Non-Gaussianity and excursion set theory: Halo bias

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We study the impact of primordial non-Gaussianity generated during inflation on the bias of halos using excursion set theory. We recapture the familiar result that the bias scales as $k^{-2}$ on large scales for local type non-Gaussianity but explicitly identify the approximations that go into this conclusion and the corrections to it. We solve the more complicated problem of nonspherical halos, for which the collapse threshold is scale-dependent.

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I. INTRODUCTION

The standard cosmological model that fits a wide variety of observations is based on inflation, in particular on the production of a nearly scale-invariant spectrum of adiabatic perturbations at very early times. Inflation also predicts, and this too is verified by the data, that the perturbations should be nearly Gaussian. While inflation is successful in explaining the current suite of observations, it has not been successfully connected to the rest of physics. Equivalently, the mechanism that drove inflation has not been identified. For these purposes, the small deviations from scale invariance or Gaussianity may prove crucial [1]. The simplest single-field slow-roll models of inflation predict negligible non-Gaussianity, so a detection has the potential to rule out a large class of models. Multiple-field models, on the other hand, often predict levels of non-Gaussianity within the range of upcoming surveys.

The cosmic microwave background is the most obvious place to search for non-Gaussianity, as the perturbations are observed when they are still small, and therefore relatively unprocessed. Large-scale structure, on the other hand, is comprised of highly evolved perturbations that are nonlinear and hence different Fourier modes have mixed with one another. Even if the primordial perturbations were perfectly Gaussian, the observed large-scale structure would be highly non-Gaussian. The trick to using large-scale structure is to identify a feature in the spectrum that can be caused by primordial non-Gaussianity but not by standard gravitational instability. Recently, such a feature has been identified [2] in the form of scale-dependent bias.1 While it has been known for some time that primordial non-Gaussianity affects the abundances of rare objects [7–9], the signal is expected to be diminished due to the gravitational evolution generically evolving the distribution away from Gaussian.

The initial argument for scale-dependent bias was based on both simulations and the statistics of high peak regions [2,10–16]. Subsequently, several groups [17–19] have applied the peak-background split in the context of excursion set theory [20]. Here we apply a recent generalization of the excursion set approach [21,22] to the problem of the clustering of halos in order to extract the scale-dependent bias term and to understand under what conditions it holds. We extract the large-scale $k^{-2}$ behavior of the bias for the case of spherical collapse and also generalize to ellipsoidal collapse. Here too the bias contains a scale-dependent $k^{-2}$ piece, but the coefficient differs from that obtained assuming spherical collapse.

This paper is organized as follows. In Sec. II, we briefly review the path integral approach to the excursion set before we derive the conditional and unconditional crossing rates in Sec. III and IV respectively, and finally, the halo bias in Sec. V. In Sec. VI we extract the bias parameters. We conclude in Sec. VII. In the Appendix, we compare our results to a case where the probability density can be evaluated exactly—where the barrier depends only linearly on time.

II. PATH INTEGRAL APPROACH TO THE EXCURSION SET

In this section we briefly describe the path integral approach to the excursion set and establish our conventions. The excursion set theory was developed in a seminal paper by Bond, Cole, Efstathiou and Kaiser [20]. For a nice, accessible review, see Zentner [23]. The path integral approach to the excursion set was developed in a series of papers by Maggiore and Riotto [21,22,24] and extended by De Simone, Maggiore and Riotto to include moving barriers and conditional probabilities [25,26].

As is usual in excursion set theory, we consider the density fluctuation with respect to the average density $\bar{\rho}$,
\[
\delta(x) = \frac{\rho(x) - \bar{\rho}}{\bar{\rho}},
\]
smoothed over a region of radius \(R\)
\[
\delta_R(x) = \int d^3x' W_R(x - x') \delta(x')
= \int \frac{d^3k}{(2\pi)^3} \tilde{W}_R(k) \delta(k) e^{i k \cdot x}.
\]

In this expression, \(W_R(x - x')\) is a window function with characteristic radius \(R\), and \(\tilde{W}_R(k)\) is its Fourier transform. Translation invariance implies we can choose \(x\), and thus for convenience we take \(x = 0\), and suppress the argument on the smoothed density field from now on, \(\delta_R \equiv \delta_R(x = 0)\). Following the notation of Ref. [21], we denote by \(S_R = \sigma^2(R)\) the variance of the smoothed field \(\delta_R\),
\[
S_R = \int_0^\infty d\ln k \frac{k^3}{2\pi^2} |\tilde{W}_R(k)|^2 P(k),
\]
where \(P(k)\) is the matter power spectrum. As shown by Bond et al. [20], if the density fluctuations, \(\delta_R\), are purely Gaussian, and the window function is a top hat in momentum space, \(\tilde{W}_R(k) = \Theta(k_R - k)\) where \(k_R = 1/R\), then the evolution of \(\delta_R\) when considered a density of "pseudotime", \(S_R\), is Markovian. That is, each time step is uncorrelated with the previous one, and the future evolution of \(\delta_R\) is independent of its history. In this case one can show that the probability density that the field \(\delta_R\) takes a value \(\delta_R\) when smoothed on a scale \(R'\) satisfies the Fokker-Planck equation, with the variance, \(S_{R'}\), playing the role of pseudotime (see Ref. [23] for a review). In order that one does not suffer from the so-called "cloud-in-cloud" problem, an absorbing barrier condition is imposed as a boundary condition on the probability density at the critical threshold for collapse. This means that the probability distribution at time \(S_R\) includes only trajectories that never crossed the critical threshold at earlier\(^2\) times \(S_{R'} < S_R\).

Obtaining the probability distribution in the Gaussian-Markovian limit simply amounts to solving the Fokker-Planck equation with various boundary conditions. However, in the general non-Markovian limit one must solve a complicated integro-differential equation for the probability density [21]. Moreover, even in the Gaussian-Markovian limit, it is difficult to obtain solutions for the probability distribution in the case where the absorbing barrier conditions are more complicated than linear in the pseudo-time variable.

Until recently it was analytically intractable to move far beyond the constant collapse threshold and sharp \(k\)-space filtering assumptions of the original formulation of the excursion set [20]. Using the original techniques, more complicated collapse thresholds have been considered in order to describe collapse scenarios beyond spherical, and the formation of ionized regions during reionization [27,28]. However, in order to capture the effects of non-Gaussian fluctuations and more physically realistic filter functions, new techniques for solving for the probability distribution when the variable \(\delta_R\) does not evolve in a Markovian fashion with \(S_R\) were needed. In recent years there have been several breakthroughs along this avenue [21,22,24–26,29–31].

In this work we will follow the path integral approach of Maggiore and Riotto [21,24]. Their formalism is general enough to account for non-Gaussian correlations between the fluctuations [22], for critical thresholds that evolve with the smoothing scale and for window functions \(\tilde{W}_R(k)\) other than \(k\)-space top hats. Rather than attempting to solve a differential equation for the probability distribution, in the approach of Maggiore and Riotto one directly constructs this quantity by summing over paths that never exceeded the threshold, that is, by performing a path integral.

Following Maggiore and Riotto, we consider the smoothed density field, \(\delta(S) = \delta(S_R)\), as a stochastic variable with zero mean \(\langle \delta(S) \rangle = 0\). This variable evolves stochastically with time \(S\), and we refer to single realization of a \(\delta(S)\) as a trajectory. We then consider an ensemble of trajectories of this stochastic variable all starting from the same initial point \(\delta(S_0 = 0) = 0\), and we follow them for some time \(S\). We discretize the time interval \([0, S]\) into steps, \(\Delta S = \epsilon\), so that \(S_k = k\epsilon\), where \(k \in [0, 1, \ldots, n]\), and \(S_n = S\). A discretized trajectory then, is defined as a set of values \(\{\delta_1, \delta_2, \ldots, \delta_n\}\) so that \(\delta(S_k) = \delta_k\).

In this path integral approach, the fundamental object is the probability density in this space of trajectories, which is given by
\[
W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \langle \delta^D(\delta(S_1) - \delta_1) \ldots \delta^D(\delta(S_n) - \delta_n) \rangle,
\]
where \(\delta^D\) is the Dirac delta, which we explicitly denote to avoid confusion with the density perturbation. Note that \(W\) is defined for all possible values of \(\delta\) including those above the threshold. The below-threshold requirement is enforced at the next stage when computing \(\Pi(\delta_n, S_n)\), the probability of arriving at point \(\delta_n\) at time \(S_n\) through trajectories that have never exceeded some threshold, \(B(S_k) \equiv B_k\)
\[
\Pi(\delta_n, S_n) = \int_{-\infty}^{B_1} d\delta_1 \ldots \int_{-\infty}^{B_{n-1}} d\delta_{n-1} W(\delta_0; \delta_1, \ldots, \delta_n; S_n).
\]

The upper limits on each \(\delta\) integral allow for the possibility of scale-dependent barriers. Although the threshold value of \(\delta\) is constant for spherical collapse (equal to 1.686 today), in nonspherical collapse and other applications, the barrier is generally scale-dependent.

The fundamental quantities, \(W\) and \(\Pi\), are the ingredients needed to connect to observations. The fraction of the universe contained in collapsed objects on scales larger

\(^2\)Recall that the variance monotonically increases with decreasing \(R\), so "early" times correspond to large scales.
than $S_n$ is found from Eq. (5) by integrating over all below-
threshold values of $\delta_n$ and taking the complement

$$f_{\text{coll}}(S_n) = 1 - \int^{0}_{-\infty} d\delta_n \pi(\delta_0; \delta_n; S_n).$$  (6)

Differentiating this gives the formation rate of objects collapsing on the scale $S_n$

$$\mathcal{F}(S_n) = \frac{\partial f_{\text{coll}}(S_n)}{\partial S_n}.  \quad (7)$$

This crossing rate is the unconditional rate that allows for trajectories passing through any intermediate values of $\delta$. It is also useful to compute a conditional rate that fixes one of the intermediate values, as $\mathcal{F}(S_n|\delta_m, S_m)$, where the second arguments dictate the time and value of the intermediate fixed point. The conditional crossing rate is found from the analogous equation to Eq. (6), where the unconditional probability that a trajectory reaches a point $\delta_n$ at time $S_n$ having had a value $\delta_m$ at intermediate time $S_m$ [26,32],

$$\Pi(\delta_n, S_n|\delta_m, S_m) = \frac{\int^{\delta_n}_{-\infty} d\delta_1 \cdots d\delta_{n-1} \int^{\delta_m}_{-\infty} d\delta_1 \cdots d\delta_m W(\delta_0; \delta_1, \ldots, \delta_n; S_n)}{\int^{\delta_m}_{-\infty} d\delta_1 \cdots d\delta_m W(\delta_0; \delta_1, \ldots, \delta_m; S_m)}.$$  (8)

The $\hat{\delta}_m$ in this expression indicates that we do not integrate over the variable $\delta_m$.

The ratio of the conditional and unconditional rates quantifies the fact that halos (on scale $S_n$) are more likely to form in overdense regions (on scale $S_m < S_n$) than in an average place in the universe. The halo overdensity on large scales $S_m$ in initial Lagrangian space is [33]

$$1 + \delta_{\text{halo}} = \frac{\mathcal{F}(S_n|\delta_m, S_m)}{\mathcal{F}(S_n)}.  \quad (9)$$

In this work, we are interested in the halo bias $b(k)$, which relates the halo overdensity to the matter overdensity,

$$\delta_{\text{halo}}(k) = b(k) \delta(k).$$  (10)

Extracting $b(k)$ from the smoothed quantities used in the excursion set formalism will require a little work, but the basic idea is that quantities smoothed on scales $R$ carry information about wavenumbers $k$ of order $R^{-1}$. So we will expand the right-hand side of Eq. (9) in $\delta_m$ and identify the coefficient of the first-order term. The halos are collapsed on scales $S_p$, but we are interested in their clustering on the much larger scales associated with $S_n$. This coefficient can then be massaged to extract $b(k)$.

As is, the expression for the probability density, Eq. (4), is not very useful. One of the insights of Maggiore and Riotto was to manipulate the terms so that the right side of that equation contains sums of $p$-point functions. The trick is to use the integral representation of the Dirac delta function,

$$\delta^0(\chi) = \int_{-\infty}^{\infty} d\lambda \frac{1}{2\pi} e^{-i\chi},$$  (11)

so that

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int_{-\infty}^{\infty} D\lambda e^{\lambda \delta_{\text{halo}}(S_n)},$$  (12)

where we have defined

$$\int_{-\infty}^{\infty} D\lambda = \int_{-\infty}^{\infty} d\lambda_1, \ldots, d\lambda_n.$$  (13)

The expectation value, $\langle e^{\lambda \delta_{\text{halo}}(S_n)} \rangle$ can be rewritten as

$$\exp\left[\sum_{p=2}^{\infty} (-1)^p p! \sum_{j_1, \ldots, j_p} \lambda_{j_1} \cdots \lambda_{j_p} \delta(S_{j_1}) \cdots \delta(S_{j_p}) \right],$$  (14)

where $\langle \delta(S_{j_1}) \cdots \delta(S_{j_p}) \rangle_c$ is the connected $p$-point function. In the Gaussian case, the $p$-point functions with $p > 2$ vanish, and the probability density reduces to the limit

$$W_{0n} = \int_{-\infty}^{\infty} D\lambda e^{\lambda \delta_{\text{halo}}(S_n)},$$  (15)

where the subscript denotes the initial and final times of the trajectories; the superscript identifies this as the Gaussian limit; and we suppress the dependence on the $\delta_1, \ldots, \delta_{n-1}$. We will also have occasion to invoke trajectories with starting time $S_m$ instead of $S_0$; these will be drawn from the distribution $W_{mn}$ and will be integrated to obtain the probability

$$\Pi_{mn}^{\text{G}} = \Pi^{\text{G}}(\delta_m, S_m; \delta_n, S_n).$$  (16)

Note that the Gaussian expression in Eq. (15) satisfies the identity

$$\delta_{k_1} \cdots \delta_{k_n} W^{\text{G}}_{0n} = \langle (i)^n \int_{-\infty}^{\infty} D\lambda \lambda_{k_1} \cdots \lambda_{k_n} e^{\lambda \sum \delta \lambda_{k_1} \cdots \delta \lambda_{k_n}} \rangle,$$  (17)

where $\partial_i \equiv \partial/\partial \delta_i$. Using this identity in Eqs. (12) and (14) leads to a general expression for the probability density in terms of the $p$-point functions and derivatives of the Gaussian density.

$$W_{0n} = W(\delta_0; \delta_1, \ldots, \delta_n; S_n)$$

$$\quad = \exp\left[\sum_{p=3}^{\infty} (-1)^p p! \sum_{j_1, \ldots, j_p} \langle \delta_{j_1} \cdots \delta_{j_p}, \delta_{j_1} \cdots \delta_{j_p} \rangle_c \right] W^{\text{G}}_{0n}.$$  (18)
The effects of non-Gaussianity primarily arise from the nonzero three-point function, so we will drop all higher-order \( p \)-point functions (and all terms quadratic and higher in the three-point function). In this case, the probability density reduces to

\[
W_{0n} = \left[ 1 - \frac{1}{6} \sum_{i,j,k=1}^n \langle \delta_i \delta_j \delta_k \rangle_c \delta_i \partial_j \partial_k \right] W_0^\xi. \tag{19}
\]

We are never interested in the raw probability density described by Eq. (19). Rather, we are interested in the probability of reaching a point \( \delta_n \) at time \( S_n \) or in the collapse fraction. Thus, rather than evaluate Eq. (19) directly, we will integrate over intermediary values of \( \delta_i \) to form \( \Pi \) and the (observable) functions that can be constructed from it.

One well-known limit is the case when the barrier is constant (all \( B \)'s in Eq. (5) are the same) and the window function in Eq. (2) is a top hat in \( k \)-space. The top hat filter in \( k \)-space means that in the three-point function). In this case, the probability density becomes identical, equal to \( B_n \). The integrand [Eq. (19)] is an operator acting on \( W_{0n}^\xi \) which now becomes

\[
W_0^\xi = \int_{-\infty}^{\infty} \mathcal{D} \lambda \exp \left\{ i \sum_{i=1}^n \lambda_i \left[ \delta_i + \frac{\infty}{p=1} \frac{B_n^{(p)}}{p!} (S_i - S_n)^p \right] - \frac{1}{2} \sum_{i,j=1}^n \lambda_i \lambda_j \langle \delta_i \delta_j \rangle_c \right\}. \tag{25}
\]

Integrating to form \( \Pi \) leads to a Gaussian and non-Gaussian term:

\[
\Pi^\xi(\delta_n, S_n) = \Pi^\xi(\delta_n, S_n) - \frac{1}{6} \sum_{i,j,k=1}^n \langle \delta_i \delta_j \delta_k \rangle_c \times \left[ \prod_{i=1}^{n-1} \int_{-\infty}^{B_i} d\delta_i \right] \delta_i \partial_j \partial_k W_0^\xi. \tag{26}
\]

### A. Gaussian limit

Consider the Gaussian part of this, the first term on the right in Eq. (26). In the limit of a constant barrier, this is straightforward to obtain and one finds the result Eq. (20).

However, for a moving barrier, things are complicated by the presence of the additional factor in the exponent. To evaluate the probability distribution in this case, we expand the exponential

\[
\exp \left\{ i \sum_{i=1}^n \lambda_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (S_i - S_n)^p \right\} = 1 + \sum_{i=1}^n \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (S_i - S_n)^p \partial_i + \frac{1}{2} \sum_{i,j=1}^n \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{p! q!} (S_i - S_n)^p (S_j - S_n)^q \partial_i \partial_j + \cdots, \tag{27}
\]

where the derivatives are understood to be acting on the constant barrier limit of \( W \). In the Sheth-Tormen
by differentiating the probability distribution, Planck equation, the formation rate can be found directly
Markovian probability distribution satisfies the Fokker-Planck equation, the unconditional probability in the presence of a moving barrier in the Gaussian limit. This is given by [25,26]

$$\Pi^\delta(\delta_n, S_n) \approx \Pi_{\text{gb}}(\delta_n, S_n) + 2 \frac{B_n - \delta_n}{\sqrt{2\pi S_n^5}} e^{-\frac{(B_n - \delta_n)^2}{2S_n^5}} \mathcal{P}_{0n} - 2 \frac{(B_n - \delta_n)^2}{\sqrt{2\pi S_n^5}} e^{-\frac{(B_n - \delta_n)^2}{2S_n^5}} \mathcal{P}_{0n}^2 + \cdots, \quad (28)$$

To obtain the formation rate, or the mass function, there are two ways of proceeding. One could directly differentiate the expression in Eq. (30) with respect to $S_n$ to obtain the formation rate. Alternatively, since the Gaussian-Markovian probability distribution satisfies the Fokker-Planck equation, the formation rate can be found directly by differentiating the probability distribution,

$$\mathcal{F}^\delta(S_n) = -\frac{1}{2} \frac{\partial \Pi}{\partial \delta} \bigg|_{\delta = B_n}. \quad (31)$$

Of course, these two prescriptions should lead to the same result. However, due to the rather crude approximation that leads to Eq. (28), this expression no longer satisfies the Fokker-Planck equation. This is an obvious drawback of the crude approximation we used in order to evaluate the probability distribution. In the Appendix, we demonstrate that enforcing the Fokker-Planck equation when calculating the formation rates [i.e., using Eq. (31)], rather than directly differentiating the collapse fraction, leads to results that are consistent with those obtained by an exact treatment of the probability distribution in the case of a linear barrier where no such approximation needs to be made. Thus, for the rest of this work, where possible we apply the Fokker-Planck equation.

Using Eq. (31) and (28) we obtain the unconditional formation rate for a general barrier

$$\mathcal{F}^\delta(S_n) = \frac{B_n + \mathcal{P}_{0n}}{\sqrt{2\pi S_n^5}} e^{-\frac{B_n^2}{2S_n^5}}. \quad (32)$$

Eq. (32) is a compact expression that agrees with that obtained in previous work, but as described above, it hides a subtlety in the Sheth-Tormen approximation. Had we simply differentiated Eq. (30), we would have arrived at a considerably more complicated looking result, however, numerically the difference is small.

where

$$\mathcal{P}_{mn} = \sum_{p=1}^{n} \frac{(S_m - S_n)^p}{p!} B_n^p, \quad (29)$$

and the ... terms higher order in an expansion in $\mathcal{P}_{0n}$, and we are assuming that the barrier is varying only slowly in time, $S_n$. It is straightforward to integrate this over $\delta_n$ to obtain the collapse fraction,

$$f^\text{NG}_{\text{coll}}(S_n) = \text{erfc}\left[\frac{B_n - \delta_n}{\sqrt{2S_n}}\right] - \frac{2S_n}{\pi} \mathcal{P}_{0n} \left[e^{-B_n^2/2S_n} - \frac{\pi B_n^2}{2S_n} \text{erfc}\left[\frac{B_n}{\sqrt{2S_n}}\right]\right] + \frac{\mathcal{P}_{0n}^2}{S_n} (B_n^2 + S_n) \left[\text{erfc}\left[\frac{B_n}{\sqrt{2S_n}}\right] - 2\right] \quad (30)$$

B. Non-Gaussian contribution

To compute the non-Gaussian piece in Eq. (26), we need to carry out the sums (integrals in the limit that $\epsilon \to 0$). Maggiore and Riotto showed that the simple approximation of evaluating the three-point function at the endpoint $i = j = k = n$ leads to the first term in an expansion in the parameter $S_n/B_n^2$. A useful way of evaluating the remaining integrals is to use the identity [22,25]

$$\frac{\partial^3 }{\partial B_n^3} f^\text{gb}_{\text{coll}}(S_n) = -\sum_{j_i,j_2,j_3=1}^{n} \int_{-\infty}^{B_n} \int_{-\infty}^{B_n} \int_{-\infty}^{B_n} \prod_{i=1}^{3} \partial_{j_i} W_{0n}^{3}. \quad (33)$$

This is almost identical to the sum and integrals in the last term in Eq. (26), with the exception that the right-hand side in Eq. (33) also includes an integral over $\delta_n$. To use the identity then, we can integrate Eq. (26) over $\delta_n$ (which we need to do anyway in order to compute the collapse fraction). This leaves

$$f^\text{NG}_{\text{coll}} = \frac{1}{6} (\delta_n^3) \frac{\partial^3}{\partial B_n^3} f^\text{gb}_{\text{coll}}(S_n). \quad (34)$$

To obtain the crossing rate, we differentiate this with respect to $S_n$. The derivative acting on $f^\text{gb}_{\text{coll}}$, we pull the derivative all the way through: $f d\delta \delta \Pi(\delta; S_n) / \delta S_n$; use the Fokker-Planck equation and then integrate by parts so that the non-Gaussian part of the collapse fraction is

$$f^\text{NG}(S_n) = -\frac{1}{2} \left(1 - \frac{1}{6} S_n^3 + \frac{1}{6} S_n^3 S_n^3 \right) \frac{\partial^3}{\delta S_n} \left|_{\delta = B_n} \right. \left. \mathcal{F}^\delta(S_n) \right. - \frac{1}{6} \left(2S_n S_n^3 + S_n^3 S_n^3 \right) \frac{\partial^3}{\partial B_n^3} f^\text{gb}_{\text{coll}}(S_n). \quad (35)$$
We have defined
\[ S_3 = \frac{\langle \delta_n^3 \rangle}{S_n^3} \quad (36) \]
and denoted derivatives with respect to \( S_n \) by primes. Using Eq. (30) and (28) we find that the full unconditional Gaussian-Markovian probability [which we evaluated in Eq. (28)] as long as we make the substitutions
\[ \delta_n \rightarrow \delta_n - \delta_m \quad B_n \rightarrow B_n - \delta_m \quad S_n \rightarrow S_n - S_m. \quad (40) \]

As depicted in Fig. 1, this expression is identical to the unconditional Gaussian-Markovian probability [which we evaluated in Eq. (28)] as long as we make the substitutions
\[ \delta_n \rightarrow \delta_n - \delta_m \quad B_n \rightarrow B_n - \delta_m \quad S_n \rightarrow S_n - S_m. \]

For example, the first term in Eq. (28)—the constant barrier piece—becomes
\[ \Pi^{gb}(\delta_n, S_n | \delta_m, S_m) = \frac{1}{\sqrt{2\pi S_m}} e^{-\frac{\delta_n^2}{2S_m}} - e^{-\frac{(2B_n - \delta_m)^2}{2(S_n - S_m)}}. \]

**B. Non-Gaussian contribution**
To evaluate the non-Gaussian corrections to the conditional collapse rate, we again rely on the approximation of considering the three-point functions only at their endpoints. In the case of unconditional collapse, this translated to setting \( \langle \delta_i \delta_j \delta_k \rangle \rightarrow \langle \delta_i \delta_j \delta_k \rangle \), thereby simplifying the sum over \( i, j, k \). Here, too, we set all three-point functions to their values at the endpoint but when \( i < m \), the endpoint is time \( S_m \), not \( S_n \). For example,
\[ \sum_{i,j,m+1}^{n} \sum_{k=1}^{m-1} \langle \delta_i \delta_j \delta_k \rangle \rightarrow \langle \delta_m \delta_n \rangle \sum_{i,k=m+1}^{n} \sum_{k=1}^{m-1}. \quad (42) \]

Working to linear order in the long wavelength mode, \( \delta_m \), we can write (see also Ref. [35]).
\[ \sum_{i,j,k=1}^{n} \langle \delta \delta \partial \rangle_{c} \partial_{i} \partial_{j} \partial_{k} W_{0,n}^{\oplus} \]

\[ = 3 \langle \delta_{m} \delta_{n}^{\oplus} \rangle_{c} \sum_{k=1}^{m-1} \sum_{i,j=m+1}^{n} \partial_{i} W_{0,m}^{\oplus} \partial_{j} W_{m,n}^{\oplus} \]

\[ + 3 \langle \delta_{m}^{2} \delta_{n} \rangle_{c} \sum_{i,j=m+1}^{n} \partial_{i} (W_{0,m}^{\oplus} \partial_{j} W_{m,n}^{\oplus}) \]

\[ + \langle \delta_{n}^{2} \rangle_{c} W_{0,m}^{\oplus} \sum_{i,j=k}^{n} \partial_{i} \partial_{j} \partial_{k} W_{m,n}^{\oplus}. \] (43)

The key result here is the appearance of the combination

\[ \langle \delta_{m} \delta_{n}^{2} \rangle_{c} \left( \frac{\partial}{\partial B_{m}^{n}} + \frac{\partial}{\partial \delta_{m}^{n}} \right) \ln \Pi_{0,m}^{\oplus} = \langle \delta_{m} \delta_{n}^{2} \rangle_{c} \frac{\delta_{m}}{S_{m}}. \] (47)

which multiplies the last line in Eq. (46). The appearance of the terms proportional to \( \langle \delta_{m} \delta_{n}^{2} \rangle_{c} \) in Eq. (46) encode the fact that the conditional probability of collapse now depends on the details of the correlation between long and short wavelength fluctuations. We will see that these correlations give rise to a scale-dependent bias. In order to extract the bias we will eventually correlate the expression in Eq. (46) (to linear order in \( \delta_{m} \)) with the underlying density fluctuations smoothed on the scale \( R_{m} \), before taking the limit that \( S_{m} \rightarrow 0 \). While the quantity in Eq. (47) remains finite after this procedure, the terms in the first line of Eq. (46) proportional to \( \langle \delta_{m} \delta_{n}^{2} \rangle_{c} \) vanish, and we thus drop them in what follows.

We can then evaluate the conditional crossing rate, which is the conditional rate at which objects form at time \( S_{m} \), given that they had an overdensity \( \delta_{m} \) at time \( S_{m} \). As we have stressed above, we do not proceed by simply differentiating Eq. (46) with respect to \( S_{m} \). Rather, we write the non-Gaussian contribution to the conditional crossing rate as

\[ f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) = \frac{\partial f_{\text{coll}}(S_{m} \mid \delta_{m}, S_{m})}{\partial B_{m}^{n}} \]

\[ \int_{-\infty}^{B_{m}^{n}} d \delta_{1} \ldots \int_{-\infty}^{B_{m}^{n}} d \delta_{m-1} \prod_{j=1}^{N} \partial_{j} W_{0,m}^{\oplus}. \] (44)

we find that the non-Gaussian part of the conditional collapse fraction can be written

\[ f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) = \frac{\partial f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m})}{\partial B_{m}^{n}} \]

\[ \int_{-\infty}^{B_{m}^{n}} d \delta_{1} \ldots \int_{-\infty}^{B_{m}^{n}} d \delta_{m-1} \prod_{j=1}^{N} \partial_{j} W_{0,m}^{\oplus}. \] (45)

The key result here is the appearance of the combination

\[ \langle \delta_{m} \delta_{n}^{2} \rangle_{c} \left( \frac{\partial}{\partial B_{m}^{n}} + \frac{\partial}{\partial \delta_{m}^{n}} \right) \ln \Pi_{0,m}^{\oplus} = \langle \delta_{m} \delta_{n}^{2} \rangle_{c} \frac{\delta_{m}}{S_{m}}. \] (47)

where \( f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) \) is the conditional collapse fraction in the Gaussian limit, which follows from the results of Sec. IVA.

Taking the derivatives and working in the limit \( S_{m} \rightarrow 0 \) with \( B_{n} \gg \delta_{m} \), we find that the non-Gaussian contribution to the conditional collapse fraction is

\[ f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) = \frac{\partial f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m})}{\partial B_{m}^{n}} \]

\[ \int_{-\infty}^{B_{m}^{n}} d \delta_{1} \ldots \int_{-\infty}^{B_{m}^{n}} d \delta_{m-1} \prod_{j=1}^{N} \partial_{j} W_{0,m}^{\oplus}. \] (45)

\[ \text{Rotating the integrals in the analogous manner as Eqs. (23) and (24) and making use of identities such as Eq. (33) as well as} \]

\[ \text{we find that the non-Gaussian part of the conditional collapse fraction can be written} \]

\[ f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) = \frac{\partial f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m})}{\partial B_{m}^{n}} \]

\[ \int_{-\infty}^{B_{m}^{n}} d \delta_{1} \ldots \int_{-\infty}^{B_{m}^{n}} d \delta_{m-1} \prod_{j=1}^{N} \partial_{j} W_{0,m}^{\oplus}. \] (44)

\[ \text{we find that the non-Gaussian part of the conditional collapse fraction can be written} \]

\[ f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) = \frac{\partial f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m})}{\partial B_{m}^{n}} \]

\[ \int_{-\infty}^{B_{m}^{n}} d \delta_{1} \ldots \int_{-\infty}^{B_{m}^{n}} d \delta_{m-1} \prod_{j=1}^{N} \partial_{j} W_{0,m}^{\oplus}. \] (45)

\[ \text{we find that the non-Gaussian part of the conditional collapse fraction can be written} \]

\[ f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m}) = \frac{\partial f_{\text{coll}}^{NG}(S_{m} \mid \delta_{m}, S_{m})}{\partial B_{m}^{n}} \]

\[ \int_{-\infty}^{B_{m}^{n}} d \delta_{1} \ldots \int_{-\infty}^{B_{m}^{n}} d \delta_{m-1} \prod_{j=1}^{N} \partial_{j} W_{0,m}^{\oplus}. \] (45)
\[ \mathcal{F}^{\text{mb,NG}}(S_n|\delta_m, S_m) = -\frac{1}{3} S_n^{\prime \prime} \frac{e^{-\frac{(B_n - \delta_m)^2}{2S_m^2}}}{\sqrt{2\pi S_m^2}} [(S_n - S_m) - (B_n - \delta_m)^2 + (B_n - \delta_m)P_{mn} - 2P_{mn}^2]
\]

\[ + \frac{S_n^3}{6} \frac{e^{-\frac{(B_n - \delta_m)^2}{2S_m^2}}}{\sqrt{2\pi S_m^2}} \left[ P_{mn}(B_n - \delta_m)((B_n - \delta_m)^2 + (S_n - S_m)) - 10(B_n - \delta_m)^2(S_n - S_m) \right] \]

\[ + (B_n - \delta_m)^4 - 8P_{mn}(S_n - S_m) + 7(S_n - S_m)^2 \right] + \sqrt{S_mS_n}m \delta_m \frac{e^{-\frac{(B_n - \delta_m)^2}{2S_m^2}}}{\sqrt{2\pi(S_n - S_m)^3}} \left[ (B_n - \delta_m)^2(B_n - \delta_m + P_{mn}) \right] \]

\[ - 3P_{mn}(B_n - \delta_m) + 2 \sqrt{\frac{2\pi}{S_n - S_m}} P_{mn} \text{erfc} \left[ \frac{B_n - \delta_m}{\sqrt{2(S_n - S_m)}} \right] \].

**V. HALO BIAS**

We now have all the ingredients with which to extract the halo bias. Taking the ratio of Eqs. (50) and (37), and expanding to linear order in \( \delta_m \), we find that the relationship between the halo overdensity and the matter overdensity is

\[ \delta_m^{\text{halo}} = \left( \frac{B_n}{S_n} - \frac{1}{B_n + P_{mn}} \right) \delta_m - \frac{1}{6(S_n + P_{mn})} \left( 20P_{mn}B_n + 10B_n^2 - 9P_{mn}^2 - 3B_n^2 \right) \delta_m m \]

\[ - \frac{S_nS_n^\prime}{3(B_n + P_{mn})} ((B_n - P_{mn})(B_n + 3P_{mn}) + S_n) \delta_m + \sqrt{S_mS_n} \left( \frac{B_n}{S_n} - \frac{1}{B_n + P_{mn}} \right) - \frac{3P_{mn}}{B_n + P_{mn}} \]

\[ + 2 \sqrt{\frac{2\pi}{S_n B_n + P_{mn}}} \text{erfc} \left[ \frac{B_n}{\sqrt{2S_n}} \right] \frac{\delta_m}{2S_m} + \frac{S_nS_n^\prime}{S_m} \left( 1 - \frac{2P_{mn}}{B_n + P_{mn}} + \frac{2\pi}{S_n B_n + P_{mn}} \text{erfc} \left[ \frac{B_n}{\sqrt{2S_n}} \right] \right) \frac{\delta_m}{S_m^2}. \]

**VI. THE SCALE-DEPENDENT BIAS FROM EXCURSION SETS**

We can now extract the scale-dependent bias from our excursion set result. For simplicity, we begin with the case of spherical collapse where the barrier is constant, \( B(S_n) = \delta_c \). Focusing on the scale-dependent parts of Eq. (51), and assuming a spherical barrier so that \( P_{mn} = 0 \), we have

\[ \delta_m^{\text{halo}} = \sqrt{S_mS_n^\prime} \frac{\delta_m}{S_m} + \sqrt{S_mS_n^\prime} \frac{\delta_m}{S_m} \left( \frac{\delta_c^2}{S_n} - 1 \right) \frac{\delta_m}{S_m}. \]

The problem is to transform this relation, between the real space halo overdensity smoothed on a large scale \( S_m \) with the real space matter overdensity smoothed on the same scale, into a Fourier space relationship

\[ \delta^{\text{halo}}(\vec{k}) = b(k) \delta(\vec{k}). \]
\[ \int \frac{d^3k}{(2\pi)^3} \delta(k) b^{SD,sph}(k) W(\tilde{k}, R_n) = \left( \frac{\sqrt{m} S^{-2}_{n} - 1}{2} + \sqrt{m} S_{n} S_{n}' \right) \times \int \frac{d^3k'}{(2\pi)^3} \delta(k') W(\tilde{k}, R_n) , \]  
\[ \text{where the} \; b^{SD,sph} \text{superscript reminds us that we are dealing} \]
\[ \text{with the scale-dependent part of the bias only here and, for} \]
\[ \text{now, focusing on the spherical collapse limit. Now correlate} \]
\[ \text{both sides of Eq. (55) with} \; \delta_m. \text{The left-hand side becomes} \]
\[ \int_{k_m}^{k_n} \frac{d^3k}{(2\pi)^3} P_\delta(k) b^{SD,sph}(k) = \int_{k_m}^{k_n} d\ln b^{SD,sph}(k) \Delta^2(k), \]
\[ \text{where} \; \Delta^2(k) = k^3 P(k)/2\pi^2, \text{and we have defined the power} \]
\[ \text{spectrum in the usual way,} \]
\[ \langle \delta(\tilde{k}) \delta(\tilde{k}) \rangle = P_\delta(k)(2\pi)^3 \delta^3(\tilde{k} - \tilde{k}). \]
\[ \text{The right-hand side becomes the set of coefficients in parenthesis in Eq. (55) multiplying} \]
\[ \frac{1}{S_m} \int_{k_m}^{k_n} \frac{d^3k}{(2\pi)^3} P_\delta(k) = 1, \]
\[ \text{as the remaining equality is} \]
\[ \int_{k_m}^{k_n} d\ln b^{SD,sph}(k) \Delta^2(k) = \left( \frac{\sqrt{m} S^{-2}_{n} - 1}{2} + \sqrt{m} S_{n} S_{n}' \right) \]  
\[ \text{To proceed we need information about the shape of} \]
\[ \text{the three-point function that enters into Eq. (59). The primordial} \]
\[ \text{three-point functions are most easily written in terms of the} \]
\[ \text{gravitational potential, so first recall that} \]
\[ \delta(k, z) = M(k, z) \Phi(k), \]
\[ \text{where} \]
\[ M(k, z) = 2 \frac{k^2 T(k) g(z)}{3 \Omega_m H^2(1 + z)} \]
\[ \text{and here} \; T(k) \text{is the matter transfer function normalized at} \]
\[ \text{unity} \; k \rightarrow 0, \; g(z) \text{is the linear growth of the gravitational} \]
\[ \text{potential during the matter-dominated epoch, and} \; \Phi(k) \]
\[ \text{denotes the primordial value of the gravitational potential} \]
\[ \text{(hence no} \; z \text{dependence). A generic three-point function of} \]
\[ \Phi \text{is characterized by the bispectrum} \]
\[ \langle \Phi(k_1) \Phi(k_2) \Phi(k_3) \rangle = B_\Phi(k_1, k_2, k_3)(2\pi)^3 \]
\[ \times \delta^3(k_1 + k_2 + k_3). \]
\[ \text{We can then evaluate the three-point function of density fluctuations,} \]
\[ \langle \delta_m \delta_m^2 \rangle = \int_{k_m}^{k_n} \frac{dk_1}{k_1} \int_{k_1}^{k_2} \frac{dk_2}{k_2} \int_{k_2}^{k_3} d\cos \theta \frac{k_1^3}{2 \pi^2} \frac{k_2^3}{4 \pi^2} \times M(k_1) M(k_2) M(k_3) B_\Phi(k_1, k_2, k_3), \]  
\[ \text{where} \; \theta \text{is the angle between} \; \tilde{k}_1 \text{and} \; \tilde{k}_2 \text{and we have} \]
\[ \text{assumed that} \; |\tilde{k}_1| = |\tilde{k}_1 + \tilde{k}_2| \leq k_n. \text{Following the notation of Ref. [12,18], we introduce the function (not to be confused with the crossing rate!)} \]
\[ F^3_n(k) = \frac{1}{S_n \Delta^2(k)} \int \frac{k_2^3 d^3k_2}{2 \pi^2} M(k_2) M(k_2 - \tilde{k}) \times B_\Phi(k_1, k_2, |\tilde{k}_2 - \tilde{k}|), \]
\[ \text{where} \; \tilde{k} = k \tilde{z}, \text{and} \; \tilde{z} \text{is an arbitrary unit vector. In the limit} \]
\[ k_m \ll k_n, \text{Eq. (59) then reduces to} \]
\[ \int_{k_m}^{k_n} d\ln \Delta^2(k) b^{SD,sph}(k) = \int_{k_m}^{k_n} d\ln \Delta^2(k) \frac{1}{2 S_n} - 1 \frac{F^3_n(k) + dF^3_n(k)}{d\ln S_n} . \]
\[ \text{At this stage, we have an equality between two sums of things. To see that we can equate each term in the sum, imagine that we measure these sums for two slightly different scales,} \]
\[ k_m \text{and} \; k_m + \delta k, \text{then we can extract information about the contribution to the sum from scale} \; k_m \text{by differencing these quantities. This then implies, as well as being an equality between sums, equality between each term in the sum. Then the scale-dependent bias can be read off} \]
\[ b^{SD,sph}(k) = M^{-1}(k) \left[ \frac{1}{2 S_n} - 1 \right] F^3_n(k) + dF^3_n(k) \]
\[ \text{which agrees with the result derived by Ref. [18]. Note that we naturally obtain the important second term, which can be thought of arising due to the fact that a scale-dependent rescaling of the variance also changes the significance interval that corresponds to a fixed mass. As we will see, this extra term vanishes for the local ansatz, but is nonzero and generally important for other bispectrum shapes [18,36] and in particular largely ameliorates the discrepancies noted by Ref. [37].} \]
\[ \text{For the local ansatz,} \]
\[ B_\Phi(k_1, k_2, k_3) = 2f_{NL}(P(k_1)P(k_2), \text{perm.}) , \]
\[ \text{so that} \]
\[ F^3_n = 4f_{NL} , \]
\[ \text{which is a constant. Therefore only the first term in Eq. (66) contributes leaving} \]
\[ b^{SD,sph,local(k)} = 2f_{NL} \frac{\Delta^2}{M(k)} \left( \frac{\delta^2}{S_n} - 1 \right) . \]
for the scale-dependent bias in the spherical collapse case, which agrees exactly with the result of Ref. [2].

Using the same techniques, we can extract the scale-dependent bias from Eq. (51) for the full ellipsoidal collapse problem. It too has pieces proportional to both \( \mathcal{F}_n^3 \) and \( d \mathcal{F}_n^3/dS_n \). We separate these out as

\[
b^{SD}(k) = \frac{\mathcal{F}_n^3(k)}{2 \mathcal{M}(k)} c_n + \frac{1}{\mathcal{M}(k)} \frac{d \mathcal{F}_n^3(k)}{d \ln S_n} d_n, \tag{70}
\]

where the coefficients are

\[
c_n = B_n b^\delta - \frac{3 \mathcal{P}_{0n}}{B_n + \mathcal{P}_{0n}}
+ 2 \sqrt{\frac{2 \pi}{S_n (B_n + \mathcal{P}_{0n})}} e^{\frac{\pi^2}{B_n}} \text{erfc} \left( \frac{B_n}{\sqrt{2 S_n}} \right), \tag{71}
\]

and

\[
d_n = 1 - \frac{2 \mathcal{P}_{0n}}{(B_n + \mathcal{P}_{0n})}
+ \frac{2 \sqrt{\frac{2 \pi}{S_n (B_n + \mathcal{P}_{0n})}} e^{\frac{\pi^2}{B_n}} \text{erfc} \left( \frac{B_n}{\sqrt{2 S_n}} \right)}{B_n}. \tag{72}
\]

These equations are the main results of the paper.

Ignoring the terms proportional to \( \partial \mathcal{F}_n^3(k)/\partial S_n \) in Eq. (70) (which vanishes for the local ansatz) we see that the coefficient of the \( k^{-2} \) term, \( c_n \), has a piece equal to the barrier times the Gaussian bias \( (B_n b^\delta) \). This is the standard coefficient. However, the full treatment here has uncovered other terms, the remaining ones on the right in Eq. (71). In Fig. 2 we plot \( c_n \) for the spherical collapse barrier [i.e., \( B(S_n) = \delta_e \)] and for the ellipsoidal collapse barrier of [38]

\[
B(S_n) = \sqrt{a} \delta_e [1 + \beta (a \nu)^{-\alpha}], \tag{73}
\]

where \( \nu = \delta_e^2/S_n, a = 0.7, \alpha = 0.6 \) and \( \beta = 0.4. \) For the spherical collapse barrier, \( c_n \) is given by the well-known expression \( c_n = \delta_e^2/S_n - 1 \) [e.g., Eq. (69)]. For the ellipsoidal collapse model, we must use the more complicated expression we have derived in Eq. (71).

The naive prediction for \( c_n \) (which holds in the case of the spherical collapse barrier) is that it is related to the scale-independent bias via \( c_n = b^\delta B_n(S_n) \). In fact, one way to try to extract constraints on \( f_{NL} \) is to fit the power spectrum with a bias of the form \( 1 + b^\delta + 2f_{NL}b^\delta \delta_e \mathcal{M}^{-1}(k) \). We have shown, however, in Eq. (70) that this simple relation does not appear to be correct when the barrier evolves with the smoothing scale. In Fig. 3 we show the ratio of \( c_n \) to the naive prediction as a function of \( S_n \) for the ellipsoidal collapse barrier given in Eq. (73). Note that our result depends heavily on the approximation of evaluating the three-point functions at their endpoint values, as in for example Eq. (42). This is known to break down as \( S_n \) becomes large, of order \( B_n^2 \). Therefore, the upturn in Fig. 3 may be an artifact of our approximation.
and almost certainly does not persist as $S_n$ increases. Techniques that move beyond this approximation [39] are clearly called for.

VII. CONCLUSIONS

Making use of the path integral approach to excursion set theory, we have derived the bias of collapsed objects. We have kept all terms linear in the long-wavelength density fluctuation and we have allowed for a completely general moving barrier. The bias contains both the scale-independent contributions, analogous to those first reported by Ref. [26] for a general barrier, and contributions that depend on the long-wavelength smoothing scale. We have demonstrated how this dependence can be interpreted as a scale-dependent biasing and shown that our result reduces to the famous Dalal et al. [2] result in the limit of spherical collapse and the local ansatz. In the spherical collapse limit for a general bispectrum shape, we reproduce the results of Desjacques, Jeong and Schmidt [18,36], including the previously overlooked additional term.

In the case where the barrier is not constant and depends on the smoothing scale, we find that the coefficient of the scale-dependent bias is no longer simply related to the Gaussian bias parameter, but rather contains additional terms [Eq. (71)] that might affect the extraction of $f_{NL}$ from upcoming surveys. While the simple relation is recovered on sufficiently large mass scales, on smaller mass scales, we find a significant departure from the expected result.

In arriving at this result, we have made several approximations. Following standard techniques in evaluating the effect of a moving barrier (which characterizes ellipsoidal collapse), we truncated the probability distribution in Eq. (28). This approximation causes $\Pi$ to no longer satisfy the Fokker-Planck equation. However, we have shown in the Appendix that when used in combination with the Fokker-Planck equation to calculate the halo bias, this approximation leads to results consistent with those found in the limit where the barrier is constant, and one restricts to the leading-order result.

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APPENDIX A: THE LINEAR BARRIER

In this Appendix we compare our results to a case where the probability distribution can be evaluated exactly. Aside from the limit where the absorbing barrier is independent of the smoothing scale, the only other exact solution to the Fokker-Planck equation with an absorbing barrier condition is in the case where the barrier has a linear dependence on the smoothing scale, $B(S) = B_0 + B_1 S$, (A1)

where $B_0$ and $B_1$ are constants. For this barrier, it is an elementary exercise to obtain the probability distribution

$$
\Pi^{LB}(\delta_n, S_n) = \frac{1}{\sqrt{2\pi S_n}} \left( \exp\left[ -\frac{\delta_n^2}{2S_n} \right] - \exp\left[ -\frac{(2B_n - \delta_n)^2}{2S_n} + 2B_n(B_n - \delta_n) \right] \right),
$$

which one can verify solves the Fokker-Planck equation, with the linear boundary condition Eq. (A1). The superscript \text{LB} here and in the rest of this Appendix denotes expressions that are only valid for the barrier in Eq. (A1).

It is straightforward to obtain the Gaussian formation rate,
which agrees exactly with the result in Eq. (31) since for the barrier in Eq. (A1)
\[ \mathcal{P}_{0n} = -S_n B_n'. \]  
(A3)

The conditional crossing rate is found as described above by simply translating the Gaussian result using the shift of variable in Eq. (40). We can also compute the halo bias analogous to Eq. (51),
\[ \delta_m^{\text{halo,LB}} = \left( \frac{B_n}{S_n} - \frac{1}{(B_n - S_n B_n')} \right) \delta_m - \frac{1}{6} \left( \frac{S_n^3}{(B_n - S_n B_n')} \right)^2 \left( -20 S_n B_n' B_n + 10 B_n^2 - 9(S_n B_n')^2 - 3 \frac{B_n^2}{S_n^3} (B_n - S_n B_n')^2 \right. \\
+ 7 S_n - 8 \sqrt{2 \pi S_n^{5/2}} B_n^3 e^{-\frac{(B_n - 2 S_n B_n')(B_n - S_n B_n')}{2 S_n}} \text{erfc} \left( \frac{B_n - 2 S_n B_n'}{\sqrt{2 S_n}} \right) \delta_m \\
- \frac{S_n S_n'}{3(B_n - S_n B_n')^2} \left( B_n + S_n B_n' (B_n - 3 S_n B_n') + S_n - 4 S_n^2 B_n^3 (B_n - S_n B_n') \right) \\
+ 2 \sqrt{2 \pi S_n^{3/2}} B_n^3 e^{-\frac{(B_n - 2 S_n B_n')(B_n - S_n B_n')}{2 S_n}} \text{erfc} \left( \frac{B_n - 2 S_n B_n'}{\sqrt{2 S_n}} \right) \delta_m S_n^3 m \\
\left. \times \left( \frac{B_n^2}{S_n} - \frac{B_n}{(B_n - S_n B_n')} + \frac{3 S_n B_n'}{(B_n - S_n B_n')} + 2 \sqrt{2 \pi S_n^{3/2}} B_n^3 e^{-\frac{(B_n - 2 S_n B_n')(B_n - S_n B_n')}{2 S_n}} \text{erfc} \left( \frac{B_n - 2 S_n B_n'}{\sqrt{2 S_n}} \right) \right) \delta_m \right. \\
+ \frac{S_n S_n'}{S_n^3 m} \left( 1 + \frac{2 S_n B_n'}{(B_n - S_n B_n')} + \sqrt{2 \pi S_n^{3/2}} B_n^3 e^{-\frac{(B_n - 2 S_n B_n')(B_n - S_n B_n')}{2 S_n}} \text{erfc} \left( \frac{B_n - 2 S_n B_n'}{\sqrt{2 S_n}} \right) \right) \delta_m. \]  
(A4)

from which we can obtain the scale-dependent part of the bias
\[ b_{\text{LB}}^{\text{LB}}(k) = \frac{d^3 \mathcal{F}_n(k)}{d^3 \ln S_n} \left( \frac{B_n}{S_n} - \frac{1}{B_n - S_n B_n'} \right) + 3 \frac{S_n B_n'}{B_n - S_n B_n'} + 2 \frac{2 \pi S_n^3 B_n^3 e^{-\frac{(B_n - 2 S_n B_n')(B_n - S_n B_n')}{2 S_n}} \text{erfc} \left( \frac{B_n - 2 S_n B_n'}{\sqrt{2 S_n}} \right)}{S_n^3 m} \left( 1 + \frac{2 S_n B_n'}{(B_n - S_n B_n')} + \sqrt{2 \pi S_n^{3/2}} B_n^3 e^{-\frac{(B_n - 2 S_n B_n')(B_n - S_n B_n')}{2 S_n}} \text{erfc} \left( \frac{B_n - 2 S_n B_n'}{\sqrt{2 S_n}} \right) \right) \]  
(A5)

and thus we find exact agreement with Eqs. (51) and (70) up to terms cubic in $B_n'$, the order to which we have evaluated the approximation in Eq. (28). We thus conclude that the deviation we find at low masses is not a consequence of the approximation of the probability distribution, but rather represents a breakdown of the approximations at Eq. (42) or possibly of the excursion set itself.