

# Residual Inflation Risk

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December 2016

## Abstract

I decompose inflation risk into (i) a component that is correlated with factors that determine investor's preferences and investment opportunities and real returns on real assets with risky cash flows (stocks, corporate bonds, real estate, commodities, etc.), and (ii) a residual inflation risk component. In equilibrium, only the first component earns a risk premium. Therefore investors should avoid exposure to the residual component. All nominal bonds, including the money-market account, have constant nominal cash flows and thus their real returns are equally exposed to residual inflation risk. In contrast, inflation-protected bonds provide a means to avoid cash flow and residual inflation risk. Hence, every investor should put 100% of her wealth in real assets (inflation-protected bonds, stocks, corporate bonds, real estate, commodities, etc.), and finance every long/short position in nominal bonds with an equal amount of other nominal bonds or by borrowing/lending cash, that is, investors should hold a zero-investment portfolio of nominal bonds and cash.

**Keywords:** Inflation Risk, Nominal Bonds, Cash, Money Market Account, Inflation-Protected Bonds, Inflation-Indexed Bonds, TIPS, Dynamic Asset Allocation, Portfolio Choice.

**JEL Classification:** G11.

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Almost every country in the world has experienced periods of high and volatile inflation rates. For instance, the United States experienced high and volatility inflation rates during the monetary experiment in the early eighties. While inflation and inflation risk came down significantly during the Great Moderation, inflation risk recently spiked in the financial crisis with some investors fearing high inflation while others are more worried about deflation.<sup>1</sup> Despite the possibility to invest in inflation-protected securities for the last 20 years in the United States, portfolios consisting of large amounts of cash and nominal bonds are still widely recommended, in particular, for very risk averse investors.

How does the availability of inflation-protected bonds effect the investment in nominal bonds and cash? To fix ideas consider a ten-year nominal Treasury bond and a ten-year Treasury inflation-protected security (TIPS) and suppose the summary statistics reported in Table 1 reflect future beliefs of investors. An investment in the ten-year TIPS protects the real purchasing power of the investment over the next ten years and earns an annual real yield of 1.67%. In contrast, an investment in the ten-year nominal bond will earn an annual nominal yield of 3.9% over the next ten years, which is an expected annualized real return of 1.70% after subtracting 2.2% expected inflation. In this case the investor is exposed to the risk that realized inflation is higher than expected for which she earns an inflation risk premium. The investor can also buy ten-year nominal or inflation-protected bonds and replace them with new ten-year bonds every year. These strategies expose the investor to additional risks (e.g. short term real interest rate risk) and thus earn higher expected returns which are comparable to the ones of stocks. Despite the well known sensitivity of optimal investment portfolios to the risk-reward trade-off and correlation structure of assets, as well as, risk preferences and investment horizons, this paper makes the very strong and robust prediction that the optimal investment in nominal bonds and cash should always be zero when inflation-protected bonds are available.

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<sup>1</sup>Annual inflation volatility estimates based on a GARCH(1,1) model exceeded 12% during the early eighties and 5% during the Great Recession.

Table 1: **Summary Statistics for Inflation, Nominal Treasury Bonds, and TIPS.** Time is measured in years and all reported numbers are in percent.  $y_{n,t}^{\$}$  denotes the continuously compound yield of an  $n$ -year nominal discount bond and  $y_{n,t}^{\text{TIPS}}$  denotes the continuously compound yield of an  $n$ -year inflation-protected discount bond.  $\text{Infl}_t$  is the log inflation rate from time  $t - 1$  until  $t$ .  $r_{n,t}^{\$}$  is the real log holding period return from buying an  $n$ -year nominal discount bond at time  $t - 1$  and selling it as an  $n - 1$  year nominal discount bond at time  $t$ . The inflation rate,  $\text{Infl}_t$ , is subtracted from the nominal return to obtain the real return.  $r_{10,t}^{\text{TIPS}}$  is the real log holding period return from buying an  $n$ -year inflation-protected discount bond at time  $t$  and selling it as an  $n - 1$  year inflation-protected discount bond at time  $t$ . Data are available at the monthly frequency from January 1999 until December 2015.

Panel A: yields			Panel B: one-year returns					
	Mean	Std		Mean	Std	Correlation		
$y_{1,t}^{\$}$	1.96	1.96	$\text{Infl}_t$	2.20	1.33	100	-20.54	26.49
$y_{10,t}^{\$}$	3.90	1.24	$r_{10,t}^{\$}$	5.36	7.05		100	67.78
$y_{10,t}^{\text{TIPS}}$	1.67	1.23	$r_{10,t}^{\text{TIPS}}$	4.57	6.72			100

To understand the portfolio choice result, it is important to recognize that nominal bonds are special because they promise a certain nominal cash flow and the government can always make good on its promise by raising taxes or printing more money which is in contrast to real assets with risky cash flows such as stocks, corporate bonds, real estate, commodities, etc. Hence, inflation can affect the real price of every security through two channels. First, inflation may affect the real economy, meaning the real stochastic discount factor and the real cash flows of real assets. Second, inflation affects the real cash flows of nominal bonds. I decompose inflation risk into (i) a component that is correlated with real returns on real assets and factors that determine investor's preferences and investment opportunities and (ii) a residual component. The first component affects security prices through both channels; however, the residual component, by definition, operates only through the second. In equilibrium, only the first component earns a risk premium, and investors should avoid exposure to the residual component.

Inflation-protected bonds provide a means to avoid real cash flow and residual inflation risk. This role for inflation-protected bonds has not been emphasized in previous literature, but it has dramatic consequences for investments in cash and

nominal bonds. Specifically, I show that: i) the real risk-free asset consists of a long position in inflation-protected bonds and a zero-investment portfolio of nominal bonds and cash, (ii) the tangency portfolio consists of long or short positions in real assets and a zero-investment portfolio of nominal bonds and cash, and (iii) the hedging portfolios consist of long or short positions in real assets and a zero-investment portfolios of nominal bonds and cash. These facts imply directly that (iv) every investor should put 100% of her wealth in real assets (inflation-protected bonds, stocks, corporate bonds, real estate, commodities, etc.) and hold a zero-investment portfolio of nominal bonds and the money-market account.

The nominal return of every discount bond is exposed to factor risk and thus the real return, defined as the difference between the nominal return and realized inflation, is exposed to factor and residual inflation risk. Nominal bonds only differ with respect to their exposure to factor risk and thus results (i)-(iv) follow from the equal exposure of nominal bonds and cash to residual inflation risk. This risk cannot be present in the real locally risk-free asset; thus (i) holds. This risk is not priced; thus, the variance-minimizing portfolio producing a given expected return has no residual inflation risk, producing result (ii). The hedging portfolios are the portfolios maximally correlated with the factors and therefore cannot include residual inflation risk; thus, (iii) holds. The conclusion that every investor should hold a zero-investment portfolio of nominal bonds and cash does not imply a zero investment in each nominal bond. For instance, investors might have a long position in a particular bond to pick up the term premium or hedge against changes in future investment opportunities. However, investors should finance this long position with a short position in other nominal bonds and/or by borrowing cash to avoid exposure to residual inflation risk.

It is well known since Merton (1971) that the optimal dynamic investment strategy is to hold a linear combination of  $k + 2$  mutual funds: two funds to form the optimal portfolio on the mean-variance frontier and  $k$  funds to hedge changes in investor's preferences and investment opportunities. I show for a broad class of preferences and asset return distributions that the optimal amount of nominal bonds and

cash invested in each mutual fund is always zero without explicitly solving for the value function. In other words, the decision to hold a zero-investment portfolio in nominal bonds and cash in each mutual fund does not depend on an investor's preferences or investment opportunities whereas the decision of how much to contribute to a specific nominal bonds in each mutual fund will depend on an investor's preferences and the return characteristic of this bond.

It is crucial for the portfolio predictions in this paper that unpriced residual inflation risk exists. Hence, I consider many different portfolio choice models and show empirically that unpriced residual inflation risk exists, that is, I document that it is almost always more than 50% of inflation risk. For instance, consider a reduced-form five-factor nominal bond pricing model. In this case more than 95% of shocks to realized inflation are not spanned by the factors and thus unpriced. Similarly, suppose investors consider a consumption based asset pricing model with the four factors—expected consumption growth, consumption growth volatility, expected inflation, and inflation volatility—which are, in addition to realized consumption growth, priced. The nominal and real return of every nominal bond may lead differently on the four factors but its real return has exactly the same exposure to residual inflation risk. Residual inflation risk is more than 65% of total inflation risk, and it is unpriced because it is by definition not correlated with realized consumption growth and factor risk. Hence, investors should hold a zero-investment portfolio in all nominal bonds and cash to avoid unpriced residual inflation risk.

What are the economic costs of following suboptimal investments in cash and nominal bonds and thus having exposure to residual inflation risk? To answer this question I consider an investor with an investment horizon of 25 years who can invest in cash, a nominal bond, an inflation-protected bond, and the stock market. There are no hedging demands and thus investors follow simple myopic strategies. Nevertheless, investors with high risk aversion and beliefs about inflation volatility between 3% and 5% are willing to give up between 15% and 50% of their wealth to be able to invest in inflation-protected bonds and hold a zero-investment portfolio consisting of cash

and the nominal bond to avoid residual inflation risk. The costs are between 2% and 5% for risk averse investors who think inflation volatility is more in line with the Great Moderation, rather than the recent high inflation volatility episode or the high inflationary period of the early eighties. Moreover, the cost of exposure to residual inflation risk is strictly increasing with the investment horizon.

The utility cost for suboptimal strategies are in general very sensitive to the choice of assets, factors, and estimated parameters. For instance, Sangvinatsos and Wachter (2005) estimate a three-factor term structure model with time varying risk premia and show that the in sample utility cost for investors who ignore bond predictability are huge. In contrast Feldhütter, Larsen, Munk, and Trolle (2012) show that even with long data sets to estimate parameters, an investor is better off following a portfolio strategy implied by a misspecified but parsimonious model than a correctly-specified but difficult-to-estimate three-factor affine model with time-varying risk premia. The portfolio advice in this paper is robust to model-misspecification and parameter uncertainty as long as residual inflation risk exists. For instance, suppose you want to optimally invest in a portfolio consisting of inflation-protected bonds, nominal bonds, and cash. You consider the first three-principal components of nominal yields as factors. We know from Cochrane and Piazzesi (2005) that the fourth and fifth PC also contain information about bond risk premia and thus your portfolio choice model is misspecified. Moreover, real returns on nominal bonds may load differently on the misspecified residual inflation risk due to possible correlation of the fourth and fifth PCs with inflation shocks. However, it is still true that you should hold a zero investment portfolio in nominal bonds and cash unless the missing factors render residual inflation risk zero.

Fischer (1975), Bodie, Kane, and McDonald (1983), and Viard (1993), assuming a constant investment opportunity set, show that (i) only the part of inflation risk that is correlated with real stock returns should earn a risk premium if the CAPM for real asset returns holds and (ii) investors should shun nominal bonds when inflation-protected bonds are available. I show that part (ii) is no longer true when the real

and nominal short rate are stochastic (the money market account and nominal bonds, as well as, the real risk-free asset and inflation-protected bonds are not perfect substitutes) because in this case it is optimal to hold long/short positions in nominal bonds that are financed by an equal amount of other nominal bonds and cash.

Studies on optimal portfolio choice with inflation-protected bonds include Campbell and Viceira (2001) and Campbell, Chan, and Viceira (2003). Campbell and Viceira (2001) and Campbell, Chan, and Viceira (2003) solve the discrete-time dynamic portfolio choice problem of an infinitely-lived investor with Epstein-Zin preferences, who can invest in equity, nominal bonds, and inflation-protected bonds, using a log linear approximation and a Gaussian investment opportunity set. While this paper employs different assumptions and a different solution method, the principal difference is that I show that real returns of cash and nominal bonds have the same exposure to unpriced residual inflation risk and thus investors should avoid this risk with a zero-investment portfolio in nominal bonds and cash.

This paper is also related to Brennan and Xia (2002) and Sangvinatsos and Wachter (2005), who discuss dynamic asset allocation decision with inflation risk and provide closed form solutions. Brennan and Xia (2002) analyze the portfolio problem of a finite-lived investor with power utility who can invest in the stock market, cash, and nominal bonds when the conditional distribution of all asset returns is Gaussian. Sangvinatsos and Wachter (2005) extend their work by adding another state variable to account for time-varying risk premia and explore the resulting predictability of nominal bond returns for portfolio choice. My paper differs from these papers in that I add inflation-protected bonds to the analysis and consider a broader class of preferences and asset return distributions. Importantly, the fact that residual inflation risk is not priced allows me to determine the optimal investment in nominal bonds and cash in each mutual fund without explicitly solving for the value function of the dynamic portfolio choice problem.

My paper is also related to studies of inflation-protected bonds by Bodie (1990), Gapen and Holden (2005), Hunter and Simon (2005), Kothari and Shanken (2004),

Roll (2004), Brynjolfsson and Fabozzi (1999), Deacon, Derry, and Mirfendereski (2004), Benaben (2005) and Cartea, Saul, and Toro (2012). These studies analyze the mean, variance, and correlation of returns on nominal bonds, inflation-protected bonds, and stocks and discuss the welfare gains of adding inflation-protected bonds to standard investment portfolios consisting of nominal bonds and stocks in a static mean-variance framework. The main conclusion is that adding inflation-protected bonds increases the welfare of investors because of the low standard deviation of real returns of inflation-protected bonds and their diversification benefits. More recently, Pflueger and Viceira (2011) document a relative high correlation between TIPS and nominal bonds over short investment horizons questioning the benefits of investing in inflation-protected bonds. On the other hand, Matthias Fleckenstein and Lustig (2014) argue that TIPS are very attractive investments, claiming even arbitrage opportunities in this market. The model in this paper is very general and can capture empirical stylized facts of inflation, real and nominal bond markets, or other asset classes. Moreover, the qualitative results that investors should hold zero-investment portfolios of nominal bonds and cash is not sensitive to different estimates of the risk-reward tradeoff and correlation structure of assets, as long as, residual inflation risk exists.

## **1 Investment Opportunities**

This section introduces a general framework to study optimal portfolio allocations to nominal bonds and cash when there is inflation risk. Specifically, I specify the conditional distribution of inflation and real assets returns and discuss the exposure of each asset to inflation shocks. The model that I consider is very general and thus I discuss the intuition by means of a simple example throughout the paper.



## 1.1 Model

Let  $X$  denote a  $k$ -dimensional vector of state variables (factors) that describe investor's preferences and investment opportunity sets and  $Z$  a  $d$ -dimensional vector of independent Brownian motions. The state vector  $X$  is Markov<sup>2</sup> with dynamics

$$dX = \mu_X(X) dt + \Sigma_X(X)' dZ, \quad (1)$$

in which  $\mu_X(X)$  is  $k$ -dimensional and  $\Sigma_X(X)$  is  $d \times k$ -dimensional.<sup>3</sup>

Prices in the economy are measured in terms of a basket of real goods. Let  $\Pi$  denote the price level,  $\mu_\Pi(X)$  the expected inflation rate, and  $\Sigma_\Pi(X)$  the  $d$ -dimensional volatility vector of  $\Pi$ . The dynamics of the price level or changes in (realized) inflation are

$$\frac{d\Pi}{\Pi} = \mu_\Pi(X) dt + \Sigma_\Pi(X)' dZ. \quad (2)$$

Assume there is no arbitrage and therefore there exists a strictly positive stochastic discount factor  $M$  that determines real prices of all assets in the economy. Let  $r(X)$  denote the (shadow) risk-free rate or real short rate and  $\Lambda(X)$  the  $d$ -dimensional vector of market prices of risk. The dynamics of the real stochastic discount factor (SDF) are

$$\frac{dM}{M} = -r(X) dt - \Lambda(X)' dZ. \quad (3)$$

The real stochastic discount factor  $M$  and the price level  $\Pi$  are sufficient to price all assets in the economy. Let  $M^\$$  denote the the nominal stochastic discount factor that is given by  $M^\$ = M/\Pi$ .<sup>4</sup> The dynamics of  $M^\$$  are

$$\frac{dM^\$(\Pi)}{M^\$(\Pi)} = -r^\$(X) dt - (\Lambda(X) + \Sigma_\Pi(X))' dZ, \quad (4)$$

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<sup>2</sup>The conditional distribution of  $X_T$  given all information at time  $t$  only depends on  $X_t$ .

<sup>3</sup>The covariance matrix of  $X$  is not necessarily invertible, e.g. time could be a state variable. An apostrophe denotes the transpose of a vector or matrix.

<sup>4</sup>I focus on U.S. investors in the empirical section and thus the price of the basket of real goods is measured in dollars. However, all portfolio predictions in this paper also hold for foreign investors who measure the price of the basket of real goods in units of their currency.

in which

$$r^{\$}(X) = r(X) + \mu_{\Pi}(X) - \Lambda(X)' \Sigma_{\Pi}(X) - \Sigma_{\Pi}(X)' \Sigma_{\Pi}(X). \quad (5)$$

The nominal short rate  $r^{\$}(X)$  is equal to the sum of the real short rate, the expected inflation rate, an inflation risk premium, and a Jensen inequality term. The Fisher equation for the nominal short rate does not hold unless the term  $-\Lambda(X)' \Sigma_{\Pi}(X)$  is zero in which case the expected real return of the money market account is equal to the real short rate (see equation (17) below).

## 1.2 Cash and Nominal Bonds

All nominal Treasury discount bonds, in short nominal bonds, and the cash or money-market account considered are default-free. A nominal bond pays one U.S. dollar at its maturity date and every dollar invested in the money market account earns the nominal risk-free rate over the next instant as interest. Denote real prices of nominal bonds by  $B$  and the real value of the money market account by  $R$ . The corresponding nominal prices are  $B^{\$} = B\Pi$  and  $R^{\$} = R\Pi$ , respectively. The nominal stochastic discount factor is Markov and thus the nominal price of a nominal bond only depends on the state vector  $X$  and time to maturity  $T - t$ .<sup>5</sup> Specifically,

$$B^{\$} = B^{\$}(T - t, X) = \mathbb{E} \left[ \frac{M^{\$}(T)}{M^{\$}(t)} \mid X(t) = X \right]. \quad (6)$$

The nominal value at time  $t$  of \$1 invested in the money market account at time 0 depends on the path of the state vector  $X$  and time  $t$ . Specifically,

$$R^{\$} = R^{\$}(t, \{X(a), 0 \leq a \leq t\}) = e^{\int_0^t r^{\$}(X(a)) da}. \quad (7)$$

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<sup>5</sup>When time  $t$  is a state variable (and thus part of the state vector  $X$ ), then  $B^{\$}$  depends on  $t$  and  $T - t$ .

### 1.3 Real Assets

Suppose in addition to the money market account and nominal bonds there are  $N$  non-redundant real assets, that is, assets that are claims on real cash flows, outstanding. Real assets include inflation-protected bonds, stocks, inflation-protected and nominal corporate bonds, real estate, commodities, and derivatives. I do not explicitly model cash-flows and other characteristics of these securities that are important to price them but instead take their prices as given. Specifically, for  $n = 1, \dots, N$ , let  $S_n$  denote the real, income reinvested price of security  $n$  and  $dS/S$  the column vector with  $dS_n/S_n$  as its  $n$ -th component.<sup>6</sup> Real returns satisfy

$$\frac{dS}{S} = \mu_S(X) dt + \Sigma_S(X)' dZ, \quad (8)$$

in which  $\Sigma_S(X)$  is  $d \times N$ -dimensional. The volatility matrix  $\Sigma_S(X)$  together with the real SDF pins down the expected excess return vector  $\mu_S(X)$ . Specifically,

$$\mu_S(X) = r(X)\mathbf{1} - \frac{dS}{S} \frac{dM}{M} = r(X)\mathbf{1} + \Sigma_S(X)' \Lambda(X), \quad (9)$$

where  $\mathbf{1}$  denotes a column vector of ones.

The state vector  $X$ , the securities  $S_1, \dots, S_N$ , and the consumer price index  $\Pi$  form a Markov system with dynamics

$$\begin{pmatrix} dX \\ dS/S \\ d\Pi/\Pi \end{pmatrix} = \begin{pmatrix} \mu_X(X) \\ \mu_S(X) \\ \mu_\Pi(X) \end{pmatrix} dt + \Sigma(X)' dZ. \quad (10)$$

Without loss of generality, one can take  $X_1$  to depend only on the Brownian motion  $Z_1$ ,  $X_2$  to depend only on  $Z_1$  and  $Z_2$ , etc.<sup>7</sup> This means that I can assume  $d = k + N + 1$

<sup>6</sup> $S_n$  denotes the real price of a portfolio consisting of security  $n$  where any income is used to buy more shares and any outflow (negative income) is financed by selling shares. For instance, any dividends are reinvested in more shares of the security and any storage cost for commodities are financed by selling shares of the security.

<sup>7</sup>For all vectors  $v$  I denote with  $v_i$  the  $i$ -th component.

and that the  $(d \times d)$ -dimensional, volatility matrix

$$\Sigma(X) = (\Sigma_X(X), \Sigma_S(X), \Sigma_\Pi(X)) \quad (11)$$

is upper diagonal.<sup>8</sup> The Markov system in equation (10) is very general. It allows for perfect or imperfect correlations of any variables, and it does not impose an affine or any other structure on the drifts and volatilities.

## 1.4 Residual Inflation Risk

**Definition 1** (Residual Inflation Risk). *Define the last component of the Brownian vector  $Z$ , that is,  $Z_d$ , which is the additional shock to  $d\Pi/\Pi$  that is uncorrelated with changes in the state variables and real returns on real assets, as residual inflation risk. Moreover, define the amount of residual inflation risk, that is,  $RIR$ , as the fraction of the total variance of inflation risk that is due to residual inflation risk  $Z_d$ . Specifically,*

$$RIR = \frac{\Sigma_{\Pi,d}^2}{\Sigma_\Pi' \Sigma_\Pi}. \quad (12)$$

All portfolio choice results in this paper, which are described in detail in the next section, are derived under the assumption that unpriced residual inflation risk exists. The fact that residual inflation risk is unpriced follows almost immediately from its definition but I nevertheless provide a formal argument in Section 3.1. Moreover, I provide empirical support for the existence of unpriced residual inflation risk in Section 3.2.

**Assumption 1.** *Shocks to realized inflation are not spanned by the shocks to factors and real returns on real assets, that is,  $\Sigma_{\Pi d}(X) \neq 0$ . Moreover, the real market price of residual inflation risk is zero, that is,  $\Lambda_d(X) = 0$ .*

Assumption 1 implies that neither the price level nor functions of the price level

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<sup>8</sup>Every vector of dependent Brownian motions can be rotated into a vector of independent Brownian motions using the Cholesky decomposition.

can be part of the state vector, but it does not rule out expected inflation and/or inflation volatility as state variables. It is possible that the price level and functions of it are correlated with state variables. Moreover, Assumption 1 does not impose any restrictions on the inflation risk premium for nominal bonds and the money market account.<sup>9</sup>

To provide intuition for the theoretical results I consider the following example to which I will come back to in the remainder of this paper.

**Example 1** (Markov system and residual inflation risk). *Suppose there is one state variable, the expected inflation rate  $x(t)$ . The dynamics of expected and realized inflation are*

$$dx(t) = \kappa(\bar{x} - x(t)) dt + \sigma_x dz_x(t), \quad (13)$$

$$d\Pi(t) = x(t)\Pi(t) dt + \sigma_\Pi \Pi(t) dz_\Pi(t), \quad (14)$$

where  $z_x(t)$  and  $z_\Pi(t)$  are Brownian motions with  $dz_x(t)dz_\Pi(t) = \rho dt$ . I can equivalently write the dynamics of expected and realized inflation in equations (13) and (14) in terms of the vector of independent Brownian motions  $Z(t) = (Z_1(t), Z_2(t))'$ .<sup>10</sup> Specifically, the Markov system is

$$\begin{pmatrix} dx(t) \\ d\Pi(t)/\Pi(t) \end{pmatrix} = \begin{pmatrix} \kappa(\bar{x} - x(t)) \\ x(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_x & 0 \\ \sigma_\Pi \rho & \sigma_\Pi \sqrt{1 - \rho^2} \end{pmatrix} dZ(t). \quad (15)$$

The loading of realized inflation on  $Z_2(t)$ , defined as residual inflation risk, is  $\Sigma_{\Pi,2} = \sigma_\Pi \sqrt{1 - \rho^2}$ , and

$$RIR = \frac{\Sigma_{\Pi,2}^2}{\Sigma'_\Pi \Sigma_\Pi} = \frac{\sigma_\Pi^2 (1 - \rho^2)}{\sigma_\Pi^2} = (1 - \rho^2) \in [0, 1]. \quad (16)$$

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<sup>9</sup>See Section 1.6.2 for details.

<sup>10</sup>The Cholesky decomposition of the covariance matrix and the rotation of the Brownian motions  $z_x(t)$  and  $z_\Pi(t)$ :

$$\begin{pmatrix} \sigma_x^2 & \rho \sigma_x \sigma_\Pi \\ \rho \sigma_x \sigma_\Pi & \sigma_\Pi^2 \end{pmatrix} = \Sigma' \Sigma, \quad \Sigma = \begin{pmatrix} \sigma_x & \sigma_\Pi \rho \\ 0 & \sigma_\Pi \sqrt{1 - \rho^2} \end{pmatrix}, \quad \begin{pmatrix} dz_x(t) \\ dz_\Pi(t) \end{pmatrix} = \Sigma' dZ(t).$$

If expected inflation is uncorrelated with realized inflation, then  $RIR = 100\%$ . Assumption 1 is violated if expected and realized inflation are perfectly correlated, in which case,  $RIR = 0\%$ .

## 1.5 Real Returns of Nominal Bonds and Cash

Suppose nominal prices of nominal bonds are sufficiently smooth (see Definition 2 in Appendix A). Then, the real return of the money market account and nominal bonds is given in the next proposition.

**Proposition 1** (Money market account and nominal bonds). *The real return of the nominal cash or money market account is*

$$\frac{dR(R^{\$}, \Pi)}{R(R^{\$}, \Pi)} = (r(X) - \Sigma_{\Pi}(X)' \Lambda(X)) dt - \Sigma_{\Pi}(X)' dZ. \quad (17)$$

The real return of a nominal bond maturing at  $T$  is

$$\frac{dB(T-t, X, \Pi)}{B(T-t, X, \Pi)} = (r(X) + \Sigma_B(T-t, X)' \Lambda(X)) dt + \Sigma_B(T-t, X)' dZ, \quad (18)$$

in which the  $d$ -dimensional local real return volatility vector is

$$\Sigma_B(T-t, X) = \Sigma_X(X) \nabla_X B^{\$}(T-t, X) / B^{\$}(T-t, X) - \Sigma_{\Pi}(X) \quad (19)$$

and  $\nabla_X B^{\$}(T-t, X)$  denotes the gradient of  $B^{\$}(T-t, X)$ .<sup>11</sup> Moreover,  $\Sigma_{Bd}(T-t, X) = -\Sigma_{\Pi d}(X)$  for all maturities  $T$ .

*Proof.* See Appendix A. □

The nominal return of the money market account is riskless and the real return is perfectly negatively correlated with realized inflation and thus not exposed to factor risk. Nominal bonds may load differently on the factors and thus both their real and

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<sup>11</sup>The nominal return of a nominal bond is given in equation (54) in Appendix A.

nominal returns have different exposure to factor risk. Residual inflation risk is not correlated with the factors and hence the real return of the money market account and every nominal bond has exactly the same exposure to residual inflation risk, that is,  $-\Sigma_{\Pi}d(X)$ , because nominal bonds and cash pick up residual inflation risk when their nominal returns are converted into real returns. Hence, it is impossible to have a long or short position in a portfolio consisting solely of nominal bonds and cash without having exposure to unpriced residual inflation risk. In contrast, the real return of real assets is not exposed to this unpriced risk source.

**Example 1** (Real return on nominal bonds and their residual inflation risk exposure). *Suppose the real short rate and the market price of risk  $\Lambda = (\lambda_x, 0)'$  is constant. The market price of residual inflation risk is zero and thus  $\Lambda_2 = 0$ . The dynamics of the real SDF are*

$$\frac{dM}{M} = -r dt - \Lambda' dZ. \quad (20)$$

*The nominal short rate is*

$$r^{\$}(t) = r + x(t) - \Lambda' \Sigma_{\Pi} - \Sigma'_{\Pi} \Sigma_{\Pi} = r + x(t) - \lambda_x \sigma_{\Pi} \rho - \sigma_{\Pi}^2, \quad (21)$$

*and the real return of the money market account is*

$$\frac{dR(t)}{R(t)} = (r - \lambda_x \sigma_{\Pi} \rho) dt - \underbrace{\sigma_{\Pi} \left( \rho dZ_1(t) + \sqrt{1 - \rho^2} dZ_2(t) \right)}_{=dz_{\Pi}(t)}. \quad (22)$$

*An investment in the money market account earns the nominal short rate for sure over the next instant and thus the real return is exposed to inflation shocks  $z_{\Pi}(t)$ . Investors are protected against changes in expected inflation,  $x(t)$ , and may earn an inflation risk premium if expected and realized inflation are correlated, that is, if  $\lambda_x \rho < 0$ . However, a large unexpected increase in inflation always reduces the real return on the money market account.*

The nominal price of a nominal bond maturing at  $T$  is

$$B^{\$}(t) = B^{\$}(x, T - t) = \mathbb{E} \left[ \frac{M^{\$}(T)}{M^{\$}(t)} \mid x(t) = x \right] = e^{a(T-t) - b(T-t)x}, \quad (23)$$

where  $b(T-t) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)})$  and  $a(T-t)$  is the solution of an ordinary differential equation.

The real return of a nominal bond maturing at  $T$  is

$$\begin{aligned} \frac{dB(t)}{B(t)} &= (r - (b(T-t)\sigma_x + \sigma_{\Pi\rho}) \lambda_x) dt \\ &\quad - (b(T-t)\sigma_x + \sigma_{\Pi\rho}) dZ_1(t) - \sigma_{\Pi} \sqrt{1 - \rho^2} dZ_2(t). \end{aligned} \quad (24)$$

The nominal  $\tau$ -year holding period return of nominal bond that yields \$1 at maturity  $T = t + \tau$  is certain when held until maturity and the corresponding real return depends on the realized inflation rate over the next  $\tau$  years. In contrast to the nominal return on the money market account, the nominal bond return is exposed to shocks to expected inflation and hence the real return is exposed to shocks to realized and expected inflation. To summarize, the money market account (equation (21)) and every nominal bond (equation (22)) have exactly the same exposure to residual inflation risk, that is,  $-\Sigma_{\Pi,2} = -\sigma_{\Pi} \sqrt{1 - \rho^2}$ , but have different exposure to shocks to expected inflation. If residual inflation risk is zero, then shocks to expected inflation and realized inflation are perfectly correlated and hence nominal bonds have the convenient but unrealistic property that they perfectly hedge against expected and realized shocks to inflation.

I conclude this section with a discussion of real returns on inflation-protected bonds and stocks.

## 1.6 Real Returns of Inflation-Protected Bonds and Stocks

Real returns of inflation-protected bonds only load on factor risk while real returns on stocks may also load on other risks. Importantly, the real return of neither inflation-



protected bonds nor stocks is exposed to residual inflation risk.

### 1.6.1 Inflation-Protected Bonds

An inflation-protected Treasury bonds is a default free zero-coupon bond that pays one unit of the basket of real goods at its maturity date  $T$ . The real stochastic discount factor is Markov and thus the real price of an inflation-protected bond maturing at  $T$  is only a function of the state vector  $X$  and time to maturity  $T - t$ .<sup>12</sup> Specifically,

$$S_T = S_T(X, T - t) = \mathbb{E} \left[ \frac{M(T)}{M(t)} \mid X(t) = X \right]. \quad (25)$$

Suppose real prices of inflation-protected bonds are sufficiently smooth (see Definition 2 in Appendix A). Then the real return of inflation-protected bonds is given in the next proposition.

**Proposition 2** (Inflation-protected bonds). *The real return of an inflation-protected bond maturing at  $T$  is*

$$\frac{dS_T(T - t, X)}{S_T(T - t, X)} = (r(X) + \Sigma_{S_T}(T - t, X)' \Lambda(X)) dt + \Sigma_{S_T}(T - t, X)' dZ, \quad (26)$$

in which the  $d$ -dimensional local real return volatility vector is

$$\Sigma_{S_T}(T - t, X) = \Sigma_X(X) \nabla_X S_T(T - t, X) / S_T(T - t, X) \quad (27)$$

and  $\nabla_X S_T(T - t, X)$  denotes the gradient of  $S_T(T - t, X)$ . Moreover, there is no exposure to residual inflation risk, that is,  $\Sigma_{S_T, d}(T - t, X) = 0$ .

*Proof.* See Appendix A. □

The real cash flow of an inflation-protected bond is constant. Hence, there is no cash flow risk and thus the yield of an inflation-protected bond may be affected by

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<sup>12</sup>When time  $t$  is a state variable (and thus part of the state vector  $X$ ), then  $S_T$  depends on  $t$  and  $T - t$ .

inflation only through the first channel: the real stochastic discount factor. If the inflation-protected bond is held until the maturity then its real return is certain and if it is sold before maturity, then its real return is exposed to factor but not cash flow risk. This is in stark contrast to assets such as nominal bonds and cash whose real cash flows and thus their real return are affected by residual inflation risk which is not correlated with factor risk.

### 1.6.2 Inflation Risk Premium

I discuss in this section how unpriced residual inflation affects the inflation risk premium. We know already that the inflation risk premium of the money market account is defined as the expected real return of the money market account in excess of the real short rate. The inflation risk premium of a nominal bond that matures in  $\tau$  years is defined as the annualized continuously compound expected real return of holding a  $\tau$ -year nominal bond until maturity in excess of the  $\tau$ -year real yield. The inflation risk premium of the money market and nominal bond is given in the next proposition.

**Proposition 3** (Inflation risk premium). *The inflation risk premium of the money market account is*

$$irp(X) = r^{\$}(X) - r(X) - \mu_{\Pi}(X) + \Sigma_{\Pi}(X)' \Sigma_{\Pi}(X) = -\Lambda(X)' \Sigma_{\Pi}(X). \quad (28)$$

*The inflation risk premium of a nominal bond that matures in  $\tau$  years is*

$$\begin{aligned} irp(\tau, X) &= y^{\$}(\tau, X) - y^{TIPS}(\tau, X) - \frac{1}{\tau} \mathbb{E} \left[ \log \left( \frac{\Pi(t + \tau)}{\Pi(t)} \right) \mid X(t) = X \right] \\ &= Cov \left( \frac{\frac{M(T)}{M(t)}}{\mathbb{E} \left[ \frac{M(T)}{M(t)} \mid X(t) = X \right]}, \log \left( \frac{\Pi(T)}{\Pi(t)} \right) \mid X(t) = X \right) \end{aligned} \quad (29)$$

*Proof.* See Appendix A. □

The inflation risk premium of the money market account is zero if the shock to realized inflation is uncorrelated with shocks to the real stochastic discount factor,

that is,  $\Lambda(X)' \Sigma_{\Pi}(X) = 0$ . Hence, unpriced residual inflation risk does not render the inflation risk premium zero unless it is a 100%. The inflation risk premium is zero if the tau-year log inflation rate is uncorrelated with changes in the real stochastic discount factor. Unpriced residual inflation risk does not imply a zero inflation risk premium. This is true even if residual inflation risk is 100% in which case  $\Lambda(X)' \Sigma_{\Pi}(X) = 0$ .<sup>13</sup>

### 1.6.3 Stock Market

Define the real (ex-dividend) price of the stock market, denoted by  $P$ , as an unlevered claim on future aggregate dividends. Let  $D$  denote the real value of aggregate dividends with dynamics

$$\frac{dD}{D} = \mu_D(X) dt + \Sigma_D(X) dZ, \quad \text{where} \quad \Sigma_{D,d}(X) = 0. \quad (30)$$

Hence, real aggregate dividend growth is not correlated with residual inflation risk. The price of the stock market is

$$P(t) = \mathbb{E}_t \left[ \int_t^\infty \frac{M(a)}{M(t)} D(a) dZ(a) \right]. \quad (31)$$

The joint distribution of changes in the SDF and aggregate dividend growth only depends on the state vector  $X$  and thus the price-dividend ratio only depends on  $X$ . Specifically,

$$\text{PD} = \text{PD}(X) = \frac{P}{D} = \mathbb{E} \left[ \int_t^\infty \frac{M(a)}{M(t)} \frac{D(a)}{D(t)} dZ(a) \mid X(t) = X \right]. \quad (32)$$

Let  $\delta = \log(\text{PD})$  denote the continuously compounded dividend yield and  $S(t) = P(t)e^{\int_0^t \delta(a) da}$ , the price of a portfolio that invests one share in the stock market at date 0 and continuously reinvests the dividends in new shares of the stock market.

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<sup>13</sup>The covariance term, and thus the inflation risk premium, is in general not zero even if there is no shock to realized inflation, that is,  $\Sigma_{\Pi}(X) = 0$ , because the tau-year log inflation rate is in general still stochastic due to factor risk.

The real return of the stock market is given in the next proposition.

**Proposition 4** (Stock market). *The real return of the stock market (including dividends) is*

$$\frac{dS}{S} = \frac{dP}{P} + \delta dt = (r(X) + \Sigma_S(X)' \Lambda(X)) dt + \Sigma_S(X)' dZ, \quad (33)$$

*in which the  $d$ -dimensional local real return volatility vector is the Malliavian derivative of  $S$ . Moreover, there is no exposure to residual inflation risk, that is,  $\Sigma_{S,d}(X) = 0$ .*

*Proof.* See Appendix A. □

The stock market may be affected by inflation through the first channel: the real stochastic discount factor and real cash flows. Specifically, the real stock market return is exposed to factor and cash flow risk. But neither factors nor aggregate dividends are correlated with residual inflation risk and thus real stock market returns are not affected by residual inflation risk.

To summarize, shocks to factors and real returns on real assets do not span shocks to inflation risk, and the orthogonal component, denoted by residual inflation risk, is unpriced. Moreover, in contrast to real returns on real assets who are not exposed to residual inflation risk, real returns on nominal bonds and the money market account have exactly the same exposure to residual inflation risk and hence any long or short position in a portfolio consisting of nominal bonds and cash is exposed to this unpriced risk. I will show in the next section that every investor should hold a zero investment portfolio in nominal bonds and cash and put all her wealth in real assets to avoid residual inflation risk. Moreover, I document empirically in Section 3 that unpriced residual inflation risk exists.

## 2 Portfolio Choice

Consider investors who can continuously trade in a frictionless security market and maximize

$$\mathbb{E} \left[ \int_0^T e^{-\int_0^t \beta(X(a)) da} u(c(t), X(t)) dt + e^{-\int_0^T \beta(X(a)) da} U(W(T), X(T)) \right] \quad (34)$$

for some investment horizon  $T$ , subjective discount factor  $\beta$ , utility function  $u$ , and bequest  $U$ .<sup>14</sup> The horizon  $T$  could be infinite in which case  $U = 0$  or it could be random in which case it is assumed to be independent of asset returns. All investors have strictly positive initial wealth and receive either no labor income or labor income that is spanned by real asset returns in which case the present value of future labor income is taken to be part of the initial wealth.

**Assumption 2.** *The market is complete.*

I provide a weaker assumption in the appendix that requires the existence of a mimicking portfolio for the real risk-free assets but the market is incomplete. Hence, investors need to have access to inflation-protected bonds in order to avoid residual inflation risk. While trading in nominal bonds and cash may help to hedge against factor risk, Assumption 1 implies that they are not sufficient to form a mimicking portfolio for the real-risk-free asset due to their exposure to residual inflation risk (see next example).

**Example 1** (Mimicking portfolio for the real risk-free asset). *Suppose you invest the fraction  $\alpha_B$  in the nominal bond and the remaining amount in the money market account, that is,  $\alpha_R = 1 - \alpha_B$ . The real return of this portfolio is*

$$\frac{dW(t)}{W(t)} - \text{“drift” } dt = -(\alpha_B b(T-t)\sigma_x + \sigma_{\Pi}\rho) dZ_1(t) - \sigma_{\Pi}\sqrt{1-\rho^2} dZ_2(t). \quad (35)$$

*The amount invested in the money market account and the nominal bond can be*

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<sup>14</sup>The expectation in equation (34) is assumed to be finite and  $u$  and  $U$  are assumed to fulfill the standard conditions for utility functions (see Karatzas and Shreve (1998)).

chosen to avoid exposure to the risk of changes in the expected inflation rate,  $x(t)$ , but every long or short position in the nominal bond and cash is exposed to residual inflation risk. If residual inflation risk is zero and hence shocks to expected and realized inflation are perfectly correlated, then the nominal bond and cash are sufficient to form a mimicking portfolio for the risk-free asset, that is, they can perfectly hedge against expected and realized inflation risk. In this case, inflation-protected bonds are redundant.

The optimal portfolio of an investor who can trade continuously in the complete security market consisting of cash, nominal bonds, and real assets (inflation-protected bonds, stocks, corporate bonds, real estate, commodities and derivatives) and who seeks to maximize the utility function given in equation (34) is described in the next theorem.<sup>15</sup>

**Theorem 1.** *Adopt Assumptions 1 and 2. Every investor should hold a linear combination of the real risk-free asset, the tangency portfolio, and hedging portfolios. Moreover,*

1. *the mimicking portfolio for the real risk-free asset consists of long positions in inflation-protected bonds and a zero-investment portfolio of nominal bonds and cash.*
2. *The tangency portfolio consists of long or short positions in real assets and a zero-investment portfolio of nominal bonds and cash.*
3. *The portfolios that hedge changes in the investment opportunity set consist of long or short positions in real assets and a zero-investment portfolios of nominal bonds and cash.*
4. *Investors should put 100% of their wealth in real assets and hold a zero-investment portfolio of nominal bonds and cash.*

*Proof.* See Appendix B. □

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<sup>15</sup>The value function  $J(\cdot)$  is defined in equation (78) in Appendix B.

A brief description of the proof is as follows. Assumption 2 implies that there exists a real risk-free asset and hence by the  $(k+2)$ -fund separation theorem of Merton (1971) the optimal portfolio is a linear combination of the mimicking portfolio for the real risk-free asset, the tangency portfolio, and  $k$  portfolios that hedge changes in investor's preferences and investment opportunities. The tangency portfolio is by definition the portfolio with maximal Sharpe ratio, the hedging portfolios are maximally correlated with the factors, and the mimicking portfolio of the real-risk free asset is riskless, and thus neither of these portfolios can be exposed to residual inflation risk.

The composition of the mimicking portfolio for the real risk-free asset, the tangency portfolio, and the hedging portfolios do not depend on the value function. But to obtain the optimal portfolio (to choose the optimal linear combination of the  $(k+2)$  funds) it is necessary to determine the marginal value of wealth, the sensitivity of the marginal value of wealth to changes in wealth and to changes in the state variables. Specifically, the optimal point on the local mean-variance frontier depends on investor's attitude towards risk as measured by the relative risk aversion coefficient  $\gamma \equiv -wJ_{ww}/J_w$ , whereas the hedging demands depend on the sensitivity of the investor's marginal value of wealth to changes in the factors measured by  $\Theta \equiv -J_{wX}/(wJ_{ww})$ .<sup>16</sup>

**Example 1** (Portfolio choice). *Suppose  $\rho = 0$  and  $\lambda_x < 0$ . Consider three securities, an inflation-protected bond, a money market account, and a nominal bond with maturity  $T_B$ . The real risk-free rate is constant and thus all inflation-protected bonds are perfect substitutes. Specifically, the real return for every inflation protected bond is*

$$\frac{dP(t)}{P(t)} = r dt. \quad (36)$$

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<sup>16</sup>Illeditsch (2007) provides closed form solutions for the value function and optimal portfolios (consisting of cash, the stock market, and nominal and inflation-protected bonds) when investors have constant relative risk aversion preferences and asset drifts are quadratic and asset volatilities are affine functions of the expected inflation rate that follows a mean reverting Ornstein-Uhlenbeck process. More generally, Liu (2007) solves the dynamic portfolio choice problem of constant relative risk averse investors (up to the solution of a system of ordinary differential equations) when asset returns are quadratic.

The real return on the money market account and nominal bond in excess of the inflation protected bond is

$$\frac{dR(t)}{R(t)} - \frac{dP(t)}{P(t)} = -\sigma_{\Pi} dZ_2(t) \quad (37)$$

$$\frac{dB(t)}{B(t)} - \frac{dP(t)}{P(t)} = -b(T-t)\sigma_x \lambda_x dt - b(T-t)\sigma_x dZ_1(t) - \sigma_{\Pi} dZ_2(t) \quad (38)$$

There is no risk premium for shocks to realized inflation and thus the real excess return on the money market account is zero. There is a positive risk premium for shocks to expected inflation that an investor can pick up by buying the nominal bond. However, any long position in the nominal bond also exposes the investor to unpriced residual inflation risk and thus the Sharpe ratio does not attain the Hansen and Jagannathan (1991) bound. The investor can increase the Sharpe ratio and attain the bound by financing the long position in the nominal bond by borrowing cash. This investment does not cost anything and thus all the money goes into the inflation-protected bond.

Specifically, let  $w_0$  denote initial wealth,  $\alpha_R$  the fraction of wealth invested in the money market account and  $\alpha_B$  the fraction of wealth invested in the nominal bond with maturity  $T_B$ . Consider an investor who chooses  $\alpha = (\alpha_R, \alpha_B)'$  to maximize

$$\mathbb{E} \left[ \frac{1}{1-\gamma} W(T)^{1-\gamma} \mid W(0) = w_0 \right] \quad (39)$$

subject to the dynamic budget constraint

$$\frac{dW}{W} = (r + \Sigma'_W \Lambda) dt + \Sigma'_W dZ, \quad \Sigma_W = \begin{pmatrix} 0 & -b(T-t)\sigma_x - \sigma_{\Pi} \\ -\sigma_{\Pi} & -\sigma_{\Pi} \end{pmatrix} \alpha. \quad (40)$$

The conditional distribution of real excess returns does not depend on expected inflation  $x(t)$  and thus expected utility in equation (39) does not depend on expected inflation. Hence, there are no hedging demands and optimal demand is

$$\alpha^* = \begin{pmatrix} 0 & -b(T-t)\sigma_x - \sigma_{\Pi} \\ -\sigma_{\Pi} & -\sigma_{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_x \\ 0 \end{pmatrix} = \frac{1}{\gamma b(T-t)} \begin{pmatrix} \frac{\lambda_x}{\sigma_x} \\ -\frac{\lambda_x}{\sigma_x} \end{pmatrix} \quad (41)$$



*Hence, the investor puts 100% in inflation-protected bonds and holds a zero investment portfolio in nominal bonds and the money market account.*

### 3 Residual Inflation Risk

The purpose of this section is threefold. I first provide a formal argument that residual inflation risk is not priced in equilibrium, then I document empirically that unpriced residual inflation risk exists, and finally I consider an example to show that it is quantitatively important.

#### 3.1 Unpriced Residual Inflation Risk

In this subsection, I formally prove that the market price of residual inflation risk is zero.

**Proposition 5** (ICAPM). *Assume that nominal bonds and the money market account are in zero-net-supply and investors have homogeneous beliefs, their endowments are spanned by real asset returns, and their initial wealth (including the present value of future labor income) is strictly positive.<sup>17</sup> Then the market price of residual inflation risk is zero, that is,  $\Lambda_d(X) = 0$ .*

*Proof.* See Appendix C. □

Intuitively, the value function of the representative investor depends on aggregate wealth which is equal to the market portfolio and on the state vector that describes changes in investors's preferences and investment opportunities. The market portfolio is a value weighted sum of all positive-net-supply securities and hence excludes nominal bonds and the money market account. Residual inflation risk is neither correlated with the state vector nor with real returns on the market portfolio and therefore it is

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<sup>17</sup>Preferences, beliefs and endowments of every investor and the security market are defined in Appendix A.

not priced. The conclusion that residual inflation risk is not priced does not require complete markets and homogeneous investors. Specifically, investors can differ with respect to endowments, preferences, and investment horizons.

Nominal-bonds are in zero-net supply and thus not part of the market portfolio.<sup>18</sup> This is still true if we consider a government that issues nominal and inflation-protected bonds and collects taxes to make interest payments on their bonds and to retire them (pay face value). In this case, they are in positive supply but the value of government bonds outstanding is equal to the total tax liability, rendering them effectively in zero-net-supply. Tax payments can differ across investors and can depend on the state of the economy and, as long as, they do not depend on wealth, residual inflation risk is unpriced. I provide a formal argument in Appendix C.2. Moreover, a zero-investment portfolio in nominal Treasury bonds, in this case, should be interpreted as inclusive of the investor's short position in nominal Treasury bonds that corresponds to her position as a taxpayer. In other words, an investor should hold just enough Treasury bonds to immunize his tax liability.

## 3.2 Existence of Unpriced Residual Inflation Risk

I describe the data used in this section before I document empirically the existence of residual inflation risk.

### 3.2.1 Data

**Monthly data.** I obtain monthly Consumer Price Index (CPI) data from the FRED Economic Data base to convert nominal asset prices into real asset prices and compute inflation rates as logarithmic changes. To proxy for expected inflation I consider the cross sectional median of one-year ahead inflation forecasts of consumers.<sup>19</sup> The

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<sup>18</sup>I do not need to assume anything about the supply of the other securities because regardless of whether they show up in the market portfolio or not their real return is by assumption not exposed to residual inflation risk.

<sup>19</sup>The results are similar if I consider the cross sectional average of one-year ahead inflation forecasts of consumers.

inflation forecasts, conducted at a monthly frequency, are available from the Michigan Surveys of Consumers (MSC) database since January 1978.<sup>20</sup> I also compute principal components (PCs) from real and nominal bond yields. Daily continuously-compounded real bond yields, extracted from U.S. Treasury Inflation Protected Securities (TIPS) prices by the method of Gürkaynak, Sack, and Wright (2010), are available from the Federal Reserve Boards webpage since January 1999 for maturities 5 to 20 years and since January 2004 for maturities 2 to 20 years.<sup>21</sup> The set of nominal bond yields consists of the one-month and three-month TBill rate obtained at a monthly frequency from CRSP and nominal bond yields with maturities ranging from 1 to 30 years from Federal Reserve Boards webpage.<sup>22</sup> Daily continuously-compounded nominal bond yields, extracted from U.S Treasury security prices by the method of Gürkaynak, Sack, and Wright (2007), are available from the Federal Reserve Boards webpage since June 1964 for maturities 1 to 7 years, since August 1971 for maturities 1 to 10 years, since November 1971 for maturities 1 to 15 years, since July 1981 for maturities 1 to 20 years, since November 1985 for maturities 1 to 30 years. I average the daily real and nominal yield observations within a month to obtain a time-series of monthly yield observations. Moreover, I consider continuously-compounded real returns of four different asset classes: stocks, corporate bonds, real estate, and commodities. The CRSP value-weighted index including dividends, which is available at the monthly frequency from CRSP represents stocks. The Barclays intermediate term corporate bond index, which is available from Bloomberg at the monthly frequency represents corporate bonds. The Case-Shiller housing price index, which is available from S&P at the monthly frequency represents real estate. The S&P GSCI Total Return commodity index, which is available from Datastream at the monthly frequency, represents commodities. I also consider monthly nominal Fama-Bliss discount bond yields from CRSP with maturities 1 to 5 years.

**Quarterly data.** I consider quarterly nominal personal consumption expendi-

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<sup>20</sup>The website [www.sca.isr.umich.edu/](http://www.sca.isr.umich.edu/) contains detailed information regarding the Michigan Surveys of Consumers.

<sup>21</sup>[www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html](http://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html)

<sup>22</sup>[www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html](http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html)

ture data from the Bureau of Economic Analysis NIPA table 2.3.5 and divide the level by the CPI to obtain quarterly real consumption data. To proxy for expected consumption growth and consumption growth volatility I consider the cross sectional median and interquartile range of one-year ahead personal consumption expenditure forecasts based on the survey of professional forecasters (SPF).<sup>23</sup> To proxy for expected inflation and inflation volatility I consider the cross sectional median and interquartile range of one-year ahead inflation forecasts based on the survey of professional forecasters (SPF).<sup>24</sup>

### 3.2.2 Derivation of Residual Inflation Risk

To determine residual inflation risk we need to choose a set of factors and real assets and then extract the shocks to (i) realized inflation, (ii) factors, and (iii) real returns on real assets. Real returns on nominal bonds are by definition exposed to residual inflation risk and hence need to be excluded from the set of real assets. Before I compute residual inflation risk, more generally, I demonstrate the procedure by means of our workhorse example where we only need to extract the shock to realized and expected inflation. Moreover, I show how to estimate all parameters of the inflation model given by equations (13) and (14) which I use in Section 3.3.

**Example 1** (Residual inflation risk derivation). *Time is measured in years. Consider the cross-sectional median of one-year ahead inflation forecasts of consumers, based on MSC, as proxy for the expected inflation rate,  $x_t$ . The discrete time counterpart of the OU-process given in equation (13) is*

$$x_t = \alpha + \beta x_{t-\Delta t} + \sqrt{\gamma} \varepsilon_t^x, \quad \varepsilon_t^x \sim N(0, 1), \quad (42)$$

where  $\beta = e^{-\kappa\Delta t}$ ,  $\alpha = \bar{x}(1 - \beta)$ , and  $\gamma = \frac{\sigma_x^2}{2\kappa}(1 - \beta^2)$ . The MSC estimator for expected inflation,  $x_t$ , is available at the monthly frequency and thus  $\Delta = \frac{1}{12}$ . Panel A of Table

<sup>23</sup><https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files/rgdp>

<sup>24</sup><https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files/cpi>

2 shows the estimated parameters  $\alpha$ ,  $\beta$ , and  $\gamma$ . Newey-West adjusted standard errors with five lags are in parenthesis.<sup>25</sup> The corresponding continuous time parameters  $\bar{x}$ ,  $\kappa$ , and  $\sigma_x$  are shown in Table 6.

Let  $\text{Infl}_t = \ln\left(\frac{\Pi_t}{\Pi_{t-1}}\right)$  denote the one-year ahead inflation rate and discretize the dynamics of the price level  $\Pi_t$ , measured by the CPI, given in equation (14). Specifically,

$$\text{Infl}_t = \alpha + \beta x_{t-1} + \sqrt{\gamma} \varepsilon_t^{\text{Infl}}, \quad \varepsilon_t^{\text{Infl}} \sim N(0, 1), \quad (43)$$

where  $\alpha = -\frac{1}{2}\sigma_{\Pi}^2$ ,  $\beta = 1$ , and  $\gamma = \sigma_{\Pi}^2$ . Hence,  $E_{t-1}[\text{Infl}_t] = \alpha + \beta x_{t-1}$ . Panel B of Table 2 shows the estimated parameters  $\alpha$ ,  $\beta$ , and  $\gamma$  with Hansen-Hodrick adjusted standard errors with twelve lags in parenthesis to account for the heteroscedasticity and serial correlation in errors due to monthly-overlapping data.<sup>26</sup> The regression results confirm that the MSC estimator is a good and unbiased predictor of inflation.<sup>27</sup> Specifically, the explanatory power is over 60% and the constant  $\alpha$  and the slope  $\beta$  are not statistically significantly different from the Jensen inequality adjustment  $-\frac{1}{2}\sigma_{\Pi}^2 \approx 0$  and 1, respectively.

Finally, we regress the innovation to realized inflation,  $\varepsilon_t^{\text{Infl}}$ , on the innovation to expected inflation,  $\varepsilon_t^x$ , to determine residual inflation risk  $\varepsilon_t^{\text{RIR}}$ . Panel C of Table 2 shows the results of this spanning regression. The  $R^2$  is only 1.9% and thus residual inflation risk is,  $\text{RIR} = 1 - R^2 = 98.10\%$ . The correlation between the shock to realized and expected inflation is, thus,  $\rho = \sqrt{R^2} = 13.77\%$ .

I now consider  $k$  factors and  $N$  real assets (excluding cash and nominal bonds) and extract innovations from realized inflation, factors, and real returns of real assets. Let  $\text{Infl}_t = \ln\left(\frac{\Pi_t}{\Pi_{t-\Delta t}}\right)$  denote the continuously compound realized inflation rate and  $r_t^S = \ln\left(\frac{S_t}{S_{t-\Delta t}}\right)$  the continuously compounded return of a real asset with real asset price  $S_t$ . The  $N$  dimensional vector of real assets is denoted by  $R_t^S$ . Let's stack all

<sup>25</sup>I follow current practice and chose  $\text{Nobs}^{\frac{1}{4}}$  as the number of lags for the Newey and West (1987) autocorrelation consistent covariance estimator (see Greene (2012) page 920).

<sup>26</sup>See Hansen and Hodrick (1980) for the derivations of autocorrelation and heteroskedasticity consistent covariance estimators.

<sup>27</sup>See Ang, Bekaert, and Wei (2007) and the reference therein.

Table 2: **Example 1: Residual Inflation Risk.** Time is measured in years. We use the cross-sectional median of one-year ahead inflation forecasts based on the Michigan Surveys of Consumers (MSC), available at the monthly frequency, as proxy for the expected inflation rate,  $x_t$ , and compute one-year realized inflation rates at the monthly frequency as logarithmic changes of the consumer price index, that is,  $\text{Infl}_t = \ln(\Pi_t/\Pi_{t-1})$ . Panel A shows the constant  $\alpha$ , the  $AR(1)$  coefficient  $\beta$ , the conditional variance  $\gamma$ , and the number of observations, Nobs, of the  $AR(1)$  process for MSC expected inflation,  $x_t$ . Panel B shows the OLS regression results of the realized one year inflation rate on a constant and the one-year expected inflation rate  $x_t$ . The predictive regression confirms that  $x_t$  is a good and unbiased predictor of future inflation. Panel C shows the OLS regression results of the innovation to realized inflation,  $\varepsilon_t^{\text{Infl}}$ , on a constant and the innovation to expected inflation,  $\varepsilon_t^{\text{RIR}}$ . The residual from this spanning regressions is  $\varepsilon_t^{\text{RIR}}$  and residual inflation risk is  $\text{RIR} = 1 - R^2 = 98.10\%$ . Standard errors in parenthesis are Newey-West adjusted with 5 lags in Panel A and C and they are Hansen-Hodrick adjusted with 12 lags in Panel B. The sample period is January 1978 until December 2015.

Panel A: $AR(1)$ process of expected inflation				
$x_t = \alpha + \beta x_{t-\frac{1}{12}} + \gamma \varepsilon_t^x, \quad \varepsilon_t^x \sim N(0, 1)$				
$\alpha \times 10^3$	$\beta$	$\gamma \times 10^4$	Nobs	
1.1608	0.9668	0.1940	455	
(0.5190)	(0.0163)	(0.0062)		
Panel B: Predictive inflation regression				
$\text{Infl}_t = \alpha + \beta x_{t-1} + \gamma \varepsilon_t^{\text{Infl}}, \quad \varepsilon_t^{\text{Infl}} \sim N(0, 1)$				
$\alpha \times 10^3$	$\beta$	$\gamma \times 10^2$	$R^2$	Nobs
-8.9452	1.1835	2.7573	0.6088	444
(5.9640)	(0.1446)			
Panel C: Spanning regression				
$\varepsilon_t^{\text{Infl}} = \alpha + \beta \varepsilon_t^x + \gamma \varepsilon_t^{\text{RIR}}, \quad \varepsilon_t^{\text{RIR}} \sim N(0, 1)$				
$\alpha \times 10^3$	$\beta$	$\gamma \times 10^2$	$R^2$	Nobs
0.0407	0.5446	2.7050	0.0190	444
(1.7024)	(0.2030)			

variables in a vector, that is,  $Y_t = (\text{Infl}_t, X_t, R_t^S)'$ . I consider an ARMA(1,1) and a VAR(1) time-series model for  $Y_t$  to extract innovations to realized inflation,  $\varepsilon_t^{\text{Infl}}$ , factors,  $\varepsilon_t^X$ , and real returns on real assets,  $\varepsilon_t^S$ . Specifically,

- ARMA(1,1):  $Y_t^i = a_0^i + a_1^i Y_{t-\Delta}^i + \varepsilon_t^{Y^i} - a_2^i \varepsilon_{t-\Delta}^{Y^i}$ ,  $\forall i \in \{k + N + 1\}$ ,
- VAR(1):  $Y_t = A_0 + A_1 Y_{t-\Delta} + \varepsilon_t^Y$ .

I take the extracted innovations for realized inflation,  $\varepsilon_t^{\text{Infl}}$ , and regress it on a constant and the factor innovations,  $\varepsilon_t^X$ , and asset return innovations,  $\varepsilon_t^S$ . Specifically,

$$\varepsilon_t^{\text{Infl}} = \alpha + \beta_X \varepsilon_t^X + \beta_S \varepsilon_t^S + \varepsilon_t^{\text{RIR}}. \quad (44)$$

The  $R^2$  of this spanning regression measures how much of the shock to realized inflation is spanned by shocks to factors and real returns on real assets. Hence,  $\text{RIR} = 1 - R^2$ , determines the amount of residual inflation risk.

To compute residual inflation risk, RIR, I consider the real return of four different asset classes, the CRSP value-weighted stock market index including dividends, the Barclays intermediate term corporate bond index, the Case-Shiller housing price index, and the S&P GSCI Total Return commodity index and two different set of factors. The first set of factors consists of the MSC estimator of expected inflation and the first five PCs extracted from the one- and three-month Tbill rate from CRSP and the GSW nominal bond yields with maturities ranging from 1 to 15 years. The first five PCs of nominal bond yields describe almost all (over 99.8%) of the variation in nominal bond yields and thus capture information about real yields, term premia, expected inflation, inflation volatility, inflation volatility, and inflation risk premia. The monthly data set starts in January 1978 and ends in December 2015 and thus the sample size is  $12 \times 38 = 456$ . The second set of factors consists of the MSC estimator of expected inflation, the first three PCs extracted from real bond yields with maturities ranging from 5 to 20 years, and the first three PCs extracted from nominal bond yields with maturities ranging from 1 to 30 years. The monthly data set starts

in January 1999 and ends in December 2015 and thus it is with  $12 \times 17 = 204$  significantly smaller. However, including inflation-protected bonds provides additional information about real rates and term premia and it addresses the concern that the introduction of inflation-protected bonds changed the correlation structure of assets.

The results for residual inflation risk are reported in Table 3 and 4. Panel A in both Tables shows results for residual inflation risk when innovations are extracted from univariate ARMA(1,1) time-series models and Panel B shows results when innovations are extracted from a VAR(1) model. The results in both tables confirm the existence of residual inflation risk. For instance, Table 3 shows that residual inflation risk is either 96.13% or 94.49% when considering the first five PCs of nominal bond yields, expected inflation, stocks, and corporate bonds. The corresponding results are 79.62% and 77.97% when considering the first three PCs of nominal and inflation-protected bond yields, as well as, expected inflation, stocks, and corporate bonds as shown in Table 4.<sup>28</sup> Both tables show that adding real estate and commodities to the portfolio reduces residual inflation risk but it remains at least a quarter of total inflation risk.<sup>29</sup>

What are the implications of the empirical findings for nominal bond portfolios in a reduced form term structure model? Consider a model where first three PCs of nominal bonds are the factors that drive all the variation in nominal bond yields. The nominal and real return of every nominal bond may load differently on the three factors but the real return of every nominal bond has the same exposure to residual inflation risk which does not correlate with the factors and is therefore unpriced. Hence, every long or short position in a nominal bond portfolio is exposed to inflation risk of which at least 97% is not priced. The conclusion does not change if we add the fourth and fifth PC which do not even explain an additional percent of total inflation risk. Specifically, Table 3 shows that residual inflation risk is 98.2% or 97.0%

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<sup>28</sup>Residual inflation risk is either 92.16% or 91.87% when restricting the sample period in Table 3 to January 1999 to December 2015 which indicates that the drop in residual inflation risk is partly due to the sample period.

<sup>29</sup>The results are very similar to the findings in Table 4 when restricting the sample to January 2004 to December 2015 where PCs for real yields are based on TIPS with maturities ranging from 2-20 years.



Table 3: **Residual inflation risk since 1978:  $RIR = 1 - R^2$** . The  $R^2$  is computed from an OLS-regression of shocks to realized inflation,  $\varepsilon_t^{\text{Infl}}$ , onto shocks to factors,  $\varepsilon_t^X$ , and real returns of real assets,  $\varepsilon_t^S$ . The factors consists of the first five PCs of nominal yields and expected inflation (Einfl). The real asset are stocks (S), corporate bonds (CB), real estate (RE), and commodities (Com). The shocks are extracted from a univariate  $ARMA(1, 1)$  time-series model in Panel A and from a VAR(1) model in Panel B. The first/last row in Panel A/B shows the results from univariate regressions, the second/second-to-last row shows results from regressions with two regressors: 1st PC and 2nd PC, 1st PC and 3rd PC, ..., 1st PC and Com, and the last/first row shows the result from the regression with all ten regressors. Sample period: January 1978 to December 2015.

Panel A: Innovations from ARMA(1,1) model									
$PC_1$	$PC_2$	$PC_3$	$PC_4$	$PC_5$	Einfl	S	CB	RE	Com
99.23	99.37	99.99	100.00	99.93	99.78	98.77	99.53	56.33	78.32
	98.76	99.14	99.19	99.22	98.99	97.85	99.20	54.65	77.81
		98.20	98.55	98.76	98.60	97.50	98.75	54.62	76.91
39.04			98.15	98.19	98.03	96.71	98.19	54.00	76.25
67.54	50.07			98.13	97.97	96.55	98.15	53.95	75.95
67.60	50.14	94.49			97.95	96.50	98.13	53.95	75.92
67.61	52.38	96.09	94.49			96.34	97.95	52.98	75.23
67.87	54.04	96.61	95.10	96.23			96.13	51.48	75.23
68.02	54.09	96.72	95.14	96.29	96.82			51.09	75.04
68.02	54.29	96.81	95.17	96.48	96.99	96.90			41.37
68.89	55.04	97.68	96.24	97.29	97.86	97.85	97.00		
69.34	55.07	97.85	96.37	97.49	98.03	98.05	97.63	97.88	
70.26	57.49	99.60	98.64	99.33	99.80	99.97	99.84	99.54	98.05
Com	RE	CB	S	Einfl	$PC_5$	$PC_4$	$PC_3$	$PC_2$	$PC_1$
Panel B: Innovations from VAR(1) model									

Table 4: **Residual inflation risk since 1999:  $RIR = 1 - R^2$ .** The  $R^2$  is computed from an OLS-regression of shocks to realized inflation,  $\varepsilon_t^{\text{Infl}}$ , onto shocks to factors,  $\varepsilon_t^X$ , and real returns of real assets,  $\varepsilon_t^S$ . The factors consists of the first three PCs of real yields, the first three PCs of nominal yields, and expected inflation (Einfl). The real asset are stocks (S), corporate bonds (CB), real estate (RE), and commodities (Com). The shocks are extracted from a univariate  $ARMA(1, 1)$  time-series model in Panel A and from a VAR(1) model in Panel B. The first/last row in Panel A/B shows the results from univariate regressions, the second/second-to-last row shows results from regressions with two regressors: 1st PC and 2nd PC, 1st PC and 3rd PC, ..., 1st PC and Com, and the last/first row shows the result from the regression with all ten regressors. Sample period: January 1999 to December 2015.

Panel A: Innovation extraction with ARMA(1,1) model										
$PC_1^{\text{TP}}$	$PC_2^{\text{TP}}$	$PC_3^{\text{TP}}$	$PC_1^{\text{NB}}$	$PC_2^{\text{NB}}$	$PC_3^{\text{NB}}$	Einfl	S	CB	RE	Com
96.58	96.31	99.81	99.58	99.59	96.97	99.92	95.00	99.92	67.52	59.74
	92.71	96.47	90.17	96.58	95.92	96.56	91.74	95.23	66.85	59.19
		92.70	89.06	89.91	92.65	92.44	88.58	91.66	62.27	59.00
28.76			89.03	89.90	92.65	92.42	88.57	91.64	61.65	57.87
45.64	44.12			81.90	88.96	88.19	87.05	88.75	53.92	57.86
45.68	44.21	77.97			81.59	81.05	80.62	81.90	49.91	53.73
46.05	44.79	78.92	78.54			80.61	80.49	81.59	47.67	53.57
46.10	44.83	79.05	78.61	79.09			79.83	80.58	47.04	53.48
46.25	45.41	79.44	78.77	79.41	79.25			79.62	46.30	53.38
49.48	48.99	86.71	85.33	86.78	86.42	79.49			46.30	53.35
49.67	55.34	91.12	88.06	91.88	91.79	89.41	86.78			32.48
49.90	55.97	91.16	88.12	91.95	91.89	89.69	86.84	91.91		
50.17	58.80	95.02	91.66	95.73	95.30	95.78	88.55	95.75	91.96	
50.54	60.15	99.80	95.48	99.90	97.54	99.60	99.98	99.94	96.67	95.80
Com	RE	CB	S	Einfl	$PC^{\text{NB}_3}$	$PC_2^{\text{NB}}$	$PC_1^{\text{NB}}$	$PC_3^{\text{TP}}$	$PC_2^{\text{TP}}$	$PC_1^{\text{TP}}$
Panel B: Innovation extraction with VAR(1) model										

when we consider the first three PCs and it is 98.1% and 96.8% if we consider the first five PCs. If we consider the first three PCs of nominal and inflation-protected bonds as factors, then nominal bonds are exposed to unpriced residual inflation risk that is, according to Table 4, 81.6% or 79.3% of total inflation risk.

I conclude this subsection with a discussion of optimal nominal bond portfolios in a consumption based asset pricing model.<sup>30</sup> In this model, we need to decompose shocks to realized inflation into shocks to (i) realized consumption growth (C), (ii) expected consumption growth (ExpC), (iii) consumption growth volatility (SigC), (iv) expected inflation (Einfl), (v) inflation volatility (SigInfl), and (vi) a residual component. Consumption growth rates are computed from nominal personal consumption expenditures deflated by the CPI, the SPF cross-sectional median of one-year ahead consumption growth and inflation rates proxies for expected consumption growth and expected inflation, and the SPF cross-sectional interquartile range of one-year ahead consumption growth and inflation rates proxies for consumption growth and inflation volatility. There are four panels in Table 5 which show results for residual inflation risk that differ w.r.t the innovation extraction model, ARMA(1,1) or VAR(1), and the return horizon, quarterly or annual.

Suppose investors have power utility and thus only shocks to realized consumption growth are priced. In this case the nominal and real return of ever nominal bond may load differently on the four factors—expected consumption growth, consumption growth volatility, expected inflation, and inflation volatility—but neither of them is priced. If shocks to realized inflation are spanned by shocks to realized consumption growth and the four factors, rendering residual inflation risk zero, then it is not clear how to optimal investment in the portfolio consisting of all nominal bonds without specifying the statistical model for consumption growth and inflation, deriving the nominal bond yields in equilibrium, and then solving for the optimal portfolio. In this case the optimal nominal bond investment will depend on the risk-return tradeoff and the correlation structure of all bonds, as well as, risk aversion and the investment

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<sup>30</sup>I thank the referee for suggesting this example.

Table 5: **Residual inflation in consumption based model:  $RIR = 1 - R^2$ .** The  $R^2$  is computed from an OLS-regression of shocks to realized inflation (Infl) onto shocks to realized consumption growth (C), SPF expected consumption growth (ExpC), SPF consumption growth volatility (SigC), SPF expected inflation (Einfl), SPF inflation volatility (SigInfl). The shocks are extracted from a univariate  $ARMA(1, 1)$  time-series model for quarterly returns in Panel A.1 and annual returns in Panel A.2. The shocks are extracted from a univariate  $VAR(1)$  time-series model for quarterly returns in Panel B.1 and annual returns in Panel B.2. The first/last row in Panel A/B shows the results from univariate regressions, the second/second-to-last row shows results from regressions with two regressors: C and ExpC, . . . , and the last/first row shows the result from the regression with all five regressors. Sample period: Q3 1981 to Q4 2015.

Panel A: Innovations from ARMA(1,1) model									
Panel A.1: Quarterly rate					Panel A.2: Annual rate				
C	ExpC	SigC	Einfl	SigInfl	C	ExpC	SigC	Einfl	SigInfl
89.48	99.44	99.20	99.36	99.99	89.15	99.41	98.13	98.54	99.55
	85.43	87.52	89.29	89.48		85.79	85.87	88.53	89.14
		84.59	85.14	85.33			83.81	85.10	85.78
78.02			84.30	84.41	65.75			83.19	83.77
79.69	78.02			84.12	83.01	65.89			83.12
80.42	80.06	79.76			83.06	68.01	85.04		
89.52	85.34	89.93	80.43		94.15	71.42	98.44	85.08	
99.78	97.68	99.93	97.79	89.95	95.95	80.53	100.00	93.23	98.47
SigInfl	Einfl	SigC	ExpC	C	SigInfl	Einfl	SigC	ExpC	C
Panel B.1: Quarterly rate					Panel B.2: Annual rate				
Panel B: Innovations from VAR(1) model									

horizon. In contrast, if residual inflation risk exists, then the real return of every nominal bond has exactly the same exposure to it and hence we know that the total investment in nominal bonds should be zero without specifying the details of the model. Table 5 confirms the existence of residual inflation risk, that is, it is more than 65% of the total. The optimal investment in nominal bonds and cash remains zero, when shocks to realized consumption growth and all four factors are priced, that is, when investors have Epstein-Zin preferences because the real return of every nominal bond has still the same exposure to residual inflation risk which is neither correlated with shocks to realized consumption growth nor shocks to the factors and thus unpriced.

### 3.3 Cost of Residual Inflation Risk Exposure

In order to assess the economic importance of hedging residual inflation risk by investing in inflation-protected bonds and holding a zero-investment portfolio in nominal bonds and cash, I compute the utility cost for investors who do not invest in inflation-protected bonds and thus hold suboptimal long/short positions in a portfolio consisting of nominal bonds and cash. Specifically, I consider Example 1, discussed in the previous two sections, and add the stock market to the set of available asset consisting of cash, one nominal bond, and one inflation-protected bond. There is almost no correlation between real stock market returns and expected and realized inflation in the data and thus I set both correlations to zero. The dynamics of the SDF are

$$\frac{dM}{M} = -r dt - \Lambda' dZ, \quad \Lambda = (\lambda_x, \lambda_S, 0)'. \quad (45)$$

The dynamics of real stock returns, and realized and expected inflation are

$$\begin{pmatrix} dx \\ dS/S \\ d\Pi/\Pi \end{pmatrix} = \begin{pmatrix} \kappa(\bar{x} - x) \\ r + \sigma_S \lambda_S \\ x \end{pmatrix} dt + \Sigma' dZ, \quad \Sigma' = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_S & 0 \\ \sigma_{\Pi\rho} & 0 & \sigma_{\Pi}\sqrt{1 - \rho^2} \end{pmatrix} \quad (46)$$

with  $\Sigma = (\Sigma_x, \Sigma_s, \Sigma_\Pi)$ . The real return for a nominal bond with maturity  $T_B$  is

$$\frac{dB}{B} = (r + \Sigma'_B \Lambda) dt + \Sigma'_B dZ, \quad \Sigma_B = (-b(T_B - t)\sigma_x - \sigma_\Pi \rho, 0, \sigma_\Pi \sqrt{1 - \rho^2})'. \quad (47)$$

I calibrate the model to inflation and asset return data and summarize the results for the estimated parameters in Table 6. The sample period is January 1978 to December 2015. The inflation parameters,  $\bar{x} = 0.0349$ ,  $\kappa = 0.4055$ ,  $\sigma_x = 0.0155$ ,  $\sigma_\Pi = 0.0166$ , and  $\rho = 0.1377$ , are estimated from CPI realized inflation and MSC expected inflation data.<sup>31</sup> The risk-free interest rate,  $r = 0.0092$ , is set to match the average of the real one-year risk-free interest rate which is defined as the nominal one-year yield minus the MSC expected inflation rate,  $x(t)$ . The stock market volatility,  $\sigma_S$ , and the market price of stock market risk (stock market Sharpe ratio),  $\lambda_S$ , are set to match the mean,  $m_S = 0.0355$ , and variance,  $v_S = 0.02634$ , of the one-year real holding period return of the CRSP value-weighted stock market index including dividends in excess of the one-year real risk-free rate, that is,  $\sigma_S = \sqrt{v_S} = 0.1623$  and  $\lambda_S = (m_S + 0.5v_S)/\sqrt{v_S} = 0.2999$ . The market price of expected inflation risk,  $\lambda_x$ , with absolute value equal to the Sharpe ratio of every nominal bond in this example, is set to match the Sharpe ratio of the one-year real holding period return of the ten-year nominal discount bond in excess of the real risk-free rate, that is,  $-\lambda_x = (m_B + 0.5v_B)/\sqrt{v_b} = 0.3997$ , where  $m_B = 0.041$  and  $v_B = 0.014596$ . The Sharpe ratio of the nominal bond is higher than the Sharpe ratio of the stock market for this sample period and thus I consider another alternative for,  $\lambda_x$ . Specifically,  $\lambda_x$ , is set to match the Sharpe ratio of the one-year real holding period return of the five-year nominal Fama-Bliss discount bond in excess of the real risk-free rate, that is,  $-\lambda_x = (m_B + 0.5v_B)/\sqrt{v_b} = 0.2124$ , where  $m_B = 0.01165$  and  $v_B = 0.004189$ . The sample period is June 1952 to December 2015 and the real risk-free rate is defined as the nominal one-year yield minus realized inflation over the next year.

Let  $\alpha_S$  denote the fraction of wealth invested in the stock market,  $\alpha_B$  denote the fraction of wealth invested in the 10-year nominal bond,  $\alpha_P$  the fraction of wealth

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<sup>31</sup>See Example 1 and Table 2 of Section 3.2 for a detailed discussion.

invested in an inflation-protected bond, and  $\alpha_R = 1 - \mathbf{1}'_3\alpha$  the fraction of wealth invested in the money market account with  $\alpha = (\alpha_R, \alpha_B, \alpha_S)'$ . The value function of an investor who holds the portfolio  $\alpha$  is<sup>32</sup>

$$\begin{aligned} V(w, T - t; \alpha) &= \mathbb{E} \left[ \frac{1}{1 - \gamma} W(T)^{1-\gamma} \mid W(t) = w \right] \\ &= \frac{1}{1 - \gamma} w^{1-\gamma} e^{-(\beta + (\gamma-1)(r + \Sigma'_W \Lambda) - \frac{1}{2}\gamma(\gamma-1)\Sigma'_W \Sigma_W)(T-t)}, \end{aligned} \quad (48)$$

where

$$\Sigma_W = -\Sigma_{\Pi} + \Omega\alpha, \quad \Omega = (\Sigma_S + \Sigma_{\Pi}, \Sigma_B + \Sigma_{\Pi}, \Sigma_{\Pi}). \quad (49)$$

Let  $\alpha^{opt}$  denote the optimal portfolio and  $\alpha^{sub}$  the suboptimal portfolio where the investors is not allowed to invest in the inflation-protected bond, that is,  $\alpha_4^{sub} = 0$ .

I measure the utility cost of following the suboptimal strategy by calculating the fraction of wealth an investor is willing to give up in order to be able to invest in inflation-protected bonds and thus avoid exposure to residual inflation risk with an zero-investment in the nominal bond cash. Hence, the utility cost  $\mathcal{C}(T - t)$  is the solution of the equation

$$V(w, T - t; \alpha^{sub}) = V(w(1 - \mathcal{C}(T - t)), T - t; \alpha^{opt}). \quad (50)$$

The investment opportunity set is constant. Hence, there are no hedging demands and the utility cost does not depend on the factor  $x(t)$ . Table 6 shows the optimal and suboptimal investment strategy for the baseline parameter case with high and low nominal bond Sharpe ratios. Specifically, for the high nominal bond Sharpe ratio the optimal investment strategy is 18.48% in stocks, 81.52% in the inflation-protected bond, 106.32% in the nominal bond, and  $-106.31\%$  in cash. The optimal portfolio when investors cannot invest in the inflation-protected bond is 19.30% in stocks, 101.42% in the nominal bond, and  $-20.72\%$  in cash. The fraction of wealth an investor is willing to give up in order to follow the optimal strategy is 2.20%. The utility cost and the optimal and suboptimal investment in stocks and the inflation

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<sup>32</sup>See Sangvinatsos and Wachter (2005) for details.

protected-bond does not change when we consider the case of a lower nominal bond Sharpe ratio. In this case the nominal bond positions are with,  $\alpha_B^{opt} = 56.49\%$ , and,  $\alpha_B^{sub} = 51.58\%$ , not as extreme as before.

The top two plots of Figure 1 show the utility cost of residual inflation risk exposure for different inflation volatility,  $\sigma_{\Pi}$ , and nominal bond Sharpe ratios,  $-\lambda_x$ , as a non-monotone function of risk aversion,  $\gamma$ . The cost is high for low and high risk aversion and zero when risk aversion is approximately 1.85. Intuitively, when risk aversion is 1.85, then it is optimal for the investor to put 100% in the stock market and nothing in the inflation-protected bond. In this case, the cost of not being able to invest in the inflation-protected bond is zero. The zero-investment portfolio in nominal bonds and cash for the high and low nominal bond Sharpe ratio is (575.49%/-575.49%) and (305.77%/-305.77%), respectively. If risk aversion goes up, then the investor puts less money in the stock market and starts investing in the inflation-protected bond. The utility costs goes up because saving in cash instead of the inflation-protected bond exposes the investor to residual inflation risk. Similarly, the utility cost goes up when risk aversion falls below 1.85 because the investor starts levering up their stock market investment which exposes her to residual inflation risk if done by borrowing cash instead of shorting inflation-protected bonds.

The bottom two plots of Figure 1 show the utility cost of residual inflation risk exposure for different inflation volatility,  $\sigma_{\Pi}$ , and different nominal bond Sharpe ratios,  $-\lambda_x$ , as a strictly increasing function of the investment horizon,  $T$ . Intuitively, investors continuously rebalance their optimal and suboptimal portfolio to keep a constant fraction in each security because there are no hedging demands and hence the portfolio weights are independent of the investment horizon. The longer the investor keeps a constant exposure to residual inflation risk, the higher the cost, and, hence the utility cost are strictly increasing in the investment horizon. The utility cost is insensitive to size of the nominal bond Sharpe ratio ( $-\lambda_x$ ) because the Sharpe ratio mainly effects the composition of the portfolio consisting of the nominal bond and cash but not the residual inflation risk exposure. The volatility of residual infla-



Table 6: **Portfolio choice example.** The table shows parameter estimates for the sample period January 1978 to December 2015. The inflation parameters are estimated from realized inflation based on the CPI and expected inflation based on MSC. The real risk-free rate, stock volatility, and the market price of stock and expected inflation risk are calibrated to moments of the real risk-free interest rate (defined as the nominal one-year yield minus MSC expected inflation), the mean and volatility of the one-year real excess return of the CRSP value-weighted stock market index including dividends, and the one-year real excess return of the ten-year nominal GSW discount bond. Another estimate for  $\lambda_x$  is considered by matching the (lower) Sharpe ratio of the one-year real excess return of the five-year nominal Fama-Bliss discount bond available since 1952. The table also shows the utility cost of residual inflation risk exposure and optimal and suboptimal portfolio strategies.

Parameter	Description	Value
<i>Stochastic discount factor:</i> $dM = -rM dt - \Lambda' dZ, \quad \Lambda = (\lambda_x, \lambda_S, 0)'$		
$r$	Real short rate	0.0092
$\lambda_x$	Market price of expected inflation risk	-0.3998/-0.2124
$\lambda_S$	Market price of stock market risk	0.2999
<i>Expected inflation</i>		
$dx = \kappa(\bar{x} - x) dt + \sigma_x dz_x$		
$\bar{x}$	Long run mean of expected inflation	0.0349
$\kappa$	Mean reversion of expected inflation	0.4055
$\sigma_x$	Volatility of expected inflation	0.0155
<i>Realized inflation:</i>		
$d\Pi(t) = x(t)\Pi(t) dt + \sigma_\Pi\Pi(t) dz_\Pi(t)$		
$\sigma_\Pi$	Inflation volatility	0.0166
$\rho$	Correlation with expected inflation	0.1377
<i>Real stock return:</i>		
$dS = (r + \sigma_S\lambda_S)S dt + \sigma_S S dz_S$		
$\sigma_S$	Stock market volatility	0.1623
<i>Investors preferences:</i>		
$E\left[\frac{1}{1-\gamma}W(T)^{1-\gamma} \mid W(0) = w_0\right]$		
$w_0$	Initial wealth	1
$\gamma$	Common risk aversion	10
$T$	Investment horizon	25
<i>Optimal investment:</i>		
$\alpha_S$	Stock market	0.1848
$\alpha_R$	Cash	-1.0632/-0.5649
$\alpha_B$	Nominal bond	1.0632/0.5649
$\alpha_P$	Inflation-protected bond	0.8152
<i>Suboptimal investment:</i>		
$\alpha_S$	Stock market	0.1930
$\alpha_R$	Cash	-0.2072/-0.2911
$\alpha_B$	Nominal bond	1.0142/0.5158
$\alpha_P$	Inflation-protected bond	0
<i>Utility cost:</i>	Fraction of wealth lost	0.0220

tion risk goes up with inflation volatility  $\sigma_{\Pi}$  and so does the utility cost. Hence, the inability to hedge against unpriced residual inflation risk when holding long or short positions in nominal bonds and cash incurs significant losses when inflation volatility is high as for instance during the late 1980s and the recent financial crisis.

## 4 Conclusion

I decompose inflation risk into (i) a component that is correlated with factors that determine investor's preferences and investment opportunities and real returns on real assets with risky cash flows (stocks, corporate bonds, real estate, commodities, etc.), and (ii) a residual inflation risk component. In equilibrium, only the first component earns a risk premium. Therefore investors should avoid exposure to the residual component. All nominal bonds, including the money-market account, have constant nominal cash flows and thus their real returns are equally exposed to residual inflation risk. In contrast, inflation-protected bonds provide a means to avoid cash flow and residual inflation risk. Hence, every investor should put 100% of her wealth in real assets (inflation-protected bonds, stocks, corporate bonds, real estate, commodities, etc.), and finance every long/short position in nominal bonds with an equal amount of other nominal bonds or by borrowing/lending cash, that is, investors should hold a zero-investment portfolio of nominal bonds and cash.

I provide empirical evidence for the existence of unpriced residual inflation risk in the United States where investors can invest in Treasury Inflation Protected Securities (TIPS) since January 1997. I also compute large utility costs of holding non-zero portfolios in nominal bonds and cash for very risk averse investors and long investment horizons in the U.S. and I find that the utility costs are small when inflation risk is low because nominal bonds are good substitutes for inflation-protected bonds in this case. Hence, the portfolio advice of holding a zero investment portfolio in nominal bonds and cash to avoid unpriced residual inflation risk is even more important for countries with high inflation risk.

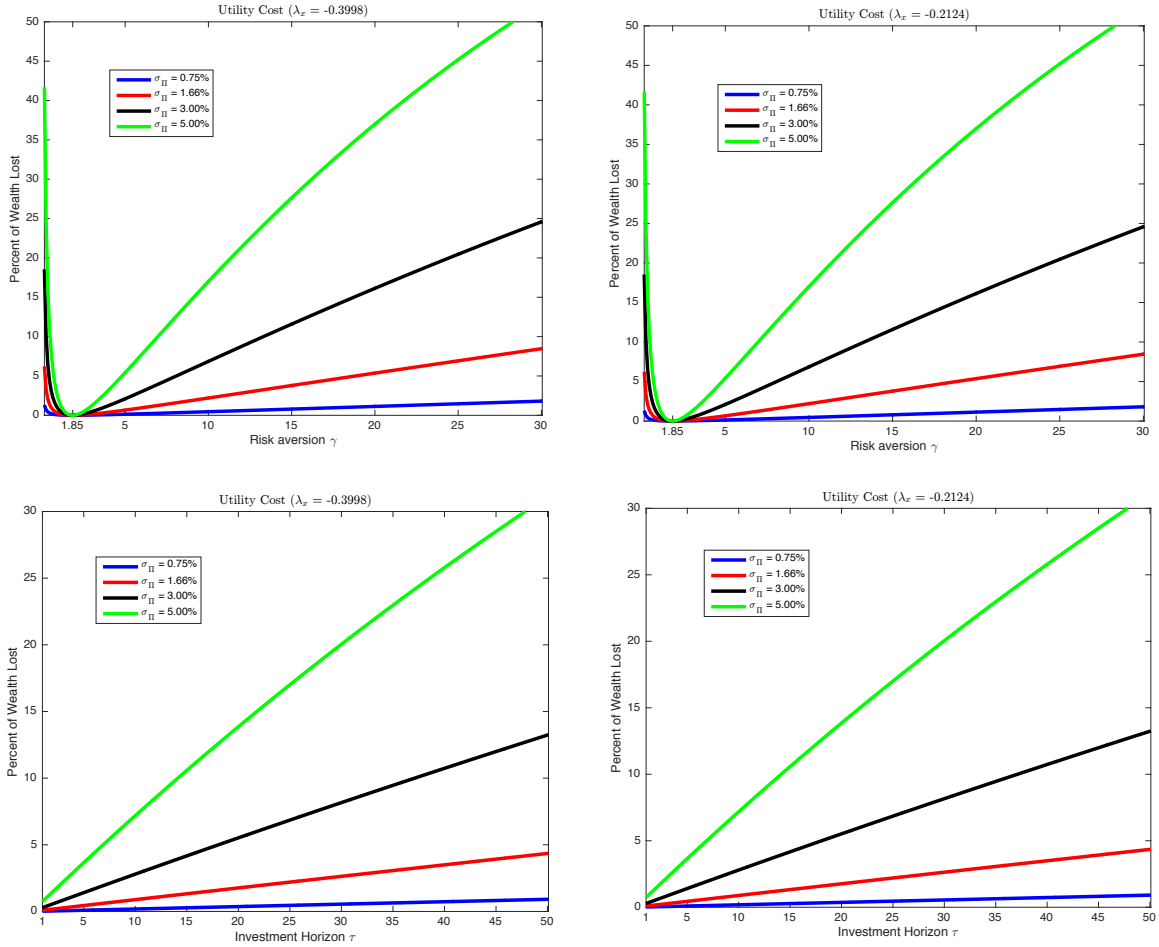


Figure 1: Quantitative importance of residual inflation risk

The top and bottom two plots show the utility cost of investment strategies without inflation-protected bonds for different inflation volatility,  $\sigma_{\Pi}$ , and different nominal bond Sharpe ratios,  $-\lambda_x$ , as a function of risk aversion,  $\gamma$ , and the investment horizon,  $T$ , respectively. The utility cost always increases with inflation volatility and is insensitive to the size of the nominal bond Sharpe ratio. The utility cost is strictly increasing in the investment horizon and it is high for very low and very high risk aversion.

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## A Investment opportunities

**Definition 2** (D: Sufficiently smooth). *A function  $f(t, X, S, \Pi)$  is sufficiently smooth if it is continuously differentiable with respect to time  $t$  and all second partial derivatives with respect to  $X$ ,  $S$ , and  $\Pi$  exist and are continuous.*

*Proof of Proposition 1 (Nominal bonds and the money market account).* The nominal value of a unit of currency invested in the money market account at time  $t$  is

$$R^{\$} = e^{\int_0^t r^{\$(X(a))} da} \quad (51)$$

and hence depends on the whole path of the nominal short rate. Specifically,  $R^{\$} = R^{\$(t, \{X(a), 0 \leq a \leq t\})}$ .

Applying Itô's Lemma to  $R(t, \{X(a) \mid 0 \leq a \leq t\}, \Pi) = R^{\$(t, \{X(a) \mid 0 \leq a \leq t\})/\Pi$  and using equation (5) for the nominal short rate leads to the real return dynamics of  $R$  given in equation (17).

The solution of the stochastic differential equation (4) is

$$M^{\$(T)} = M^{\$(t)} e^{-\int_t^T (r^{\$(X(a)) + \frac{1}{2} \Lambda^{\$(X(a))' \Lambda^{\$(X(a))}) da - \int_t^T \Lambda^{\$(X(a))' dZ(a)}, \quad (52)$$

in which  $\Lambda^{\$(X)} = \Lambda(X) + \Sigma_{\Pi}(X)$  denotes the nominal market price of risk. The state vector  $X$  is Markov and therefore the conditional distribution of  $M^{\$(T)}/M^{\$(t)}$  at time  $t$  only depends on the value of  $X$  at time  $t$  and time to maturity  $T-t$ . Hence, the nominal price of a nominal bond at time  $t$  is given by

$$B^{\$} = E_t \left[ \frac{M^{\$(T)}{M^{\$(t)}} \right] \quad (53)$$

depends only on  $X$  and  $T-t$ , i.e.  $B^{\$} = B^{\$(T-t, X)}$ .

$B^{\$(T-t, X)}$  is sufficiently smooth and hence applying Itô's Lemma to  $B^{\$(T-t, X)$  and using the continuous time pricing equation  $E[dB^{\$}/B^{\$}] - r^{\$} dt = -dB^{\$}/B^{\$} dM^{\$}/M^{\$}$  for nominal assets leads to the nominal return dynamics of nominal bonds. Specifically,

$$\frac{dB^{\$(T-t, X, \Pi)}}{B^{\$(T-t, X, \Pi)}} = (r^{\$(X) + \Sigma_B^{\$(T-t, X)'(\Lambda(X) + \Sigma_{\Pi}(X))}) dt + \Sigma_B^{\$(T-t, X)' dZ \quad (54)$$

with

$$\Sigma_{B^s}(T-t, X) = \frac{\Sigma_X(X) \nabla_X B^s(T-t, X)}{B^s(T-t, X)}.$$

Applying Itô's Lemma to  $B(T-t, X, \Pi) = B^s(T-t, X)/\Pi$  and using the continuous time pricing equation  $E[dB/B] - r dt = -dB/B dM/M$  for real assets leads to the real return dynamics of nominal bonds given in equation (18).

Moreover, the upper diagonal form of the volatility matrix  $\Sigma_X(X)$  (see equation (11)) implies that the last column of  $\Sigma_X(X)$  is zero and hence  $\Sigma_{Bd}(T-t, X) = -\Sigma_{\Pi d}(X)$ .

□

*Proof of Proposition 2 (Inflation-protected bonds).* The solution of the stochastic differential equation (3) is

$$M(T) = M(t) e^{-\int_t^T (r(X(a)) + \frac{1}{2} \Lambda(X(a))' \Lambda(X(a))) da - \int_t^T \Lambda(X(a))' dZ(a)}. \quad (55)$$

The state vector  $X$  is Markov and therefore the conditional distribution of  $M(T)/M(t)$  at time  $t$  only depends on the value of  $X$  at time  $t$  and time to maturity  $T-t$ . Hence, the real price of an inflation-protected bond at time  $t$  given by

$$P = E_t \left[ \frac{M(T)}{M(t)} \right] \quad (56)$$

depends only on  $X$  and  $T-t$ , i.e.  $P = P(T-t, X)$ .

$P(T-t, X)$  is sufficiently smooth and hence applying Itô's Lemma to  $P(T-t, X)$  and using the continuous time pricing equation  $E[dP/P] - r dt = -dP/P dM/M$  for real assets leads to the local return dynamics in equation (33) with  $\Sigma_P(T-t, X)$  given in equation (27).<sup>33</sup>

Moreover, the upper diagonal form of the volatility matrix  $\Sigma_X(X)$  (see equation

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<sup>33</sup>In the remainder of this section I abuse notation and denote with  $E[dY]$  the drift of the stochastic process  $Y$ 's dynamics.



(11)) implies that the last column of  $\Sigma_X(X)$  is zero and hence  $\Sigma_{Pd}(T - t, X) = 0$ .

□

## B Portfolio choice

We know that the expected rate of return of every traded asset in a frictionless economy that allows for continuous trading is equal to the real risk-free rate plus the local volatility of the asset times the market price of risk and hence every continuously traded asset is uniquely defined by its local volatility vector.<sup>34</sup>

Define local excess returns of all real assets introduced in Section 1 as the difference between the real local return of an asset minus the real local return of the money market account.<sup>35</sup> Specifically, the real local excess return of a nominal zero-coupon bond maturing at  $T$  is<sup>36</sup>

$$\frac{dB_T}{B_T} - \frac{dR}{R} = \Sigma'_{B_T} \Lambda dt + \Sigma'_{B_T} dZ. \quad (57)$$

The real local excess return of an inflation-protected zero-coupon bond maturing at  $T$  is

$$\frac{dP_T}{P_T} - \frac{dR}{R} = (\Sigma_{P_T} + \Sigma_{\Pi})' \Lambda dt + (\Sigma_{P_T} + \Sigma_{\Pi})' dZ. \quad (58)$$

The real local excess return of positive-net-supply security  $n$  is

$$\frac{dS_n}{S_n} - \frac{dR}{R} = (\Sigma_{S_n} + \Sigma_{\Pi})' \Lambda dt + (\Sigma_{S_n} + \Sigma_{\Pi})' dZ. \quad (59)$$

Let  $\Omega(X)$  denote the  $(d \times l)$ -dimensional local real excess return volatility matrix with  $l = h_B + h_P + N + 1$ . Specifically, the first  $h_B$  columns of  $\Omega(X)$  are the volatility

<sup>34</sup>See equation (8), Proposition 2, and Proposition 1 in Section 1.

<sup>35</sup>There is no loss in generality to choose the money market account as reference asset. The money market (the nominal risk-free rate) is usually chosen as reference asset in the literature. However, I consider real returns in which case the money market account is in general not risk-free because of its exposure to inflation risk.

<sup>36</sup>I sometimes suppress arguments of functions for notional simplicity.

vectors of  $(dB_1/B_1 - dR/R), \dots, (dB_{h_B}/B_{h_B} - dR/R)$ , the next  $h_P + 1$  columns are the volatility vectors of  $(dP_1/P_1 - dR/R), \dots, (dP_{h_P+1}/P_{h_P+1} - dR/R)$ , and the last  $N$  columns are the volatility vectors of  $(dS_1/S_1 - dR/R), \dots, (dS_N/S_N - dR/R)$ .

Let  $\mathcal{M}(X)$  denote the asset return or asset space that consists of  $l + 1$  assets:  $h_B$  nominal bonds,  $h_P + 1$  inflation-protected bonds,  $N$  positive-net-supply securities, and the money market account. Moreover, let  $\mathcal{E}(X)$  denote the excess return space consisting of the same assets. Geometrically,  $\mathcal{E}(X)$  is a  $l$ -dimensional vector space that is spanned by the columns of  $\Omega(X)$  and  $\mathcal{M}(X) = (-\Sigma_P(X), \mathcal{E}(X))$  is a  $l$ -dimensional affine space.<sup>37</sup> The dimension of  $\mathcal{E}(X)$  and hence  $\mathcal{M}(X)$  is equal to the number of non redundant assets (linearly independent columns of  $\Omega(X)$ ).

We prove Theorem 1 under the weaker Assumption 3 that allows for incomplete markets.

**Assumption 3** (Spanning condition). *Let  $X = (X^a, X^b)$  in which  $X^a$  is spanned by real returns of inflation-protected bonds and nominal returns of nominal bonds. Either (i) the market is complete, or (ii) the part of inflation risk that is not spanned by  $X^a$  is orthogonal to  $X^b$  and to real returns on positive-net-supply securities.*

Neither condition (i) nor (ii) of Assumption 3 implies the other.<sup>38</sup> Assumption 3 implies that there is a mimicking portfolio for the real risk-free asset.<sup>39</sup> Intuitively, a long position in inflation-protected bonds avoids exposure to residual inflation risk, which is not possible with a long or short position in nominal bonds and the money market account because of their equal exposure to residual inflation risk. On the other hand, the exposure of the long position in inflation-protected bonds to factor risk (components of  $X^a$ ) can be hedged, because  $X^a$  is spanned by real returns of inflation protected bonds and real returns of zero investment portfolios of nominal

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<sup>37</sup>If the money market account is locally riskless, then  $\mathcal{E}(X)$  and  $\mathcal{M}(X)$  coincide.

<sup>38</sup>It is equivalent to say in Assumption 3 that  $X^a$  is spanned by real returns of inflation-protected bonds and real returns of zero-investment portfolios of nominal bonds and the money market account because the additional exposure of real returns of nominal bonds to (i) residual inflation risk and (ii) to factor risk (if the factor is correlated with inflation) is offset by borrowing/lending in the money market account. A formal discussion of the spanning condition is provided in Proposition 6 in Appendix B.

<sup>39</sup>The proof is given in Theorem 1.

bonds and the money market account. Moreover, every claim that solely depends on the state vector  $X^a$  can be perfectly replicated with a portfolio consisting of inflation-protected bonds and zero-investment portfolios of nominal bonds and the money market account. Hence, Assumption 3 implies that the nominal and inflation-protected bond market is complete.

I show in Claim III of Proposition 6 below that if Assumption 1 and Assumption 3 are true and if returns on each positive-net-supply securities (it is assumed that all  $N$  positive-net-supply securities are non-redundant) are not spanned by returns of the money market account, nominal bonds, and inflation-protected bonds, then the number of linearly independent nominal and inflation-protected bonds is equal to the dimension of  $X^a$  and  $l = k_1 + N + 1$ , in which  $k_1$  denotes the dimension of  $X^a$ .<sup>40</sup>

Let  $W$  denote the real value of a self financing investment portfolio  $\alpha \in \mathbb{R}^l$ . Specifically, the first  $h_B$  elements of  $\alpha$  denote the fraction of  $W$  invested in the  $h_B$  nominal bonds, the second  $h_P + 1$  elements denote the fraction of  $W$  invested in the  $h_P + 1$  inflation-protected bonds, the last  $N$  element denote the fraction of  $W$  invested in the  $N$  positive-net-supply securities, and  $1 - \mathbf{1}'\alpha$  denotes the fraction of  $W$  invested in the money market account.<sup>41</sup> The real local return of the portfolio  $\alpha$  is uniquely defined by the local return volatility

$$\Sigma_W(X) = -\Sigma_\Pi(X) + \Omega(X)\alpha. \quad (60)$$

Hence, the real local return of  $W$  is

$$\frac{dW}{W} = (r(X) + \Sigma_W(X)'\Lambda(X)) dt + \Sigma_W(X)' dZ \quad (61)$$

and the volatility  $\Sigma_W(X)$  is an element of the affine space  $\mathcal{M}(X)$ .

We will see below that the geometric interpretation of any self financing portfolio

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<sup>40</sup>If the real returns of  $n$  positive-net-supply securities are spanned by returns of the money market account and nominal and inflation-protected bonds, then all  $n$  assets are excluded and the whole analysis that follows holds true with  $l = k_1 + N - n + 1$ .

<sup>41</sup> $\mathbf{1}_l$  denotes the  $l$ -dimensional vector of ones.

with dynamics given in (61) as an element of  $\mathcal{M}(X)$  is very useful in determining the optimal investment portfolio. Specifically, I show in the proof of Theorem 1 that the mimicking portfolio for the real risk-free asset, the tangency portfolio, and the hedging portfolios are uniquely determined by the (affine) projection of the null vector, the market price of risk  $\Lambda(X)$ , and the local covariance matrix of the state vector  $\Sigma_X(X)$  onto the asset space  $\mathcal{M}(X)$ .

Recall that  $X = (X^a, X^b)$  with  $\Sigma_X(X) = (\Sigma_{X^a}(X), \Sigma_{X^b}(X))$  and let  $X^a$  be  $k_1$ - and  $X^b$  be  $k_2$ -dimensional. If time  $t$  is a state variable, then redefine the state vector  $X$  as  $(t, X)$ . Moreover, exclude any state variable that can be written as a linear combination of other state variables from the definition of the state vector  $X$ . Finally, if some positive-net-supply securities are state variables or can be written as a linear combination of some state variables, then the state vector is redefined. Specifically, let  $\{S_1, \dots, S_n\}$  with  $n \leq N$  denote the set (which is empty if no positive-net-supply security can be written as a linear combination of the state vectors) of all positive-net-supply securities which can be written as a linear combination of the state variables. Then the state vector can be defined by  $Y = (X, S_1, \dots, S_n)$ . Hence, we can without loss of generality assume that the local covariance matrix of  $X$  and  $S_1, \dots, S_N$  which is  $(\Sigma_X(X), \Sigma_S(X))'(\Sigma_X(X), \Sigma_S(X))$  has full rank.<sup>42</sup> Implications of the spanning condition of the economy – Assumption 3 – are provided in the next proposition.

**Proposition 6.** *Let  $\Sigma_{\Pi}^{\perp}$  denote the part of inflation risk that is not spanned by  $X^a$ . Then,*

$$\Sigma_{\Pi}^{\perp}(X) = (0, \dots, 0, \Sigma_{\Pi k_1+1}(X), \dots, \Sigma_{\Pi k+N+1}(X))'. \quad (62)$$

*Adopt Assumption 3. Then the following six claims are true.*

*Claim I: The part of inflation risk that is not spanned by  $X^a$  is orthogonal to  $X^b$*

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<sup>42</sup>If Assumption 1 holds, then the local covariance matrix of the Markov system given in equation (11) has full rank.

and real returns on all positive-net-supply securities if and only if

$$\Sigma_{X^b}(X)' \Sigma_{\Pi}^{\perp}(X) = 0 \quad (63)$$

$$\Sigma_S(X)' \Sigma_{\Pi}^{\perp}(X) = 0. \quad (64)$$

*Claim II: The part of inflation risk that is not spanned by  $X^a$  is orthogonal to  $X^b$  if and only if it is orthogonal to  $X$ .*

*Claim III: The part of inflation risk that is not spanned by  $X^a$  is orthogonal to  $X$  and real returns on all positive-net-supply securities if and only if*

$$\Sigma_{\Pi_i} = 0 \quad \forall i = k_1 + 1, \dots, k + N. \quad (65)$$

*Claim IV:  $X^a$  is spanned by real returns of inflation protected bonds and nominal returns of nominal bonds if and only if  $X^a$  is spanned by real returns of inflation protected bonds and real returns of zero-investment portfolios of nominal bonds and the money market account.*

*Claim V: If residual inflation risk is not zero, then real returns of inflation-protected bonds and zero investment portfolios of nominal bonds and the money market account span  $X^a$  if  $h_B + h_P \geq k_1$ . Moreover, the dimension of the asset space is equal to  $l = k_1 + N + 1$ .*

*Claim VI: Neither Condition (i) nor (ii) of Assumption 3 implies the other.*

*Proof.* It follows directly from the upper diagonal form of the local covariance matrix of the Markov system  $\Sigma(X)$  given in equation (11) in Section 1 that the  $k_1$  linearly independent columns of  $\Sigma_{X^a}(X)$  span the vector  $(\Sigma_{\Pi_1}(X), \dots, \Sigma_{\Pi_{k_1}}(X), 0, \dots, 0)'$ . Hence, the part of inflation risk that is not spanned by  $X^a$  is given in equation (62).

Moreover, two stochastic processes are locally uncorrelated if their local volatility vectors are orthogonal to each other and hence the part of inflation risk that is not spanned by  $X^b$  and real returns on the market portfolio is orthogonal to  $X^b$  and real

returns on all positive-net-supply securities if and only if equations (63) and (64) hold. This proves Claim I.

Similarly, the upper diagonal form  $\Sigma(X)$  and equation (62) imply that  $\Sigma_{X^a}(X)' \Sigma_{\Pi}^{\perp}(X) = 0$  and hence  $\Sigma_{X^b}(X)' \Sigma_{\Pi}^{\perp}(X) = 0$  if and only if  $\Sigma_X(X)' \Sigma_{\Pi}^{\perp}(X) = 0$ . This proves Claim II.

The “if part” of Claim III follows directly from equation (65). For the “only if part” we rewrite conditions (63) and (64) and drop the zero identities. This leads to

$$\begin{pmatrix} \Sigma_{X^{b_1 k_1+1}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \Sigma_{X^{b_{k_2 k_1+1}}} & \ddots & \Sigma_{X^{b_{k_2 k}} } & 0 & \cdots & \cdots & 0 \\ \Sigma_{S_1 k_1+1} & \cdots & \Sigma_{S_1 k} & \Sigma_{S_1 k+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \Sigma_{S_N k_1+1} & \cdots & \Sigma_{S_N k} & \Sigma_{S_N k+1} & \cdots & \cdots & \Sigma_{S_N k+N} \end{pmatrix} \cdot \begin{pmatrix} \Sigma_{\Pi k_1+1} \\ \vdots \\ \Sigma_{\Pi k} \\ \Sigma_{\Pi k+1} \\ \vdots \\ \Sigma_{\Pi k+N} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad (66)$$

The solution given in equation (65) is the trivial solution of the system of linear equation given in (66). The columns of  $\Sigma_{X^b}(X)$  and  $\Sigma_S(X)$  are linearly independent implying that the coefficient matrix of the system of linear equations in (66) is non-singular and hence the trivial solution is the unique solution. This proves the “only if part” of Claim III.

The span of nominal returns of nominal bonds (see equation (54)) coincides with real returns of zero investment portfolios of nominal bonds (see equation (18)) and the money market account (see equation (17)) because the difference between the local volatility vector of nominal and real returns of every nominal bond and the money market account is equal to the volatility vector  $\Sigma_{\Pi}$  and hence this difference vanishes if the total investment in nominal bonds and the nominal money market account is zero. This proves Claim IV.

Let  $\mathcal{E}_{\text{bonds}}(X)$  denote the space spanned by real excess returns of  $h_P + 1$  inflation-protected and  $h_B$  nominal bonds. The  $h_B + h_P + 1$  bonds are linearly independent.

Moreover, elementary column transformations lead to a set of  $h_B + h_P + 1$  unit vectors  $\{e_1, \dots, e_{h_B+h_P}, e_d\}$  which span  $\mathcal{E}_{\text{bonds}}(X)$ .<sup>43</sup> From the upper diagonal form of the local covariance matrix of the Markov system given in equation (10) in Section 1 follows that the local volatility matrix of  $X^a$  only loads on the first  $k_1$  components and hence it is spanned if  $h_B + h_P = k_1$ . If  $h_B + h_P > k_1$ , then  $X^a$  is still spanned but in this case  $h_B + h_P - k_1$  bonds are redundant.

Moreover, if real returns on all positive-net-supply securities are not spanned by real returns of the money market account and inflation-protected and nominal bonds, then the dimension of the asset space is  $l = k_1 + N + 1$ . If  $0 \leq n \leq N$  positive-net-supply securities are spanned, then there is no need to add them to the investment opportunity set and hence we drop their excess return volatility vectors from  $\Omega(X)$  and let  $l = k_1 + N - n + 1$ . This proves Claim V.

I provide two counter examples to prove Claim VI. Let  $k = 0$ ,  $N = 1$ ,  $\Sigma_{\Pi 1} \neq 0$ ,  $\Sigma_{\Pi 2} \neq 0$ , and  $\Sigma_{S1} \neq 0$ . Then, the money market account, the market portfolio (there is only one positive-net-supply security), and an inflation-protected bond (which is in this case the real risk-free asset) complete the market. But  $\Sigma_{\Pi 1} \neq 0$  and hence part (i) of Assumption 3 is satisfied but part (ii) is violated.

Assume that part (ii) of Assumption 3 is satisfied and consider a state variable that is locally not perfectly correlated with real returns on all positive-net-supply securities and is not spanned by real returns of inflation protected bonds and nominal returns of nominal bonds – e.g. stochastic volatility of the market portfolio. In this case the market is incomplete and hence part (ii) of Assumption 3 is satisfied but part (i) is violated.  $\square$

Let  $\mathcal{P}_{\mathcal{M}}$  denote the projector onto the asset space  $\mathcal{M}$  and  $\mathcal{P}_{\mathcal{E}}$  the projector onto the excess return space  $\mathcal{E}$ .<sup>44</sup> Both projectors are given in the next lemma.

**Lemma 1.** *[Projector onto the asset space]*

<sup>43</sup> $e_i$  denotes a  $d$ -dimensional vector with  $i$ -th component equal to one and remaining components zero. See Lemma 1 for details on the basis change.

<sup>44</sup>See Kreyszig (1989) Chapter 3 and Brockwell and Davis (2006) Chapter 2 for properties of Hilbert spaces and projectors.

The projector onto the asset space  $\mathcal{M}$  is

$$\mathcal{P}_{\mathcal{M}}(X) = -\mathcal{P}_{\mathcal{E}^\perp}(X)\Sigma_{\Pi}(X) + \mathcal{P}_{\mathcal{E}}(X) \quad (67)$$

with projector on the excess return space  $\mathcal{E}$  and the orthogonal complement of the excess return space  $\mathcal{E}^\perp$  given by<sup>45</sup>

$$\begin{aligned} \mathcal{P}_{\mathcal{E}}(X) &= \Omega(X) (\Omega(X)' \Omega(X))^{-1} \Omega(X)', \\ \mathcal{P}_{\mathcal{E}^\perp}(X) &= I_d - \mathcal{P}_{\mathcal{E}}(X), \end{aligned} \quad (68)$$

respectively. Specifically, the projection of the  $d$ -dimensional vector  $v$  onto  $\mathcal{M}$  is

$$\mathcal{P}_{\mathcal{M}}(X)v = -\mathcal{P}_{\mathcal{E}^\perp}(X)\Sigma_{\Pi}(X)v + \mathcal{P}_{\mathcal{E}}(X)v. \quad (69)$$

Adopt Assumption 1 and 3. If the market is complete, then the projector onto  $\mathcal{E}$  simplifies to

$$\mathcal{P}_{\mathcal{E}}(X) = I_d. \quad (70)$$

If the market is incomplete, then the projector simplifies to

$$\mathcal{P}_{\mathcal{E}}(X) = \mathcal{P}_{\mathcal{E}_{\text{bonds}}}(X) + \mathcal{P}_{\mathcal{E}_S}(X), \quad (71)$$

in which  $\mathcal{P}_{\mathcal{E}_{\text{bonds}}}$  denotes the projector onto the space spanned by the real excess returns of the  $h_B$  nominal bonds and the  $h_P + 1$  inflation-protected bonds and  $\mathcal{P}_{\mathcal{E}_S}(X)$  denotes the projector onto the space spanned by the part of real excess returns on all positive-net-supply securities that is uncorrelated with real excess returns of nominal

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<sup>45</sup>Let  $I_k$  denote the  $k$ -dimensional unit matrix.



and inflation-protected bonds.<sup>46</sup> Specifically,

$$\mathcal{P}_{\mathcal{E}_{bonds}}(X) = \begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (72)$$

and

$$\begin{aligned} \mathcal{P}_{\mathcal{E}_S}(X) &= \bar{\Sigma}_S(X) (\bar{\Sigma}_S(X)' \bar{\Sigma}_S(X))^{-1} \bar{\Sigma}_S(X)' \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (73)$$

in which  $\bar{\Sigma}_S(X)$  equals  $\Sigma_S(X)$  except for the first  $k_1$  rows which are equal to zero.

*Proof.* If the market is complete, then  $\mathcal{R}^d = \mathcal{E}$  and hence  $\mathcal{P}_{\mathcal{E}}(X) = I_d$ .

If the market is incomplete, then Claim III of Proposition 6 implies that the volatility of inflations is

$$\Sigma_{\Pi} = (\Sigma_{\Pi 1}, \dots, \Sigma_{\Pi k_1}, 0, \dots, 0, \Sigma_{\Pi d})'. \quad (74)$$

Hence,  $\Omega(X) =$

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<sup>46</sup>In the special case when real returns of all positive-net-supply securities are uncorrelated with real returns of nominal and inflation protected bond, then  $\mathcal{E}_S$  is spanned by real returns of all positive-net-supply securities.

$$\left( \begin{array}{ccc|ccc|ccc}
\Sigma_{B1}^1 & \dots & \Sigma_{B1}^h & \Sigma_{P1}^1 + \Sigma_{\Pi1} & \dots & \Sigma_{P1}^{l+1} + \Sigma_{\Pi1} & \Sigma_{S11} + \Sigma_{\Pi1} & \dots & \Sigma_{S_{N1}} + \Sigma_{\Pi1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Sigma_{Bk_1}^1 & \dots & \Sigma_{Bk_1}^h & \Sigma_{Pk_1}^1 + \Sigma_{\Pi k_1} & \dots & \Sigma_{Pk_1}^{l+1} + \Sigma_{\Pi k_1} & \Sigma_{S_1 k_1} + \Sigma_{\Pi k_1} & \vdots & \Sigma_{S_{Nk_1}} + \Sigma_{\Pi k_1} \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & \Sigma_{S_1 k_1 + 1} & \dots & \Sigma_{S_{Nk_1 + 1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & \Sigma_{S_1 k} & \dots & \Sigma_{S_{Nk}} \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & \Sigma_{S_1 k + 1} & \dots & \Sigma_{S_{Nk + 1}} \\
\vdots & \ddots & \vdots & \vdots & \dots & \vdots & 0 & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \Sigma_{S_{Nk + N}} \\
\hline
0 & \dots & 0 & \Sigma_{\Pi d} & \dots & \Sigma_{\Pi d} & \Sigma_{\Pi d} & \dots & \Sigma_{\Pi d}
\end{array} \right),$$

in which the first block of columns denotes the excess return volatility vectors of the  $h_B$  nominal bonds, the second block denotes the excess return volatility vectors of the  $h_P + 1$  inflation-protected bonds, and the last column denotes the excess return volatility vectors of the  $N$  positive-net-supply securities. The first block of rows denotes excess return exposure to the first  $k_1$  components of  $Z$ , the second block of rows denotes excess return exposure to the next  $k_2$  components of  $Z$ , the third block denotes excess return exposure to next  $N$  components of  $Z$ , and the last row denotes excess return exposure to residual inflation risk  $Z_{k+N+1}$ .

The first  $h_b + h_P + 1$  columns span  $\mathcal{E}_{\text{bonds}}$  by definition. Moreover, the  $h_B$  nominal bonds and the  $h_P + 1$  inflation-protected bonds are non-redundant and hence

elementary column transformations lead to

$$\left( \begin{array}{cc|ccc} I_{k_1} & 0 & 0 & \dots & 0 \\ 0 & \vdots & \Sigma_{S_1 k_1+1} & \dots & \Sigma_{S_N k_1+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \Sigma_{S_1 k+1} & \dots & \Sigma_{S_N k+1} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \Sigma_{S_N k+N} \\ 0 & 1 & 0 & \dots & 0 \end{array} \right). \quad (75)$$

The second block which I define as  $\bar{\Sigma}_S(X)$  is the part of real returns on all positive-net-supply securities that is not spanned by real returns of inflation-protected bonds and real returns of zero investment portfolios of nominal bonds and the money market account and hence  $N$  columns of the matrix  $\bar{\Sigma}_S(X)$  span  $\mathcal{E}_S$ . It is clear from equation (75) that  $\mathcal{E}_{\text{bonds}}$  and  $\mathcal{E}_S$  are orthogonal and hence  $\mathcal{E}$  can be written as direct sum of the two spaces. Hence, the projector onto  $\mathcal{E}$  is equal to the projector onto  $\mathcal{E}_{\text{bonds}}$  plus the projector onto  $\mathcal{E}_S$ .

Moreover, the space  $\mathcal{E}_{\text{bonds}}(X)$  is spanned by the  $k_1+1$  unit vectors  $\{e_1, \dots, e_{k_1}, e_d\}$  and thus  $\mathcal{P}_{\mathcal{E}_{\text{bonds}}}(X)$  is given in equation (72). The projector onto the space  $\mathcal{E}_S(X)$  that is spanned by the column vectors of the matrix  $\bar{\Sigma}_S(X)$  is

$$\mathcal{P}_{\mathcal{E}_S}(X) = \bar{\Sigma}_S(X) (\bar{\Sigma}_S(X)' \bar{\Sigma}_S(X))^{-1} \bar{\Sigma}_S(X)'. \quad (76)$$

$\mathcal{E}_{\text{bonds}}(X)$  and  $\mathcal{E}_S(X)$  are orthogonal and hence

$$\mathcal{P}_{\mathcal{E}_S}(X) = \begin{pmatrix} 0_{k_1 \times k_1} & 0_{k_1 \times (N+k_2)} & 0_{k_1 \times 1} \\ 0_{(N+k_2) \times k_1} & * & 0_{(N+k_2) \times 1} \\ 0_{1 \times k_1} & 0_{1 \times (N+k_2)} & 0 \end{pmatrix}, \quad (77)$$

in which  $0_{i \times j}$  denotes the  $i \times j$ -dimensional null matrix.  $\square$

*Proof of Theorem 1.* The value function of investors who can continuously trade in

the money market account,  $h_B$  nominal zero-coupon bonds,  $h_P + 1$  inflation-protected zero-coupon bonds, and  $N$  positive-net-supply securities and who seek to maximize the utility function in equation (34) is

$$J(t, W, X) = \sup_{\{c(a), \alpha(a) | t \leq a \leq T\}} \mathbb{E} \left[ \int_t^T e^{-\int_t^b \beta(X(a)) da} u(c(b), X(b)) db + e^{-\int_t^T \beta(X(a)) da} U(W(T), X(T)) \mid W(t) = W, X(t) = X \right]. \quad (78)$$

Assume that the value function satisfies all regularity condition. Hence, the value function  $J(t, W, X)$  solves the HJB equation

$$\sup_{c > 0, \alpha \in \mathbb{R}^l} (\mathcal{A}^\alpha J(t, W, X)) = 0, \quad J(T, W(T), X(T)) = U(W(T), X(T)), \quad (79)$$

in which the characteristic operator is given by<sup>47</sup>

$$\begin{aligned} \mathcal{A}^\alpha J = & J_t + J'_X \mu_X + (rW + W \Sigma_W (\alpha)' \Lambda - c) J_W + \frac{1}{2} \text{trace} (J_{XX} \Sigma'_X \Sigma_X) \\ & + \Sigma_W (\alpha)' \Sigma_X W J_{WX} + \frac{1}{2} \Sigma_W (\alpha)' \Sigma_W (\alpha) W^2 J_{WW} + u - \beta J. \end{aligned} \quad (80)$$

If the investment horizon is infinite, then the value function does not depend on time  $t$  and hence  $J_t = 0$ .

Investors prefer more to less and are strictly risk averse which implies that  $J_W > 0$  and  $J_{WW} < 0$ . Hence, the characteristic operator given in equation (80) can be rewritten as

$$\mathcal{A}^\alpha J = W^2 J_{WW} \cdot \frac{1}{2} \left\| \Sigma_W (\alpha) - \left( \frac{1}{\gamma} \Lambda + \Sigma_X \Theta \right) \right\|^2 + K, \quad (81)$$

in which  $\gamma = -W J_{WW} / J_W$  denotes the relative risk aversion coefficient,  $\Theta = -J_{WX} / (W J_{WW})$  denotes the sensitivity of the marginal value of real wealth with respect to changes

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<sup>47</sup>I sometimes suppress arguments for notional convenience.

in the state vector,  $\|\cdot\|$  denotes the Euclidian norm, and  $K$  is given by

$$K = J_t + J'_X \mu_X + (rW - c)J_W + \frac{1}{2} \text{trace}(J_{XX} \Sigma'_X \Sigma_X) - \frac{1}{2} W^2 J_{WW} \left\| \frac{1}{\gamma} \Lambda + \Sigma_X \Theta \right\|^2 + u - \beta J \quad (82)$$

and hence does not depend on the portfolio weight  $\alpha$ .

The local volatility of the real wealth portfolio is  $\Sigma_W(\alpha) = -\Sigma_\Pi + \Omega\alpha$  and  $W^2 J_{WW} < 0$  and hence the optimal portfolio demand  $\alpha^*$  of the maximization problem given in equation (79) is

$$\alpha^* = \underset{\alpha \in \mathbb{R}^l}{\text{argmin}} \left( \frac{1}{2} \left\| \Sigma_W(\alpha) - \left( 0 + \frac{1}{\gamma} \Lambda + \Sigma_X \Theta \right) \right\|^2 \right). \quad (83)$$

Hence, the solution of the quadratic optimization problem in equation (83) is given by the projection of  $\left( 0 + \frac{1}{\gamma} \Lambda + \Sigma_X \Theta \right)$  onto the asset space  $\mathcal{M}$ .<sup>48</sup> Specifically, (i) the projection of 0 onto  $\mathcal{M}$  is the portfolio with minimum distance to the origin – i.e. the minimum variance portfolio which in this case is equal to the mimicking portfolio of the real risk-free asset because 0 is spanned by real asset returns ( $0 \in \mathcal{M}$ ), (ii) the projection of  $\Lambda(X)$  onto  $\mathcal{M}$  is the portfolio with maximum local Sharpe ratio – i.e. the tangency portfolio, and (iii) the projection of  $\Sigma_X(X)$  onto  $\mathcal{M}$  are the portfolios that are maximally correlated with the state variables – i.e. the hedging portfolios.

Let  $\hat{\Lambda}(X) \equiv \mathcal{P}_{\mathcal{M}}(X)\Lambda(X)$  and  $\hat{\Sigma}_X(X) \equiv \mathcal{P}_{\mathcal{M}}(X)\Sigma_X(X)$ . The market price of residual inflation risk is zero – i.e.  $\Lambda_d(X) = 0$  – and the state variables are uncorrelated with residual inflation risk – i.e.  $\Sigma_{Xd}(X) = 0$  – and hence it follows from Lemma 1 that  $\hat{\Lambda}_d(X) = 0$  and  $\hat{\Sigma}_{Xd}(X) = 0$ .<sup>49</sup> Moreover, real returns of inflation-protected bonds and all positive-net-supply securities are not exposed to residual inflation risk and real returns of nominal bonds and the money market account have exactly the same exposure to this risk source and hence the total investment in nominal bonds and the money market account in (i) the mimicking portfolio for the real risk-free

<sup>48</sup>See Bertsekas, Nedić, and Ozdaglar (2003) chapter 2.2 for applications of the projection theorem to quadratic optimization problems.

<sup>49</sup> $v_i$  denotes the  $i$ -th element of the vector  $v$  and  $M_d$  denotes the  $i$ -th row of the matrix  $M$ .

asset, (ii) the tangency portfolio, and (iii) the hedging portfolio is zero.<sup>50</sup> □

## C Residual inflation risk

I show in this section that the market price of residual inflation risk is zero in equilibrium. Specifically, I prove Proposition 5 with and without a government that issues bonds and collects taxes to pay interest and face value of their bonds.

### C.1 ICAPM

*Proof of Proposition 5 (ICAPM).* Suppose that there are  $I$  individuals in the economy that share the same beliefs and can continuously trade in a frictionless security market described below. Each individual makes investment decisions and consumption choices to maximize

$$\mathbb{E} \left[ \int_0^{T^i} u^i(t, c^i(t), X(t)) dt + U^i(T^i, W^i(T), X(T)) \mid X(0) = x \right] \quad (86)$$

for some horizon  $T^i$ , utility function  $u^i$ , and bequest function  $U^i$ .<sup>51</sup> The horizon  $T^i$  could be infinite in which case  $U = 0$  or it could be random in which case it is assumed to be independent of asset returns.

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<sup>50</sup>The optimal demand  $\alpha^*$  given in equation (83) is the solution of the system of linear equations

$$\Omega(X)\alpha^* = \Sigma_{\Pi}(X) + \frac{\mathcal{P}_{\mathcal{E}}(X)\Lambda(X)}{\gamma(W, X)} + \mathcal{P}_{\mathcal{E}}(X)\Sigma_X(X)\Theta(W, X), \quad (84)$$

with  $\mathcal{P}_{\mathcal{E}}(X)$  given in Lemma 1. One could get  $\alpha^*$  directly from the first order condition of the HJB equation. Specifically,

$$\alpha^* = (\Omega(X)' \Omega(X))^{-1} \Omega(X)' \left( \Sigma_{\Pi}(X) + \frac{\Lambda(X)}{\gamma(W, X)} + \Sigma_X(X)\Theta(W, X) \right). \quad (85)$$

It is straightforward to verify that the solution for equation (84) and (85) are the same by multiplying both sides of equation (84) with  $(\Omega(X)' \Omega(X))^{-1} \Omega(X)'$  and using the general formula for  $\mathcal{P}_{\mathcal{E}}(X)$  given in equation (68).

<sup>51</sup>The expectation in equation (86) is assumed to be finite and  $u$  and  $U$  are assumed to fulfill the standard conditions for utility functions (see Karatzas and Shreve (1998)).

Suppose that for any  $i$  there exist a stochastic discount factor process  $M^i(t)$  such that investor  $i$ 's static budget constraint can be written as

$$w^i \geq \mathbb{E} \left[ \int_0^{T^i} M^i(t) c^i(t) dt \right] + \mathbb{E} [M^i(T^i) W^i(T^i)]. \quad (87)$$

It is assumed that the labor income of every investor is spanned by real asset returns and hence it can be taken as part of an investor's initial wealth  $w^i$ .

For  $i = 1, \dots, I$ . Optimal consumption of investor  $i$  who maximizes (86) subject to the static budget constraint (87) has to satisfy the first order condition

$$u_c^i(t, c^i(t), X(t)) = \lambda^i M^i(t), \quad (88)$$

in which  $u_c^i$  denotes the partial derivative of  $u^i$  with respect to consumption and  $\lambda^i$  denotes the Lagrange multiplier for the budget constraint (87). Let  $U_W^i$  denote the partial derivative of  $U^i$  with respect to wealth. If the investment horizon is finite, then  $M^i(t)$  has to satisfy the FOC,  $U_W^i(T, W^i(T^i), X(T^i)) = \lambda^i M^i(T^i)$ , and if the investment horizon is infinite, then it has to satisfy,  $\lim_{T^i \rightarrow \infty} M^i(T^i) = 0$ .

Consider a frictionless security market consisting of  $N + 1$  risky assets, such as stocks, real estate, inflation-protected bonds, nominal bonds, a money market account, etc. The money market account and all bonds are in zero-net-supply. Moreover, assume w.l.o.g. that the number of shares of each positive-net-supply security outstanding is normalized to one.

For  $n = 0, 1, \dots, N+1$ , let  $S_0(t)$  denote the real price of the money market account,  $Y_n(t)$  the real ex-dividend price of risky asset  $n$ ,  $\delta_n(t)$  the real dividend paid by asset  $n$ , and  $S_n(t)$  the dividend reinvested price of risky asset  $n$ . If asset  $n$  doesn't pay dividends (e.g. a nominal zero-coupon bond), then  $\delta_n(t) \equiv 0$ . The price of the market portfolio is

$$Y = \sum_{n \in \{\text{positive-net-supply securities}\}} Y_n.$$

Moreover, let  $dS(t)/S(t)$  denote the  $N$ -dimensional column vector with  $(dY_n(t) + \delta_n(t) dt)/Y_n(t)$  as its  $n$ -th component. The dynamics of all assets are

$$\begin{aligned}\frac{dY(t)}{Y(t)} &= \mu(t) dt + \Sigma(t)' dZ(t) \\ \frac{dS_0(t)}{S_0(t)} &= \mu_0(t) dt + \Sigma_0(t)' dZ(t),\end{aligned}\tag{89}$$

in which  $\mu_0$  is one-dimensional,  $\mu$  is  $N$ -dimensional,  $\Sigma_0$  is  $d$ -dimensional, and  $\Sigma$  is  $(d \times N)$ -dimensional.

Let  $\alpha_n^i(t)$  the fraction of wealth investor  $i$  holds in the  $n$ -th risky asset at time  $t$ , and  $\alpha^i(t)$  denote the column vector with  $n$ -th component equal to  $\alpha_n^i(t)$ . The remaining wealth of investor  $i$  is put in the money market account; i.e.  $\alpha_0^i(t) = 1 - \mathbf{1}'\alpha^i(t)$ .<sup>52</sup>

The intertemporal budget constraint of each investor is

$$dW^i + c^i dt = W^i \left( (\mu_0 + (\mu - \mu_0 \mathbf{1})' \alpha^i) dt + (\Sigma_0 + (\Sigma(t) - \Sigma_0 \mathbf{1}') \alpha^i)' dZ(t) \right). \tag{90}$$

The value function of each investor is

$$\begin{aligned}J^i(t, w^i, x) &= \sup_{\{\alpha^i(a), c^i(a) | t \leq a \leq T^i\}} \left( \mathbb{E} \left[ \int_t^{T^i} u^i(a, c^i(a), X(a)) da \right. \right. \\ &\quad \left. \left. + U^i(T^i, W^i(T^i), X(T^i)) \mid W^i(t) = w^i, X(t) = x \right] \right).\end{aligned}\tag{91}$$

The envelope condition and the boundary condition of the HJB-equation together with the FOC of the static optimization problem in equation (88) imply that

$$\lambda^i M^i(t) = J_w^i(t, w^i(t), X(t)) \quad \forall 0 \leq t \leq T^i, \quad \forall i = 1, \dots, I, \tag{92}$$

in which  $J_w^i(\cdot)$  denotes the partial derivative of investor  $i$ 's value function w.r.t. his wealth.

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<sup>52</sup> $\mathbf{1}$  denotes the  $N$ -dimensional vector of ones.



Applying Itô's Lemma to equation (92) leads to<sup>53</sup>

$$\frac{dM^i}{M^i} - \mathbb{E} \left[ \frac{dM^i}{M^i} \right] = -A^i (dW^i - \mathbb{E}[dW^i]) - \sum_{l=1}^k \Psi_l^i (dX_l - \mathbb{E}[dX_l]) \quad \forall i, \quad (93)$$

in which  $A^i = -J_{ww}^i/J_w^i$  denotes individual  $i$ 's coefficient of absolute risk aversion and  $\Psi_l^i = -J_{wX_l}^i/J_w^i$  denotes the sensitivity of individual  $i$ 's marginal value of wealth with respect to changes in the state vector.

For  $i = 1, \dots, I$  and  $n = 1, \dots, N$ . The following pricing equation for asset  $n$  has to hold at an optimum for investor  $i$ :

$$\begin{aligned} (\mu_n(t) - \mu_0(t)) dt &= - \left( \frac{dS_n}{S_n} - \frac{dS_0}{S_0} \right) \frac{dM^i}{M^i} \\ &= \left( \frac{dS_n}{S_n} - \frac{dS_0}{S_0} \right) \left( A^i dW^i + \sum_{l=1}^k \Psi_l^i dX_l \right) \end{aligned} \quad (94)$$

Rearranging terms and summing over all investors leads to

$$(\mu_n(t) - \mu_0(t)) dt = \left( \frac{dS_n}{S_n} - \frac{dS_0}{S_0} \right) \left( AdW + \sum_{l=1}^k \Psi_l dX_l \right) \quad (95)$$

in which  $W = \sum_{i=1}^I W^i$  denotes aggregate wealth,  $A = 1/(\sum_{i=1}^I 1/A^i)$ , and  $\Psi = \sum_{i=1}^I (A/A^i) \Psi^i$ .

Market clearing implies that  $Y = W$ . Residual inflation risk is by definition not correlated with factors and positive-net-supply securities (and thus with the market portfolio) and hence it is not priced.  $\square$

## C.2 ICAPM with taxes

I show in this section that the market price of residual inflation risk is still zero when investors are subject to nominal lump-sum tax payments and Treasury bonds

<sup>53</sup>In the remainder of this section I abuse notation and denote with  $\mathbb{E}[dY]$  the drift of the stochastic process  $Y$ 's dynamics.

are in positive supply. Suppose that there are  $I$  individuals in the economy that share the same beliefs and can continuously trade in the frictionless security market specified below. Each individual makes investment decisions and consumption choices to maximize

$$\mathbb{E} \left[ \int_0^\infty u^i(t, c^i(t), X(t)) dt \mid X(0) = x \right] \quad (96)$$

for some utility function  $u^i$ .<sup>54</sup>

It is assumed that the labor income of every investor is spanned by real asset returns and hence it can be taken as part of an investor's initial wealth  $w^i$ . Moreover, each individual has to continuously pay the nominal lump-sum tax  $\tau^{\$i}(t)$ . Real tax payments are denoted without \$ sign ( $\tau^{\$i}(t) = \tau^i(t)\Pi(t)$ ).

Suppose that for any  $i$  there exist a stochastic discount factor process  $M^i(t)$  such that investor  $i$ 's static budget constraint can be written as

$$w^i - \mathbb{E} \left[ \int_0^\infty M^i(t) \tau^i(t) dt \right] \geq \mathbb{E} \left[ \int_0^\infty M^i(t) c^i(t) dt \right] \quad (97)$$

and each investor's initial wealth exceeds his tax liability (the left hand side of equation (97) is positive).

Each individual can invest in a well diversified asset portfolio (consisting of stocks, inflation-protected and nominal corporate bonds, real estate, etc., but excluding nominal and inflation-protected Treasury bonds) and two Treasury bonds (a real consol that continuously pays the real constant coupon  $\nu$  and a nominal consol that continuously pays the nominal constant coupon  $\kappa^{\$}$ ). Let  $Y(t)$  denote the real ex-dividend price per share of the asset portfolio,  $\delta^{\$}(t)$  the continuous nominal dividend payment per unit of time  $dt$ , and  $S(t)$  the real dividend-reinvested price per share of the asset portfolio. The total number of shares with price  $Y(t)$  outstanding is normalized to one. Moreover, denote the real price of the real consol by  $P_\nu(t)$  and the real price of the nominal consol by  $B_\kappa(t)$ . The total real return of  $S(t)$  is  $(dY(t) + \delta(t) dt)/Y(t)$ ,

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<sup>54</sup>The expectation in equation (96) is assumed to be finite and  $u^i$  is assumed to fulfill the standard conditions for utility functions (see Karatzas and Shreve (1998)).

the total real return of the inflation-protected consol is  $(dP_\nu(t) + \nu dt)/P_\nu(t)$ , and the total real return of the nominal consol is  $(dB_\kappa(t) + \kappa(t) dt)/B_\kappa(t)$ . \$ signs indicate nominal dividend or coupon payments ( $\delta^\$(t) = \delta(t)\Pi(t)$ ,  $\nu^\$(t) = \nu\Pi(t)$ , and  $\kappa^\$ = \kappa(t)\Pi(t)$ ).

At any time  $t$  the government has one inflation-protected and one nominal consol outstanding and it collects continuously the nominal lump-sum tax  $\tau^{\$i}(t) = f^i \cdot \tau^\$(t)$  from each investor. The constant  $f^i$  captures the heterogeneity in tax liabilities across investors and satisfies  $\sum_{i=1}^I f^i = 1$ . Assume that aggregate tax payments are used to pay the interest on both consols; i.e.  $\tau^\$(t) = \nu\Pi(t) + \kappa^\$$ .

The tax liability of an investor is the present value of his future tax payments. It is determined in the next lemma.

**Lemma 2** (Individual tax liabilities). *The real value of investor  $i$ 's tax liability is*

$$L_\tau^i(t) = f^i(P_\nu(t) + B_\kappa(t)), \quad \forall 0 \leq t < \infty. \quad (98)$$

*Proof.* For  $i = 1, \dots, I$ . The individual real tax liability of investor  $i$  at time  $t$  is

$$\begin{aligned} L_\tau^i(t) &= \mathbb{E}_t \left[ \int_t^\infty \frac{M^i(a)}{M^i(t)} \tau^i(a) da \right] \\ &= \mathbb{E}_t \left[ \int_t^\infty \frac{M^i(a)}{M^i(t)} f^i(\nu + \kappa^\$/\Pi(a)) da \right] \\ &= f^i \mathbb{E}_t \left[ \int_t^\infty \frac{M^i(a)}{M^i(t)} \nu da \right] + \frac{f^i}{\Pi(t)} \mathbb{E}_t \left[ \int_t^\infty \frac{M^{\$i}(a)}{M^{\$i}(t)} \kappa^\$ da \right] \\ &= f^i(P_\nu(t) + B_\kappa(t)) \quad \forall 0 \leq t \leq \infty, \end{aligned} \quad (99)$$

in which  $M^{\$i}(t) = M^i(t)/\Pi(t)$  is investor  $i$ 's nominal stochastic discount factor.  $\square$

Lemma 2 implies that every investor can immunize his tax liability by holding a constant share of Treasury consols. Hence, the initial wealth of every investor has to exceed the cost of this strategy; i.e.  $w^i > f^i(P_\nu(0) + B_\kappa(0))$ . I show in the next Proposition that the market price of residual inflation risk is zero.

**Proposition 7** (ICAPM with taxes). *Assume that investors have homogeneous beliefs and their endowments are spanned by real asset returns. Each investor is subject to continuous lump-sum tax payments  $f^i \tau^S(t)$  and is initially endowed (including the present value of future labor income) with  $\alpha_{S0}^i > 0$  shares of the asset portfolio and  $f^i$  shares of both the inflation-protected and nominal consol. Moreover, the aggregate tax payment  $\tau^S(t)$  is used by the government to pay the interest on their two Treasury consols outstanding (one inflation-protected and one nominal). Then the market price of residual inflation risk is zero.*

*Proof.* For each individual  $i = 1, \dots, I$ . Let  $\alpha_S^i(t)$  denote the number of shares invested in the asset portfolio at time  $t$ ,  $\alpha_P^i(t)$  the number of shares invested in the inflation-protected consol at time  $t$ ,  $\alpha_B^i(t)$  the number of shares invested in the nominal consol at time  $t$ ,  $W^i(t)$  the real wealth at time  $t$ ,  $c^i(t)$  real consumption at time  $t$ , and  $L_\tau^i(t)$  the real tax liability at time  $t$ . Moreover, let  $W(t) = \sum_{i=1}^I W^i(t)$  denote aggregate wealth,  $c(t) = \sum_{i=1}^I c^i(t)$  aggregate consumption, and  $L_\tau(t) = \sum_{i=1}^I L_\tau^i(t)$  the aggregate real tax liability.

Each investor is initially endowed with  $\alpha_S^i(0) = \alpha_{S0}^i > 0$  shares of the asset portfolio and  $f^i$  shares of the inflation-protected and nominal consol; i.e.  $\alpha_P^i(0) = \alpha_B^i(0) = f^i$ . Hence, investor  $i$ 's initial wealth is equal to

$$w^i = \alpha_{S0}^i S(0) + f^i (P_\nu(0) + B_\kappa(0)). \quad (100)$$

Lemma 2 implies that every investor can always immunize his tax liability by holding the constant share  $f^i$  in the inflation-protected and nominal consol. This strategy is affordable for every investor because  $w^i - L_\tau^i(0) = \alpha_{S0}^i S(0) > 0$ .

Moreover, Lemma 2 implies that

$$L_\tau(t) = P_\nu(t) + B_\kappa(t) \quad \forall 0 \leq t \leq \infty. \quad (101)$$

because  $\sum_{i=1}^I f^i = 1$ .

The intertemporal budget constraint of each investor is

$$dW^i + c^i dt + (dL_\tau^i + \tau^i dt) = \alpha_S^i(dY + \delta dt) + \alpha_P^i(dP_\nu + \nu dt) + \alpha_B^i(dB_\kappa + \kappa dt) \quad (102)$$

$$W^i(0) + L_\tau^i(0) = w^i \quad (103)$$

In equilibrium markets clear. Specifically,

$$\sum_{i=1}^I \alpha_S^i(t) = 1, \quad \sum_{i=1}^I \alpha_P^i(t) = 1, \quad \sum_{i=1}^I \alpha_B^i(t) = 1, \quad c(t) = \delta(t). \quad (104)$$

Summing over all individuals in equations (102) and (103), and using equations (101) and (104) leads to

$$dW = dY \quad \text{with} \quad W(0) = S(0). \quad (105)$$

Hence, aggregate wealth equals the market portfolio; i.e.  $W(t) = Y(t)$ .

The two consols outstanding do not appear in the market portfolio because their positive cash flows are offset by the negative cash flows of investor's tax liabilities. Only the part of inflation risk that is correlated with factors and real stock returns is priced and hence the market price of residual inflation risk is zero.<sup>55</sup>

□

The two consols outstanding do not appear in the market portfolio because their positive cash flows are offset by the negative cash flows of investor's tax liabilities. Residual inflation risk is by definition not correlated with real returns on the market portfolio and changes in factors and hence it is not priced.

The conclusion that every investor, not just the representative investor, should hold exactly enough Treasury bonds to cover his tax liability does not require complete markets or homogeneous investors. In particular, investors can be subject to different tax payments.

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<sup>55</sup>The derivation of the pricing equation (95) is similar to the derivation in the proof of Proposition 7 and thus omitted.