Optimal Control of Spatially Distributed Systems

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***Abstract*—In this paper, we study the structural properties of optimal control of spatially distributed systems. Such systems consist of an infinite collection of possibly heterogeneous linear control systems that are spatially interconnected via certain distant-dependent coupling functions over arbitrary graphs. We study the structural properties of optimal control problems with infinite-horizon linear quadratic criteria, by analyzing the spatial structure of the solution to the corresponding operator Lyapunov and Riccati equations. The key idea of the paper is the introduction of a special class of operators called spatially decaying (SD). These operators are a generalization of translation invariant operators used in the study of spatially invariant systems. We prove that given a control system with a state-space representation consisting of SD operators, the solution of operator Lyapunov and Riccati equations are SD. Furthermore, we show that the kernel of the optimal state feedback for each subsystem decays in the spatial domain, with the type of decay (e.g., exponential, polynomial or logarithmic) depending on the type of coupling between subsystems.**

***Index Terms*—Distributed control, infinite-dimensional systems, networked control, optimal control, spatially decaying systems.**

I. Introduction

Analysis and synthesis of distributed coordination and control algorithms for networked dynamic systems have become a vibrant part of control theory research. From consensus and agreement problems to formation control and sensing and coverage problems, researchers have been interested in the development and analysis of control protocols that are "localized" and spatially distributed and designed to achieve a global objective, such as consensus or coverage, using only local interactions. Despite some success, a general theory of optimal control for linear systems with information constraints on the optimal feedback law is lacking. This is not surprising as it is well known that [1] very simple-looking linear optimal control problems with sparsity or decentralization constraints on the feedback structure can have complicated nonlinear solutions.

A canonical decentralized optimal control problem with linear-quadratic (LQ) criteria can be cast as an LQR problem in which the stabilizing controller is restricted to lie in a particular subspace $\mathcal{S}$. This subspace of admissible controllers is often referred to as the information constraint set [2]. For a general linear system and subspace $\mathcal{S}$, there is no known tractable algorithm for computing the optimum. In fact, certain cases have been shown to be intractable [3], [4].

Manuscript received September 19, 2006; revised July 25, 2007. Current version published September 10, 2008. This research was supported in part under Grants ARO/MURI W911NF-05-1-0381, ONR/YIP N00014-04-1-0467, and NSF-ECS-0347285. Reviewed by Associate Editor Chaouki Abdallah.

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Digital Object Identifier 10.1109/TAC.2008.929366

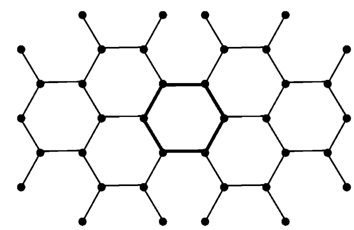

Fig. 1. Spatial invariance on a hexagonal array.

Recently, certain special cases of the general canonical problem with particular symmetries in the spatial structure (cf. Fig. 1) have been successfully studied in the literature. In [5], Bamieh *et al.* used spatial Fourier transforms and operator theory to study the optimal control of linear spatially invariant systems with standard LQ criteria. It was shown that if subspace $\mathcal{S}$ is the set of all bounded translation-invariant operators whose Fourier transforms have analytic continuation to some annulus around the unit circle in the complex domain and the state-space operators and weighing operators in the LQ cost are all in $\mathcal{S}$, then the canonical problem is equivalent to the standard LQR problem without information constraint.

In [2], the authors introduce the notion of quadratic invariance for the constraint set $\mathcal{S}$. Using this notion, the authors show that the problem of finding optimal controllers for an information constraint set that has the quadratic invariance property can be cast as a convex optimization problem, although the resulting controller might have a very high order. It turns out that many (but not all) tractable decentralized optimal control problems do indeed satisfy the quadratic invariance property.

Other authors have used a synthesis-based approach to develop a control method which yields a distributed controller with possibly the same architecture as the underlying plant. In [6], the authors developed linear matrix inequality (LMI) conditions for well-posedness, stability, and performance of spatially interconnected systems consisting of homogeneous units interconnected over a discrete group (e.g., a 1-D or 2-D lattice or ring). These results were later extended to systems with certain types of boundary conditions [10], and with arbitrary discrete symmetry groups [8], [9]. Heterogeneous spatially distributed systems are studied in [7] and [11], where the authors use operator-theoretic tools to design optimal controllers for heterogeneous systems which are not shift invariant with respect to spatial or temporal variables. Other recent results in this area include [12]–[14]. Another much older but related work on this subject was reported in [15] where homogeneous interconnected systems are studied using $\mathcal{Z}$-transform analysis.

Among all of the aforementioned results, this paper is closest in spirit to [5]. Our goal here is to analyze the spatial structure of the optimal control of spatially distributed systems with arbitrary interconnection topologies. The spatial structures studied in [5] are locally compact Abelian (LCA) groups [16], such as $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, \oplus)$, where the group operation naturally induces a translation operator. However, when the dynamics of individual subsystems are not identical and the spatial structure does not necessarily enjoy the symmetries of LCA groups, standard tools, such as Fourier analysis, cannot be used to analyze the system.

To address this issue, we introduce a new class of linear operators, called spatially decaying (SD) operators. These operators are a natural generalization of the class of linear translation-invariant operators. Roughly speaking, an operator is SD if a certain auxiliary operator formed by a block-wise exponential (or algebraic) "inflation" of the operator remains bounded with respect to all induced norms. We show that such operators exhibit a localized behavior in the spatial domain (i.e., the norm of blocks in the matrix representation of the operator decay exponentially or algebraically in space). It turns out that many well-known cooperative control and networked control problems can be characterized by SD operators.

A linear control system is called spatially decaying if the operators in its state-space representation are SD. It is shown that the space of SD operators $\mathcal{S}_\tau^\infty(\mathscr{C})$ is a normed vector space with respect to a specific operator-norm which is not induced and is denoted by $|||\cdot|||$. Furthermore, such operators equipped with this norm form a B^*-algebra. A succinct definition is that a B^*-algebra is a *-algebra that is also a Banach algebra. Using this result, we prove that the unique solution of *Lyapunov* and algebraic *Riccati* equations (ARE) corresponding to an SD system are indeed SD themselves. As a result, the corresponding optimal controllers are SD and, thus, spatially localized.

The implications of this result are quite far reaching. It essentially means that the contribution of a "far away" subsystem to the optimal feedback gains of a given subsystem are negligible. More precisely, for SD systems, the size of the feedback decays as a function of spatial distance between subsystems and as controllers are inherently localized. It should be mentioned that the machinery developed in this paper can be also used to analyze the spatial structure of a broader class of optimal control problems, such as constrained, finite horizon control, or model predictive control problems. This problem has been analyzed in detail in [17] and [18].

This paper is organized as follows. We introduce the notation and mathematical preliminaries in Section II. The optimal control of linear spatially distributed systems is discussed in Section III. In Section IV, we introduce the notion of SD operators. Our main results on the structure of solutions of Lyapunov and Riccati equations are given in Section V. Simulation results are included in Section VI, and our concluding remarks are presented in Section VII.

II. Preliminaries, Notation, and Definitions

The notation used in this paper is fairly standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, $\mathbb{C}$ is the set of complex numbers, and $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$. The inner product on $\mathbb{R}^n$ is denoted by $\langle , \rangle$ with corresponding norm $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{R}^n$. For notational simplicity, the matrix norm induced by $\|\cdot\|$ is also denoted by $\|\cdot\|$. The maximum singular value of a matrix A is denoted $\bar{\sigma}(A)$. A subset $\mathbb{G}$ of $\mathbb{Z}^d$ or $\mathbb{R}^d$ is referred to as the spatial domain if it consists of countably many d-tuples $i = (i_1, \ldots, i_d)$. In this paper, we restrict ourselves to spatial domains with infinite cardinality.

Definition 1: A distance function on a discrete topology with a set of nodes $\mathbb{G}$ is defined as a single-valued function $\mathrm{dis} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}^+$ which has the following properties for all $k, i, j \in \mathbb{G}$:

1) $\mathrm{dis}(k, i) = 0$ iff $k = i$;
2) $\mathrm{dis}(k, i) = \mathrm{dis}(i, k)$;
3) $\mathrm{dis}(k, i) \le \mathrm{dis}(k, j) + \mathrm{dis}(j, i)$.

Throughout this paper, $\mathrm{dis}(k, i)$ represents the spatial distance between two nodes (e.g. subsystems) with indices $k, i \in \mathbb{G}$. The Banach space $\ell_p(\mathbb{G})$ for $1 \le p < \infty$ is defined to be the set of all sequences $x = (x_i)_{i\in\mathbb{G}}$ in which $x_i \in \mathbb{R}^{n_i}$ for some $n_i \ge 1$ satisfies

$$\sum_{i\in\mathbb{G}} \|x_i\|^p < \infty$$

endowed with the norm

$$\|x\|_p^p := \sum_{i\in\mathbb{G}} \|x_i\|^p.$$

The Banach space $\ell_\infty(\mathbb{G})$ denotes the set of all bounded sequences endowed with the norm

$$\|x\|_\infty := \sup_{i\in\mathbb{G}} \|x_i\|.$$

Throughout this paper, we will use the shorthand notation ℓ_p for $\ell_p(\mathbb{G})$. The space ℓ_2 is a Hilbert space with inner product

$$\langle x, y\rangle := \sum_{i\in\mathbb{G}} \langle x_i, y_i\rangle$$

for all $x, y \in \ell_2$. An operator $\mathcal{A} : \ell_p \to \ell_q$ is bounded if it has a finite induced norm, that is, the following quantity:

$$\|\mathcal{A}\|_{p,q} := \sup_{\|x\|_p=1} \|\mathcal{A}x\|_q \tag{1}$$

is bounded. The identity operator is denoted by $\mathcal{I}$. The set of all bounded linear operators of ℓ_p into ℓ_q is denoted by $\mathscr{B}\ (\ell_p, \ell_q)$. The space $\mathscr{B}\ (\ell_p, \ell_q)$ equipped with norm (1) is a Banach space (cf. [19]). When and the initial and target spaces are both ℓ_p, we use the notation $\mathscr{B}\ (\ell_p)$. An operator $\mathcal{A} \in \mathscr{B}\ (\ell_p)$ has an *algebraic* inverse if it has an inverse $\mathcal{A}^{-1}$ in $\mathscr{B}\ (\ell_p)$ [19]

$$\mathcal{A}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{A} = \mathcal{I}.$$

The adjoint operator of $\mathcal{A} \in \mathscr{B}\ (\ell_2)$ is the operator $\mathcal{A}^*$ in $\mathscr{B}\ (\ell_2)$ such that $\langle \mathcal{A}x, y\rangle = \langle x, \mathcal{A}^*y\rangle$ for all $x, y \in \ell_2$. An operator $\mathcal{A}$ is self-adjoint if $\mathcal{A} = \mathcal{A}^*$. An operator $\mathcal{A} \in \mathscr{B}\ (\ell_2)$ is positive definite, shown as $\mathcal{A} \succ \mathbf{0}$, if a number $\alpha > 0$ exists such that

$$\langle x, \mathcal{A}x\rangle > \alpha\|x\|_2^2$$

for all nonzero $x \in \ell_2$.

Throughout this paper, we are interested in linear operators $\mathcal{A} : \ell_p \rightarrow \ell_p$ which have a matrix representation as

$$\mathcal{A} \longmapsto \begin{bmatrix} & \vdots & \\ \ldots & [\mathcal{A}]_{ki} & \ldots \\ & \vdots & \end{bmatrix}$$

where the block element $[\mathcal{A}]_{ki} \in \mathbb{R}^{n_k \times n_i}$ for all $k, i \in \mathbb{G}$.

When $\mathbb{G}$ is a group of integer numbers $(\mathbb{Z}, +)$, a translation-invariant operator on ℓ_p is defined to be a linear operator whose matrix representation is Toeplitz. In order to study the properties of translation-invariant operators using Fourier analysis, we define the unit translation operator to the left with respect to the group operation $+$ as follows:

$$\mathbf{T}u = \mathbf{T}(\ldots, |u_i, u_{i+1}, \ldots) = (\ldots, |u_{i+1}, u_{i+2}, \ldots).$$

Note that it is assumed that $u_i \in \mathbb{R}^n$ for all $i \in \mathbb{Z}$. One can verify that $\|\mathbf{T}\|_{p,p} = 1$ for all $1 \leq p \leq \infty$. Higher order translation operators can be defined iteratively by $\mathbf{T}^s = \mathbf{T}^{s-1}\mathbf{T}$ for all $s \geq 0$ and by $\mathbf{T}^s = \mathbf{T}^{s+1}\mathbf{T}^{-1}$ for $s < 0$. We now define a translation-invariant operator.

Definition 2: Suppose that $\mathscr{D}(\mathcal{A}) \subseteq \ell_p$ is translation invariant. Operator $\mathcal{A} : \mathscr{D}(\mathcal{A}) \rightarrow \ell_p$ is translation invariant if it commutes with every translation operator $\mathbf{T}^s : \mathscr{D}(\mathcal{A}) \rightarrow \mathscr{D}(\mathcal{A})$, that is, $\mathbf{T}^s\mathcal{A} = \mathcal{A}\mathbf{T}^s$ for all $s \in \mathbb{Z}$.

It can be shown that all linear translation-invariant operators on ℓ_p can be characterized by forming linear combinations of higher order translation operators of the form

$$\mathcal{A} = \sum_{k \in \mathbb{Z}} A_k \mathbf{T}^k \tag{2}$$

with $A_k \in \mathbb{R}^{n \times n}$. Note that the matrix representation of (2) is defined blockwise as $[\mathcal{A}]_{ki} = A_{i-k}$. For every $x \in \ell_2$, the discrete Fourier transform (DFT) is defined by

$$\hat{x}(z) = \sum_{k \in \mathbb{Z}} x_k z^{-k}$$

where $z \in \mathbb{S}^1$. Using this definition, one can compute the DFT of a translation-invariant operator. We will assume that the Fourier transform of all operators is continuous. A translation-invariant operator $\mathcal{A}$ is bounded on ℓ_2 [16] if and only if

$$\|\mathcal{A}\|_{2,2} = \sup_{z \in \mathbb{S}^1} \bar{\sigma}(\hat{\mathcal{A}}(z)) < \infty. \tag{3}$$

For translation-invariant operators defined on group $(\mathbb{Z}, +)$, the existence of a region of analyticity around the unit circle in $\mathbb{C}$ is equivalent to boundedness on all ℓ_p spaces (see proposition 1 in Section IV).

Definition 3: A spatially distributed LTI system is called spatially invariant if all operators in its state-space representation are translation invariant.

The following decay result for spatially invariant systems over discrete group $\mathbb{Z}$ is similar to that of [20, Theor 7.4.2] for continuous group $\mathbb{R}$ (see also [5, Theor. 5] for the continuous space version).

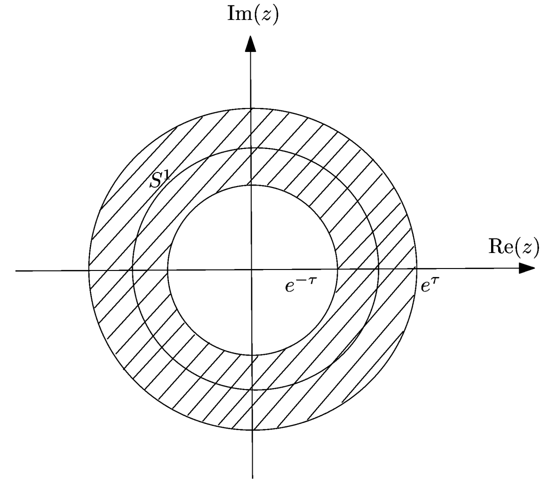

Fig. 2. Analytic continuation to annulus Ω when $\mathbb{G} = \mathbb{Z}$.

Theorem 1: Let $\mathcal{A}$ be defined by (2) and $\mathcal{A} \in \mathscr{B}(\ell_2)$ with DFT $\hat{\mathcal{A}}(z)$. If $\hat{\mathcal{A}}(z)$ has analytic continuation to some annulus

$$\Omega = \{z \in \mathbb{C} : e^{-\tau} < |z| < e^{\tau}, \tau > 0\} \tag{4}$$

(see Fig. 2) when the norm of matrix coefficients of operator $\mathcal{A}$ decay exponentially in the spatial domain, that is, for all $k \in \mathbb{Z}$

$$\|A_k\| \leq \mu e^{-\alpha|k|} \tag{5}$$

for some $\mu > 0$ and $0 < \alpha < \tau$.

Proof: According to our assumptions, $\hat{\mathcal{A}}(z)$ has an analytic continuation to some annulus Ω. Now consider the modified operator

$$\tilde{\mathcal{A}} = \sum_{k \in \mathbb{Z}} \tilde{A}_k \mathbf{T}^k$$

with $\tilde{A}_k := A_k e^{\alpha k}$. One can see that $\tilde{\mathcal{A}}$ is also a translation-invariant operator. From (3), it follows that:

$$\|\tilde{\mathcal{A}}\|_{2,2} = \sup_{z \in \mathbb{S}^1} \bar{\sigma}(\hat{\mathcal{A}}(e^{\alpha}z)) < \infty$$

for all $-\tau < \alpha < \tau$. Therefore, by using the inequality

$$\|\tilde{A}_k\| \leq \|\tilde{\mathcal{A}}\|_{2,2}$$

for all $k \in \mathbb{Z}$, it immediately follows that:

$$\|A_k\| \leq \mu e^{-\alpha k}$$

where $-\tau < \alpha < \tau$ and $\mu = \|\tilde{\mathcal{A}}\|_{2,2}$. By analyzing the two different cases, the decay result of (5) can be derived, with $0 < \alpha < \tau$ when $k \geq 0$ and $-\tau < \alpha < 0$ for $k < 0$. ■

In summary, given a bounded translation-invariant operator on ℓ_2, analytic continuity of its Fourier transform guarantees spatial locality of the operator by guaranteeing that the operator decays exponentially in space. We will use this result in Section III to study spatially invariant systems.

The set of all functions from $D \subseteq \mathbb{R}$ into $\mathbb{R}$ is a vector space $\mathscr{F}$ over $\mathbb{R}$. For $\chi', \chi'' \in \mathscr{F}$, the notation $\chi' \preceq \chi''$ will be used to mean the pointwise inequality $\chi'(x) \leq \chi''(x)$ for all $x \in D$. A family of seminorms on $\mathscr{F}$ is defined as $\{\|\cdot\|_T | T \in \mathbb{R}^+\}$ in which

$$\|\chi\|_T := \sup_{x \leq T} |\chi(x)|$$

for all $\chi \in \mathscr{F}$. The topology generated by all open $\|\cdot\|_T$-balls is called the topology generated by the family of seminorms and is denoted by $\|\cdot\|_T$-topology. Continuity of a function χ in this topology is equivalent to continuity in every seminorm in the family. More precisely, at any given point x and for all $T > 0$, for any given $\varepsilon > 0$, $\delta > 0$ exists such that

$$\|\chi(x) - \chi(y)\|_T < \varepsilon$$

for all $|x - y| < \delta$.

Next, we will define the notion of a coupling characteristic function which will then be used in Section IV-B.

Definition 4: A nondecreasing continuous function $\chi : \mathbb{R}^+ \to [1, \infty)$ is called a coupling characteristic function if $\chi(0) = 1$ and $\chi(x + y) \leq \chi(x)\chi(y)$ for all $x, y \in \mathbb{R}^+$.

Examples of coupling characteristic functions are $e^x, (1 + \lambda x)^n$, and $\ln(e + x)$. The constant coupling characteristic function with a unit value everywhere is denoted by $\mathbf{1}$. In order to characterize rates of decay, we define a one-parameter family of coupling characteristic functions as follows.

Definition 5: A one-parameter family of coupling characteristic functions $\mathscr{C}$ is defined to be an ordered set of all coupling characteristic functions χ_α for $\alpha \in \mathbb{R}^+$ such that

1) $\chi_0(x) = 1$ for all $x \in \mathbb{R}^+$;
2) $\chi_\alpha(x)\chi_\beta(x) = \chi_{\alpha+\beta}(x)$ for all $x \in \mathbb{R}^+$;
3) for $\alpha \leq \beta$, relation $\chi_\alpha \preceq \chi_\beta$ holds;
4) χ_α is a continuous function of α in the $\|\cdot\|_T$-topology.

Two simple examples of such one-parameter families are the family of exponential functions $e^{\alpha x}$ and polynomial functions $(1 + \lambda x)^\alpha$ for all $x, \alpha \in \mathbb{R}^+$.

Remark 1: The space of all coupling characteristic functions is closed under function multiplication. We can therefore construct more complicated coupling characteristic functions from simpler ones by combining them using multiplication operation. For example, the following function:

$$\chi(x) = e^{\alpha x}(1 + \lambda x)^\beta$$

can be obtained by point-wise multiplication of exponential and algebraical coupling characteristic functions. It can be shown that χ satisfies definition 4 and, therefore, it is a coupling characteristic function.

III. Stability and Optimal Control of Linear Spatially Distributed Systems

We begin this section by considering a continuous-time linear model for spatially distributed systems over a discrete spatial domain $\mathbb{G}$ described by

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t) \tag{6}$$

$$y(t) = (\mathcal{C}\psi)(t) + (\mathcal{D}u)(t) \tag{7}$$

with the initial condition $\psi(0) = \psi_0$. All signals are assumed to be in $L_2([0, \infty); \ell_2)$ space (i.e., at each time instant $t \in [0, \infty)$, signals $\psi(t), u(t), y(t)$ are assumed to be in ℓ_2). The state-space operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are assumed to be constant functions of time and linear from ℓ_2 to itself. The following assumption guarantees the existence and uniqueness of solutions of the system given by (6)–(7) (see [21, Ch. 3] for more details). Throughout this paper, we assume that the semigroup generated by $\mathcal{A}$ is strongly continuous on ℓ_2. The following example is a spatially distributed system on $\mathbb{G} = \mathbb{Z}$.

Example 1: Consider the general 1-D heat equation for a bi-infinite bar [22]

$$\frac{\partial}{\partial t}\psi(x, t) = \frac{\partial}{\partial x}\left(c(x)\frac{\partial}{\partial x}\psi(x, t)\right) + b(x)u(x, t)$$

where x is the spatial independent variable, t is the temporal independent variable, $\psi(x, t)$ is the temperature of the bar, and $u(x, t)$ is a distributed heat source. The thermal conductivity c is only a function of x and is differentiable with respect to x. The boundary conditions are assumed to be $\psi(\infty, t) = \psi(-\infty, t) = 0$. By replacing the partial derivatives with their finite-difference approximations, we can obtain the following continuous-time, discrete-space model:

$$\begin{aligned}\frac{\partial}{\partial t}\psi(x_k, t) = c'(x_k)&\left(\frac{\psi(x_{k-1}, t) - \psi(x_k, t)}{\delta}\right)\\ &+ c(x_k)\left(\frac{\psi(x_{k-1}, t) - 2\psi(x_k, t) + \psi(x_{k+1}, t)}{\delta^2}\right)\\ &+ b(x_k)u(x_k, t)\end{aligned}$$

where $c'(x) = \frac{d}{dx}c(x)$. The discretization is performed with equal spacing of the points x_k such that there is an integer number of points in space (i.e., $\delta = x_k - x_{k-1}$). Hence, after discretization, the spatial domain becomes $\mathbb{G} = \mathbb{Z}$. This model can be represented as

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t)$$

in which the infinite-tuples $\psi(t) = (\psi(x_k, t))_{k\in\mathbb{G}}$ and $u(t) = (u(x_k, t))_{k\in\mathbb{G}}$ are the state and control input variables of the infinite-dimensional system and the block elements of the state-space operators $\mathcal{A}$ and $\mathcal{B}$ are defined as follows for every:

$$[\mathcal{A}]_{ki} = \begin{cases} \frac{c'(x_k)\delta + c(x_k)}{\delta^2}, & \text{if } i = k - 1 \\ -\frac{c'(x_k)\delta + 2c(x_k)}{\delta^2}, & \text{if } i = k \\ \frac{c(x_k)}{\delta^2}, & \text{if } i = k + 1 \\ 0, & \text{if otherwise} \end{cases}$$

and

$$[\mathcal{B}]_{ki} = \begin{cases} b(x_k), & \text{if } i = k \\ 0, & \text{if otherwise} \end{cases}$$

for all $k, i \in \mathbb{G}$. One can show that $\mathcal{A}$ is an unbounded operator on ℓ_2. The generated semigroup generated by $\mathcal{A}$ is, however, strongly continuous on ℓ_2 (cf. [21]).

A. Operator Lyapunov and Riccati Equations

In what follows, we study the exponential stability of autonomous systems of the form (8) as well as the LQ optimal control problem for systems described by (6)–(7). While the main focus of this paper is on LQR problems, results are valid for general $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems as well.

Consider the following autonomous system over $\mathbb{G}$

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) \tag{8}$$

with initial condition $\psi(0) = \psi_0$. Suppose that $\mathcal{A}$ generates a strongly continuous C_0-semigroup on ℓ_2, denoted by $\mathcal{T}(t)$. The exponential stability can be defined as follows.

Definition 6: The system (8) is *exponentially stable* if

$$\|\mathcal{T}(t)\|_{2,2} \leq Me^{-\alpha t} \text{ for } t \geq 0$$

for some $M, \alpha > 0$.

Similar to the finite dimensional case, one can define a similar Lyapunov equation in an operator-theoretic framework for infinite-dimensional systems. The following theorem from [21] is standard and provides such an extension.

Theorem 2: Let $\mathcal{A}$ be the infinitesimal generator of the C_0-semigroup $\mathcal{T}(t)$ on ℓ_2 and $\mathcal{Q}$ a positive definite operator. Then, $\mathcal{T}(t)$ is exponentially stable if and only if the *Lyapunov* equation

$$\langle \mathcal{A}\phi, \mathcal{P}\phi\rangle + \langle \mathcal{P}\phi, \mathcal{A}\phi\rangle + \langle \phi, \mathcal{Q}\phi\rangle = 0 \tag{9}$$

for all $\phi \in \mathscr{D}(\mathcal{A})$, has a positive definite solution $\mathcal{P} \in \mathscr{B}(\ell_2)$.

We now review the basics of linear-quadratic regulator theory for infinite-dimensional systems. Such problems have been addressed in the literature for general classes of distributed parameter systems [21], [24]. A complete and elegant analysis for the spatially invariant case can be found in [5]. Similar to the finite-dimensional case, optimal solutions to infinite-dimensional LQR can be formulated in terms of an operator Riccati equation. Consider the quadratic cost functional given by

$$\mathfrak{J} = \int_0^\infty \langle \psi(t), \mathcal{Q}\psi(t)\rangle + \langle u(t), \mathcal{R}u(t)\rangle dt. \tag{10}$$

The system (6)–(7) with cost (10) is said to be *optimizable* if for every initial condition, $\psi(0) = \psi_0 \in \ell_2$, an input function $u \in L_2([0,\infty);\ell_2)$ exists such that the value of (10) is finite [21]. Note that if $(\mathcal{A}, \mathcal{B})$ is *exponentially stabilizable*, then the system (6)–(7) is optimizable. The following text is a standard result from [21].

Theorem 3: Let operators $\mathcal{Q} \succeq \mathbf{0}$ and $\mathcal{R} \succ \mathbf{0}$ be in $\mathscr{B}(\ell_2)$. If the system (6)–(7) with cost functional (10) is optimizable and $(\mathcal{A}, \mathcal{Q}^{1/2})$ is exponentially detectable, a unique nonnegative, self-adjoint operator $\mathcal{P} \in \mathscr{B}(\ell_2)$ exists, satisfying the ARE

$$\langle \varphi, \mathcal{P}\mathcal{A}\phi\rangle + \langle \mathcal{P}\mathcal{A}\varphi, \phi\rangle + \langle \varphi, \mathcal{Q}\phi\rangle - \langle \mathcal{B}^*\mathcal{P}\varphi, \mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}\phi\rangle = 0$$

for all $\varphi, \phi \in \mathscr{D}(\mathcal{A})$ such that $\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}$ generates an exponentially stable C_0-semigroup. Moreover, the optimal control $\tilde{u} \in L_2([0,\infty);\ell_2)$ is given by the feedback law

$$\tilde{u}(t) = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}\tilde{\psi}(t)$$

where $\tilde{\psi}$ is the solution of

$$\frac{d}{dt}\tilde{\psi}(t) = (\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P})\tilde{\psi}(t) \tag{11}$$

with initial condition ψ_0.

In general, solving the operator Lyapunov equation and ARE can be a tedious task. However, the complexity of the problem is reduced significantly if the underlying system is spatially invariant with respect to $\mathbb{G}$ [5, Sec. III–B]). In order to motivate our results on the structure of optimal control for general spatially distributed systems, we first consider the important class of spatially invariant systems on discrete groups $(\mathbb{Z}, +)$. Note that this problem has been studied extensively in [5] with an emphasis on continuous group $(\mathbb{R}, +)$. We will mention these results and modify them when necessary for the discrete group $\mathbb{G} = \mathbb{Z}$. As shown in [5], for a spatially invariant system, operator ARE reduces to the following parameterized equation:

$$\hat{\mathcal{A}}^*\hat{\mathcal{P}} + \hat{\mathcal{P}}\hat{\mathcal{A}} - \hat{\mathcal{P}}\hat{\mathcal{B}}\hat{\mathcal{R}}^{-1}\hat{\mathcal{B}}^*\hat{\mathcal{P}} + \hat{\mathcal{Q}} = \mathbf{0} \tag{12}$$

which is evaluated on $\mathbb{S}^1$, where the spatial frequency-domain variable z has been dropped from the aforementioned equation for notational simplicity. Assuming that all conditions of theorem 3 are satisfied, (12) has a unique bounded solution $\mathcal{P}$ on ℓ_2. Furthermore, if the Fourier transform of all operators $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{R}$ has an analytic continuation to some annulus around $\mathbb{S}^1$, a similar argument as in [5, Sec. V-B-1] can be used to show that the Fourier transform of $\mathcal{P}$ also has an analytic continuation to the same annulus. Therefore, the Fourier transform of the solution of a standard LQR problem

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}$$

has an analytic continuation to the same annulus. This, in combination with Theorem 1, guarantees that the coefficients of the translation-invariant operator $\mathcal{K}$, decay exponentially in the spatial domain, that is

$$\|K_k\| \leq \mu e^{-\lambda|k|} \tag{13}$$

for some $\mu, \lambda > 0$ and all $k \in \mathbb{Z}$. Note that the spatial decay of the solution in (13) is identical to that of [5] for continuous group $\mathbb{R}$ with the minor difference being that additional assumptions on growth bounds for $\hat{\mathcal{K}}(z)$ are not required; see [5, App. B] for more details). This is due to the fact that the annulus is a compact set in $\mathbb{C}$, and $\hat{\mathcal{K}}(z)$ is a continuous function (in the case of a continuous group, a strip around the imaginary axis is not bounded). Therefore, the extreme points are attained on the set.

The applicability of this result is limited to systems that are highly symmetric, such as identical dynamics on a lattice. The main question that we are trying to answer here is whether these concepts can be extended to a larger class of operators which are not necessarily spatially invariant.

This question is answered in a rigorous fashion in the next section. It turns out that the notion of spatial locality can be extended from translation invariant operators to a larger class of linear operators. This requires extending the notion of spatial decay in a natural way from linear translation invariant operators to a larger class of linear operators called spatially decaying or SD for short.

IV. Spatially Decaying Operators

The main difficulty in extending the results of previous section to arbitrary interconnection structures is lack of a transform theory for systems that are not spatially invariant. Recall that the notion of spatial invariance was critical in our use of Fourier methods which greatly simplified the analysis. Simply put, if we replace "space" with "time", we get a more familiar analogue of this problem: Fourier methods can not be used directly for analysis of linear time-varying systems.

A. Boundedness of Translation Invariant Operators on ℓ_p

The key in extending the results of the previous section to general spatially varying systems is to somehow extend the notion of analytic continuity without resorting to the transform domain. Consider the bounded translation invariant operator $\mathcal{A}$ of form (2) with discrete Fourier transform $\hat{\mathcal{A}}(z)$ which has analytic continuation to some annulus (4) around $\mathbb{S}^1$. Suppose that Γ is a circle with radius e^{α} around the origin where $0 < \alpha < \tau$. By analytic continuity, it follows that

$$\sup_{z\in\Gamma} \bar{\sigma}(\hat{\mathcal{A}}(z)) < \infty$$

Now consider the following inequality

$$\begin{aligned}\sup_{z\in\Gamma} \bar{\sigma}(\hat{\mathcal{A}}(z)) &= \sup_{z\in\Gamma} \|\sum_{k\in\mathbb{Z}} A_k z^k\| \\ &= \sup_{z\in\mathbb{S}^1} \|\sum_{k\in\mathbb{Z}} A_k e^{\alpha k} z^k\| \\ &\leq \sum_{k\in\mathbb{Z}} \|A_k\| e^{\alpha|k|}. \end{aligned} \tag{14}$$

By applying the results of theorem 1 to (14), it follows that there exists a number $\alpha < \beta < \tau$ such that

$$\|A_k\| \leq \mu e^{-\beta|k|}$$

and that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \|A_k\| e^{\alpha|k|} &\leq \mu \sum_{k\in\mathbb{Z}} e^{-\beta|k|} e^{\alpha|k|} \\ &\leq \mu \sum_{k\in\mathbb{Z}} e^{-(\beta-\alpha)|k|} < \infty\end{aligned}$$

where $\beta - \alpha > 0$. This implies that

$$\sum_{k\in\mathbb{Z}} \|A_k e^{\alpha|k|}\| < \infty \tag{15}$$

for all $0 < \alpha < \tau$ if and only if $\hat{\mathcal{A}}(z)$ has analytic continuity on annulus Ω. Consider the matrix representation of a translation invariant operator $\mathcal{A}$ and define a new linear operator $\tilde{\mathcal{A}}(\alpha)$ by

$$[\tilde{\mathcal{A}}(\alpha)]_{ki} := A_{k-i} e^{\alpha|k-i|}.$$

One can see that the modified operator $\tilde{\mathcal{A}}(\alpha)$ is also translation invariant. If condition (15) holds, from (3) and (14) we see that $\tilde{\mathcal{A}}(\alpha) \in \mathscr{B}(\ell_2)$. Therefore, we have the following result.

Proposition 1: For a translation invariant operator $\mathcal{A} \in \mathscr{B}(\ell_2)$, the discrete Fourier transform $\hat{\mathcal{A}}(z)$ has analytic continuation to some annulus

$$\Omega = \{z \in \mathbb{C} : e^{-\tau} < |z| < e^{\tau}, \tau > 0\}$$

if and only if $\tilde{\mathcal{A}}(\alpha) \in \mathscr{B}(\ell_p)$ for all $0 < \alpha < \tau$ and $1 \leq p \leq \infty$.

The above proposition suggests that analytic continuity is equivalent to boundededness of an auxiliary operator $\tilde{\mathcal{A}}(\alpha)$, which is the exponentially weighted version of the original operator.

B. Spatially Decaying Operators

In the following, we will generalize this idea to a larger class of linear operators by first forming an auxiliary *weighted* operator and imposing boundedness of the modified operator on ℓ_p.

Definition 7: Suppose that a distance function $\mathrm{dis}(\cdot,\cdot)$ and a one-parameter family of parameterized coupling characteristic functions $\mathscr{C}$ are given. A linear operator $\mathcal{A}$ is SD with respect to $\mathscr{C}$ if there exists $\tau > 0$ such that the auxiliary operator $\tilde{\mathcal{A}}(\alpha)$, defined block-wise as

$$[\tilde{\mathcal{A}}(\alpha)]_{ki} := [\mathcal{A}]_{ki} \chi_\alpha(\mathrm{dis}(k,i))$$

where $\chi_\alpha \in \mathscr{C}$, is bounded on all spaces $\ell_p (1 \leq p \leq \infty)$ for all $0 \leq \alpha < \tau$. The number τ is referred to as the *decay margin*.

In general, determining the boundedness of the auxiliary operator is considered to be difficult and depends greatly on the choice of p. Lemma 1 provides a simple necessary and sufficient condition for the boundedness problem on all spaces ℓ_p. Under the assumptions of definition 7, we make the following assumption.

Assumption 1: For all $\chi_\alpha \in \mathscr{C}$ with $\alpha > 0$, the following condition holds

$$\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \chi_\alpha(\mathrm{dis}(k,i))^{-1} < \infty.$$

Lemma 1: A linear operator $\mathcal{A}$ is SD with respect to $\mathscr{C}$ and decay margin $\tau > 0$ if and only if the following holds

$$\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)) < \infty \tag{16}$$

for all $0 \leq \alpha < \tau$.

Proof ($\Leftarrow$): We will show that the auxiliary operator $\tilde{\mathcal{A}}(\alpha)$ is simultaneously in $\mathscr{B}(\ell_1)$ and $\mathscr{B}(\ell_\infty)$. For a fixed $\alpha \in [0,\tau)$, it is straightforward to verify the following relations:

$$\|\tilde{\mathcal{A}}(\alpha)\|_{\infty,\infty} \leq \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i))$$

and

$$\|\tilde{\mathcal{A}}(\alpha)\|_{1,1} \leq \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)).$$

By assuming (16), it is concluded that $\tilde{\mathcal{A}}(\alpha) \in \mathscr{B}(\ell_1) \cap \mathscr{B}(\ell_\infty)$. Finally, using the Riesz–Thorin theorem, it follows that $\tilde{\mathcal{A}}(\alpha)$ is also bounded on all intermediate spaces ℓ_p for $1 \leq p \leq \infty$.

$(\Rightarrow)$: Since $\mathcal{A}$ is SD with respect to $\mathscr{C}$ and decay margin $\tau > 0$, from the definition, it follows that $\tilde{\mathcal{A}}(\alpha) \in \mathscr{B}(\ell_p)$ for all p and $0 \leq \alpha < \tau$. Thus, $\tilde{\mathcal{A}}(\alpha) \in \mathscr{B}(\ell_\infty)$, and that it can be shown that a number $c > 0$ exists such that

$$\|[\mathcal{A}]_{ki}\| \leq c\, \chi_\beta(\mathrm{dis}(k,i))^{-1} \tag{17}$$

for all $0 \leq \beta < \tau$. Pick any α where $0 \leq \alpha < \tau$, then a number $\alpha < \beta < \tau$ exists such that inequality (17) holds. By using assumption 1, for any $k \in \mathbb{G}$, it follows that:

$$\sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)) \leq c \sum_{i\in\mathbb{G}} \frac{\chi_\alpha(\mathrm{dis}(k,i))}{\chi_\beta(\mathrm{dis}(k,i))} = c \sum_{i\in\mathbb{G}} \chi_{\beta-\alpha}(\mathrm{dis}(k,i))^{-1} < \infty.$$

Therefore, condition (16) holds for all $0 \leq \alpha < \tau$. ∎

Examples of SD operators appear naturally in many applications. Intuitively, we may interpret the norm of each block element $[\mathcal{A}]_{ki}$ as the coupling strength between subsystems k and i. Given the one-parameter family of coupling characteristic functions $\mathscr{C}$, fix a value for $\alpha \in [0, \tau)$. For an infinite graph, if we fix a node k and move on the graph away from node k, the coupling strength decays proportional to the inverse of the coupling characteristic function χ_β for some $\alpha < \beta < \tau$ so that relation (16) holds. For example, if the coupling characteristic function is chosen to be exponential, the coupling strength will decay exponentially. We finish this section by introducing the notion of an SD system using the concept of SD operators.

Definition 8: The system (6)–(7) is called SD with respect to $\mathscr{C}$ if the state-space operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are SD with respect to $\mathscr{C}$.

As we will see in Section IV-C, all spatially invariant systems are indeed SD with respect to exponential coupling characteristic functions.

C. Examples of SD Operators

The following class of operators, which are used extensively in cooperative and distributed control, consists of interesting special classes of SD operators.

1) Exponentially Decaying Operators: The parameterized family of coupling characteristic functions of exponential type with one-parameter can be defined as follows:

$$\mathscr{C}_e := \{\chi_\alpha : \chi_\alpha(x) = e^{\alpha x} \text{ for all } \alpha \in \mathbb{R}^+\}.$$

Operator $\mathcal{A}$ is said to be *exponentially* SD if a number $\tau > 0$ exists such that condition (16) holds with respect to $\chi_\alpha \in \mathscr{C}_e$ for all $\alpha \in [0, \tau)$. Here, τ is the decay margin.

An important example of exponentially SD operators is the class of translation-invariant operators with analytic Fourier transforms. The result of theorem 1, along with the immediate application of lemma 1, shows that a translation-invariant operator with an analytic Fourier transform is exponentially SD. In this case, since the spatial domain is assumed to be $\mathbb{G} = \mathbb{Z}$, the suitable choice of a distance function is $\mathrm{dis}(k,i) = |k - i|$. According to proposition 1 and theorem 1, the decay margin of a translation-invariant operator $\mathcal{A}$ is equal to r, the distance of the nearest pole to its Fourier transform $\hat{\mathcal{A}}(z)$ to the unit circle $\mathbb{S}^1$.

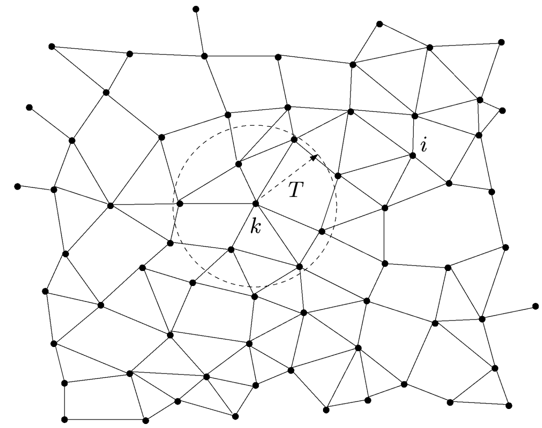

Fig. 3. Interconnection topology of a spatially distributed system on an arbitrary connected graph. The coupling between two agents is shown by an undirected edge between them.

2) Algebraically Decaying Operators: Operator $\mathcal{A}$ is said to be *algebraically* SD if a number $\tau > 0$ exists such that condition (16) holds with respect to $\chi_\alpha \in \mathscr{C}_a$ for all $\alpha \in [0, \tau)$, where

$$\mathscr{C}_a := \{\chi_\alpha : \chi_\alpha(x) = (1 + \lambda x)^\alpha \text{ for all } \alpha \in \mathbb{R}^+\}$$

for some given $\lambda > 0$. The number $\tau > 0$ is the decay margin. Such functions have been recently used as pair-wise potentials among agents in flocking and cooperative control problems [23] where the adjacency operator is defined by the following weight function:

$$w_{ki} = \frac{1}{\chi_\alpha(\mathrm{dis}(k,i))}$$

for some $\alpha \geq 0$ and $\lambda > 0$. Another example of such coupling functions arises in loss functions in wireless networks. The coupling between nodes, which is considered as the power of the communication signal between agents, decays with the inverse of the fourth power of distance.

3) Banded Operators: Given a natural notion of distance on $\mathbb{G}$, operator $\mathcal{A}$ is banded if a number $T > 0$ exists such that

$$[\mathcal{A}]_{ki} = \begin{cases} A_{ki}, & \text{if } \mathrm{dis}(k,i) \leq T \\ \mathbf{0}_{n\times n}, & \text{if } \mathrm{dis}(k,i) > T \end{cases} \tag{18}$$

where $A_{ki} \in \mathbb{R}^{n\times n}$. These operators have a finite-range coupling (see Fig. 3) and are trivially SD with respect to all coupling characteristic functions. Some common choices for the distance function are $\mathrm{dis}(k,i) = |k - i|$ if $\mathbb{G} \subseteq \mathbb{Z}$ and Euclidean distance when $\mathbb{G} \subset \mathbb{R}^n$. For such operators and every given node $k \in \mathbb{G}$, we have

$$\sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)) \leq \sum_{k\sim i} \|[\mathcal{A}]_{ki}\| \chi_\alpha(T) < \infty.$$

The relation $\sim$ is the neighborhood relation defined as $k \sim i$ if and only if $\mathrm{dis}(k,i) \leq T$. The aforementioned inequality shows that $\mathcal{A}$ is SD with respect to all $\mathscr{C}$ and the decay margin is $\tau = +\infty$.

Banded operators, such as adjacency and the graph *Laplacian* [25], are pervasive in graph theory. Given a connected proximity graph $\mathscr{G}$ with the set of nodes $\mathbb{G}$ and the set of edges $\mathbb{E}$, suppose that edges are weighted with a given weighting function $w : \mathbb{E} \to \mathbb{C}^{n\times n}$. Let $f : \mathbb{G} \to \mathbb{C}$ be a function mapping vertices to complex numbers. Then, the discrete Laplacian operator $\mathcal{L}_w$ is defined as

$$(\mathcal{L}_w f)(k) = \sum_{k\sim i} w_{ik}(f(i) - f(k)).$$

The matrix representation of the Laplacian operator will be

$$[\mathcal{L}_w]_{ki} = \begin{cases} -d_k w_{kk}, & \text{if } k = i \\ w_{ik}, & \text{if } k \sim i \text{ and } k \neq i \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

in which d_k is the degree of node k. Such operators also arise in machine-learning problems, such as image segmentation and dimensionality reduction.

D. Properties of SD Operators

Suppose that a parameterized family of coupling characteristic functions $\mathscr{C}$ is given. For an SD operator $\mathcal{A}$ with respect to $\mathscr{C}$ with decay margin $\tau > 0$, we define the operator norm

$$\|\mathcal{A}\|_{(\mathscr{C},\tau)} := \max\left\{ \sup_{\alpha\in[0,\tau)} \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i)), \sup_{\alpha\in[0,\tau)} \sup_{i\in\mathbb{G}} \sum_{k\in\mathbb{G}} \|[\mathcal{A}]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i)) \right\}$$

and the normed vector space

$$\mathcal{S}_\tau^\infty(\mathscr{C}) := \{\mathcal{A} : \|\mathcal{A}\|_{(\mathscr{C},\tau)} < \infty.\}$$

For notational simplicity, we will drop the subscript in the operator norm and adopt the notation $\|\cdot\|$ for the operator norm in the rest of this paper.

It can be shown that the operator norm satisfies the following properties for all $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ and $c \in \mathbb{C}$:

1) $\|\mathcal{A}\| \geq 0$ and $\|\mathcal{A}\| = 0$ iff $\mathcal{A} \equiv \mathbf{0}$;
2) $\|c\mathcal{A}\| = |c|\|\mathcal{A}\|$;
3) $\|\mathcal{A} + \mathcal{B}\| \leq \|\mathcal{A}\| + \|\mathcal{B}\|$;
furthermore, it satisfies the submultiplicative property
4) $\|\mathcal{A}\mathcal{B}\| \leq \|\mathcal{A}\|\|\mathcal{B}\|$.

Theorem 4: Given a one-parameter family of coupling characteristic functions $\mathscr{C}$ and $\tau > 0$, the operator space $\mathcal{S}_\tau^\infty(\mathscr{C})$ forms a B^*-algebra (with $*$ acting as a matrix transposition) with respect to $\|\cdot\|$ under the operator composition operation.

Proof: See Appendix IX-A for a proof. ■

For a comprehensive discussion on Banach algebras, we refer the reader to any functional analysis and operator theory textbook, for example, [19].

Corollary 1: Let $\mathscr{C}$ be a one-parameter family of coupling characteristic functions. Consider the one-parameter family of operator-valued functions $\mathcal{P}(t) : \mathbb{R}^+ \to \mathscr{B}(\ell_p)$ for some $1 \leq p \leq \infty$ with the following properties:

1) $\lim_{t\to\infty} \|\mathcal{P}(t) - \mathcal{P}\|_{p,p} = 0$.
2) $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$ and some $\tau > 0$.

Then, $\lim_{t\to\infty} \|\mathcal{P}(t) - \mathcal{P}\| = 0$. Furthermore, $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$.

Proof: See Appendix IX-B for a proof. ■

To summarize, we have shown that operator space $\mathcal{S}_\tau^\infty(\mathscr{C})$ is closed under addition, multiplication, and taking a limit properties.

Remark 2: It can be shown that given an operator $\mathcal{A} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ with decay margin $\tau > 0$, which has an algebraic inverse on $\mathscr{B}(\ell_2)$, the inverse operator $\mathcal{A}^{-1}$ is in $\mathcal{S}_{\hat{\tau}}^\infty(\mathscr{C})$ where $0 < \hat{\tau} < \tau$ (see [26] for details) (i.e., inverse of SD operators are also SD). Furthermore, the SD notion is quadratically invariant [2] (i.e., given two SD operators G and K, the product KGK is also SD).

Remark 3: Using the aforementioned results, it is straightforward to check that the serial and parallel composition of two SD systems are SD. Furthermore, a well-posed feedback interconnection of two SD systems is also SD.

In the next section, using the closure under taking a limit property of SD operators proven in Corollary 1, we show that the solution of differential Lyapunov and Riccati equations converge to an SD operator.

V. Structure of Quadratically Optimal Controllers

As discussed in Section III, our aim is not to solve the operator Lyapunov equation and ARE explicitly but to study the spatial structure of the solution of these algebraic equations by means of tools developed in the previous sections. We now state our main results which state that the solution of the operator Lyapunov equation and ARE have an inherent spatial locality and the characteristics of the coupling function will determine the degree of localization.

A. Operator Lyapunov Equations

We now prove that for stable SD systems described by (8), the solution of the operator Lyapunov equation $\mathcal{P}$ is also SD.

Theorem 5: Assume that a B^*-algebra $\mathcal{S}_\tau^\infty(\mathscr{C})$ is given and operators $\mathcal{A}, \mathcal{Q} \in \mathcal{S}_\tau^\infty(\mathscr{C})$, where $\mathcal{Q}$ is positive definite. If $\mathcal{A}$ is the infinitesimal generator of an exponentially stable C_0-semigroup $\mathcal{T}(t)$ on ℓ_2, then the unique positive definite solution of the Lyapunov equation

$$\langle \mathcal{A}\phi, \mathcal{P}\phi\rangle + \langle \mathcal{P}\phi, \mathcal{A}\phi\rangle + \langle \phi, \mathcal{Q}\phi\rangle = 0 \tag{19}$$

for all $\phi \in \ell_2$, satisfies $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$.

Proof: First, we will show that the C_0-semigroup $\mathcal{T}(t)$ with infinitesimal generator $\mathcal{A}$ is SD with respect to $\mathscr{C}$. The following is a standard result from [21]:

$$\frac{d}{dt}\mathcal{T}(t)\phi = \mathcal{A}\mathcal{T}(t)\phi \tag{20}$$

with $\mathcal{T}(0) = \mathcal{I}$ and for all $\phi \in \ell_2$ and $t > 0$. Therefore, for all $k, i \in \mathbb{G}$, we have

$$\frac{d}{dt}[\mathcal{T}(t)]_{ki} = \sum_{j\in\mathbb{G}} [\mathcal{A}]_{kj}[\mathcal{T}(t)]_{ji}. \tag{21}$$

For a differentiable matrix $X(t) \in \mathbb{C}^{n\times n}$ for $t \geq 0$, we have the following inequality:

$$\begin{aligned}\frac{d}{dt}\|X(t)\| &= \lim_{\delta\to 0}\frac{\|X(t+\delta)\| - \|X(t)\|}{\delta}\\ &\leq \lim_{\delta\to 0}\left\|\frac{X(t+\delta)-X(t)}{\delta}\right\|\\ &\leq \left\|\frac{d}{dt}X(t)\right\|. \end{aligned}\tag{22}$$

Assume that $\mathcal{T}(t)$ is a solution of (20), using inequality (22), we have

$$\begin{aligned}&\frac{d}{dt}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|[\mathcal{T}(t)]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i))\\ &\quad\leq \sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\left\|\frac{d}{dt}[\mathcal{T}(t)]_{ki}\right\|\chi_\alpha(\mathrm{dis}(k,i))\\ &\quad\leq \sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\left\|\sum_{j\in\mathbb{G}}[\mathcal{A}]_{kj}[\mathcal{T}(t)]_{ji}\right\|\chi_\alpha(\mathrm{dis}(k,i)).\end{aligned}$$

Using the fact that $\|\cdot\|$ is submultiplicative, from the aforementioned inequality, we can conclude that

$$\frac{d}{dt}|\!|\!|\mathcal{T}(t)|\!|\!| \leq |\!|\!|\mathcal{A}|\!|\!|\,|\!|\!|\mathcal{T}(t)|\!|\!|$$

and it follows that:

$$|\!|\!|\mathcal{T}(t)|\!|\!| \leq |\!|\!|\mathcal{T}(0)|\!|\!| e^{|\!|\!|\mathcal{A}|\!|\!| t} \tag{23}$$

for all $t \geq 0$. Note that $|\!|\!|\mathcal{T}(0)|\!|\!| = 1$. Since operator $\mathcal{A} \in \mathcal{S}_\tau^\infty(\mathscr{C})$, according to (23), the family of one-parameter operators satisfies $\mathcal{T}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$. Now consider the differential form of the Lyapunov function

$$\langle\phi, \frac{d}{dt}\mathcal{P}(t)\phi\rangle = \langle\mathcal{A}\phi,\mathcal{P}\phi\rangle + \langle\mathcal{P}\phi,\mathcal{A}\phi\rangle + \langle\phi,\mathcal{Q}\phi\rangle$$

with $\mathcal{P}(0) = \mathcal{Q}$ for all $\phi \in \ell_2$. This equation has a solution of the following form [21]:

$$\mathcal{P}(t)\phi = \int_0^t \mathcal{T}(\sigma)^*\mathcal{Q}\mathcal{T}(\sigma)\phi d\sigma \tag{24}$$

for all $\phi \in \ell_2$. Therefore, for every $k, i \in \mathbb{G}$, we have

$$[\mathcal{P}(t)]_{ki} = \int_0^t [\mathcal{T}(\sigma)^*\mathcal{Q}\mathcal{T}(\sigma)]_{ki} d\sigma. \tag{25}$$

According to inequality (23) and (25) and using the submultiplicative property of $|\!|\!|\cdot|\!|\!|$, we obtain

$$|\!|\!|\mathcal{P}(t)|\!|\!| \leq |\!|\!|\mathcal{Q}|\!|\!|\frac{(e^{2|\!|\!|\mathcal{A}|\!|\!| t}-1)}{2|\!|\!|\mathcal{A}|\!|\!|}. \tag{26}$$

Therefore, $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$. On the other hand, the solution of the differential Lyapunov (24) converges to the unique solution of (19), i.e.,

$$\lim_{t\to\infty}\|\mathcal{P}(t) - \mathcal{P}\|_{2,2} = 0.$$

According to Corollary 1, it follows that:

$$\lim_{t\to\infty}\mathcal{P}(t) = \mathcal{P}$$

uniformly in $\mathcal{S}_\tau^\infty(\mathscr{C})$. Therefore, $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$. ∎

B. Operator Algebraic Riccati Equation

Here, we show that the solution of the Riccati equation for SD systems as well as the kernel of the associated optimal feedback

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P} \tag{27}$$

is an SD operator. For simplicity, we will assume that $\mathcal{R} = \mathcal{I}$. Otherwise, by only assuming that $\mathcal{R}$ is SD and has an algebraic inverse on $\mathscr{B}(\ell_2)$, it can be shown that $\mathcal{R}^{-1}$ is SD [26]. According to the closure under multiplication property of SD operators, if $\mathcal{P}$ is SD, then $\mathcal{K}$ will also be an SD operator.

Theorem 6: Assume that a B*-algebra $\mathcal{S}_\tau^\infty(\mathscr{C})$ is specified. Let $\mathcal{A},\mathcal{B},\mathcal{Q} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ and $\mathcal{Q} \succeq 0$. Moreover, assume that conditions of theorem 3 hold. Then, the unique positive definite solution of the following ARE:

$$\langle\varphi,\mathcal{P}\mathcal{A}\phi\rangle + \langle\mathcal{P}\mathcal{A}\varphi,\phi\rangle + \langle\varphi,\mathcal{Q}\phi\rangle - \langle\mathcal{B}^*\mathcal{P}\varphi,\mathcal{B}^*\mathcal{P}\phi\rangle = 0$$

for all $\varphi,\phi \in \ell_2$, satisfies $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$.

Proof: Consider the following differential Riccati equation:

$$\begin{aligned}\frac{d}{dt}\langle\varphi,\mathcal{P}(t)\phi\rangle &= \langle\varphi,\mathcal{P}(t)\mathcal{A}\phi\rangle + \langle\mathcal{P}(t)\mathcal{A}\varphi,\phi\rangle\\ &\quad + \langle\varphi,\mathcal{Q}\phi\rangle - \langle\mathcal{B}^*\mathcal{P}(t)\varphi,\mathcal{B}^*\mathcal{P}(t)\phi\rangle\end{aligned}$$

with $\mathcal{P}(0) = 0$. We denote the unique solution of this differential Riccati equation in the class of strongly continuous, self-adjoint operators in $\mathscr{B}(\ell_2)$ by the one-parameter family of operator-valued function $\mathcal{P}(t)$ for $t \geq 0$. The nonnegative operator $\mathcal{P}$, the unique solution of ARE, is the strong limit of $\mathcal{P}(t)$ on ℓ_2 as $t \to \infty$ [21, Theor. 6.2.4]. Therefore, we have

$$\lim_{t\to\infty}\|\mathcal{P}(t) - \mathcal{P}\|_{2,2} = 0. \tag{28}$$

From the differential Riccati equation, it follows that:

$$\frac{d}{dt}[\mathcal{P}(t)]_{ki} = [\mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t) + \mathcal{Q}]_{ki}$$

for all $k, i \in \mathbb{G}$. Using inequality (22), we have

$$\begin{aligned}\frac{d}{dt}|\!|\!|\mathcal{P}(t)|\!|\!| &= \frac{d}{dt}\sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|[\mathcal{P}(t)]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i))\\ &\leq \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|\frac{d}{dt}[\mathcal{P}(t)]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i))\\ &\leq |\!|\!|\mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t) + \mathcal{Q}|\!|\!|.\end{aligned}$$

For simplicity in notations, denote $\pi(t) = \|\mathcal{P}(t)\|$. Using the triangle inequality and the submultiplicative property of norm $\|\cdot\|$, we have the following differential inequality:

$$\dot{\pi}(t) \leq 2\|\mathcal{A}\|\pi(t) + \|\mathcal{B}\|^2\pi(t)^2 + \|\mathcal{Q}\| \tag{29}$$

with initial condition $\pi(0) = 0$ and constraint $\pi(t) \geq 0$ for all $t \geq 0$. Note that $\|\mathcal{A}\| = \|\mathcal{A}^*\|$. From our assumptions, all coefficients $\|\mathcal{A}\|, \|\mathcal{B}\|, \|\mathcal{Q}\|$ in the right-hand side of the inequality (29) are finite numbers. It is straightforward to verify that if $\pi(t)$ for $t \geq 0$ is a solution of the differential inequality (29), then it is also a solution of the following differential inequality:

$$\dot{\pi}(t) \leq \lambda(\pi(t) + 1)^2 \tag{30}$$

with initial condition $\pi(0) = 0$, in which

$$\lambda = \max\{\|\mathcal{A}\|, \|\mathcal{B}\|^2, \|\mathcal{Q}\|\}.$$

In other words, the set of feasible solutions of (29) is a subset of solutions of (30). From (30), we have

$$-\frac{d}{dt}\left(\frac{1}{\pi(t)+1}\right) \leq \lambda$$

which has the following set of solutions:

$$\frac{1}{\pi(t)+1} \geq \frac{e^{-\lambda t}}{\pi(0)+1}.$$

Using the fact that $\pi(t) \geq 0$ for all $t \geq 0$ and $\pi(0) = 0$, it follows that:

$$\pi(t) \leq e^{\lambda t} - 1.$$

The above inequality is feasible (that is, at least one sequence of solutions satisfying $\pi(t) \geq 0$ and the above inequality for all $t \geq 0$ exist. The above inequality also proves that $\pi(t) < \infty$ for all $t \geq 0$. Therefore, we have $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$. According to Corollary 1, we can use this result and (28) to conclude that $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$. ■

Remark 4: When the operators are finite-dimensional matrices, condition (16) holds trivially. Therefore, in the finite-dimensional case, condition (16) does not provide any information about the spatial decay of the corresponding matrix. However, the results of this section can be extended to finite-dimensional operators by appropriately adjusting the notion of an SD operator to the finite-dimensional case as follows: For a given spatial domain $\mathbb{G}$ with cardinality N, we define the set of SD matrices with decay margin $\tau > 0$ to be the subspace $\mathcal{S}_\tau^N(\mathscr{C})$ of all matrices $\mathcal{A}$ of which a constant $C_\mathcal{A} > 0$ and $0 < \alpha < \tau$ exist such that each block submatrix of $\mathcal{A}$ satisfies

$$\|[\mathcal{A}]_{ki}\| \leq C_\mathcal{A}\chi_\alpha(\text{dis}(k,i))^{-1} \tag{31}$$

for all $k, i \in \mathbb{G}$. It can be shown that $\mathcal{S}_\tau^N(\mathscr{C})$ is closed under addition, multiplication, and matrix inversion operations. The proof techniques in the finite-dimensional case are different and out of the scope of this paper (see [27] for more details).

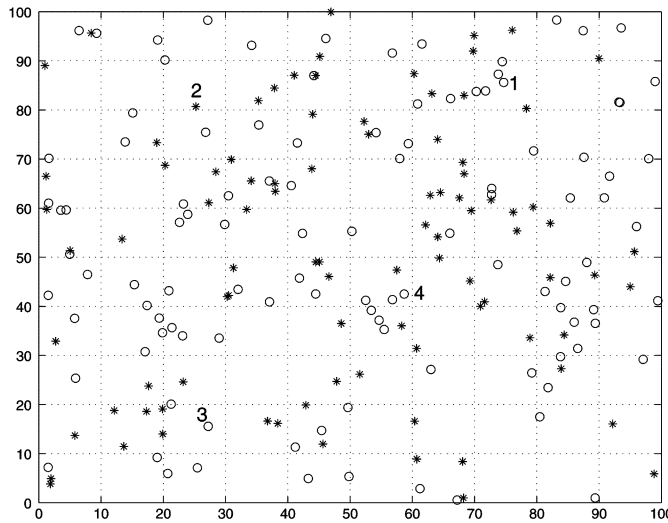

Fig. 4. N = 200 nodes are randomly and uniformly distributed in a region of area 100×100 (units)^2. Each node is a linear subsystem which is coupled to other subsystems through their dynamic and a central cost function by a given coupling characteristic function.

VI. SIMULATIONS

In the following simulations, $N = 200$ nodes are randomly distributed (with a uniform distribution) in a region of area 100×100 (units)^2 (see Fig. 4). Each node is assumed to be a linear system which is coupled through its dynamics and the LQ cost functional to other subsystems. The aggregate dynamics of the N linear subsystems can be described as

$$\frac{d}{dt}\psi_k(t) = [\mathcal{A}]_{kk}\psi_k(t) + \sum_{\substack{i=1 \\ i\neq k}}^{N}[\mathcal{A}]_{ki}\psi_i(t) + [\mathcal{B}]_{kk}u_k(t) \tag{32}$$

for all $k \in \mathbb{G} = \{1, \ldots, N\}$. In Fig. 4, the state-space matrices of agents marked by "$*$" are given by

$$[\mathcal{A}]_{kk} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, [\mathcal{B}]_{kk} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the state-space matrices of those agents marked by "o" are given by

$$[\mathcal{A}]_{kk} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, [\mathcal{B}]_{kk} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The coupling characteristic function χ_α is given and the coupling matrices in (32) are defined as follows:

$$[\mathcal{A}]_{ki} = \frac{1}{\chi_\alpha(\text{dis}(k,i))}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

and $[\mathcal{B}]_{ki} = \mathbf{0}$ for all $k, i \in \mathbb{G}$. The distance function is Euclidean. An undirected graph with N nodes can be associated with the system described by (32) where there is an edge between nodes k and i if $\|[\mathcal{A}]_{ki}\| \neq 0$. We will study the LQR problem discussed in Section III with $\mathcal{R} = \mathcal{I}$ and $\mathcal{Q}$ chosen as the graph Laplacian. The corresponding ARE is given by

$$\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} - \mathcal{P}\mathcal{B}\mathcal{B}^*\mathcal{P} + \mathcal{Q} = \mathbf{0}. \tag{34}$$

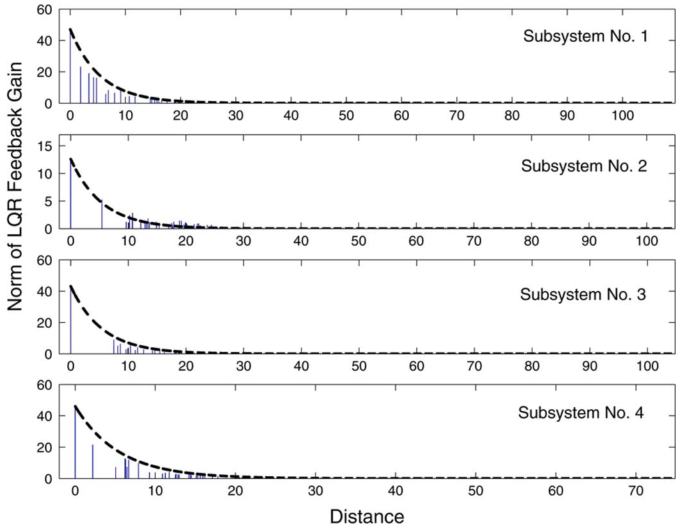

Fig. 5. Norm of the LQR feedback gain $\|[\mathcal{K}]_{ki}\|$ and $\|[\mathcal{K}]_{kk}\|/e^{\alpha \, \mathrm{dis}(k,i)}$ when $\alpha = 0.1823$ (dashed) for subsystems $k = 1, 2, 3, 4$, respectively, from top to bottom.

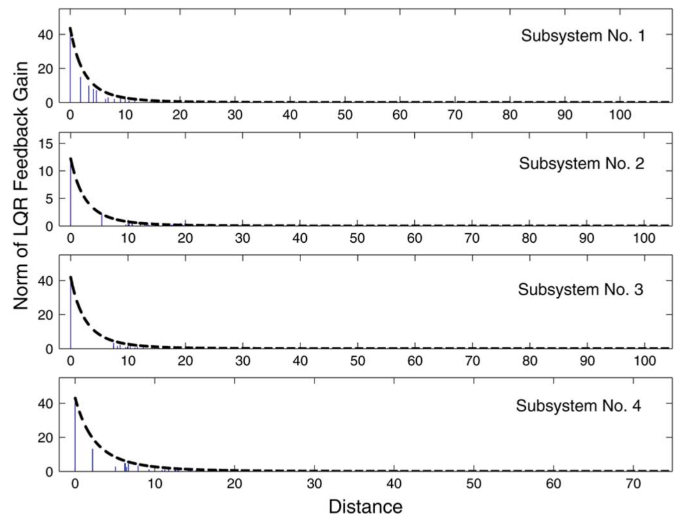

Fig. 6. Norm of the LQR feedback gain $\|[\mathcal{K}]_{ki}\|$ (bar) and $\|[\mathcal{K}]_{kk}\|/(1 + \lambda \mathrm{dis}(k,i))^{\alpha}$ when $\lambda = 0.1$ and $\alpha = 4$ (dashed) for subsystems $k = 1, 2, 3, 4$, respectively, from top to bottom.

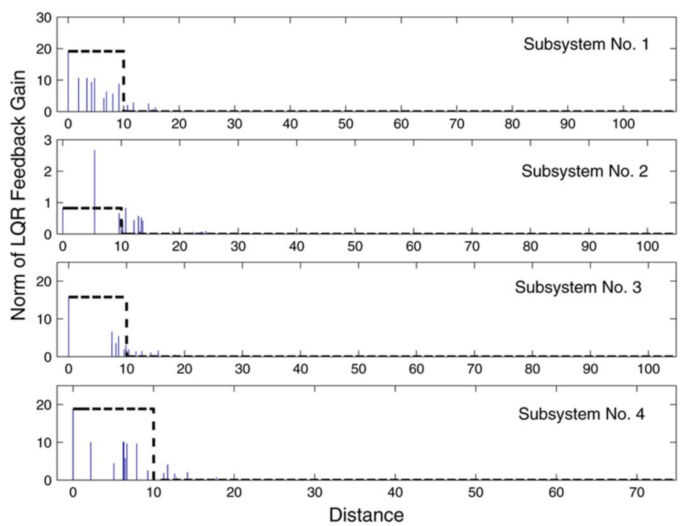

Fig. 7. Norm of the LQR feedback gain $\|[\mathcal{K}]_{ki}\|$ (bar) and $\|[\mathcal{K}]_{ki}\| \times$ pulse function with length $T = 10$ (dashed) for subsystems $k = 1, 2, 3, 4$ as a function of $\mathrm{dis}(k, i)$.

Then, the LQR optimal feedback is given by

$$\mathcal{K} = -\mathcal{B}^* \mathcal{P}. \tag{35}$$

In the sequel, three different scenarios are considered for the coupling characteristic function.

A. *Locality Features of LQR Control*

The first simulation is done based on the exponential coupling characteristic functions (see Section IV-C) with parameter $\alpha = 0.1823$. Fig. 5 shows the norm of the LQR feedback gains (35), corresponding to agents $k = 1, 2, 3, 4$ versus the distance of other subsystems to subsystem k. In the next simulation, the coupling characteristic functions of the algebraical type with parameters $\lambda = 0.1$ and $\alpha = 4$ are investigated. In Fig. 6, the norm of the LQR feedback gains (35) corresponding to agents $k = 1, 2, 3, 4$ is depicted versus the distance of other subsystems to subsystem k. In the last simulation, the nearest neighbor coupling case is studied where the coupling matrices (33) are now defined as follows:

$$[\mathcal{A}]_{ki} = \begin{cases} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, & \text{if } \mathrm{dis}(k,i) \le 10 \\ \mathbf{0}, & \text{otherwise} \end{cases}. \tag{36}$$

Fig. 7 represents the norm of LQR feedback gains (35) corresponding to agents $k = 1, 2, 3, 4$ versus the distance of other subsystems to subsystem k.

As seen from these simulations, for every subsystem k, the norm of the optimal feedback kernel $[\mathcal{K}]_{ki}$ is almost enveloped by the function $\|[\mathcal{K}]_{kk}\|/\chi_\alpha(\mathrm{dis}(k,i))$. Therefore, the spatial decay rate of the optimal controller can be determined *a priori*, only using the information of the coupling characteristic function $\chi_\alpha(\mathrm{dis}(k,i))$. As seen in Figs. 5–7, for each subsystem k, the corresponding optimal controller is effectively coupled only to those subsystems (with index i's) for which $\mathrm{dis}(k,i) \le 20$ (units). This suggests the possibility of formulating the optimal control problem in a distributed fashion, rather than solving a centralized high-dimension algebraic equation such as (34) (see [11]). Simulation results affirm that the optimal controller inherits the same architecture as the underlying system.

B. *Spatial Truncation*

Let $\mathcal{K}_T$ be the spatially truncated operator defined by

$$[\mathcal{K}_T]_{ki} = \begin{cases} [\mathcal{K}]_{ki} & \text{if } \mathrm{dis}(k,i) \le T \\ \mathbf{0} & \text{if } \mathrm{dis}(k,i) > T. \end{cases}$$

Using simulations, we obtain the maximum stabilizing truncation length T_s for the example problem. The following stabilizing truncation lengths are obtained by running different simulations:

- $T_s = 7.9785$ for exponential decay.
- $T_s = 2.9603$ for algebraical decay.
- $T_s = 15.0934$ for nearest neighbor coupling.

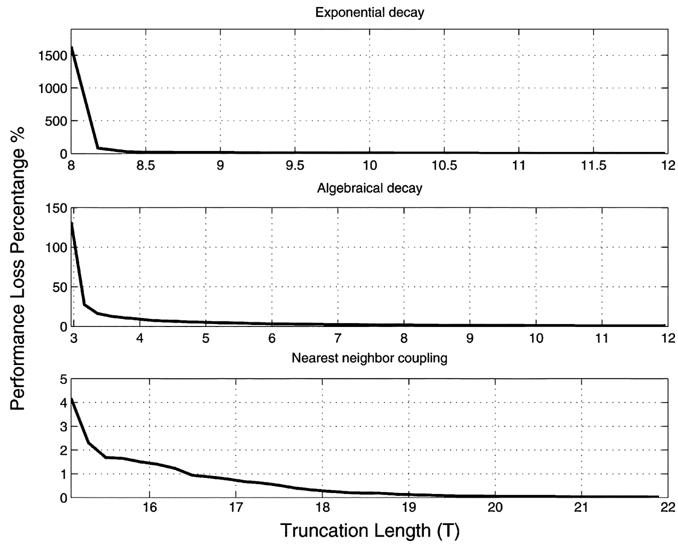

Fig. 8. Performance Loss percentage of LQR controller after spatial truncation for different types of couplings: (i) exponential decay (ii) algebraical decay (iii) nearest neighbor coupling.

One could also use small-gain arguments to find the truncation length T_s for which $\mathcal{K}_T$ is stabilizing for all $T \geq T_s$ (cf. Section V.B in [5]).

We use the cost-to-go, averaged over zero mean, unit variance random initial conditions to quantify relative performance deterioration of the closed-loop system under the spatially truncated feedback law $\mathcal{K}_T$. This is represented by:

$$\left|\frac{\mathrm{Trace}(\mathcal{P}_T) - \mathrm{Trace}(\mathcal{P})}{\mathrm{Trace}(\mathcal{P})}\right| \times 100$$

where $\mathcal{P}_T$ satisfies

$$(\mathcal{A}+\mathcal{B}\mathcal{K}_T)^*\mathcal{P}_T + \mathcal{P}_T(\mathcal{A}+\mathcal{B}\mathcal{K}_T) + \mathcal{Q} + \mathcal{K}_T^*\mathcal{R}\mathcal{K}_T = \mathbf{0}.$$

Fig. 8 illustrates the performance loss percentage versus different values of $T \geq T_s$ for different coupling characteristic functions. As seen from Fig. 8, the larger values of truncation length T ensue better closed-loop performance.

VII. Conclusion

In this paper we studied the spatial structure of infinite horizon optimal controllers for spatially distributed systems. By introducing the notion of SD operators, we extended the notion of analytic continuity to operators that are not spatially invariant. Furthermore, we proved that SD operators form a B^*-algebra. This was then used to prove that solutions of Lyapunov and Riccati equations for SD systems are themselves SD. As a result, the kernel of optimal LQ feedback (or the feedback gain operator) is also SD. Although these results were proven for LQ problems, they can be easily extended to general $\mathcal{H}_2$, and $\mathcal{H}_\infty$ optimal control problems, since the key enabling property is the spatial decay of solution of the corresponding Riccati equations. One major implication of these results is that the optimal control problem for spatially decaying systems lends itself to distributed solutions without too much loss in performance, as even the centralized solutions for such systems are inherently localized. These results have been extended to the case of constrained finite horizon optimal control problems by blending the ideas developed here with Multi Parametric Quadratic Programing [17], [18]. One important future research direction is to further study the case of SD operators with finite support (e.g., systems with nearest neighbor coupling). It would be interesting to find out under what extra conditions the optimal solutions are themselves finite support, as opposed to just being spatially decaying. This would provide an interesting connection between our results and those of [2].

Appendix

A. *Proof of Theorem 4*

Properties (1) and (2) are immediate from the definition. To prove (3), we use the following chain of inequalities:

$$\begin{aligned}\|\mathcal{A}+\mathcal{B}\| &= \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|[\mathcal{A}+\mathcal{B}]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i))\\ &\leq \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}(\|[\mathcal{A}]_{ki}\|+\|[\mathcal{B}]_{ki}\|)\chi_\alpha(\mathrm{dis}(k,i))\\ &\leq \|\mathcal{A}\|+\|\mathcal{B}\|\end{aligned}$$

To show property (4), we proceed as follows

$$\begin{aligned}\|\mathcal{A}\mathcal{B}\| &\leq \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|[\mathcal{A}\mathcal{B}]_{ki}\|\chi_\alpha(\mathrm{dis}(k,i))\\ &= \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|\sum_{j\in\mathbb{G}}[\mathcal{A}]_{kj}[\mathcal{B}]_{ji}\|\chi_\alpha(\mathrm{dis}(k,i))\end{aligned}$$

Using the fact that the induced norm of linear maps is submultiplicative, we obtain the following:

$$\|\mathcal{A}\mathcal{B}\| \leq \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\sum_{j\in\mathbb{G}}\|[\mathcal{A}]_{kj}\|\|[\mathcal{B}]_{ji}\|\chi_\alpha(\mathrm{dis}(k,i)).$$

For every $j\in\mathbb{G}$, we have $\mathrm{dis}(k,i)\leq\mathrm{dis}(k,j)+\mathrm{dis}(j,i)$. Applying this inequality and using definition 4, the following chain of inequalities holds:

$$\begin{aligned}\chi_\alpha(\mathrm{dis}(k,i)) &\leq \chi_\alpha(\mathrm{dis}(k,j)+\mathrm{dis}(j,i))\\ &\leq \chi_\alpha(\mathrm{dis}(k,j))\chi_\alpha(\mathrm{dis}(j,i))\end{aligned}$$

therefore

$$\begin{aligned}&\|\mathcal{A}\mathcal{B}\|\\ &\leq \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\sum_{j\in\mathbb{G}}\|[\mathcal{A}]_{kj}\|\|[\mathcal{B}]_{ji}\|\chi_\alpha(\mathrm{dis}(k,j)+\mathrm{dis}(j,i))\\ &\leq \sup_{\alpha\in[0,\tau)}\sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}}\sum_{j\in\mathbb{G}}\|[\mathcal{A}]_{kj}\|\|[\mathcal{B}]_{ji}\|\chi_\alpha(\mathrm{dis}(k,j))\chi_\alpha(\mathrm{dis}(j,i)).\end{aligned}$$

Finally, we can write

$$\|\mathcal{A}\mathcal{B}\| \leq \sup_{\alpha\in[0,\tau)}\left(\sup_{k\in\mathbb{G}}\sum_{j\in\mathbb{G}}\|[\mathcal{A}]_{kj}\|\chi_\alpha(\mathrm{dis}(k,j))\right.\\ \left.\times\sup_{j\in\mathbb{G}}\sum_{i\in\mathbb{G}}\|[\mathcal{B}]_{ji}\|\chi_\alpha(\mathrm{dis}(j,i))\right).$$

From this, we obtain the final result

$$|\!|\!|\mathcal{AB}|\!|\!| \leq |\!|\!|\mathcal{A}|\!|\!| |\!|\!|\mathcal{B}|\!|\!|.$$

According to the definition, it is easy to check that

$$|\!|\!|\mathcal{A}|\!|\!| = |\!|\!|\mathcal{A}^*|\!|\!|.$$

The last step of the proof is to show that $\mathcal{S}_\tau^\infty(\mathscr{C})$ is complete. Consider the Cauchy sequence $\{\mathcal{A}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C}) : t \geq 0\}$ and the corresponding sequence of continuous functions

$$\varphi_t(\alpha) = \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}(t)]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i))$$

defined on interval $[0, \tau)$. According to definition

$$\lim_{t,s\to\infty} |\!|\!|\mathcal{A}(t) - \mathcal{A}(s)|\!|\!| = 0. \tag{37}$$

Since $\mathcal{S}_\tau^\infty(\mathscr{C}) \subset \mathscr{B}(\ell_\infty)$, we may assume that $\lim_{t\to\infty} \mathcal{A}(t) = \mathcal{A}$ in which $\mathcal{A} \in \mathscr{B}(\ell_\infty)$. It follows that:

$$\lim_{t\to\infty} \|[\mathcal{A}(t)]_{ki}\| = \|[\mathcal{A}]_{ki}\|.$$

Hence, $\varphi_t(\alpha) \to \varphi(\alpha)$ pointwise as $t \to \infty$ where

$$\varphi(\alpha) = \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)).$$

By applying the triangle inequality, we have

$$\begin{aligned} &\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}(t)]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)) \\ &\quad - \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}(s)]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)) \\ &\quad\quad \leq \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}(t) - \mathcal{A}(s)]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i)) \end{aligned}$$

therefore

$$|\varphi_t(\alpha) - \varphi_s(\alpha)| \leq \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}(t) - \mathcal{A}(s)]_{ki}\| \chi_\alpha(\mathrm{dis}(k,i))$$

for all $\alpha \in [0, \tau)$. Hence

$$\|\varphi_t - \varphi_s\|_\infty = \sup_{\alpha\in[0,\tau)} |\varphi_t(\alpha) - \varphi_s(\alpha)| \leq |\!|\!|\mathcal{A}(t) - \mathcal{A}(s)|\!|\!|.$$

By applying (37), we have

$$\lim_{t,s\to\infty} \|\varphi_t - \varphi_s\|_\infty = 0. \tag{38}$$

According to the Cauchy criteria [28, Theor. 7.3.1], (38) is equivalent to the fact that φ_t converges uniformly to the limit function φ on $[0, \tau)$, i.e.,

$$\lim_{t\to\infty} \|\varphi_t - \varphi\|_\infty = 0.$$

Hence, we have$\|\varphi\|_\infty < \infty$. Furthermore, φ is continuous on $[0, \tau)$ (see [28, Theor. 7.3.2]). This proves that $\mathcal{A} \in \mathcal{S}_\tau^\infty(\mathscr{C})$.

B. *Proof of Corollary 1*

Proof: From property (i), it follows that:

$$\lim_{t,s\to\infty} \|\mathcal{P}(t) - \mathcal{P}(s)\|_{p,p} = 0. \tag{39}$$

In the following, we will prove that:

$$\lim_{t,s\to\infty} |\!|\!|\mathcal{P}(t) - \mathcal{P}(s)|\!|\!| = 0.$$

In Section IV-D, it is shown that $\mathcal{S}_\tau^\infty(\mathscr{C})$ is a normed vector space. The norm $|\!|\!|\cdot|\!|\!|$ is a continuous function on $\mathcal{S}_\tau^\infty(\mathscr{C})$. Consider the sequence of operators $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for $t \in \mathbb{R}^+$. From (39), we have

$$\lim_{t,s\to\infty} \|(\mathcal{P}(t) - \mathcal{P}(s))\phi\|_p = 0$$

for all $\phi \in \ell_p$. Therefore, we have

$$\lim_{t\to\infty} \|[\mathcal{P}(t) - \mathcal{P}(s)]_{ki}\| = 0$$

for all $k, i \in \mathbb{G}$. From the continuity property of the norm $|\!|\!|\cdot|\!|\!|$ on $\mathcal{S}_\tau^\infty(\mathscr{C})$, it follows that:

$$\begin{aligned} \lim_{t,s\to\infty} |\!|\!|\mathcal{P}(t) - \mathcal{P}(s)|\!|\!| &= |\!|\!| \lim_{t,s\to\infty} (\mathcal{P}(t) - \mathcal{P}(s)) |\!|\!| \\ &= \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \left(\lim_{t,s\to\infty} \|[\mathcal{P}(t) - \mathcal{P}(s)]_{ki}\| \right) \chi_\alpha(\mathrm{dis}(k,i)) = 0. \end{aligned}$$

This result shows that the sequence $\{\mathcal{P}(t) : t \in \mathbb{R}^+\}$ is a Cauchy sequence in $\mathcal{S}_\tau^\infty(\mathscr{C})$. Therefore, using the fact that $\mathcal{S}_\tau^\infty(\mathscr{C})$ is a Banach Algebra, we conclude that $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$. ■

Acknowledgment

The authors would like to thank B. Bamieh for his valuable comments and suggestions which led to the proof of theorem 6.

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