(\bar{t}_m) \mid \mathcal{L}_{j_1} \mid \cdots \mid \mathcal{L}_{j_l}} \, ew$$

In comparison to the initial label σ_m in $\mathfrak{G}$

$$\sigma_m = \eta_m \frown (\mathcal{G}_m \mid \Gamma_m, \exists y_m A_m \Rightarrow \Delta_m) \frown (\mathcal{G}_m \mid \Gamma_m, A_m[f_m(\bar{x}_m)/y_m] \Rightarrow \Delta_m)$$

we replace $\mathfrak{G}$ with

$$\mathfrak{L} \mid (((\mathcal{G}_m)_{\frac{1}{n}} \mid \Gamma_m, \exists y_m A_m, \Rightarrow_{\frac{1}{n}} \Delta_m)(\bar{t}_m))^{\eta_m \frown (\mathcal{G}_m \mid \Gamma_m, \exists y_m A_m \Rightarrow \Delta_m)} \mid \mathcal{L}_{j_1}^{\sigma_{j_1}} \mid \cdots \mid \mathcal{L}_{j_l}^{\sigma_{j_l}}$$

For other elements $\mathfrak{R}$ in $\mathbb{X}$, there are two cases.

- $\mathfrak{R}$ *is of the form* $\mathfrak{L}^* \mid (\mathcal{L}_m^*)^{\sigma_m}{}^*$ *where* σ_m^* *has the form*

$$\eta_m^* \frown (\mathcal{G}_m^* \mid \Gamma_m^*, \exists y_m A_m \Rightarrow \Delta_m^*) \frown (\mathcal{G}_m^* \mid \Gamma_m^*, A_m[f_m(\bar{x}_m)/y_m] \Rightarrow \Delta_m^*)$$

Applying ec to components j *that are identical to component* m *and notice that then* $f_m(\bar{x}_m)[\bar{t}_m/\bar{x}]$ *now still satisfies the eigen-condition because* $\mathfrak{R}$ *does not contains more terms than* $\mathfrak{G}$ *due to Lemma 4.4.3. Hence, we can apply* $\exists \Rightarrow_{\frac{1}{n}}$ *and ew to obtain*

$$\mathfrak{L}^* \mid (((\mathcal{G}_m^*)_{\frac{1}{n}} \mid \Gamma_m^*, \exists y_m A_m \Rightarrow_{\frac{1}{n}} \Delta_m^*)(\bar{t}_m))^{\eta_m^* \frown (\mathcal{G}_m^* \mid \Gamma_m^*, \exists y_m A_m \Rightarrow \Delta_m^*)}$$

and replace $\mathfrak{R}$ *with it.*

- $\mathfrak{R}$ *is of the form* $\mathfrak{L}^* \mid (\mathcal{L}_m^*)^{\sigma_m}{}^*$ *where* σ_m^* *has the form*

$$\eta_m^* \frown (\mathcal{G}_m^* \mid \Gamma_m^* \Rightarrow \Delta_m^*) \frown (\mathcal{G}_m^* \mid \Gamma_m^* \Rightarrow \Delta_m^*)$$

We replace $\mathfrak{R}$ *with the following*

$$\mathfrak{L}^* \mid (((\mathcal{G}_m^*)_{\frac{1}{n}} \mid \Gamma_m^* \Rightarrow_{\frac{1}{n}} \Delta^*)(\bar{t}_i))^{\eta_m^* \frown (\mathcal{G}_m^* \mid \Gamma_m^* \Rightarrow \Delta_m^*)}$$

Define $F(\mathbb{X})$ *to be the result of substituting* $\mathfrak{G}$ *and other elements in* $\mathbb{X}$*. Similar to the case in (1) and (2),* $F(\mathbb{X})$ *remains to be well-formed.*

(5): *If all previous conditions fail, then* $F(\mathbb{X}) = \mathbb{X}$*. Notice that by the construction of* $T_{\mathcal{H}}$*, the left-most element* $\mathcal{G}$ *in* $\mathbb{X}$ *must have all its labels to be* $(\mathcal{H})$ *in this case.*

Lemma 4.4.14. *Given any well-formed set* $\mathbb{X}$*, either* $\mu(F(\mathbb{X})) < \mu(\mathbb{X})$ *or the left-most element in* $\mathbb{X}$ *only has labels taking form of* $(\mathcal{H})$*.*

Proof. Proof by induction on the construction of F, since F always reduce labels except in case (5) where the left-most element in $\mathbb{X}$ only has labels as $(\mathcal{H})$. □

Theorem 4.4.15 (Strong Completeness of GŁ∀)**.** *If $\mathcal{H}$ is a valid first-order hypersequent sequent, then $\mathcal{H}_{\frac{1}{n}}$ is derivable in GŁ∀ for all n.*

Proof. Let $\mathcal{H}$ and n be given. We consider its tree $T_{\mathcal{H}}$ with free variables $\bar{x}$ and its leaf set X_h. By Lemma 4.4.4 and Lemma 4.4.5 applying to $T_{\mathcal{H}}$ and X_h, there exist terms $\bar{t}_1, \ldots, \bar{t}_k$ such that

$$\mathcal{G}^1_{\frac{1}{n}}(\bar{t}_1) \mid \cdots \mid \mathcal{G}^k_{\frac{1}{n}}(\bar{t}_k)$$

is valid hypersequent for any $G^1, \ldots, G^k \in X_h$.

Define $\mathbb{X}_0$ as in Definition 4.4.11 and it is well-formed by Lemma 4.4.12. We now apply F as in Definition 4.4.13 recursively to $\mathbb{X}_0$. By Lemma 4.4.14, there exists some finite t such that the left-most element $\mathfrak{G}$ of $F^t(\mathbb{X})$ has only labels as $(\mathcal{H})$. By well-formedness of $F^t(\mathbb{X})$, we have a derivation of

$$\mathfrak{G} = (\mathcal{H}_{\frac{1}{n}}(\bar{t}_1))^{(\mathcal{H})} \mid \cdots \mid (\mathcal{H}_{\frac{1}{n}}(\bar{t}_k))^{(\mathcal{H})} = \mathcal{H}^{(\mathcal{H})}_{\frac{1}{n}} \mid \cdots \mid \mathcal{H}^{(\mathcal{H})}_{\frac{1}{n}}$$

since $\mathcal{H}$ has no free variables. We then derive $\mathcal{H}_{\frac{1}{n}}$ by *ec*.

$$\frac{\mathcal{H}_{\frac{1}{n}} \mid \cdots \mid \mathcal{H}_{\frac{1}{n}}}{\mathcal{H}_{\frac{1}{n}}}\, ec$$

□

The main claim in Baaz and Metcalfe (2010) is now a corollary by noticing $\Rightarrow A \oplus \overbrace{A \odot \cdots \odot A}^{n}$ is derivable from $\Rightarrow_{\frac{1}{n}} A$.

Corollary 4.4.16. *The followings are equivalent for arbitrary first-order formula A:*

1. *A is a valid formula in the standard $[0,1]$-semantics.*
2. *$\Rightarrow A \oplus \overbrace{A \odot \cdots \odot A}^{n}$ is derivable in Ł∀ for all n.*
3. *A is a valid formula in the affine $[0,1]$-semantics.*

4. $\Rightarrow_{\frac{1}{n}} A$ *is derivable in GŁ∀ for all* n.

5. $\Rightarrow A \oplus \overbrace{A \odot \cdots \odot A}^{n}$ *is derivable in GŁ∀ for all* n.

Corollary 4.4.17. *GŁ∀ together with the following infinitary rule*

$$\frac{\mathcal{H}_{\frac{1}{n}} \text{ for all } n}{\mathcal{H}} \, apx$$

is complete with respect to the affine $[0,1]$*-semantics.*

Corollary 4.4.18. *If* $\mathcal{H}$ *is derivable from GŁ∀ with cut, then* $\mathcal{H}_{\frac{1}{n}}$ *is derivable from GŁ∀ without cut for all* $n \in \mathbb{N}$.

Proof. The *cut* rule is sound with respect to the affine $[0,1]$-semantics. □

Corollary 4.4.19. *The infinitary GŁ∀ with the apx rule admits cut.*

CHAPTER 5

FURTHER DIRECTIONS

We end this thesis by pointing out a few future research directions.

First, we can generalize the affine $[0, 1]$-semantics and hypersequent calculi to continuous logic to obtain similar approximate completeness, as well as other proof-theoretic results. It might also show directions to other desired properties for continuous logic, such as Herbrand's Theorem and Skolemization. We might expect a stronger form of Herbrand's Theorem since an infinimum over a compact metric space is always attained.

Second, it should be possible to strengthen the cut admissibility in Corollary 4.4.18 to a cut elimination procedure. The main obstacle is that Herbrand witnesses from Theorem 4.4.5 are obtained from a proof by contradiction and Tychonoff's theorem, an equivalent form of countable Choice, and thus bear no information. We might be able to find an more effective proof of this theorem by the means of proof mining similar to Gerhardy and Kohlenbach (2005).

Lastly, Łukasiewicz logic and continous logic are non-constructive logics in the sense that $(A \to B) \vee (B \to A)$ is derivable. Another source of non-constructiveness comes from contraction in the Hilbert-style system and the *ec* rule in Gentzen-style system. One might expect an "intuitionistic" Łukasiewicz logic by removing *ec* from GŁ. A development of the intuitionistic proof system and semantics, together with some form of the double-negation translation and realizability might be possible and have great potentials in proof mining for continuous model theory. Notice that a system of intuitionistic Łukasiewicz logic was already investigated in Arthan and Oliva (2019) where their removed the double-negation elimination from Ł, which is a rather different approach.

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