ESSAYS IN TWO-SIDED MARKETS WITH INTERMEDIARIES

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This thesis is dedicated to my parents, Qiulin and Xiaolan, who always support and believe in me with endless and unconditional love and make me who I am.

To my grandpa in heaven, who will be missed, always.

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ABSTRACT

ESSAYS IN TWO-SIDED MARKETS WITH INTERMEDIARIES

Jing Xu

Rakesh V. Vohra

In this thesis, I study the two-sided marketplaces with intermediaries that can facilitate matching, search and trades.

The first chapter considers the welfare and distributional consequences of introducing the student-proposing deferred acceptance mechanism in a model where schools have exogenous qualities and the benefit from attending a school is supermodular in school quality and student type. Unlike neighborhood assignment, deferred acceptance induces non-positive assortative matching where higher-type students do not necessarily choose neighborhoods with better schools. Student types are more heterogeneous within neighborhoods under deferred acceptance. Assuming an elastic housing supply, deferred acceptance benefits residents in lower-quality neighborhoods with more access to higher quality schools. Moreover, more parents will 'vote with their feet' for deferred acceptance, other things equal, than for neighborhood assignment.

The second chapter studies a search platform in a setting where buyers search for sellers directly or through a platform with lower search costs, and the platform charges both sides for the transactions it facilitates. While many intermediaries attract as many users as possible by lowering search cost, potential buyers also care about how attractive the sellers available via the intermediary are, not just the number. A search platform's strategy is determined by the coexisting positive and negative cross-group externalities: (i) while buyers appreciate more choices of sellers available on the platform, (ii) increasing the number of available sellers makes the search for low-priced and high-value sellers harder due to an unfavorable price dispersion. A platform optimally adopts a threshold strategy of targeting

sellers with lower costs to balance the competing externalities.

The third chapter studies intermediation in a buyer-seller network with sequential bargaining. An intermediary matches traders connected in a network to bargain over the price of heterogeneous goods and has the freedom to charge each side commission. A profit-maximizing middleman can help eliminate trading delays but limits trade executed that are not surplus maximizing. When the middleman competes with the buyers and sellers being matched through an exogenous search process, she matches buyer and seller pairs that are selected less often by the exogenous search process.

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PREFACE

The thesis is centered on pricing and matching in the two-sided marketplaces with intermediaries. Chapter 1 focuses on school choice, an active research area that studies the problem of how to assign students to public schools, which has led to application in major cities in the US like New York, Chicago, San Francisco, etc. The effectiveness of school choice mechanisms is a different issue in practice. As parents can 'vote with their feet', housing markets and student assignment to public schools are all profoundly influenced by introducing school choice mechanisms. The question on whether school choice is effective in improving the student assignment in practice can be fundamental to the design of school choice.

Chapter 2 &3 focuse on the decentralized two-sided markets with intermediaries, such as e-commerce platforms, labor markets, search platforms, dating websites, advertisement exchange platforms where intermediaries can create values primarily by facilitating search and matching between two or more distinct types of customers. These marketplaces focus on a different design problem. To internalize the network externalities and the surpluses from bilateral transactions, the intermediaries must choose the right prices and the right matching in a two-sided network.

CHAPTER 1 : Housing Choices, Sorting and the Distribution of Educational Benefits under Deferred Acceptance

1.1. Introduction

In many U.S. school districts, students are assigned to schools within the neighborhood they reside in. This is called neighborhood assignment. As the qualities of schools vary, this is reflected in home prices. Black (1999), for example, finds that on average, households are willing to pay an extra $3948 for a 5% increase in the average elementary school test score. Since affluent families can afford houses in more expensive locations with higher quality schools and poorer families can not, critics of neighborhood assignment are concerned about the inequitable distribution of access to high quality schools.

In response, there has been a move to delink student assignment from residential location so that students have the opportunity to attend schools outside their neighborhoods. This is called school choice and it is widely advocated as providing more equitable access to high quality schools, especially for disadvantaged families. In this paper I examine the effect of school choice on residential choices and the redistribution of educational benefits. I study the impact not just within the community that adopts school choice but on neighboring communities that don't.

I model school choice as being implemented by one of the most widely adopted school choice mechanisms: the student-proposing deferred acceptance. It assigns students to public schools based on submitted preferences subject to neighborhood priorities. Student-proposing deferred acceptance was introduced by Abdulkadiroğlu and Sönmez (2003) and has been adopted in Boston, Chicago, Denver, New York City (NYC), and Washington DC. In practice, neighborhood priorities appear in various versions: home-address-based choice lists,[1] 50-50 seat split (BPS), attendance zone priority (NYC) etc. I abstract away

[1] For instance, Boston Public Schools (BPS) gives priority to students who live within one mile of an elementary school, within 1.5 miles of a middle school, and within two miles of a high school in attending those schools.

from these specific forms and assume that neighborhood priority refers to resident students enjoying higher priorities at their neighborhood schools than non-residents.

The model is inspired by Avery and Pathak (2015) which examines residential choices when one of two towns switches from neighborhood assignment to school choice. Assuming students care only about other students with whom they attend school (pure peer effect), all schools and neighborhoods are identical under school choice. What underlies these results in Avery and Pathak (2015) is the equalization of peer group qualities under school choice.

Unlike them I assume: school qualities are exogenously given, and students have preferences for school quality. While peer effects in schooling are important, the recent literature has found little evidence on the direct causal effects of average peer group qualities on students' academic achievements (Greenwald et al., 1996; Burke and Sass, 2013; Lavy et al., 2009; Imberman et al., 2009; Abdulkadiroğlu et al., 2014).[2] Moreover, there is mounting empirical evidence that schools and teachers have significant impact on student achievement (Rivkin et al., 2005; Barrow and Rouse, 2005; Rockoff, 2004; Jackson, 2010). Rivkin et al. (2005), for example, demonstrate the causal effects of pure teacher qualities on student achievement in primary schools in Texas and find significant and systematic differences between schools and teachers in their abilities to raise student achievement. Schools have exogenous qualities that influence parental decisions on which schools to select (Hoxby, 2003; MacLeod and Urquiola, 2009; Hatfield et al., 2015; Barseghyan et al., 2014; McMillan, 2005) and this is what motivates this paper.

Consider a town where each family has one child of school age that differs in student type. The benefit of attending a school is supermodular in student type and school quality. One can interpret the student type as the ability of the child or the wealth of the family or a combination of the two. Households interested in enrolling in one of the town's public

[2] Abdulkadiroğlu et al. (2014) for example, find evidence against the importance of peer effects and racial composition in the education production function of the students from 6 public exam high schools in New York and Boston. In particular, they develop an empirical analysis that embeds deferred acceptance, the school choice mechanism of interest in this paper.

schools have to own a house in the town. In the student-proposing deferred acceptance, students submit their preferences over schools to the central planner and are assigned to one of the public schools subject to neighborhood priority. Students are not required to attend a school in the town and they have a payoff from an outside option that is monotone in student type. Those unassigned or unwilling to accept the assigned schools can also opt for their outside option, which can include private or home schooling.

As a benchmark, consider a multi-district town that adopts deferred acceptance. Under neighborhood assignment, with supermodular utility that provides incentives for Positive Assortative Matching (PAM), households self-select themselves into quality-ranked neighborhoods, while residents within each neighborhood share similar types. If the town adopts deferred acceptance with neighborhood priority, its residential distribution differs dramatically.

1. Deferred acceptance can generate a non-PAM residential pattern: students of higher types do not necessarily live in better school districts.

2. There is greater heterogeneity in student type distribution within neighborhood: student types within the same neighborhood are more diverse and spread out.

3. Deferred acceptance increases access to higher quality schools for students living in lower-quality school districts.

Even if deferred acceptance admits neighborhood priority, the residential pattern can still be non-PAM because of uncertainty about what schools children are assigned to under deferred acceptance. A top-quality school district may have more residents than its school capacity, thus students are rationed via neighborhood priorities and random lotteries. In this case, some students may choose a lower quality district with less rationing. This paper is, to the best of my knowledge, the first to characterize non-PAM in residential choices under deferred acceptance with priority when students value school quality and the benefit of attending a school is supermodular in student type and school quality.[3]

Next, unlike neighborhood assignment under which students living in the same neighborhood share similar types (Hoxby, 2003; Calsamiglia et al., 2015), under deferred acceptance, student types within the same neighborhood are more spread out and heterogeneous. In Example 1.2.2, neighborhood T_2 under deferred acceptance accommodates students of types from both the lowest 20 percent quantile and the highest 20 percent quantile, whereas students with 'in-between' types opt for a different neighborhood. It turns out that parents' risk attitudes lead to this residential pattern. If families share the same risk attitude towards uncertainty about student assignment, those of similar types self-select into the same neighborhood under deferred acceptance. Otherwise if their risk attitudes vary, the heterogeneity in student type within neighborhoods arises.

What drive the non-PAM across neighborhoods and increasing heterogeneity within each neighborhood under deferred acceptance are the conflicts between limited school capacity and over-demanded high-quality schools, a key issue school choice mechanisms try to resolve. In my model this conflict arises because the supply of housing is elastic but not school capacity. While school capacity is elastic, it changes much more slowly than housing supply. For example, in the Greater Center City area of Philadelphia, the population has increased 19% and over 20,000 housing units have been added since 2000. About half of the schools exceed their rated capacity, and the remainder have seen double digit growth.[4] Yet, the most recent master plan of the Philadelphia School district (2011) has focused on reducing school capacity![5] At least two reasons account for why school capacity fails to adjust quickly enough to changes in enrollment: some districts are chronically underfunded; and

[3] Calsamiglia et al. (2015) show student assignment is non-PAM under school choice with no priorities, i.e., higher-type students are not necessarily assigned to better schools, yet student assignment and residential choices are always PAM under school choice with priority in their paper.

[4] http://www.centercityphila.org/uploads/attachments/ciz8shha90i339eqdeyw4c63n-ccr17-housing.pdf;
http://www.openinfophilly.com/blog-1/2016/2/1/center-city-schools-may-soon-collapse-due-to-their-own-success

[5] http://thenotebook.org/sites/default/files/FMP_summary_of_recommendations.pdf

many conduct capacity plans on a very long time scale.[6]

Even with limited school capacity, deferred acceptance still generates more equitable access to better schools. I find that among districts that adopt deferred acceptance, living in a lower quality school district provides a higher chance of attending a school better than its neighborhood school. When the residential pattern is PAM, this implies that the lowest-type students in the town can benefit the most from the highest opportunities of attending a public school better than their neighborhood school.

The discussion above considers a situation where there is a switch to deferred acceptance in a single town. In reality, a student can choose to live in a town that operates deferred acceptance or neighborhood assignment. Consider two multi-district towns with public schooling. One town adopts deferred acceptance with neighborhood priority while the other still implements neighborhood assignment. As households' preferences and risk attitudes towards uncertainty may vary, they can 'vote with their feet' for one town over the other. Results 1-3 under the one town model still hold for the town that adopts deferred acceptance. In addition, deferred acceptance impacts the town implementing neighborhood assignment in the following ways:

4. There is more heterogeneity in student type within neighborhoods that implements neighborhood assignment.

5. For two towns implementing different assignment rules, everything equal (distribution of school qualities and school capacities), the town adopting deferred acceptance attracts more residents than the one implementing neighborhood assignment.

[6]Consider Clarksville and surrounding Montgomery County which is one of the fastest growing regions in the state of Tennessee. Under its twenty-year enrollment and capacity analysis, capacities of public middle schools and high schools in most of the 5 zoning regions will see a one-time capacity adjustment for the following 20 years. See `https://www.cmcss.net/documents/operations/10yearplan.pdf` for further details

1.1.1. Previous Literature

My work builds on the inter-district student assignment models of Epple et al. (2001) and Epple and Romano (2003) and is close to Avery and Pathak (2015). However it differs from prior work in the following ways.

1. Exogenous school quality.

   Avery and Pathak (2015), Calsamiglia et al. (2015) assume that the quality of a school is purely a function of the students who attend it. Assuming students only care about other students with whom they attend school (pure peer effect), all schools are identical under school choice without residential priority. This paper studies the polar opposite case with empirical evidence on the importance of exogenous school qualities (Rivkin et al., 2005; Barrow and Rouse, 2005; Rockoff, 2004; Jackson, 2010).

2. Priorities.

   Unlike Epple and Romano (2003), Barseghyan et al. (2014), Avery and Pathak (2015), I incorporate school priorities in the school choice mechanism. Without priority, all districts are equalized and all households are indifferent to living in those districts (Epple and Romano, 2003; Avery and Pathak, 2015). As emphasized by Abdulkadiroğlu and Sönmez (2003), student assignment mechanisms should be flexible enough to give students different priorities at different schools and neighborhood priority is among 7 factors that student assignment decisions should be based on, especially when a town considers shifting from neighborhood assignment to deferred acceptance.

3. More general outside option.

   Calsamiglia et al. (2015) model how the availability of a private school can alter the outcome of school choice. They assume a private school with quality y_p and price of admission p is available after students get their assignment decisions under school choice. In other papers, the outside option is modeled as being 0 (Abdulkadiroğlu et al., 2015), He et al. (2012), negative infinity (Pathak and Shi, 2013; Miralles, 2009;

Epple and Romano, 2003) or a set of schools of all possible qualities (Avery and Pathak, 2015). I require only that the payoff from the outside option be monotone in student type.

4. Elastic housing supply.

   Calsamiglia et al. (2015) also have a model examining the effects of deferred acceptance with capacity constraints and priorities in a three-district single town model, but the supply of housing is inelastic, and equal to the inelastic school capacity in each district. In my model, the supply of housing is elastic.

5. Costless choice.

   Barseghyan et al. (2014) assume a cost of exercising choice if attending an inter-district school and the benefit of attending a school is linear in student type and school quality in a two-district model. This paper assumes that all students are perfectly mobile with the cost of exercising choice being zero as in Avery and Pathak (2015), and increasing marginal utility as in most of the school choice literature (Epple and Romano, 2003; Avery and Pathak, 2015; Hoxby, 2003).

The second strand of literature related to this paper is empirical. Black (1999), Reback (2005), Brunner et al. (2012) explore appreciation of home values by households and the significant effects of switching to school choice on housing values and population density. My work also demonstrates how access to good schools are rationed by home prices under deferred acceptance with neighborhood priority.

The sorting effect across neighborhoods and schools have been tested in a number of papers such as Epple and Sieg (1999), Epple et al. (2001), Rothstein (2006), Bayer et al. (2007). They find strong sorting effects in student type across neighborhoods. Calsamiglia et al. (2015) define the concept of partial sorting where the distribution of student types in one school first-order stochastically dominates that in another. In their model the partial sorting appears in the school composition instead of residential choices. My paper focuses on the households' differentiation in their residential choices, and the ensuing diversity in school composition as a consequence of residential priorities.

The paper is organized as follows. Section 2 introduces the definition of equilibrium under deferred acceptance with neighborhood priority and proves its existence. To characterize equilibrium outcomes, I introduce the 2 examples that help distinguish deferred acceptance from neighborhood assignment. Section 3 describes the model where students choose between one town adopting deferred acceptance and the other running neighborhood assignment. I formalize the analysis of PAM and heterogeneity within neighborhoods under deferred acceptance in this section and also discuss on home prices and distribution of educational benefits. Section 4 discusses the implications and extensions of this paper.

1.2. One Town Model

1.2.1. Setting

A town is divided into D districts. Each district has one school that offers tuition-free education. A unit mass of households each with one school-aged child are interested in public schools in the town. Households are distinguished by a one-dimensional type x, which could be interpreted either as wealth of the family or ability of their child or some combination of the two. μ is the non-atomic measure of types with support $X = [\underline{x}, \bar{x}]$. $\mu(X) = 1$. Assume a unitary actor for each household, so I can refer interchangeably to households and students as decision makers.

Each student applying to the public schools must own a house in the town. Suppose the housing price in district d is p_d. A student of type x living in district d and enrolled in a school of quality y, receives a utility of,

$$u(x, y, p_d) = v(x, y) - p_d.$$

For a child of type x who does not get a seat under school choice, an outside option is available with payoff denoted by $\pi : X \to \mathbb{R}$. One can interpret $\pi(x)$ as the outside option for type x if she attends a private school or home education. Assume that $\pi(x)$ is continuous and non-decreasing in type x.

The following are some major assumptions that hold throughout the paper.

Assumption 1 (A1) The school in district d is of quality y_d. Then school qualities are strictly ordered: $y_1 < y_2 < \ldots < y_D$.

Assumption 2 (A2) $v(x, y)$ is twice differentiable, strictly increasing in both arguments and $\frac{\partial^2 v}{\partial x \partial y} > 0$ for all $(x, y) \in X \times [y_1, y_D]$.

(A1) assumes that schools have exogenously given qualities. (A2) requires that the utility is supermodular in student type x and school quality y so that households with higher types are willing to pay more for an increase in school quality.

Assume that the school capacity in district d is k_d.

Housing supply is derived from profit maximization by price-taking builders, who choose the optimal density of labor, construction cost, and quantity of houses built, subject to certain constraints. In particular, this paper assumes the following specific form for the elastic housing supply,

$$H^A(l_d, p) = l_d p^r,$$

as in Epple et al. (2001), Calabrese et al. (2011), Epple and Zelenitz (1981).[7] Here l_d is the land capacity of district d, p is the home price in district d, $r > 0$ is the price elasticity of supply. Here I make the assumption that building densities or the persons per square mile for school constructions are proportional to that of residential housing, therefore the land capacity for public schools is proportional to the land use for residential constructions,[8] i.e., $l_d = \alpha k_d$ for some $\alpha > 0$.

[7]For example, Epple et al. (2001) assume for price-taking firms, the optimum of maximizing a constant return-to-scale production function without local constraints is $H(l_j, p_j, t_j) = l_j(\frac{p_j}{1+t_j})^r$ with t_j the tax rate.

[8]See `http://www.devon.gov.uk/education-section-106-policy-jan-2013.pdf` for how land acquisition is determined for school construction.

Denote by the tuple $(\mathbf{y}, \mathbf{k}, \mu)$ the economy for the one town model.

1.2.2. Deferred Acceptance Equilibrium

The timing of the model is described below.

Stage 1 Households as price takers, first choose which district to live in.

Stage 2 Students submit preferences over schools. Each school ranks its all applicants by priority: residents from the district where the school is located are ranked higher than applicants outside the district. Within each priority class, break ties at random.[9] Apply deferred acceptance to assigning students to schools.

We implement school choice as deferred acceptance first introduced by Gale and Shapley (1962). Abdulkadiroğlu and Sönmez (2003) describe the mechanism with students proposing to schools as follows,

Step 1 Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

Step k Each student rejected in the previous step proposes to her next favorite choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students following their priority order. Remaining proposers are rejected.

School priority in this paper follows Abdulkadiroğlu and Sönmez (2003): resident students share the same higher priority of admission to their neighborhood school, while non-resident students share the same lower priority of admission. A student's expected utility is therefore

[9] In previous papers, the coarse priority with tie breaking is implemented as follows. For each x, denote by $l^x = (l^x_1, l^x_2, \dots, l^x_D)$ the scores for type x at all D schools. $l^x \in [0,1]^D$ is i.i.d. drawn from a distribution $\mathscr{F}$ for each x. Each school d ranks students by $e^x_d = l^x_d + \mathbb{1}_{h_d(x)=1}$: the higher e^x_d is, the higher the priority is. Here $h_d(x) = 1$ if and only if type x lives in district d. Obviously, resident students from the district enjoy higher priorities than non-residents. The two most common tie-breaking rules are single tie-breaking (STB) and multiple tie-breaking (MTB). My results do not depend on which one of STB and MTB is implemented.

determined by: (1) housing prices, (2) her residential choice and submitted preference over schools and (3) others' residential choices and submitted preference profile. We use subgame perfect equilibrium as the solution concept. Then deferred acceptance ensures the following.

Lemma 1.2.1. *In the subgame of Stage 2, truth-telling is a dominant strategy for any student.*

From now on, we assume every applicant reports truthfully in the second stage and restrict attention to their residential strategies only.

Let $\mathbf{h}$ be the mapping from student type x to her choice of districts. Each entry $h_d(x)$ is equal to the probability that a student of type x assigns to living in district d. Given $\mathbf{h}$ the residential strategies of all households, the demand for houses in district d is,

$$m(h_d) = \int h_d(x) d\mu.$$

Let $U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h})$ be the expected payoff of a student of type x living in district d given prices $\mathbf{p}$ and the residential choices of others $\mathbf{h}$. Here $\mathbf{e}_d$ is the vector with a 1 in the dth coordinator and 0's elsewhere. Then,

$$U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) = \sum_{s \geq j_x} \Pr(s|d, \mathbf{h}) v(x, y_s) + \big(1 - \sum_{s \geq j_x} \Pr(s|d, \mathbf{h})\big)\pi(x) - p_d,$$

where $\Pr(s|d, \mathbf{h})$ is the probability of being assigned to school s conditional on living in district d and the strategy profile $\mathbf{h}$. If $v(x, y_D) \geq \pi(x)$, denote by

$$j_x = \min\{1 \leq j \leq D : v(x, y_j) \geq \pi(x)\},$$

the least favorite school that is acceptable to type x compared to her outside option. Otherwise, let $j_x = D + 1$. Then $1 - \sum_{s \geq j_x} \Pr(s|d, \mathbf{h})$ is the probability of opting out (being unassigned by the mechanism or unwilling to attend due to more attractive outside options). Under truthful reporting, $\Pr(s|d, \mathbf{h})$ can be determined explicitly by $\mathbf{h}$ (See A.2.2

for detailed expression).

When a student of type x plays mixed strategy ϕ, her expected payoff is,

$$U(x, \phi; \mathbf{p}, \mathbf{h}) = \sum_d \phi_d \cdot U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}).$$

Then the deferred acceptance equilibrium is defined as follows,

Definition 1. *Given* $(\mathbf{y}, \mathbf{k}, \mu)$*, a* ***deferred acceptance equilibrium*** *in a town of* D *districts is* $(\mathbf{p}, \mathbf{h})$*, where* $\mathbf{p}$ *is the vector of home prices in all districts and* $\mathbf{h}$ *is the mapping from student types to residential strategies such that,*

1. *for each student of type* $x \in X$*,*

$$U(x, \mathbf{h}(x); \mathbf{p}, \mathbf{h}) = \max_{1 \le d \le D} U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}),$$

2. *housing supply should match demand in each district,*

$$H^A(l_d, p_d) = m(h_d).$$

Condition (1) is the incentive compatibility constraint: each household is maximizing expected utility when deciding where to live. Condition (2) is the housing market clearing condition for each district. Based on Schmeidler (1973), I show the existence of a pure residential strategy equilibrium.

Theorem 1.2.2. *For each* $(\mathbf{y}, \mathbf{k}, \mu)$*, there always exists a pure strategy deferred acceptance equilibrium that admits truth-telling in the second stage.*

To capture the idea of what a deferred acceptance equilibrium looks like, one must understand first how home prices are affected by deferred acceptance.

Proposition 1.2.3. *In any deferred acceptance equilibrium* $(\mathbf{p}, \mathbf{h})$*, housing prices ascend with school quality, i.e.,* $p_1 \le p_2 \le \ldots \le p_{D-1} \le p_D$*.*

It is interesting to note that, under elastic housing supply, Proposition 1.2.3 does not imply that better school districts have larger housing supply, since the housing supply is also restricted by its limited land capacity, specially in those richest ones (partly due to restricted building density).

The ensuing question is whether $\mathbf{h}$, the households' residential choices, are monotone in x, i.e., the higher the student type is, the better school district she lives in. In neighborhood assignment, supermodularity in student's payoff often implies monotone residential choices and perfect stratification across neighborhoods. Under deferred acceptance with rationing, supermodularity can be offset by uncertainty about the student assignment outcomes, therefore non-monotonicity may arise. Below is an example illustrating this difference between deferred acceptance and neighborhood assignment.

Example 1.2.1. *There are* 3 *school districts with qualities* $y_1 = 0.4$*,* $y_2 = 2.5y_1, y_3 = 13.192101y_1$*. School capacities are* $k_1 = 0.6399921, k_2 = 0.2842510, k_3 = 0.07575686$*. Housing prices are* $p_1 = 0.1, p_2 = 1, p_3 = 3.06$*. The elastic housing supply is* $H(k_d, p_d) = k_d p_d^2$*. Student types are uniformly distributed on* $[0, 0.5]$ *with unit mass. The benefit of attending schools is* $v(x, y) = (x + y)^2 + 4$*. Assume all public schools are acceptable. Below is a deferred acceptance equilibrium.*

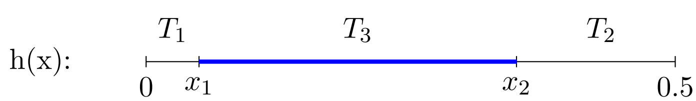

Figure 1: Example of Non-monotone Residential Pattern

In the equilibrium, students of type $[0, x_1]$ *live in district* 1*, while* $[x_1, x_2]$ *live in district* 3*, and students of the highest types live in district* 2 *as shown in Figure 1, where* $x_1 = 0.0032, x_2 = 0.3579$*. The numbers of residents in each district are* $m(h_1) = 0.01k_1$*,* $m(h_2) =$

k_2, $m(h_3) = 9.3636k_3$ with equilibrium payoffs,

$$U(x, \mathbf{e}_3; \mathbf{p}, \mathbf{h}) = \frac{1}{9.3636}(x + y_3)^2 + (1 - \frac{1}{9.3636})(x + y_1)^2 + 4 - 3.06,$$

$$U(x, \mathbf{e}_2; \mathbf{p}, \mathbf{h}) = (x + y_2)^2 + 4 - 1,$$

$$U(x, \mathbf{e}_1; \mathbf{p}, \mathbf{h}) = (x + y_1)^2 + 4 - 0.1.$$

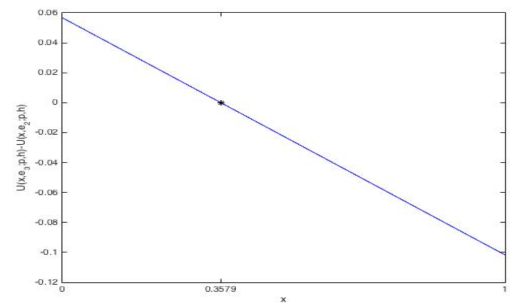

Figure 2: $U(x, \mathbf{e}_3; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_2; \mathbf{p}, \mathbf{h})$ for the non-PAM matching example

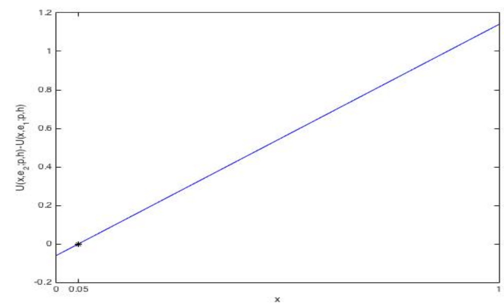

Figure 3: $U(x, \mathbf{e}_2; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_1; \mathbf{p}, \mathbf{h})$ for the non-PAM matching example

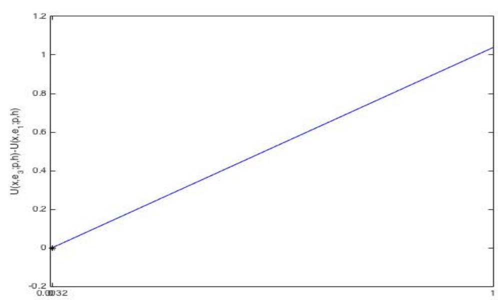

Figure 4: $U(x, \mathbf{e}_3; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_1; \mathbf{p}, \mathbf{h})$ for the non-PAM matching example

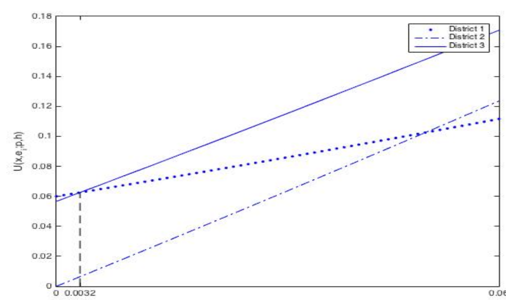

Figure 5: $U(x, \mathbf{e}_i; \mathbf{p}, \mathbf{h})$ for $x \in [0, 0.06], i = 1, 2, 3$ for the non-PAM matching example

Figures 2, 3, 4 show how residential preferences change as student types vary and Figure 5 zooms in the equilibrium payoffs near x_1. For example in Figure 2, $\Delta U(x) = U(x, \mathbf{e}_3; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_2; \mathbf{p}, \mathbf{h})$ is decreasing in student type x. When $x < 0.3579, \Delta U(x) > 0$ implies students of type x receive higher payoff living in district 3. As x grows, living in district 2 generates higher utility for higher types.

Obviously, living in district 3 is risky: with probability $1 - \frac{1}{9.3636}$ residents may end up in the worst school in the town. Yet the marginal benefit of taking the risky option is not large enough, therefore students of highest type would rather opt for neighborhood 2 with less rationing. On the other side, Students of type $x \in [x_1, x_2]$ benefit from the high school quality y_3 the most thus they are willing to pay more.

In Example 1.2.1, high-risk school district 3 can drive away high-type households unwilling to bear the risk of being unassigned to its neighborhood school. Suppose however that $v(x, y) = x^2 + y^2 + 4 + 100xy$, one could verify that the non-monotonicity result in Example 1.2.1 no longer holds. The underlying difference between the two payoff functions is the stronger supermodularity condition in the last term xy which offsets parents' risk aversion to living in an over-subscribed school district with good neighborhood school. The following Proposition 1.2.4 extends this idea of stronger supermodualiry condition under deferred acceptance. First let ∂_+ be the operator of right derivative with respect to x.

Condition 1. $\partial_+(\max\{v(x, y_d) - \pi(x), 0\}) \geq 0$ for all d.

Condition 1 implies that, if a student of type x prefers the school in district d to her outside option, then so does any student with higher type $x' > x$. In other words, the set of acceptable schools grows larger as type increases. In previous literature, (i) $\pi(x) = 0$, (ii) $\pi(x) = -M$ for M sufficiently large, (iii) $\pi(x) = v(x, x)$ in Avery and Pathak (2015), and (iv) $\pi(x) = u(x - p_d) + h(x, y_p)$ for private schools with p the price of admission and y_p the school quality in Calsamiglia et al. (2015) under their Assumption 1, are all examples of Condition 1.

Denote by $T_d = \{x : h_d(x) = 1\}$ the set of student types that live in district d given a pure strategy deferred acceptance equilibrium.

Proposition 1.2.4. *Assuming Condition 1, if for all x, d,*

$$k_d \cdot \frac{\partial v}{\partial x}(x, y_d) \geq \frac{\partial v}{\partial x}(x, y_{d-1}), \tag{1.1}$$

then for each deferred acceptance equilibrium, there exists $d^* \geq 1$ *and* $x_{d^*} \leq x_{d^*+1} \leq \ldots \leq x_D$*, s.t.,*

1. *for all* $d \geq d^* + 1$*,* $T_d = [x_{d-1}, x_d]$*,*

2. *for all* $d \leq d^*$*,* $T_d \subseteq [\underline{x}, x_{d^*}]$*, and* $p_d = \min\{p_j : 1 \leq j \leq D\}$*.*

Proposition 1.2.4 is a relaxed Positive Assortative Matching (PAM) result under deferred acceptance with the presence of outside options. There are three ways of interpreting the inequality (1.1): (a) when capacities k_d in high-quality schools are large enough, top quality schools have enough capacities to accommodate all their resident students. (b) For all x, d, $\frac{\partial^2 v}{\partial x \partial y}(x, y_d) \geq M$ for some M large enough, that is if student type and school quality are strong welfare complements, households increasingly prefer higher quality district regardless of how small the chances of admission are. (c) School qualities are more spread out: $y_d \gg y_{d-1}$. All these imply under deferred acceptance, higher-type students will choose better quality school districts.

The following captures another residential pattern appeared in previous papers (Hoxby, 2003; Epple and Romano, 2003; Avery and Pathak, 2015; Calsamiglia et al., 2015). In these papers, neighborhoods are a stratified partition of the student type space where different intervals represent distinct neighborhoods.

Definition 2. (**Connectedness**) *Neighborhood* T_d *is connected in type if it is a single interval. A town is connected if all its neighborhoods are connected.*

In a connected town, neighborhoods are partitions of student type space where each interval corresponds to a distinct neighborhood and only households of similar types live in the same neighborhood. On the other hand, if a neighborhood is disconnected, student types are more heterogeneous and spread out within the neighborhood. Example 1.2.2 illustrates a disconnected neighborhood in a deferred acceptance equilibrium.

Example 1.2.2. (**Disconnected Equilibrium**) *There are* 3 *school districts.* $y_1 = 0.4$*,* $y_2 = 2y_1, y_3 = 3y_1$*. School capacities are* $k_1 = 0.452, k_2 = 0.3255, k_3 = 0.2225$*. Housing* *prices are* $p_1 = 0.2926, p_2 = 1, p_3 = 1.6903$*. The elastic housing supply is* $H(k_d, p_d) = k_d p_d^2$*. Student types are uniformly distributed on* $[0,1]$*, with benefit of attending schools* $v(x,y) = (8xy + y^2)^2 + 5$*. Assume that all schools are acceptable to any student. Below is a deferred acceptance equilibrium.*

Figure 6: Example of a Disconnected Neighborhood T_2

Here $x_1 = 0.0387, x_2 = 0.2071, x_3 = 0.8431$*. In equilibrium, households of types* $[0, x_1]$ *live in district* 1*, while* $[x_1, x_2] \cup [x_3, 1]$ *live in district* 2*, and* $[x_2, x_3]$ *live in district* 3*. The numbers of residents in each district are respectively,* $m(h_1) = 0.0856k_1, m(h_2) = k_2, m(h_3) = 2.8571k_3$*. In this example, district* 2 *is disconnected.*

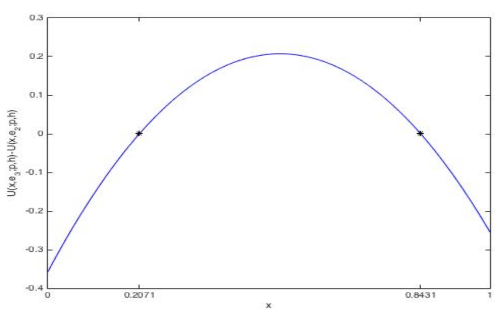

Figure 7: $U(x, \mathbf{e}_3; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_2; \mathbf{p}, \mathbf{h})$ for the disconnected neighborhood example

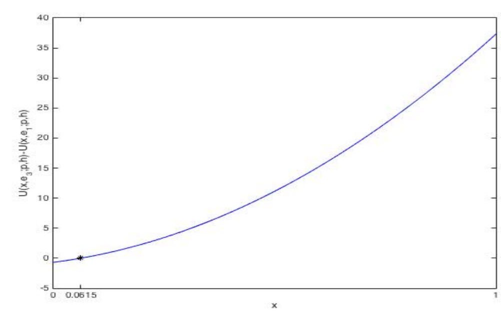

Figure 8: $U(x, \mathbf{e}_3; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_1; \mathbf{p}, \mathbf{h})$ for the disconnected neighborhood example

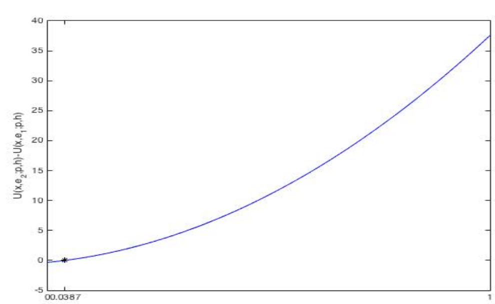

Figure 9: $U(x, \mathbf{e}_2; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_1; \mathbf{p}, \mathbf{h})$ for the disconnected neighborhood example

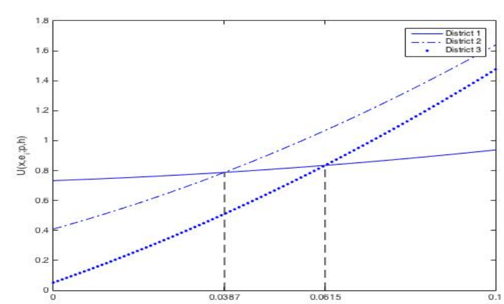

Figure 10: $U(x, \mathbf{e}_i; \mathbf{p}, \mathbf{h})$ for $x \in [0, 0.1]$ for the disconnected neighborhood example

Figures 7, 8, 9 indicate how residential preferences change as types vary and Figure 10 zooms *in details on equilibrium payoffs near x_1. The reason for neighborhood 2 being disconnected is shown in Figure 7. The lowest-type students prefer district 2 to 3 because of lower price. As student types increase, risk-seeking households increasingly favor district 3 because of supermodularity in student type and high school quality. However, as district 3 is indeed a risky option for the highest-type students compared to the safer choice of district 2, those of the highest types eventually choose to reside in district 2 with a guarantee to enroll in its neighborhood school.*

Notice that neighborhood T_2 has widely dispersed student type distribution: half of its student types lie below the lower 20% quantile, while the other half fall in the top 20% quantile. Figure 7 in Example 1.2.2 captures how households make tradeoff between less risky and lower-cost option (T_2) and the risky option with higher marginal benefit from better schools (T_3), and how their residential choices vary because of that. In the next section I will point out how heterogeneity in parents' risk attitudes in this example leads to disconnectedness.

1.3. Two Town Model

The one-town model paves the way for a two-town model where one town adopts deferred acceptance and the other still runs neighborhood assignment. In the two-town model, the outside options of one town consist of exogenous options with payoff $\pi(x)$[10] and the other town that adopts a distinct student assignment rule. Parents can choose which town and which assignment rule they prefer. How will they decide between school choice and neighborhood assignment in equilibrium? Who benefits from school choice? To answer these questions, one must understand how housing prices in both towns will be affected by introducing school choice.

[10] This allows for opting out from schools in both towns when outside options are more attractive than existing public schooling. It could be that private schools spring up when capacity is constrained.

1.3.1. Setup

The model is similar to Avery and Pathak (2015) with major differences. Two towns A and B each has D districts, with one public school in each district. Town A adopts deferred acceptance while town B adopts neighborhood assignment to assign students. District d in town t is denoted as d_t, $d = 1, 2, \ldots, D$, $t = A, B$. And the school in district d has exogenous given quality y_d and capacity k_{d_t}. Assume that $k_{d_t} \geq 0$ and $y_1 < y_2 < \ldots < y_D$.[11]

Student of type x with partisanship for town t is denoted by x_t. Partisanship denotes households' special (geographic) preferences to living in the town. Households with such special preferences are 'partisans' of the town. The distribution of all partisans of town t is given by a non-atomic measure μ_t ($t = A, B$) with compact support on $X_t = [\underline{x}, \bar{x}]$. Normalize the total mass of all partisans from both towns to be one.

As in the one town case, if a student of type x and a partisan to town t, pays the home price $p_{d_{t'}}$ of district $d_{t'}$ and enrolls in a school in down t' of quality y, she receives a utility of,

$$u(x_t, y, p_{d_{t'}}) = v(x, y) + \theta \cdot \mathbb{1}_{t=t'} - p_{d_{t'}}.$$

If the student chooses to live in the town she prefers as a partisan, she receives a bonus benefit of θ. The larger θ is, the more favorable it is, other things equal, to live in the town one is a partisan of.

If a student is unassigned by the school choice mechanism, she can certainly opt for outside options. Let $\pi : [\underline{x}, \overline{x}] \to \mathbb{R}$ be the utility of her outside options.[12] Assume $\pi(x)$ is continuous and non-decreasing in type x.

[11]Instead of assigning y_{d_t} to each district d_t, I assume that $y_{d_A} = y_{d_B} = y_d$. Otherwise, if there is a unique quality y_{d_t}, one could add a dummy school in the other town t' with the same quality y_{d_t} and 0 capacity.

[12] One could also assume $\pi_t(x)$ the payoff of outside option that depends on town t, which does not alter the main results of this paper.

Similar to the one town case, the housing supply under deferred acceptance is

$$H^A(l_{d_A}, p_{d_A}) = l_{d_A} p_{d_A}^r.$$

For town B that implements neighborhood assignment, I assume that housing supply is in addition, subject to a local (zoning) constraint to ensure that all resident students are guaranteed seats in their neighborhood schools,

$$H^B(l_{d_B}, p_{d_B}) = \min\{k_{d_B}, l_d p_{d_B}^r\}.$$

See previous literature for similar zoning constraints.[13] Moreover, the main results in this section on sorting do not depend on the housing supply functional assumption in town B as long as it guarantees that in each neighborhood, enough slots in its neighborhood school are reserved for the resident students. Similarly, assume that the school capacity is proportional to the land capacity $l_{d_t} = \alpha k_{d_t}$ for some $\alpha > 0$.

Denote the economy by $(\mathbf{y}, \mathbf{k}, \mu_A, \mu_B)$. The timing of the two-town model is similar to the one-town model.

Stage 1 Given housing prices, households simultaneously choose which town and which district to live in.

Stage 2 (Town A running deferred acceptance) Students submit preferences over schools in town A. Each school ranks all applicants by priority: residents from the district where the school is located are ranked higher than applicants outside the district. Within each priority class, break ties at random.[14] Apply deferred acceptance to assigning students to all schools.

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[13] Avery and Pathak (2015), Calsamiglia et al. (2015) model housing supply the same as school capacity k_d. Fischel (1999) explains the relationship between housing supply and local municipal constraints. In Bunten (2015), firms maximize profits at each location subject to a local zoning constraint that restricts the number of houses produced complying to local amenities, i.e., $\min\{k_d, H(l, p, t)\}$ with k_d the local zoning constraints and $H(l, p, t)$ the production function for a profit maximizer.

[14] Same tie-breaking rule as in the one-town model.

(Town B running neighborhood assignment) Assign applicants to their neighborhood schools.

Applying Lemma 1.2.1 to the two-town case, one could assume that truth-telling is a Nash equilibrium in the subgame of Stage 2.

Let $\mathbf{h}$ map a partisan of town t with type x to her mixed strategy $\mathbf{h}(x_t) \in \mathbb{R}^{2D}$. Entry $h_{d_{t'}}(x_t)$ is the probability that she assigns to living in district d in town t'. Given $\mathbf{h}$ the residential choices of all households, the housing demand in district d_t is,

$$m(h_{d_t}) = \int h_{d_t}(x_A) d\mu_A + \int h_{d_t}(x_B) d\mu_B.$$

Denote by $U(x_t, \mathbf{e}_{d_{t'}}; \mathbf{p}, \mathbf{h})$ the expected payoff for a student of type x with partisanship of town t to live in district $d_{t'}$, where $\mathbf{e}_{d_{t'}}$ is the $2D$-dimensional vector with a 1 in the $d_{t'}$th coordinator and 0's elsewhere. Then,

$$U(x_t, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) = \sum_{s \geq j_x} \Pr(s|d_A, \mathbf{h}) v(x, y_s) + \big(1 - \sum_{s \geq j_x} \Pr(s|d_A, \mathbf{h})\big) \pi(x) + \theta \cdot \mathbb{1}_{t=A} - p_{d_A},$$

where $\Pr(s|d_A, \mathbf{h})$ is the probability of being assigned to school s in town A conditional on living in district d_A. $1 - \sum_{s \geq j_x} \Pr(s|d_A, \mathbf{h})$ is the probability of opting out (unassigned under school choice or unwilling to attend the assigned school). j_x is the same as in the one-town model.

Student x_t applies to her neighborhood school in town B if and only if attending her neighborhood school yields higher benefit than her outside option, i.e.,

$$U(x_t, \mathbf{e}_{d_B}; \mathbf{p}, \mathbf{h}) = \max\{v(x, y_d), \pi(x)\} + \theta \cdot \mathbb{1}_{t=B} - p_{d_B}.$$

If student x_t plays mixed residential strategy ϕ and submits truthful preference in the school choice process, her expected payoff is,

$$U(x_t, \phi; \mathbf{p}, \mathbf{h}) = \sum_{d_{t'}} \phi_{d_{t'}}(x_t) \cdot U(x_t, \mathbf{e}_{d_{t'}}; \mathbf{p}, \mathbf{h}).$$

Definition 3. *A* ***two-town school choice equilibrium*** *consists of* $(\mathbf{p}, \mathbf{h})$*, where* $\mathbf{p}$ *is the vector of housing prices of all district* d_t*,* $\mathbf{h}$ *is the mapping from student type and partisanship to her residential strategy, such that,*

(1) for each student of type x_t*,*

$$U(x_t, \mathbf{h}(x_t); \mathbf{p}, \mathbf{h}) = \max_{d_{t'}} U(x_t, \mathbf{e}_{d_{t'}}; \mathbf{p}, \mathbf{h}),$$

(2) housing supply should match demand in each district,

$$m(h_{d_t}) = H^t(l_{d_t}, p_{d_t}), \quad \text{for all } d_t.$$

Condition (1) is the incentive compatibility constraint. In the two town model, the outside options of each town include: the exogenous option $\pi(x)$ and the endogenous option of living in the other town implementing a different student assignment rule. Condition (2) is the housing market clearing condition.

Theorem 1.3.1. *For any* $(\mathbf{y}, \mathbf{k}, \mu_A, \mu_B)$*, there exists a pure strategy two-town school choice equilibrium that admits truth-telling.*

1.3.2. Residential Pattern under School Choice

Now we only focus on pure strategy two-town school choice equilibria that admit truth-telling. Let $T_d = \{x_t \in X_A \cup X_B | h_{d_T}(x_t) = 1\}$ the set of households that live in district d_T in the two-town school choice equilibrium with strategy profile $\mathbf{h}$.

To understand how households redistribute themselves in equilibrium, one must understand equilibrium home prices in both towns.

Proposition 1.3.2. (Home Prices) *In any two-town school choice equilibrium, housing prices in each town increase in school quality, i.e. for* $t = A, B$*,* $p_{1_t} \leq p_{2_t} \leq \ldots \leq p_{D_t}$*.*

The following definition of positive assortative matching (PAM) extends the sorting results in Proposition 1.2.4 to a two-town case.

Definition 4. **(PAM)** *Given town T, let* $\tilde{d} = \max\{d : p_{d_T} = p_{1_T}\}$*. Then town T is PAM in residential pattern if the following two conditions hold,*

1. *For* $d_1 > d_2 > \tilde{d}$*,* $\; x'_{t'} \in T_{d_1},\; x_t \in T_{d_2} \Rightarrow x' \geq x$*.*

2. *For* $d_1 > \tilde{d} \geq d_2$*,* $\; x'_{t'} \in T_{d_1},\; x_t \in T_{d_2} \Rightarrow x' \geq x$*.*

Definition 4 implies that students of higher types live in more expensive neighborhoods. In particular, student type space can be partitioned into 2 groups: one subset corresponds to those of lowest types that live in the cheapest school districts T_d where $d \leq \tilde{d}$, and a subset of higher-type households in T_d where $d > \tilde{d}$ that match assortatively with districts of increasing school qualities. This is a generalization of PAM between student types and school districts in the presence of outside options. Moreover, the monotonicity of residential choices does not depend on partisanship (t is not necessarily equal to t' in the statement of the definition).

Proposition 1.3.3. *Assuming Condition 1, town B is PAM in any two-town school choice equilibrium. Moreover, if for any* x, d*,*

$$k_{d_A} \cdot \frac{\partial v}{\partial x}(x, y_d) > \frac{\partial v}{\partial x}(x, y_{d-1}),$$

then town A is also PAM in any two-town school choice equilibrium.

Notice that PAM may not hold in general (see Example 1.2.1, 1.2.2 from the previous section) except for some strong complementarity condition such as those in Proposition 1.3.3. In non-PAM equilibria, neighborhoods are less stratified by student type. Mediumtype students opt for a top quality school district while the highest types choose a medium-quality school district. As a consequence of neighborhood priority, the composition of student body at each school is less stratified as well.

The following is the second characterization of the residential patterns after adopting school choice, analogous to Definition 2 in the one-town case.

Definition 5. (**Connected Neighborhood**) *Given* $(\mathbf{p}, \mathbf{h})$ *a pure strategy two-town school choice equilibrium, neighborhood* T_d *is connected if each of* $T_d \bigcap X_A$*,* $T_d \bigcap X_B$ *is a single interval. A town* t *is connected if all its neighborhoods are connected.*

The definition of connected neighborhoods attempts to capture the homogeneity of a neighborhood: students of similar types live in the same neighborhood. Notice that in any equilibrium, the set of non-partisan student types x who live in T_d is a subset of its partisan student types in T_d. Therefore, the partisans and non-partisans who live in the same neighborhood must share similar types in a connected neighborhood based on Definition 5.

The connectedness property is a major difference distinguishing deferred acceptance from neighborhood assignment. Unlike neighborhood assignment where seats are guaranteed to resident students, students are subject to randomized assignments under school choice. Therefore, attitude towards risk and uncertainty about student assignment matters. The following proposition shows that if students share the same risk attitudes, neighborhoods are always connected under school choice.

Proposition 1.3.4. *If households' preferences satisfy,*

(i) there exists some $g(x), c(x), q(y) \geq 0$*,s.t.* $v(x, y) = g(x)q(y) + c(x)$*.*

(ii) $\pi(x) = \mathbb{E}v(x, Y)$*, for some random variable* Y*,*

then in any two-town school choice equilibrium $(\mathbf{p}, \mathbf{h})$*,* $U(x_t, \mathbf{e}_{d'_{t'}}; \mathbf{p}, \mathbf{h}) - U(x_t, \mathbf{e}_{d''_{t''}}; \mathbf{p}, \mathbf{h})$*, the difference in equlibrium payoffs between any pair of residential choices* $d'_{t'}$ *and* $d''_{t''}$*, crosses each type space* X_t *at most once.*

Condition (i) in Proposition 1.3.4 is equivalent to the fact that the Arrow-Pratt measure of absolute risk aversion $r(x,y) = \frac{-\frac{\partial^2 v}{\partial y^2}(x,y)}{\frac{\partial v}{\partial y}(x,y)} = r(y)$ is constant in x, i.e., all households share the same absolute risk aversion. Cobb-Douglas utility function in Epple and Romano (2003) is an example of condition (i). Condition (ii) implies that students also share the same $j_x = j$ for all x_t, that is, their least acceptable local school is the same, which also implies that their risk aversion to being unassigned by the school choice mechanism is the same. Under those assumptions, Proposition 1.3.4 states that $U(x_t, \mathbf{e}_{d_t}; \mathbf{p}, \mathbf{h})$ satisfies a single crossing condition, which ensures that all neighborhoods to be connected except for some indifferent cases where two districts generate identical expected payoffs for all households.

Suppose however that households are increasingly absolute risk averse towards uncertainty about student assignment, the following result guarantees that town A is connected in any two-town school choice equilibrium.

Denote by $CE(x,P) : X \times \Delta_{2D-1} \to \mathbb{R}$ the certainty equivalent of any random variable Y with CDF P, i.e., $v(x, CE(x,P)) = \mathbb{E}_P[v(x,Y)]$.

Proposition 1.3.5. *Assuming Condition 1, if $\theta > 0$, and*

(i) $k_{d_A} \frac{\partial v}{\partial x}(x, y_d) > \frac{\partial v}{\partial x}(x, y_{d-1})$ *for all d,*

(ii) $\frac{\partial^2 v}{\partial y^2} < 0$, $\frac{\partial^3 v}{\partial x^2 \partial y} < 0$, *and certainty equivalent $CE(x,P)$ is a decreasing and concave function of x for all P,*

then in any two-town school choice equilibrium, all neighborhoods in town A are connected except for those with the lowest home price in town A. When $\theta = 0$, Condition 1 and (i)-(ii) guarantee that all neighborhoods in town A are connected except for those with the lowest price, and neighborhood d for which $p_{d_B} = p_{d_A}$.

Proposition 1.3.5 investigates the case where households are increasingly absolute risk averse. Condition 1 states that the set of acceptable schools is expanding as student type increases, which suggests that students of higher types are increasingly risk averse to being unassigned. Condition (i) can be interpreted as strong complements of student type and school quality or widely dispersed school qualities in town A, thus students of higher types are more risk averse to be assigned to schools of low qualities.

Condition (ii) eliminates the scenario where two extreme types prefer risky options in town A, but the 'in-between' types favor safer choices in town B, thus implying connectedness in town A. Consider some risky option in town A implementing deferred acceptance with an attractive school quality equivalent $CE(x, P)$. Non-increasing $CE(x, P)$ implies that for students with higher types, the school quality equivalent becomes less attractive. Moreover, $\frac{\partial}{\partial x}[\frac{\partial^2}{\partial x \partial y}v(x, y)] \leq 0$ suggests that the supermodularity between student type and school quality is diminishing. Under these assumptions, switching to a riskier choice with higher $CE(x, P)$ which is monotone decreasing in x generates decreasing surplus for higher types. Therefore, if some student type gains higher payoffs from the safer choice under neighborhood assignment in town B than from some risky option under deferred acceptance in town, so does anyone with higher student type. The concavity of certainty equivalent CE implies convexity of absolute risk aversion (Gollier and Pratt, 1996), an indicator of how their risk aversion varies as student type increases.

One class of utility functions satisfying conditions in Proposition 1.3.5 is $v(x, y) = w(y+xy)$ where,

(i) $w'(z) > 0$, $w''(z) < 0$,

(ii) $w'(z) + zw''(z) > \kappa$,

(iii) $w'''(z) \leq \frac{(zw''(z)-w'(z))w''(z)}{zw'(z)}$.

For example, $w(z) = \frac{z^{1-\eta}-1}{1-\eta}$ for some $\eta < 1$ satisfies conditions (i)-(iii). This utility functional is derived from McElroy (2007).[15]

[15] Let $M_U(x) = U^{-1}[\mathbb{E}U(\tilde{z}_1 + x\tilde{z}_2)]$ for all discrete-valued random variables $\tilde{z}_1$ and $\tilde{z}_2$ where $\tilde{z}_1$ is non-negative. When $U' > 0$ and $U'' < 0$, $M_U(x)$ is concave for all $\tilde{z}_1$ and $\tilde{z}_2$ if and only if U'/U'' is convex. Here we take $\tilde{z}_1 = \tilde{z}_2 = Y$.

Suppose households are increasingly risk seeking.

Condition 2 $\partial_+(\max\{\pi(x) - v(x, y_d), 0\}) \geq 0$ for all d.

Condition 2 implies the following preference structure: if a student of type x prefers her outside option to the school in district d, so does any student with higher type $x' > x$. In other words, the set of acceptable schools is narrowing down as type increases. Notice that Condition 2 is the opposite of Condition 1. Then Proposition 1.3.6 states that under risk seeking environment, town B can be connected.

Proposition 1.3.6. *Assuming Condition 2, if $\theta > 0$, and*

(i) $\frac{\partial^2 v}{\partial y^2} > 0$, $\frac{\partial^3 v}{\partial x^2 \partial y} < 0$ and certainty equivalent $CE(x, P)$ is an increasing and convex function of x for all P,

(ii) for each d, $\frac{d^2}{dx^2}[\pi(x) - v(x, y_d)] \geq 0$ at $x \in \{x | \pi(x) > v(x, y_d)\}$,

then in any two-town school choice equilibrium, all neighborhoods in town B are connected except for those with the lowest home price. When $\theta = 0$, Condition 2 and (i)-(ii) ensure all neighborhoods in town B are connected except for those with the lowest price, and district d_B for which $p_{d_B} = p_{d_A}$.

For risk seeking households, one can interpret Condition 2 as students being increasingly open to outside options. Condition (ii) in Proposition 1.3.6 on the convexity of the difference $\pi(x) - v(x, y_d)$ also implies that 'risk seeking' households are increasingly willing to opt for outside options. A non-decreasing certainty equivalent $CE(x, P)$ implies that when switching to a potentially safer choices, the increment in school quality equivalent is decreasing as student type increases, and so is the increment in expected payoff if $\frac{\partial}{\partial x}[\frac{\partial^2}{\partial x \partial y} v(x, y)] \leq 0$ holds. Hence for risk seeking households, conditions (i) and (ii) combined imply that risk seeking households with higher types prefer riskier options from school choice assignment in town A to deterministic yet less attractive options in town B.

Notice that in both cases $\frac{\partial^3 v}{\partial x^2 \partial y}(x, y) \leq 0$, which suggests that although students increasingly favor higher quality schools ($\frac{\partial^2 v}{\partial x \partial y} > 0$), the supermodularity in student type and school quality has negative impact on the local homogeneity within each neighborhood, and restricting the growth of supermodularity (i.e., $\frac{\partial^3 v}{\partial x^2 \partial y}(x, y) \leq 0$) can generate a more locally 'connected' neighborhood. Notice that in Example 1.2.2 where T_2 is disconnected, this third-order condition fails.

To sum up, all neighborhoods are sorted not just by type but risk attitude. This paper introduces the notion of disconnected neighborhood to characterize heterogeneity within neighborhood, a major distinction from many previous papers that focus solely on cross-neighborhood sorting and heterogeneity. When school choice and outside options are introduced, districts in the town implementing neighborhood assignment may no longer be connected. Recall that in Example 1.2.2, student types are more diversified and spread out in a disconnected neighborhood. Moreover, the heterogeneity within each neighborhood contributes to the heterogeneous student composition in each public school under deferred acceptance as well as under neighborhood assignment implemented by the neighboring town, a sorting effect not only across and within neighborhoods but across and within schools in both towns.

1.3.3. Home Price and Distribution of Educational Benefits

The redistribution of residential patterns influences the redistribution of educational benefits, in a good way.

Proposition 1.3.7. *In the town that is implementing deferred acceptance, students living in the lower-quality school district have higher opportunities of enrolling in schools better than their neighborhood schools, i.e.,* $\sum_{s>d} \Pr(s|d, h)$ *are decreasing in* d *in any equilibrium.*

$\sum_{s>d} \Pr(s|d, h)$ is the probability of being admitted to a school better than their neighborhood school by deferred acceptance. A similar assertion can be found in a field report by the Chicago Tribune,[16]

[16] The report is available from http://www.chicagotribune.com/news/ct-chicago-schools-choice-neighborhood-enrollment-met-20160108-story.html

... that belies a common assumption that poor kids in low-performing schools are trapped in those schools. In fact, students who live within the boundaries of the city's worst schools have the highest rate of going elsewhere.

One could also prove that $\sum_{s \geq d} \Pr(s|d,h)$ the probability of being admitted to a school at least as good as her neighborhood schools is also decreasing in d. Indeed as advocated, deferred acceptance provides more equitable opportunities for students living in the low-quality school districts. When equilibrium is PAM, this implies more equitable opportunities for low-type students since the probability of going to a school better than neighborhood schools under neighborhood assignment is alway 0.

The following results aim to compare parental choices over the two towns and the two assignment rules.

Proposition 1.3.8. (Home Prices) *For any* $(\mathbf{y}, \mathbf{k}, \mu_A, \mu_B)$*, housing prices in each town ascend. Moreover, if* $\theta = 0$*, there exists* d^**, s.t. for all* $d > d^*$*,* $p_{d_B} \geq p_{d_A}$*;* $p_{d^*_B} = p_{d^*_A}$*; and for all* $d < d^*$*,* $p_{d_B} \leq p_{d_A}$*.*

Proposition 1.3.8 echoes Avery and Pathak (2015), who argue that introducing school choice inflates the home prices of low-quality school districts, while deflates housing prices of high-quality ones, thus producing incentives for types at both extremes to opt for neighborhood assignment in town B. Nevertheless, unlike Avery and Pathak (2015) where the town implementing school choice drives away low and high types and attracts only those with types in between, in this paper, town A implementing deferred acceptance also attracts those of low and high types, if not the very extreme ones, because of differentiated school qualities.

Next, suppose the two towns share the same school capacities and distribution of school qualities, but adopt distinct student assignment rules, which town will the parents choose?

Proposition 1.3.9. *Suppose* $k_{d_A} = k_{d_B}$*,* $\theta = 0$*, that is everything equal except for distinct assignment rules, then in any equilibrium* $(\mathbf{p}, \mathbf{h})$*, the numbers of residents satisfy,*

$$m(h_{d_A}) \geq m(h_{d_B}), \quad \forall d.$$

To rephrase, under the same educational resources and economic environment, more households will 'vote with their feet' for deferred acceptance over neighborhood assignment not just in total but in every single school district. [17]

1.4. Discussion

1. Probabilistic serial and random priority mechanisms.

   This paper studies deferred acceptance with neighborhood priorities. We can also apply Probabilistic Serial (PS) to student assignment with neighborhood priorities (Che and Kojima, 2010). Since PS is strategy-proof in the large market (Azevedo and Budish, 2013) and by the asymptotic equivalence between PS and RP (Che and Kojima, 2010), we can apply similar argument to both RP and PS to analyze an approximately truth-telling school choice equilibrium. For school priority, associate PS (or RP) with $\{g_c(t)\}$, where each $g_c : [0, 1] \to \mathbb{R}$ is the eating speed of priority class c. When $g_c(t)$ is uniform distributed, it is equivalent to PS with no residential priority; for some specific $g_c(t)$, it is equivalent to deferred acceptance with neighborhood priority.

2. Boston mechanism and deferred acceptance

   Boston Mechanism is undeniably one of the most popular school choice mechanisms. Calsamiglia et al. (2015) study a model of 3 school districts each has a neighborhood school in a single town, with inelastic housing supplies equal to the school capacities. They find that both Boston Mechanism and deferred acceptance yield the same unique PAM equilibrium where student types are stratified into 3 groups and all students get assigned to their neighborhood schools. The results are driven by the fact that there are sufficient seats for resident students in every single neighborhood and no risk of uncertainty about student assignment to bear. Results can differ when there is potential gain from gaming the Boston Mechanism. However, Boston Mechanism is

[17] Proposition 1.3.9 does not imply that every school in town A is over-subscribed since not all schools in town B are fully enrolled, which is often the case in many school districts that run neighborhood assignment.

known not strategyproof in the large (Azevedo and Budish, 2017) which complicates the analysis of parental reports of their preferences.

3. Combination of exogenous school quality and peer effects.

   This paper considers the polar opposite case of exogenous school qualities. If students value qualities as well as peer effects, school choice equilibrium can be imperfectly-assortative matching as in my paper, or PAM depending on how much a student values school quality relative to peer effects and how strong the supermodularity between student type and school quality is. Some techniques in this paper such as proving the existence of pure strategy Nash equilibria can be adapted to the combination of exogenous quality and peer effects, which is of future interest.

1.5. Conclusion

School choice is designed to offer more equitable access to high quality schools. As households value quality schools, I assume schools do have qualities that drive most of the results in this paper. Compared to neighborhood assignment, deferred acceptance can provide more equitable access to higher quality schools for low-type students.

This paper also finds how deferred acceptance reshapes neighborhoods in a PAM or a non-PAM manner and a connected or a disconnected manner, due to households' risk attitudes. When equilibrium is non-PAM, there is less stratification across neighborhoods. When neighborhoods are disconnected, student distribution within each neighborhood is more diverse. Moreover, the heterogeneity in households' preferences and risk aversion attitudes redistribute the neighborhoods not just in the town that adopts deferred acceptance but also the neighboring town that still implement neighborhood assignment.

CHAPTER 2 : Price Dispersion in a Buyer-Seller Search Platform

2.1. Introduction

Many marketplaces, such as online platforms (eBay, Amazon), dating websites (eHarmony.com) and video game platforms (Nitendo) feature facilitating search and matching for agents that would otherwise have difficulty finding each other. A common practice for such platforms is to lower the cost of search and offer convenience in transactions to attract as many subscribers as possible. This is driven by (positive) cross-group externalities in multi-sided platforms, see Rochet and Tirole (2003, 2006); Armstrong (2006): as the number of users on one side increases, users on the other side are more willing to join, thus increasing the value of the platform.

However, many platforms limit participation due to two types of negative externalities: (1) congestion effects that arise when sellers have limited capacity and users on the other side must compete for a match partner, and (2) cross-side negative consumption externalities such as browsers' aversion to advertisements. Arnosti et al. (2015), for example, identify the negative congestion externality in dating websites: while a male applicant feels he is more likely to find an attractive match when more female candidates participating in the dating website, he may be less likely to be accepted by his chosen match because females also have a larger pool to choose from and profit from a higher premium. This congestion externality incentivizes dating websites such as eHarmony.com to limit the set of candidate choices while its customers are willing to pay a premium of 25%.[1] To resolve negative consumption externalities in advertising markets where the audience dislikes advertisements, Gomes and Pavan (2016) show that intermediaries such as newspapers or TV stations limit participation by matching subsets of advertisers with eyeballs.

Interestingly, limiting participation can be found in other marketplaces where both congestion and cross-group negative consumption externalities are absent. One such case is a

[1] See Piskorski et al. (2008) for details on eHarmony.com .

buyer-seller platform where each seller sets the prices according to its demand curve on the platform. Dinerstein et al. (2014) investigated eBay's redesign of limiting consideration sets in 2011. Prior to the redesign, consumers entering a search query were shown a substantial number of potential matches. The redesign of the search algorithm after 2011 directs buyers to a specific match (for example, "Space Grey iPhone 6 32GB (AT&T) " instead of anything relevant to "iPhone") and a smaller fraction of offers from low-priced and top-rated sellers. While Dinerstein et al. (2014) estimate the effects of limiting choices on firms' decreasing retail prices, markup, and consumers' browsing behavior, the reason for such a redesign is unclear from the platform's perspective.

The main question of the paper is the following: is it optimal for a buyer-seller search platform to include as many buyers and sellers as possible? If not, is it better to strategically limit choices and target participation to a smaller subset? To answer this question, one must understand the nature of cross-group externalities in buyer-seller platforms, and that the value of a platform not only results from the number of users but the mix of users available on the platform.

Consider a two-sided market where buyers search for sellers directly or through a middleman (i.e. platform). Firms with unlimited capacity decide on whether to join the platform and set prices for both on-platform and off-platform transactions. Buyers decide whether to subscribe to the platform and search for the optimal products through platform or directly. Each search reveals the price and value of a firm's product but at some cost to the buyer. After each search, a buyer must decide whether to proceed with another search, or to stop and purchase the best alternative observed so far, or switch to an off-platform transaction. As a middleman who facilitates search, the platform lowers search costs for the buyers but can charge both sides for the transactions it facilitates.

In the two-sided marketplace, absent congestion and negative consumption externalities, this paper proposes a novel explanation for platforms to limit choices: to balance two competing positive and negative cross-group externalities. Intuitively, attracting more sellers on the platform generates positive externalities (choice effect): buyers have more choices and therefore, higher transaction values when the search cost is low on the platform. On the other hand, increasing the number of subscribed sellers may impose negative externalities on the buyers: search for low-priced and high-value sellers can be more difficult because of the increased price dispersion, offsetting the positive externalities from having a variety of sellers.

To formalize the idea of the negative externalities under price dispersion, I model firms as selling horizontally differentiated goods and differing in their marginal costs, which produces price dispersion similar to Reinganum (1979). The existence of search frictions explains how retailers selling identical goods can enjoy positive mark-ups and why price dispersion is ubiquitous even on online platforms.[2] In the presence of price dispersion, firms with heterogeneous marginal costs differ in the profits gained from subscribing to the platform. I find that the optimal membership fees set by the platform satisfy a threshold structure: only firms with marginal costs lower than a threshold are willing to join the platform. Lowering the membership fee on the sellers' side attracts sellers with higher posted prices, subsequently making search for low-priced sellers less efficient. In this sense, a platform acts as a price gatekeeper and decides on not only how many sellers but which sellers to target, to maximize its profits.

The horizontal differentiation in buyers' valuation for sellers, contributes to the positive network externalities due to more sellers. In this setting where the search cost on the platform is relatively low compared to searching directly, buyers search until they settle on optimal alternatives using a reservation strategy introduced by Weitzman (1979); Kohn and Shavell (1974). With horizontal differentiation for each seller's product, there is an incentive for buyers to search more frequently on the platform for high-value low-priced sellers and for the platform to attract a larger variety of sellers.

[2] Brynjolfsson and Smith (2000) found that the variation across retailers selling books and CDs online are significant, up to 30% of the average price. Baye et al. (2004) also found similarly large posted price variation for consumer electronics in data obtained from a price search engine.

The major contribution of the paper is to identify the coexistence of positive and negative cross-group externalities due to the number of sellers subscribed, distinct from the linear approximation of cross-group network externalities in the previous literature (Rochet and Tirole, 2003, 2006; Armstrong, 2006; Weyl, 2010). Moreover, I show that whether positive or negative externalities dominate depends on the distribution of buyers' valuation for sellers. One could interpret the distribution of buyer's valuation as related to the price elasticity of demand. Under some conditions when the negative externalities from attracting additional sellers always outweigh the choice benefit, the platform gains more profits targeting sellers with lower marginal costs.

Another economic insight the paper offers, is the impact of lowering search costs on social welfare. The search platform can promote efficiency in two ways. The first, which is standard, is the lower costs of search for buyers to find satisfying deals. The second is that when the search platform acts as a gatekeeper who restricts the set of sellers available on the platform to those with low marginal costs, the surpluses from trades increase as demands accrue to the firms with low production costs.

2.2. Literature Review

This paper studies the network externalities and pricing theory for a buyer-seller search platform. As such it relates to three strands of literature: price theory of multi-sided platforms, the theory of search and the ensuing price dispersion.

1. Pricing theory of multi-sided platform.

The model of search platforms belongs to a recent strand of two-sided markets with intermediaries, pioneered by Rochet and Tirole (2003, 2006); Armstrong (2006); Caillaud and Jullien (2003); Weyl (2010). Assuming a linear approximation of network externalities and exogenous valuation over the other side, Rochet and Tirole (2003, 2006); Armstrong (2006) derived a cross-subsidy pricing structure for platforms to attract as many users as possible and internalize the cross-group network effects. Weyl (2010) relaxed the assumptions of Rochet and Tirole (2006) and introduced a model of linear externalities but heterogeneous users. He derived platform's imperfect internalization over heterogeneous agents in two-sided markets. I study the coexistence of both positive and negative network effects: buyers care about the variety of sellers; however, the price dispersion endogenized by the size of the buyer-seller network on the platform can cause inefficient results of searching and congestions. Gomes and Pavan (2016) studied a platform that match eyeballs with advertisements, and intrinsic to the negative network effect in their paper is the negative consumption externalities from eyeballs averse to too many ads. The paper closest to mine is Wang and Wright (2016) which study a search platform with identical marginal cost across all sellers. In this paper, the law of one price breaks down because of heterogenous production costs.

2. Costly search and price dispersion.

This paper focuses on a particular type of platform that facilitates buyers' search for best products. While abundant papers have advanced the theory of price dispersion (Stigler, 1961; Reinganum, 1979), recent literature focuses on this topic in the context of e-commerce and platforms where search cost is reduced (Bar-Isaac et al., 2012; Baye and Morgan, 2001; Ellison and Ellison, 2009; Levin, 2011; Ellison and Wolitzky, 2012; Ellison and Ellison, 2018). Without the role of a platform, Bar-Isaac et al. (2012) studied the effect of reducing search costs on product design and the ensuing price dispersion of firms. In Baye and Morgan (2001), the platform can offer transparent information on low prices therefore the cost of search is zero on the platform. Other papers focuses on platform's techniques not considered in this paper, such as search obfuscation (Ellison and Ellison, 2009), ordered search engine. Rather I look into the pricing strategies of a platform and how the results in pricing, price dispersion and efficiency differ from that of the benchmark case with no intermediary. Finally, recent literature has drawn attention to a platform's strategic play of limiting choice to eliminate search friction, due to congestion and negative consumption externalities in matching markets (Casadesus-Masanell and Hałaburda, 2014; Halaburda et al., 2017; Kanoria and Saban, 2017). Unlike them, this paper studies a buyer-seller search platform where buyers have unit demand and sellers post prices, and negative consumption externalities are negligible.

3. Search strategy for optimal alternative.

Starting with Stigler (1961); Weitzman (1979); Kohn and Shavell (1974); Rothschild (1978), the theory of search has extended to a wide collection of problems such as sampling, switch point, adaptive belief update etc. I model buyer-seller search as a Pandora model in Weitzman (1979), rather than to take at most n samples and then stop. Sequential search for price quotes has been studied in Burdett and Judd (1983); Athey and Ellison (2011), especially when buyers can search more frequently and adaptively update their prior belief about price distribution with low search cost. As an simplified extension, I consider the case of non-adaptive beliefs endogenized by the price-setting firms subscribed to the platform.

2.3. Model

There is a continuum of buyers and sellers (firms) on each side with mass B and S. Each firm j produces a horizontally differentiated product at a marginal cost of m_j. For now I assume the supply is unlimited. Suppose the marginal costs of all firms are i.i.d distributed according to the cdf F with compact support $[\underline{m}, \overline{m}]$ and pdf f. Each firm posts a sale price for its product and can price discriminate by posting different prices for different transaction channels.

A buyer visits each firm and decides whether or not to buy from the firm or keep looking. Each buyer i has different values for products sold by different firms and the valuation v_{ij} are i.i.d. distributed according to cdf G with compact support $[\underline{v}, \overline{v}]$[3].

Consumer search. In the benchmark model without any intermediary, firms are ex ante identical before buyers' search and buyers share common belief over the value and price

[3]Bounded valuation is not necessary

distribution of products sold in the marketplace. All participating buyers search sequentially for firms and receive a price quote p_j and value assessment v_{ij} each time she visits a firm j, at a search cost of $c_D > 0$. After each search she decides whether to proceed with search or stop to trade with some firm she has visited so far. Assume there is no discounting on values and consumers can perfectly recall prices they have observed as in opening Pandora's Box model in Weitzman (1979). If consumer i decides to purchase from firm j at a posted price of p_j^D after k direct searches, she receives a utility of

$$v_{ij} - p_j^D - kc_D,$$

i.e., the consumer surplus $v_{ij} - p_j^D$ less the total search costs kc_D.

A platform can facilitate search by lowering search cost to $c_D > c_L > 0$.[4] A buyer i not subscribed to the platform can reach any seller directly with search cost c_D, regardless of whether the seller is available on the platform or not (that is, sellers can multi-home). For buyer i subscribed to the platform, each time after sampling firm j on the platform and observing v_{ij} and the posted price p_j^L for transaction through the platform, and p_j^D for an off-platform transaction, the consumer has to decide whether to switch to an offline purchase and incurs a switching cost of w_j. One can interpret w_j as transportation costs that depends on each individual seller. The flexibility of transactions with or without the platform reflects the non-exclusiveness policy of the platform. If a subscribed buyer i searches k times on the platform until she purchases from firm j, she enjoys a payoff,

$$V_b + v_{ij} - \min\{p_j^L, p_j^D + w_j\} - kc_L - P_b,$$

where V_b is the fixed subscription benefit as in Weyl (2010), and P_b the membership fee set by the platform. $\min\{p_j^L, p_j^D + w_j\}$ is the transaction cost spent on firm j, and kc_L is the aggregate search costs on the platform.

[4] I do not assume $c_L = 0$ since it is still reasonable to assume that consumers need to spend time and some effort in investigating the quality and price of each products. For the case of $c_L = 0$, refer to Baye and Morgan (2001).

Firms post prices. Any firm j not on the platform can still trade with any buyer who reach them directly at the posted price p_j^D for off-line transactions. The firm's profit from all off-line transactions is $(p_j^D - m_j)X_j^D$ with X_j^D being the total demand for off-line transactions. On the other hand, any seller j available on the platform can accept transactions both on and off the platform at different posted prices: p_j^D for off-line transactions and p_j^L for on-line transaction. A seller j with numbers of transactions X_j^D off the platform and X_j^L on the platform, enjoys a profit of

$$V_s + (p_j^L - m_j)X_j^L + (p_j^D - m_j)X_j^D - P_s,$$

where V_b is the fixed subscription benefit as in Weyl (2010) and P_s is the membership fee charged by the platform if the firm decides to register.

Platform. A platform facilitates search but incurs costs for the convenience and service it provides. Assume that $C(b, s)$ is the fixed cost of lowering the search costs between a measure of b buyers and s sellers subscribed on the platform. $C(b, s)$ is non-decreasing in b, s. Moreover assume that $V_b B_L \geq C(0, B_L), V_s S_L \geq C(S_L, 0)$, i.e., the platform cannot profit by attracting only one side of the market.

The profit of the platform with B_L subscribed buyers and S_L subscribed sellers is

$$\pi_0 = P_s S_L + P_b B_L - C(B_L, S_L).$$

For now we only consider charging subscription fees on both sides.

The following assumption holds throughout.

Assumption 1. G is twice differentiable with pdf g, and $\frac{1-G(v)}{g(v)}$ is non-increasing.

Assumption 1 is a standard monotone hazard rate condition, which ensures that the equilibrium prices posted by the firms are uniquely determined by a first-order condition.

Below is the timing of the search game.

Timing

Step 1 The platform announces membership fees (P_b, P_s).

Step 2 Sellers and buyers simultaneously decide whether to join the platform. Firms subscribed to the platform pay the subscription fee P_s in advance while for buyers the membership fee is P_b.

Each firm j joining the platform posts price p_j^L for purchases on the platform and p_j^D for transactions off the platform. Firms subscribed to the platform pay the subscription fee P_s in advance.

Any firm j not on the platform posts one price p_j^D for transactions off the platform.

Step 3 Each buyer subscribed to the platform searches sequentially for the optimal deal. During each search, after the posted prices p_j^L, p_j^D and switch cost w_j of the firm j visited are observed, the buyer has to decide whether to proceed with another search or stop and purchase from some seller visited so far. The transaction with a seller can happen either on or off the platform. If the trade occurs on the platform, the buyer pays p_j^L to the firm, and p_j^D otherwise, in addition to the switch cost w_j.

Consumers who search directly also conduct sequential search until they stop and make purchases with some seller j at price p_j^D. While the switch from on-the-platform to off-the-platform transactions are available to any subscribers of the platform, I assume that the reverse isn't true. In other words, non-subscribers can not switch to transactions on the platform.

If a platform makes zero or negative profit whatever membership fees it adopts, then, we assume that it does not operate in the first place. The solution concept is Bayesian Nash Equilibrium and all sellers and buyers are risk neutral.

Denote by t_j^I the virtual price spent on firm j through channel I, then,

$$t_j^L = \min\{p_j^D + w_j, p_j^L\},$$

for each seller j on the platform and $t_j^D = p_j^D$ for those off the platform. Since posted prices and switching costs of each individual firm are unobservable to buyers unless they search, sellers are assumed ex ante homogenous before the search and buyers are assumed to hold common beliefs $\tilde{F}_L$ over the virtual prices t_j^L on the platform, as in Wang and Wright (2016). We consider the case where beliefs are unadaptive as buyers search on the platform. Similarly, buyers who search directly also hold some common belief $\tilde{F}_D$ over posted p_j^D off the platform before prices are observed.

The following lemma by Weitzman (1979) characterizes the optimal sequential searching strategies of buyers: setting a reservation price as stopping rule.

Lemma 2.3.1 (Weitzman (1979))**.** *Each buyer choose a value z_I, if $z_I \geq 0$ consumers starts searching via I ($I = L$ if search on the platform, and $I = D$ if search directly), and stop if the surplus $v_{ij} - t_j$ from trading with firm j exceeds z_I, and continue searching otherwise. Moreover, z_I satisfies,*

$$c_I = \int \int_{v \geq t + z_I} (v - t - z_I) dG(v) d\tilde{F}_I(t), \quad I = L, D.$$

z_I is the reservation price for sequential search via $I = L, D$.

I consider the symmetric Bayesian Nash Equilibrium where agents of the same type adopt the same strategies on pricing, search and entry. In this model, buyers share the same searching strategies because of their common ex ante valuations of the product sold in each firm and common belief over the price distribution on and off the platform. The focus is on the seller side. After each search and inspection, the subscribed buyer i learns the value, prices and switching cost of firm j, and decide to continue searching if $v_{ij} - t_j < z_I$ or stop and purchase from firm j.

If the firm set prices (p_j^L, p_j^D) respectively for transactions on and off the platform, its total demand via the platform is,

$$X_L(p_j^L, p_j^D) = \frac{B_L}{S_L(1-\rho_L)} \Pr(v \geq t_j^L + z_L), \tag{2.1}$$

where B_L is the measure of buyers searching on the platform, S_L is the number of registered sellers, i.e., her competing peers, on the platform. Here $\rho_L = \Pr(v - t < z_L)$ denotes the probability that the search on the platform terminates at each time period, i.e., with probability $1 - \rho_L$ a consumer stop searching after the current search. Notice that if $p_j^L \leq p_j^D + w_j$, all demands from the subscribed buyers are completed on the platform.

Similarly, the demand from unsubscribed buyers who search directly is

$$X_D(p_j^D) = \frac{B_D}{S_D(1-\rho_D)} \Pr(v - p_j^D \geq z_D),$$

where B_D are those who only search directly off the platform and ρ_D the probability of an unsuccessful transaction from a direct search. Hence if firm j does not enter the platform, its expected profit is

$$\pi_d(p_j^D; m_j) = (p_j^D - m_j)X_D(p_j^D).$$

If firm j decides to participate on the platform, his expected profit is

$$\pi_L(p_j^L, p_j^D; m_j) = \begin{cases} (p_j^L - m_j)X_L(p_j^D, p_j^L) + (p_j^D - m_j)X_D(p_j^D) & \text{if } p_j^L \leq p_j^D + w_j \\ (p_j^D - m_j)X_L(p_j^D, p_j^L) + (p_j^D - m_j)X_D(p_j^D) & \text{if } p_j^L > p_j^D + w_j \end{cases} \tag{2.2}$$

Then firm j is willing to join the platform if the benefit from subscribing exceeds that from not, i.e.,

$$\max_{p^D, p^L} \pi_L(p^L, p^D; m_j) + V_s - P_s \geq \max_{p^D} \pi_D(p^D; m_j). \tag{2.3}$$

For buyers, the expected benefit of searching on the platform is:

$$
\begin{aligned}
\mathbb{E}[\text{gain from the platform}] &= \sum_k \mathbb{E}[\text{ gain from } k^{th} \text{ search}] - P_b \\
&= \sum_k \mathbb{E}[(v-t)\mathbb{1}(v-t \geq z_L)](\rho^L)^{k-1} - \frac{c^L}{1-\rho^L} + V_b - P_b \\
&= \frac{\int\int_{v-t\geq z^L}(v-t)dGd\tilde{F}_L - c_L}{1-\rho_L} + V_b - P_b \\
&= \frac{\int\int_{v-t\geq z^L} z_L dGd\tilde{F}_L}{1-\rho^L} + V_b - P_b \\
&= z^L + V_b - P_b
\end{aligned}
\tag{2.4}
$$

Similarly, the expected benefit of searching off the platform is z_D. Hence a buyer is willing to pay the membership fee and search on the platform if,

$$z^L + V_b - P_b \geq z_D.$$

2.4. Equilibrium Targeting Strategy

The equilibrium of the search model is defined as follows.

Definition 6 (Search Equilibrium)**.** *For a buyer-seller search model with platform facilitating search, given* (P_s, P_b) *the membership fee announced by the platform, a search equilibrium is* (p^L, p^D, z^L, z_D)*, where* $p^L : [\underline{m}, \overline{m}] \to \mathbb{R} \cup \{\infty\}$ *is a mapping from firms' marginal costs to its posted prices on the platform, and* $p^D : [\underline{m}, \overline{m}] \to \mathbb{R} \cup \{\infty\}$ *a mapping from firm's marginal costs to its posted prices off the platform,*[5] *such that*

(i) Each firm with marginal production cost m *who is subscribed to the platform, posts the optimal prices to maximize its aggregate profit from transaction on and off the platform,*

$$(p_L(m), p_D(m)) = \arg\max_{p_1,p_2} \pi^L(p_1, p_2; m_j).$$

[5] $p_j^L = \infty$ if firm j is not subscribed to the platform

For those not subscribed to the platform, the optimal prices are selected to maximize profit from direct search, i.e.,

$$p_D(m_j) = \arg\max_p \pi_D(p; m_j),$$

(ii) Firm j joins the platform if

$$\max_{p_1,p_2} \pi^L(p_1, p_2; m_j) + V_s - P_s \geq \max_p \pi_D(p; m_j).$$

(iii) Buyers join the platform if

$$z^L + V_b - P_b \geq z^D.$$

and adopt the optimal stopping rule stated in Lemma 2.3.1

One incentive for buyers to search online is lower prices. The following result confirms that on the buyer-seller search platform, prices charged are lower than prices through direct search, thus eliminating the showroom issues in Wang and Wright (2016) where buyers search for sellers on the platform but complete their transaction off the platform.

Lemma 2.4.1. *In any search equilibrium when the platform operates, $p^L(m) \leq p^D(m), \forall m$.*

Based on the proof of Lemma 2.4.1, in any equilibrium when the platform operates, the equilibrium prices p_j^D, p_j^L posted by firm j satisfy:

$$p_j^D = m_j + \frac{1 - G(p_j^D + z_D)}{g(p_j^D + z_D)},$$

$$p_j^L = m_j + \frac{1 - G(p_j^L + z_L)}{g(p_j^L + z_L)}.$$

with demand from on the off the platform being:

$$X_D(p_j^D) = \frac{B_D}{S_D(1 - \rho_D)}(1 - G(p_j^D + z_D)),$$

$$X_D(p_j^L, p_j^D) = \frac{B_L}{S_L(1-\rho_L)}(1 - G(p_j^L + z_L)).$$

Therefore, the profits for firm j from transaction on or off the platform are respectively,

$$\pi_L(p_j^L, p_j^D; m_j) = \frac{(1 - G(p_j^L + z_L))^2}{g(p + z_L)} \frac{B_L}{S_L(1-\rho_L)} + \frac{(1 - G(p_j^D + z_D))^2}{g(p + z_D)} \frac{B_D}{S(1-\rho_D)}$$

$$\pi_D(p_j^D; m_j) = \frac{(1 - G(p_j^D + z_D))^2}{g(p + z_D)} \frac{B_D}{S(1-\rho_D)}$$

The incentive condition for firms' entry thus depends on:

$$\frac{B_l}{S_L(1-\rho_L)} \frac{(1 - G(p_j^L + z_L))^2}{g(p_j^L + z_L)} + V_s \geq P_s \tag{2.5}$$

Since $\frac{(1-G(p+z))^2}{g(p+z)}$ is non-increasing in p, the lemma above yields an upper bound for posted prices on the platform for given V_s, P_s. The equilibrium strategy profile exhibits the following threshold structures in any search equilibrium.

Theorem 2.4.2 (Threshold Strategy)**.** *In any buyer-seller search equilibrium,*

(i) the equilibrium prices for firms with marginal costs m *is* $p_L(m) = p(m, z_L), p_D(m) = p(m, z_D)$*, where* $p(m, z)$ *satisfies,*

$$p(m, z) = m + \frac{1 - G(p(m, z) + z)}{g(p(m, z) + z)}$$

(ii) there exists m_L*, s.t., any firms with production cost* $m < m_L$ *join the platform while those with higher production cost* $m > m^L$ *choose to stay off the platform. Moreover,*

$$\frac{B_L}{S_L(1-\rho_L)} \frac{(1 - G(p(m_L, z_L) + z_L))^2}{g(p(m_L, z_L) + z_L)} + V_s = P_s.$$

(iii) $S_L = S \cdot F(m_L)$

(iv) z_L *satisfies,*

$$c_L = \frac{1}{F(m_L)} \int_{\underline{m}}^{m_L} \int_{v \geq p(m,z_L)+z_L} (v - p - z_L) dG(v) dF(m).$$

The success of the platform is determined by the amount of membership fees P_s, P_b charged on both side, which in term are determined by the optimal threshold m_L. By including more sellers with high threshold m_L, buyers enjoy positive externalities of the variety of choices and higher values of trades v_{ij} in equilibrium. On the other hand, the price dispersion on the platform can be detrimental for the success of the platform, as buyers have to search extensively to find sellers with low prices p_j^L. Therefore, it is not obvious that the middleman should attract all sellers to the platform in equilibrium.

The following theorem characterizes the platform's optimal choice of thresholds m_L by limiting the set of sellers available on the platform.

Theorem 2.4.3. *Suppose*

(i) $f(m)$ *is non-increasing in* m*,*

(ii) $\frac{\partial C}{\partial s}(B_L, S_L) \geq V_s$*,*

then the profit of the platform is a non-decreasing function of the threshold m_L*, i.e.,* $\frac{\partial \pi_0}{\partial m_L} \leq 0$.

Condition (i) states that the density of sellers is a monotone function of its production costs. In particular, more sellers are equipped with production technologies with lower marginal costs. Condition (ii) implies that the cost of serving one more seller is at least that of the subscription benefit the seller received after joining the platform. For example, it can be satisfied if $\frac{\partial C}{\partial s}(b, s) = V_s$. What is non-standard is that under the two conditions, the middleman can profit more by targeting fewer sellers. In other words, the middleman profits by acting as a gatekeeper that only attract the set of sellers with low marginal costs.

2.5. Welfare Analysis

In this section, we study how limiting choice can affect the welfare of search under the intermediary. Denote by $W^L(c_L)$ the welfare from search and trade on the platform.

$$\begin{aligned} W^L(c_L) = \mathbb{E}[\text{welfare from the platform}] =& \frac{\int\int_{v-t\geq z_L} v - m dG d\tilde{F}_L}{1-\rho_L} - \frac{c_L}{1-\rho_L} \\ =& \frac{\int\int_{v-t\geq z_L} t - m dG d\tilde{F}_L}{1-\rho_L} + z_L \end{aligned} \tag{2.6}$$

$W^L(c_L)$ consists of the transaction surpluses, less the total search costs, a function of c_L in any search equilibrium. Denote by m_L^* the optimal threshold selected by the platform such that

$$m_L^* = \arg\max \pi_0(m_L).$$

Then m_L^* is a function of c_L. The following theorem analyzes the social welfare results under the intermediation of a search platform.

Theorem 2.5.1. *Suppose that*

$$\frac{\partial m_L^*}{c_L} \geq 0, \tag{2.7}$$

then $\frac{\partial W^L}{\partial c_L} \leq 0$.

Inequality (2.7) states that the optimal threshold strategy m_L^* is non-increasing in c_L, that is the lower the search cost a platform offers, the fewer firms it attracts to the platform in its optimal strategy. Theorem 2.5.1 then implies the social welfare generated by the middleman increases with lower search costs it provides to the buyer. The middleman can promotes efficiency in two ways. The first, which is standard, is to lower the costs of search on the platform. The second, which is less trivial, is to accrue demands from all buyers to the subscribed sellers with low production costs. Therefore, the surpluses from transactions increase as the middleman acts as a gatekeeper that restricts the set of available sellers to those more efficient ones.

2.6. Conclusion

This paper studies the search platform that facilitates search between buyers and sellers by lowering search costs. The price dispersion is intrinsic to the platform's threshold strategy that by setting membership fees, limits the set of available firms subscribed to the platform. Buyers can then search more efficiently with fewer searches and the price competition between those low-cost sellers is more fierce, which adds the benefit of lower prices to buyers subscribed to the platform. Moreover, the search platform can also promote social welfare by lowering buyers' search costs and accrues demands to sellers with lower production costs.

CHAPTER 3 : Intermediated Bargaining In Networks with a Matchmaker

3.1. Introduction

Many goods and services (real estate, automobiles, jobs, etc) are traded in decentralized markets where traders search or wait for their counterparties and prices are determined via bargaining. Trading opportunities are largely restricted by social relationships, geographic dispersion, information opportunities, technical compatibilities, etc, which can generate considerable search costs. Traders often rely on intermediaries (brokers) to reduce the search costs. In the U.S. labor market for example, more than 90% of companies and 40% of employees looking for their first job (or those reentering the job market) use staffing agencies, generating annual revenues of $129.6 billion in 2014 according to the American Staffing Association.

Matchmaking middlemen also offer their professional expertise in proposing high-quality matches. Wine brokerage, a poorly known yet quite regulated activity, accounts for about 60% of bulk table and local wine transactions and for about 80% of the AOC exchanges in France. Wine is not a standardized product. A wine broker will regularly visit growers he is in contact with to taste and collect samples and once the sample fits the needs[1] of the wine buyer who hires him, the broker will bring together the grower and the buyer for negotiation. A broker's expertise is based on his specialization in a specific wine production area and on knowledge of their customers. On average, a broker maintains more or less strong ties with 158 buyers and 49 growers, according to a survey conducted on Languedoc-Roussillon wine brokers (Baritaux et al., 2006).

In other businesses, matchmaking middlemen such as talent agents intermediate most transactions. In the U.S. in 2010, 664 out of 1728 NFL professional players were represented by top 7 sports agents in the field. Moreover, the concentration of the matchmaking business in those fields endows the matchmaking intermediaries with significant market power. For

[1] Such as quality, flavor and geographic regions. In France, depending on the region where they are situated, wine buyers don't have the same trade practices.

example, the distribution of footballer representations are highly skewed between intermediaries. In five major European championships, half of over 2400 football players from the main European leagues are managed by 83 agents or agencies and a quarter are represented by just 24 of them (Poli and Rossi, 2012).

Common to the examples above is an intermediary whose business is to match buyers and sellers who subsequently negotiate the price. Unlike intermediaries in other settings, these

- do not hold inventories or set transaction prices. They profit from facilitating a match between buyers and sellers. The sale price is determined by negotiation between the buyer and the seller.
- They do not act as marketmakers who set bid and ask prices at which traders can buy and sell for their own account.

In contrast to the extensive literature on intermediated bargaining or trading models where middlemen act as traders or marketmakers, there has been little discussion on the role of a matchmaking middleman in a decentralized bargaining setting. This paper is, to my knowledge, the first to introduce a finite model of matchmaking middleman in a buyer-seller bargaining network after Yavaş (1994, 1992a,b), which discuss a one-buyer-one-seller search model with a middleman facilitating search.

Given the prevalence of networks in modeling bilateral bargaining, I consider a bipartie graph that determines which pairs of buyer and seller can bargain over the prices of heterogeneous goods. Each seller has a single unit of an indivisible good to trade and each buyer has unit demand. Feasible trading opportunities are represented by links. Associated with each link is a probability that corresponds to the chances of that buyer-seller pair finding each other without an intermediary. In each time period a feasible buyer-seller pair is selected at random (referred to as being matched by nature) or they are matched by a middleman to negotiate a deal. Among them, one trader is designated to make a take-it-or-leave-it offer. If the offer is accepted, the pair exit the market with the share agreed, less commission fees paid to the middleman if matched by the middleman. If the offer is rejected, the pair remain in the market for the next period without any payments to anyone including the middleman. Moreover, a feasible buyer-seller pair can always walk away from the middlemen's match, and wait for trading opportunities to arrive at random. Before the start of the next time period, the exiting pair are replaced by their clones at the exact same positions of the network or they are never replaced in the following periods. The two extremes are referred to as the with-replacement assumption (Manea, 2011) and no-replacement assumption (Abreu and Manea, 2012a,b) in prior literature. This paper generalizes both cases by allowing exiting players to be replaced by clones with positive probability.

As is well known from prior work, bargaining outcomes largely depend on network structure and the matching process associated with it: how many suppliers with higher quality materials are accessible to the buyer, how often an impatient landlord receive a purchase offer, etc. In my setting with a strategic matchmaking middleman, the following questions arise: which pairs will the middleman select and what is the impact on the surplus of buyers and sellers; how would intermediation in network affect the efficiency of bargaining outcomes? To answer these, one must understand what commission fees the middleman posts to both sides and what matching processes the middleman selects under the chosen commission.

Similar to the two-sided platform literature (Rochet and Tirole, 2003, 2006; Armstrong, 2006), which side should pay and how much should be paid to the middleman determine the middlemen's success. The mainbody of the paper assumes that the middleman has the freedom to charge a percentage of the transaction price on both sides. That is, both buyer and seller pay a percentage of the sale price. In labor markets, headhunters are paid a percentage of employee's salary by the employer (buyers) and job seekers (sellers) get access to the recruiters' services free of charge. In real estate, depending on whom the brokers represent, a percentage of the sale price is paid mostly by the sellers but sometimes by buyers if they sign a buyer broker agreement containing clauses that will compensate the brokerage for the fee it is due less the amount paid by the seller. In the professional sports, payment rules to the intermediaries vary a lot (Brocard et al., 2016). For example, the players associations in the North-American closed sport leagues have introduced Collective Bargarining Agreements that prohibit payments from clubs to sport agents (Shropshire et al., 2016). Thus, sport agents are paid by their principals - the players in the North American professional leagues, such as NBA and NFL. On the other hand, a study by the European Commission in 2009 shows that the existing mechanisms for sports agents are fairly heterogeneous (agents paid by the player, agents paid by the club, or a mixed commission payment)[2]. To sum up, a commission appears to be the most popular scheme employed by an intermediary and it can take one of two forms.

1. "$\alpha + \beta$" commission fees: Charge α of the sales price to the seller and β to the buyer for each successful transaction, denoted by "$\alpha + \beta$". This includes $\alpha = 0$, or $\beta = 0$ as a special case.

2. "α *or* β" commission fees: Whoever makes an offer that is accepted, pays the middleman's commission. If the seller, the commission is α of the price, if the buyer it is β of the price.

The "$\alpha+\beta$" pricing where middleman charges on one or both sides of the market regardless of who makes the accepted offers are more commonly seen in pratice compared to "α *or* β". This paper focuses on the discussion of "$\alpha + \beta$", and compares the two pricing models at the end.

The other critical design problem for the intermediary is which buyer-seller pairs to match. I assume that buyer-seller links are given and not created by the intermediary. Rather, by selecting which pair of buyers and sellers to match, a middleman can determine how quickly or often a buyer (seller) can meet specific counter-parties. Prior work assumes exogenously specified matching probabilities(Abreu and Manea, 2012a), or endogenously determined by

[2] "Study on sports agents in the European Union". A study commissioned by the European Commission (Directorate-General for Education and Culture), November 2009.

the mass of traders of the same type (Diamond and Maskin, 1979; Gale, 1987; Nöldeke and Tröger, 2009). In this paper, agents can choose between an exogenously specified matching process (nature) or matches determined by an intermediary.

The analysis of bargaining games under intermediation relies on Markov Perfect Equilibria (MPEs) introduced by Fudenberg and Tirole (1991); Maskin and Tirole (2001), where the strategy profile is a function of the Markov states of the game. The state in this model is the network structure, along with the selection of a link and a proposer, and in my context, the commission. Despite a multiplicity of MPEs under commission rates α, β, this paper characterizes the unique MPE revenue of a monopoly matchmaking middleman with the optimal choice of commission.

The first economic insight the paper offers is how a middlemen affects the efficiency through redistributing the trading opportunities across agents. The efficiency of trade is the total discounted sum of surpluses. If v_{bs} is buyer b's value for seller s's good and c_s the opportunity cost of s, the surplus from trade will be $v_{bs} - c_s$. In the benchmark case without middlemen, efficiency is compromised by adverse selection and trading delays (Abreu and Manea, 2012a,b). In my model the presence of a middleman reduces trading delays relative to nature. Moreover, when exiting players are replaced with clones, the middleman always selects the links with the highest surplus in any equilibrium without the competition from nature. This fails to be true when the middleman competes with nature.

Relative to a middleman focused on maximizing efficiency, the profit-maximizing middleman distorts trades in two ways under the no-replacement setting. The first is to restrict trading opportunities by never selecting feasible buyer-seller pairs whose "value-to-cost" ratio $\frac{v_{bs}}{c_s}$ is too small. The second, is among the feasible pairs with sufficiently large $\frac{v_{bs}}{c_s}$, favoring those buyer-seller pairs whose v_{bs} is large. Thus, the profit-maximizing middleman does not necessarily favor pairs for which the gains from trade, $v_{bs} - c_s$, are large. The intermediation inefficiency stems from imperfect price discrimination of the matchmaking middleman. Under the with-replacement assumption, the monopoly middleman can perfectly price discriminate by calculating the efficient links and the commission rates α, β accordingly in order to extract all the surpluses from trade, and she is able to do so because the equilibria are stationary and the network structure remains unchanged forever. Under the no-replacement assumption however, the middleman matches sequentially until the market clears and charges the same percentages of sales prices to all matched pairs. Because of her lack of ability to perfectly price discriminate, the middleman cannot extract all surpluses from trade and can deviate from inefficient matchings to the profit-maximizing ones.

The interplay between efficiency, matching and competition with nature is novel to the literature on intermediation in networks. Blume et al. (2009) show the efficiency of intermediaries that announce bid prices for sellers and ask prices for buyers simultaneously. Choi et al. (2017) prove the existence of efficient equilibria in a network where all nodes along the path of intermediaries post prices simultaneously, in addition to showing examples of inefficient equilibria where trade does not occur. Polanski (2007) studies a model in which a maximum number of pairs of connected players are selected to bargain every period. As a consequence, efficiency is not an issue in Polanski's model and in equilibria, all matched pairs reach agreement immediately. In another recent paper on intermediation in networks, Manea (2015) characterizes the intermediation inefficiency that stems from hold-ups created by competition between layers and paths of the network where middlemen form chains of traders. Unlike them, intrinsic to the inefficiency results of my model is the matching technology endogenized by the profit-maximizing middleman and the imperfect price discrimination toward the commission the middleman must commit to for all trades before bargaining starts.

This paper also characterizes the impact of intermediation on players' surpluses from trade. Under the "$\alpha + \beta$" pricing, assuming that traders have no outside options but to search and trade through the middleman, the expected surplus of all buyers is zero in all networks. Thus, all of the gains of trade accrue to either the middleman and the sellers. This is true even if supply exceeds demand or if buyers are more patient than sellers. Under the "α *or* β" pricing, depending on the underlying network structures, buyers can enjoy positive gains from intermediated bargaining.

3.1.1. Literature Reviews

My work builds on the intersection between bargaining models (Rubinstein and Wolinsky, 1985; Gale, 1987; Binmore and Herrero, 1988; Abreu and Manea, 2012b; Manea, 2015) and two-sided marketplaces with intermediaries (Rochet and Tirole, 2003, 2006). It differs from prior work in the following ways.

1. Matchmaking middlemen

The literature on decentralized trade in networks has focused on the exchange of goods without intermediation (Abreu and Manea, 2012a,b; Manea, 2011; Corominas-Bosch, 2004; Kranton and Minehart, 2001; Polanski, 2007). In particular, the two-sided versions of Abreu and Manea (2012a,b); Manea (2011) are special cases of my paper under the with-replacement and no-placement assumption.

Moreover, most of the literature on intermediation in networks investigates intermediaries as traders or resellers in the network consisting of paths where sellers are linked to buyers directly, or indirectly via a chain of intermediaries (Manea, 2015; Kotowski and Leister, 2018; Manea, 2017a, 2015; Choi et al., 2017; Nguyen et al., 2016).

Brocard et al. (2016) study a sports agent facilitating the negotiation between a buyer (club) and a seller (professional athelete) with fixed bargaining payoffs, whereas in my setting, the middleman can facilitate searches across a network of buyers and sellers and the outcomes of bargaining are endogenized by matching and competition with nature.

2. Matching technologies

Abreu and Manea (2012a,b) assume exogenously specified matching probabilities $p_{ij}(G')$ for any subnetwork $G' \subseteq G$ with full support. In particular, the two-sided versions of Abreu and Manea (2012a,b) are special cases of this paper when the middleman is entitled with zero influence over the matching. Shimer and Smith (2000); Manea (2017b); Nöldeke and Tröger (2009); Diamond and Maskin (1979); Gale (1987) assume the probability of player i meeting a type j is given by the proportion of traders j in the market. In non-cooperative group bargaining models, Nguyen (2015) assumes that the probability of a coalition S being selected at random with i being the proposer is $\Pr(S, i) = \alpha_i \cdot P(S)$.

Abreu and Manea (2012a) also introduces an alternative matching technology where each individual is endowed with a bargaining opportunity and then proceed to contact a partner to negotiate over a deal. In reality, the set of neighbors visible to each trader subject to manipulation by the middleman (such as eBay, headhunters, wine brokers), which is closer to the setting discussed in this paper.

3. Steady state and new entry

As shown in the paper, the efficiency results are profoundly different under the with-replacement and the no-replacement assumptions. The replacement assumption is ubiquitous in matching and bargaining models to main the steady states of the network structure (Rubinstein and Wolinsky, 1985; Diamond and Maskin, 1979; Shimer and Smith, 2000; Duffie et al., 2005; Atakan, 2006; Satterthwaite and Shneyerov, 2007; Manea, 2011; Lauermann, 2013; Lauermann and Nöldeke, 2015). Among those, some assume players are repalced by clones at the exact same position of networks (Manea, 2011), while some assume an inflow of new players with costly entry (Manea, 2017b), or costless entry and the probability of being matched with counterparties is proportional to the size of her peers with the same type (Shimer and Smith, 2000). Abreu and Manea (2012a) study the bargaining networked game under the no-replacement assumption.

4. Heterogenous goods.

Prior worksAbreu and Manea (2012a,b); Manea (2011); Corominas-Bosch (2004) study the exchange of goods with unit surplus and focus on the impact of network structure on the efficiency of bargaining (the maximum cardinality of matchings in the network) as agents become patient. The distinguishing feature of this paper is the endogenous matching technology and distorted equilibrium matchings selected by the profit-maximizing middleman who favors trades with higher values.

3.2. Model

3.2.1. Setup

Let B denote the set of buyers and S the set of sellers. Each seller s has a single unit of indivisible good for sale, with opportunity cost $c_s \geq 0$. Each buyer b has unit demand and her value for the good sold by seller s is v_{bs}. If buyer b pays a price p to seller s, the payoffs are $v_{bs} - p$ for the buyer and $p - c_s$ for the seller.

Buyers and sellers are linked by a network $G_0 = (V, E)$ with the set of vertices $V \subseteq B \cup S$ and the set of edges (or links) $E \subseteq \{(b, s)|b \in B, s \in S\}$ such that $(b, s) \in E$ whenever $(s, b) \in E$. A link (b, s) is interpreted as the feasible pair for trade. A network $G' = (V', E')$ is induced by V' if $E' = E \cap (V' \times V')$. We write $G \ominus V''$ for the subnetwork of G induced by the vertices $V \setminus V''$.

Exogenous matching technologies in the benchmark. In the benchmark case, every network has an associated probability distribution over links $(p_{ij}(G))$, $\forall(i, j) \in E$. Given G, $p_{ij}(G)$ is the probability that player i in the network G meets with one of her neighboring traders j and propose a division over the surplus $v_{ij} - c_j$ (if i is a buyer, or $v_{ji} - c_i$ if i is a seller) to j. The random matching technologies can be arbitrary, for example, the uniform distribution over all links $p_{ij}(G) = \frac{1}{2|E|}$.

Matching process with middleman. A long-lived middleman can match buyers and sellers that would otherwise have difficulty finding each other. Consider the following dynamic matching process. Each period $t = 1, 2, \ldots$, with probability λ, the middleman selects a buyer-seller pair from the current network G_t at time t and assigns one of them to make a take-it-or-leave-it offer to the other trader; with probability $1-\lambda$, a buyer-seller pair is selected at random according $\{p_{ij}(G_t)\}_{ij\in E_t}$. λ can be interpreted the market power of a middleman: λ of matching and bargaining opportunities are brought by the matchmaking middleman while $1-\lambda$ of them are by chance. $\lambda = 0$ corresponds to the the benchmark case without the middleman in prior literature.

Commission fees. Both the buyer and the seller matched by the middleman pay a percentage of the sale price to the middleman upon reaching an agreement. α is the commission rate on the seller side and β on the buyer side, announced before any bargaining game starts. Moreover, no one gains from no trade, including the middleman.

Replacement assumption. In prior work, two polar cases are studied: either exiting players are replaced by clones at the exact same positions of the network, or they exit the market without any replacement. This paper makes the following extension: with probability $0 \leq \gamma \leq 1$ the exiting pair who have reached an agreement are replaced at the end of this period, by a new pair at the same positions in the network. Otherwise, they are never replaced by any clones in the following periods. Abreu and Manea (2012b) is a special case of $\gamma = 0$ and Manea (2011) a special case of $\gamma = 1$.

Assume all sellers share a common discount factor $0 < \delta_s < 1$, while all buyers share a common $0 < \delta_b < 1$. The middleman's discount factor is $0 < \delta_M < 1$.

3.2.2. Timing

We consider the following infinite horizon bargaining game generated by the network and the middleman's choice of commission and matchings. Denote this pricing model by **"$\alpha+\beta$" commission**.

Stage 1 At time $t = 0$, the middleman announces a non-discriminating percentage commission rates $\alpha, \beta \geq 0$.

Stage 2 At time $t = 1, 2, \ldots$, if no feasible pairs exist, the game ends. Otherwise,

(i) Matching a buyer-seller pair: With probability λ, the middleman selects a feasible pair b, s and assigns one of them to make a take-it-or-leave-it offer. Otherwise, a single link is selected at random according to $\{p_{ij}(G_t)\}_{ij \in E_t}$.

(ii) Bargaining: The matched pair (b, s) meet, with one of them proposing to the other player (the responder) specifying a transfer (or price) p. If the offer is accepted, they trade and exit the game with buyer's payoff being $v_{bs} - (1 + \beta)p$ and the seller's being $(1 - \alpha)p - c_s$ if matched by the middleman. If matched by nature, buyer b gets $v_{bs} - p$ and the seller $p - c_s$.

(iii) When the pair exit the market with an agreement, with probability γ they are replaced by clones at the same positions in the network and there are no clones otherwise. If the offer is rejected, the matched pair dissolves and the two traders return in the next period.

Under the "$\alpha + \beta$" commission pricing, the middleman commits to non-discriminating commission rates α, β on both sides. This includes special cases where the commission rate on one side is set to 0: eBay charges a 3% of sale prices on seller side and 0% on buyer side; the real estate broker mostly charges a 6.5% of the sale prices on seller side and 0% on buyer side. For headhunters, regardless of who makes the take-it-or-leave-it offer, the commission fee is always charged on the buyer side (the employers).

3.2.3. Markov Perfect Equilibrium

There are three types of histories for any $t \geq 2$. Denote by h_t a complete history of the game up to (not including) time t, which includes: the choice of commission α and β, a sequence of $t - 1$ pairs of proposers and responders in G with corresponding proposals and responses, payment to the middleman if applicable and whether the pair exiting the market are replaced with clones. The history h_t uniquely determines the set of buyers and sellers $V(h_t)$ still present at time t. Denote by $G(h_t)$ the subnetwork of G_0 induced by $V(h_t)$. Let $\mathscr{H}$ be the set of any possible complete histories and $\mathscr{G}$ the set of subnetworks of G_0 induced by any complete histories, i.e., $\mathscr{G} = \cup_{h \in \mathscr{H}} G(h)$. The history $(h_t; \mathbb{1}_M)$ denotes the history h_t followed by the identity of the matching maker ($\mathbb{1}_M = 1$ if it is the middleman's turn to select the pair at time t and 0 otherwise). The history $(h_t; \mathbb{1}_M; i \to j)$ consists of h_t followed by nature or the middleman selecting i to propose to j, and $(h_t; \mathbb{1}_M; i \to j; x)$ includes additionally the proposed transfer $x \in R$ made by i to j.

The equilibrium analysis is restricted to Markov Perfect Equilibria (MPEs) in Stage 2. A strategy σ_i for player i specifies, for all histories $(h_t; \mathbb{1}_M; i \to j)$ the offer $\sigma_i(h_t; \mathbb{1}_M; i \to j)$ i makes to j conditional on the history $(h_t; \mathbb{1}_M; i \to j)$, and the response $\sigma_i(h_t; \mathbb{1}_M; j \to i; x)$ that i gives to j conditional on $(h_t; \mathbb{1}_M; i \to j)$ and the offer x proposed by j. A matching strategy σ_M for the middleman specifies the pair to select for bargaining given the histories $(h_t; \mathbb{1}_M = 1)$. A Markov strategy is defined on the payoff relevant Markov states: the subnetwork of players who did not reach agreement by that time, and announced commission α, β, the matchmaker's identity (nature or the middleman), the selection of a link and the proposer as well as the offer made by the proposer. In particular, for all complete histories h_t, and $(i, j) \in E(h_t)$, $\sigma_i(h_t; \mathbb{1}_M; i \to j)$ only depends on the network $G(h_t)$, the identity of the proposer i and responder j, and the identity of the matchmaker $\mathbb{1}_M$, and i's response $\sigma_i(h_t; \mathbb{1}_M; j \to i; x)$ depends additionally on the offer x made by the proposer j. A Markov perfect equilibrium (MPE) is a subgame perfect equilibrium in stationary Markov strategies[3]. We first establish the existence of MPEs.

Denote by $\Gamma^{\alpha+\beta}(G)$ the Stage 2 bargaining and matching games under the choice of "$\alpha+\beta$" commission.

Theorem 3.2.1. *For any $\alpha, \beta \geq 0$, there exists a Markov perfect equilibrium for both the*

[3]For $t, t' = 1, 2, \ldots, N$, $\sigma_i|_{h_t} = \sigma_i|_{h'_{t'}}$ if $G(h_t) = G(h'_{t'})$.

bargaining and matching game $\Gamma^{\alpha+\beta}(G)$*.*

Since the two-sided version of Abreu and Manea (2012b) can be seen as a special case of mine when $\lambda = 0$, MPE payoffs might not be unique and can be sensitive to the network structure, the discount factors etc. The following proposition however, characterizes the uniqueness of buyers' payoffs under certain conditions.

Proposition 3.2.2. *For any* $\alpha + \beta > 0$*, if*

$$(1 - \lambda)p_{bs}(G) = 0, \forall G \subseteq G_0,$$

then the expected payoff of any buyer b is zero in any MPEs of $\Gamma^{\alpha+\beta}(G)$*. In particular, if one of the following two conditions is satisfied,*

1. $\lambda = 1$*, that is middleman M is the monopoly matchmaker;*

2. $p_{bs}(G) = 0$ *for any buyers in any subnetwork of* G_0*.*

buyers gain zero surplus in any MPEs.

Proposition 3.2.2 describes a scenario when all surpluses from trades accrue to either the middleman and the sellers: if buyers have no bargaining power under nature, neither would they under the perfect intermediation by the middleman. This is true even if the buyers are sufficiently patient with δ_b approximating 1 or buyers are on the short side of the market. What underlies the result is the difference between prices proposed by buyers and sellers: the ask-prices proposed by the sellers are often higher than the purchase-prices offered by the buyers. Therefore, the middleman treats buyer and seller asymmetrically when deciding on assigning which side to make a take-it-or-leave-it offer. Before jumping to the proof of Proposition 3.2.2, we first present a description of the MPE strategy profile under $\lambda = 1$, in particular the equilibrium transfers proposed by both sides.

Lemma 3.2.3. *Given the network G, in any subgames with the initial network G, there exist* $u_b, u_s \geq 0, \forall b, s \in G$*, s.t. the MPE strategy profile is in the form of the following:*

- *Buyer b offers a bid-price $p = \min\{\frac{c_s+\delta_s u_s}{1-\alpha}, \frac{v_{bs}-\delta_b u_b}{1+\beta}\}$ whenever chosen by the middleman to propose to seller s.*

- *Seller s proposes an ask-price $p = \max\{\frac{c_s+\delta_s u_s}{1-\alpha}, \frac{v_{bs}-\delta_b u_b}{1+\beta}\}$ whenever chosen by the middleman to propose to buyer b.*

- *When buyer b responds to the offer p from seller s: she accepts any ask-price $p < \frac{v_{bs}-\delta_b u_b}{1+\beta}$, rejects any $p > \frac{v_{bs}-\delta_b u_b}{1+\beta}$, and is indifferent between accepting and rejecting a price of $p = \frac{v_{bs}-\delta_b u_b}{1+\beta}$.*

- *Similarly, when seller s responds to the purchase offer p from buyer b: he accepts any offer s.t., $p > \frac{c_s+\delta_s u_s}{1-\alpha}$, rejects any $p < \frac{c_s+\delta_s u_s}{1-\alpha}$, and is indifferent between accepting and rejecting an offer $p = \frac{c_s+\delta_s u_s}{1-\alpha}$.*

Following Lemma 3.2.3, it is straight forward to notice that within any buyer-seller pair (b, s), the ask-price proposed by seller s is at least the amount offered by buyer b, even if there is less demand than supply or sellers are much more anxious to close the trade. Moreover, no matter who is assigned as the proposer, the necessary condition for trade between buyer b and seller s to occur is the same:

$$v_{bs} - \delta_b u_b \geq \frac{1+\beta}{1-\alpha}(c_s + \delta_s u_s), \tag{3.1}$$

i.e., the value v_{bs} is sufficiently large. Here u_b, u_s are the expected payoffs for buyer and seller given the network. Therefore, within any pair that can potentially reach an agreement, assigning the seller to be the proposer generates at least the same profit to the middleman as choosing the buyer side. Proposition 3.2.2 implies buyers gain no additional bargaining power from the intermediary except that some of them might be selected to bargain more often with the middleman than without. Still, their payoffs are 0 as any positive gains from bilateral bargaining result from the ability to make a take-it-or-leave-it offer.[4]

[4] If a trader can only respond to a offer but is unable to make one, any equilibrium offer indifferent between accepting and rejecting it, will eventually amount to 0 when the discount factor is strictly positive.

3.3. Inefficiency of MPEs

This section focuses on the optimal choice of α, β at time 0 and the resulting equilibrium matching and bargaining outcomes. Moreover, what is the impact of intermediation by a profit-maximizing middleman on the social welfare especially when the middleman acts as a matchmaker ?

Eficiency is defined as the optimal matching created by a welfare-maximizing central planner who in each time period, selects the pair that maximizes the expected total discounted sum of surpluses from trades with some discount factor $\delta > 0$. If $\gamma = 0$ where exiting players are never replaced by clones, the set of efficient matchings in the bargaining game is any maximum weight matching on the entire network G_0, with the weight of each link being the transaction surplus $v_{bs} - c_s$ (Abreu and Manea, 2012a,b; Polanski, 2007). Assuming $\gamma = 1$, the MPEs are stationary and the efficiency of the intermediated bargaining game is defined as the selection of efficient links with highest surplus $v_{bs} - c_s$ (Manea, 2011).

Theorem 3.3.1. *Suppose* $\lambda = 1$*.*

1. *If* $\gamma = 0$ *(no-replacement assumption),*
   - *The highest possible revenue the middleman can achieve is,*

     $$\max_{\alpha,\beta} \frac{\alpha + \beta}{1 + \beta} \Pi(G_{\alpha,\beta}),$$

     where $G_{\alpha,\beta} = (V, E_{\alpha,\beta})$ *is the subnetwork of* G_0 *with* $E_{\alpha,\beta} = \{(b,s) : v_{bs} \geq \frac{1+\beta}{1-\alpha} c_s\}$*.* $\Pi(G_{\alpha,\beta})$ *is the maximum total discounted sum of buyers' values* v_{bs} *of any feasible matchings in network* $G_{\alpha,\beta}$*.*
   - *For any* $\epsilon > 0$*, the middleman can choose* α^*, β^* *at time* 0*, s.t., her revenue is at least* ϵ*-close to the maximum possible revenue the middleman could possibly attain, i.e.,*

     $$\pi^{\alpha^*+\beta^*}(G_0) \geq \max_{\alpha,\beta} \frac{\alpha + \beta}{1 + \beta} \Pi(G_{\alpha,\beta}) - \epsilon.$$

The middleman's matching outcomes are the maximum weight bipartite matching of $G_{\alpha,\beta}$ with weight of each link being v_{bs} rather than $v_{bs} - c_s$.

2. *If $\gamma = 1$ (with-replacement assumption), then for any $\epsilon > 0$, the middleman can choose α^*, β^* at time 0, s.t., her revenue in any MPEs of Stage 2 satisfies*

$$\pi^{\alpha^*+\beta^*}(G_0) \geq \max_{b,s}(v_{bs} - c_s) - \epsilon,$$

with the optimal matched pairs always being the most efficient links (the highest $v_{bs} - c_s$). The optimal choice of α^*, β^* satisfies $v_{bs} - c_s = \frac{\alpha+\beta}{1+\beta} v_{bs}$ where (b, s) is the most efficient link.

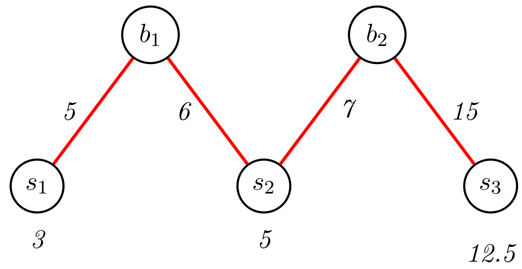
Moreover, there are no trading delays along the equilibrium paths in both scenarios.

Under the with-replacement assumption, Theorem 3.3.1 shows that for the optimal choice of α^*, β^*, middleman promotes efficiency by selecting links with the highest surplus. This efficiency result holds for any $\delta_M, \delta_b, \delta_s \in (0, 1)$, a sharp constrast to the prior work (Abreu and Manea, 2012a,b) where efficiency of MPEs is either unattainable, or can be attained by a modification of punishment regimes adopted by the traders. What drives the efficiency result under the with-replacement assumption is perfect price discrimination. In the stationary world where only efficient links matters rather than the entire network structures, the middleman can always select links with the highest surplus under the "customized" commission rates α, β such that trade always occurs and the surplus from trade can be fully extracted by the middleman.

When $\gamma = 0$, the middleman distorts trade in two ways. First, the middleman only targets matchings on a subnetwork with higher "value-over-cost" ratio $\frac{v_{bs}}{c_s}$. $v_{bs} \geq \frac{1+\beta}{1-\alpha} c_s$ is implied by Inequality (3.1): for trades to occur under intermediation, the ratio must be high enough for both sides to afford the two-sided "tariffs" posed by the middleman. Unlike the stationary network structures under the with-replacement assumption $\gamma = 0$, the network evolves and the value of matchings feasible under the commission rates determines the success of the middleman. If matching pairs with large surplus $v_{bs} - c_s$ yet small ratio $\frac{v_{bs}}{c_s}$ (this is possible when the production cost c_s is also large), for the efficient trade to be feasible under intermediation, the commission rates must be restrained from setting too high. As α, β are uniform for all trades, the middleman can potentially generate less revenue from matching pairs with lower surplus $v_{bs} - c_s$ but high ratio $\frac{v_{bs}}{c_s}$ than she could have earned by setting higher commission rates α, β. Sometimes targeting fewer feasible pairs under higher commission rates can be more profitable than enabling more trades (including the more efficient ones) to occur under a low commission (see Example 3.3.2). What underlies targeting a subnetwork is the inability to perfectly extract all surplus from trades. Example 3.3.2 illustrates the targeting strategy in a two-buyer-two-seller network where in equilibrium, only one link is selected with all of its surplus extracted by the profit-maximizing middleman. Secondly, the middleman selects the matchings on the subnetwork with high buyers' values v_{bs} rather than the transaction surplus $v_{bs} - c_s$. This is implicitly derived from the fact that in MPEs with a monopoly matchmaker, the transfer of each trade is proportional to the value v_{bs}. Therefore among all feasible trades under intermediation, the middleman favors those with higher transfers, i.e., higher values v_{bs}. The ϵ error in the statement of Theorem 3.3.1 is due to the fact that, by sacrificing a small ϵ in revenue, the middleman can avoid any possible trading delays where the responder is indifferent between accepting and rejecting an offer.

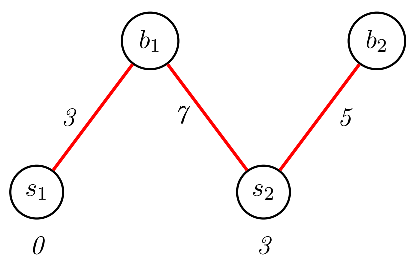
The following two examples illustrate how distorted matchings can deviate from the efficient ones when matching pairs with higher values v_{bs} in Example 3.3.1 and when restricting trades in Example 3.3.2.

Example 3.3.1 (Ineffiency induced by middleman)**.** *Consider the bargaining game with* $\lambda = 1, \gamma = 0$ *in the following two-buyer-three-seller network. The values associated with each link are* $v_{b_1s_1} = 5, v_{b_1s_2} = 6, v_{b_2s_2} = 7, v_{b_2s_3} = 15$*. Sellers' productions costs are respectively* $c_{s_1} = 3, c_{s_2} = 5, c_{s_3} = 12.5$*.*



There are 3 candidates $\frac{6}{5}, \frac{7}{5}, \frac{5}{3}$ *for setting* $\frac{1+\beta}{1-\alpha}$*. For any subnetwork with* $\{(b,s) : v_{bs} \geq \frac{6}{5}c_s\} = E$*,* $\frac{\alpha+\beta}{1+\beta}\Pi(G) = \frac{1}{6}(6+15) = 3.5$*. With* $\{(b,s) : v_{bs} \geq \frac{7}{5}c_s\} = \{(b_1,s_1),(b_2,s_2)\}$*, the surplus from its maximum weight matching is* $\frac{\alpha+\beta}{1+\beta}\Pi(G) = \frac{2}{7}(5+7) \approx 3.4$*. Among* $\{(b,s) : v_{bs} \geq \frac{5}{3}c_s\} = \{(b_1,s_1)\}$*, the surplus from its maximum weight matching is* $\frac{\alpha+\beta}{1+\beta}\Pi(G) = \frac{2}{5}(5) \approx 2$*. In any profit-maximizing equilibrium outcome,* $(b_1,s_2),(b_2,s_3)$ *are matched by the middleman. However, the most efficient matching is* $(b_1,s_1),(b_2,s_3)$*.*

Example 3.3.2 (Ineffiency induced by middleman)**.** *Consider the bargaining game with* $\lambda = 1, \gamma = 0$ *in the following two-buyer-two-seller network. The values associated with each link is* $v_{b_1s_1} = 3, v_{b_1s_2} = 7, v_{b_2s_2} = 5$*. Sellers' productions costs are respectively* $c_{s_1} = 0, c_{s_2} = 3$.



There are three threshold $\frac{5}{3}, \frac{7}{3}, \infty($ *or* $\alpha = 1)$*. For any subnetwork with* $\{(b,s) : v_{bs} \geq \frac{5}{3}c_s\} = E$*,* $\frac{\alpha+\beta}{1+\beta}\Pi(G) = \frac{2}{5}(3+5) = 3.2$*. If however* $\{(b,s) : v_{bs} \geq \frac{7}{3}c_s\} = \{(b_1,s_1),(b_2,s_2)\}$*, and the maximum surplus extract by the middleman is* $\frac{\alpha+\beta}{1+\beta}\Pi(G) = \frac{4}{7}(7) = 4$*. If setting* $\alpha = 1$*, the maximum surplus extract from* (b_1,s_1) *is 3. In any profit-maximizing equilibrium outcome,* (b_1,s_2) *is matched by the middleman. However, the most efficient matching is* $(b_1,s_1),(b_2,s_2)$*. In this case, there is a tradeoff between low commission rates* α, β *and high total sum of values* v_{bs}*. While the aggregate buyers' valuation of the matching* $(b_1,s_1),(b_2,s_2)$

is larger than that of (b_1, s_2)*, for trades in the former to occur, the middleman must commit to lower commission rates that those necessary for trade to occur in the latter. Moreover, setting lower commission rates does not necessarily yields higher revenue for the middleman.*

Examples 3.3.1 and 3.3.2 illustrate two sources of mismatch a monopoly matchmaking middleman can introduce by intermediation under the no-replacement assumption. In contrast, under the with-replacement assumption, a link with the highest surplus $v_{bs} - c_s$ in the network is always selected by the monopoly middleman in each time period. However, this no longer holds when the middleman competes with nature when $0 < \lambda < 1$. The following example illustrates an alternative distortion the middleman introduces to matchings on a star network when competing with nature.

Example 3.3.3 (Star-n Network)**.** *Consider a one-buyer-n-selller network. Each seller* s*'s production cost is* c_s *and buyer* b *values the product sold by seller* s *at* $v_{bs} \geq c_s > 0$*, with at least one pair generating positive surplus. Suppose* $p_{bs} = 0, \forall s$*, i.e., under nature the seller proposes to the middleman and the probability of seller* s *meeting with buyer* b *is* p_{sb}*. Assuming* $\gamma = 1$*, that is any exiting players are replaced with clones. The middleman announces the commission rates* α, β *to be charged on both sides.*

In the following, we will show that, the optimal choice of α, β *and the stationary matching strategy adopted by the middleman in the MPE is to select the pair* (b, s^*) *such that,*

$$s^* = \arg\max \frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s).$$

The middleman's profit-maximization problem is to solve:

$$\begin{aligned}\pi_M(G) &= \lambda \max\{\max_{b,s} q_{bs}\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s), \max_{b,s} q_{sb}\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b)\} \\ &= \lambda \max_{b,s} q_{sb}\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b),\end{aligned}$$

where q_{bs} *is the probability of buyer* b *and seller* s *reaching an agreement under the intermediaton. Let* w_{bs} *be the probability of matching* s *and* b *and assigning* b *to be the proposer* *by the middleman. For each pair of* (b, s)*, if* $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > \delta_b u_b$*, the difference* Δ *between selecting* b *proposing to* s *(*$b \to s$*) and selecting* $s \to b$ *is,*

$$\begin{aligned}
\Delta =& q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (q_{bs}\gamma + 1 - q_{bs})\delta_M \pi_M(G) \\
&- q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] - \big(q_{sb}\gamma + 1 - q_{sb}\big)\delta_M \pi_M(G) \\
=& \frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) - \frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) \\
<& 0
\end{aligned}$$

Hence $w_{bs} = 0$*. If* $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) < \delta_b u_b$*, no trades are feasible thus* $q_{bs} = 0$*. Therefore, in all cases we have*

$$w_{bs} q_{bs}(v_{bs} - \delta_b u_b - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) = 0.$$

It can be calculated that buyers' and sellers' equilibrium payoffs u_b, u_s *and the middleman's revenue satisfy:*

$$\begin{aligned}
\big(1-\delta_b\big)u_b =& \sum_s \lambda w_{bs} q_{bs}(v_{bs} - \delta_b u_b - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) \\
&+ \sum_s (1-\lambda) p_{bs} \max(v_{bs} - \delta_b u_b - (\delta_s u_s + c_s), 0) \\
=& (1-\lambda) \sum_s p_{bs} \max(v_{bs} - \delta_b u_b - (\delta_s u_s + c_s), 0) \\
=& 0,
\end{aligned}$$

$$\begin{aligned}
\big(1-\delta_s\big)u_s =& \lambda \sum_b w_{sb} q_{sb}(\frac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s - \delta_s u_s) \\
&+ \sum_b (1-\lambda) p_{sb} \max(v_{bs} - \delta_b u_b - c_s - \delta_s u_s, 0), \\
\pi_M(G) =& \lambda \max_{b,s} \frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) = \lambda \max_{b,s} \frac{\alpha+\beta}{1+\beta} v_{bs}.
\end{aligned}$$

Moreover, for any s such that $w_{sb} = 0$,

$$u_s = \frac{(1-\lambda)p_{sb}(v_{bs} - c_s - \delta_b u_b)_+}{1 - \delta_s + (1-\lambda)p_{sb}\delta_s}.$$

For s s.t. $w_{sb} > 0$,

$$u_s = \frac{\lambda w_{sb}(\frac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s) + (1-\lambda)p_{sb}(v_{bs} - c_s - \delta_b u_b)}{1 - \delta_s + (1-\lambda)p_{sb}\delta_s + \lambda w_{sb}\delta_s}.$$

And one can verify that for both cases,

$$\frac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s - \delta_s u_s \geq 0 \iff \frac{1-\delta_s}{1-\delta_s + (1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s) \geq \frac{\alpha+\beta}{1+\beta}v_{bs},$$

which implies the middleman's revenues in any MPEs are bounded $\pi_M(G) \leq \frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s)$*. We show that the upper bound can be arbitrarily approximated by the choice of* α, β*. Let*

$$s^* = \arg\max_s \frac{1-\delta_s}{1-\delta_s + (1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s)$$

For any $\epsilon > 0$*, let* α, β *satisfy*

$$\frac{1-\delta_s}{1-\delta_s + (1-\lambda)\delta_s p_{s^*b^*}}(v_{b^*s^*} - c_{s^*}) - \epsilon = \frac{\alpha+\beta}{1+\beta}v_{b^*s^*}$$

Then from middleman's perspective, given the choice of α, β*, the optimal seller to select is:*

$$\begin{aligned} s' = \arg\max_s & q_{sb}v_{bs} \\ s.t. \ & \frac{1-\delta_s}{1-\delta_s + (1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s) \geq \frac{\alpha+\beta}{1+\beta}v_{bs} \end{aligned}$$

And

$$\frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{s^*b^*}}(v_{b^*s^*} - c_{s^*}) \geq \frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{s'b}}(v_{bs'} - c_{s'}) \geq \frac{\frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{s^*b^*}}(v_{b^*s^*} - c_{s^*}) - \epsilon}{v_{b^*s^*}}v_{bs'}$$

hence

$$\frac{v_{bs'}}{v_{bs^*}} \leq 1 + \frac{\epsilon}{\frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{s^*b^*}}(v_{b^*s^*} - c_{s^*}) - \epsilon},$$

thus for sufficiently small $\epsilon > 0$*, we have* $s' = s^*$*.*

As shown in Theorem 3.3.1 and Example 3.3.1, the middleman can effectively eliminate trading delays but not necessarily select the most efficient matchings. As nature comes into play with $\lambda < 1$ and competes with the middleman, the middleman has to take into account the random matching technology p_{sb} implemented by nature when selecting which pair to match more often. As shown in Example 3.3.3 of the star-n network, the matched pair (b, s^*) is the one with the highest distorted welfare contaminated by nature's random matching p_{sb},

$$s^* = \arg\max \frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s).$$

When $v_{bs} = v$ for all s, the middleman selects the link with the smallest probability p_{sb} of being matched under nature. In other words, the middleman profits most from matching the disadvantageous seller in the homogenous case. When v_{bs} is heterogenous, middleman does not necessarily select the pair with highest $v_{bs} - c_s$. The distortion vanishes as middleman's market power $\lambda \to 1$. In particular, one can prove that in the Example 3.3.3, there exists λ^*, such that, for any $\lambda > \lambda^*$, i.e.,

$$\arg\max_s \frac{1-\delta_s}{1-\delta_s+(1-\lambda)\delta_s p_{sb}}(v_{bs} - c_s) = \arg\max_s v_{bs} - c_s.$$

For small λ however, the middleman can deviate from the inefficient matching.

The following corollary shows that under the "$\alpha + \beta$" pricing, the revenues from charging on both sides can be realized by charging on only one side of the market.

Corollary 3.3.2. *Suppose* $\lambda = 1, \gamma = 0$*. Given* ϵ *in Theorem 3.3.1. Let*

$$\theta = \arg\max \theta\Pi(G_\theta) - \epsilon$$

where $G_\theta = (V, E_\theta), E_\theta = \{(b,s) : v_{bs} \geq \frac{1}{1-\theta} c_s\}$*. Then for any* α, β *combination such that* $\theta = \frac{\alpha+\beta}{1+\beta}$*, the revenue in MPEs under* α, β *is* ϵ*-close to the maximum revenue a middleman could attain.*

Corollary 3.3.2 can be derived directly from Theorem 3.3.1 by letting $\theta = \frac{\alpha+\beta}{1+\beta}$. Moreover, it implies that, given the maximizer $\theta = \arg\max \theta\Pi(G_\theta) - \epsilon$, setting $\alpha = \theta, \beta = 0$ or $\alpha' = 0, \beta' = \frac{\theta}{1-\theta}$ generate the same revenue for the middleman. In particular, $\alpha < \beta'$, which means, if deciding to charge on one side of market, the commission rate for buyer is alway larger than the seller side. Suppose however that the middleman would announce the same commission rate charged on either side of the market. Each buyer, in addition to paying the price to the seller, is subject to a commission fee paid to the middleman. Therefore the buyers are less willing to pay a higher price than he would have offered under 0 commission rate, which in term implies that the middleman can extract lower surplus from trade.

3.4. "α or β" Pricing Model

The "α+β" pricing is implemented by many intermediaries where the commission is charged on one or both sides of the market, regardless of who proposes the offer. In real estate, sellers are responsible for paying the commission even though buyers submit purchase offers. In Airbnb the accommodation sharing platform, both customers and hosts are charged commission fees, at $6-12\%$ and $3-5\%$ respectively despite the fact that only sellers proposes the accommodation prices. During salary negotiations, whenever the compensation packages are proposed by either the employers or the candidates, headhunters are paid solely by the employers. In this model where the middleman can assign one side of the matched pair to make a take-it-or-leave-it offer, in the following I propose an alternative commission pricing "α or β" where the middleman only charges the proposers she assigns. It is less commonly seen in practice and to the best of my knowledge, some sports agents in Europe are paid by whoever they are representing and proposing the offer.

Below is the timing of the "α or β" pricing model.

Choice 2. α or β Commission Pricing

Stage 1 At time $t = 0$, the middleman announces non-discriminating percentage commission rates $\alpha, \beta \geq 0$.

Stage 2 At time $t = 1, 2, \ldots$, if no feasible matches exist in the currently network G_t, the game ends. Otherwise,

(i) Matching a buyer-seller pair: With probability λ, the middleman selects a feasible pair b, s and assigns one of them to make a take-it-or-leave-it offer. Otherwise, a single link is selected at random according to $\{p_{ij}(G_t)\}_{ij \in E_t}$.

(ii) Bargaining: The matched pair (b, s) meet, with one of them proposing to the other player (the responder) specifying a transfer (or price).

If buyer b offers a bid-price p and s accepts, they trade and exit the game with payoffs $v_{bs} - (1 + \beta)p, p - c_s$ respectively if matched by the middleman and $v_{bs} - p, p - c_s$ if matched by nature. Here βp is the commission fee paid to the middleman by buyer b.

If seller s proposes an ask-price p and b accepts, they trade and exit the game with payoffs $v_{bs} - p, (1 - \alpha)p - c_s$ respectively if matched by the middleman and $v_{bs} - p, p - c_s$ if matched by nature. Here αp is the commission fee paid to the middleman by seller s.

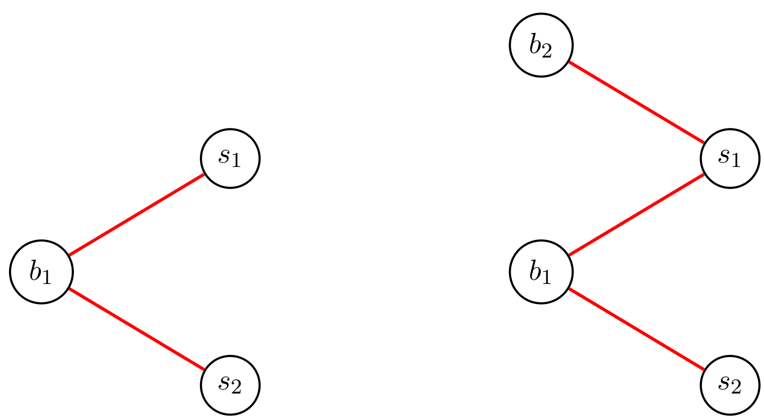
(iii) When the pair exit the market with an agreement, with probability γ they are replaced by clones at the same positions in the network and there are no clones otherwise. If the offer is rejected, the matched pair dissolves and the two traders return in the next period.

Denote by $\Gamma^{\alpha/\beta}(G)$ the game under the choice of "α or β" commission fees.

Theorem 3.4.1. *For any $\alpha, \beta \geq 0$, there exists a Markov perfect equilibrium for both the* *bargaining and matching games* $\Gamma^{\alpha/\beta}(G)$*.*

Unlike the "$\alpha + \beta$ choice where buyers gain 0 in any MPEs under the monopoly matchmaking middleman, the "α or β" payment scheme allows for more positive gains for the buyers even if the middleman is the monopoly matchmaker. The following example illustrates a MPE where buyers earns positive gains from trade.

Example 3.4.1 ($\Gamma^{\alpha/\beta}(G)$)**.** *Consider the following lefthand-side, a one-buyer-two-seller network with values* $v_{b_1s_1}, v_{b_1s_2}$ *and production costs* $c_1 > c_2 > 0$*.* $v_{b_1s_1} - c_1 > v_{b_1s_2} - c_2$*. Consider the case where* $\gamma = 0, \lambda = 1$*. The middleman sets* $\alpha = 0$*, that is, he is always representing the buyer and charges him* β *of the sale price.*



For any $\beta < \frac{v_{b_1s_1} - c_1}{c_1}$*, the only MPE is to match* (b_1, s_1) *with revenue* $\pi_M = \beta c_1$*, and buyer 1's equilibrium payoff is,*

$$u_{b_1} = v_{b_1s_1} - (1+\beta)c_1 > 0$$

Suppose now there is a new coming buyer b_2 *(in the righthand-side network), with* $v_{b_2s_1} > c_1$*,* $v_{b_2s_2} = 0$*. Then under the same* β*, the middleman's revenue is still* βc_1*. However, the MPE payoffs for buyer 1 and 2 change:*

$$u_{b_1} = \frac{w_{b_1s_1}(v_{b_1s_1} - (1+\beta)c_1)}{1 - \delta_b + \delta_b w_{b_1s_1}},$$

$$u_{b_2} = \frac{w_{b_2 s_1}(v_{b_2 s_1} - (1+\beta)c_1)}{1 - \delta_b + \delta_b w_{b_2 s_1}},$$

here $w_{b_1 s_1} + w_{b_2 s_1} = 1$, and $0 \leq w_{b_1 s_1} \leq 1$. In particular, buyer 1's payoff has decreased with the presence of buyer 2.

As shown in Example 3.4.1, the buyers' payoffs are more sensitive to the network structures under $\Gamma^{\alpha/\beta}(G)$ compared to $\Gamma^{\alpha+\beta}(G)$. In the lefthand-side network in Example 3.4.1, buyer b_1 is the monopoly buyer with positive gains from trade, whereas in the righthand-side, buyer b_1 has a competing peer b_2, therefore less profit earned from bargaining under the same commission rate. This is in sharp contrast to Proposition 3.2.2 for which regardless of the network structure, buyers earn 0 from intermediated bargaining when the middleman takes full control of the matchmaking ($\lambda = 1$).

Although the choice of "$\alpha + \beta$" is most commonly adopted by the middleman, the following Proposition 3.4.2 shows the choices of "α or β" is dominated by the choice "$\alpha + \beta$" in terms of revenue.

Proposition 3.4.2. *For any choices of $1 > \alpha \geq 0, \beta \geq 0$, let $\alpha' = \frac{\alpha+\beta}{1+\beta}, \beta' = \frac{\alpha+\beta}{1-\alpha}$, then the middleman's revenue from $\Gamma^{\alpha'/\beta'}(G)$ is at least that from $\Gamma^{\alpha+\beta}(G)$.*

In particular, Proposition 3.4.2 implies that the maximum revenue attained by "α or β" pricing is at least that from "$\alpha + \beta$" pricing model.

3.5. Discussion

1. Matching technology: sequential matching.

This paper studies a decentralized sequential bargaining game where each time only one link is chosen for bargaining. The one-match-per-period assumption is standard in decentralized markets(Rubinstein and Wolinsky, 1985; Gale, 1987; Abreu and Manea, 2012a,b; Manea, 2011). A rationale is that, important economic transactions such as buying houses, trading over-the-counter assets, and hiring employees are decentralized, in the sense that they typically involve extensive bilateral negotiations. Moreover, it also costs the middleman efforts and time to intermediate the negotiation, thus multiple agreements are not reached at the same instant.

Kranton and Minehart (2001) adopt an ascending-bid auction, analogous to the fictional Walrasian auctioneer in the buyer-seller network setting that determines unique equilibrium payoffs. Polanski (2007) implements the simultaneous matching technology in which a maximum number of pairs of connected players are selected to bargain every period. The analysis in the paper can be extended to the settings where more than one link is chosen for bargaining at each time.

2. Membership fees:

In this paper, the middleman adopts the variable fees charged upon agreement and any traders can walk away from the intermediation without any payment to the middleman and wait for nature to match. Some intermediaries also charge fixed membership fees upon registration beforehand. The techniques applied in this paper can be viewed as the first step of computing the expected payoffs from intermediated bargaining given the entry of all the members. In particular, given the zero payoff on the buyer side when middleman is the monopoly matchmaker, it can be inferred that the membership charges on the buyer side should also be 0 unless other modification of the setting is made.

3.6. Conclusion

This paper investigates how intermediation through matchmaking in a bargaining network can affect the efficiency of the matching and bargaining outcomes. We show 3 novel ways of distortion a profit-maximizing middleman introduces to the matching: to restrain matchings on a subnetwork; to select the maximum weight matching weighted by the buyers' values; and to match pairs selected less often by nature. As one of the first few attempts to characterize the matchmaking intermediaries on a networked model, I introduce a general framework based on a number of flexible settings such as (1) the exchange of heterogenous goods, (2) arbitrary network structures, (3) impatient traders with arbitrary discount factor smaller than 1, (4) exogenous matching technology by nature with arbitrary probabilistic distribution. Intermediated bargaining still contributes to an active area of research and in the future, one can restrict the attention to more specific settings and derive further interplay between the network structure and the intermediation

APPENDIX TO CHAPTER 1

A.1. Neighborhood Assignment

A paramount difference between neighborhood assignment and school choice is the strict zoning constraint on student enrollment: whether outsiders of neighborhoods are allowed to attend the neighborhood schools. Zoning and land use regulations are ubiquitous in the United States. Many local governments will seek to limit housing development for otherwise they must raise taxes to fund schools and other needed public services (Schill, 2005).[1] Therefore in this paper, I assume that housing supply is also subject to a local (zoning) constraint when the town runs neighborhood assignment to ensure that there are enough slots in each neighborhood school for its resident students,

$$H^B(l_d, p) = \min\{k_d, l_d p^r\}.$$

A.1.1. Neighborhood Assignment Equilibrium

Under the neighborhood assignment rule, students are assigned to schools in the districts they live in. Given the vector $\mathbf{p}$ of equilibrium prices for all school districts, student of type x applies to her neighborhood school in district d if and only if it yields higher surplus than her outside option. In other words, the payoff for student of type x to live in district d is, $U(x, d; \mathbf{p}) = \max\{v(x, y_d), \pi(x)\} - p_d$.

Definition 7 (Neighborhood Assignment Equilibrium)**.** *Given* $(\mathbf{y}, \mathbf{k}, \mu)$*, a* ***neighborhood assignment equilibrium*** *is* $(\mathbf{p}, \{T_d\})$*, where* $\mathbf{p}$ *is the vector of equilibrium home prices,* $T_d \subseteq X$ *is the set of student types that live in district d, such that,*

[1] For example, according to the school impact fee study by Lee County in January 2012 (see `https://www.leegov.com/dcd/Documents/Studies_Reports/ImpactFees/SchoolImpactFee2012.pdf`), the average total capital cost per student of opening a new school is \$25,184, among which, \$302 is covered by state funding credit and \$11,442 by discounted future tax revenues within the next 20 years. Therefore, the net capital cost per student would be \$13,440 on average, which will possibly be passed on to the residents.

1. *for each $x \in T_d$, residential choice is optimal,*

$$U(x, d; \mathbf{p}) = \max_{1 \le d' \le D} U(x, d'; \mathbf{p}),$$

2. *housing supply should match demand in each district,*

$$H^B(l_d, p_d) = \mu(T_d).$$

Condition (1) is an incentive compatibility condition, that each student is maximizing her payoff given equilibrium prices. Condition (2) is a market clearing condition for the housing market. $\mu(T_d)$ is the measure of resident students in district d.

The supermodularity assumption on $v(x, y)$ implies the marginal benefit for taking a higher school quality y increases with higher student type x. When students are price takers, households of higher types are willing to pay more on the margin for increases in school quality. Therefore, equilibrium residential choices are assortative as a consequence of Topkis' Monotonicity Theorem.

Lemma A.1.1. (Topkis' Monotonicity Theorem) *For any pair of lattices $(Y, \preceq)$ and $(T, \preceq)$, let $f : Y \times T \to \mathbb{R}$ be a supermodular function (with coordinate-wise order). For each $t \in T$, let $\gamma(t) = \arg\max_{\gamma \in \Gamma(t)} f(\gamma, t)$, where $\Gamma(t) \subseteq Y$. If $t' \ge t$ and $\Gamma(t') \supseteq \Gamma(t)$, then $\gamma(t') \ge \gamma(t)$.*

Let $Y = \{(y_d, p_d)\}_d$, $T = [\underline{x}, \bar{x}]$. It is easy to verify $U(y, x)$ is supermodular on (Y, T). Let the choice set $\Gamma(t) = Y$ for all $t \in T$. Then Topkis' Monotonicity Theorem implies the following result appeared in a number of previous papers on neighborhood assignment.

Corollary A.1.2. *Suppose $\pi(x) = -\infty$ for all x, then each T_d is in the form of $[\underline{x}_d, \overline{x}_d]$, s.t., $\underline{x}_1 \le \overline{x}_1 \le \underline{x}_2 \le \overline{x}_2 \le \ldots \le \underline{x}_D \le \overline{x}_D$, that is, in any neighborhood assignment equilibrium, higher-type students live in higher-quality school districts.*

This strict stratification result may not hold, however, for arbitrary $\pi(x)$ when the surplus between attending a local school and opting for the outside option does not satisfy some single-crossing condition.

Proposition A.1.3. *If Condition 1 holds, then for any (one-town) neighborhood assignment equilibrium, there exists* d^* *and* $x_{d^*} \leq x_{d^*+1} \leq \ldots \leq x_D$*, s.t.*

1. *for all* $d \geq d^* + 1$, $T_d = [x_{d-1}, x_d]$,
2. *for all* $d \leq d^*$, $T_d \subseteq [\underline{x}, x_{d^*}]$, *and* $p_d = \min\{p_j : 1 \leq j \leq D\}$.

Proposition A.1.3 shows a relaxed stratification result under neighborhood assignment with outside options. Condition 1 implies that the set of acceptable schools is expanding with higher type, which is exactly the assumption $\Gamma(t') \supseteq \Gamma(t), \forall t' \geq t$ in Topkis' Monotonicity Theorem ($\Gamma(t)$ is the choice set type t is optimizing over). Therefore, the optimal residential choice is monotone in student type. The second statement in Proposition A.1.3 takes care of the extreme cases when some local schools are unfavorable compared to the outside option. Students are thus indifferent to any of these districts as they can always opt out.

As indicated above, neighborhood assignment is criticized for quality-ordered stratification across neighborhoods and schools. Only students of high types live in top quality school districts, and subsequently attend top quality schools. To offer more equitable access to good schools, school choice is introduced.

A.2. Mathematical Appendix

A.2.1. Proof of Proposition 1.2.1

The argument applies to deferred acceptance with any tie breaking rule independent of students' reports and types. A continuum of students are matched to a finite set of schools $S = \{1, 2, \ldots, D\}$. In school choice, a student t differs in her profile $(\mathbf{u^t}, \mathbf{e^t})$, where $\mathbf{u^t}$ is the vector of utilities of attending public schools and $\mathbf{e^t} \in [0,2]^D$ describe the school priorities of student t, with sth entry e^t_s indicating student t's score at school s. $e^t_s > e^{t'}_s$ if t has higher priority than t' at school s. Let R^t be a corresponding strict preference of student t. Denote by $\mathscr{R}$ the set of strict preferences. The deferred acceptance mechanism is a stable matching mechanism , and can be described using cutoffs $\mathbf{P} = (P_1, P_2, \ldots, P_D) \in [0,2]^D$ such that, student t is admitted to school s if

$$D_s^{(R^t,e^t)}(\mathbf{P}) = \mathbb{1}\{e^t_s \geq P_s, sR^t\emptyset\} \prod_{s'R^ts} \mathbb{1}\{e^t_{s'} < P_{s'}\} = 1,$$

and let η is measure over $\mathscr{R} \times [0,2]^D$. Then the aggregate demand for school s is $D_s(P|\eta) = \eta(\{t : D_s^{(R^t,e^t)}(P) = 1\})$, satisfies $D_s(P|\eta) \leq k_s$ for all s. Next we need to specify η. Denote by H the set of $\{0,1\}^D$, each with only one non-zero entry. For $h^t \in H$, $h^t_d = 1$ if student t lives in district d. Denote by $\sigma : X \times H \to \Delta(\mathscr{R})$ households' strategy of preference submission. Each entry $\sigma_R(t,h)$ is the probability of student of type t with her choice of residence $\mathbf{h}$ reporting R. For each $(R, \mathbf{h}) \in \mathscr{R} \times H$, $m^\sigma(R, \mathbf{h}) = \int_X \sigma_R(t, \mathbf{h}) d\mu$. Since $\mathscr{R} \times H$ is finite, and m^σ is a discrete measure over student preference and priority profiles. Each school s assigns a lottery number $l^t_s \in [0,1]$ to student t. L is the measure on $\{l^t_s\}_{t,s}$. More specifically, consider L to be the product measure on s independent Markov processes $\{l_s(t) : t \in T\}$ with initial continuous and bounded distribution π and transition probabilities the same as π. Under deferred acceptance with neighborhood priority, the assigned priority scores for student t is $\mathbf{e^t} = \mathbf{h^t} + \mathbf{l^t}$. Then,

$$\begin{aligned} D_s(P|\eta) &= \eta(\{t : D_s^{(R^t,e^t)}(P) = 1\}) \\ &= (m^\sigma \times L)(\{(R^t, h^t, l^t) : h_s^t + l_s^t \geq P_s, \text{and } h_{s'}^t + l_{s'}^t < P_{s'} \text{ for } s'R^t s\}). \end{aligned}$$

Suppose student of type t misreports. Denote by σ' the new strategy profile. Then, $\mu(\{t : \sigma'(t, h^t) \neq \sigma(t, h^t)\}) = 0$, and $m^\sigma = m^{\sigma'}$. Hence the market clearing thresholds P under σ and the market clearing threshold $\hat{P}$ under σ' must satisfy: $\hat{P} = P$. Then for any fixed e^t and for any $R \in \mathscr{R}$, $D_s^{(R^t,e^t)}(P) \cdot u^t \geq D_s^{(R,e^t)}(P) \cdot u^t$, by definition of $D_s^{(R^t,e^t)}(P)$. Since $P = \hat{P}$, we have,

$$D_s^{(R^t,e^t)}(P) \cdot u^t \geq D_s^{(R,e^t)}(\hat{P}) \cdot u^t.$$

Therefore the expected utility of reporting R^t satisfies, for any $R \in \mathscr{R}$,

$$\mathbb{E}_\pi[D_s^{(R^t,e^t)}(P) \cdot u^t] \geq \mathbb{E}_\pi[D_s^{(R,e^t)}(\hat{P}) \cdot u^t].$$

Thus we prove that truthful reporting is a Nash Equilibrium.

A.2.2. Expression and its Proof of $\Pr(s|d, h)$

When everyone reports her preference truthfully in the second stage and admission is rationed by neighborhood priority, $\Pr(s|d, \mathbf{h})$ can be determined by the aggregate information of applicants in each district, i.e.,

$$\Pr(s|d, \mathbf{h}) = \begin{cases} \min\{1 - \sum\limits_{j>d} \Pr(j|d, \mathbf{h}), \frac{k_d}{\tilde{m}_d}\}, & \text{if } s = d, \\ \frac{k_s - \Pr(s|s,\mathbf{h})\tilde{m}_s}{a_s - \Pr(s|s,\mathbf{h})\tilde{m}_s}(1 - \sum\limits_{j>s} \Pr(j|d, \mathbf{h})), & \text{if } s \neq d, \end{cases}$$

where $\tilde{m}_d$ is the number of residents in district d that find their neighborhood school d acceptable. a_s is the aggregate number of applicants applying to school s who have been rejected in the previous rounds. The expression incorporates two facts about deferred acceptance: schools first assign seats to previously rejected resident students up to the school capacity; if there are slots left over, non-resident students are admitted up to the school capacity. Applicants from the same priority class are equally treated by the assignment rule regardless of their types.

Denote by $X^s = \{x \in X : v(x, y_s) \geq \pi(x) > v(x, y_{s-1})\}$, then the measure of residents in district d with school s being the least favored acceptable is $m_d^s = \int_{X^s} h_d(x) d\mu$. Denote by $\Pr(s|d,h)$ the probability of enrolling in school s conditional on living in district d. $\Pr(D|D,h) = \min\{\frac{k_D}{\sum\limits_{s \leq D} m_D^s}, 1\}$, $\Pr(D|d,h) = \frac{k_D - \sum\limits_{j \leq D} m_D^s \Pr(D|D,h)}{\sum\limits_{d,s \leq D} m_d^s - \sum\limits_{s \leq D} m_D^s}$, $\forall d < D$. For $d < D$,

$$
\begin{aligned}
\Pr(s|s,h) &= \Pr(\text{admitted to } s| \text{ rejected by all } j > s,\ s,\ \text{h}) \cdot \Pr(\text{rejected by all } j > s|\ s, h) \\
&= \min\{1, \frac{k_s}{(1 - \sum\limits_{j>s} \Pr(j|s,h))\tilde{m}_s}\} \cdot (1 - \sum_{j>s} \Pr(j|s,h)) \\
&= \min\{1 - \sum_{j>s} \Pr(j|s,h), \frac{k_s}{\tilde{m}_s}\}.
\end{aligned}
$$

Here $\tilde{m}_s = \sum_{r \leq s} m_s^r$ the measure of residents living in district s who also finds their neighborhood school acceptable. The third equality is due to the fact that, the conditional probability of x being rejected to $r > s$ depends on the event that x has lower priority scores than those previously admitted, while the conditional probability of x being admitted to d depends on her relative rank among those previously rejected. Here $(1 - \sum\limits_{j>s} \Pr(j|s,h))\tilde{m}_s$ is the number of applicants rejected from previous rounds. Similarly for $d \neq s$,

$$
\begin{aligned}
\Pr(s|d,h) =& \Pr(\text{admitted to } s| \text{ rejected by all } j > s,\ d, h) \cdot \Pr(\text{rejected by all } j > s|\ d, h) \\
=& \frac{k_s - \Pr(s|s,h)\tilde{m}_s}{a_s - \Pr(s|s,h)\tilde{m}_s} (1 - \sum_{r>s} \Pr(r|d,h)).
\end{aligned}
$$

Here $a_s = \sum\limits_{b \leq D, j \leq s} m_b^j (1 - \sum\limits_{k>s} \Pr(k|b,h))$ the number of applicants to school s.

A.2.3. Proof of Theorem 1.2.2

Suppose households are truth-telling in the second stage. To apply Theorem 2 from Schmeidler (1973), we need to modify some of the definitions.

1. $X = [\underline{x}, \bar{x}]$ is set of student types with non-atomic measure μ.

2. $\mathbf{h} : X \to \{(\sigma_d^s)_{s,d} : \sum_{s,d} \sigma_d^s = 1\}$. Each entry $h_d^s(x)$ is the probability of type x choosing district d to live in and reporting preferences as if $x \in X_s$. $\mathscr{H}$ is set of all Lebesgue integrable $\mathbf{h}$ with L_1 weak topology.

3. $p = (p_1, p_2, \ldots, p_D) : \mathscr{H} \to \mathbb{R}_+^D$ vector of housing prices. $p_d(\mathbf{h}) = (\frac{\int \sum_s h_d^s(x) d\mu}{\alpha k_d})^{\frac{1}{r}}$.

4. $\hat{\mathbf{u}} : X \times \mathscr{H} \to \mathbb{R}^{D^2}$, $\hat{u}_d^s(x, \mathbf{h}) = \sum_{j \geq s} \Pr(j|d, \mathbf{h}) v(x, y_j) + \big(1 - \sum_{j \geq s} \Pr(j|d, \mathbf{h})\big)\pi(x) - p_d$.

Then $\mathbf{h}$ is an equilibrium strategy if and only if, for all x and $\sigma, \mathbf{h}(x) \cdot \hat{\mathbf{u}}(x, \mathbf{h}) \geq \sigma \cdot \hat{u}(x, \mathbf{h})$. First by truth-telling of deferred acceptance, we restrict the attention to such equilibrium strategy $\mathbf{h}$ that, for any $x \in X_s$, $h_d^j(x) = 0$, for $j \neq s$, and for any d. Theorem 1.2.2 then is a direct corollary of Theorem 2 in Schmeidler (1973). To be more specific, it's easy to verify that (i) for all $x \in X$, $\hat{\mathbf{u}}(x, \cdot)$ is continuous on $\mathscr{H}$, (ii) for $\mathbf{h} \in \mathscr{H}$ and i, j, k, m, the set $\{x \in X | \hat{u}_j^i(x, \mathbf{h}) > \hat{u}_k^j(x, \mathbf{h})\}$ is measurable, (iii) $\hat{\mathbf{u}}(x, \mathbf{h})$ depends on $\mathbf{p}$ and $\Pr(s|d, \mathbf{h})$ through $m_d^s = \int_X h_d^s(x) d\mu$. Therefore by Theorem 2 in Schmeidler (1973), a pure strategy NE exists.

A.2.4. Proof of Proposition 1.2.3

Let $m_d = \int_X h_d(x) d\mu$ be the number of residents in district d, $m_d^s = \int_{X_s} h_d(x) d\mu$ be the number of residents in district d with least favored acceptable school j.

Claim 1. If $m_d \leq k_d$, then $\forall d' < d$, $m_{d'} \leq k_{d'}$, moreover $\frac{m_{d'}}{k_{d'}} \leq \frac{m_d}{k_d}$.

Proof of Claim 1: Proof by contradiction. Suppose $\exists d' < d$, $m_{d'} > k_{d'}$. Then $p_{d'} > p_d$, and for any x s.t. $j_x \leq d$,

$$\begin{aligned} U(x,\mathbf{e}_d;\mathbf{p},\mathbf{h}) - U(x,\mathbf{e}_{d'};\mathbf{p},\mathbf{h}) =& \big(1-\sum_{r>d}\Pr(r|d,h)\big)v(x,y_d) - \sum_{d\geq r\geq j_x}\Pr(r|d',h)v(x,y_r) \\ &- \big(1-\sum_{r\geq j_x}\Pr(r|d',h)\big)\pi(x) + p_{d'} - p_d \\ >& 0. \end{aligned}$$

For $j_x > d$, $U(x,\mathbf{e}_d;\mathbf{p},\mathbf{h}) - U(x,\mathbf{e}_{d'};\mathbf{p},\mathbf{h}) = p_{d'} - p_d > 0$. Hence $m_{d'} = 0$, contradicting with $m_{d'} > k_{d'}$. Moreover, the argument above proves that $p_d \geq p_{d'}$, i.e., $\frac{m_{d'}}{k_{d'}} \leq \frac{m_d}{k_d}$. Let$p_{\min} = \min_{1\leq j\leq D}\{p_j\}$ the lowest home price. $\tilde{m}_d = \sum_{j\leq d} m_d^j$.

Claim 2. If $p_d \neq p_{\min}$, then $m_d^s = 0$ for all $s > d$.

This is obvious since for type x, the benefit of living in any district $d' < j_x$ is the same and she would prefer the one with the lowest home price.

Claim 3. Let $d = \max\{1 \leq j \leq D : p_j = p_{\min}\}$, then $p_{d'} = p_{\min}, \quad \forall d' \leq d$.

Proof of Claim 3: By the definition of d, for any $d' > d$, $p_{d'} > p_{\min}$, thus $m_{d'} = \tilde{m'}_d$ by Claim 2. If $k_d \geq \tilde{m}_d$, then $\forall d' < d$ and x s.t. $j_x \leq d$, $U(x,\mathbf{e}_d;\mathbf{p},\mathbf{h}) > U(x,\mathbf{e}_{d'};\mathbf{p},\mathbf{h})$. Hence $m_{d'}^j = 0$, for all $j \leq d$. By Claim 2, $p_{d'} = p_{\min}$. If $k_d < \tilde{m}_d$, suppose $p_{d-1} > p_{\min}$. Then by Claim 2, $\tilde{m}_{d-1} = m_{d-1} > k_{d-1}$. For x s.t. $j_x \geq d$, $U(x,\mathbf{e}_d;\mathbf{p},\mathbf{h}) - U(x,\mathbf{e}_{d-1};\mathbf{p},\mathbf{h}) \geq p_{d-1} - p_d > 0$. For x s.t. $j_x \leq d-1$, since $k_{d'} < m_{d'} = \tilde{m}_{d'}$,

$$\begin{aligned} U(x,\mathbf{e}_d;\mathbf{p},\mathbf{h}) - U(x,\mathbf{e}_{d-1};\mathbf{p},\mathbf{h}) =& \frac{k_d}{\tilde{m}_d}(v(x,y_d)-w(x)) - p_d - \frac{k_{d-1}}{m_{d-1}}(v(x,y_{d-1})-w(x)) \\ &+ p_{d-1} \\ >& \frac{k_d}{m_d}[v(x,y_d)-w(x)] - \frac{k_{d-1}}{m_{d-1}}[v(x,y_{d-1})-w(x)] \\ >& 0. \end{aligned}$$

Here $w(x) = \sum_{d-2\geq r\geq j_x}\frac{k_r-\Pr(r|r,h)\tilde{m}_r}{a_r-\Pr(r|r,h)\tilde{m}_r}v(x,y_r) + \big(1-\sum_{d-2\geq r\geq j_x}\frac{k_r-\Pr(r|r,h)\tilde{m}_r}{a_r-\Pr(r|r,h)\tilde{m}_r}\big)\pi(x)$. Hence $m_{d-1} = 0$, contradicting with $p_{d-1} > p_{\min}$. Apply the same argument above to show $p_{d-2} = p_{d-3} = \ldots p_1 = p_{\min}$, thus conclude the induction. We are ready to prove the theorem.

Case 1. There exists $d^* = \max\{1 \leq d \leq D : m_d \leq k_d\}$.

By Claim 1, for all $d \leq d^*$, $m_d \leq k_d$. For $d' < d \leq d^*$, $U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_{d'}; \mathbf{p}, \mathbf{h}) \geq p_{d'} - p_d$. Hence $p_d \geq p_{d'}$ otherwise leading to a contradiction. For any $d > d^*$, $m_d = \tilde{m}_d$,

$$\begin{aligned} &U(x, \mathbf{e}_{d+1}; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) \\ =&\frac{k_{d+1}}{m_{d+1}}[v(x, y_{d+1}) - w(x)] - p_{d+1} - \frac{k_d}{m_d}[v(x, y_d) - w(x)] + p_d, \end{aligned} \tag{A.1}$$

where $w(x) = \sum_{d^* \geq r \geq j_x} \frac{k_r - \Pr(r|r,h)\tilde{m}_r}{a_r - \Pr(r|r,h)\tilde{m}_r} v(x, y_r) + \big(1 - \sum_{d^* \geq r \geq j_x} \frac{k_r - \Pr(r|r,h)\tilde{m}_r}{a_r - \Pr(r|r,h)\tilde{m}_r}\big)\pi(x)$. Suppose $\frac{k_{d+1}}{m_{d+1}} \geq \frac{k_d}{m_d}$, then $U(x, \mathbf{e}_{d+1}; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) > 0$ for all $j_x \leq d$, which implies $m_d = 0$, a contradiction. Hence $p_{d+1} > p_d$ for all $d > d^*$. And $p_{d^*+1} > p_{d^*}$.

Case 2. $m_d > k_d$ for all d.

By Claim 3, there exists $d^* = \max\{1 \leq d \leq D : p_d = p_{\min}\}$, s.t. $p_d = p_{\min}$, $\forall d < d^*$. For all $d' > d > d^*$, by similar argument as in Eqn. (A.1), $p_{d'} > p_d$. Therefore housing prices ascend in quality under Case 1 and Case 2.

A.2.5. Proof of Theorem 1.3.1

Let $P^j = \{x \in [\underline{x}, \bar{x}] : v(x, y_j) \geq \pi(x) > v(x, y_{j-1})\}$. $J = |\{j : P_j \neq \emptyset\}|$. M is the simplex in $\mathbb{R}^{D \times J}$. $\mathbf{m} = (\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_D}) \in M$, s.t. $\mathbf{m_d} = (m_d^j)_{j \in [J]}$, where m_d^j is the number of residents with $x \in P^j$ that live in district d_A .

Let $\mathbf{e}_i$ be the $2D$-dimensional unit vector with ith non-zero entry. Let $V(x_t, \mathbf{e}_i, \mathbf{m})$ be the expected benefit of living in district i given $\mathbf{m}$. A price vector $\mathbf{p} \in \mathbb{R}^D$ specifies housing prices for each district in town B. The expected utility of living in district d in town A given $\mathbf{m}$ is,

$$U(x_t, \mathbf{e}_{d_A}; \mathbf{m}, \mathbf{p}) = V(x_t, \mathbf{e}_{d_A}, \mathbf{m}) - (\frac{\sum_j m_d^j}{\alpha k_d})^{1/r},$$

and the expected utility of living in district d in town B given $\mathbf{p}$,

$$U(x_t, \mathbf{e}_{d_B}; \mathbf{m}, \mathbf{p}) = V(x_t, \mathbf{e}_{d_B}, \mathbf{m}) - p_d.$$

Define student x_t's demand correspondence $D(x_t, p, m)$ as the set of all mixed strategies that are best responses to $\mathbf{p}, \mathbf{m}$, i.e., $D(x_t, \mathbf{m}, \mathbf{p}) = \arg\max_{\phi \in \Phi} U(x_t, \phi; \mathbf{m}, \mathbf{p})$. Define the aggregate demand correspondence $D : M \times \mathbb{R}^D \to\to M \times \mathbb{R}^D$. D is a correspondence that maps a price vector $\mathbf{p}$ and $\mathbf{m}$ the pre-assumed numbers of residents in each district of town A to number of residents who choose each district as best response. $D(\mathbf{m}, \mathbf{p}) = \{(\{D^j_{d_A}\}, \{D_{d_B}\}) : \mathbf{h} \in \mathscr{H}, \mathbf{h}(x_t) \in D(x_t, \mathbf{m}, \mathbf{p}), D^j_{d_A}(\mathbf{m}, \mathbf{p}) = \int_{x \in P^j} h_d(x_A) d\mu_A + \int_{x \in P^j} h_d(x_B) d\mu_B, D_{d_B}(\mathbf{m}, \mathbf{p}) = \int_{X_A} h_d(x_A) d\mu_A + \int_{X_B} h_d(x_B) d\mu_B.\}$. It is a correspondence since people may be indifferent between different districts. For simplicity, for each $\mathbf{h} \in \mathscr{H}$, denote by the vector

$$\int \mathbf{h} d\mu = \big((\int_{x \in P^j} h_{d_A}(x) d\mu_A + \int_{x \in P^j} h_{d_A}(x) d\mu_B)^j_d, (\int h_{d_B}(x) d\mu_A + \int h_{d_A}(x) d\mu_B)_d \big).$$

Define the tatonnement correspondence $T : M \times \mathbb{R}^D \to\to M \times \mathbb{R}^D$.

$T(\mathbf{m}, \mathbf{p}) = (D_A(\mathbf{m}, \mathbf{p}), \mathbf{g^r}(\mathbf{m}, \mathbf{p}))$, where $D_A(\mathbf{m}, \mathbf{p})$ is the vector of house demand in town B specifically among $x \in P_j$, r is the price elasticity of supply, $\mathbf{g^r} = (g^r_1, g^r_2, \ldots, g^r_D) : M \times \mathbb{R}^D \to \mathbb{R}^D$, s.t., if $0 < r < 1$,

$$g^r_d(\mathbf{m}, \mathbf{p}) = \begin{cases} p_d + D_{d_B}(\mathbf{m}, \mathbf{p}) - \min\{l_{d_B} p^r_d, k_{d_B}\}, & \text{if } p_d \geq 0, \\ -\frac{p_d}{2} + D_{d_B}(\mathbf{m}, (|p_d|, \mathbf{p_{-d}})), & \text{if } p_d < 0. \end{cases}$$

If $r \geq 1$,

$$g^r_d(\mathbf{m}, \mathbf{p}) = p_d + \frac{D_{d_B}(\mathbf{m}, (p_d, \mathbf{p_{-d}})) - \min\{l_{d_B} p^r_d, k_{d_B}\}}{N}.$$

N is a constant to be specified later. $D_B(\mathbf{m}, \mathbf{p})$ is the vector of house demand in town B, $\mathbf{k_B}$ the vector of capacity in town B. When $\mathbf{p} \geq 0$, $\mathbf{p} + D_B(\mathbf{m}, \mathbf{p}) - \min\{\mathbf{l}_B \cdot \mathbf{p}^r, \mathbf{k}_B\}$ is precisely the tatonnement process. It's easy to verify that $\mathbf{g}$ is continuous function of $\mathbf{p}$ for any fixed $\mathbf{m}$.

Claim 1. each $T(\mathbf{m}, \mathbf{p})$ is non-empty and convex.

The set of $D(\mathbf{m}, \mathbf{p})$ is nonempty since each x_t has at least one optimal district. Therefore, there exists at least one satisfying $\mathbf{h}$. To show $T(\mathbf{m}, \mathbf{p})$ is convex, it suffices to show $\mathbf{D}(\mathbf{m}, \mathbf{p})$

is convex. For $\mathbf{z}, \mathbf{z}' \in D(\mathbf{m}, \mathbf{p})$, there exists $\mathbf{h}, \mathbf{h}'$ s.t. $\mathbf{z} = \int \mathbf{h} d\mu$ and $\mathbf{h}(x) \in D(x, \mathbf{m}, \mathbf{p})$ for all x. Similarly $\mathbf{z} = \int \mathbf{h}' d\mu$ and $\mathbf{h}'(x) \in D(x, \mathbf{m}, \mathbf{p})$. For any $\alpha \in (0, 1)$, since $D(x, \mathbf{m}, \mathbf{p})$ is convex, we have $\alpha \mathbf{h}(x) + (1-\alpha)\mathbf{h}'(x) \in D(x, \mathbf{m}, \mathbf{p})$ for all x. Hence $\alpha \mathbf{z} + (1-\alpha)\mathbf{z}' = \int \alpha \mathbf{h}(x) + (1-\alpha)\mathbf{h}'(x) d\mu \in D(\mathbf{m}, \mathbf{p})$.

Claim 2.1 For $0 < r < 1$, there exists $K > 0$ such that T maps $M \times [-K, K]^D$ to itself.

Pick $K > 1$ so that for all $1 \leq d \leq D$, $v(x, y_d) - K + \theta < k_{d_B} v(x, y_d) - (\frac{1}{\alpha k_{d_A}})^{1/r}$, and $\alpha K^r \geq 1$. If $0 \leq p_d \leq K$, $p_d - k_{d_B} \leq g_d^r(\mathbf{m}, \mathbf{p}) \leq p_d + 1 \leq K + 1$. If $K \leq p_d \leq K + 1$, $p_d - k_{d_B} \leq g_d^r(\mathbf{m}, \mathbf{p}) \leq p_d - k_{d_B}$. If $-K \leq p_d \leq 0$, $-p/2 \leq g_d^r(\mathbf{m}, \mathbf{p}) \leq -p/2 + 1$ Pick $K + 1$ is the bound for the domain of T.

Claim 2.2 For $r \geq 1$,there exists $K > 0$ such that T maps $M \times [0, K]^D$ to itself.

Pick the same K as Claim 2.1, then N s.t. $\alpha(K+1)^{r-1} < N$. If $0 \leq p_d \leq K$, $g_d^r(\mathbf{m}, \mathbf{p}) \leq p_d + 1 \leq K + 1/N$. If $K \leq p_d \leq K + 1$, $g_d^r(\mathbf{m}, \mathbf{p}) \leq p_d - k_{d_B}/N$. And for all $\mathbf{p}$, $g_d^r(\mathbf{m}, \mathbf{p}) \geq p_d - \frac{\alpha k_{d_B} p_d^r}{N} = p_d(1 - \frac{\alpha k_{d_B} p_d^{r-1}}{N}) \geq 0$. Therefore K is the upper bound for the domain.

Claim 3: $T(\cdot, \cdot)$ has a closed graph.

Consider a sequence $(\mathbf{m}^n, \mathbf{p}^n, z^n)$ with $z^n \in D(\mathbf{m}^n, \mathbf{p}^n)$ and $(\mathbf{m}^{\mathbf{n}}, \mathbf{p}^{\mathbf{n}}, z^n) \to (\mathbf{m}, \mathbf{p}, z)$ as $n \to \infty$. $(\mathbf{m}, \mathbf{p}) \in M \times \mathbb{R}^D$. Need to show $z \in D(\mathbf{m}, \mathbf{p})$. Suppose not, we want to prove a contradiction. Since $\mathbf{z}^n \in D(\mathbf{m}^n, \mathbf{p}^n)$, there exists $\mathbf{h}^n \in \mathscr{H}$ such that $\mathbf{z}^n = \int \mathbf{h}^n d\mu$ and $\mathbf{h}^n(x) \in D(x, \mathbf{m}^n, \mathbf{p}^n)$. Since the set of all allocation function $\mathscr{H}$ is compact w.r.t. L_1 norm (Azevedo et al., 2013), $\{\mathbf{h}^n\}$ has a convergent subsequence in L_1 norm. Without loss of generality, suppose that $\mathbf{h}^n \to \mathbf{h}$ in L_1 norm. $\mathbf{h} \in \mathscr{H}$ is the limit. By dominant convergence theorem,

$$\int \mathbf{h} d\mu = \int \lim_n \mathbf{h}^n d\mu = \lim_n \int \mathbf{h}^n d\mu = \lim_n \mathbf{z}^n = \mathbf{z}.$$

Now that $\mathbf{z} \notin D(\mathbf{m}, \mathbf{p})$, $\mu(\{x \in X : \mathbf{h}(x) \notin D(x, \mathbf{m}, \mathbf{p})\}) > 0$. In particular, there exists

$\mathbf{h}' \in \mathscr{H}$ and $\epsilon_0 > 0$, such that

$$\int V(x, \mathbf{h}', \mathbf{m}) - (\tilde{p}(\mathbf{m}), \mathbf{p}) \cdot \mathbf{h}' d\mu \geq \int V(x, \mathbf{h}, \mathbf{m}) - (\tilde{p}(\mathbf{m}), \mathbf{p}) \cdot \mathbf{h} d\mu + \epsilon_0. \tag{A.2}$$

where $\tilde{p} : M \to \mathbb{R}^D$ s.t. $\tilde{p}_d(\mathbf{m}) = (\frac{\sum_j m_d^j}{\alpha k_d})^{1/r}$. Define for each n, $\mathbf{w}^n(x) \equiv \mathbf{m}^n$, $\mathbf{w}(x) \equiv \mathbf{m}$. $\mathbf{w}^n \to \mathbf{w}$ a.e. therefore $\mathbf{w}^n \to \mathbf{w}$ in L_1 since X is compact. Then the vectors of function $(\mathbf{id}, \mathbf{h}^n, \mathbf{w}^n) \to (\mathbf{id}, \mathbf{h}, \mathbf{w})$ in L_1, where $\mathbf{id}$ is identity map from X to X. Denote by $\mathbf{f}^n = (\mathbf{id}, \mathbf{h}^n, \mathbf{w}^n)$, $\mathbf{f} = (\mathbf{id}, \mathbf{h}, \mathbf{w})$.

Since V is continuous and compactly supported, it's uniformly continuous. the composition $V \circ \mathbf{f}$ is a continuous mapping from $L_1(\mathbb{R})^{2D+1}$ to $L_1(\mathbb{R})^{2D+1}$ by boundedness of f and Markov inequality, i.e. for any $\epsilon > 0$, there exists $\delta > 0$, s.t., for all $\|\mathbf{f} - \mathbf{f}'\|_1 < \delta$, $\|V \circ \mathbf{f} - V \circ \mathbf{f}'\|_1 < \epsilon$. Hence, there exists N_1 s.t. $\forall n > N_1$,

$$\int |V(x, \mathbf{h}, \mathbf{m}) - V(x, \mathbf{h}^n, \mathbf{m}^n)| d\mu < \epsilon,$$
$$\int |V(x, \mathbf{h}', \mathbf{m}) - V(x, \mathbf{h}', \mathbf{m}^n)| d\mu < \epsilon,$$

and since $\tilde{p}$ is continuous in m, there exists N_2 s.t. $\forall n > N_2$,

$$\int |(\tilde{p}(\mathbf{m}), \mathbf{p}) \cdot \mathbf{h} - (\tilde{p}(\mathbf{m}^n), p^n) \cdot \mathbf{h}^n| d\mu < \epsilon,$$
$$\int |(\tilde{p}(\mathbf{m}), \mathbf{p}) \cdot \mathbf{h}' - (\tilde{p}(\mathbf{m}^n), \mathbf{p}^n) \cdot \mathbf{h}'| d\mu < \epsilon,$$

From (A.2), the following must holds $\forall n > \max\{N_1, N_2\}$,

$$\int V(x, \mathbf{h}', \mathbf{m}^n) - (\tilde{p}(\mathbf{m}^n), \mathbf{p}^n) \cdot \mathbf{h}' d\mu \geq \int V(x, \mathbf{h}^n, \mathbf{m}^n) - (\tilde{p}(\mathbf{m}^n), \mathbf{p}^n) \cdot \mathbf{h}^n d\mu + \epsilon_0 - 4\epsilon.$$

In particular, $\exists x \in X$ s.t.

$$V(x, \mathbf{h}', \mathbf{m}^n) - (\tilde{p}(\mathbf{m}^n), \mathbf{p}^n) \cdot \mathbf{h}'(x) > V(x, \mathbf{h}^n, \mathbf{m}^n) - (\tilde{p}(\mathbf{m}^n), \mathbf{p}^n) \cdot \mathbf{h}^n(x),$$

contradicting with $\mathbf{h}^n(x) \in D(x, \mathbf{p}^n, \mathbf{m}^n)$. By Kakutani's fixed point theorem, there exists a fixed point $(\mathbf{m}^*, \mathbf{p}^*) \in T(\mathbf{m}^*, \mathbf{p}^*)$, in particular it must satisfy, (i) $\mathbf{p}^* \geq 0$, (ii) $D_A(\mathbf{m}^*, \mathbf{p}^*) = \mathbf{m}^*$, (iii) $D_{d_B}(\mathbf{m}^*, \mathbf{p}^*) = \min\{k_{d_B}, \alpha k_{d_B}(p_d^*)^r\}$ for all d. The first is due to the fact that when $0 < r < 1$, negative p_d can't be in the fixed point entry since $p_d = -p_d/2 + D_{d_B}(p, m)$ will yield $0 > p_d = \frac{2}{3} D_{d_B}(p, m) \geq 0$ a contradiction. For $r \geq 1$ any fixed point must consist of non-negative price vector by the domain of $\mathbf{g}^r$. Fix the equilibrium price $\mathbf{p}^*$, and the same argument of purification in Schmeidler (1973) Theorem 2 proves the existence of pure strategy equilibrium.

A.2.6. Proof of Proposition 1.3.3

For town B, proof by contradiction. Suppose there exists $x' < x$, $x' \in B_{d'}$, $x \in B_d$, s.t. $d' > d$. Since $p_{d'_B} > p_{1_B}$, $j' = j_{x'} \leq d'$ and by Condition 1, $j_x = j \leq j' \leq d'$. If $j \leq j' \leq d$, by supermodularity of $v(x, y)$, $d \leq d'$ a contradiction. If $j \leq d < j' \leq d'$, $v(x', y_{d'}) - \pi(x') \geq p_{d'} - p_d \geq v(x, y_{d'}) - v(x, y_d)$, implies $v(x', y_{d'}) - v(x', y_d) > v(x, y_{d'}) - v(x, y_d)$, a contradiction. If $d < j \leq j' \leq d'$, by monotonicity of $v(x, y_{d'}) - \pi(x)$, it's a contradiction.

For town A, we discuss by case. Let $\Delta V_1 = V(x', \mathbf{e}_{d'_A}; \mathbf{p}, \mathbf{h}) - V(x', \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h})$, $\Delta V_2 = V(x, \mathbf{e}_{d'_A}; \mathbf{p}, \mathbf{h}) - V(x, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h})$. $q_r = \Pr(r|1_A, \mathbf{h})$.

Case 1. There exists $d^* = \max\{1 \leq d \leq D : m_{d_A} \leq k_{d_A}\}$

Proof by contradiction. Suppose there exists $x \in A_d$, $x' \in A_{d'}$, s.t. $d' > d$ and $x' < x$. Denote by $j = j_x, j' = j_{x'}$, then $j' \geq j$ by Condition 1.

(i) $d > d^*$. By Claim 2 of Proposition 1.2.3, $j \leq d, j' \leq d'$.

$$
\begin{aligned}
\Delta V_2 =& \frac{k_{d'}}{m_{d'}} v(x, y_{d'}) - \frac{k_d}{m_d} v(x, y_d) + (\frac{k_d}{m_d} - \frac{k_{d'}}{m_{d'}})[\sum_{r \geq j} q_r v(x, y_r) + (1 - \sum_{r \geq j} q_r)\pi(x)]. \\
\Delta V_1 =& \frac{k_{d'}}{m_{d'}} v(x', y_{d'}) - \frac{k_d}{m_d} \max\{v(x', y_d), \pi(x')\} \\
& + (\frac{k_d}{m_d} - \frac{k_{d'}}{m_{d'}})[\sum_{r \geq j'} q_r v(x', y_r) + (1 - \sum_{r \geq j'} q_r)\pi(x')] \\
\leq & \frac{k_{d'}}{m_{d'}} v(x', y_{d'}) - \frac{k_d}{m_d} v(x', y_d) + (\frac{k_d}{m_d} - \frac{k_{d'}}{m_{d'}})[\sum_{r \geq j'} q_r v(x', y_r) + (1 - \sum_{r \geq j'} q_r)\pi(x')],
\end{aligned}
$$

since $\frac{k_{d'}}{m_{d'}}\frac{\partial v}{\partial x}(x, y'_d) \geq \frac{\partial v}{\partial x}(x, y_d)$, $\frac{k_{d'}}{m_{d'}}v(x, y_{d'}) - \frac{k_d}{m_d}v(x, y_d) > \frac{k_{d'}}{m_{d'}}v(x', y_{d'}) - \frac{k_d}{m_d}v(x', y_d)$.

Claim 5. $\sum\limits_{r\geq j} q_r v(x, y_r) + (1 - \sum\limits_{r\geq j} q_r)\pi(x) \geq \sum\limits_{r\geq j'} q_r v(x', y_r) + (1 - \sum\limits_{r\geq j'} q_r)\pi(x')$.

Proof of Claim 5: $LHS - RHS \geq \sum_{r\geq j'} q_r[v(x, y_r) - v(x', y_r)] + (1 - \sum_{r\geq j'} q_r)[\pi(x) - \pi(x')] > 0$, therefore $\Delta V_2 > \Delta V_1$, which implies x prefers d'_A to d_A, a contradiction.

(ii) $d \leq d^* < d'$. By Claim 2 of Proposition 1.2.3, $j \leq j' \leq d'$.

(a) $j \leq j' \leq d$.

$$\Delta V(z) = \frac{k_{d'}}{m_{d'}}v(z, y_{d'}) + (1 - \frac{k_{d'}}{m_{d'}})\big(\sum_{r\geq j_z} q_r v(z, y_r) + (1 - \sum_{r\geq j_z} q_r)\pi(z)\big) - \sum_{r>d} q_r v(z, y_r) - (1 - \sum_{r>d} q_r)v(z, y_d),$$

since $\frac{d}{dz}[\frac{k_{d'}}{m_{d'}}v(z, y_{d'}) - \sum\limits_{r>d} q_r v(z, y_r) - (1 - \sum\limits_{r>d} q_r)v(z, y_d)] > 0$, $\forall z \geq x'$. $\Delta V(x) > \Delta V(x')$.

(b) $j \leq d < j' \leq d'$.

$$\begin{aligned}
\Delta V_2 =& \frac{k_{d'}}{m_{d'}}v(x, y_{d'}) + (1 - \frac{k_{d'}}{m_{d'}})\big(\sum_{r\geq j} q_r v(x, y_r) + (1 - \sum_{r\geq j} q_r)\pi(x)\big) - \sum_{r>d} q_r v(x, y_r) \\
& - (1 - \sum_{r>d} q_r)v(x, y_d), \\
\Delta V_1 =& \frac{k_{d'}}{m_{d'}}v(x', y_{d'}) + (1 - \frac{k_{d'}}{m_{d'}})\big(\sum_{r\geq j'} q_r v(x', y_r) + (1 - \sum_{r\geq j'} q_r)\pi(x')\big) \\
& - \sum_{r\geq j'} q_r v(x', y_r) - (1 - \sum_{r\geq j'} q_r)\pi(x') \\
<& \frac{k_{d'}}{m_{d'}}v(x', y_{d'}) + (1 - \frac{k_{d'}}{m_{d'}})\big(\sum_{r\geq j'} q_r v(x', y_r) + (1 - \sum_{r\geq j'} q_r)\pi(x')\big) - \sum_{r>d} q_r v(x', y_r) \\
& - (1 - \sum_{r>d} q_r)v(x', y_d),
\end{aligned}$$

By part (a) of (ii), $\Delta V_2 > \Delta V_1$.

(c) $d < j \leq j' \leq d'$.

$V(z,\mathbf{e}_{d'};\mathbf{p},\mathbf{h}) - V(z,\mathbf{e}_d;\mathbf{p},\mathbf{h}) = \frac{k_{d'}}{m_{d'}}[v(z,y_{d'}) - \sum_{r\geq j_z} q_r v(z,y_r) - (1-\sum_{r\geq j_z} q_r)\pi(z)]$,

$$\begin{aligned}\Delta V_2 - \Delta V_1 =& \frac{k_{d'}}{m_{d'}}[v(x,y_{d'}) - \sum_{r\geq j} q_r v(x,y_r) - (1-\sum_{r\geq j} q_r)\pi(x) \\ & - v(x',y_{d'}) + \sum_{r\geq j'} q_r v(x',y_r) + (1-\sum_{r\geq j'} q_r)\pi(x')] \\ >& \frac{k_{d'}}{m_{d'}}[v(x',y_{d'}) - \sum_{r\geq j} q_r v(x',y_r) - (1-\sum_{r\geq j} q_r)\pi(x') - v(x',y_{d'}) \\ & + \sum_{r\geq j'} q_r v(x',y_r) + (1-\sum_{r\geq j'} q_r)\pi(x')],\end{aligned}$$

since $\frac{d}{dz}[v(z,y_{d'}) - \sum_{r\geq j} q_r v(z,y_r) - (1-\sum_{r\geq j} q_r)\pi(z)] \geq 0$, $\forall z \geq x'$.

(iii) $d < d' \leq d^*$.

(a) $p_{d'} \geq p_d > p_{\min}$. By Claim 2 of Proposition 1.2.3, $j \leq d, j \leq j' \leq d'$.

If $j \leq j' \leq d < d'$,

$$\begin{aligned}\Delta V_2 - \Delta V_1 =& \sum_{r>d'} q_r v(x,y_r) + (1-\sum_{r>d'} q_r)v(x,y_{d'}) - \sum_{r>d} q_r v(x,y_r) - (1-\sum_{r>d} q_r)v(x,y_d) \\ & - \sum_{r>d'} q_r v(x',y_r) - (1-\sum_{r>d'} q_r)v(x',y_{d'}) \\ & + \sum_{r>d} q_r v(x',y_r) + (1-\sum_{r>d} q_r)v(x',y_d) \\ =& (1-\sum_{r>d'} q_r)v(x,y_{d'}) - \sum_{d'\geq r>d} q_r v(x,y_r) - (1-\sum_{r>d} q_r)v(x,y_d) \\ & - (1-\sum_{r>d'} q_r)v(x',y_{d'}) + \sum_{d'\geq r>d} q_r v(x',y_r) + (1-\sum_{r>d} q_r)v(x',y_d).\end{aligned}$$

Since $\frac{d}{dz}(1-\sum_{r>d'} q_r)v(z,y_{d'}) - \sum_{d'\geq r>d} q_r v(z,y_r) - (1-\sum_{r>d} q_r)v(z,y_d) > 0$, $\Delta V_2 - \Delta V_1 > 0$. If

$j \le d < j' \le d'$,

$$\begin{aligned}\Delta V_2 - \Delta V_1 =&(1 - \sum_{r>d'} q_r)v(x, y_{d'}) - \sum_{d' \ge r > d} q_r v(x, y_r) - (1 - \sum_{r>d} q_r)v(x, y_d) \\ &- (1 - \sum_{r>d'} q_r)v(x', y_{d'}) + \sum_{d' \ge r \ge j'} q_r v(x', y_r) + (1 - \sum_{r \ge j'} q_r)\pi(x') \\ \ge&(1 - \sum_{r \ge j'} q_r)\pi(x') - \sum_{j' > r > d} q_r v(x', y_r) - (1 - \sum_{r>d} q_r)v(x', y_d)\end{aligned}$$

(b) $p_{d'} > p_d = p_{\min}$. By Claim 2 of Proposition 1.2.3, $j \le j' \le d'$.

If $j \le j' \le d$ or $j \le d < j' \le d'$, similar argument of part (a) in (iii) applies.

If $d < j \le j' \le d'$,

$$\begin{aligned}\Delta V_2 - \Delta V_1 =&(1 - \sum_{r \ge d'+1} q_r)v(x, y_{d'}) - \sum_{d' \ge r \ge j} q_r v(x, y_r) - (1 - \sum_{r \ge j} q_r)\pi(x) \\ &- (1 - \sum_{r \ge d'+1} q_r)v(x', y_{d'}) + \sum_{d' \ge r \ge j'} q_r v(x', y_r) + (1 - \sum_{r \ge j'} q_r)\pi(x') \\ >& \sum_{j' > r \ge j} q_r[\pi(x') - v(x', y_r)],\end{aligned}$$

since $\frac{d}{dx}[v(z, y_{d'}) - \pi(z)] \ge 0$, $\forall z \ge x'$.We have proved $\forall d' > d$, $p_{d'} > p_{\min}$ in Case 1.

Case 2 $\forall d, k_d < m_d$. Let $d^* = \max\{d : p_d = p_{\min}\}$, $\tilde{d} = \max\{d : \tilde{m}_d \le k_d\}$.

(i) $d^* < d < d'$: similar argument as (i) in Case 1.

(ii) $\tilde{d} < d \le d^* < d'$: by Claim 2, $j' \le d'$.If $j \le d$, apply similar argument in (i) of Case 1. If $d < j \le j' \le d'$. $\Delta V_2 - \Delta V_1 = \frac{k_{d'}}{m_{d'}}(v(x, y_{d'}) - \pi(x)) - \frac{k_{d'}}{m_{d'}}(v(x', y_{d'}) - \pi(x')) > 0$.

(iii) $d \le \tilde{d} \le d^* < d'$: by Claim 2, $j' \le d'$. We can prove each scenario $j \le j' \le d$, $j \le d < j' \le d'$, $d < j \le j' \le d'$ by similar argument as (a) - (c) of (ii).

Therefore we have proved for $\forall d' > d$, s.t. $p_{d'} > p_{\min}$ in Case 2.

A.2.7. Proof of Proposition 1.3.4

First need to show that $\frac{\frac{\partial^2 v}{\partial y^2}(x,y)}{\frac{\partial v}{\partial y}(x,y)} = r(y)$ if and only if $v(x,y) = g(x)w(y) + c(x)$, for some continuous function g, w, c.

$$\begin{aligned}
\frac{\partial}{\partial y}[\log \frac{\partial v}{\partial y}(x,y)] =& r(y), \\
\frac{\partial v}{\partial y}(x,y) =& \exp[\int r(z)dz + C(x)], \\
v(x,y) =& \int \exp(\int r(z)dz) \exp[C(x)] + c(x).
\end{aligned}$$

For any district d_A, the benefit of living is

$$V(x, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) = g(x)\big(\sum_{r \geq j} \Pr(r|d,h) w(y_d) + (1 - \sum_{d \geq j} \Pr(r|d,h)\big)\mathbb{E}[w(Y)]\big) + c(x),$$

where j is identical for all x by the assumption on $\pi(x)$. $\frac{d}{dx}[V(x_t, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) - V(x_t, \mathbf{e}_{d'_A}; \mathbf{p}, \mathbf{h})]$ has the identical sign for all x, therefore $V(x_t, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) - V(x_t, \mathbf{e}_{d'_A}; \mathbf{p}, \mathbf{h})$ satisfies single crossing condition.

A.2.8. Proof of Proposition 1.3.6

Neighborhood $B_{d'}$ can be possibly disconnected by A_d as in Figure (11), where students of

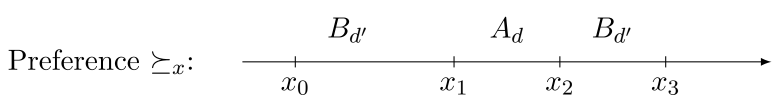


Figure 11: Disconnected neighborhood $B_{d'}$

types $[x_0, x_1] \cup [x_2, x_3]$ prefers $B_{d'}$ to A_d, students of types $[x_1, x_2]$ prefers A_d to $B_{d'}$. Since

$p_{d'B} > p_{\min}$, $j_x \leq d'$ for all $x \in [x_0, x_3]$.

$$\begin{aligned} D(x) =& U(x, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_{d'_B}; \mathbf{p}, \mathbf{h}) \\ =& \sum_{r \geq j_x} \Pr(r|d, \mathbf{h}) v(x, y_r) + \big(1 - \sum_{r \geq j_x} \Pr(r|d, \mathbf{h})\big) v(x, y_{j_x - 1}) - v(x, y_{d'}) \\ &+ \big(1 - \sum_{r \geq j_x} \Pr(r|d, \mathbf{h})\big)\big(\pi(x) - v(x, y_{j_x - 1})\big) + \Delta p \\ =& v(x, CE(x)) - v(x, y_{d'}) + \big(1 - \sum_{r \geq j_x} \Pr(r|d, \mathbf{h})\big)(\pi(x) - v(x, y_{j_x - 1})) + \Delta p, \end{aligned}$$

where $\Delta p = \pm\theta + p_{d'_B} - p_{d_A}$, and $CE(x)$ is the certainty equivalent s.t.,

$$v(x, CE(x)) = \sum_{r \geq j_x} \Pr(r|d, \mathbf{h}) v(x, y_r) + \big(1 - \sum_{r \geq j_x} \Pr(r|d, \mathbf{h})\big) v(x, y_{j_x - 1}).$$

Need to show (i) $CE(x)$ is non-decreasing function of x, (ii) $CE(x) \leq y_{d'}$, for all $x \leq x_2$, (iii) $\partial_+ D(x)$ is non-decreasing. Take $D(x)$ as a piecewise twice differentiable function. Let $X = X_1 \cup X_2 \cup \ldots \cup X_D \cup X_{D+1}$, where $X_r = \{x \in X : v(x, y_r) \geq \pi(x) > v(x, y_{r-1})\}$. In the interior of X_r, $CE'(x) \geq 0$; at the boundeferred acceptancery $\partial X_r = \{x : v(x, y_r) = \pi(x)\}$, $CE(x)$ is non-decreasing by Condition 2. For (ii), prove by contradiction. If $CE(x_2) > y_{d'}$, $\partial_+ D(x_2) = \frac{\partial v}{\partial x}(x_2, CE(x_2)) - \frac{\partial v}{\partial x}(x_2, y'_d) + \frac{\partial v}{\partial y}(x, CE(x_2)) CE'(x_2) + \big(1 - \sum_{r \geq j_x} \Pr(r|d, \mathbf{h})\big)(\pi'(x_2) - \frac{\partial v}{\partial x}(x_2, y_{j_{x_2} - 1}))$, thus $\partial_+ D(x_2) > 0$ a contradiction. Hence $CE(x_2) \leq y_{d'}$. For (iii), in the interior of each X_r where $D(x)$ is differentiable, and for all

$x \leq x_2$,

$$
\begin{aligned}
D'(x) =& \frac{\partial v}{\partial x}(x, CE) + \frac{\partial v}{\partial y}(x, CE)CE'(x) - \frac{\partial v}{\partial x}(x, y_{d'}) \\
&+ \big(1 - \sum_{r \geq j_x} \Pr(r|d)\big)(\pi'(x) - \frac{\partial v}{\partial x}(x, y_{j_x - 1})), \\
D''(x) =& \frac{\partial^2 v}{\partial x^2}(x, CE(x)) + 2\frac{\partial^2 v}{\partial x \partial y}(x, CE(x))CE'(x) + \frac{\partial^2 v}{\partial y^2}(x, CE(x))[CE'(x)]^2 \\
&+ \frac{\partial v}{\partial y}(x, CE(x))CE''(x) - \frac{\partial^2 v}{\partial x^2}(x, y_{d'}) + \big(1 - \sum_{r \geq j_x} \Pr(r|d)\big)(\pi''(x) - \frac{\partial^2 v}{\partial x^2}(x, y_{j_x - 1})) \\
&> \frac{\partial^2 v}{\partial x^2}(x, CE(x)) - \frac{\partial^2 v}{\partial x^2}(x, y_{d'}) \\
&\geq 0.
\end{aligned}
$$

On the boundeferred acceptancery, $\partial_+ D(x) - \partial_- D(x) = \Pr(j|d, \mathbf{h})\big(\pi'(x) - \frac{\partial v}{\partial x}(x, y_j)\big) \geq 0$. Hence $\partial_+ D(x)$ is non-decreasing and $\partial_+ D(x_2) > \partial_+ D(x_1) \geq 0$, a contradiction.

Second, $B_{d'}$ can be possibly disconnected by B_d, $d \leq d^* = \max\{d : p_{d_B} = p_{1_B}\}$.

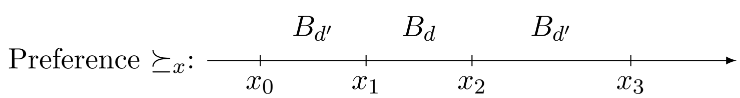


Figure 12: Disconnected neighborhood $B_{d'}$

By Condition 2, $j_3 \leq d'$. Denote by $D(x) = v(x, y_{d'}) - \max\{v(x, y_d), \pi(x)\}$. In the interior of each X_r, s.t. $r \leq d$, $D''(x) = \frac{\partial^2 v}{\partial x^2}(x, y_{d'}) - \frac{\partial^2 v}{\partial x^2}(x, y_d) < 0$. In the interior of each X_r, s.t. $r > d$, $D''(x) = [\frac{\partial^2 v}{\partial x^2}(x, y_{d'}) - \frac{\partial^2 v}{\partial x^2} v(x, y_d)] - [\pi''(x) - \frac{\partial^2 v}{\partial x^2}(x, y_d)] \leq 0$. At the discontinuous point, $\partial_+ D(x) - \partial_- D(x) = \frac{\partial v}{\partial x}(x, y_d) - \pi'(x) \leq 0$. Therefore $\partial_+ D(x_2) < \partial_+ D(x_1) \leq 0$, a contradiction.

A.2.9. Proof of Proposition 1.3.5

Prove by contradiction. Suppose indifference curve crosses the type space at least twice, as illustrated in Figure 13.

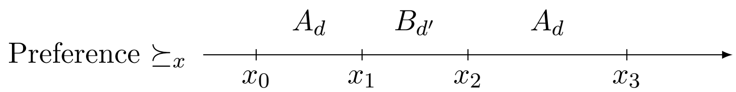


Figure 13: Disconnected Neighborhood A_d

Students of types $[x_0, x_1] \cup [x_2, x_3]$ prefers A_d to $B_{d'}$, while students of types in $[x_2, x_3]$ prefers $B_{d'}$ to A_d. $d^* = \max\{d : k_{d_A} \geq m_{d_A}\}$.

(i) $d > d^*$.

Since $m_{d_A} = \tilde{m}_{d_A}$, assume without loss of generality $[x_0, x_1] \bigcup [x_2, x_3] \subseteq \bigcup_{r \leq d} X_r$.

$$D(x) = \frac{k_{d_A}}{m_{d_A}} v(x, y_d) + (1 - \frac{k_{d_A}}{m_{d_A}}) \big(\sum_{r \geq j_x} q_r v(x, y_r) + (1 - \sum_{r \geq j_x} q_r) \pi(x) \big) - \max\{v(x, y_{d'}), \pi(x)\} + \Delta p.$$

If $d' < d$, then for all $x < x_2$, either $\partial_+ D(x) > \frac{k_{d_A}}{m_{d_A}} \frac{\partial v}{\partial x}(x, y_d) - \frac{\partial v}{\partial x}(x, y_{d'}) \geq 0$, or $\partial_+ D(x) = \frac{k_{d_A}}{m_{d_A}} (\frac{\partial v}{\partial x}(x, y_d) - \pi'(x)) + (1 - \frac{k_{d_A}}{m_{d_A}}) \big(\sum_{r \geq j_x} q_r \frac{\partial v}{\partial x}(x, y_r) - \pi'(x) \big)$. By assumption of the proposition, $\partial_+ D(x) \geq 0$ for all $x \leq x_2$ and at least one non-zero right derivative. Since $D(x_1) = 0$, $D(x) > 0$ for all $x > x_1$, a contradiction. If $d' \geq d$, then $j \leq d \leq d'$, and

$$-\partial_+ D(x) > (1 - \frac{k_{d_A}}{m_{d_A}}) [\sum_{r \geq j_x} q_r \big(\frac{\partial v}{\partial x}(x, y_{d'}) - \frac{\partial v}{\partial x}(x, y_r) \big) + (1 - \sum_{r \geq j_x} q_r) \big(\frac{\partial v}{\partial x}(x, y_{d'}) - \pi'(x)] \geq 0,$$

contradicting with $\partial_+ D(x_2) \geq 0$.

(ii) $d \leq d^*$.

Consider d, s.t. $p_{d_A} > p_{1_A}$, then $m_{d_A} = \tilde{m}_{d_A}$. Without loss of generality, assume $j_{x_0} \leq d$, hence $j_x \leq d$ for all $x \geq x_0$ by Condition 1.

$$\begin{aligned} D(x) =& U(x, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_{d'_B}; \mathbf{p}, \mathbf{h}) \\ =& \sum_{r \geq d+1} q_r v(x, y_r) + (1 - \sum_{r \geq d+1} q_r) v(x, y_d) - \max\{v(x, y_{d'}), \pi(x)\} + \Delta p \\ =& v(x, CE(x)) - \max\{v(x, y_{d'}), \pi(x)\} + \Delta p, \end{aligned}$$

If $d > d'$, $\partial_+ D(x) > 0$. Suppose $d \leq d'$, by similar argument we can show that (i) $CE(x) \geq y_{d'}$, for all $x \leq x_2$, (ii) $D''(x) \leq 0$.

$$\begin{aligned} D'(x) =& \frac{\partial v}{\partial x}(x, CE(x)) + \frac{\partial v}{\partial y}(x, CE(x))CE'(x) - \frac{\partial v}{\partial x}(x, y_{d'}), \\ D''(x) =& \frac{\partial^2 v}{\partial x^2}(x, CE(x)) + 2\frac{\partial^2 v}{\partial x \partial y}(x, CE(x))CE'(x) + \frac{\partial^2 v}{\partial y^2}(x, CE(x))[CE'(x)]^2 \\ &+ \frac{\partial v}{\partial y}(x, CE(x))CE''(x) - \frac{\partial^2 v}{\partial x^2}(x, y_{d'}) \\ &\leq \frac{\partial^2 v}{\partial x^2}(x, CE(x)) - \frac{\partial^2 v}{\partial x^2}(x, y_{d'}) + \frac{\partial v}{\partial y}(x, CE(x))CE''(x). \end{aligned}$$

Therefore $D''(x) \leq 0$,$\partial_+ D(x_1) \geq \partial_+ D(x_2) \geq 0$ and $\partial_+ D(x_1) = 0$ if and only if $CE(x) = y_{d'}$ for all $x \leq x_2$.

A.2.10. Proof of Proposition 1.3.7

Case 1. There exists $d^* = \max\{1 \leq d \leq D : m_d \leq k_d\}$.

For $d > d^*$, if $j_x > d$, $U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) = \pi(x) - p_d$. If $j_x \leq d$,

$$U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) + p_d = \frac{k_d}{\tilde{m}_d} v(x, y_d) + (1 - \frac{k_d}{\tilde{m}_d})[\sum_{d^* \geq r \geq j_x} \Pr(r|1, h)v(x, y_r) + (1 - \sum_{d^* \geq r \geq j_x} \Pr(r|1, h))\pi(x)].$$

For $d \leq d^*$,

$$U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) = \begin{cases} \sum_{d^* \geq r \geq j_x} \Pr(r|1, h)v(x, y_r) + (1 - \sum_{d^* \geq r \geq j_x} \Pr(r|1, h))\pi(x) - p_d, & \text{if } j_x > d, \\ \sum_{d^* \geq r > d} \Pr(r|1, h)v(x, y_r) + (1 - \sum_{d^* \geq r > d} \Pr(r|1, h))v(x, y_d) - p_d, & \text{if } j_x \leq d. \end{cases}$$

Case 2. $m_d > k_d$ for all d.

Let $p_{\min} = \min p_d$, $d^* = \max\{1 \leq d \leq D, p_d = p_{\min}\}$.

Claim 4. If $k_d \geq \tilde{m}_d$ for some $d \leq d^*$, then $k_{d'} > \tilde{m}_{d'}$, for all $d' < d$.

Proof of Claim 4: Suppose not and there exists some $d' < d$, $k_{d'} > \tilde{m}_{d'}$. For any $j_x \leq d$, $U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) > U(x, \mathbf{e}_{d'}; \mathbf{p}, \mathbf{h})$. Therefore $\tilde{m}_{d'} = \sum_{j \leq d'} m^j_{d'} = 0$, which is a contradiction.

Let $\tilde{d} = \max\{1 \leq r \leq D : \tilde{m}_d \leq k_d\}$. $\tilde{d} \leq d^*$, and by Claim 4, $\forall d' < \tilde{d}$, $\tilde{m}_{d'} = 0$. Consider

$d \geq \tilde{d}$, if $j_x > d$, $U(x, e_{d_A}; p, h) = \pi(x) - p_{d_A}$. If $j_x \leq d$,

$$U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) = \frac{k_d}{\tilde{m}_d} v(x, y_d) + (1 - \frac{k_d}{\tilde{m}_d})[\sum_{\tilde{d} \geq r \geq j_x} q_r v(x, y_r) + (1 - \sum_{\tilde{d} \geq r \geq j_x} q_r)\pi(x)] - p_d.$$

For $d < \tilde{d}$,

$$U(x, \mathbf{e}_d; \mathbf{p}, \mathbf{h}) = \begin{cases} \sum\limits_{\tilde{d} > r \geq j_x} \Pr(r|1, h) v(x, y_r) + (1 - \sum\limits_{\tilde{d} > r \geq j_x} \Pr(r|1, h))\pi(x) - p_d, & \text{if } j_x > d, \\ \sum\limits_{\tilde{d} > r > d} \Pr(r|1, h) v(x, y_r) + (1 - \sum\limits_{\tilde{d} > r > d} \Pr(r|1, h)) v(x, y_d) - p_d, & \text{if } j_x \leq d. \end{cases}$$

And $p_d = p_{\min}$ for each $d \leq d^*$.

A.2.11. Proof of Proposition 1.3.9

Only consider Case 1, since in Case 2, $m_{d_A} > k_{d_A} \geq m_{d_B}$. For $d > d^*$, $m_{d_A} > k_d \geq m_{d_B}$. Suppose $d \leq d^*$, for $j_x \leq d$, $U(x, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_{d_B}; \mathbf{p}, \mathbf{h}) \geq p_{d_B} - p_{d_A}$; for $j_x > d$, $U(x, \mathbf{e}_{d_A}; \mathbf{p}, \mathbf{h}) - U(x, \mathbf{e}_{d_B}; \mathbf{p}, \mathbf{h}) \geq p_{d_B} - p_{d_A}$. Hence $p_{d_B} \leq p_{d_A}$, i.e., $\frac{m_{d_A}}{\alpha k_d} \geq p^r_{d_B} \geq \min\{p^r_{d_B}, \frac{1}{\alpha}\} = \frac{m_{d_B}}{\alpha k_d}$. Hence $m_{d_A} \geq m_{d_B}$.

APPENDIX TO CHAPTER 2

B.1. Proof of Eqn (2.1)

To easily understand, consider the discrete case with infinitely many shops. Suppose there are S_L registered sellers in the platform and B_L subscribed buyers. Consider the probability of purchasing from firm j for a fixed buyer i,

$$
\begin{aligned}
&\Pr(\text{ a buyer purchases from firm } j \text{ in } k^{th} \text{ search })\\
=&\frac{(k-1)!\binom{S_L-1}{k-1}}{k!\binom{S_L}{k}}(\Pr(v-t<z_L))^{k-1}\Pr(v-t_j\geq z_L)\\
=&\frac{1}{S_L}\rho_L^{k-1}\bar{G}(t_j+z_L)
\end{aligned}
$$

where $\rho_L = \Pr(v-t<z_L)$ is the probability of an unsuccessful shopping experience and $\bar{G}(x) = 1-G(x)$. Hence the expected demand from B_L consumers, by posting p_j^D, p_j^L is,

$$
\begin{aligned}
X_L(p_j^D,p_j^L) &= B_L[\sum_k \frac{1}{S_L}\rho_L^{k-1}\bar{G}(t_j+z_L)]\\
&= \frac{B_L}{S_L(1-\rho_L)}\bar{G}(t_j+z_L)
\end{aligned}
\tag{B.1}
$$

B.2. Proof of Lemma 2.4.1

Notice that if $z^L < z^D$, then subscribed buyers never search online thus $S_L P_s = 0$. In this case, the platform does not operate in the first place.

Consider the case $z^L \geq z^D$ and will later confirm that it is an equilibrium. For firm j with marginal cost m_j,

Case 1 $p_j^L \le p_j^D + w_j$. Then

$$\begin{aligned}(p_j^L, p_j^D) =& \arg\max_{p_1 \le p_2 + w_j} \big[(p_1 - m_j)(1 - G(p_1 + z_L))\frac{B_L}{S_L(1-\rho_L)} \\ &+ (p_2 - m_j)(1 - G(p_2 + z_D))\frac{B_D}{S(1-\rho_D)}\big] \\ =& \arg\max_{p_1} \big[(p_1 - m_j)(1 - G(p_1 + z_L))\frac{B_L}{S_L(1-\rho_L)}\big] \\ &+ \arg\max_{p_2} \big[(p_2 - m_j)(1 - G(p_2 + z_D))\frac{B_d}{S(1-\rho_D)}\big] \\ =& (p(m_j; z_L), p(m_j; z_D))\end{aligned}$$

The second equation is due to first-order condition,

$$\frac{d}{dp}[(p-m)(1-G(p+z)] = 1 - G(p+z) - g(p+z)(p-m),$$

the right-hand side monotone in p, hence the unique global maximizer $p = p(m, z)$ satisfies,

$$p = m + \frac{1 - G(p+z)}{g(p+z)},$$

and $p(m; z)$ is non-increasing function of z due to Assumption 1[1]. Hence $p(m_j, z_L) \le p(m_j, z_D)$ and this holds for $\forall m_j$.

[1] $\frac{\partial}{\partial m} p(m; z) = \frac{1}{1 - \frac{\partial}{\partial x}(\frac{1-G(x)}{g(x)})}$

Case 2 $p_j^L > p_j^D + w_j$.

$$\begin{aligned}
\pi_L(p_j^L, p_j^D; m_j) =& \max_p \big[(p - m_j)(1 - G(p + z_L + w_j))\frac{B_l}{S_l(1-\rho_l)} \\
&+ (p - m_j)(1 - G(p + z_D))\frac{B_D}{S(1-\rho_D)}\big] \\
\leq& \max_p (p - m_j)(1 - G(p + z_L + w_j))\frac{B_l}{S_l(1-\rho_l)} \\
&+ \max_p (p - m_j)(1 - G(p + z_D))\frac{B_D}{S(1-\rho_D)} \\
\leq& \max_p (p - m_j)(1 - G(p + z_L))\frac{B_l}{S_l(1-\rho_l)} \\
&+ \max_p (p - m_j)(1 - G(p + z_d))\frac{B_d}{S(1-\rho_D)} \\
=& \pi_L(p(m_j; z_L), p(m_j; z_D); m_j),
\end{aligned}$$

which is not incentive compatible. Hence $(p_j^L, p_j^D) = (p(m_j; z_L), p(m_j; z_D))$.

B.3. Proof of Theorem 2.4.3

Now let $p_L = p(m_L, z_L), p = p(m, z_L)$. Some facts :

$$\begin{aligned}
\frac{\partial p_L}{\partial m_L} &= \frac{1 + \frac{\partial}{\partial t}\frac{1-G}{g}\frac{\partial z_L}{\partial m_L}}{1 - \frac{\partial}{\partial t}\frac{1-G}{g}} \\
\frac{\partial p_L}{\partial m_L} + \frac{\partial z_L}{m_L} &= \frac{1 + \frac{\partial z_L}{\partial m_L}}{1 - \frac{\partial}{\partial t}\frac{1-G}{g}} \\
\frac{\partial p}{\partial m_L} &= \frac{\frac{\partial}{\partial t}\frac{1-G}{g} \cdot \frac{\partial z_L}{\partial m_L}}{1 - \frac{\partial}{\partial t}\frac{1-G}{g}} \\
\frac{\partial p}{m_L} + \frac{\partial z_L}{\partial m_L} &= \frac{\frac{\partial z_L}{\partial m_L}}{1 - \frac{\partial}{\partial t}\frac{1-G}{g}}
\end{aligned}$$

The platform's revenue is given by,

$$
\begin{aligned}
\pi_0 =& P_s S_L + P_b B_L - C(B_L, S_L) \\
=& (\frac{B_L}{S_L(1-\rho_L)} \frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)} + V_s) S_L + (z_L - z_D + V_b) B_L - C(S_L, B_L) \\
=& B_L [\frac{(1-G(p_L+z_L))^2}{(1-\rho_L) g(p_L+z_L)} + z_L - z_D + V_b] + V_s S_L - C(B_L, S_L) \\
=& B_L \pi_{01} + V_s S F(m_L) - C(B_L, S F(m_L)).
\end{aligned}
$$

Denote by

$$
\begin{aligned}
\pi_{01} =& \frac{(1-G(p_L+z_L))^2}{(1-\rho_L) g(p_L+z_L)} + z_L - z_D + V_b \\
=& \frac{(1-G(p_L+z_L))^2}{\Pr(v - t \geq z_L) g(p_L+z_L)} + z_L - z_D + V_b \\
=& \frac{F(m_L)}{\int_{\underline{m}}^{m_L} 1 - G(p(m, z_L) + z_L) f(m) dm} \frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)} + z_L - z_D + V_b.
\end{aligned}
$$

Then the partial derivative is,

$$
\begin{aligned}
&\frac{\partial \pi_{01}}{\partial m_L}\\
=&\frac{f(m_L)}{\int_{\underline{m}}^{m_L} 1-G(p(m,z_L)+z_L)dF}\frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)}\\
&-\frac{F(m_L)}{[\int_{\underline{m}}^{m_L} 1-G(p(m,z_L)+z_L)dF]^2}\frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)}[(1-G(p_L+z_L))f(m_L)\\
&-\int_{\underline{m}}^{m_L} g(p(m,z_L)+z_L)(\frac{\partial}{\partial m_L}p(m,z_L)+\frac{\partial z_L}{\partial m_L})f(m)dm]\\
&+\frac{F(m_L)}{\int_{\underline{m}}^{m_L} 1-G(p(m,z_L)+z_L)f(m)dm}(\frac{\partial}{\partial x}[\frac{1-G}{g}(x)]\\
&\cdot(1-G(x))-\frac{1-G(x)}{g(x)}g(x))|_{x=p_L+z_L}(\frac{\partial p_L}{\partial m_L}+\frac{\partial z_L}{\partial m_L})+\frac{\partial z_L}{\partial m_L}\\
=&\frac{f(m_L)(1-G(p_L+z_L))^2}{g(p_L+z_L)\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF}-\frac{(1-G(p_L+z_L))^3F(m_L)f(m_L)}{g(p_L+z_L)(\int_{\underline{m}}^{m_L} 1-G(p(m,z_L)+z_L)dF)^2}\\
&+\frac{(1-G(p_L+z_L))^2F(m_L)}{g(p_L+z_L)(\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF)^2}\\
&\cdot\int_{\underline{m}}^{m_L} g(p+z_L)\frac{1}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dF\frac{\partial z_L}{\partial m_L}\\
&-\frac{F(m_L)(1-G(p_L+z_L))}{\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF}(1+\frac{\partial z_L}{\partial m_L})+\frac{\partial z_L}{\partial m_L}\\
=&I_1+I_2\cdot\frac{\partial z_L}{\partial m_L},
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=&\frac{f(m_L)(1-G(p_L+z_L))^2}{g(p_L+z_L)\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF}\frac{\int_{\underline{m}}^{m_L} G(p_L+z_L)-G(p+z_L)dF}{\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF}\\
&-\frac{F(m_L)(1-G(p_L+z_L))}{\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF},\\
I_2=&1+\frac{(1-G(p_L+z_L))^2F(m_L)\int_{\underline{m}}^{m_L} g(p+z_L)\frac{f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm}{g(p_L+z_L)(\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF)^2}\\
&-\frac{F(m_L)(1-G(p_L+z_L))}{\int_{\underline{m}}^{m_L} 1-G(p+z_L)dF}.
\end{aligned}
$$

From

$$c_L = \frac{1}{F(m_L)} \int_{\underline{m}}^{m_L} \int_{p(m,z_L)+z_L}^{\bar{v}} (v - p - z_L) dG(v) dF(m),$$

we can calculate its derivative with respect to m_L,

$$\begin{aligned}
c_L f(m_L) =& f(m_L) \int_{p_L+z_L}^{\bar{v}} (v - p_L - z_L) dG(v) \\
&+ \int_{\underline{m}}^{m_L} \int_{p(m,z_L)+z_L}^{\bar{v}} -(\frac{\partial p(m,z_L)}{\partial m_L} + \frac{\partial z_L}{\partial m_L}) dG(v) dF(m) \\
=& f(m_L) \int_{p_L+z_L}^{\bar{v}} (v - p_L - z_L) dG \\
&- \int_{\underline{m}}^{m_L} \int_{p(m,z_L)+z_L}^{\bar{v}} \frac{1}{1 - \frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dG(v) dF(m) \cdot \frac{\partial z_L}{\partial m_L} \\
=& f(m_L) \int_{p_L+z_L}^{\bar{v}} (v - p_L - z_L) dG - \int_{\underline{m}}^{m_L} \frac{1 - G(p + z_L)}{1 - \frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p+z_L}} dF \cdot \frac{\partial z_L}{\partial m_L}.
\end{aligned}$$

The second equality is due to, $\frac{\partial p(m,z_L)}{\partial m_L} = \frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}(\frac{\partial p(m,z_L)}{\partial m_L} + \frac{\partial z_L}{\partial m_L})$. Hence,

$$\frac{\partial p(m,z_L)}{\partial m_L} + \frac{\partial z_L}{\partial m_L} = \frac{1}{1 - \frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} \frac{\partial z_L}{\partial m_L}.$$

To calculate the partial derivative of the second term:

$$\begin{aligned}
\frac{\partial z_L}{\partial m_L} =& \frac{f(m_L)(c_L - \int_{p_L+z_L}^{\bar{v}} (v - p_L - z_L) dG)}{- \int_{\underline{m}}^{m_L} \frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dF} \\
=& \frac{\frac{f(m_L)}{F(m_L)} [\int_{\underline{m}}^{m_L} \int_{p(m,z_L)+z_L}^{\bar{v}} (v - p - z_L) dG dF - \int_{\underline{m}}^{m_L} \int_{p_L+z_L}^{\bar{v}} (v - p_L - z_L) dG dF]}{- \int_{\underline{m}}^{m_L} \frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dF} \\
=& \frac{\frac{f(m_L)}{F(m_L)} [\int_{\underline{m}}^{m_L} \int_{p_L+z_L}^{\bar{v}} p_L - p(m,z_L) dG dF + \int_{\underline{m}}^{m_L} \int_{p(m,z_L)}^{p_L} (x - p) dG(x) dF]}{- \int_{\underline{m}}^{m_L} \frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dF} \\
=& \frac{f(m_L)[\int_{\underline{m}}^{m_L} p_L - p(m,z_L) dF \cdot (1 - G(p_L + z_L)) + \int_{\underline{m}}^{m_L} \int_{p(m,z_L)}^{p_L} (x - p) dG dF]}{-F(m_L) \int_{\underline{m}}^{m_L} \frac{1-G(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dF} \\
=& - \frac{f(m_L)[\int_{\underline{m}}^{m_L} \frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}} dm \cdot (1 - G(p_L + z_L))]}{F(m_L) \int_{\underline{m}}^{m_L} \frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dF} \\
&- \frac{f(m_L) \int_{\underline{m}}^{m_L} \int_{p(m,z_L)}^{p_L} (x - p(m,z_L)) dG(x) dF]}{F(m_L) \int_{\underline{m}}^{m_L} \frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dF}
\end{aligned}$$

the last equality is due to

$$\begin{aligned}\int_{\underline{m}}^{m_L} p(m_L,z_L)-p(m,z_L)dF &= \int_{\underline{m}}^{m_L}\int_{m}^{m_L}\frac{f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(t,z_L)+z_L}}dtdm\\ &= \int_{\underline{m}}^{m_L}\int_{\underline{m}}^{t}\frac{f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(t,z_L)+z_L}}dmdt\\ &= \int_{\underline{m}}^{m_L}\frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm\end{aligned}$$

And

$$\int_{\underline{m}}^{m_L} G(p(m_L,z_L)+z_L)-G(p+z_L)dF = \int_{\underline{m}}^{m_L}\frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm.$$

Therefore

$$\frac{\partial\pi_0}{\partial m_L} = B_L(I_1+I_2\frac{\partial z_L}{\partial m_L})+V_sSf(m_L)-\frac{\partial C}{\partial y}(B_L,S_L)Sf(m_L),$$

and

$$\begin{aligned}&I_1+I_2\frac{\partial z_L}{\partial m_L}\\ =&\frac{f(m_L)(1-G(p_L+z_L))^2\int_{\underline{m}}^{m_L}\frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm}{g(p(m_L,z_L)+z_L)[\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)f(m)dm]^2}-\frac{F(m_L)(1-G(p(m_L,z_L)+z_L))}{\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)dF}\\ &-[\frac{f(m_L)(1-G(p_L+z_L))\int_{\underline{m}}^{m_L}\frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm}{F(m_L)\int_{\underline{m}}^{m_L}\frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}}dF}\\ &+\frac{f(m_L)\int_{\underline{m}}^{m_L}\int_{p(m,z_L)}^{p_L}(x-p)dG(x)dF}{F(m_L)\int_{\underline{m}}^{m_L}\frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}}dF(m)}]\\ &\cdot[\frac{(1-G(p_L+z_L))^2F(m_L)\int_{\underline{m}}^{m_L}\frac{g(p(m,z_L)+z_L)f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm}{g(p(m_L,z_L)+z_L)(\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)dF)^2}+1-\frac{F(m_L)(1-G(p_L+z_L))}{\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)dF}]\\ \leq&\frac{f(m_L)(1-G(p_L+z_L))^2\int_{\underline{m}}^{m_L}\frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm}{g(p_L+z_L)[\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)f(m)dm]^2}-\frac{F(m_L)(1-G(p_L+z_L))}{\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)dF}\\ &-\frac{f(m_L)(1-G(p_L+z_L))\int_{\underline{m}}^{m_L}\frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm}{F(m_L)\int_{\underline{m}}^{m_L}\frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}}dF}\\ &\cdot[\frac{(1-G(p_L+z_L))^2F(m_L)\int_{\underline{m}}^{m_L}\frac{g(p(m,z_L)+z_L)f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm}{g(p_L+z_L)(\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)dF)^2}+\frac{\int_{\underline{m}}^{m_L}\frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm}{\int_{\underline{m}}^{m_L}1-G(p(m,z_L)+z_L)dF}]\end{aligned}$$

where $p_L = p(m_L, z_L)$. The second inequality is due to

$$1 - \frac{F(m_L)(1-G(p_L+z_L))}{\int_{\underline{m}}^{m_L} 1 - G(p(m,z_L)+z_L)dF} = \frac{\int_{\underline{m}}^{m_L} G(p_L+z_L) - G(p(m,z_L)+z_L)dF}{\int_{\underline{m}}^{m_L} 1 - G(p(m,z_L)+z_L)dF} \geq 0$$

Need to show that $I_1 + I_2 \frac{\partial z_L}{\partial m_L}$, which suffices to show that

$$\begin{aligned}
&\frac{(1-G(p_L+z_L))\int_{\underline{m}}^{m_L} \frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm}{g(p_L+z_L)\int_{\underline{m}}^{m_L} 1-G(p+z_L)f(m)dm} \\
\leq & \frac{F(m_L)}{f(m_L)} + \frac{(1-G(p_L+z_L))^2 \int_{\underline{m}}^{m_L} \frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm \int_{\underline{m}}^{m_L} \frac{g(p+z_L)f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm}{g(p_L+z_L)\int_{\underline{m}}^{m_L} \frac{1-G(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p+z_L}}dF \int_{\underline{m}}^{m_L} 1-G(p+z_L)dF} \\
&+ \frac{\int_{\underline{m}}^{m_L} \frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm \int_{\underline{m}}^{m_L} \frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm}{F(m_L)\int_{\underline{m}}^{m_L} \frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}}dF}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{1-G(p_L+z_L)}{g(p_L+z_L)}\int_{\underline{m}}^{m_L} \frac{g(p+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p(m,z_L)+z_L}}dm \int_{\underline{m}}^{m_L} \frac{1-G(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}}dF \\
\leq & \frac{F(m_L)}{f(m_L)}\int_{\underline{m}}^{m_L} \frac{1-G(p)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p+z_L}}dF \int_{\underline{m}}^{m_L} 1-G(p+z_L)dF \\
&+ \frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)}\int_{\underline{m}}^{m_L} \frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm \int_{\underline{m}}^{m_L} \frac{g(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dF \\
&+ \frac{1}{F(m_L)}\int_{\underline{m}}^{m_L} \frac{F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm \\
&\cdot \int_{\underline{m}}^{m_L} \frac{g(p(m,z_L)+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm \int_{\underline{m}}^{m_L} 1-G(p+z_L)dF
\end{aligned}$$

Notice under the condition $f'(m) \le 0$, we have

$$
\begin{aligned}
&\frac{1-G(p_L+z_L)}{g(p_L+z_L)}\int_{\underline{m}}^{m_L}\frac{g(p+z_L)F(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm\int_{\underline{m}}^{m_L}\frac{1-G(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p+z_L}}dF\\
\le& F(m_L)\int_{\underline{m}}^{m_L}\frac{1-G(p+z_L)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm\int_{\underline{m}}^{m_L}\frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}}dF\\
\le&\frac{F(m_L)}{f(m_L)}\int_{\underline{m}}^{m_L}\frac{(1-G(p(m,z_L)+z_L))f(m)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})|_{x=p+z_L}}dm\int_{\underline{m}}^{m_L}\frac{1-G(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p+z_L}}dF\\
\le&\frac{F(m_L)}{f(m_L)}\int_{\underline{m}}^{m_L}(1-G(p(m,z_L)+z_L))dF\int_{\underline{m}}^{m_L}\frac{1-G(p(m,z_L)+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p+z_L}}dF
\end{aligned}
$$

Hence $I_1 + I_2\frac{\partial z_L}{\partial m_L} \le 0$ Therefore

$$\frac{\partial \pi_0}{\partial m_L} = B_L(I_1 + I_2\frac{\partial z_L}{\partial m_L}) + Sf(m_L)(V_s - \frac{\partial C}{\partial y}(B_L, S_L)) \le 0.$$

B.4. Proof of Theorem 2.5.1

$$
\begin{aligned}
W^L(c_L) =& \frac{\int\int_{v-t\ge z_L} v - m dGd\tilde{F}_L}{1-\rho_L} - \frac{c_L}{1-\rho_L}\\
=& \frac{\int\int_{v-t\ge z_L} t - m dGd\tilde{F}_L}{1-\rho_L} + z_L\\
=& \frac{\int \frac{1-G(p+z_L))^2}{g(p+z_L)}d\tilde{F}_L}{1-\rho_L} + z_L
\end{aligned}
\tag{B.2}
$$

Suppose that m_L^* is the optimal threshold strategy selected by the platform such that

$$m_L^* = \arg\max \pi_0(m_L).$$

Then m_L^* is a function of c_L, and

$$\begin{aligned}
\frac{\partial p}{\partial c_L} =& \frac{\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L} \cdot \frac{\partial z_L}{\partial c_L}}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} \\
\frac{\partial p}{\partial c_L} + \frac{\partial z_L}{\partial c_L} =& \frac{\frac{\partial z_L}{\partial c_L}}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} \\
\frac{\partial p_L}{\partial c_L} + \frac{\partial z_L}{\partial c_L} =& \frac{\frac{\partial m_L^*}{\partial c_L} + \frac{\partial z_L}{\partial c_L}}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p_L+z_L}}
\end{aligned}$$

From

$$c_L F(m_L^*) = \int_{\underline{m}}^{m_L^*} \int_{p(m,z_L)+z_L}^{\bar{v}} (v-p-z_L) dG(v) dF(m),$$

we can calculate its derivative with respect to c_L,

$$\begin{aligned}
c_L f(m_L^*) \frac{\partial m_L^*}{c_L} + F(m_L^*) =& \int_{\underline{m}}^{m_L^*} \int_{p(m,z_L)+z_L}^{\bar{v}} -(\frac{\partial p}{\partial c_L} \\
&+ \frac{\partial z_L}{\partial c_L}) dG dF + \int_{p_L+z_L}^{\bar{v}} (v-p_L-z_L) dG \cdot f(m_L^*) \frac{\partial m_L^*}{\partial c_L} \\
=& \int_{\underline{m}}^{m_L^*} \int_{p(m,z_L)+z_L}^{\bar{v}} -\frac{\frac{\partial z_L}{\partial c_L}}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{t=p(m,z_L)+z_L}} dG dF \\
&+ \int_{p_L+z_L}^{\bar{v}} (v-p_L-z_L) dG \cdot f(m_L^*) \frac{\partial m_L^*}{\partial c_L} \\
=& \int_{\underline{m}}^{m_L^*} -\frac{1-G(x)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})}|_{x=p(m,z_L)+z_L} dF \cdot \frac{\partial z_L}{\partial c_L} \\
&+ \int_{p_L+z_L}^{\bar{v}} (v-p_L-z_L) dG \cdot f(m_L^*) \frac{\partial m_L^*}{\partial c_L}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial z_L}{\partial c_L} =& \frac{(\int_{p_L+z_L}^{\bar{v}} (v-p_L-z_L) dG - c_L) f(m_L^*) \frac{\partial m_L^*}{\partial c_L} - F(m_L^*)}{\int_{\underline{m}}^{m_L^*} \frac{1-G(x)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})}|_{x=p(m,z_L)+z_L} dF} \\
=& -\frac{\frac{\partial m_L^*}{\partial c_L} \frac{f(m_L^*)}{F(m_L^*)} [(1-G(p_L+z_L)) \int_{\underline{m}}^{m_L^*} (p_L-p) dF + \int_{\underline{m}}^{m_L^*} \int_{p}^{p_L} (v-p) dG dF] + F(m_L^*)}{\int_{\underline{m}}^{m_L^*} \frac{1-G(x)}{1-\frac{\partial}{\partial x}(\frac{1-G}{g})}|_{x=p(m,z_L)+z_L} dF}
\end{aligned}$$

For seller registered on the platform

$$
\begin{aligned}
\frac{\partial W^L}{\partial c_L} =& \frac{\partial z_L}{\partial c_L} + \frac{\frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)} f(m_L^*)}{\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF} \frac{\partial m_L^*}{\partial c_L} + \frac{\int \frac{\partial}{\partial x}[\frac{(1-G(x))^2}{g(x)}]|_{x=p+z_L} \cdot (\frac{\partial p}{\partial c_L} + \frac{\partial z_L}{\partial c_L})dF}{\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF} \\
&- \frac{\int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF}{(\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF)^2} [(1-G(p_L+z_L))f(m_L^*)\frac{\partial m_L^*}{\partial c_L} \\
&- \int_{\underline{m}}^{m_L^*} g(p+z_L)(\frac{\partial p}{\partial c_L} + \frac{\partial z_L}{\partial c_L})dF] \\
=& \frac{\partial m_L^*}{\partial c_L}[\frac{\frac{(1-G(p_L+z_L))^2}{g(p_L+z_L)} f(m_L^*)}{\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF} - \frac{\int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF}{(\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF)^2}(1-G(p_L+z_L))f(m_L^*)] \\
&+ \frac{\partial z_L}{\partial c_L}[\frac{\int \frac{\partial}{\partial x}[\frac{(1-G(x))^2}{g(x)}]|_{x=p+z_L} \frac{1}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{x=p+z_L}} dF}{\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF} \\
&+ \frac{\int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF}{(\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF)^2} \int_{\underline{m}}^{m_L^*} \frac{g(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{x=p+z_L}} dF + 1] \\
=& \frac{\partial m_L^*}{\partial c_L} \frac{(1-G(p_L+z_L))f(m_L^*)}{(\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF)^2}[\frac{1-G(p_L+z_L)}{g(p_L+z_L)} \int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF \\
&- \int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF] \\
&+ \frac{\partial z_L}{\partial c_L}[\frac{\int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF}{(\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF)^2} \int_{\underline{m}}^{m_L^*} \frac{g(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{x=p+z_L}} dF]
\end{aligned}
$$

Here,

$$
\frac{\int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF}{(\int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF)^2} \int_{\underline{m}}^{m_L^*} \frac{g(p+z_L)}{1-\frac{\partial}{\partial t}(\frac{1-G}{g})|_{x=p+z_L}} dF > 0.
$$

Moreover by monotonicity of $\frac{1-G(x)}{g(x)}$,

$$
\frac{1-G(p_L+z_L)}{g(p_L+z_L)} \int_{\underline{m}}^{m_L^*} 1-G(p+z_L)dF - \int \frac{(1-G(p+z_L))^2}{g(p+z_L)} dF \leq 0.
$$

If $\frac{\partial m_L^*}{\partial c_L} \geq 0$, we have $\frac{\partial z_L}{\partial c_L} < 0$, therefore

$$
\frac{\partial W^L}{\partial c_L} \leq 0.
$$

APPENDIX TO CHAPTER 3

C.1. Proof of Theorem 3.2.1 under the choice "$\alpha+\beta$"

Under $\alpha+\beta$ commission fees:

b propose to s : $(v_{bs}-(1+\beta)p,(1-\alpha)p-c_s)$

s proposes to b: $(v_{bs}-(1+\beta)p,(1-\alpha)p-c_s)$.

I provide a characterization of MPE payoffs and strategies. Suppose agent of type i has discounting factor $\delta_i \in (0,1)$. The Markov state in this model is the network induced by players who did not reach agreement, and the commission fees, the matchmaker's identity and the selection of a link and a proposer.

Denote by h_t a complete history of the game up to (not including) time t, which is the choice of payment schemes, a sequence of $t-1$ pairs of proposers and responders in G with corresponding proposals and responses, payment to the middleman if possible and whether the matched pair exiting the market with an agreement are replaced by an exactly new pair of traders at the same position. The history h_t uniquely determines the set of players $V(h_t)$ still active at time t. Denote by $G(h_t)$ the subnetwork of G induced by $V(h_t)$. Denote by $\mathscr{H}$ the set of any possible complete histories at $t \geq 1$ and $\mathscr{G}$ the set of subnetworks of G induced by any complete histories, i.e., $\mathscr{G} = \cup_{h\in\mathscr{H}} G(h)$. The history $(h_t; \mathbb{1}_M)$ denotes the history h_t followed by the matching maker being middleman M or nature ($\mathbb{1}_M = 1$ if it's middleman who selects the pair at time t and 0 otherwise). The history $(h_t; \mathbb{1}_M; i \to j)$ consists of h_t followed by nature/middleman selecting i to propose to j, and $(h_t; \mathbb{1}_M; i \to j; x)$ consists of additionally the proposed transfer $x \in R$ made by i to j.

A strategy profile $\{\sigma\}$ is a **Markov (stationary) strategy profile** if for any $1 \leq i \leq N$, any $h_t, h'_{t'} \in \mathscr{H}$ s.t. $G(h_t) = G(h'_{t'})$,

$$\sigma_i^{\alpha+\beta}(h_t; \mathbb{1}_M; i \to j) = \sigma_i^{\alpha+\beta}(h'_{t'}; \mathbb{1}_M; i \to j),$$

$$\sigma_i^{\alpha+\beta}(h_t; \mathbb{1}_M; j \to i; x) = \sigma_i^{\alpha+\beta}(h'_{t'}; \mathbb{1}_M; j \to i; x)$$

$$\sigma_M^{\alpha+\beta}(h_t; \mathbb{1}_M) = \sigma_M^{\alpha+\beta}(h'_{t'}; \mathbb{1}_M)$$

A Markov strategy σ is Markov Perfect Equilibrium if it is Markov (stationary) strategy and subgame perfect.

Given a set of network $\mathscr{G}$, a collection of Markov strategy profile σ for the respective game $\Gamma^{\alpha+\beta}(G)$ is **subgame consistent** if for any pair of networks $G, G' \in \mathscr{G}$, the Markov strategy profiles $\sigma^{\alpha+\beta}(G)$ and $\sigma^{\alpha+\beta}(G')$, conditional the initial state G and $\alpha+\beta$, induce the same behavior in any pair of identical subgames of $\Gamma^{\alpha+\beta}(G')$ and $\Gamma^{\alpha+\beta}(G)$. Formally, subgame consistency of $(\Gamma^{\alpha+\beta}(G))_{G\in\mathscr{G}}$ if for any pair of networks $G, G' \in \mathscr{G}$, and any $h_t, h'_{t'} \in \mathscr{H}$, s.t., $G(h_t) = G'(h'_{t'})$,

$$\sigma_i(G)(h_t; \mathbb{1}_M; i \to j) = \sigma_i(G')(h'_{t'}; \mathbb{1}_M; i \to j),$$

$$\sigma_i(G)(h_t; \mathbb{1}_M; j \to i; x) = \sigma_i(G')(h'_{t'}; \mathbb{1}_M; j \to i; x)$$

$$\sigma_M(G)(h_t; \mathbb{1}_M) = \sigma_M(G')(h'_{t'}; \mathbb{1}_M)$$

I use a fixed point argument to give an non-constructive proof of the existence of MPEs. Denote $\mathbf{u}$ the vector of equilibrium payoff induced by an Markov strategy profile σ of the game $\Gamma^{\alpha+\beta}(G)$. By definition, any MPE σ^* must belong to a subgame consistent collection of MPEs $\sigma|_{G'}$ of the respective subgames $G' \in \mathscr{G}$. In particular, when $\Gamma^{\alpha+\beta}(G)$ is played according to MPE $\sigma^*(G)$, every player $i \in B \cup S$ has ex ante payoffs $u_i(G, \sigma^*)$ before any pairs are selected, and $u_k(G \ominus \{i,j\}, \sigma^*)$ at the beginning of any subgame before which only i and j reached an agreement ($k \neq i, j$).

Fix a history h_t along which no agreement has been reached, that is $G(h_t) = G_0$. Suppose now that it is middleman's turn to select a bargaining pair $e \in E$. After the matched pair is realized, suppose is (b, s), with b being the proposer, in the subgame following $(h_t; b \to s, x)$, it must be that the strategy $\sigma_s(h_t; b \to s, x)$ specifies that player s accepts any offer s.t. $(1-\alpha)p - c_s > \delta_s u_s$, and reject any offer s.t. $(1-\alpha)p - c_s < \delta_s u_s$, and may accept with positive probability the offer $p = \frac{c_s+\delta_s u_s}{1-\alpha}$. Then by subgame perfection, player b does not offer to pay a price more than $\frac{c_s+\delta_s u_s}{1-\alpha}$. On the other hand, the offer p proposed by b must satisfy $v_{bs} - (1+\beta)p \geq \delta_b u_s$, i.e., $p \leq \frac{v_{bs}-\delta_b u_b}{1+\beta}$.

Let q_{bs} be the probability (conditional on $(h_t; \mathbb{1}_M = 1; b \to s)$) of the joint event that b offers $\min\{\frac{c_s+\delta_s u_s}{1-\alpha}, \frac{v_{bs}-\delta_b u_b}{1+\beta}\}$ to s and the offer is accepted. The payoff of player $k \neq b, s$ at the beginning of the next period conditional on $(h_t; \mathbb{1}_M = 1; b \to s))$ is $q_{bs}(1-\gamma)\delta_k u_k(G_0 \ominus \{b, s\}) + [q_{bs}\gamma + (1-q_{bs})]\delta_k u_k(G_0)$.

Case 1. $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > \delta_b u_b$.

Conditional on $(h_t; b \to s)$, it must be that $q_{bs} = 1$. To see this, if $\tilde{q}_{bs} < 1$, then b's expected payoff conditional on offering $p = \frac{c_s+\delta_s u_s}{1-\alpha}$ is $q_{bs}(v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) + (1 - q_{bs})\delta_b u_s < v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)$, while it is more profitable to offer $p = \frac{c_s+\delta_s u_s}{1-\alpha} + \epsilon$ with payoff $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) - \epsilon(1+\beta)$, for sufficiently small $\epsilon > 0$. And it's obvious that offers p smaller than $\frac{c_s+\delta_s u_s}{1-\alpha}$ are not optimal for i since they are rejected with certainty and yields $\frac{c_s+\delta_s u_s}{1-\alpha} < v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)$. Hence s has no best response to b's equilibrium strategy when $q_{ij} < 1$, a contradiction. Hence $q_{bs} = 1$. Moreover $\Pr(\text{`}Yes'|(h_t; \mathbb{1}_M = 1; b \to s; p)) = q_{bs} = 1$, for $p = \frac{c_s+\delta_s u_s}{1-\alpha}$.

Case 2. $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) < \delta_b u_b$.

By a similar fashion, one could show that $q_{bs} = 0 = \Pr(\text{`}Yes'|(h_t; b \to s; p))$ since b's offer is no more than $\frac{v_{bs}-\delta_b u_b}{1+\beta} < \frac{c_s+\delta_s u_s}{1-\alpha}$.

Case 3. $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) = \delta_b u_b$.

When b is chosen to propose a bid-price to s, whether they reach an agreement or not, the payoff for both agents are respectively $\delta_b u_b$ and $\delta_s v_s$, i.e., any value $q_{bs} \in [0, 1]$ support the $(h_t; \mathbb{1}_M = 1; b \to s; p)$. Since b and s are indifferent between any $p \leq \frac{c_s+\delta_s u_s}{1-\alpha}$,WOLG assume that $q_{bs} = \Pr(\text{`Yes'}|(h_t; \mathbb{1}_M = 1; b \to s; p))$.

Similarly, suppose now that it is middleman's turn to pick a bargaining pair $e \in E$. After the matched pair is realized, suppose is (s, b), with s being the proposer, in the subgame following $(h_t; \mathbb{1}_M = 1; s \to b; p)$, it must be that the strategy $\sigma_s(G_0)(h_t; \mathbb{1}_M = 1; b \to s; p)$ specifies that player b accepts any price p s.t. $v_{bs} - (1+\beta)p > \delta_b u_b$, and reject any price p s.t. $v_{bs} - (1+\beta)p < \delta_b u_b$, and may accept with some probability the offer $p = \frac{v_{bs} - \delta_b u_b}{1+\beta}$. Similar argument can be applied to the induction of q_{sb}, with $\Pr(\text{`}Yes'|(h_t; \mathbb{1}_M = 1; b \to s; p)) = q_{sb}$ where $p = \max\{\frac{c_s + \delta_s u_s}{1-\alpha}, \frac{v_{bs} - \delta_b u_b}{1+\beta}\}$.

Hence for any mixed matching strategy $\mathbf{w}$ by the middleman M over matched pair E, the continuation payoff $\{u_i(G_0)\}$ satisfy,

$$\begin{aligned} u_b =& \sum_s \lambda w_{bs} q_{bs} (v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) + (1-\lambda) p_{bs} \tilde{q}_{bs} (v_{bs} - (\delta_s u_s + c_s)) \\ &+ \sum_s \big((\lambda w_{bs}(1 - q_{bs}) + (1-\lambda) p_{bs}(1 - \tilde{q}_{bs})\big)\delta_b u_b \\ &+ \sum_s (\lambda w_{sb} + (1-\lambda) p_{sb}) \delta_b u_b \\ &+ \sum_{j,k \neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})(1-\gamma) \delta_b u_b (G_0 \ominus \{j,k\}) \\ &+ \big((\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})\gamma + \lambda w_{kj}(1 - q_{kj}) + (1-\lambda) p_{kj}(1 - \tilde{q}_{kj})\big)\delta_b u_b] \end{aligned} \tag{C.1}$$

$$\begin{aligned} u_s =& \sum_b \lambda w_{sb} q_{sb} (\frac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s) + (1-\lambda) p_{sb} \tilde{q}_{sb} (v_{bs} - \delta_b u_b - c_s) \\ &+ \sum_b \big((\lambda w_{sb}(1 - q_{sb}) + (1-\lambda) p_{sb}(1 - \tilde{q}_{sb})\big)\delta_s u_s \\ &+ \sum_b (\lambda w_{bs} + (1-\lambda) p_{bs}) \delta_s u_s \\ &+ \sum_{j,k \neq s} [(\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})(1-\gamma) \delta_s u_s (G_0 \ominus \{j,k\}) \\ &+ \big((\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})\gamma + \lambda w_{kj}(1 - q_{kj}) + (1-\lambda) p_{kj}(1 - \tilde{q}_{kj})\big)\delta_s u_s] \end{aligned} \tag{C.2}$$

where

$$q_{bs} = \begin{cases} 1 & \text{if } v_{bs} - \dfrac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > \delta_b u_b \\ [0,1] & \text{if } v_{bs} - \dfrac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) = \delta_b u_b, \\ 0 & \text{if } v_{bs} - \dfrac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) < \delta_b u_b. \end{cases} \tag{C.3}$$

And

$$q_{sb} = \begin{cases} 1 & \text{if } \dfrac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s > \delta_s u_s \\ [0,1] & \text{if } \dfrac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s = \delta_s u_s, \\ 0 & \text{if } \dfrac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s < \delta_s u_s. \end{cases} \tag{C.4}$$

Notice that

$$v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > \delta_b u_b \iff \frac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s > \delta_s u_s\}.$$

Denote by $\tilde{q}_{ij} = \Pr(`Yes'|(h_t; \mathbb{1}_M = 0; i \to j; p))$ where $p = \min\{c_s + \delta_s u_s, v_{bs} - \delta_b u_b\}$ if $i \in B$ and $p = \max\{c_s + \delta_s u_s, v_{bs} - \delta_b u_b\}$ if $i \in S$, then by similar arguments,

$$\tilde{q}_{sb} = \begin{cases} 1 & \text{if } v_b - c_s > \delta_b u_b + \delta_s u_s \\ [0,1] & \text{if } v_b - c_s = \delta_b u_b + \delta_s u_s, \\ 0 & \text{if } v_b - c_s < \delta_b u_b + \delta_s u_s. \end{cases} \tag{C.5}$$

$$\tilde{q}_{bs} = \begin{cases} 1 & \text{if } v_b - c_s > \delta_b u_b + \delta_s u_s, \\ [0,1] & \text{if } v_b - c_s = \delta_b u_b + \delta_s u_s, \\ 0 & \text{if } v_b - c_s < \delta_b u_b + \delta_s u_s. \end{cases} \tag{C.6}$$

Since the middleman is maximizing her discounted aggregate profit when deciding which link to pick from G,

$$\begin{aligned}
&\pi_M(G)\\
=&(1-\delta_M)\lambda\max_{b,s}\max\{q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s+c_s)+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]+(q_{bs}\gamma+1-q_{bs})\delta_M\pi_M(G),\\
&q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs}-\delta_b u_b)+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]+\big(q_{sb}\gamma+1-q_{sb}\big)\delta_M\pi_M(G)\}\\
&+(1-\delta_M)(1-\lambda)\sum_{i,j\in B\cup S}p_{ij}[\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})+(\tilde{q}_{ij}\gamma+1-\tilde{q}_{ij})\delta_M\pi_M(G)],
\end{aligned}$$

where

$$\begin{aligned}
&w_{bs}+w_{sb}>0\Rightarrow\pi_M(G)\\
=&(1-\delta_M)\lambda\max\{q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s+c_s)+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]+(q_{bs}\gamma+1-q_{bs})\delta_M\pi_M(G),\\
&q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs}-\delta_b u_b)+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]+\big(q_{sb}\gamma+1-q_{sb}\big)\delta_M\pi_M(G)\}\\
&+(1-\delta_M)(1-\lambda)\sum_{i,j\in B\cup S}p_{ij}[\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})+(\tilde{q}_{ij}\gamma+1-\tilde{q}_{ij})\delta_M\pi_M(G)].
\end{aligned}$$

This yields:

$$\begin{aligned}
&\pi_M(G)/(1-\delta_M)\\
=&\max\{\max_{b,s}\frac{\lambda q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s+c_s)+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]+(1-\lambda)\sum_{i,j\in B\cup S}p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})}{1-\lambda\delta_M(q_{bs}\gamma+1-q_{bs})-(1-\lambda)\delta_M\sum_{i,j\in B\cup S}p_{ij}(\tilde{q}_{ij}\gamma+1-\tilde{q}_{ij})}\\
&\ldots,\max_{b,s}\frac{\lambda q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs}-\delta_b u_b)+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]+(1-\lambda)\sum_{i,j\in B\cup S}p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})}{1-\lambda\delta_M(q_{bs}\gamma+1-q_{sb})-(1-\lambda)\delta_M\sum_{i,j\in B\cup S}p_{ij}(\tilde{q}_{ij}\gamma+1-\tilde{q}_{ij})}\}
\end{aligned}$$

Let $g_{ij}(\mathbf{u})=\frac{\alpha+\beta}{1-\alpha}(\delta_j u_j+c_j)$ if $i\in B, j\in S$, and $g_{ij}(\mathbf{u})=\frac{\alpha+\beta}{1+\beta}(v_{ij}-\delta_j u_j)$ if $i\in S, j\in B$.

$$
\begin{aligned}
&f(\mathbf{u},\mathbf{q},\tilde{\mathbf{q}})\\
=&\{(\mathbf{u}',\mathbf{q}',\tilde{\mathbf{q}}')|\tilde{q}'_{bs}=\tilde{q}'_{sb}=1(0) \text{ if } v_{bs}-c_s>(<)\delta_b u_b+\delta_s u_s,\\
&\text{ and } \tilde{q}'_{sb},\tilde{q}'_{bs}\in[0,1] \text{ if } v_{bs}-c_s=\delta_b u_b+\delta_s u_s,\\
&q'_{sb}=q'_{bs}=1(0) \text{ if } v_{bs}-\frac{1+\beta}{1-\alpha}(\delta_s u_s+c_s)>(<)\delta_b u_b, \text{ and } q'_{sb},q'_{bs}\in[0,1] \text{ otherwise };\\
&\exists \mathbf{w}\in\Delta, \text{ s.t. } \sum_{i,j\in B\cup S} w_{ij}\le 1, \text{ and}\\
&\{ij|w_{ij}>0\}\\
\subseteq&\frac{\lambda q_{ij}[g_{ij}(\mathbf{u})+(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})]+(1-\lambda)\sum_{i,j\in B\cup S}p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})}{1-\lambda\delta_M(q_{ij}\gamma+1-q_{ij})-(1-\lambda)\delta_M\sum_{k,l\in B\cup S}p_{kl}(\tilde{q}_{kl}\gamma+1-\tilde{q}_{kl})};\\
&u'_b=\sum_s \lambda w_{bs}q_{bs}\max\{v_b-\frac{1+\beta}{1-\alpha}(\delta_s u_s+c_s),\delta_b u_b\}+(1-\lambda)p_{bs}\tilde{q}_{bs}\max\{v_b-(\delta_s u_s+c_s),\delta_b u_b\}\\
&+\sum_s\big((\lambda w_{bs}(1-q_{bs})+(1-\lambda)p_{bs}(1-\tilde{q}_{bs})\big)\delta_b u_b\\
&+\sum_s(\lambda w_{sb}+(1-\lambda)p_{sb})\delta_b u_b\\
&+\sum_{j,k\neq b}(\lambda w_{kj}q_{kj}+(1-\lambda)p_{kj}\tilde{q}_{kj})\delta_b u_b(G_0\ominus\{j,k\})+(\lambda w_{kj}(1-q_{kj})+(1-\lambda)p_{kj}(1-\tilde{q}_{kj}))\delta_b u_b\\
u'_s=&\sum_b \lambda w_{sb}q_{sb}\max\{\frac{1-\alpha}{1+\beta}(v_b-\delta_b u_b)-c_s,\delta_s u_s\}+(1-\lambda)p_{sb}\tilde{q}_{sb}\max\{v_b-\delta_b u_b-c_s,\delta_s u_s\}\\
&+\sum_b\big((\lambda w_{sb}(1-q_{sb})+(1-\lambda)p_{sb}(1-\tilde{q}_{sb})\big)\delta_s u_s\\
&+\sum_b(\lambda w_{bs}+(1-\lambda)p_{bs})\delta_s u_s\\
&+\sum_{j,k\neq s}[(\lambda w_{kj}q_{kj}+(1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_s u_s(G_0\ominus\{j,k\})\\
&+\big((\lambda w_{kj}q_{kj}+(1-\lambda)p_{kj}\tilde{q}_{kj})\gamma+\lambda w_{kj}(1-q_{kj})+(1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_s u_s]\}
\end{aligned}
$$

First need to show that f has fixed point. Let $\bar{v}=\max_{b,s}(v_{bs}-c_s)$.

Lemma C.1.1. *$f:[0,\bar{v}]^{B\cup S}\times[0,1]^E\times[0,1]^E\to[0,\bar{v}]^{B\cup S}\times[0,1]^E\times[0,1]^E$ has a fixed point.*

Proof.

Claim 1. $\forall \mathbf{q},\tilde{\mathbf{q}},\mathbf{u}, f(\mathbf{u},\mathbf{q},\tilde{\mathbf{q}})$ is non-empty and convex.

Proof of Claim 1: It's easy to see that $f(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}})$ is non-empty.

$\forall \mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}$, and $(\mathbf{u}', \mathbf{q}', \tilde{\mathbf{q}}'), (\mathbf{u}'', \mathbf{q}'', \tilde{\mathbf{q}}'') \in f(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}})$, $\eta \in [0, 1]$,

$\eta \tilde{q}'_{bs} + (1-\eta)\tilde{q}''_{bs} = \eta \tilde{q}'_{sb} + (1-\eta)\tilde{q}''_{sb} = 1(0)$ if $v_{bs} > (<) \delta_b u_b + \delta_s u_s$.

$\eta q'_{sb} + (1-\eta) q''_{sb} = \eta q'_{bs} + (1-\eta) q''_{bs} = 1(0)$ if $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > (<) \delta_b u_b$, and $\eta q'_{sb} + (1-\eta) q''_{sb}, \eta q'_{bs} + (1-\eta) q''_{bs} \in [0, 1]$ otherwise.

Suppose $\mathbf{w}', \mathbf{w}''$ are corresponding $\mathbf{w}$s in the mapping that maps to $\mathbf{u}', \mathbf{u}''$,

then $supp(\mathbf{w}'), supp(\mathbf{w}'')$ all belong to

$$\arg\max_{i,j} \frac{\lambda q_{ij}[g_{ij}(\mathbf{u}) + (1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M(G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{ij}\gamma + 1 - q_{ij}) - (1-\lambda)\delta_M \sum_{k,l \in B \cup S} p_{kl}(\tilde{q}_{kl}\gamma + 1 - \tilde{q}_{kl})},$$

hence $\{ij | \eta w'_{ij} + (1-\eta) w''_{ij} > 0\}$ belong to the set

$$\arg\max_{i,j} \frac{\lambda q_{ij}[g_{ij}(\mathbf{u}) + (1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M(G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{ij}\gamma + 1 - q_{ij}) - (1-\lambda)\delta_M \sum_{k,l \in B \cup S} p_{kl}(\tilde{q}_{kl}\gamma + 1 - \tilde{q}_{kl})}.$$

Therefore $(\eta \mathbf{u}' + (1-\eta)\mathbf{u}'', \eta \mathbf{q}' + (1-\eta)\mathbf{q}'', \eta \tilde{\mathbf{q}}' + (1-\eta)\tilde{\mathbf{q}}'') \in \Gamma(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}})$

Claim 2. f has closed graph.

Proof of Claim 2: For any $(\mathbf{x}^n, \mathbf{y}^n, \mathbf{z}^n) \in f(\mathbf{u}^n, \mathbf{q}^n, \tilde{\mathbf{q}}^n)$, where

$$(\mathbf{u}^n, \mathbf{q}^n, \tilde{\mathbf{q}}^n) \to (\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}), \quad (\mathbf{x}^n, \mathbf{y}^n, \mathbf{z}^n) \to (\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

If $v_{bs} - c_s > \delta_b u_b + \delta_s u_s$, there exists some $n_1 > 0$, s.t.

$$v_{bs} - c_s > \delta_s u_s^n + \delta_b u_b^n, \quad \forall n > n_1$$

thus $z_{sb}^n = z_{bs}^n = 1, \forall n > n_1$,and $z_{bs} = z_{sb} = 1$.

If $v_{bs} - c_s < \delta_b u_b + \delta_s u_s j$, there exists some $n_2 > 0$, s.t.

$$v_{bs} - c_s < \delta_b u_b^n + \delta_s u_s^n, \quad \forall n > n_2,$$

and $z_{bs}^n = z_{sb}^n = 0, \forall n > n_2$, thus $z_{bs} = z_{sb} = 0$.

If $v_{bs} - c_s = \delta_b u_b + \delta_s u_s$, $z_{bs}, z_{sb} \in [0, 1]$ is trivial. Similarly, we can prove that $y_{sb} = y_{bs} = 1(0)$ if $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > (<) \delta_b u_b$, and $y_{sb}, y_{bs} \in [0, 1]$ otherwise .

Next let

$$\zeta_{ij}(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}) = \frac{\lambda q_{ij}[g_{ij}(\mathbf{u}) + (1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{ij}\gamma + 1 - q_{ij}) - (1-\lambda)\delta_M \sum_{k,l \in B \cup S} p_{kl} (\tilde{q}_{kl}\gamma + 1 - \tilde{q}_{kl})}.$$

It's obvious that for each $(\mathbf{u}^n, \mathbf{q}^n, \tilde{\mathbf{q}}^n)$, the corresponding $\mathbf{w}^n$ satisfies the following

$$w_{ij}^n (\max_{i,j \in B \cup S} \zeta_{ij}(\mathbf{u}^n, \mathbf{q}^n, \tilde{\mathbf{q}}^n) - \zeta_{ij}(\mathbf{u}^n, \mathbf{q}^n, \tilde{\mathbf{q}}^n)) = 0, \forall n, i, j \in B \cup S.$$

Suppose that a subsequence $\mathbf{w}^{n_k}$ converges to $\mathbf{w}$. Since $\max \zeta_{ij}(\cdot), \zeta_{ij}(\cdot)$ are all continuous, let $k \to \infty$, we have

$$w_{ij} (\max_{i,j \in B \cup S} \zeta_{ij}(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}) - \zeta_{ij}(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}})) = 0, \forall i, j \in B \cup S.$$

Thus $\{ij | w_{ij} > 0\} \subseteq$

$$\arg\max_{ij} \frac{\lambda q_{ij}[g_{ij}(\mathbf{u}) + (1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{ij}\gamma + 1 - q_{ij}) - (1-\lambda)\delta_M \sum_{k,l \in B \cup S} p_{kl} (\tilde{q}_{kl}\gamma + 1 - \tilde{q}_{kl})}.$$

Notice that $\mathbf{x}^n$ is a continuous function of $(\mathbf{u}^n, \mathbf{q}^n, \tilde{\mathbf{q}}^n, \mathbf{w}^n)$, hence $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in f(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}})$.

Claim 3. Γ maps the compact set $[0, \bar{v}]^{B \cup S} \times [0, 1]^E \times [0, 1]^E$ to itself (given that $\forall i, j$, $u_i(G \ominus i, j) \in [0, \bar{v}]$)

Therefore by Kakutatni's fixed point theorem, f has a fixed point

$$(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}) \in f(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}).$$

□

At the begining of the subgame h_t when no one has reached an agreement, given $\mathbf{u}$ the players' expected payoff starting at t, f computes agents' best response $\mathbf{q}, \tilde{\mathbf{q}}$ to $\mathbf{u}$ and middleman's best response $\mathbf{w}$ to agents' strategies $\mathbf{q}$, and the corresponding new expected payoff $\mathbf{u}'$ since they depend on the strategy of the middleman. Formally,

Lemma C.1.2. $\mathbf{u}^*$ *is a Markov Perfect equilibrium payoff vector if there exists a collection of subgame consistent MPE of the game* $\{\Gamma^{\alpha+\beta}(G)\}_{G\subset G_0}$ *with payoffs* $\{\mathbf{u}^*(G)\}_{G\subset G_0}$ *and* $\mathbf{q}, \tilde{\mathbf{q}}^*$*, such that* $(\mathbf{u}^*, \mathbf{q}^*, \tilde{\mathbf{q}}^*) \in f(\mathbf{u}^*, \mathbf{q}^*, \tilde{\mathbf{q}}^*)$.

Proof. Suppose that the collection of subgame consistent MPE of the game $\{\Gamma^{\alpha+\beta}(G)\}_{G\subset G_0}$ has payoffs $\{\mathbf{u}^*(G)\}_{G\subset G_0}$. If $(\mathbf{u}^*, \mathbf{q}^*, \tilde{\mathbf{q}}^*) \in f(\mathbf{u}^*, \mathbf{q}^*, \tilde{\mathbf{q}}^*)$, then there exists $\mathbf{w}^*$, s.t.

(i) $\mathbf{q}, \tilde{\mathbf{q}}$ satisfies Eq(C.3), (C.4), (C.5), (C.6).

(ii) $\mathbf{w}$ is middleman's best response to $\mathbf{q}, \tilde{\mathbf{q}}$, proposals and responses. given $\sigma^*(G), \forall G \subset G_0$.

(iii) $\mathbf{u}$ satisfies Eq(C.1), (C.2) , that is, $\mathbf{u}$ are the corresponding continuation payoffs (ex ante payoffs for network G_0).

Then we construct the following strategy profile and prove it is an MPE with corresponding MPE payoff $\mathbf{u}^*$. First define the strategies for histories h_t along which no agreement has occurred. Recall that $G(h_t)$ denotes the network induced by the players remaining after the ex post history h_t. Construct time-t strategy of each player according to the time-0 behavior specified by $\sigma^*(G_0)$.[1] For histories along which no agreement has occurred, $\sigma^*(G(h_t))$ specifies that

- Middleman choose matching pair (i, j) with probability w^*_{ij}.
- For buyer b, when chosen by the middleman to propose to seller s, he offers a bid-price $p = \min\{\frac{c_s+\delta_s u_s}{1-\alpha}, \frac{v_{bs}-\delta_b u_b}{1+\beta}\}$. If chosen by nature to propose to seller s, he offers

[1] Formally, $(\sigma(G))_i(h_t; i \to j) = (\sigma(G))_i(h_0; i \to j), (\sigma(G))_j(h_t; i \to j, x) = (\sigma(G))_i(h_0; i \to j, x)$, where $h_0 = \emptyset$, $G = G(h_t)$.

$p = \min\{c_s + \delta_s u_s, v_{bs} - \delta_b u_b\}$.

- Similarly, for seller s, when chosen by the middleman to propose to buyer b, he offers an ask-price $p = \max\{\frac{c_s+\delta_s u_s}{1-\alpha}, \frac{v_{bs}-\delta_b u_b}{1+\beta}\}$. If chosen by nature to propose to seller s, he offers $p = \max\{c_s + \delta_s u_s, v_{bs} - \delta_b u_b\}$.

- When buyer b responds to the offer p from seller s: if intermediated by the middleman, he accepts any offer s.t., $p < \frac{v_{bs}-\delta_b u_b}{1+\beta}$, and rejects any $p > \frac{v_{bs}-\delta_b u_b}{1+\beta}$, and accept an offer of $p = \frac{v_{bs}-\delta_b u_b}{1+\beta}$ with probability q^*_{sb}; if intermediated by nature, he accepts any offer s.t., $p < v_{bs} - \delta_b u_b$, and rejects any $p > v_{bs} - \delta_b u_b$, and accept an offer of $p = v_{bs} - \delta_b u_b$ with probability $\tilde{q}^*_{sb}$.

- Similarly, when seller s responds to the purchase offer p from buyer b: if intermediated by the middleman, he accepts any offer s.t., $p > \frac{c_s+\delta_s u_s}{1-\alpha}$, and rejects any $p < \frac{c_s+\delta_s u_s}{1-\alpha}$, and accept an offer of $p = \frac{c_s+\delta_s u_s}{1-\alpha}$ with probability q^*_{bs}; if intermediated by nature, he accepts any offer s.t., $p > c_s + \delta_s u_s$, and rejects any $p < c_s + \delta_s u_s$, and accept an offer of $p = c_s + \delta_s u_s$ with probability $\tilde{q}^*_{bs}$.

Given the collection of subgame consistency guarantees that under $(\sigma^*(G))_{G\subset G_0}$ the expected payoffs of any subgames are $\mathbf{u}(G)$.

Lemma C.1.3 (Mailath & Samuelson (2006) Proposition 5.7.1)**.** *A strategy profile is subgame perfect in a dynamic game if and only if there are no profitable one-shot deviations.*

Based on the definition of f, everyone is best responding at period t: there is no profitable deviation for all players in this stage game. It's then easy to verify that $\mathbf{u}^*$ are indeed the equilibrium payoff by the strategy profile conditional on h_t.

We now need to show that Lemma C.1.2 implies the existence of MPEs. We prove a subgame consistent collection of MPEs for the game $\{\Gamma^{\alpha+\beta}(G)\}_{G\subseteq\mathscr{G}(n)}$, where $\mathscr{G}(n)$ denotes the subset of subnetworks in $\mathscr{G}$ with at most n vertices. The proof proceed by induction on n. For $n = 0, 1$ it's trivial. Suppose we proved the statement for all values smaller than n, and proceed to proving the case n. By induction hypothesis, there exists a subgame consistent collection of MPEs $(\sigma^*(G'))_{G' \in \mathscr{G}(n-1)}$ of the corresponding games $\{\Gamma^{\alpha+\beta}(G)\}_{G \subseteq \mathscr{G}(n-1)}$. Fix a network $G \in \mathscr{G}(n) \setminus \mathscr{G}(n-1)$, $S(G) = \{G' : G' \subset G\} \subseteq \mathscr{G}(n-1)$. Therefore there exist MPEs $(\sigma^*(G'))_{G' \in S(G)}$ for the games $\{\Gamma^{\alpha+\beta}(G)\}_{G' \in S(G)}$ that are subgame consistent, hence we can use their MPE payoffs to define f. Suppose now that f has a fixed point $(\mathbf{u}^*, \mathbf{q}^*, \tilde{\mathbf{q}}^*)$, with induced $\sigma^*(G)$ of the game $\Gamma^{\alpha+\beta}(G)$ so that $(\sigma^*(G'))_{G' \in \mathscr{G}(n-1) \cup \{G\}}$ is subgame consistent. If we append all MPE $\sigma^*(G)$ of $G \in \mathscr{G}(n) \setminus \mathscr{G}(n-1)$ to $(\sigma^*(G'))_{G' \in \mathscr{G}(n-1)}$, the resulting collection of MPEs $(\sigma^*(G'))_{G' \in \mathscr{G}(n)}$ is also subgame consistent. The the collection of subgame consistent Markovian strategy profile are a MPE of the game $\Gamma^{\alpha+\beta}(G)$. □

C.2. Proof of Theorem 3.4.1 under the choice "α or β"

This time:

b propose to s: $(v_{bs} - (1+\beta)p, p - c_s)$

s proposes to b: $(v_{bs} - p, (1-\alpha)p - c_s)$.

$$
\begin{aligned}
u_b = &\sum_s \lambda w_{bs} q_{bs}(v_b - (1+\beta)(c_s + \delta_s u_s)) + (1-\lambda) p_{bs} \tilde{q}_{bs}(v_b - (\delta_s u_s + c_s)) \\
&+ \sum_s \big((\lambda w_{bs}(1-q_{bs}) + (1-\lambda)p_{bs}(1-\tilde{q}_{bs})\big)\delta_b u_b \\
&+ \sum_s (\lambda w_{sb} + (1-\lambda)p_{sb})\delta_b u_b \\
&+ \sum_{j,k \neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) \\
&+ \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_b u_b]
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
u_s = &\sum_b \lambda w_{sb} q_{sb}((1-\alpha)(v_b - \delta_b u_b) - c_s) + (1-\lambda) p_{sb} \tilde{q}_{sb}(v_b - \delta_b u_b - c_s) \\
&+ \sum_b \big((\lambda w_{sb}(1-q_{sb}) + (1-\lambda)p_{sb}(1-\tilde{q}_{sb})\big)\delta_s u_s \\
&+ \sum_b (\lambda w_{bs} + (1-\lambda)p_{bs})\delta_s u_s \\
&+ \sum_{j,k \neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_s u_s(G_0 \ominus \{j,k\}) \\
&+ \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_s u_s]
\end{aligned}
\tag{C.8}
$$

and

$$
q_{bs} = \begin{cases} 1 & \text{if } v_{bs} - (1+\beta)(c_s + \delta_s u_s) > \delta_b u_b \\ [0,1] & \text{if } v_{bs} - (1+\beta)(c_s + \delta_s u_s) = \delta_b u_b, \\ 0 & \text{if } v_{bs} - (1+\beta)(c_s + \delta_s u_s) < \delta_b u_b. \end{cases} \tag{C.9}
$$

And $q_{bs} = \Pr(`Yes'|(h_t; \mathbb{1}_M = 1; b \to s; p))$ where $p = \min\{c_s + \delta_s u_s, \frac{v_{bs} - \delta_b u_b}{1+\beta}\}$. Similarly we have,

$$
q_{sb} = \begin{cases} 1 & \text{if } (1-\alpha)(v_b - \delta_b u_b) - c_s > \delta_s u_s \\ [0,1] & \text{if } (1-\alpha)(v_b - \delta_b u_b) - c_s = \delta_s u_s, \\ 0 & \text{if } (1-\alpha)(v_b - \delta_b u_b) - c_s < \delta_s u_s. \end{cases} \tag{C.10}
$$

where $q_{sb} = \Pr(`Yes'|(h_t; \mathbb{1}_M = 1; s \to b; p))$ where $p = \max\{\frac{c_s + \delta_s u_s}{1-\alpha}, v_{bs} - \delta_b u_b\}$. When nature selects bargaining pair,

$$
\tilde{q}_{sb} = \begin{cases} 1 & \text{if } v_b - c_s > \delta_b u_b + \delta_s u_s \\ [0,1] & \text{if } v_b - c_s = \delta_b u_b + \delta_s u_s, \\ 0 & \text{if } v_b - c_s < \delta_b u_b + \delta_s u_s. \end{cases} \tag{C.11}
$$

$$
\tilde{q}_{bs} = \begin{cases} 1 & \text{if } v_b - c_s > \delta_b u_b + \delta_s u_s, \\ [0,1] & \text{if } v_b - c_s = \delta_b u_b + \delta_s u_s, \\ 0 & \text{if } v_b - c_s < \delta_b u_b + \delta_s u_s. \end{cases} \tag{C.12}
$$

From the middle man's perspective: the middleman must maximize its total discounted payoff: that is

$$\begin{aligned}
&\pi_M(G)\\
=&(1-\delta_M)\lambda \max_{b,s} \max\{q_{bs}[\beta(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (q_{bs}\gamma + 1 - q_{bs})\delta_M \pi_M(G),\\
&q_{sb}[\alpha(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + \big(q_{sb}\gamma + 1 - q_{sb}\big)\delta_M \pi_M(G)\}\\
&+ (1-\delta_M)(1-\lambda) \sum_{i,j \in B\cup S} p_{ij}[\tilde{q}_{ij}(1-\gamma)\delta_M \pi_M(G \ominus \{i,j\}) + (\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})\delta_M \pi_M(G)],
\end{aligned}$$

And

$$\begin{aligned}
w_{bs} > 0 \Rightarrow \pi_M(G) =& \lambda q_{bs}[\beta(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + \lambda(q_{bs}\gamma + 1 - q_{bs})\delta_M \pi_M(G)\\
&+ (1-\lambda) \sum_{i,j \in B\cup S} p_{ij}[\tilde{q}_{ij}(1-\gamma)\delta_M \pi_M(G \ominus \{i,j\}) + (\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})\delta_M \pi_M(G)]\\
w_{sb} > 0 \Rightarrow \pi_M(G) =& \lambda q_{sb}[\alpha(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + \lambda(q_{sb}\gamma + 1 - q_{sb})\delta_M \pi_M(G)\\
&+ (1-\lambda) \sum_{i,j \in B\cup S} p_{ij}[\tilde{q}_{ij}(1-\gamma)\delta_M \pi_M(G \ominus \{i,j\}) + (\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})\delta_M \pi_M(G)]
\end{aligned}$$

This yields:

$$\begin{aligned}
&\pi_M(G)/(1-\delta_M)\\
=&\max\{\max_{b,s} \frac{\lambda q_{bs}[\beta(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (1-\lambda)\sum_{i,j\in B\cup S} p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})}{1 - \lambda\delta_M(q_{bs}\gamma + 1 - q_{bs}) - (1-\lambda)\delta_M \sum_{i,j\in B\cup S} p_{ij}(\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})}\\
&\ldots, \max_{b,s} \frac{\lambda q_{sb}[\alpha(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (1-\lambda)\sum_{i,j\in B\cup S} p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})}{1 - \lambda\delta_M(q_{bs}\gamma + 1 - q_{sb}) - (1-\lambda)\delta_M \sum_{i,j\in B\cup S} p_{ij}(\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})}\}
\end{aligned}$$

Let $g_{ij}(\mathbf{u}) = \beta(\delta_j u_j + c_j)$ if $i \in B, j \in S$, and $g_{ij}(\mathbf{u}) = \alpha(v_{ij} - \delta_j u_j)$ if $i \in S, j \in B$.

$$
\begin{aligned}
& f(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}) \\
=& \{(\mathbf{u}', \mathbf{q}', \tilde{\mathbf{q}}') | \tilde{q}'_{bs} = \tilde{q}'_{sb} = 1(0) \text{ if } v_{bs} - c_s > (<) \delta_b u_b + \delta_s u_s, \\
& \text{ and } \tilde{q}'_{sb}, \tilde{q}'_{bs} \in [0,1] \text{ if } v_{bs} - c_s = \delta_b u_b + \delta_s u_s, \\
& q'_{bs} = 1(0) \text{ if } v_{bs} - (1+\beta)(\delta_s u_s + c_s) > (<) \delta_b u_b, \text{ and } q'_{bs} \in [0,1] \text{ otherwise ;} \\
& q'_{sb} = 1(0) \text{ if } (1-\alpha)(v_{bs} - \delta_b u_b) > (<) \delta_s u_s + c_s, \text{ and } q'_{bs} \in [0,1] \text{ otherwise ;} \\
& \exists \mathbf{w} \in \Delta, \text{ s.t. } \sum_{i,j \in B \cup S} w_{ij} \leq 1, \text{ and} \\
& \{ij | w_{ij} > 0\} \subseteq \\
& \frac{\lambda q_{ij}[g_{ij}(\mathbf{u}) + (1-\gamma)\delta_M \pi_M(G \ominus \{i,j\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{ij}\gamma + 1 - q_{ij}) - (1-\lambda)\delta_M \sum_{k,l \in B \cup S} p_{kl} (\tilde{q}_{kl}\gamma + 1 - \tilde{q}_{kl})}; \\
& \text{and } u'_b = \sum_s \lambda w_{bs} q_{bs} \max\{v_b - (1+\beta)(\delta_s u_s + c_s), \delta_b u_b\} + (1-\lambda) p_{bs} \tilde{q}_{bs} \max\{v_b - (\delta_s u_s + c_s), \delta_b u_b\} \\
& + \sum_s \big((\lambda w_{bs}(1-q_{bs}) + (1-\lambda) p_{bs}(1-\tilde{q}_{bs})\big)\delta_b u_b + \sum_s (\lambda w_{sb} + (1-\lambda) p_{sb}) \delta_b u_b \\
& + \sum_{j,k \neq b} (\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj}) \delta_b u_b (G_0 \ominus \{j,k\}) + (\lambda w_{kj}(1-q_{kj}) + (1-\lambda) p_{kj}(1-\tilde{q}_{kj})) \delta_b u_b \\
u'_s =& \sum_b \lambda w_{sb} q_{sb} \max\{(1-\alpha)(v_b - \delta_b u_b) - c_s, \delta_s u_s\} + (1-\lambda) p_{sb} \tilde{q}_{sb} \max\{v_b - \delta_b u_b - c_s, \delta_s u_s\} \\
& + \sum_b \big((\lambda w_{sb}(1-q_{sb}) + (1-\lambda) p_{sb}(1-\tilde{q}_{sb})\big)\delta_s u_s + \sum_b (\lambda w_{bs} + (1-\lambda) p_{bs}) \delta_s u_s \\
& + \sum_{j,k \neq s} [(\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})(1-\gamma) \delta_s u_s (G_0 \ominus \{j,k\}) \\
& + \big((\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda) p_{kj}(1-\tilde{q}_{kj})\big)\delta_s u_s]\}
\end{aligned}
$$

By similar argument, one can show that

Lemma C.2.1. *$f : [0, \bar{v}]^{B \cup S} \times [0,1]^E \times [0,1]^E \to [0, \bar{v}]^{B \cup S} \times [0,1]^E \times [0,1]^E$ has a fixed point $(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}}) \in f(\mathbf{u}, \mathbf{q}, \tilde{\mathbf{q}})$.*

By the definition of Γ, there exists $\mathbf{w}$, s.t.

1. $\mathbf{q}, \tilde{\mathbf{q}}$ satisfies Eq (C.9), (C.10), (C.11), (C.12).

2. $\mathbf{w}$ is middleman's best response to $\mathbf{q}, \tilde{\mathbf{q}}$ and proposals.

3. $\mathbf{u}$ satisfies Eq(C.15), (C.16) , that is, $\mathbf{u}$ is the corresponding continuation payoff vector.

C.3. Proof of Proposition 3.2.2

Since the middleman is maximizing her discounted aggregate profit when deciding which link to pick from G,

$$
\begin{aligned}
&\pi_M(G) \\
=&(1-\delta_M)\lambda \max\{q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (q_{bs}\gamma + 1 - q_{bs})\delta_M \pi_M(G), \\
&q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + \big(q_{sb}\gamma + 1 - q_{sb}\big)\delta_M \pi_M(G)\} \\
&+ (1-\delta_M)(1-\lambda) \sum_{i,j \in B \cup S} p_{ij}[\tilde{q}_{ij}(1-\gamma)\delta_M \pi_M(G \ominus \{i,j\}) + (\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})\delta_M \pi_M(G)].
\end{aligned}
$$

For each pair of (b,s), if $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > \delta_b u_b$, the difference Δ between selecting $b \to s$ and selecting $s \to b$ is,

$$
\begin{aligned}
\Delta =&q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (q_{bs}\gamma + 1 - q_{bs})\delta_M \pi_M(G) \\
&- q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] - \big(q_{sb}\gamma + 1 - q_{sb}\big)\delta_M \pi_M(G) \\
=&\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) - \frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) \\
<&0
\end{aligned}
$$

If $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) = \delta_b u_b$, $\Delta = (q_{bs} - q_{sb})[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\}) - (1-\gamma)\delta_M \pi_M(G)]$.

If $v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) < \delta_b u_b$, $\Delta = 0$, $q_{bs} = 0$

In any of the situations, we have

$$
w_{bs} q_{bs}(v_{bs} - \delta_b u_b - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) = 0.
$$

Therefore

$$
\begin{aligned}
u_b =& \sum_s \lambda w_{bs} q_{bs}(v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) + (1-\lambda) p_{bs}\tilde{q}_{bs}(v_{bs} - (\delta_s u_s + c_s)) \\
&+ \sum_s \big((\lambda w_{bs}(1-q_{bs}) + (1-\lambda)p_{bs}(1-\tilde{q}_{bs})\big)\delta_b u_b + \sum_s (\lambda w_{sb} + (1-\lambda)p_{sb})\delta_b u_b \\
&+ \sum_{j,k\neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) \\
&+ \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_b u_b] \\
=& \sum_s \lambda w_{bs} q_{bs}(v_{bs} - \delta_b u_b - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s)) + (1-\lambda) p_{bs}\tilde{q}_{bs}(v_{bs} - \delta_b u_b - (\delta_s u_s + c_s)) \\
&+ \sum_s (\lambda w_{bs} + (1-\lambda)p_{bs}\big)\delta_b u_b + \sum_s (\lambda w_{sb} + (1-\lambda)p_{sb})\delta_b u_b \\
&+ \sum_{j,k\neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) \\
&+ \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_b u_b] \\
=& \sum_s (1-\lambda) p_{bs}\tilde{q}_{bs}(v_{bs} - \delta_b u_b - (\delta_s u_s + c_s)) \\
&+ \sum_s (\lambda w_{bs} + (1-\lambda)p_{bs}\big)\delta_b u_b + \sum_s (\lambda w_{sb} + (1-\lambda)p_{sb})\delta_b u_b \\
&+ \sum_{j,k\neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) \\
&+ \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_b u_b]
\end{aligned}
\tag{C.13}
$$

Now suppose $(1-\lambda)p_{bs}(G) = 0, \forall s, G \subseteq G_0$. We prove by induction on n the number of vertices of the subnetwork that

$$u_b(G) = 0, \forall G \subseteq G_0, \text{ s.t. } b \in G.$$

For any subnetworks $G \subseteq G_0$ of $n \leq 3$, $u_b(G \ominus \{j,k\}) = 0, \forall j,k \neq b$. Suppose $u_b(G) =$

$0, \forall G \subseteq G_0$ with at most $n-1$ vertices. For n, by Eq(C.13), we have

$$(1-\ldots)u_b = \sum_{j,k\neq b}[(\lambda w_{kj}q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) = 0,$$

where $1-\ldots \geq 1-\delta_b$. Therefore, $u_b(G) = 0$.

C.4. Proof of Theorem 3.3.1

Suppose that $\lambda = 1$ and by Proposition 3.2.2, $u_b = 0$, let

$$N_1 = \{(b,s) : v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > 0\},$$

$$N_2 = \{(b,s) : v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) = 0\},$$

$$N_3 = \{(b,s) : v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) < 0\}.$$

From the previous argument in Proof of Proposition 3.2.2,

$$\forall (b,s) \in N_1, \quad q_{bs} = q_{sb} = 1, w_{bs} = 0.$$

$$\forall (b,s) \in N_1, \quad q_{bs}, q_{sb} \in [0,1].$$

$$\forall (b,s) \in N_3, \quad q_{bs} = q_{sb} = 0.$$

Hence, each time middleman is selecting the pair (b,s) such that,

$$\begin{aligned}
&\pi_M(G)\\
=&(1-\delta_M)\max\{q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})] + (q_{bs}\gamma + 1 - q_{bs})\delta_M\pi_M(G),\\
&q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})] + \big(q_{sb}\gamma + 1 - q_{sb}\big)\delta_M\pi_M(G)\}\\
=&(1-\delta_M)\max\{\frac{\max_{(b,s)\in N_1}[\frac{\alpha+\beta}{1+\beta}v_{bs} + (1-\gamma)\delta_M\pi_M(G\ominus\{b,s\})]}{1-\gamma\delta_M},\\
&\max_{(b,s)\in N_2}\frac{\max(q_{bs},q_{sb})(\frac{\alpha+\beta}{1+\beta}v_{bs} + (1-\gamma)\delta_M\pi_M(G\ominus\{b,s\}))}{1-\delta_M + (1-\gamma)\delta_M\max(q_{bs},q_{sb})}\}
\end{aligned}$$

The last inequality is due to $\frac{\partial}{\partial q}[\frac{q(\frac{\alpha+\beta}{1+\beta}v_{bs}+(1-\gamma)\delta_M\pi_M(G\ominus\{b,s\}))}{1-\delta_M+(1-\gamma)\delta_M q}] > 0$ as long as $v_{bs} > 0$, $\delta_M < 1$.

$$\begin{aligned} u_s =& \sum_b w_{sb}q_{sb}(\frac{1-\alpha}{1+\beta}v_{bs} - c_s - \delta_s u_s) + \sum_{j,k\neq s} w_{kj}q_{kj}(1-\gamma)\delta_s u_s(G_0 \ominus \{j,k\}) \\ &+ [\sum_b (w_{sb}+w_{bs}) + \sum_{j,k\neq s} \big(w_{kj}q_{kj}\gamma + w_{kj}(1-q_{kj})\big)]\delta_s u_s \end{aligned} \tag{C.14}$$

Assume $\gamma = 1$:

$$(1-\delta_s)u_s = \sum_b w_{sb}|\frac{1-\alpha}{1+\beta}v_{bs} - c_s - \delta_s u_s|$$

$$\pi_M(G) = \frac{\alpha+\beta}{1+\beta}\max(\max_{(b,s)\in N_1} v_{bs}, \max_{(b,s)\in N_2} \max(q_{bs}, q_{sb})v_{bs}),$$

where

$$N_1(s) = \{b \in N(s) : v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > 0\},$$

$$N_2(s) = \{b \in N(s) : v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) = 0\},$$

$$N_3(s) = \{b \in N(s) : v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) < 0\}.$$

Notice that, if $v_{bs} \leq \frac{1+\beta}{1-\alpha}c_s, \forall b \in N(s)$, we have $N_1(s) = \emptyset$. Moreover,

$$N_1(s) = \emptyset \Rightarrow u_s = 0 \Rightarrow v_{bs} \leq \frac{1+\beta}{1-\alpha}c_s, \forall b \in N(s).$$

If $N_1(s) \neq \emptyset$, $\max_{b\in N(s)} v_{bs} = \max_{b\in N_1(s)} v_{bs}$, and

$$[1-\delta_s(1-\sum_{b\in N_1(s)} w_{sb})]u_s = \sum_{b\in N_1(s)} w_{sb}(\frac{1-\alpha}{1+\beta}v_{bs} - c_s) = (\sum_{b\in N_1(s)} w_{sb})(\frac{1-\alpha}{1+\beta}\max_{b\in N(s)} v_{bs} - c_s)$$

Let

$$N_1 = \{s : \exists b \in N(s), \text{ s.t. } v_{bs} > \frac{1+\beta}{1-\alpha}c_s\},$$

$$N_2 = \{s : v_{bs} \le \frac{1+\beta}{1-\alpha} c_s, \forall b \in N(s),\ \text{and } \exists b \in N(s) \text{ s.t. “=” holds}\}$$

Hence,

$$\begin{aligned}
\pi_M(G) =& \frac{\alpha+\beta}{1+\beta} \max_s \big(\max_{b \in N_1(s)} v_{bs}, \max_{b \in N_2(s)} \max(q_{bs}, q_{sb}) v_{bs} \big) \\
=& \frac{\alpha+\beta}{1+\beta} \max\big(\max_{s \in N_2} \big(\max_{b \in N(s)} (q_{bs}, q_{sb}) v_{bs}\big), \max_{s \in N_1} \max_{b \in N(s)} v_{bs}\big)
\end{aligned}$$

Notice that, in any MPEs where $q_{bs}, q_{sb} \in \{0,1\}$, in particular, $q_{bs} = q_{sb} = 1, \forall s \in N_2, b \in N_2(s)$ we have

$$\begin{aligned}
\pi_M(G) =& \frac{\alpha+\beta}{1+\beta} \max_{v_{bs} \ge \frac{1+\beta}{1-\alpha} c_s} v_{bs} \\
=& \max_{v_{bs} - c_s \ge \frac{\alpha+\beta}{1+\beta} v_{bs}} \frac{\alpha+\beta}{1+\beta} v_{bs} \\
=& \max_{b,s} v_{bs} - c_s
\end{aligned}$$

Assume that

$$\max_{b,s} v_{bs} - c_s = v_{b^* s^*} - c_{s^*}.$$

Then, for sufficiently small $\epsilon > 0$, we can find α, β, s.t., $v_{b^*s^*} - c_{s^*} - \epsilon = \frac{\alpha+\beta}{1+\beta} v_{b^*s^*}$, and one can easily verify that

$$v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) > 0 \iff v_{bs} - \frac{1+\beta}{1-\alpha} c_s > 0,$$

hence $q_{b^*s^*} = q_{b^*s^*} = 1$ and give such choice of α, β,

$$v_{b^*s^*} - c_{s^*} \ge \pi_M(G) \ge v_{b^*s^*} - c_{s^*} - \epsilon$$

Assume $\gamma = 0$:

$$\pi_M(G) =(1-\delta_M)\max\{\max_{(b,s)\in N_1}[\frac{\alpha+\beta}{1+\beta}v_{bs}+\delta_M\pi_M(G\ominus\{b,s\})],$$
$$\max_{(b,s)\in N_2}\frac{\max(q_{bs},q_{sb})(\frac{\alpha+\beta}{1+\beta}v_{bs}+\delta_M\pi_M(G\ominus\{b,s\}))}{1-\delta_M+\delta_M\max(q_{bs},q_{sb})}\}$$

$$u_s = \sum_b w_{sb}q_{sb}(\frac{1-\alpha}{1+\beta}v_{bs}-c_s-\delta_s u_s)+\sum_{j,k\neq s}w_{kj}q_{kj}\delta_s u_s(G_0\ominus\{j,k\})$$
$$+[\sum_b w_{sb}+\sum_{j,k\neq s}w_{kj}(1-q_{kj})]\delta_s u_s$$

Hence,

$$\big(1-(\sum_b w_{sb}(1-q_{sb})+\sum_{j,k\neq s}w_{kj}(1-q_{kj}))\delta_s\big)u_s = \sum_b w_{sb}q_{sb}(\frac{1-\alpha}{1+\beta}v_{bs}-c_s)+\sum_{j,k\neq s}w_{kj}q_{kj}\delta_s u_s(G_0\ominus\{j,k\})$$

$$\begin{aligned}
&\frac{1-\alpha}{1+\beta}v_{bs}-c_s-\delta_s u_s\\
=&\frac{1-\alpha}{1+\beta}v_{bs}-c_s-\frac{\delta_s\sum_b w_{sb}q_{sb}(\frac{1-\alpha}{1+\beta}v_{bs}-c_s)+\delta_s\sum_{j,k\neq s}w_{kj}q_{kj}\delta_s u_s(G_0\ominus\{j,k\})}{1-(\sum_b w_{sb}(1-q_{sb})+\sum_{j,k\neq s}w_{kj}(1-q_{kj}))\delta_s}\\
=&\frac{(1-\delta_s+\delta_s\sum_{j,k\neq s}w_{kj}q_{kj})(\frac{1-\alpha}{1+\beta}v_{bs}-c_s)-\delta_s\sum_{j,k\neq s}w_{kj}q_{kj}\delta_s u_s(G_0\ominus\{j,k\})}{1-(\sum_b w_{sb}(1-q_{sb})+\sum_{j,k\neq s}w_{kj}(1-q_{kj}))\delta_s}\\
=&\frac{(1-\delta_s)(\frac{1-\alpha}{1+\beta}v_{bs}-c_s)+(\delta_s\sum_{j,k\neq s}w_{kj}q_{kj})(\frac{1-\alpha}{1+\beta}v_{bs}-c_s-\delta_s u_s(G_0\ominus\{j,k\}))}{1-(\sum_b w_{sb}(1-q_{sb})+\sum_{j,k\neq s}w_{kj}(1-q_{kj}))\delta_s}
\end{aligned}$$

Denote by $\mathscr{M}_{B,S}$ be the collection of possible matching outcomes, i.e.,

$$\mathscr{M}_{B,S} = \{\text{ a sequence of buyer-seller pairs: } (b_1s_1,b_2s_2,\ldots,b_Ts_T) \text{ s.t., } v_{b_ts_t}-c_{s_t}\geq 0,\forall t\ \},$$

$\Pi : \mathscr{G} \to \mathbb{R}$ is the mapping from a bipartie subnetwork G to its maximum discounted surplus,

$$\Pi(G) = (1-\delta_M)\sum \delta_M^t v_{b_ts_t}$$

For each (b,s) pair, one can show by induction that if any subnetwork $G'\subset G$, $\delta_s u_s(G')\leq$

$\frac{1-\alpha}{1+\beta}v_{bs} - c_s$, then $\delta_s u_s(G) \le \frac{1-\alpha}{1+\beta}v_{bs} - c_s$. And for the initial state G', either $u_s(G') = 0$, or $\{(j,k) : j,k \neq s\} = \emptyset$, which yields $u_s(G') = \sum_b w_{sb} q_{sb}(\frac{1-\alpha}{1+\beta}v_{bs} - c_s - \delta_s u_s(G')) + \sum_b w_{sb}\delta_s u_s(G')$. Hence $\delta_s u_s(G') \le \frac{1-\alpha}{1+\beta}v_{bs} - c_s \iff 0 \le \frac{1-\alpha}{1+\beta}v_{bs} - c_s$, and in any MPEs where $q_{bs}, q_{sb} \in \{0,1\}$, in particular, $q_{bs} = q_{sb} = 1, \forall s \in N_2, b \in N_2(s)$.

$$\begin{aligned}\pi_M(G) =& (1-\delta_M) \max_{v_{bs} \ge \frac{1+\beta}{1-\alpha}c_s} [\frac{\alpha+\beta}{1+\beta}v_{bs} + \delta_M \pi_M(G \ominus \{b,s\})] \\ =& \frac{\alpha+\beta}{1+\beta}\Pi(G_{\alpha,\beta})\end{aligned}$$

where $G_{\alpha,\beta}$ is a subnetwork of G_0 where the set of links $E_{\alpha,\beta} = \{(b,s) : v_{bs} \ge \frac{1+\beta}{1-\alpha}c_s\}$. Assume that

$$\alpha^*, \beta^* = \arg\max \frac{\alpha+\beta}{1+\beta}\Pi(G_{\alpha,\beta}).$$

By a similar ϵ argument, one can show that even if $q_{bs} = q_{sb} < 1, \forall s \in N_2, b \in N_2(s)$, one could find α', β', s.t., $\{(b,s) : v_{bs} \ge \frac{1+\beta^*}{1-\alpha^*}c_s\} = \{(b,s) : v_{bs} \ge \frac{1+\beta'}{1-\alpha'}c_s + \epsilon\}$. Therefore for some bounded $C > 0$

$$\pi_M(G_0) \ge \frac{\alpha'+\beta'}{1+\beta'}\Pi(G_{\alpha',\beta'}) \ge \frac{\alpha^*+\beta^*}{1+\beta^*}\Pi(G_{\alpha^*,\beta^*}) - C\epsilon,$$

which concludes the proof.

C.5. Proof of Proposition 3.4.2

$$\begin{aligned}u_b =& \sum_s \lambda w_{bs} q_{bs}(v_b - (1+\beta)(c_s + \delta_s u_s)) + (1-\lambda)p_{bs}\tilde{q}_{bs}(v_b - (\delta_s u_s + c_s)) \\ &+ \sum_s \big((\lambda w_{bs}(1-q_{bs}) + (1-\lambda)p_{bs}(1-\tilde{q}_{bs})\big)\delta_b u_b \\ &+ \sum_s (\lambda w_{sb} + (1-\lambda)p_{sb})\delta_b u_b \\ &+ \sum_{j,k\neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) \\ &+ \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_b u_b]\end{aligned} \tag{C.15}$$

$$
\begin{aligned}
u_s =& \sum_b \lambda w_{sb} q_{sb}((1-\alpha)(v_b - \delta_b u_b) - c_s) + (1-\lambda) p_{sb} \tilde{q}_{sb} (v_b - \delta_b u_b - c_s) \\
&+ \sum_b \big((\lambda w_{sb}(1-q_{sb}) + (1-\lambda) p_{sb}(1-\tilde{q}_{sb})\big)\delta_s u_s \\
&+ \sum_b (\lambda w_{bs} + (1-\lambda) p_{bs}) \delta_s u_s \\
&+ \sum_{j,k \neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})(1-\gamma)\delta_s u_s(G_0 \ominus \{j,k\}) \\
&+ \big((\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda) p_{kj}(1-\tilde{q}_{kj})\big)\delta_s u_s]
\end{aligned}
\tag{C.16}
$$

$$
\begin{aligned}
&\pi_M(G)/(1-\delta_M) \\
&= \max\{\max_{b,s} \frac{\lambda q_{bs}[\beta(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{bs}\gamma + 1 - q_{bs}) - (1-\lambda)\delta_M \sum_{i,j \in B \cup S} p_{ij}(\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})} \\
&\ldots, \max_{b,s} \frac{\lambda q_{sb}[\alpha(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{sb}\gamma + 1 - q_{sb}) - (1-\lambda)\delta_M \sum_{i,j \in B \cup S} p_{ij}(\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})}\}
\end{aligned}
$$

For $\lambda = 1$:

$$
\begin{aligned}
u_b =& \sum_s w_{bs} q_{bs} (v_b - (1+\beta)(c_s + \delta_s u_s)) + \sum_s w_{bs}(1-q_{bs})\delta_b u_b + \sum_s w_{sb} \delta_b u_b \\
&+ \sum_{j,k \neq b} [w_{kj} q_{kj} (1-\gamma) \delta_b u_b (G_0 \ominus \{j,k\}) + \big(w_{kj} q_{kj} \gamma + w_{kj}(1-q_{kj})\big)\delta_b u_b] \\
u_s =& \sum_b w_{sb} q_{sb}((1-\alpha)(v_b - \delta_b u_b) - c_s) + \sum_b w_{sb}(1-q_{sb})\delta_s u_s + \sum_b w_{bs} \delta_s u_s \\
&+ \sum_{j,k \neq b} [w_{kj} q_{kj} (1-\gamma) \delta_s u_s (G_0 \ominus \{j,k\}) + \big(w_{kj} q_{kj} \gamma + w_{kj}(1-q_{kj})\big)\delta_s u_s]
\end{aligned}
$$

$$
\begin{aligned}
\pi_M(G) =& (1-\delta_M) \max\{\max_{b,s} \frac{q_{bs}[\beta(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M (G \ominus \{b,s\})]}{1 - \delta_M (q_{bs}\gamma + 1 - q_{bs})}, \\
&\max_{b,s} \frac{q_{sb}[\alpha(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G \ominus \{b,s\})]}{1 - \delta_M(q_{sb}\gamma + 1 - q_{sb})}\}
\end{aligned}
$$

For $\gamma = 1$,

$$(1-\delta_b)u_b = \lambda \sum_s w_{bs} q_{bs}(v_{bs} - (1+\beta)(c_s + \delta_s u_s) - \delta_b u_b) + (1-\lambda) p_{bs} \tilde{q}_{bs}(v_{bs} - \delta_b u_b - \delta_s u_s - c_s)$$

$$(1-\delta_s)u_s = \lambda \sum_b w_{sb} q_{sb}((1-\alpha)(v_{bs} - \delta_b u_b) - c_s - \delta_s u_s) + (1-\lambda) p_{sb} \tilde{q}_{sb}(v_{bs} - \delta_b u_b - c_s - \delta_s u_s)$$

$$\pi_M(G) = \lambda \max\{\max_{b,s} q_{bs}\beta(\delta_s u_s + c_s), \max_{b,s} q_{sb}\alpha(v_{bs} - \delta_b u_b)\}$$

If $\frac{\beta}{1+\beta} < \alpha$, which is $\frac{\beta}{\alpha} < 1+\beta < \frac{1}{1-\alpha}$, then for any pair (b,s),

$$\beta(\delta_s u_s + c_s) \geq \alpha(v_{bs} - \delta_b u_b) \iff v_{bs} - \delta_b u_b \leq \frac{\beta}{\alpha}(\delta_s u_s + c_s) < \min((1+\beta), \frac{1}{1-\alpha})(\delta_s u_s + c_s),$$

thus $q_{bs} = q_{sb} = 0$. Thus $\frac{\beta}{1+\beta} \geq \alpha$, which is equivalent to $\frac{\beta}{\alpha} \geq 1+\beta \geq \frac{1}{1-\alpha}$. Let:

$$B_1 = \{b \in B : \exists s \in N(b), \text{ s.t. } w_{bs} > 0\}$$

$$B_2 = \{b \in B : \forall s \in N(b), \text{ s.t. } w_{bs} = 0\}$$

$$S_1 = \{s \in S : \exists b \in N(s), \text{ s.t. } w_{sb} > 0\}$$

$$S_2 = \{s \in S : \forall b \in N(s), \text{ s.t. } w_{sb} = 0\}$$

For any $b \in B_2, s \in S_2$, $u_b = 0, u_s = 0$, and

$$\forall b \in B_1, \quad u_b = \frac{\sum_s w_{bs}(v_{bs} - (1+\beta)(c_s + \delta_s u_s))_+}{1 - \delta_b + \delta_b \sum_s w_{bs}} = \frac{\sum_s w_{bs}(v_b - \frac{1+\beta}{\beta}\pi_M)_+}{1 - \delta_b + \delta_b \sum_s w_{bs}}$$

$$\forall s \in S_1, \quad u_s = \frac{\sum_b w_{sb}((1-\alpha)(v_{bs} - \delta_b u_b) - c_s)_+}{1 - \delta_s + \delta_s \sum_b w_{sb}} = \frac{\sum_b w_{sb}(\frac{1-\alpha}{\alpha}\pi_M - c_s)_+}{1 - \delta_s + \delta_s \sum_b w_{sb}}$$

$$\forall s \in S_1, \quad \delta_s u_s + c_s = \frac{\delta_s \sum_b w_{sb} \frac{1-\alpha}{\alpha}\pi_M + (1-\delta_s)c_s}{1 - \delta_s + \delta_s \sum_b w_{sb}}$$

$$\forall b \in B_1, \quad v_{bs} - \delta_b u_b = \frac{\delta_b \sum_s w_{bs} \frac{1+\beta}{\beta} \pi_M + (1-\delta_b) v_b}{1 - \delta_b + \delta_b \sum_s w_{bs}}$$

Claim: For any optimal choices of α, β under $\Gamma^{\alpha+\beta}(G)$, let $\alpha' = \frac{\alpha+\beta}{1+\beta}, \beta' = \frac{\alpha+\beta}{1-\alpha}$, choose α or β instead, then the middleman can achieve as much as the revenue from $\Gamma^{\alpha+\beta}(G)$.

Under $\Gamma^{\alpha+\beta}(G)$,

$$\begin{aligned}
u_b = & \sum_s \lambda w_{bs} q_{bs} (v_{bs} - \frac{1+\beta}{1-\alpha}(\delta_s u_s + c_s) - \delta_b u_b) + (1-\lambda) p_{bs} \tilde{q}_{bs} (v_{bs} - (\delta_s u_s + c_s) - \delta_b u_b) \\
& + \sum_s \big((\lambda w_{bs} + (1-\lambda) p_{bs}\big) \delta_b u_b + \sum_s (\lambda w_{sb} + (1-\lambda) p_{sb}) \delta_b u_b \\
& + \sum_{j,k \neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})(1-\gamma) \delta_b u_b (G_0 \ominus \{j,k\}) \\
& + \big((\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj}) \gamma + \lambda w_{kj} (1 - q_{kj}) + (1-\lambda) p_{kj} (1 - \tilde{q}_{kj})\big) \delta_b u_b]
\end{aligned}$$

$$\begin{aligned}
u_s = & \sum_b \lambda w_{sb} q_{sb} (\frac{1-\alpha}{1+\beta}(v_{bs} - \delta_b u_b) - c_s - \delta_s u_s) + (1-\lambda) p_{sb} \tilde{q}_{sb} (v_{bs} - \delta_b u_b - c_s - \delta_s u_s) \\
& + \sum_b \big((\lambda w_{sb} + (1-\lambda) p_{sb}\big) \delta_s u_s + \sum_b (\lambda w_{bs} + (1-\lambda) p_{bs}) \delta_s u_s \\
& + \sum_{j,k \neq s} [(\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj})(1-\gamma) \delta_s u_s (G_0 \ominus \{j,k\}) \\
& + \big((\lambda w_{kj} q_{kj} + (1-\lambda) p_{kj} \tilde{q}_{kj}) \gamma + \lambda w_{kj} (1 - q_{kj}) + (1-\lambda) p_{kj} (1 - \tilde{q}_{kj})\big) \delta_s u_s]
\end{aligned}$$

$$\begin{aligned}
& \pi_M(G)/(1-\delta_M) \\
= & \max\{\max_{b,s} \frac{\lambda q_{bs} [\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M (G \ominus \{b,s\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{bs}\gamma + 1 - q_{bs}) - (1-\lambda)\delta_M \sum_{i,j \in B \cup S} p_{ij} (\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})} \\
& \ldots, \max_{b,s} \frac{\lambda q_{sb} [\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M (G \ominus \{b,s\})] + (1-\lambda) \sum_{i,j \in B \cup S} p_{ij} \tilde{q}_{ij} (1-\gamma) \delta_M \pi_M (G \ominus \{i,j\})}{1 - \lambda \delta_M (q_{bs}\gamma + 1 - q_{sb}) - (1-\lambda)\delta_M \sum_{i,j \in B \cup S} p_{ij} (\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})}\}
\end{aligned}$$

Let

$$\frac{1+\beta}{1-\alpha} = 1+\beta',$$

$$\frac{1-\alpha}{1+\beta} = 1-\alpha'.$$

Then under the choice of α', β' and in the $\Gamma^{\alpha/\beta}(G)$ game,

$$\begin{aligned} u_b = & \sum_s \lambda w_{bs} q_{bs}(v_b - \frac{1+\beta}{1-\alpha}(c_s + \delta_s u_s) - \delta_b u_b) + (1-\lambda)p_{bs}\tilde{q}_{bs}(v_b - (\delta_s u_s + c_s) - \delta_b u_b) \\ & + \sum_s \big((\lambda w_{bs} + (1-\lambda)p_{bs}\big)\delta_b u_b + \sum_s (\lambda w_{sb} + (1-\lambda)p_{sb})\delta_b u_b \\ & + \sum_{j,k\neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_b u_b(G_0 \ominus \{j,k\}) \\ & + \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_b u_b] \end{aligned}$$

$$\begin{aligned} u_s = & \sum_b \lambda w_{sb} q_{sb}(\frac{1-\alpha}{1+\beta}(v_b - \delta_b u_b) - c_s - \delta_s u_s) + (1-\lambda)p_{sb}\tilde{q}_{sb}(v_b - \delta_b u_b - c_s - \delta_s u_s) \\ & + \sum_b \big((\lambda w_{sb} + (1-\lambda)p_{sb}\big)\delta_s u_s + \sum_b (\lambda w_{bs} + (1-\lambda)p_{bs})\delta_s u_s \\ & + \sum_{j,k\neq b} [(\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})(1-\gamma)\delta_s u_s(G_0 \ominus \{j,k\}) \\ & + \big((\lambda w_{kj} q_{kj} + (1-\lambda)p_{kj}\tilde{q}_{kj})\gamma + \lambda w_{kj}(1-q_{kj}) + (1-\lambda)p_{kj}(1-\tilde{q}_{kj})\big)\delta_s u_s] \end{aligned}$$

$$\begin{aligned} & \pi'_M(G)/(1-\delta_M) \\ = & \max\{\max_{b,s} \frac{\lambda q_{bs}[\frac{\alpha+\beta}{1-\alpha}(\delta_s u_s + c_s) + (1-\gamma)\delta_M \pi_M(G\ominus\{b,s\})] + (1-\lambda)\sum_{i,j\in B\cup S} p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})}{1-\lambda\delta_M(q_{bs}\gamma + 1 - q_{bs}) - (1-\lambda)\delta_M \sum_{i,j\in B\cup S} p_{ij}(\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})} \\ & \ldots, \max_{b,s} \frac{\lambda q_{sb}[\frac{\alpha+\beta}{1+\beta}(v_{bs} - \delta_b u_b) + (1-\gamma)\delta_M \pi_M(G\ominus\{b,s\})] + (1-\lambda)\sum_{i,j\in B\cup S} p_{ij}\tilde{q}_{ij}(1-\gamma)\delta_M\pi_M(G\ominus\{i,j\})}{1-\lambda\delta_M(q_{sb}\gamma + 1 - q_{sb}) - (1-\lambda)\delta_M \sum_{i,j\in B\cup S} p_{ij}(\tilde{q}_{ij}\gamma + 1 - \tilde{q}_{ij})}\} \end{aligned}$$

Therefore $\pi'_M(G) \geq \pi_M(G)$.

C.6. Proof of the existence of MPEs under fixed commission fee

I provide a characterization of MPE payoffs and strategies. Suppose agent of type i has discounting factor $\delta_i \in (0,1)$.

We use a fixed point argument to (implicitly) prove the existence of MPEs σ^* of $\Pi(\delta, v, G)$.

Denote $v(G,\sigma)$ the vector of equilibrium payoff induced by an Markov strategy profile σ of the game $\Pi(\delta, \mathbf{v}, G)$. By definition, any MPE σ^* of $\Pi(\delta, v, G)$ must belong to a subgame consistent collection of MPEs $\sigma|_{G'}$ of the respective games $(\Pi(\delta, v, G'))_{G' \in \mathscr{G}}$. In particular, when $\Pi(\delta, \mathbf{v}, G)$ is played according to MPE $\sigma^*(G)$, every player $1 \leq i \leq N$ has ex ante payoffs $v_i(G, \sigma^*(G))$ before any pairs are selected, and $v_k(G \ominus \{i,j\}, \sigma^*(G \ominus \{i,j\})$ at the beginning of any subgame before which only i and j reached an agreement $(k \neq i, j)$.

Fix a history h_t along which no agreement has been reached, that is $G(h_t) = G_0$. In the subgame with active middleman, middleman M first picks a bargaining pair $e \in E$. After the matched pair is realized, suppose is (i,j), with i being the proposer, in the subgame following $(h_t; i \to j, x)$, it must be that the strategy $(\sigma(G_0))_j(h_t; i \to j, x)$ specifies that player j accepts any offer $x > \delta_j v_j(G_0, \sigma^*(G_0))$, and reject any offer $x < \delta_j v_j(G_0, \sigma^*(G_0))$, and may accept with positive probability offers of $\delta_j v_j(G_0, \sigma^*(G_0))$. Then by subgame perfection, player i does not offer more than $\delta_j v_j(G_0, \sigma^*(G_0))$, i.e., $(\sigma^*(G_0))_i(h_t, i \to j) \leq \delta_j v_j(G_0, \sigma^*(G_0))$.

Let q_{ij} be the probability (conditional on $(h_t, i \to j)$) of the joint event that i offers $\delta_j v_j(G_0, \sigma^*(G_0))$ to j and the offer is accepted. The payoff of player $k \neq i, j$ at the beginning of the next period conditional on $(h_t; i \to j)$ is $q_{ij}\delta_k v_k(G_0 \ominus \{i,j\}, \sigma^*(G_0 \ominus \{i,j\})) + (1 - q_{ij})\delta_k v_k(G_0, \sigma^*(G_0))$.

Case 1. $\delta_i v_i(G_0, \sigma^*(G_0)) + \delta_j v_j(G_0, \sigma^*(G_0)) < v_{ij} - \beta$.
Conditional on $(h_t; i \to j)$, it must be that in any equilibrium, $q_{ij} = 1$. To see this, if $q_{ij} < 1$, then i's expected payoff conditional on offering $x = \delta_j v_j(G_0, \sigma^*(G_0))$ is $q_{ij}(v_{ij} -$

$\beta - x) + (1 - q_{ij})\delta_i v_i(G_0, \sigma^*(G_0)) < (v_{ij} - \beta - x)$, while it is more profitable to offer $x = \delta_j v_j(G_0, \sigma^*(G)) + \eta(\eta > 0)$ with payoff $v_{ij} - \beta - \delta_j v_j(G_0, \sigma^*(G_0))) - \eta$, for sufficiently small $\eta > 0$. And it's obvious that offers smaller than $\delta_j v_j(G_0, \sigma^*(G_0))$ are not optimal for i since they are rejected with certainty and yields $\delta_i v_i(G_0, \sigma^*(G_0) < v_{ij} - \beta - \delta_j v_j(G_0, \sigma^*(G_0))$. Hence i has no best response to j's equilibrium strategy when $q_{ij} < 1$, a contradiction. Hence $q_{ij} = 1$. Moreover $\Pr(`Yes?|(h_t; i \to j) = q_{ij} = 1$.

Case 2. $\delta_i v_i(G_0, \sigma^*(G_0)) + \delta_j v_j(G_0, \sigma^*(G_0))) > v_{ij} - \beta$.
By the similar fashion, one could show that $q_{ij} = 0 = \Pr(`Yes'|(h_t; i \to j))$ since i's offer is no more than $v_{ij} - \beta - \delta_i v_i(G_0, \sigma^*(G_0)) < \delta_j v_j(G_0, \sigma^*(G_0))$.

Case 3. $\delta_i v_i(G_0, \sigma^*(G_0)) + \delta_j v_j(G_0, \sigma^*(G_0))) = v_{ij} - \beta$.
When i is chosen to propose to j, whether the bargain reach an agreement or not, the payoff for both agents are $\delta_i v_i(G_0, \sigma^*(G_0))$ and $\delta_j v_j(G_0, \sigma^*(G_0))$. The subgame perfection conditional on $(h_t, i \to j)$ allows any value $q_{ij} \in [0, 1]$. Since i and j are indifferent between any $x \leq \delta_j v_j^*(G_0, \sigma^*)$,WOLG assume that $q_{ij} = \Pr(\text{`Yes'}|(h_t; i \to j))$.

Hence given any probability distribution $\mathbf{w}$ over matched pair A_m, the continuation payoff $\{v_i(G_0, \sigma^*(G_0))\}$ satisfy,

$$
\begin{aligned}
v_i = &\sum_j (\lambda w_{ij} q_{ij}(v_{ij} - \beta - \delta_j v_j) + (1-\lambda) p_{ij} q_{ij}(v_{ij} - \delta_j v_j) + (\lambda w_{ij} + (1-\lambda)p_{ij})(1 - q_{ij})\delta_i v_i] \\
&+ \sum_j (\lambda w_{ji} + (1-\lambda)p_{ji})\delta_i v_i \\
&+ \sum_{j,k \neq i} (\lambda w_{jk} + (1-\lambda)p_{jk})[q_{jk}\delta_i v_i(G_0 \ominus \{j,k\}, \sigma^*(G_0 \ominus \{j,k\})) + (1 - q_{jk})\delta_i v_i]
\end{aligned}
\tag{C.17}
$$

Since the continuation payoff does not depends on the current matchmaking state (whether matched by the middleman or nature), so one can assume that the agent behaves the same under both cases. However, it's worth notice that, when varying λ, the threshold would change, so agents have different expectations for their continuation payoff. and

$$q_{ij} = \begin{cases} 1 & \text{if } \delta_i v_i + \delta_j v_j < v_{ij} - \beta \\ [0,1] & \text{if } \delta_i v_i + \delta_j v_j = v_{ij} - \beta, \\ 0 & \text{if } \delta_i v_i + \delta_j v_j > v_{ij} - \beta. \end{cases} \tag{C.18}$$

And $q_{ij} = \Pr(`Yes'|(h_t; i \to j))$.

From the middle man's perspective: its ex ante payoff at subnetwork G is,

$$\begin{aligned} \pi_0(G) =& \lambda \sum_{ij} w_{ij}[q_{ij}(\beta + \delta_0\pi_0(G \ominus \{i,j\})) + (1-q_{ij})\delta_0\pi_0(G)] \\ &+ (1-\lambda)\sum_{ij} p_{ij}[q_{ij}\delta_0\pi_0(G \ominus \{i,j\}) + (1-q_{ij})\delta_0\pi_0(G)] \end{aligned}$$

Since the middleman is maximizing it's discounted profit when deciding which link to pick,

$$\begin{aligned} \pi_0(G) =& \lambda \max_{ij}[q_{ij}(\beta + \delta_0\pi_0(G \ominus \{i,j\})) + (1-q_{ij})\delta_0\pi_0(G)] \\ &+ (1-\lambda)\sum_{ij} p_{ij}[q_{ij}\delta_0\pi_0(G \ominus \{i,j\}) + (1-q_{ij})\delta_0\pi_0(G)], \end{aligned}$$

where

$w_{ij} > 0 \Rightarrow$

$$q_{ij}(\beta + \delta_0\pi_0(G \ominus \{i,j\})) + (1-q_{ij})\delta_0\pi_0(G) = \max_{k,s} q_{ks}(\beta + \delta_0\pi_0(G \ominus \{i,j\})) + (1-q_{ks})\delta_0\pi_0(G)$$

This yields:

$$\pi_0(G) = \max_{i,j} \frac{\lambda q_{ij}\beta + \delta_0\big(\lambda q_{ij}\pi(G \ominus \{i,j\}) + (1-\lambda)\sum_{k,s} p_{ks}q_{ks}\pi(G \ominus \{k,s\})\big)}{1 - \delta_0\big(\lambda(1-q_{ij}) + (1-\lambda)\sum_{k,s} p_{ks}(1-q_{ks})\big)}$$

Define the following mapping $\Gamma(\mathbf{v},\mathbf{q})$. $\Gamma : [0,\bar{v}]^{|V|} \times [0,1]^{2|E|} \to [0,\bar{v}]^{|V|} \times [0,1]^{2|E|}$.

$$
\begin{aligned}
\Gamma(\mathbf{v},\mathbf{q}) =&\{(\mathbf{v}',\mathbf{q}')|q'_{ij} = 1(0) \text{ if } v_{ij} - \beta > (<) \delta_i v_i + \delta_j v_j, \\
&\quad \text{and } q'_{ij} \in [0,1] \text{ if } v_{ij} - \beta = \delta_i v_i + \delta_j v_j, \\
&\exists \mathbf{w} \in \Delta, s.t. \{ij|w_{ij} > 0\} \\
&\subseteq \arg\max_{ij} \frac{\lambda q_{ij}\beta + \delta_0\big(\lambda q_{ij}\pi(G \ominus \{i,j\}) + (1-\lambda)\sum_{k,s} p_{ks} q_{ks} \pi(G \ominus \{k,s\})\big)}{1 - \delta_0\big(\lambda(1-q_{ij}) + (1-\lambda)\sum_{k,s} p_{ks}(1-q_{ks})\big)}, \\
&\text{and } v'_i = \sum_j [\lambda w_{ij} q_{ij} \max\{v_{ij} - \beta - \delta_j v_j, \delta_i v_i\} + (1-\lambda) p_{ij} q_{ij} \max\{v_{ij} - \delta_j v_j, \delta_i v_i\} \\
&+ (\lambda w_{ij} + (1-\lambda) p_{ij})(1 - q_{ij}) \delta_i v_i] \\
&+ \sum_j (\lambda w_{ji} + (1-\lambda) p_{ji}) \delta_i v_i \\
&+ \sum_{j,k \neq i} (\lambda w_{jk} + (1-\lambda) p_{jk}) [q_{jk} \delta_i v_i (G_0 \ominus \{j,k\}, \sigma^*(G_0 \ominus \{j,k\})) + (1 - q_{jk}) \delta_i v_i]
\end{aligned}
$$

First need to show that Γ has fixed point.

Lemma C.6.1. $\Gamma : [0,\bar{v}]^N \times [0,1]^E \to [0,\bar{v}]^N \times [0,1]^E$ *has a fixed point.*

Proof. **Claim 1.** $\forall \mathbf{q}, \mathbf{v}, \Gamma(\mathbf{v},\mathbf{q})$ is non-empty and convex.

Proof: It's easy to see that $\Gamma(\mathbf{v},\mathbf{q})$ is non-empty. $\forall \mathbf{v}, \mathbf{q}$, and $(\mathbf{v}',\mathbf{q}'), (\mathbf{v}'',\mathbf{q}'') \in \Gamma(\mathbf{v},\mathbf{q})$, $\eta \in [0,1]$, $\eta q'_{ij} + (1-\eta) q''_{ij} = 1(0)$ if $v_{ij} - \beta > (<) \delta_i v_i + \delta_j v_j$, and $\eta q'_{ij} + (1-\eta) q''_{ij} \in [0,1]$ for any pair of (i,j). Suppose $\mathbf{p}', \mathbf{p}''$ are corresponding $\mathbf{w}$s in the mapping, then $\mathbf{p}', \mathbf{p}''$ have the same support, hence

$$\eta \mathbf{p}' + (1-\eta)\mathbf{p}'' \in \arg\max \frac{\lambda q_{ij}\beta + \delta_0\big(\lambda q_{ij}\pi(G \ominus \{i,j\}) + (1-\lambda)\sum_{k,s} p_{ks} q_{ks} \pi(G \ominus \{k,s\})\big)}{1 - \delta_0\big(\lambda(1-q_{ij}) + (1-\lambda)\sum_{k,s} p_{ks}(1-q_{ks})\big)}.$$

Thus $(\eta \mathbf{v}' + (1-\eta)\mathbf{v}'', \eta \mathbf{q}' + (1-\eta)\mathbf{q}'') \in \Gamma(\mathbf{v},\mathbf{q})$

Claim 2. Γ has closed graph.

Proof: For any $(\mathbf{x}^n, \mathbf{y}^n) \in \Gamma(\mathbf{v}^n, \mathbf{q}^n)$, where

$$(\mathbf{v}^n, \mathbf{q}^n) \to (\mathbf{v}, \mathbf{q}), \quad (\mathbf{x}^n, \mathbf{y}^n) \to (\mathbf{x}, \mathbf{y}).$$

If $v_{ij} - \beta > \delta_i v_i + \delta_j v_j$, there exists some $n_1 > 0$, s.t.

$$v_{ij} - \beta > \delta_i v_i^n + \delta_j v_j^n, \quad \forall n > n_1,$$

and $y_{ij}^n = 1$, thus $y_{ij} = 1$.

If $v_{ij} - \beta < \delta_i v_i + \delta_j v_j$, there exists some $n_2 > 0$, s.t.

$$v_{ij} - \beta < \delta_i v_i^n + \delta_j v_j^n, \quad \forall n > n_2,$$

and $y_{ij}^n = 0$, thus $y_{ij} = 0$.

If $v_{ij} - \beta = \delta_i v_i + \delta_j v_j$, $y_{ij} \in [0, 1]$ is trivial.

Next let

$$\gamma_{ij}(\mathbf{q}) = \frac{\lambda q_{ij}\beta + \delta_0\big(\lambda q_{ij}\pi(G \ominus \{i,j\}) + (1-\lambda)\sum_{k,s} p_{ks} q_{ks} \pi(G \ominus \{k,s\})\big)}{1 - \delta_0\big(\lambda(1-q_{ij}) + (1-\lambda)\sum_{k,s} p_{ks}(1-q_{ks})\big)}.$$

It's obvious that for each $(\mathbf{v}^n, \mathbf{q}^n)$, the corresponding $\mathbf{w}^n$ satisfies the following

$$p_{ij}^n(\max_{ij} \gamma_{ij}(\mathbf{q}^n) - \gamma_{ij}(\mathbf{q}^n)) = 0$$

Suppose that a subsequence $\mathbf{w}^{n_k}$ converges to $\mathbf{w}$. Since $\max \gamma_{ij}(\cdot), \gamma_{ij}(\cdot)$ are all continuous, let $n_k \to \infty$, we have

$$w_{ij}(\max_{ij} \gamma_{ij}(\mathbf{q}) - \gamma_{ij}(\mathbf{q})) = 0.$$

Thus

$$\{ij | w_{ij} > 0\} \subseteq \arg\max \frac{\lambda q_{ij}\beta + \delta_0\big(\lambda q_{ij}\pi(G \ominus \{i,j\}) + (1-\lambda)\sum_{k,s} p_{ks} q_{ks} \pi(G \ominus \{k,s\})\big)}{1 - \delta_0\big(\lambda(1-q_{ij}) + (1-\lambda)\sum_{k,s} p_{ks}(1-q_{ks})\big)}.$$

Notice that $\mathbf{x}^n = g(\mathbf{v}^n, \mathbf{q}^n, \mathbf{w}^n)$ where g is a continuous function of $(\mathbf{w}, \mathbf{q})$. Hence

$$\mathbf{x} = \lim \mathbf{x}^{n_k} = \lim g(\mathbf{v}^{n_k}, \mathbf{q}^{n_k}, \mathbf{w}^{n_k}) = g(\mathbf{v}, \mathbf{q}, \mathbf{w})$$

Hence

$$(\mathbf{x}, \mathbf{y}) \in \Gamma(\mathbf{v}, \mathbf{q}).$$

Claim 3. Γ maps the compact set $[0, \bar{v}]^|V| \times [0, 1]^{2|E|}$ to itself.

Therefore by Kakutatni's fixed point theorem, Γ has a fixed point

$$(\mathbf{v}, \mathbf{q}) \in \Gamma(\mathbf{v}, \mathbf{q}).$$

By the definition of Γ, there exists $\mathbf{w}$, s.t.

1. $\mathbf{q}$ satisfies Eq(C.18).

2. $\mathbf{w}$ is middleman's best response to $\mathbf{q}$.

3. $\mathbf{v}$ satisfies Eq(C.17), that is, $\mathbf{v}$ is the corresponding continuation payoff.

□

At the begining of the subgame h_t when no one has reached an agreement, given $\mathbf{v}$ the players' expected payoff starting at t, Γ compute agent's best response $\mathbf{q}$ to $\mathbf{v}$, along with the payoff $\mathbf{v}'$ induced by $\mathbf{q}$, and middleman's best response $\mathbf{w}$ to agents' strategies $\mathbf{q}$, and the corresponding new expected payoff $\mathbf{v}'$ since they depend on the strategy of the middleman. Formally,

Lemma C.6.2. $\mathbf{v}^*$ *is a Markov Perfect equilibrium payoff, if there exists a collection of subgame consistent MPE of the game* $\{\Gamma(\delta, \mathbf{v}, G)\}_{G \subset G_0}$ *with payoffs* $\{v^*(G)\}_{G \subset G_0}$ *and* p^**, such that* $(\mathbf{v}^*, \mathbf{q}^*) \in \Gamma(\mathbf{v}^*, \mathbf{q}^*)$.

Proof. Suppose that the collection of subgame consistent MPE of the game $\{(\delta, \mathbf{v}, G)\}_{G \subset G_0}$ has payoffs $\{v^*(G, \sigma^*(G))\}_{G \subset G_0}$. If $(\mathbf{w}^*, \mathbf{v}^*) \in \Gamma(\mathbf{w}^*, \mathbf{v}^*)$, then there exists $\mathbf{q}^*$, s.t.

(i) $\mathbf{q}^*$ satisfies (C.18).

(ii) $\mathbf{w}$ is best response to $\mathbf{q}$ given $\sigma^*(G)$, for any $G \subset G_0$.

(iii) $\mathbf{v}^*$ is the corresponding ex ante payoff for network G_0.

Then we construct the following strategy profile and prove it is an MPE with corresponding MPE payoff $\mathbf{v}^*$. First define the strategies for histories h_t along which no agreement has occurred. Recall that $G(h_t)$ denotes the network induced by the players remaining after the ex post history h_t. Construct time-t strategy of each player according to the time-0 behavior specified by $\sigma^*(G_0)$.[2] For histories along which no agreement has occurred, $\sigma^*(G(h_t))$ specifies that

- Middleman choose matching pair (i, j) with probability p^*_{ij}.
- when i is chosen to propose to j, he offers $\min(v_{ij} - \beta - \delta_i v^*_i, \delta_j v^*_j)$.
- when i responds to the offer x, he accepts any offer $x > \delta_i v^*_i$, and rejects any $x < \delta_i v^*_i$, and accept an offer of $\delta_i v^*_i$ with probability q^*_{ij}.

Given the collection of subgame consistency guarantees that under $(\sigma^*(G))_{G \subset G_0}$ the expected payoffs of any subgames are $v(G, \sigma(G))$.

Lemma C.6.3 (Mailath & Samuelson (2006) Proposition 5.7.1)**.** *A strategy profile is subgame perfect in a dynamic game if and only if there are no profitable one-shot deviations.*

Based on the definition of Γ, everyone is best responding at period t no profitable deviation for all players in this stage game. It's then easy to verify that $\mathbf{v}^*$ are indeed the equilibrium payoff by the strategy profile conditional on h_t.

[2] Formally, $(\sigma(G))_i(h_t; i \to j) = (\sigma(G))_i(h_0; i \to j), (\sigma(G))_j(h_t; i \to j, x) = (\sigma(G))_i(h_0; i \to j, x)$, where $h_0 = \emptyset$, $G = G(h_t)$.

□

We now need to show that Lemma C.1.2 implies the existence of MPEs. We prove a subgame consistent collection of MPEs for the game $(\delta, v, G)_{G \subseteq \mathscr{G}(n)}$, where $\mathscr{G}(n)$ denotes the subset of subnetworks in $\mathscr{G}$ with at most n vertices. The proof proceed by induction on n. For $n = 0, 1$ it's trivial. Suppose we proved the statement for all values smaller than n,and proceed to proving the case n. By induction hypothesis, there exists a subgame consistent collection of MPEs $(\sigma^*(G'))_{G' \in \mathscr{G}(n-1)}$ of the corresponding games $(\Pi(\delta, v, G'))_{G' \in \mathscr{G}(n-1)}$. Fix a network $G \in \mathscr{G}(n) \setminus \mathscr{G}(n-1)$, $S(G) = \{G' : G' \subset G\} \subseteq \mathscr{G}(n-1)$. Therefore there exist MPEs $(\sigma^*(G'))_{G' \in S(G)}$ for the games $(\Gamma(\delta, v, G'))_{G' \in S(G)}$ that are subgame consistent, hence we can use their MPE payoffs to define Γ. Suppose now that Γ has a fixed point $p^*, \mathbf{v}^*$, with induced $\sigma^*(G)$ of the game $\Pi(\delta, v, G)$ so that $(\sigma^*(G'))_{G' \in \mathscr{G}(n-1) \cup \{G\}}$ is subgame consistent. If we append all MPE $\sigma^*(G)$ of $G \in \mathscr{G}(n) \setminus \mathscr{G}(n-1)$ to $(\sigma^*(G'))_{G' \in \mathscr{G}(n-1)}$, the resulting collection of MPEs $(\sigma^*(G'))_{G' \in \mathscr{G}(n)}$ is also subgame consistent.

BIBLIOGRAPHY

A. Abdulkadiroğlu and T. Sönmez. School choice: A mechanism design approach. *The American Economic Review*, 93(3):729–747, 2003.

A. Abdulkadiroğlu, J. Angrist, and P. Pathak. The elite illusion: Achievement effects at boston and new york exam schools. *Econometrica*, 82(1):137–196, 2014.

A. Abdulkadiroğlu, Y.-K. Che, and Y. Yasuda. Expanding 'choice' in school choice. *American Economic Journal: Microeconomics*, 7(1):1–42, 2015.

D. Abreu and M. Manea. Bargaining and efficiency in networks. *Journal of Economic Theory*, 147(1):43–70, 2012a.

D. Abreu and M. Manea. Markov equilibria in a model of bargaining in networks. *Games and Economic Behavior*, 75(1):1–16, 2012b.

M. Armstrong. Competition in two-sided markets. *The RAND Journal of Economics*, 37 (3):668–691, 2006.

N. Arnosti, R. Johari, and Y. Kanoria. Managing congestion in matching markets. 2015.

A. E. Atakan. Assortative matching with explicit search costs. *Econometrica*, 74(3):667–680, 2006.

S. Athey and G. Ellison. Position auctions with consumer search. *The Quarterly Journal of Economics*, 126(3):1213–1270, 2011.

C. Avery and P. A. Pathak. The distributional consequences of public school choice. Technical report, National Bureau of Economic Research, 2015.

E. M. Azevedo and E. Budish. Strategy-proofness in the large. Technical report, National Bureau of Economic Research, 2017.

E. M. Azevedo and E. B. Budish. Strategy-proofness in the large. *Chicago Booth Research Paper*, (13-35), 2013.

E. M. Azevedo, E. G. Weyl, and A. White. Walrasian equilibrium in large, quasilinear markets. *Theoretical Economics*, 8(2):281–290, 2013.

H. Bar-Isaac, G. Caruana, and V. Cuñat. Search, design, and market structure. *American Economic Review*, 102(2):1140–60, 2012.

V. Baritaux, M. Aubert, E. Montaigne, and H. Remaud. Matchmakers in wine marketing channels: the case of french wine brokers. *Agribusiness*, 22(3):375–390, 2006.

L. Barrow and C. E. Rouse. Causality, causality, causality: The view of education inputs and outputs from economics. 2005.

L. Barseghyan, D. Clark, and S. Coate. Public school choice: An economic analysis. Working Paper 20701, National Bureau of Economic Research, November 2014.

M. R. Baye and J. Morgan. Information gatekeepers on the internet and the competitiveness of homogeneous product markets. *American Economic Review*, 91(3):454–474, 2001.

M. R. Baye, J. Morgan, and P. Scholten. Price dispersion in the small and in the large: Evidence from an internet price comparison site. *The Journal of Industrial Economics*, 52(4):463–496, 2004.

P. Bayer, F. Ferreira, and R. McMillan. A unified framework for measuring preferences for schools and neighborhoods. Technical report, National Bureau of Economic Research, 2007.

K. G. Binmore and M. J. Herrero. Matching and bargaining in dynamic markets. *The review of economic studies*, 55(1):17–31, 1988.

S. E. Black. Do better schools matter? parental valuation of elementary education. *The Quarterly Journal of Economics*, 114(2):577–599, May 1999.

L. E. Blume, D. Easley, J. Kleinberg, and E. Tardos. Trading networks with price-setting agents. *Games and Economic Behavior*, 67(1):36–50, 2009.

J.-F. Brocard, M. Cavagnac, et al. Who should pay the sports agent's commission? an economic analysis of setting the legal rules in the regulation of matchmakers. Technical report, Toulouse School of Economics (TSE), 2016.

E. J. Brunner, S.-W. Cho, and R. Reback. Mobility, housing markets, and schools: Estimating the effects of inter-district choice programs. *Journal of Public Economics*, 96(7), 2012.

E. Brynjolfsson and M. D. Smith. Frictionless commerce: A comparison of internet and conventional retailers. *Management science*, 46(4):563–585, 2000.

D. Bunten. Is the rent too high? aggregate implications of local land-use regulation. Technical report, UCLA Working Paper, 2015.

K. Burdett and K. L. Judd. Equilibrium price dispersion. *Econometrica: Journal of the Econometric Society*, pages 955–969, 1983.

M. A. Burke and T. R. Sass. Classroom peer effects and student achievement. *Journal of Labor Economics*, 31(1):51–82, 2013.

B. Caillaud and B. Jullien. Chicken & egg: Competition among intermediation service providers. *RAND journal of Economics*, pages 309–328, 2003.

S. Calabrese, D. Epple, and R. Romano. Inefficiencies from metropolitan political and fiscal decentralization: Failures of tiebout competition. *The Review of Economic Studies*, page rdr048, 2011.

C. Calsamiglia, F. Martinez-Mora, and A. Miralles. School choice mechanisms, peer effects and sorting. Discussion Papers in Economics 15/01, Department of Economics, University of Leicester, 2015.

R. Casadesus-Masanell and H. Hałaburda. When does a platform create value by limiting choice? *Journal of Economics & Management Strategy*, 23(2):259–293, 2014.

Y.-K. Che and F. Kojima. Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica*, pages 1625–1672, 2010.

S. Choi, A. Galeotti, and S. Goyal. Trading in networks: Theory and experiments. *Journal of the European Economic Association*, 15(4):784–817, 2017.

M. Corominas-Bosch. Bargaining in a network of buyers and sellers. *Journal of Economic Theory*, 115(1):35–77, 2004.

P. A. Diamond and E. Maskin. An equilibrium analysis of search and breach of contract, i: Steady states. *The Bell Journal of Economics*, pages 282–316, 1979.

M. Dinerstein, L. Einav, J. Levin, and N. Sundaresan. Consumer price search and platform design in internet commerce. Technical report, National Bureau of Economic Research, 2014.

D. Duffie, N. Gârleanu, and L. H. Pedersen. Over-the-counter markets. *Econometrica*, 73 (6):1815–1847, 2005.

G. Ellison and S. F. Ellison. Search, obfuscation, and price elasticities on the internet. *Econometrica*, 77(2):427–452, 2009.

G. Ellison and S. F. Ellison. Match quality, search, and the internet market for used books. Technical report, National Bureau of Economic Research, 2018.

G. Ellison and A. Wolitzky. A search cost model of obfuscation. *The RAND Journal of Economics*, 43(3):417–441, 2012.

D. Epple and H. Sieg. Estimating equilibrium models of local jurisdictions. *Journal of Political Economy*, 107(4):645–681, 1999.

D. Epple and A. Zelenitz. The implications of competition among jurisdictions: does tiebout need politics? *The Journal of Political Economy*, pages 1197–1217, 1981.

D. Epple, T. Romer, and H. Sieg. Interjurisdictional sorting and majority rule: an empirical analysis. *Econometrica*, 69(6):1437–1465, 2001.

D. N. Epple and R. Romano. Neighborhood schools, choice, and the distribution of educational benefits. In *The economics of school choice*, pages 227–286. University of Chicago Press, 2003.

W. A. Fischel. Does the american way of zoning cause the suburbs of metropolitan areas to be too spread out? *Governance and opportunity in metropolitan America*, 15:1–19, 1999.

D. Fudenberg and J. Tirole. Game theory, 1991. *Cambridge, Massachusetts*, 393(12):80, 1991.

D. Gale. Limit theorems for markets with sequential bargaining. *Journal of Economic Theory*, 43(1):20–54, 1987.

D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

C. Gollier and J. W. Pratt. Risk vulnerability and the tempering effect of background risk. *Econometrica: Journal of the Econometric Society*, pages 1109–1123, 1996.

R. Gomes and A. Pavan. Many-to-many matching and price discrimination. *Theoretical Economics*, 11(3):1005–1052, 2016.

R. Greenwald, L. V. Hedges, and R. D. Laine. The effect of school resources on student achievement. *Review of educational research*, 66(3):361–396, 1996.

H. Halaburda, M. Jan Piskorski, and P. Yıldırım. Competing by restricting choice: The case of matching platforms. *Management Science*, 2017.

J. W. Hatfield, F. Kojima, and Y. Narita. Promoting school competition through school choice: a market design approach. *Available at SSRN 1984876*, 2015.

Y. He et al. Gaming the boston school choice mechanism in beijing. *Manuscript, Toulouse School of Economics*, 2012.

C. M. Hoxby. School choice and school productivity. could school choice be a tide that lifts all boats? In *The economics of school choice*, pages 287–342. University of Chicago Press, 2003.

S. Imberman, A. D. Kugler, and B. Sacerdote. Katrina's children: evidence on the structure of peer effects from hurricane evacuees. Technical report, National Bureau of Economic Research, 2009.

C. K. Jackson. *Peer Quality or Input Quality?: Evidence from Trinidad and Tobago*, volume 16598. National Bureau of Economic Research, 2010.

Y. Kanoria and D. Saban. Facilitating the search for partners on matching platforms: Restricting agents' actions. 2017.

M. G. Kohn and S. Shavell. The theory of search. *Journal of Economic Theory*, 9(2): 93–123, 1974.

M. H. Kotowski and C. M. Leister. Trading networks and equilibrium intermediation. 2018.

R. E. Kranton and D. F. Minehart. A theory of buyer-seller networks. *American economic review*, 91(3):485–508, 2001.

S. Lauermann. Dynamic matching and bargaining games: A general approach. *American Economic Review*, 103(2):663–89, 2013.

S. Lauermann and G. Nöldeke. Existence of steady-state equilibria in matching models with search frictions. *Economics Letters*, 131:1–4, 2015.

V. Lavy, O. Silva, and F. Weinhardt. The good, the bad and the average: Evidence on the scale and nature of ability peer effects in schools. Technical report, National Bureau of Economic Research, 2009.

J. D. Levin. The economics of internet markets. Technical report, National Bureau of Economic Research, 2011.

W. B. MacLeod and M. Urquiola. Anti-lemons: School reputation and educational quality. Working Paper 15112, National Bureau of Economic Research, June 2009.

M. Manea. Bargaining in stationary networks. *American Economic Review*, 101(5):2042–80, 2011.

M. Manea. Intermediation and resale in networks. *Journal of Political Economy*, 2015.

M. Manea. Bottleneck links, essential intermediaries, and competing paths of diffusion in networks. 2017a.

M. Manea. Steady states in matching and bargaining. *Journal of Economic Theory*, 167: 206–228, 2017b.

E. Maskin and J. Tirole. Markov perfect equilibrium: I. observable actions. *Journal of Economic Theory*, 100(2):191–219, 2001.

F. W. McElroy. Probability of risk aversion and other applications of derivatives of the certainty equivalent. *International Review of Economics*, 54(4):429–444, 2007.

R. McMillan. Erratum to 'competition, incentives, and public school productivity'. *Journal of Public Economics*, 89(5):1133–1154, 2005.

A. Miralles. School choice: The case for the boston mechanism. In *International Conference on Auctions, Market Mechanisms and Their Applications*, pages 58–60. Springer, 2009.

T. Nguyen. Coalitional bargaining in networks. *Operations Research*, 63(3):501–511, 2015.

T. Nguyen, V. Subramanian, and R. Berry. Delay in trade networks. *Operations Research*, 64(3):646–661, 2016.

G. Nöldeke and T. Tröger. Matching heterogeneous agents with a linear search technology. 2009.

P. A. Pathak and P. Shi. Simulating alternative school choice options in boston-technical appendix. Technical report, Tech. rep., MIT School Effectiveness and Inequality Initiative, 2013.

M. J. Piskorski, H. Halaburda, and T. Smith. Eharmony. 2008.

A. Polanski. Bilateral bargaining in networks. *Journal of Economic Theory*, 134(1):557–565, 2007.

R. Poli and G. Rossi. Football agents in the biggest five european football markets an empirical research report. *CIES Football Observatory*, 2012.

R. Reback. House prices and the provision of local public services: Capitalization under school choice programs. *Journal of Urban Economics*, 57(2):275–301, 2005.

J. F. Reinganum. A simple model of equilibrium price dispersion. *Journal of Political Economy*, 87(4):851–858, 1979.

S. G. Rivkin, E. A. Hanushek, and J. F. Kain. Teachers, schools, and academic achievement. *Econometrica*, 73(2):417–458, 2005.

J.-C. Rochet and J. Tirole. Platform competition in two-sided markets. *Journal of the european economic association*, 1(4):990–1029, 2003.

J.-C. Rochet and J. Tirole. Two-sided markets: a progress report. *The RAND journal of economics*, 37(3):645–667, 2006.

J. E. Rockoff. The impact of individual teachers on student achievement: Evidence from panel data. *The American Economic Review*, 94(2):247–252, 2004.

M. Rothschild. Searching for the lowest price when the distribution of prices is unknown. In *Uncertainty in Economics*, pages 425–454. Elsevier, 1978.

J. M. Rothstein. Good principals or good peers? parental valuation of school characteristics, tiebout equilibrium, and the incentive effects of competition among jurisdictions. *American Economic Review*, 96(4):1333–1350, 2006.

A. Rubinstein and A. Wolinsky. Equilibrium in a market with sequential bargaining. *Econometrica: Journal of the Econometric Society*, pages 1133–1150, 1985.

M. Satterthwaite and A. Shneyerov. Dynamic matching, two-sided incomplete information, and participation costs: Existence and convergence to perfect competition. *Econometrica*, 75(1):155–200, 2007.

M. H. Schill. Regulations and housing development: What we know. *Cityscape*, pages 5–19, 2005.

D. Schmeidler. Equilibrium points of nonatomic games. *Journal of Statistical Physics*, 7 (4):295–300, 1973.

R. Shimer and L. Smith. Assortative matching and search. *Econometrica*, 68(2):343–369, 2000.

K. L. Shropshire, T. Davis, and N. J. Duru. *The business of sports agents*. University of Pennsylvania Press, 2016.

G. J. Stigler. The economics of information. *Journal of political economy*, 69(3):213–225, 1961.

C. Wang and J. Wright. Search platforms: Showrooming and price parity clauses. Technical report, working paper, 2016.

M. L. Weitzman. Optimal search for the best alternative. *Econometrica: Journal of the Econometric Society*, pages 641–654, 1979.

E. G. Weyl. A price theory of multi-sided platforms. *American Economic Review*, 100(4): 1642–72, 2010.

A. Yavaş. Marketmakers versus matchmakers. *Journal of Financial Intermediation*, 2(1): 33–58, 1992a.

A. Yavaş. A simple search and bargaining model of real estate markets. *Real estate economics*, 20(4):533–548, 1992b.

A. Yavaş. Middlemen in bilateral search markets. *Journal of Labor Economics*, 12(3): 406–429, 1994.