Optimal Distributed Dynamic Advertising

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Abstract

We propose a novel approach to modeling advertising dynamics for a firm operating over distributed market domain based on controlled partial differential equations of diffusion type. Using our model, we consider a general type of finite-horizon profit maximization problem in a monopoly setting. By reformulating this profit maximization problem as an optimal control problem in infinite dimensions, we derive sufficient conditions for the existence of its optimal solutions under general profit functions, as well as state and control constraints, and provide general characterization of the optimal solutions. Sharper, feedback-form, characterizations of the optimal solutions are obtained for two variants of the general problem.

Key Words: optimal advertising, new product introduction, distributed control, infinite dimensional analysis, linear-quadratic control.

AMS Classification: 90B60, 91B72, 49K27, 93C25, 93C30

1 Introduction

In the last two decades, dynamic optimal control models have become a mainstay in the marketing literature focused on advertising. While the empirical marketing research pays increasingly growing attention to the multiple-market aspects of the advertising strategy ([7], [10]), the modeling support for the studies of advertising policies distributed over multiple geographic regions has been lacking (see [11] for the most comprehensive review of the related literature). In the present paper, we propose a simple model which deals with both the dynamic as well as the spatial effects of advertising.

The classical paper by Nerlove and Arrow [16] introduced the following model for the dynamics of goodwill stock under the influence of advertising for a monopolist in a single market-single product environment:

$$\frac{dx(t)}{dt} = u(t) - \rho x(t), \tag{1}$$

where $x(t)$ is the goodwill level at time $t \geq 0$, u is the rate of advertising (in monetary terms), and $\rho > 0$ is a constant factor describing the deterioration of the goodwill in the absence of advertising. Our analysis extends the Nerlove-Arrow (NA hereafter) model by considering spatially distributed advertising and by modeling geographic fluctuations in the goodwill stock. The treatment of problems of optimal advertising in the space-time setting is not new: to the best of our knowledge it was considered for the first and only time in [18] as a special case of a very general class of distributed-parameter optimal advertising models, where the parameter space is modeled as subset of a measurable space. The aim of [18] was to establish a general,

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abstract framework and essentially proved the well-posedness of the problem and the existence of optimal strategies under suitable assumptions. Our purpose here is different, as we focus instead on a simple NA model with a diffusive component, for which optimal strategies can be better characterized, sometimes even explicitly.

In the first part of this work we address a modeling question focused on describing the dynamics of goodwill which depends on both space and time coordinates. In particular, we propose to model the dynamics of goodwill through the following controlled partial differential equation (PDE)

$$\frac{\partial x}{\partial t}(t,\xi) = -\rho x(t,\xi) + \Delta_\xi x(t,\xi) + b(\xi)u(t,\xi) \tag{2}$$

(with appropriate initial and boundary conditions – see (4) below), where $x : [0,T] \times \Xi \to \mathbb{R}$, $\Xi \subset \mathbb{R}^2$ is the goodwill "density", $u(\cdot,\cdot)$ is the rate of advertising effort (expressed in Gross Rating Points, or GRPs), and $b(\cdot)$ is the coefficient of advertising effectiveness. The second term on the right-hand side of (2) is introduced to capture the effect of the spatial diffusion of goodwill (more detailed rationale for the use of this term is described in the next section) and reflects the main modeling difference between our analysis and that of Nerlove and Arrow [16]. The last term on the right-hand side of (2) represents another new feature of our model: we describe advertising effort using GRP parameter, which recent trend in advertising literature ([19], [10]) defines as more appropriate than the traditional expenditure rate. Note that (2) reduces to the NA model if x does not depend on the spatial coordinate ξ and if $b(\xi) \equiv 1$. Moreover, we shall see that the NA model is also recovered by averaging in space our model.

In the second part, we formulate and solve the space–time analogs of some of the optimal control problems studied earlier for the cases where x depends only on time. Most of the results will follow using the tools of optimal control in infinite dimensional spaces. We should clearly state at this point that the emphasis in the paper is not on deriving new results in infinite dimensional optimal control, but rather on showing that models and techniques of analysis and control in Hilbert spaces could prove useful to extend existing advertising models to account for geographic fluctuations. Moreover, we devote some effort to show that in some simple, but still interesting situations, optimal policies can be obtained in closed form, and are hence amenable to practical implementation. Finally, we discuss how the diffusive term influences the optimal strategy by a direct comparison with the standard NA model.

In particular, we consider the following types of problems: in the most general case our aim is to maximize a monopolistic firm's finite horizon profit expressed as a combination of a functional of the terminal goodwill stock and the cumulative cost of advertising. We approach this problem in the abstract setting of control of infinite-dimensional systems via Pontryagin's maximum principle: under mild assumptions on the structure of the problem, one can obtain quite general, although abstract, results on the existence and characterization of optimal policies. For some specific choices of cost and reward functions, optimal policies can be expressed in closed form. A dynamic programming approach allows one to characterize the value function as solution of an Hamilton-Jacobi equation, and to express optimal strategies in feedback form. In the special case of quadratic cost and reward functions, a linear-quadratic (LQ) regulator approach yields explicit expressions for optimal strategies in feedback form. The LQ approach is also used to solve a related problem, i.e. the minimization of a weighted sum of the distance of $x(T,\cdot)$ from a target goodwill level and the total cost of reaching the target.

Our analysis follows a classical scheme: first we establish that the controlled PDE (2) can be written as an abstract linear control system of the type

$$\frac{dx}{dt} = Ax(t) + Bu(t) \tag{3}$$

in a suitable Hilbert space of functions X. Then we show that the control problem for the PDE (2) is equivalent to a control problem on X for the abstract differential equation (3). In the general case, the optimal advertising problem is solved (although only in an abstract way) using the weak maximum principle in infinite dimensions, the theory of which can be found, e.g., in [2]. The simpler case of quadratic utility and cost functions is reduced to the study of a Riccati equation by an application of the dynamic programming principle. The solution of such an equation yields, in particular, a feedback-type optimal policy for the original problem. For the infinite-dimensional LQ problem we refer to [5], [6], [8] for the standard case of positive definite costs (see also [13]), and to [14] for the general case with indefinite costs.

The rest of the paper is organized as follows: section 2 formally defines and motivates the model for the dynamics of goodwill as a function of space and time, as well as the associated optimal advertising problems. The model as well as the optimization problems are recast in the framework of control systems in Hilbert spaces in section 3. Sections 4 to 7 deal with the optimal control problems mentioned above, and section 8 concludes.

2 Model for spatial advertising dynamics

Let us first introduce some notation used throughout the paper. Let X be a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. We denote by $L_2([0,T],X)$ the Hilbert space of functions $f:[0,T]\to X$ with finite norm $\|f\|$,

$$\|f\|^2 = \int_0^T |f(t)|^2\,dt < \infty.$$

When there will be no possibility of confusion, we shall still denote by $\langle\cdot,\cdot\rangle$ the inner product of spaces like $L_2([0,T],X)$. The space of linear bounded operators between the Hilbert spaces X and Y will be denoted as $\mathcal{L}(X,Y)$, or simply as $\mathcal{L}(X)$ when $X=Y$.

Given an operator $A: D(A)\subset X\to X$, we shall say that A is *uniformly positive definite* (denoted $A\gg 0$) if there exists $\varepsilon>0$ such that $A-\varepsilon I\geq 0$, i.e. $\langle Ax,x\rangle\geq\varepsilon|x|^2$ for all $x\in D(A)$. The adjoint of an operator A is denoted by A^*.

We shall need some standard functional spaces: for a bounded open set $\Omega\subset\mathbb{R}^d$, $H^k(\Omega)$ is the Sobolev space of functions with (generalized) derivatives up of order k in $L_2(\Omega)$, and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the topology of $H^1(\Omega)$. Here $C_0^\infty(\Omega)$ stands for the space of infinite differentiable functions with compact support on Ω.

The following elementary notions from convex analysis will be used: given a Hilbert space X and convex lower semicontinuous function $f: X\to]-\infty,+\infty]$, the mapping $\partial f: X\to X$ defined by

$$\partial f(x) = \{w\in X : \langle x-y,w\rangle \geq f(x)-f(y), \forall y\in X\}$$

is called the subdifferential of f. Moreover, the conjugate (or Fenchel-Legendre transform) of f is the function $f^*: X\to]-\infty,+\infty]$ defined by

$$f^*(p) = \sup_{x\in X}(\langle p,x\rangle - f(x)).$$

Let Ξ be a bounded open set of $\mathbb{R}^2$ with regular boundary $\partial\Xi$. This set will be the model for the geographic region of interest for the advertising campaign. Let $x:[0,T]\times\Xi\to\mathbb{R}$ be the goodwill "density" of a given product, specified as a function of time t and location $\xi\in\Xi$.

The model for the controlled dynamics of x will be given by the following equation:

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = (-\rho + \Delta_\xi)x(t,\xi) + b(\xi)u(t,\xi), & (t,\xi) \in]0,T] \times \Xi, \\ \dfrac{\partial x}{\partial n}(t,\xi) = 0, & (t,\xi) \in]0,T] \times \partial\Xi, \\ x(0,\xi) = x_0(\xi), & \xi \in \Xi, \end{cases} \tag{4}$$

where $T > 0$ is a fixed time (which could be thought, for instance, as the time of market introduction for the new product), $\rho > 0$ is a natural deterioration factor of the product image in absence of advertising, $\Delta_\xi = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$ is the Laplacian with respect to the spatial variable ξ, $u : [0,T] \times \Xi \to \mathbb{R}_+$ is the advertising effort, which also depends on time and location, and $b : \Xi \to \mathbb{R}_+$ is a (bounded) factor of advertising effectiveness. $x_0 : \Xi \to \mathbb{R}$ represent the initial level of goodwill as a function of location in the region of interest (for example, one may assume $x_0 = 0$ if at time $t = 0$ the advertised product is unknown).

The introduction of the diffusion term $\Delta_\xi x(t,\xi)$ is motivated by an analogy with the classical diffusion equation, used to model the evolution in time of the concentration of a substance in an homogeneous medium. The idea is to regard a consumer population in a bounded region Ξ as the medium in which the goodwill is spreading. The Neumann boundary condition in (4) is equivalent to assuming that there is no exchange of goodwill through the boundary of the region Ξ, i.e. that the consumer population inside Ξ does not communicate with the outside world. This assumption is natural (at least when considering large enough populations, such as a nation) and allow us to recover the classical NA by averaging our model in space. Indeed, integrating (4) we get

$$\int_\Xi \frac{\partial x}{\partial t}(t,\xi)\,d\xi = -\rho \int_\Xi x(t,\xi)\,d\xi + \int_\Xi \Delta x(t,\xi)\,d\xi + \int_\Xi b(\xi)u(t,\xi)\,d\xi.$$

The total goodwill in the region Ξ is given by $\bar{x}(t) := \int_\Xi x(t,\xi)\,d\xi$. Set $\bar{u}(t) = \int_\Xi b(\xi)u(t,\xi)\,d\xi$ (in the special case $b = 1$ this would be the total advertising effort). Note that, by the Gauss-Stokes formula and the Neumann boundary conditions in (4), one has

$$\int_\Xi \Delta x(t,\xi)\,d\xi = \int_\Xi \Delta x(t,\xi) 1(\xi)\,d\xi = \int_{\partial\Xi} \nabla x(t,\xi) \nabla 1(\xi)\,d\xi = 0.$$

Hence, by changing the order of differentiation and integration on the left hand side, we obtain

$$\frac{d\bar{x}}{dt} = -\rho\bar{x} + \bar{u},$$

which represents the classical NA dynamics for the total goodwill $\bar{x}(\cdot)$.

Remark 1. (i) Equations like (4) have also been successfully used to model the dynamics of populations, as well as the dynamics of infective diseases, in bounded regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let us also recall that simple epidemics models (without spatial dimension) were the starting point for Bass' model of the diffusion of new goods, which still is a reference model in the marketing and management science literature. These considerations, together with the above ones, would hopefully give a plausible foundation to our model.

(ii) The concept of "goodwill density" is not at all difficult to interpret nor too abstract. In fact, it can be (roughly) interpreted as the ratio of total goodwill over a region and the (2-dimensional Lebesgue) measure of the region, provided the region is small enough.

(ii) In the case of a non-homogeneous consumer population the term $\Delta x(t,\xi)$ modeling goodwill diffusion would be replaced by a more general second-order elliptical operator, adding only some technical complications.

Below we will use the controlled dynamic model (4) to identify the optimal advertising strategy for a monopolistic firm. Traditional advertising literature provides several choices in selecting the objective to be maximized in such setting. In the present work we adopt the perspective of a firm which is prepared to launch a new product or service at a pre-determined time T in the future and which aims at maximize its total pre-launch profits. Therefore, a rather general distributed advertising problem one would like to solve can be described as follows:

(P) maximize the functional

$$J_c(u) = \int_\Xi \phi_0(x(T,\xi))\, d\xi - \int_0^T \int_\Xi h_0(u(t,\xi))\, d\xi\, dt \tag{5}$$

over all controls $u(t,\xi) \in [0,R]$ (t,ξ)-a.e., subject to the dynamics (4) and the additional constraint $x(t,\xi) \geq 0$ (t,ξ)-a.e.

Here ϕ_0, $h_0 : \mathbb{R} \to \mathbb{R}$ are such that the integrals in (5) are finite, and can be thought as utility (profit projection) of final goodwill and unit cost of pre-launch advertising, respectively.

We shall see that if the cost function h_0 is quadratic, an alternative approach based on Hamilton-Jacobi equations allows one to obtain a feedback characterization of the optimal strategy. Quadratic functions provide the simplest (strictly) convex cost functions and are commonly used (see e.g. [15]).

We shall also consider two simpler variants of the general advertising problem for which sharper characterizations of the optimal advertising policy can be obtained. The first one, which we denote as (P1), is formulated as follows:

(P1) maximize the functional

$$J_i(u) = \gamma \int_\Xi |x(T,\xi)|^2\, d\xi - \int_0^T \int_\Xi |u(t,\xi)|^2\, d\xi\, dt \tag{6}$$

over all controls $u \in L_2([0,T] \times \Xi, \mathbb{R})$, subject to the dynamics (4). Here the weight coefficient $\gamma > 0$ represents the relative impact of the profit contribution of the final goodwill level vs. that of the advertising effort.

While a quadratic cost of advertising effort is a common and plausible assumption, the choice of this specific form for the profit contribution of the final pre-launch goodwill level in (6) is not an obvious one. In particular, it is not concave, and as such it can be interpreted as the utility function of a *risk-loving* firm. However, in some situations this assumption is reasonable (see e.g. [12] for a related discussion) and can also be justified by considerations of (local) second-order approximations and of analytical tractability.

In addition to (P1) we also study a "targeting" problem denoted as (P2):

(P2) minimize the functional

$$J_h(u) = \gamma \int_\Xi |x(T,\xi) - k(\xi)|^2\, d\xi + \int_0^T \int_\Xi |u(t,\xi)|^2\, d\xi\, dt \tag{7}$$

over all controls $u \in L_2([0,T] \times \Xi, \mathbb{R})$, subject to the dynamics (4), where $\gamma > 0$ is a weight coefficient as above, and $k : \Xi \to \mathbb{R}$ is the target distribution of goodwill at time T.

The targeting problem may arise in settings where a firm would like to evaluate the resources required for establishing a particular goodwill profile $k(\xi)$ for the product to be launched. In particular, if a firm is interested in enforcing a uniform good will profile $k(\xi) = k_0$, setting γ appropriately high and solving the targeting problem repeatedly for several values of k_0 will provide an assessment of necessary advertising spending as a function of k_0. Thus, in essence, the targeting problem serves as a surrogate penalty-function formulation of the problem in which the final goodwill profile is set to equal $k(\xi)$.

3 Reformulating the spatial advertising problem as a control problem in infinite dimensions

Let X be the Hilbert space of square integrable functions defined on the domain Ξ, i.e. $X = L_2(\Xi)$, equipped with the natural inner product

$$\langle f, g\rangle := \int_\Xi f(\xi)g(\xi)\, d\xi$$

and norm

$$|f| := \left(\int_\Xi f^2(\xi)\, d\xi\right)^{1/2}.$$

Denote by A the following linear operator in X:

$$\begin{cases} Ay = (\Delta - \rho)y, \\ D(A) = \left\{v \in H^1(\Xi) : \Delta v \in L_2(\Xi),\ \int_\Xi \Delta v \phi\, dx = -\int_\Xi \langle \nabla v, \nabla \phi\rangle\ dx\right\}. \end{cases} \tag{8}$$

Note that, in view of Green's formula, the second condition defining the domain of A in (8) can be seen as a weak formulation of the Neumann boundary condition $\partial v/\partial n = 0$ on $\partial\Xi$.

Setting (with a slight abuse of notation) $x(t) = x(t, \cdot)$ and $u(t) = u(t, \cdot)$, we can equivalently write (4) as an abstract linear system on the Hilbert space X:

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t), \\ x(0) = x_0 \in X \end{cases} \tag{9}$$

for $t \in [0, T]$, with $A : D(A) \subset X \to X$ as in (8), $u \in L_2([0, T], X)$, and $B : X \to X$ is the linear bounded operator defined by

$$B : y(\xi) \mapsto b(\xi)y(\xi). \tag{10}$$

We shall need the following important features of the Neumann Laplacian, which we collect in the form of a lemma. The proof can be found e.g. in [17] and [20].

Lemma 2. *The linear operator A defined in (8) is self-adjoint, negative, and generates a strongly continuous positivity-preserving semigroup of contractions. Moreover, the Cauchy problem (9) admits a unique mild solution x given by the variation of constants formula:*

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\, ds, \tag{11}$$

where e^{tA} denotes the strongly continuous semigroup generated by A.

Problem (P) can now be written as

$$\inf_{u\in\mathfrak{U}} \left(\phi(x(T)) + \int_0^T h_1(u(t))\, dt\right) \tag{12}$$

subject to the dynamics (9), where ϕ, $h_1 : X \to \mathbb{R}$ are defined as

$$\begin{aligned} \phi(x) &= -\int_\Xi \phi_0(x(\xi))\, d\xi, \\ h_1(u) &= \int_\Xi h_0(u(\xi))\, d\xi, \end{aligned} \tag{13}$$

and

$$\mathfrak{U} = \left\{ u : [0,T] \to X \,\middle|\, u(\cdot)(\xi) \in [0,R],\, x(\cdot)(\xi) \geq 0 \ \xi\text{-a.e.} \right\}.$$

Similarly, the objective functionals (6) and (7) can be respectively written as

$$J_i(u) = -\gamma |x(T)|^2 + \int_0^T |u(t)|^2 \, dt$$

and

$$J_h(u) = \gamma |x(T) - k|^2 + \int_0^T |u(t)|^2 \, dt.$$

The aim is to find an optimal control, i.e. a function $u_* \in \mathfrak{U}_{ad}$ such that

$$J(u_*) \leq J(u) \quad \forall u \in \mathfrak{U}_{ad}.$$

Here J is either J_c, J_i, or J_h, and $\mathfrak{U}_{ad}$ is the class of admissible controls: $\mathfrak{U}_{ad} = \mathfrak{U}$ for problem (P), and $\mathfrak{U}_{ad} = L_2([0,T], X)$ for problems (P1) and (P2). The pair (x_*, u_*), where x_* is the solution of (9) with $u \equiv u_*$, is often called the *optimal pair* for the corresponding optimal control problem.

Remark 3. Problem (P1) is a linear-quadratic (LQ) optimal control problem with indefinite costs, while problem (P2) is an LQ problem with positive costs similar to those encountered in the study of target tracking. While problem (P2) is always well-posed and always admits an optimal control, problem (P1) will be in general only locally well-posed, and global well-posedness will follow from additional assumptions on the parameters of the problem. For more details on LQ problems with indefinite costs in infinite dimensions, see [14].

Remark 4. The objective functional of problem (P1) could be taken, more generally, as

$$J_i(u) = -\langle P_0 x(T), x(T) \rangle + \int_0^T |u(t)|^2 \, dt,$$

with $P_0 : X \to X$. For example, if the goodwill is more valued in a region $\Xi_1 \subset \Xi$, P_0 could be an operator such that $P_0 = \gamma_1 I$ on $\Xi \setminus \Xi_1$, with $\gamma < \gamma_1$. Even more generally, one could fix a bounded function $p : \Xi \to \mathbb{R}_+$ modeling the importance of goodwill at each point of Ξ, and define $P_0 = pI$, I being the identity function on X. Note that the above also applies to problem (P2).

4 Solution of the general constrained problem

The basic idea is to embed the state and control constraints into the structure of the problem. Observe that the non-negativity of the intial condition and of the control in (9) implies the non-negativity of the state variable at any point in time. Using this observation and assigning infinite cost to the controls that do not satisfy the constraint $0 \leq u \leq R$ we enforce the required state and control constraints. Next we show that the problem admits a solution, i.e. that the optimal control exists, and we write a weak maximum principle that gives a (abstract) characterization of the sought optimal advertising policy. Finally we discuss two special cases of the problem for which we can find explicit solutions from the abstract maximum principle.

Let us define $h : X \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$h(u) = \begin{cases} h_1(u), & u \in [0,R] \text{ a.e.}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{14}$$

Proposition 5. *Assume that*

(i) $h_0 : \mathbb{R} \to \mathbb{R}$ *is convex and lower semicontinuous;*

(ii) ϕ_0 *is continuous and concave;*

(iii) x_0 *is nonnegative and* $x_0 \in H_0^1(\Xi)$.

Then there exists at least one optimal pair (x_*, u_*) *for problem (P), with* $x_* \in C([0,T], X)$ *and* $u_* \in \mathfrak{U}$.

The proof of this proposition is essentially classical, but we report it for the reader's convenience and for non-specialists in infinite dimensional control theory.

Proof. First note that $\mathfrak{U} \subset L_1([0,T], X) \cap L_2([0,T], X)$ because Ξ is a bounded set in $\mathbb{R}^2$. For every $u \in L_1([0,T], X)$ the state system (9) admits a unique mild solution $x^u \in C([0,T], X)$. Define the function

$$\begin{aligned} \Phi : L_2([0,T], X) \ &\to \ \mathbb{R} \\ u \ &\mapsto \ \int_0^T h(u(t))\, dt + \phi(x^u(T)). \end{aligned}$$

Problem (P) can be equivalently written as

$$\inf_{u \in L_2([0,T],X)} \Phi(u).$$

Let us now show that Φ is convex and lower semicontinous: for $\lambda \in [0,1]$ and u_1, $u_2 \in L_2([0,T], X)$, the convexity of h implies

$$\int_0^T h(\lambda u_1(t) + (1-\lambda)u_2(t))\, dt \le \lambda \int_0^T u_1(t)\, dt + (1-\lambda) \int_0^T u_2(t)\, dt. \tag{15}$$

One also has

$$\begin{aligned} x^{\lambda u_1 + (1-\lambda)u_2}(T) &= e^{-AT}x_0 + \int_0^T e^{-A(T-t)} B(\lambda u_1(t) + (1-\lambda)u_2(t))\, dt \\ &= \lambda e^{-AT}x_0 + (1-\lambda)e^{-AT}x_0 \\ &\quad + \lambda \int_0^T e^{-A(T-t)} B u_1(t)\, dt + (1-\lambda) \int_0^T e^{-A(T-t)} B u_2(t)\, dt \\ &= \lambda \Big(e^{-AT}x_0 + \int_0^T e^{-A(T-t)} B u_1(t)\, dt \Big) \\ &\quad + (1-\lambda) \Big(e^{-AT}x_0 + \int_0^T e^{-A(T-t)} B u_2(t)\, dt \Big) \\ &= \lambda x^{u_1}(T) + (1-\lambda)x^{u_2}(T), \end{aligned}$$

hence, by the convexity of ϕ,

$$\phi\Big(x^{\lambda u_1 + (1-\lambda)u_2}(T)\Big) \le \lambda \phi(x^{u_1}(T)) + (1-\lambda)\phi(x^{u_2}(T)). \tag{16}$$

Inequalities (15) and (16) imply

$$\Phi(\lambda u_1 + (1-\lambda)u_2) \le \lambda \Phi(u_1) + (1-\lambda)\Phi(u_2),$$

i.e. Φ is convex. In order to prove lower semicontinuity, let $(u_n)_{n\geq 0}$ be a sequence which converges strongly to u in $L_2([0,T]\times\Xi)$. Then $u_n \to u$ in measure, and in particular there exists a subsequence u_{n_k} such that $u_{n_k} \to u$ (t,ξ)-a.e.. By well-known properties of convex functions, h is bounded from below by an affine function, i.e. there exists a, $b \in \mathbb{R}$ such that $h(x) \geq ax + b$, hence

$$\lim_{n_k\to\infty} h(u_{n_k}) - au_{n_k} - b = h(u) - au - b \quad (t,\xi)\text{-a.e.},$$

and by Fatou's lemma

$$\liminf_{n_k\to\infty} \int_{[0,T]\times X} (h(u_{n_k}) - au_{n_k} - b)\, d\xi\, dt \geq \int_{[0,T]\times X} (h(u) - au - b)\, d\xi\, dt,$$

i.e.

$$\liminf_{n_k\to\infty} \int_{[0,T]\times X} (h(u_{n_k} - h(u))\, d\xi\, dt + a \liminf_{n_k\to\infty} \int_{[0,T]\times X} (u - u_{n_k})\, d\xi\, dt \geq 0.$$

But

$$\begin{aligned}
\left| \liminf_{n_k\to\infty} \int_{[0,T]\times X} (u - u_{n_k})\, d\xi\, dt \right| &\leq \liminf_{n_k\to\infty} \int_{[0,T]\times X} |u - u_{n_k}|\, d\xi\, dt \\
&\leq (T \operatorname{vol}(\Xi))^{1/2} \liminf_{n_k\to\infty} \left[\int_{[0,T]\times X} |u - u_{n_k}|^2\, d\xi\, dt \right]^{1/2} \\
&= 0,
\end{aligned}$$

hence

$$\liminf_{n_k\to\infty} \int_0^T h(u_{n_k}(t))\, dt \geq \int_0^T h(u(t))\, dt.$$

Completely similar arguments show that $\liminf_{n_k\to\infty} \phi(x^{u_{n_k}}(T) \geq \phi(x^u(T))$, thus we have proved that Φ is lower semicontinuous.
Moreover, one has $\lim_{|u|\to\infty} \Phi(u) = +\infty$. Let now $a \in \mathbb{R}$ be any (fixed) number. Set $E = \{x \in X : \Phi(x) \leq a\}$. Then one has

$$\inf_{x\in X} \Phi(x) = \inf_{x\in E} \Phi(x). \tag{17}$$

The lower semicontinuity of Φ implies that the level set E is closed, hence weakly closed. Moreover, since $\lim_{|u|\to\infty} \Phi(u) = +\infty$, one also has that E is bounded. Therefore E is also compact in the weak topology of X. By a well known result, every lower semicontinuous function on a compact subset of a topological space attains its infimum. The existence of a minimizer now follows immediately from (17).
Let u_* be the minimizer of Φ. Then it is clear that it must be $u_* \in \mathfrak{U}$. By lemma 2 we also have that e^{tA} is a positivity preserving semigroup. If $u_* \geq 0$ a.e., then one also has $Bu_* \geq 0$ a.e. by the assumptions on $\xi \mapsto b(\xi)$. This implies, together with (11), assumption (iii), and the positivity preserving property of e^{tA}, that $x_* \geq 0$ a.e., which concludes the proof. □

Note that this proof of existence of an optimal pair can be adapted to allow for more general type of constraints. Consider for instance the problem

$$\sup_u \phi(x^u(T)),$$

where u satisfies the budget constraint

$$\int_0^T \int_\Xi u^2(t,\xi)\, d\xi \leq M,$$

with M a fixed positive number, and ϕ is a concave function. Here we have followed [15] in using the quadratic form for the cost as a function of advertising effort. Then we define the set of admissible controls $\mathcal{U} \subset L_2([0,T],X)$ as

$$\mathcal{U} = \left\{ u : \int_0^T \int_\Xi u^2(t,\xi)\, d\xi \le M \right\},$$

and the functional $\Phi : L_2([0,T],X) \to \mathbb{R}$ as

$$\Phi : u \mapsto -\phi(x^u(T)).$$

Now we can reformulate the problem as

$$\inf_{u\in\mathcal{U}} \Phi(u).$$

In complete analogy to the proof of proposition 5, one could show that the set $\mathcal{U}$ is convex, closed, and bounded in $L_2([0,T],X)$, that the function Φ is convex lower semicontinuous in $L_2([0,T],X)$, and that these conditions are enough to ensure that an optimal pair (x_*, u_*) exists, with $u_* \in \mathcal{U}$.

Optimal pairs in many cases can be characterized by the first-order necessary optimality conditions, through the introduction of a Lagrange multiplier. In particular, one has to solve

$$\inf_{u\in L_2([0,T],X)} \left[-\phi(x(T)) + \lambda \int_0^T u^2(t)\, dt \right].$$

Let u_λ be the optimal control for this problem. Then u_* is given by a u_λ such that

$$\int_0^T u_\lambda^2(t)\, dt = M.$$

Once the existence of an optimal solution is established, we can characterize optimal pairs using the maximum principle in Hilbert spaces.

Proposition 6. *Let (x_*, u_*) be an optimal pair for problem (P). Then there exists $p \in C([0,T],X)$ such that x_*, u_*, p satisfy the following two-point boundary value problem:*

$$\begin{cases} x_*' = Ax_* + Bu_*, \\ p' + Ap = 0, \\ x_*(0) = x_0, \\ p(T) \in -\partial\phi(x_*(T)), \\ u_*(t) \in \partial h^*(Bp(t)), \end{cases} \tag{18}$$

where h^* is the conjugate of h, and $\cdot'$ stands for differentiation with respect to time.

Proof. In order to apply the sufficient and necessary optimality system of Theorem 4.2.1 of [3], we need to check some conditions. In particular, A is the generator of a strongly continuous semigroup (lemma 2), and B is a linear bounded operator, as follows from its definition (10) and the boundedness of b. The continuity and convexity of ϕ follow immediately by its definition (13), and similarly for the convexity and lower semicontinuity of h, cf. (14). Therefore the results of the theorem hold, from which we get our optimality conditions (18) by obvious modifications of those in [3]. □

In the next section we consider two special cases of the general spatial advertising problem for which we can provide more specific characterizations of the optimal advertising policies.

4.1 Two cases with explicit solutions

Deriving explicit expressions for the optimal pair from the maximum principle is in general a very challenging task. On the other hand, for some specific choices of the objective function, these closed-form solutions can be obtained. Consider, for instance, the case where $\phi(x) = -x$, and $h(u) = |u|^2/2$, for $u \in [0, R]$ a.e., and $h(u) = +\infty$ otherwise. Then one has $-\partial\phi = 1$, and

$$h^*(\zeta) = \begin{cases} 0, & \zeta < 0, \\ \zeta^2/2, & \zeta \in [0, R], \\ \zeta R - R^2, & \zeta > R. \end{cases}$$

Therefore the maximum principle (18) yields

$$p' + Ap = 0, \qquad p(T) = 1,$$

hence, by well known properties of the heat equation with Neumann boundary conditions,

$$p(t,\xi) = e^{-\rho(T-t)}, \quad (t,\xi) \in [0,T] \times \Xi.$$

From this representation of the dual arc p we immediately obtain that the optimal strategy is given by

$$u_*(t,\xi) = b(\xi)e^{-\rho(T-t)} \wedge R, \qquad \text{a.e.-}(t,\xi) \in [0,T] \times \Xi. \tag{19}$$

In particular, u_* is always nonnegative and is zero where the advertising effectiveness b is zero. Finally, the optimal trajectory x_* is given by

$$x_*(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} B u_*(s)\, ds.$$

An equivalent expression for the optimal trajectory that is easier to implement numerically can be given by projecting on a basis of $L_2(\Xi)$ of eigenvectors of A. In fact, it is well known (see e.g. [1]) that there exists a complete orthonormal (ONC) system $(e_k)_{k\in\mathbb{N}}$ in $L_2(\Xi)$ and a sequence of positive numbers $(\lambda_k)_{k\in\mathbb{N}} \uparrow +\infty$ such that

$$Ae_k = -(\lambda_k + \rho)e_k, \quad \forall k \in \mathbb{N}.$$

Then we have

$$x_*(t,\xi) = \sum_{k\in\mathbb{N}} x_*^k(t) e_k(\xi),$$

a.e. in Ξ_T, where

$$x_*^k(t) = e^{-(\lambda_k+\rho)t}x^k(0) + \int_0^t e^{-(\lambda_k+\rho)(t-s)}\tilde{u}^k(s,\xi)\, ds,$$

and $x^k(0) = \langle x_0, e_k\rangle_{L_2(\Xi)}$, $\tilde{u}^k(t) = \langle b(\cdot)u(t,\cdot), e_k\rangle_{L_2(\Xi)}$.

An analogous procedure allows one to obtain the optimal advertising policy and the optimal state with linear cost, i.e. with $h(u) = u$ for $u \in [0, R]$ a.e., $h(u) = +\infty$ otherwise. In particular

$$\partial h^*(p) = \begin{cases} 0, & p < 1, \\ R, & p > 1, \end{cases} \tag{20}$$

hence

$$u_*(t,\xi) = \begin{cases} 0, & b(\xi)p(t,\xi) < 1, \\ R, & b(\xi)p(t,\xi) > 1, \end{cases} \tag{21}$$

which solves the problem.

Unfortunately we were unable to obtain closed form expressions for the modified version of problem (P2) with linearized cost of control. While the general existence criterion can still be applied, the two-point boundary value problem that characterizes the optimal pair cannot be solved explicitly. In fact, the maximum principle yields

$$\begin{cases} x_*' = Ax_* + Bu_*, \\ p' + Ap = 0, \\ x_*(0) = x_0, \\ p(T) = -2(x_*(T) - k), \\ u_*(t) \in \partial h^*(Bp(t)), \end{cases} \tag{22}$$

where ∂h^* is given by (20).

Remark 7. Note that in the absence of the diffusion term, i.e. for $A = -\rho$, the optimal strategies (19) and (21) would remain unchanged (because the dual arc p would not change), even though the corresponding optimal trajectory do not coincide, as a simple calculation reveals. However, this if of course not a general phenomenon, but just a consequence of the equality $\partial\phi = -1$ in the above examples. For instance, such simplification does not happen in (22).

In the next section we employ the dynamic programming principle to give a characterization of optimal strategies in feedback form. Even though we cannot obtain, in general, completely explicit representation for the optimal strategy, we shall at least indicate some schemes to approximate the value function associated to the problem.

5 Optimal feedback strategies

Let us define the value function

$$V(t, y) = \inf_{u \in \mathfrak{U}} \Big(\phi(x(T)) + \int_t^T h(u(s))\, ds \Big), \tag{23}$$

subject to

$$x' = Ax + Bu, \quad s \in [t, T], \quad x(t) = y. \tag{24}$$

A mild solution of (24) will be denoted by $x^{y,u}$. Note that, strictly speaking, the value function of problem (P) is $-V(t, y)$. However, for consistency with the previous sections, we prefer to study the minimization problem (23), which is clearly equivalent to problem (P).

The existence of an optimal pair (x_*, u_*) for (23) is guaranteed by proposition 5. We shall now see that the dynamic programming approach allows one to obtain a characterization of (x_*, u_*) in terms of the solution of an associated Hamilton-Jacobi equation.

We first prove two simple but important qualitative properties of the value function.

Proposition 8. *The value function $V(t, y)$ is convex and decreasing in y.*

Proof. Let $y_1, y_2 \in X$, and assume that u_1, u_2 are optimal for $V(t, y_1)$ and $V(t, y_2)$, respectively. Then we have, for any $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda V(t, y_1) + (1-\lambda)V(t, y_2) &= \lambda\phi(x^{y_1,u_1}(T)) + (1-\lambda)\phi(x^{y_2,u_2}(T)) \\ &\quad + \int_t^T \lambda h(u_1(s))\, ds + \int_t^T (1-\lambda)h(u_2(s))\, ds \\ &\geq \phi(\lambda x^{y_1,u_1}(T) + (1-\lambda)x^{y_2,u_2}(T)) \end{aligned}$$

$$+\int_t^T [\lambda h(u_1(s)) + (1-\lambda)h(u_2(s))]\,ds,$$

where the inequality holds because ϕ is convex. A simple calculation reveals that

$$\lambda x^{y_1,u_1}(T) + (1-\lambda)x^{y_2,u_2}(T) = x^{\lambda y_1+(1-\lambda)y_2,u}(T),$$

where $u = \lambda u_1 + (1-\lambda)u_2 \in \mathfrak{U}$. By the convexity of the cost function h we then have

$$\begin{aligned}
\lambda V(t,y_1) + (1-\lambda)V(t,y_2) &\geq \phi(x^{\lambda y_1+(1-\lambda)y_2,u}(T)) + \int_t^T h(u(s)) \\
&\geq \inf_{u\in\mathfrak{U}} \Big(\phi(x^{\lambda y_1+(1-\lambda)y_2,u}(T)) + \int_t^T h(u(s))\Big) \\
&= V(t, \lambda y_1 + (1-\lambda)y_2).
\end{aligned}$$

Let us now show that $V(t,y)$ is increasing in y: let u_* be an optimal control, so that

$$V(t,y) = \phi(x^{y,u_*}(T)) + \int_t^T h(u_*(s))\,ds,$$

and take $\bar{y} \geq y$. Then we have

$$\begin{aligned}
J(t,\bar{y},u_*) &= \phi(x^{\bar{y},u_*}(T)) + \int_t^T h(u_*(s))\,ds \\
&= \phi\Big(e^{(T-t)A}\bar{y} + \int_t^T e^{(T-t-s)A}Bu_*(s)\,ds\Big) + \int_t^T h(u_*(s))\,ds \\
&\leq \phi\Big(e^{(T-t)A}y + \int_t^T e^{(T-t-s)A}Bu_*(s)\,ds\Big) + \int_t^T h(u_*(s))\,ds \\
&= V(t,y),
\end{aligned}$$

where the inequality holds because the semigroup generated by A is positivity preserving (cf. lemma 2), $b(\xi) \geq 0$ ξ-a.e., and ϕ_0 is increasing. The proof is completed observing that $V(t,\bar{y}) \geq J(t,\bar{y},u_*)$. □

A combination of the dynamic programming principle and of the abstract maximum principle allows one to write a feedback representation of the optimal strategy in terms of the value function V.

Proposition 9. *Let (x_*, u_*) an optimal pair for (23), and $V : [0,T]\times X \to \mathbb{R}$ the corresponding value function. Then the following inclusion holds:*

$$u_*(s) \in \partial h^*(-B^*\partial V(s, x_*(s))),$$

where ∂V stands for the subdifferential of the function $y \mapsto V(s,y)$.

Proof. Let $v \in L_1([0,T],X)$, and let z be a (mild) solution of $z' = Az + Bv$. Then we can write

$$\begin{aligned}
&\langle x_*(T) - z(T), p(T)\rangle - \langle x_*(s) - z(s), p(s)\rangle \\
&= \int_s^T \big\langle x_*'(r) - z'(r), p(r)\big\rangle\,dr + \int_s^T \big\langle x_*(r) - z(r), p'(r)\big\rangle\,dr \\
&= \int_s^T \langle A(x(r) - z(r)), p(r)\rangle\,dr + \int_s^T \langle B(u_*(r) - v(r)), p(r)\rangle\,dr \\
&\quad + \int_s^T \big\langle x_*(r) - z(r), p'(r)\big\rangle\,dr \\
&= \int_s^T \langle B(u_*(r) - v(r)), p(r)\rangle\,dr,
\end{aligned}$$

where the second equality follows by definitions of y and z, and the third equality follows by the identity $p' \in -A^*p$. By definition of subdifferential we have

$$\langle B(u_*(r) - v(r)), p(r)\rangle \geq h(u_*(r)) - h(v(r)),$$

because $u_*(t) \in \partial h^*(B^*p(t))$ implies $B^*p(t) \in \partial h(u_*(t))$. Recalling that $p(T) \in -\partial\phi(x_*(T))$ we obtain

$$-\langle x_*(s) - z(s), p(s)\rangle \geq \phi(x_*(T)) + \int_s^T h(u_*(r))\,dr - \varphi(z(T)) - \int_s^T h(v(r))\,dr.$$

The dynamic programming principle now implies

$$V(s, x_*(s)) = \phi(x_*(T)) + \int_s^T h(u_*(r))\,dr,$$

which together with the previous inequality yields $-p(s) \in \partial V(s, x_*(s))$. Recalling that $u_*(s) \in \partial h^*(B^*p(s))$ we finally obtain

$$u_*(s) \in \partial h^*(-B^*\partial V(s, x_*(s))). \qquad \square$$

One would expect that the value function V solves, in a suitable sense, the Hamilton-Jacobi equation associated to problem (23), which can be written as

$$v_s(s,y) + \langle Ay, v_y(s,y)\rangle + F(v_y(s,y)) = 0, \quad v(T,y) = \phi(y), \tag{25}$$

for $s \in [t,T]$ and $y \in D(A)$, where $F(q) := -h^*(-B^*q)$. In fact (see [4]), assuming that $F \in C^1(H)$, V satisfies (25) in the sense that there exists $\eta(s,y) \in \partial V(s,y)$ for all $(s,y) \in [0,T] \times X$ such that

$$\begin{cases} v_s(s,y) + \langle Ay, \eta(s,y)\rangle + F(\eta(s,y)) = 0, & \text{a.e. } t \in [0,T], \quad y \in D(A), \\ v(T,y) = \phi(y). \end{cases}$$

In order to approximate the solution of this terminal value problem, one could apply the Crandall-Liggett theorem, which ensures that V is the limit of a sequence of implicit difference approximations (time-discretization), under some extra assumptions. For instance, if $B = I$ and $h(z) = |z|^2$, then $V_\varepsilon(t,\cdot) \to V(t,\cdot)$ uniformly in t as $\varepsilon \to 0$, where $V_\varepsilon(t,\cdot)$ solves

$$\begin{cases} \frac{1}{\varepsilon}(V_\varepsilon(s,y) - V_\varepsilon(s-\varepsilon,y)) + \langle Ay, \nabla V_\varepsilon(s,y)\rangle + F(\nabla V_\varepsilon(s,y)) = 0, & s \in]0,T], \quad y \in D(A), \\ V_\varepsilon(s,y) = \phi(y), & s = 0, \quad y \in D(A). \end{cases}$$

A formal justification can be given following [4], sec. 4.

6 Quadratic cost and reward functions

In this section we solve the optimal control problem (P1) through the dynamic programming approach, that is, first we solve the associated operator Riccati equations, and then we show that the optimal control u_* can be written as a linear feedback of the optimal trajectory x_*. Let us recall that problem (P1) amounts to minimizing the objective functional

$$J_i(u) = -\gamma|x(T)|^2 + \int_0^T |u(t)|^2\,dt,$$

subject to the dynamics $x' = Ax + Bu$, $x(0) = x_0$.

In this and the following section we also assume $B = I$, for simplicity. The analysis of the more general case $B \in \mathcal{L}(X)$ follows essentially the same steps.

An abstract solution of problem (P1) is given by the following proposition.

Proposition 10. *Suppose that the parameters of problem (P1) satisfy the inequality*

$$1 - \frac{\gamma}{2\rho}\left(1 - e^{-2\rho T}\right) > 0.$$

Then for any $x_0 \in X$, J_i admits a unique minimizer u_* given by

$$u_* = \gamma(I - \gamma L_T^* L_T)^{-1} L_T^* e^{-AT} x_0,$$

with L_T defined in (26).

Proof. Let us define the following linear operator:

$$\begin{aligned} L_T : L_2([0,T],X) &\rightarrow X \\ u &\mapsto \int_0^T e^{(T-s)A} u(s)\, ds. \end{aligned} \tag{26}$$

Then we have $x(T) = e^{-AT}x_0 + L_T u$, and, denoting by $\|\cdot\|$ the norm in $L_2([0,T],X)$,

$$\begin{aligned} J_i(u) &= \langle P_0 x(T), x(T)\rangle + \|u\|^2 \\ &= \left\langle P_0(e^{-AT}x_0 + L_T u), e^{-AT}x_0 + L_T u\right\rangle + \|u\|^2 \\ &= \left\langle e^{-A^*T} P_0 e^{-AT} x_0, x_0 \right\rangle + 2\left\langle P_0 e^{-AT}x_0, L_T u\right\rangle + \langle P_0 L_T u, L_T u\rangle + \|u\|^2 \\ &= \left\langle e^{-AT} P_0 e^{-AT} x_0, x_0\right\rangle + 2\left\langle L_T^* P_0 e^{-AT} x_0, u\right\rangle + \langle (I + L_T^* P_0 L_T)u, u\rangle. \end{aligned} \tag{27}$$

By (27) it immediately follows that if $I + L_T^* P_0 L_T \gg 0$, then the functional $J_i : L_2([0,T],X) \to \mathbb{R}$ is convex, lower semi-continuous, and such that $\lim_{|u|\to\infty} J_i(u) = +\infty$, hence it attains its minimum in $L_2([0,T],X)$. Moreover, again as a consequence of (27), if u_* is a minimizer of J, then one must have

$$(I + L_T^* P_0 L_T)u_* + L_T^* P_0 e^{-AT} x_0 = 0. \tag{28}$$

If $\Psi := (I + L_T^* P_0 L_T)$ is an homeomorphism of $L_2([0,T],X)$ (i.e. if Ψ^{-1} is also a linear continuous operator on $L_2([0,T],X)$), then we deduce from (28) that the unique optimal control u_* is given by

$$u_* = -\Psi^{-1} L_T^* P_0 e^{-AT} x_0.$$

From the above discussion, in order to prove the proposition, we need to show that under the stated hypotheses Ψ is uniformly positive definite, and Ψ^{-1} is continuous. In order to prove the former, write

$$\begin{aligned} \langle \Psi v, v\rangle &= \|v\|^2 + \langle L_T^* P_0 L_T v, v\rangle = \|v\|^2 - \gamma\langle L_T^* L_T v, v\rangle \\ &= \|v\|^2 - \gamma\langle L_T v, L_T v\rangle \\ &= \|v\|^2 - \gamma |L_T v|^2, \end{aligned}$$

where we have used the fact that $P_0 = -\gamma I$. We also have

$$\begin{aligned} |L_T v| &= \left| \int_0^T e^{-\rho(T-s)} e^{\Delta(T-s)} v(s)\, ds \right| \\ &\leq \int_0^T e^{-\rho(T-s)} |v(s)|\, ds \\ &\leq \left[\int_0^T e^{-2\rho(T-s)}\, ds\right]^{1/2} \left[\int_0^T |v(s)|^2\, ds\right]^{1/2} \\ &= \left(C_{\rho,T}\right)^{1/2} \|v\|. \end{aligned}$$

We have used (in order) a standard estimate, the contractivity of the heat semigroup, the Cauchy-Schwarz inequality, and Fubini's theorem for positive integrands. By

$$C_{\rho,T} = \int_0^T e^{-2\rho(T-s)} ds = \frac{1}{2\rho}\left(1 - e^{-2\rho T}\right)$$

one has $\langle \Psi v, v\rangle \geq \varepsilon \|v\|^2$, with $\varepsilon = 1 - \frac{\gamma}{2\rho}\left(1 - e^{-2\rho T}\right)$.
In order to prove the continuity of Ψ, it is enough to recall the following simple fact: if $\langle \Psi v, v\rangle \geq \varepsilon\|v\|^2$ for all $v \in D(\Psi)$ and Ψ is self-adjoint, then $\|\Psi^{-1}\| \leq \varepsilon^{-1}$. But $\Psi = I - \gamma L_T^* L_T$ is clearly self-adjoint, and the proof is finished. □

This optimal control suffers of two major drawbacks: it is of the open-loop type, and it is difficult to write explicitly (in particular, the computation of Ψ^{-1} does not seem to be a straightforward task). Appealing to the dynamic programming principle, one can obtain a more explicit feedback characterization of the optimal policy:

Proposition 11. *If*

$$1 - \frac{\gamma}{2\rho}\left(1 - e^{-2\rho T}\right) > 0, \tag{29}$$

then there exists $P \in C\left([0,T]\,; \mathcal{L}\left(X\right)\right)$, $P(t) = P(t)^*$ for all $t \in [0,T]$ which solves the operator Riccati equation

$$\begin{cases} P' = 2AP - PBB^*P, \\ P(0) = P_0 = -\gamma I. \end{cases} \tag{30}$$

Moreover, the optimal control is given by $u_*(t) = -B^*P\left(T-t\right)x_*(t)$, where $x_*(t)$ is the unique (mild) solution of the closed loop equation

$$\begin{cases} x' = Ax(t) - BB^*P(T-t)x(t), \\ x(0) = x_0, \end{cases} \tag{31}$$

and the value function can be written as $V(t,y) = J_i(u_*) = \langle P(T-t)y, y\rangle$.

Proof. The proof follows the classical scheme based on solving the Riccati equation and applying a verification theorem (see e.g. [6] or [9]), but one needs to take into account that P_0 is *negative* in our case. However, the coercivity of J_i on $L_2([0,T],X)$ implied by (29), "saves" the argument (see also theorem 9.4.3 of [14]), and in fact already gives existence and uniqueness of the optimal pair (u_*, x_*). Here we limit ourselves to sketch the proof only, pointing out where the essential difference is with respect to the usual case. The first step consists in proving that the Riccati equation (30) admits a local solution in the space $C([0,\tau];\mathcal{L}(X))$, $\tau < T$ (in the usual case one has $\Sigma(X)$, the space of continuous linear symmetric operators, instead of $\mathcal{L}(X)$), that global uniqueness holds, and finally that local solutions can be extended to the whole interval $[0,T]$. The second step is to show that the closed-loop equation (31) admits a unique mild solution $x \in C([0,T],X)$, which follows as in the classical case. Again following the classical case, in the third step one obtains the representation

$$J_i(u) = \int_0^T \left|u(t) + B^*P(T-t)x(t)\right|^2 dt + \left\langle P(T)x_0, x_0\right\rangle,$$

hence $J_i(u) \geq \langle P(T)x, x\rangle$, for any $u \in L_2([0,T],X)$. Therefore, setting $u_* = -B^*P(T-t)x_*(t)$, with x_* the solution of the closed-loop equation (31), the proposition is proved. □

6.1 An example with explicit solution

In this section we assume that $B = I$, for simplicity, and we show that the Riccati equation of Proposition 4 can actually be solved explicitly in this case. In fact, it is well known (see e.g. [1]) that there exists a complete orthonormal (ONC) system $(e_k)_{k\in\mathbb{N}}$ in $L_2(\Xi)$ and a sequence of positive numbers $(\lambda_k)_{k\in\mathbb{N}} \uparrow +\infty$ such that

$$Ae_k = -\lambda_k e_k, \quad \forall k \in \mathbb{N}.$$

Then we can "project" the Riccati equation on this ONC system, obtaining the infinite set of Cauchy problems

$$\begin{cases} \dfrac{dp_k}{dt} = -2\lambda_k p_k(t) - p_k^2(t), \\ p_k(0) = -\gamma, \end{cases} \tag{32}$$

where $p_k(\cdot) := P(\cdot)e_k$. It is immediate that $q_k(t) \equiv -2\lambda_k$ is a particular solution for the k-th problem. Then set

$$z_k(t) = \frac{1}{p_k(t) - q(t)} = \frac{1}{p_k(t) + 2\lambda_k}.$$

One easily obtains that z_k satisfies the linear equation

$$\frac{dz_k}{dt} = -2\lambda_k z_k(t) + 1,$$

whose general solution is given by

$$z_k(t) = \frac{1}{2\lambda_k} + C_k e^{-2\lambda_k t}.$$

This in turn implies that the general solution for (32) is given by

$$p_k(t) = -2\lambda_k + \frac{1}{(2\lambda_k)^{-1} + C_k e^{-2\lambda_k t}}, \tag{33}$$

and by the initial condition $C_k = \frac{\gamma}{2\lambda_k(2\lambda_k - \gamma)}$.

An explicit expression for the optimal trajectory can also be obtained, again using a projection on the orthonormal system $(e_k)_{k\in\mathbb{N}}$. In particular, one has

$$\frac{dx^k}{dt} = -\lambda_k x^k(t) + p_k(T-t)x^k(t)$$

for each k, hence

$$x_*^k(t) = x^k(0)e^{-\lambda_k t} e^{\int_0^t p_k(T-s)\,ds}, \tag{34}$$

and

$$x_*(t,\xi) = \sum_{k=0}^{\infty} x_*^k(t) e_k(\xi).$$

We can now write explicitly, in terms of the basis $(e_k)_{k\in\mathbb{N}}$, the optimal distributed strategy. Namely,

$$u_*^k(t) = p_k(T-t)x_*^k(t),$$

with p_k as in (32), and x_*^k is given by (34), hence

$$u_*(t,\xi) = \sum_{k=1}^{\infty} p_k(T-t)x_*^k(t)e_k(\xi).$$

In general it is not possible to determine explicitly the eigenvalues and the corresponding eigenfunctions of A for a generic bounded domain $\Xi \subset \mathbb{R}^2$. However, for particular shapes of Ξ the eigensystem is well-known, e.g. for Ξ being a rectangle. If $\Xi = [0, L] \times [0, H]$, one has

$$\Delta e_{m,n} = -\lambda_{m,n} e_{m,n}$$

with

$$e_{m,n}(\xi_1, \xi_2) = \cos \frac{m\pi\xi_1}{L} \sin \frac{n\pi\xi_2}{H}$$

and

$$\lambda_{m,n} = \Big(\frac{m^2}{L^2} + \frac{n^2}{H^2}\Big)\pi^2.$$

Moreover, $\lambda_{0,0} = 0$ is an eigenvalue with unit eigenvector $e_{0,0} = (LH)^{-1/2}$. Hence $Ae_{m,n} = -(\lambda_{m,n} + \rho)e_{m,n}$.

7 Analysis of the targeting formulation

In the Hilbert space setting introduced in Section 3, assuming $B = I$ for simplicity, let us define the distance from the target $y : [0, T] \to X$ as

$$y(t, \xi) = h(\xi) - x(t, \xi),$$

where $h \in L_2(\Xi)$ is the desired configuration of goodwill to reach at time T. Then y is the unique mild solution of the following non-homogeneous linear Cauchy problem in X:

$$\begin{cases} y' = Ay(t) - u(t) + f, \\ y(0) = h - x_0, \end{cases} \tag{35}$$

with $f = -Ah$.

Problem (P2) can be rewritten as

$$\inf_{u \in L_2([0,T];E)} \left[\gamma |y(T)|^2 + \int_0^T |u(t)|^2 \, dt \right], \tag{36}$$

subject to (35). Appealing again to the dynamic programming principle, we can write the Riccati equation associated to problem (36) as follows:

$$\begin{cases} P'(t) = 2AP(t) - P^2(t), \\ P(0) = \gamma I, \end{cases} \tag{37}$$

and its adjoint backward equation as

$$\begin{cases} r'(t) + (A - P(t))r(t) + P(t)f = 0, \\ r(T) = 0. \end{cases} \tag{38}$$

Then one can uniquely solve the tracking problem in terms of the solution to (37) and (38).

Proposition 12. *If the target function h is such that $Ah \in L_2(\Xi)$, then the optimal control problem (P2) admits a unique optimal control u_* given by the feedback law*

$$u_*(t) = P(T - t)\, y_*(t) + r(t),$$

where y_* is the unique mild solution of the closed-loop equation

$$\begin{cases} y'(t) = (A - P(T-t))y(t) - r(t) + f, \\ y(0) = h - x_0, \end{cases} \tag{39}$$

and $P(\cdot)$, $r(\cdot)$ solve, respectively, equations (37) and (38). Moreover, the value function J^* is given by

$$\begin{aligned} J^* = J(u_*) &= \langle P(T)y(0), y(0)\rangle + 2\langle r(0), y(0)\rangle \\ &\quad + \int_0^T \left[2\langle r(t), f\rangle - |r(t)|^2\right] dt. \end{aligned}$$

Proof. It follows essentially the same lines of the "classical" proof in [9], with the only difference that in this case P_0 is positive definite (hence we do not need conditions on the data of the problem to ensure well-posedness), and we have to take into account a non-homogeneous part in the state equation, and consequently of the adjoint equation (38) accompanying the Riccati equation. The assumption $Ah \in L_2(\Xi)$ is equivalent to $f \in X$, so that a (unique) mild solution of (35) exists. For more details we refer to [6]. □

Note that, in contrast to (P1), there always exists an optimal solution for problem (P2), for any choice of the parameters and any initial condition. Below we obtain an expression for the optimal trajectory and the optimal control in terms of a basis of $L_2(\Xi)$, as we have done for (P1). In particular, by projecting the Riccati equation (37) on the system $(e_k)_{k\in\mathbb{N}}$, we obtain the infinite set of Cauchy problems

$$\begin{cases} \dfrac{dp_k}{dt} = -2\lambda_k p_k(t) - p_k^2(t), \\ p_k(0) = \gamma, \end{cases}$$

each of which admits the explicit solution

$$p_k(t) = -2\lambda_k + \frac{1}{(2\lambda_k)^{-1} + C_k e^{-2\lambda_k t}},$$

with $C_k = -\gamma(2\lambda_k(2\lambda_k + \gamma))^{-1}$. As before, we have set $p_k(\cdot) := P(\cdot)e_k$.

The adjoint backward Cauchy problem (38) can be solved similarly: projecting on the system $(e_k)_{k\in\mathbb{N}}$ we get

$$\begin{cases} \dfrac{dr_k}{dt} - \lambda_k r_k(t) - p_k(T-t)r_k(t) + p_k(T-t)f_k = 0, \\ r_k(T) = 0, \end{cases}$$

where $r_k(\cdot) := r(\cdot)e_k$ and $f_k := fe_k$. Setting $\eta_k(t) := r_k(T-t)$, one has

$$\begin{cases} \dfrac{d\eta_k}{dt} = \lambda_k \eta_k(t) - p_k(t)\eta_k(t) + p_k(t)f_k, \\ \eta_k(0) = 0, \end{cases}$$

These Cauchy problems can be solved explicitly, yielding

$$\eta_k(t) = \gamma f_k e^{\lambda_k t + \int_0^t p_k(s)\,ds} + f_k \int_0^t e^{\lambda_k(t-\tau) - \int_\tau^t p_k(s)\,ds} p_k(\tau),$$

and finally $r_k(t) = \eta_k(T-t)$.

The optimal trajectory can also be written explicitly by computing the solution of the closed-loop equation. We again project the equation on the system $(e_k)_{k\in\mathbb{N}}$, obtaining

$$\begin{cases} \dfrac{dy^k}{dt} = (-\lambda_k - p_k(T-t))y^k(t) - r_k(t) + f_k, \\ y^k(0) = h_k - x_k(0), \end{cases}$$

hence

$$y_*^k(t) = (h_k - x_k(0))e^{\int_0^t a_k(s)\,ds} + \int_0^t e^{\int_\tau^t a_k(s)\,ds}(f_k - r_k(s))\,ds,$$

where we have set $a_k(s) := -\lambda_k - p_k(T-s)$. The optimal trajectory can now be written as

$$y_*(t,\xi) = \sum_{k=0}^{\infty} y_*^k(t)e_k(\xi).$$

Similarly, the optimal policy is given by

$$u_*(t,\xi) = \sum_{k=0}^{\infty} u_*^k(t)e_k(\xi) = \sum_{k=0}^{\infty}(p_k(T-t)y_*^k(t) + r_k(t))e_k(\xi).$$

8 Discussion

Our paper extends traditional marketing models of goodwill dynamics to allow for spatially distributed advertising. Using an appropriate Hilbert space reformulation, we map a problem of profit maximization for a monopoly firm into an optimal control problem in infinite dimensions and discuss existence and characterization of its optimal solutions. In some simple, but still realistic situations, the optimal strategy can be obtained in closed form. Our analysis provides a tractable model to investigate how advertising funds should be distributed over multiple markets.

Some important issues to be considered as follow-up investigations are, for instance, an assessment of the empirical adequacy of our model, as well as the analysis of dynamic interaction between firms advertising in multiple markets. Such analysis would present an important complement to the existing literature on single-market advertising competition.

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References

[1] S. Agmon. *Lectures on elliptic boundary value problems.* D. Van Nostrand Co., Princeton, 1965.

[2] V. Barbu. *Analysis and control of nonlinear infinite-dimensional systems.* Academic Press, Boston, MA, 1993.

[3] V. Barbu. *Mathematical methods in optimization of differential systems.* Kluwer, Dordrecht, 1994.

[4] V. Barbu and G. Da Prato. Hamilton-Jacobi equations in Hilbert spaces: variational and semigroup approach. *Ann. Mat. Pura Appl. (4)*, 142:303–349 (1986), 1985.

[5] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite-dimensional systems. Vol. I.* Birkhäuser, Boston, MA, 1992.

[6] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite-dimensional systems. Vol. II.* Birkhäuser, Boston, MA, 1993.

[7] B. J. Bronnenberg and V. Mahajan. Unobserved retailer behavior in multi-market data: Joint spatial dependence in market shares and promotion variables. *Marketing Science*, 20:284–299, 2001.

[8] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory.* Springer-Verlag, New York, 1995.

[9] G. Da Prato. Linear quadratic control theory for infinite dimensional systems. In *Mathematical control theory*, ICTP Lect. Notes, VIII, pages 59–105. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[10] J. P. Dube and P. Manchanda. Differences in dynamic brand competition across markets: An empirical analysis. *Marketing Science*, forthcoming.

[11] G. Feichtinger, R. Hartl, and S. Sethi. Dynamical Optimal Control Models in Advertising: Recent Developments. *Management Sci.*, 40:195–226, 1994.

[12] L. Grosset and B. Viscolani. Advertising for a new product introduction: a stochastic approach. *Top*, 12(1):149–167, 2004.

[13] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations: continuous and approximation theories. I.* Cambridge UP, Cambridge, 2000.

[14] X. J. Li and J. M. Yong. *Optimal control theory for infinite-dimensional systems.* Birkhäuser, Boston, MA, 1995.

[15] E. Muller. Trial/awareness advertising decision: A control problem with phase diagrams with non-stationary boundaries. *J. Econ. Dynamics Control*, 6:333–350, 1983.

[16] M. Nerlove and J. K. Arrow. Optimal advertising policy under dynamic conditions. *Economica*, 29:129–142, 1962.

[17] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, New York, 1978.

[18] T. I. Seidman, S. P. Sethi, and N. A. Derzko. Dynamics and optimization of a distributed sales-advertising model. *J. Optim. Theory Appl.*, 52(3):443–462, 1987.

[19] N. J. Vilcassim, V. Kadiyali, and P. K. Chintagunta. Investigating dynamic multifirm market interactions in price and advertising. *Management Sci.*, 45:499–518, 1999.

[20] I. I. Vrabie. *C_0-semigroups and applications.* North-Holland, Amsterdam, 2003.