SUBEXPONENTIALS IN NONASSOCIATIVE LAMBEK CALCULUS

Eben Blaisdell

A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2024

Supervisor of Dissertation

Andre Scedrov, Professor of Mathematics, University of Pennsylvania

Graduate Group Chairperson

Ron Donagi, Professor of Mathematics, University of Pennsylvania

Dissertation Committee

Andre Scedrov, Professor of Mathematics, University of Pennsylvania
Henry Towsner, Professor of Mathematics, University of Pennsylvania
Steve Zdancewic, Professor of Computer and Information Science, University of Pennsylvania

ACKNOWLEDGEMENT

I am deeply grateful to my advisor, Andre Scedrov. From our first Zoom call during the pandemic, Andre has been exceptionally motivating. Andre continually provides an exceptionally clear vision of the field while remaining radically open to exploring new ideas. I am eternally grateful for the freedom and confidence Andre afforded me as his student.

I want to also thank my coauthors, Stepan Kuznetsov, Elaine Pimentel, and Max Kanovich. Working with them has been the most exciting and energizing academic experience I have every had the pleasure of participating in. I am indebted to them for their kindness, patience, and selfless mentorship.

I would also like to thank Henry Towsner for his mentorship. Without his masterfully taught Introduction to Mathematical Logic class, I would likely never have experienced the beauty of logic. Well after chairing my oral exam committee, Henry has always been available for any questions, logical or otherwise.

Finally, I express my most sincere to my parents, Michael and Karen, and my siblings Margaret and Ely, whose perpetual love and support provide the bedrock for everything I do.

ABSTRACT

SUBEXPONENTIALS IN NONASSOCIATIVE LAMBEK CALCULUS

Eben Blaisdell

Andre Scedrov

We focus on a family on noncommutative nonassociative substructural logics enriched with subexponentials, unary modalities licensing local application of structural rules. Primarily, we prove upper and lower bounds on complexity of provability in a broad class of these logics. To uniformly frame these results, we present two sets of rules, one intuitionistic and one classical, with the intention of capturing a broad subset of existing logics as fragments. We also prove equivalences of these systems in natural settings.

TABLE OF CONTENTS

CHAPTER 1

BACKGROUND

1.1. Substructural Logic

Logical calculi have been wildly successful for modelling systems well beyond classical truth.

A particularly surprising example came from Lambek (1958). Building on work of Adjukewicz Ajdukiewicz (1935) and Bar-Hillel Bar-Hillel (1953), Lambek introduced a noncommutative *syntactic calculus* for modelling natural language, now known as *Lambek calculus*.

Lambek realized that this noncommutative calculus overgenerated; that is, the natural typing of words lead to ungrammatical sentences being marked as grammatical. Specific problematic examples are handled in Lambek (1961), where a nonassociative, noncommutative logic is introduced, now known as the nonassociative Lambek calculus.

From another direction, Girard (1987), while studying semantic models of the polymorphic λ-calculus, discovered a logical system that refined both classical and intuitionistic logic. This logic, known as linear logic, achieves this via *resource awareness*. By tracking the usage of assumptions and enforcing that each assumption is used exactly one, linear logic successfully models phenomena from quantum mechanics (Pratt (1992)) to deadlock detection (Engberg and Winskel (1994)).

These logics are *substructural logics*; so-called because they can almost all be attained from the sequent calculus of Gentzen (1969) by the removal of *structural rules*; namely, exchange, weakening, and contraction. Notably, even with these rules removed, the resultant sequent calculi all admit the (cut) rule, just as in Gentzen (1969).

There is a notable exception to substructural systems arising from the removal of structural rules from Gentzen's so-called structural system, specifically nonassociative Lambek calculus. Gentzen's system is inherently associative as it considers lists of assumptions. Practically, this means that even when all structural rules are removed more sequents are still provable than in nonassociative

Lambek calculus.

However, we can mostly successfully carry out the maneuver in the reverse direction. By adding in an explicit rule for associativity to the nonassociative Lambek calculus, we obtain almost exactly Gentzen's system with the structural rules removed. There are however two distinctions between these systems. The first is minor; we need to massage the context bureaucracy in Gentzen's system to get a meaningful noncommutative logic. The second, more serious, distinction is the inclusion of what is known as *Lambek's restriction*.

Lambek's restriction, introduced with linguistic motivations in Lambek (1958) and Lambek (1961) is the requirement that the context of a sequent anywhere in a proof be nonempty. If this restriction is not applied, then Lambek calculus overgenerates grammaticity on a large class of natural language examples. However, no such restriction exists in linear logic or Gentzen's original system.

Noting this, by selectively adding in the structural rules of associativity, exchange, contraction, and weaking to nonassociative Lambek calculus without Lambek's restriction, we obtain logics which are sound and complete with respect to many well-known logical systems, including

- (associative) Lambek calculus without Lambek's restriction,
- multiplicative linear logic,
- multiplicative affine logic, and
- (nondistributive) relevant logic.

For this reason, nonassociative Lambek calculus without Lambek's restriction is a natural choice of foundation on which to start adding structural rules.

1.2. Global Structural Axioms

Independently of a sequent system, the structural rules have natural formulations as (implicitly) universally quantified second order nonlogical axioms.

These axioms are relatively straightforward, and have the expected effect on the semantics, but are listed here for later reference.

$$\begin{aligned}
\mathsf{A1} &: A \otimes (B \otimes C) \Rightarrow (A \otimes B) \otimes C \\
\mathsf{A2} &: (C \otimes B) \otimes A \Rightarrow C \otimes (B \otimes A) \\
\mathsf{E} &: A \otimes B \Rightarrow B \otimes A \\
\mathsf{C} &: A \Rightarrow A \otimes A \\
\mathsf{W} &: A \Rightarrow 1
\end{aligned}$$

Specifically, $\mathsf{A1}$ and $\mathsf{A2}$ enforce associativity of the product, and E enforces commutativity. These are very familiar algebraic conditions.

The axioms C and W are less familiar algebraically. The contraction axiom C alone is equivalent to the descriptively-named *square-increasing* property $x \leq x \cdot x$, a condition weaker than idempotency. The weakening axiom W simply states that $1 \equiv \top$; that is, the unit of the product is the upper bound of all propositions.

1.3. The Exponential

One major feature of (intuitionistic) linear logic missing from full nonassociative Lambek Calculus is the so-called exponential, written as a prefixed ! and often pronounced "bang" or sometimes "of course".

There are many ways to differentiate the three classes of connective in linear logic. Here are a few high-level descriptions that we will elaborate upon:

- *additives*, $\&, \oplus, \top$: context-sharing, lattice-like, truth-like
- *multiplicatives*, $\otimes, \rightarrow, 1$: context-splitting, groupoid- (i.e., magma-) like, resource-like
- *exponential*, !: translates between additives and multiplicatives, structural

This description of the exponential help illuminates this peculiar choice of names. That is, the exponential translates between additives and multiplicatives as seen in the following linear equiprovablity,

$$!(A \mathbin{\&} B) \iff !A \otimes !B,$$

which is intended to be reminiscent of the real number equality

$$e^{a+b} = e^{a} \cdot e^{b}$$

because & and $\otimes$ are *additive* and *multiplicative* conjunction respectively.

Interestingly, this is achieved is by licensing structural rules! Let us draw a very informal analogy. We can obtain the truth-like intuitionistic propositional logic from the resource-aware intuitionistic linear logic by adding in contraction and weakening, so perhaps we should expect to obtain the truth-like additive connectives from the multiplicative resource-like connectives by locally adding contraction and weakening.

We can make this statement much more formally. In an appropriate proof system (which defines the behavior of the involved connectives), we have that

$$!(A \mathbin{\&} B) \Rightarrow !A \otimes !B$$

follows from the exponential contraction axiom $!\mathsf{C} : !A \Rightarrow !A \otimes !A$, and monotonicity of exponential

$$\frac{A \Rightarrow B}{!A \Rightarrow !B}\ \mathsf{mon}$$

However, the reverse direction,

$$!A \otimes !B \Rightarrow !(A \mathbin{\&} B)$$

does not follow from exponential weakening, i.e. $\mathsf{!W} : !A \Rightarrow 1$ and monotonicity. The straightforward Hilbert-style proof would require another axiom, axiom K, explained below.

1.3.1. The Exponential as a Modality

The axiom K, usually presented $!(A \to B) \Rightarrow !A \to !B$ is a cornerstone of *modal logic*. From a very literal vantage, a modal logic is a logic with one or more unary connectives called *modalities*. Modalities are often monotonic as defined previously, indicating that their truth is covariant in the truth of their parameter. There is a well-explored zoo of modal logics, each defined by a collection of modalities, often coming in DeMorgan- or redisual-pairs, and axioms controlling their behavior.

Modal logics satisfying axiom K, known as *normal* modal logics are particularly important as they admit a particularly beautiful many-worlds semantics known as *Kripke semantics* which unfortunately lies outside the scope of this thesis.

For our purposes, we choose the following more convenient presentation of K.

$$\mathsf{K} : !A \otimes !B \Rightarrow !(A \otimes B)$$

Note that this is equivalent in any reasonable sequent system with monotonicity by the following two deductions.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\mathsf{init} \quad \dfrac{}{B \Rightarrow B}\,\mathsf{init}}{A, B \Rightarrow A \otimes B}\,\otimes R}{B \Rightarrow A \to (A \otimes B)}\,\to R}{!B \Rightarrow !(A \to (A \otimes B))}\,\mathsf{mon} \quad !(A \to (A \otimes B)) \Rightarrow !A \to !(A \otimes B) \quad \dfrac{\dfrac{}{!A \Rightarrow !A}\,\mathsf{init} \quad \dfrac{}{!(A \otimes B) \Rightarrow !(A \otimes B)}\,\mathsf{init}}{!A, !A \to !(A \otimes B) \Rightarrow !(A \otimes B)}\,\to L}{!A, !B \Rightarrow !(A \otimes B)}\,\mathsf{cut}}{!A \otimes !B \Rightarrow !(A \otimes B)}\,\otimes L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{!A \Rightarrow !A}\,\text{init} \qquad \dfrac{}{!(A \to B) \Rightarrow !(A \to B)}\,\text{init}}{!A, !(A \to B) \Rightarrow !A \otimes !(A \to B)}\,\otimes R \qquad !A \otimes !(A \to B) \Rightarrow !(A \otimes (A \to B)) \qquad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{init} \quad \dfrac{}{B \Rightarrow B}\,\text{init}}{A \otimes (A \to B) \Rightarrow B}\,\to L}{!(A \otimes (A \to B)) \Rightarrow !B}\,\text{mon}}{}}{!A, !(A \to B) \Rightarrow !B}\,\text{cut}}{!(A \to B) \Rightarrow !A \to !B}\,\to R$$

These deductions are not easy to read, but notice that the only unclosed assumptions are instances of the other form of the axiom.

Importantly, Girard's original linear logic (Girard (1987)) does in fact admit K. Moreover, Girard's choice of modal axioms for the exponential comprise the very familiar modal logic $\mathsf{S4}$.

The modal logic $\mathsf{S4}$ (whose modality is usually written as the box $\Box$ rather than !) is axiomatized be the following three sequents:

$$\begin{aligned} \mathsf{K} &: \; !A \otimes !B \Rightarrow !(A \otimes B) \\ \mathsf{T} &: \; !A \Rightarrow A \\ 4 &: \; !A \Rightarrow !!A \end{aligned}$$

So to summarize, the linear logic exponential is a modality governed by the modal axioms K, T, and 4 that licenses the (local) structural axioms $!\mathsf{C}$ and $!\mathsf{W}$.

1.4. Subexponentials

Linear logic proves the structural axioms $\mathsf{E}, \mathsf{A1}$, and $\mathsf{A2}$, so the only two standard structural axioms missing are C and W, both of which are licensed by the exponential.

To introduce the exponential into a system with fewer structural rules, it is unclear which should be licensed. All of the considered structural axioms have very natural exponential counterparts:

$$
\begin{aligned}
\mathsf{!E1} &: \ !A \otimes B \Rightarrow B \otimes !A \\
\mathsf{!E2} &: \ B \otimes !A \Rightarrow !A \otimes B \\
\mathsf{!A1} &: \ !A \otimes (B \otimes C) \Rightarrow (!A \otimes B) \otimes C \\
\mathsf{!A2} &: \ (C \otimes B) \otimes !A \Rightarrow C \otimes (B \otimes !A)
\end{aligned}
$$

In fact, exchange has two counterparts, one for moving in each direction.

To eliminate cut in noncommutative systems, the following nonlocal contraction exponential axioms were also introduced.

$$
\begin{aligned}
\mathsf{!nC1} &: \ !A \otimes B \Rightarrow (!A \otimes B) \otimes !A \\
\mathsf{!nC2} &: \ B \otimes !A \Rightarrow !A \otimes (B \otimes !A)
\end{aligned}
$$

Paying closer attention to the previous proofs of the exponential translation, we notice that !C and !W (and K) are sufficient to prove both directions, even in nonassociative noncommutative systems. Thus, there is no a priori reason to add a different subset of the structural axioms to the exponential.

However, local control of structural rules is very valuable in many contexts.

For example, it is important that logics modelling natural language be noncommutative, however, in some sentences a word is used far from where it is introduced. Consider the following example from Morrill and Valentín (2017).

“John praises, likes, and will love London.”

In cases like these, the application of contraction and/or exchange only to specific formulas would be quite useful.

Exponentials license all structural rules, but Danos et al. (1993) first introduced *subexponentials* which license some subset of the structural rules.

You may want to locally license multiple subsets of structural rules in the same system. In this case, we have multiple modalities, and we can also consider their relative strength. This was also introduced by Danos et al. (1993).

We introduce a host of subexponentials $!^i$ each with some label, and preorder them by strength as would be familiar in preexisting multimodal systems.

It is worth noting that the possibility of a family of subexponentials relies on the noncanonicity of the linear logic exponential. That is, if two copies of the exponential are added into the usual proof system with two copies of the usual rules, the two copies are not interderivable. The other connectives, for example $\&$, $\rightarrow$, and $\otimes$, are canonical.

1.5. Weaker Modalities

We may also weaken the subexponentials on the modal side, considering subsets of the axioms K, T, and 4. This is well-precedented in linear logic, where Light Linear Logic (Girard (1998)) and Soft Linear Logic (Lafont (2004)), which were both introduced for the study of computational complexity classes, validate the modal rules $\{\mathsf{T}\}$ and $\{\mathsf{K}, \mathsf{T}\}$ respectively.

In the domain of subexponentials, these axioms have different names. Specifically, T is known as *dereliction* and 4 is known as *digging*.

These modally restricted subexponentials are known also to be useful on the noncommutative linguistic realm. For example Kanovich et al. (2020) uses a subexponential not validating 4 so that *multiplexing*, the axiom schema $!^i A \Rightarrow A^{\otimes n}$ does not collapse to equivalence with contraction. Further, their subexponential does not validate K for reasons of cut admissibility.

Most excitingly on the linguistic semantics front, Moot and Retoré (2019) suggests that *soft subexponentials*, those not licensing 4, and the consequent computational restrictions, are inherent to human interpretation of natural language. One perspective on categorial grammars holds that the semantics of a sentence is the Curry-Howard translation of the syntactic parse. Said differently, the term language of a semantics lives in a system that corresponds to the addition of (unused) structural rules to the syntactic system. Applying this philosophy to the semantics of Moot and Retoré (2019) points towards a yet-to-be-defined noncommutative system with soft subexponentials.

CHAPTER 2

INUITIONISTIC SYSTEM

There is a great diversity of substructural logics. This remains true even when restricting our attention to intuitionistics substructural logics with subexponentials.

Firstly, different systems include different subsets of the binary connectives. Many systems, including Lambek's original syntactic calculus, restrict their attention to the multiplicative connectives, while other "full" systems include both the multiplicative and additive connectives.

The existing systems are even more varied with respect to exponentials and subexponentials. Subexponentials have two levers that can be adjusted, namely their behavior as modalities and their licensing of structural rules. Girard's original linear logic included one exponential.

We present a system with the intention of capturing as many of these varied systems as possible. We start with some preliminaries, specifically structures and (intuitionistic) sequents.

Definition 1 (Structured sequents)**.** Structures *are either the empty structure* Λ*, a formulae, or ordered pairs containing structures:*

$$\Gamma \quad ::= \quad \Lambda \mid F \mid (\Gamma, \Gamma).$$

Structures are considered up to the following notion of equivalence: (Λ, Γ) *and* (Γ, Λ) *are the same as* Γ*, including inside some larger structure. In particular, any structure other than* Λ *itself has a unique representation without* Λ*.*

A context with holes, $\Gamma\{\,\}\ldots\{\,\}$ *is a structure with designated placeholders. Given a context with holes, we write* $\Gamma\{\Delta_1\}\ldots\{\Delta_n\}$ *for the structure with the placeholders in* $\Gamma\{\,\}\ldots\{\,\}$ *replaced with structures* $\Delta_1, \ldots, \Delta_n$*.*

Note that the placeholders are distinct, so the order of $\Delta_1, \ldots, \Delta_2$ *is important. The structures*

$\Gamma\{\Delta\}\{\Pi\}$ and $\Gamma\{\Pi\}\{\Delta\}$ are not in general equivalent. However, the order in which the holes is presented is not important. In $\Gamma\{\,\}\{\,\}$ it is possible that the second placeholder appears to the left of the first placeholder in the actual tree.

In the case that any Δ_i are empty, we reduce (Λ, Γ) and (Γ, Λ) to Γ recursively until the structure is either empty or is represented without Λ.s

An (intuitionistic) structured sequent *has the form $\Gamma \Rightarrow C$ where Γ is a structure and C is a formula.*

We will consider sequent systems with these structures as contexts. To do this, we parameterize the set of modal labels, relative strength of the modalities, the structural rules licensed by the modalities, and the ambient structural rules.

Definition 2. *We start by restricting our attention to a finite set of axioms.*

- *Let $\mathcal{S}_g$ be the set of global structural axioms $\{\mathsf{C}, \mathsf{W}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$,*
- *Let $\mathcal{S}_l$ be the set of local structural axioms $\{!\mathsf{nC1}, !\mathsf{nC2}, !\mathsf{C}, !\mathsf{W}, !\mathsf{E1}, !\mathsf{E2}, !\mathsf{A1}, !\mathsf{A2}\}$,*
- *Define an abstract set of structural 'names' $\mathcal{S} := \{\mathsf{nC}, \mathsf{C}, \mathsf{W}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$ implicitly referencing both $\mathcal{S}_g$ and $\mathcal{S}_l$, and*
- *Let $\mathcal{M}$ be the set of modal axioms $\{\mathsf{K}, \mathsf{T}, 4\}$.*

A multimodal substructural specification *is a quintuple $\Sigma = (I, \preceq, m, s, S)$, where*

- *I is a set of modal labels,*
- *$(I, \preceq)$ is a preorder indicating relative strength of modalities,*
- *$m : I \to 2^{\mathcal{S}}$ is an upward closed assignment of modal axioms to modal labels; that is, $m(i) \subseteq m(j)$ for all $i \preceq j$,*
- *$s : I \to 2^{\mathcal{S}}$ is an upward closed assignment of structural axioms to modal labels; that is, $s(i) \subseteq s(j)$ for all $i \preceq j$, and*

- *$S \subseteq \mathcal{S}$ is a set of ambient axioms.*

We say that a logic is complete *with respect to a multimodal substructural specification if*

- *there is a unary connective, that is modality, $!^i$ for each $i \in I$,*
- *it proves $!^j A \Rightarrow !^i A$ when $i \preceq j$,*
- *it proves all of the modal axioms $m(i)$ for the modality $!^i$ for all $i \in I$,*
- *it proves $!^i\mathcal{A}$ or $!^i\mathcal{A}1$ and $!^i\mathcal{A}2$ for each $\mathcal{A} \in s(i)$ for all $i \in I$, and*
- *it proves all of the structural axioms in S for all formulas.*

We now present a large set of sequent rules and then parameterize subsets of these rules to yield sequent systems complete over any multimodal substructural specification.

Propositional rules

$$\frac{\Gamma\{(A,B)\} \Rightarrow C}{\Gamma\{A \otimes B\} \Rightarrow C} \otimes L \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{(\Gamma,\Delta) \Rightarrow A \otimes B} \otimes R$$

$$\frac{\Delta \Rightarrow A \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{(\Delta, A \to B)\} \Rightarrow C} \to L \qquad \frac{(A,\Gamma) \Rightarrow B}{\Gamma \Rightarrow A \to B} \to R$$

$$\frac{\Delta \Rightarrow A \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{(B \leftarrow A, \Delta)\} \Rightarrow C} \leftarrow L \qquad \frac{(\Gamma, A) \Rightarrow B}{\Gamma \Rightarrow B \leftarrow A} \leftarrow R$$

$$\frac{\Gamma\{A_1\} \Rightarrow C \quad \Gamma\{A_2\} \Rightarrow C}{\Gamma\{A_1 \oplus A_2\} \Rightarrow C} \oplus L \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \oplus A_2} \oplus R_i$$

$$\frac{\Gamma\{A_i\} \Rightarrow C}{\Gamma\{A_1 \,\&\, A_2\} \Rightarrow C} \&L_i \qquad \frac{\Gamma \Rightarrow A_1 \quad \Gamma \Rightarrow A_2}{\Gamma \Rightarrow A_1 \,\&\, A_2} \&R$$

$$\frac{\Gamma\{\,\} \Rightarrow A}{\Gamma\{1\} \Rightarrow A} 1L \qquad \frac{}{\Rightarrow 1} 1R$$

Structural Rules

$$\frac{\Gamma\{((\Delta_1,\Delta_2),\Delta_3)\} \Rightarrow C}{\Gamma\{(\Delta_1,(\Delta_2,\Delta_3))\} \Rightarrow C} \mathsf{A1} \qquad \frac{\Gamma\{(\Delta_1,(\Delta_2,\Delta_3))\} \Rightarrow C}{\Gamma\{((\Delta_1,\Delta_2),\Delta_3)\} \Rightarrow C} \mathsf{A2}$$

$$\frac{\Gamma\{(\Delta_2,\Delta_1)\} \Rightarrow C}{\Gamma\{(\Delta_1,\Delta_2)\} \Rightarrow C} \mathsf{E} \qquad \frac{\Gamma\{\,\} \Rightarrow C}{\Gamma\{\Delta\} \Rightarrow C} \mathsf{W}$$

$$\frac{\Gamma\{(\Delta,\Delta)\} \Rightarrow C}{\Gamma\{\Delta\} \Rightarrow C} \mathsf{C} \qquad \frac{\Gamma\{\Delta\}\{\,\} \Rightarrow C}{\Gamma\{\Delta\}\{\Delta\} \Rightarrow C} \mathsf{nC}$$

Initial and cut rules

$$\frac{}{A \Rightarrow A} \mathsf{init} \qquad \frac{\Delta \Rightarrow A \quad \Gamma\{A\} \Rightarrow C}{\Gamma\{\Delta\} \Rightarrow C} \mathsf{cut}$$

Figure 2.1: Propositional, structural, inital, and cut rules for the set of intuitionistic systems defined.

Subexponential Modal Rules

$$\frac{\Gamma\{!^{\succeq i}A\} \Rightarrow C}{\Gamma\{!^iA\} \Rightarrow C}\ !^iL \qquad \frac{\Gamma\{A\} \Rightarrow C}{\Gamma\{!^iA\} \Rightarrow C}\ !^iL\mathsf{T}$$

$$\frac{A \Rightarrow C}{!^iA \Rightarrow !^iC}\ !^iR \qquad \frac{\Gamma \Rightarrow C}{!^i\Gamma \Rightarrow !^iC}\ !^iR\mathsf{K}$$

$$\frac{!^iA \Rightarrow C}{!^iA \Rightarrow !^iC}\ !^iR4 \qquad \frac{!^{i*}\Gamma \Rightarrow C}{!^i\Gamma \Rightarrow !C}\ !^iR\mathsf{K}4$$

Subexponential Structural Rules

$$\frac{\Gamma\{!^iA\}\{!^iA\} \Rightarrow C}{\Gamma\{!^iA\}\{\,\} \Rightarrow C}\ !^i\mathsf{nC} \qquad \frac{\Gamma\{\,\} \Rightarrow C}{\Gamma\{!^iA\} \Rightarrow C}\ !^i\mathsf{W}$$

$$\frac{\Gamma\{(!^iA, !^iA)\} \Rightarrow C}{\Gamma\{!^iA\} \Rightarrow C}\ !^i\mathsf{C} \qquad \frac{\Gamma\{(!^i\Delta, !^i\Delta)\} \Rightarrow C}{\Gamma\{!^i\Delta\} \Rightarrow C}\ !^i\mathsf{CK}$$

$$\frac{\Gamma\{((!^iA_1, A_2), A_3)\} \Rightarrow C}{\Gamma\{(!^iA_1, (A_2, A_3))\} \Rightarrow C}\ !^i\mathsf{A1} \qquad \frac{\Gamma\{((!^i\Delta_1, \Delta_2), \Delta_3)\} \Rightarrow C}{\Gamma\{(!^i\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow C}\ !^i\mathsf{A1K}$$

$$\frac{\Gamma\{(A_1, (A_2, !^iA_3))\} \Rightarrow C}{\Gamma\{((A_1, A_2), !^iA_3)\} \Rightarrow C}\ !^i\mathsf{A2} \qquad \frac{\Gamma\{(\Delta_1, (\Delta_2, !^i\Delta_3))\} \Rightarrow C}{\Gamma\{((\Delta_1, \Delta_2), !^i\Delta_3)\} \Rightarrow C}\ !^i\mathsf{A2K}$$

$$\frac{\Gamma\{(A_2, !^iA_1)\} \Rightarrow C}{\Gamma\{(!^iA_1, A_2)\} \Rightarrow C}\ !^i\mathsf{E1} \qquad \frac{\Gamma\{(\Delta_2, !^i\Delta_1)\} \Rightarrow C}{\Gamma\{(!^i\Delta_1, \Delta_2)\} \Rightarrow C}\ !^i\mathsf{E1K}$$

$$\frac{\Gamma\{(!^iA_2, A_1)\} \Rightarrow C}{\Gamma\{(A_1, !^iA_2)\} \Rightarrow C}\ !^i\mathsf{E2} \qquad \frac{\Gamma\{(!^i\Delta_2, \Delta_1)\} \Rightarrow C}{\Gamma\{(\Delta_1, !^i\Delta_2)\} \Rightarrow C}\ !^i\mathsf{E2K}$$

Figure 2.2: Subexponential rules for the set of intuitionistic systems defined.

Definition 3. *The* multimodal substructural logic associated to a multimodal substructural specification $\Sigma = (I, \preceq, m, s, S)$ *written* MSL_Σ*, is the logic on intuitionistic structured sequents with the following rules.*

- *All propositional rules and* (init)*,*
- *The rule* $(\mathcal{A})$ *if* $\mathcal{A} \in S$*,*
- *The rule* $(!^i\mathcal{A})$ *or* $(!^i\mathcal{A}1)$ *and* $(!^i\mathcal{A}2)$ *for* $i \in I$ *if* $\mathcal{A} \in s(i)$*,*

- *The rule* $(!^i\mathcal{A}\mathsf{K})$ *or* $(!^i\mathcal{A}1\mathsf{K})$ *and* $(!^i\mathcal{A}2\mathsf{K})$ *for* $i \in I$ *if* $\mathcal{A} \in s(i)$*,* $\mathsf{K} \in m(i)$*, and such a rule exists,*
- *The rule* $(!^iL)$ *for* $i \in I$*,*
- *The rule* $(!^iL\mathsf{T})$ *if* $\mathsf{T} \in m(i)$ *for* $i \in I$*,*
- *The rule* $(!^iR)$ *if* $\mathsf{K} \notin m(i)$ *for* $i \in I$*,*
- *The rule* $(!^iR4)$ *if* $\mathsf{K} \notin m(i), 4 \in m(i)$ *for* $i \in I$*,*
- *The rule* $(!^iR\mathsf{K})$ *if* $\mathsf{K} \in m(i), 4 \notin m(i)$ *for* $i \in I$*, and*
- *The rule* $(!^iR\mathsf{K}4)$ *if* $\mathsf{K}, 4 \in m(i)$ *for* $i \in I$*.*

This definition is very terse, so we take some time to expound upon it.

Firstly, all of the propositional rules are in every system MSL_Σ. The subexponentials should not effect the behavior of the propositional connectives, and we will formalize this later via cut elimination.

Also note that there is no asymmetry in the (nC) and $(!^i\mathsf{nC})$ rules. The holes in the context can appear in either order in the tree.

For every label we have $(!^iL)$, as this encodes $!^jA \Rightarrow !^i$ for $i \preceq j$, which holds regardless of the modal (or structural for that matter) rules licensed by the label i.

With respect to the modal axiom T, this determines exactly the inclusion or exclusion of the rule $(!^iL\mathsf{T})$. That is,

$$(!^i\mathsf{T}) \text{ is a rule of } \mathsf{MSL}_\Sigma \qquad \text{iff} \qquad \mathsf{T} \in m(i).$$

The case of the modal axioms $\mathsf{K}, 4$ and the right subexponential rules are more intricate. For a given label i, we have the follwing table showing the included right subexponential rules.

	$\mathsf{K} \notin m(i)$	$\mathsf{K} \in m(i)$
$4 \notin m(i)$	$(!^i R)$	$(!^i R\mathsf{K})$
$4 \in m(i)$	$(!^i R), (!^i R4)$	$(!^i R\mathsf{K}4)$

The Right Rules for $s(i) = \{4\}$

One, at first puzzling, feature of this table is the existence of a unique right subexponential rule for every subset of $\{\mathsf{K}, 4\}$ excepting only $\{4\}$. This is purely a matter of personal choice rather than mathematical content. The axiom 4 allows for some formulas on the left to retain their bangs, while logics with T may require that some formulas lose their bangs during a promotion, i.e. an application of a right subexponential rule. In logics with K and 4, this is handled by the rule $(!^i R\mathsf{K}4)$.

$$\frac{!^{i*}\Gamma \Rightarrow C}{!^i\Gamma \Rightarrow !C}\ !^i R\mathsf{K}4$$

Note that this requires a new notation $!^{i*}\Gamma$, representing the structure Γ with some, but maybe not all or even any, formulas wrapped in $!^i$. In logics with T, this rule can be deduced by a weaker promotion rule that leaves all formulas banged, as in the standard presentation of linear logic, and several applications of $(!^i L)$, as in

$$\dfrac{\dfrac{!^{i*}\Gamma \Rightarrow C}{!^i\Gamma \Rightarrow C}\ !^i L}{!^i\Gamma \Rightarrow !C}\ !^i R\mathsf{K}4'$$

However, this deduction cannot take place in MSL_Σ where $\mathsf{T} \notin m(i)$, so we need this more flexible $(!^i R\mathsf{K}4)$ rule.

What if we add this same flexibility to the $(!^i R4)$ rule? Let's call it $(!^i R4')$.

$$\frac{!^{i*}A \Rightarrow C}{!^{i}A \Rightarrow !C}\ !^{i}R4'$$

Here A is just a single formula, so there are only two possibilities for $!^{i*}A$, namely, just $!^{i}A$ and A. Substituting these two exhaustive cases into ($!^{i}R4'$) returns exactly the rules ($!^{i}R\mathsf{4}$) and ($!R$).

$$\frac{!^{i}A \Rightarrow C}{!^{i}A \Rightarrow !^{i}C}\ !^{i}R\mathsf{4} \qquad\qquad \frac{A \Rightarrow C}{!^{i}A \Rightarrow !^{i}C}\ !R$$

The rule ($!^{i}R$) is already named because it is the correct promotion rule for a purely monotonic modality (one not containing K or 4). Thus, instead of using the more notationally awkward ($!^{i}R4'$) rule, we give both cases separately with the two rules ($!^{i}R4$) and ($!^{i}R$).

Such a collapse is not possible for the $2^{|\Gamma|}$-many cases of ($!^{i}R\mathsf{K}4$).

Modal Soundness

To informally justify our choice of rules, we will deduce ($!^{i}L$), ($!^{i}L\mathsf{T}$), ($!^{i}R4$), ($!^{i}R\mathsf{K}$), and ($!^{i}R\mathsf{K}4$) from their corresponding axioms and ($!^{i}R$). We start with the simpler cases of ($!^{i}L$), ($!^{i}L\mathsf{T}$), and ($!^{i}R4$).

$$\frac{\Gamma\{!^{j}A\} \Rightarrow C}{\Gamma\{!^{i}A\} \Rightarrow C}\ !^{i}L \quad \rightsquigarrow \quad \frac{\dfrac{}{!^{i}A \Rightarrow !^{j}A}\ j \preceq i \qquad \Gamma\{!^{j}A\} \Rightarrow C}{\Gamma\{!^{i}A\} \Rightarrow C}\ \mathsf{cut}$$

$$\frac{\Gamma\{A\} \Rightarrow C}{\Gamma\{!^{i}A\} \Rightarrow C}\ !^{i}L\mathsf{T} \quad \rightsquigarrow \quad \frac{\dfrac{}{!^{i}A \Rightarrow A}\ \mathsf{T} \qquad \Gamma\{A\} \Rightarrow C}{\Gamma\{!^{i}A\} \Rightarrow C}\ \mathsf{cut}$$

$$\frac{!^{i}A \Rightarrow C}{!^{i}A \Rightarrow !^{i}C}\ !^{i}4 \quad \rightsquigarrow \quad \frac{\dfrac{}{!^{i}A \Rightarrow !^{i}!^{i}A}\ 4 \qquad \dfrac{!^{i}A \Rightarrow C}{!^{i}!^{i}A \Rightarrow !^{i}C}\ !^{i}R}{!^{i}A \Rightarrow !^{i}C}\ \mathsf{cut}$$

For $(!^i R\mathsf{K})$ and $(!^i R\mathsf{K}4)$, we need the following lemma, where $\bigotimes \Gamma$ means the in-tree-order $\otimes$-product of a structure.

Lemma 2.0.1. *For any nonempty structure* Γ*, we can deduce*

$$!^i\Gamma \Rightarrow !^i \bigotimes \Gamma$$

from

$$\mathsf{K} : !^i A \otimes !^i B \Rightarrow !^i (A \otimes B)$$

Proof. Firstly, note that the sequent $!^i A, !^i B \Rightarrow !^i(A \otimes B)$ follows straightforwardly from K (using $(\mathsf{cut}), (\otimes R)$, and (init)).

We then proceed by induction on Γ. If Γ is a single formula the claim is trivial. If $\Gamma \equiv (\Delta, \Pi)$, then we appeal to the induction hypothesis twice in

$$\dfrac{!^i\Delta \Rightarrow !^i \bigotimes \Delta \quad !^i\Pi \Rightarrow !^i \bigotimes \Pi \quad \dfrac{}{!^i \bigotimes \Delta, !^i \bigotimes \Pi \Rightarrow !^i(\bigotimes \Delta \otimes \bigotimes \Pi)}\ \mathsf{K}}{\begin{array}{c} !^i\Delta, !^i\Pi \Rightarrow !^i(\bigotimes \Delta \otimes \bigotimes \Pi) \\ \equiv !^i\Gamma \Rightarrow !^i \bigotimes \Gamma \end{array}}\ \mathsf{cut}$$

∎

Using this, we can now deduce $(!^i R\mathsf{K})$.

$$\dfrac{\Gamma \Rightarrow C}{!^i\Gamma \Rightarrow !^i C}\ !^i\mathsf{K} \quad \rightsquigarrow \quad \dfrac{\dfrac{}{!^i\Gamma \Rightarrow !^i \bigotimes \Gamma}\ \mathsf{K} \quad \dfrac{\dfrac{\Gamma \Rightarrow C}{\bigotimes \Gamma \Rightarrow C}\ \otimes L}{!^i \bigotimes \Gamma \Rightarrow !^i C}\ !^i R}{!^i\Gamma \Rightarrow !^i C}\ \mathsf{cut}$$

For $(!^i R\mathsf{K}4)$, let $\Gamma \equiv (A_k)_k$. We will cut in 4 for some, but maybe not all or even any, formulas. The notation $!^i!^{i*}(A_k)_k$ indicates that some formulas are wrapped in two bangs while others are only wrapped in one. In the example below, say below that J is the set of indices with two bangs.

$$\dfrac{!^{i*}\Gamma \Rightarrow C}{!^{i}\Gamma \Rightarrow !^{i}C}\ !^{i}\mathsf{K} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{}{(!^{i}A_k \Rightarrow !^{i}!^{i}A_k)_{k\in J}}\ 4 \qquad \dfrac{\dfrac{}{!^{i}!^{i*}(A_k)_k \Rightarrow !^{i}\bigotimes !^{i*}(A_k)_k}\ \mathsf{K} \qquad \dfrac{\dfrac{\begin{array}{c}!^{i*}\Gamma \Rightarrow C \equiv \\ !^{i*}(A_k)_k \Rightarrow C\end{array}}{\bigotimes !^{i*}(A_k)_k \Rightarrow C}\ \otimes L}{!^{i}\bigotimes !^{i*}(A_k)_k \Rightarrow !^{i}C}\ !^{i}R}{!^{i}!^{i*}(A_k)_k \Rightarrow !^{i}C}\ \mathsf{cut}}{\begin{array}{c}!^{i}(A_k)_k \Rightarrow !^{i}C \\ \equiv !^{i}\Gamma \Rightarrow !^{i}C\end{array}}\ \mathsf{cut}$$

Thus, we have deduced all of the modal subexponential axioms from their corresponding axioms.

Failure of Structural Soundness with K

The above justification intuitively corresponds to a completeness completeness of MSL_Σ over a chosen set of modal axioms. That is, everything provable in MSL_Σ was already deducible in the propositional system from the appropriate axioms. We however, do not pose, nor even define, a formal notion of soundness, because MSL_Σ fails a straightforward notion of soundness in the case of structural subexponential rules in the presence of K.

Consider as an example a MSL_Σ with one label i such that $s(i) = \{\mathsf{C}\}$ and $m(i) = \{\mathsf{K}\}$. In this case, we have the subexponential rules $(!^{i}L), (!^{i}R\mathsf{K})$, and $(!^{i}\mathsf{CK})$. In this setup, $!a\otimes !b \Rightarrow (!a\otimes !b) \otimes (!a\otimes !b)$ is provable with the following proof.

$$\dfrac{\dfrac{\dfrac{}{!a\otimes !b \Rightarrow !a\otimes !b}\ \mathsf{init} \qquad \dfrac{}{!a\otimes !b \Rightarrow !a\otimes !b}\ \mathsf{init}}{!a\otimes !b, !a\otimes !b \Rightarrow (!a\otimes !b) \otimes (!a\otimes !b)}\ \otimes R}{!a\otimes !b \Rightarrow (!a\otimes !b) \otimes (!a\otimes !b)}\ \mathsf{CK}$$

However, $!a\otimes !b \Rightarrow (!a\otimes !b) \otimes (!a\otimes !b)$ is not deducible from axioms K and C.

It is tempting then to restrict (CK) to single formulas, as in (C). In fact the MSL_Σ rule (C) is deducible from the axiom C. However, this would unfortunately break any chance of admitting (cut). The similar-looking sequent $!a\otimes !b \Rightarrow !(a \otimes b)\otimes !(a \otimes b)$, has the following proof with (cut) and

(C), but has no cut-free proof with just (C).

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{a \Rightarrow a}\,\textsf{init}
\qquad
\dfrac{}{b \Rightarrow b}\,\textsf{init}
}{(a, b) \Rightarrow a \otimes b}\,\otimes R
}{(!a, !b) \Rightarrow !(a \otimes b)}\,!R\textsf{K}
}{!a \otimes !b \Rightarrow !(a \otimes b)}\,\otimes L
\qquad
\dfrac{
\dfrac{
\overline{\overline{!(a \otimes b) \Rightarrow !(a \otimes b)}}\,\textsf{init}
\quad
\overline{\overline{!(a \otimes b) \Rightarrow !(a \otimes b)}}\,\textsf{init}
}{!(a \otimes b), !(a \otimes b) \Rightarrow !(a \otimes b) \otimes !(a \otimes b)}\,\otimes R
}{!(a \otimes b) \Rightarrow !(a \otimes b) \otimes !(a \otimes b)}\,\textsf{C}
}{!a \otimes !b \Rightarrow !(a \otimes b) \otimes !(a \otimes b)}\,\textsf{cut}
$$

This problem can be resolved by introducing a structural unary punctuation for the subexponentials, as with the multimodalities in Kurtonina and Moortgat (1997). However, we choose this particular unsound presentation for coherence with related literature. Even though unsound with respect to the subsets of modal and structural axioms, this system is still consistent as will be emphasized via cut admissibility.

Completeness

We now show that $\textsf{MSL}_\Sigma$ is complete with respect to its multimodal substructural specification as previously defined.

Theorem 2.0.1. $\textsf{MSL}_\Sigma$ *is sound with respect to the specification* $\Sigma = (I, \preceq, m, s, S)$.

Proof. By definition, $\textsf{MSL}_\Sigma$ contains a modality $!^i$ for each $i \in I$.

More interestingly, there is a simple cut-free proof of each of the required axioms in the defined systems. We go through exemplary cases and note on the right the hypotheses we are using about the specification Σ.

$$
\dfrac{\overline{\overline{!^i A \Rightarrow !^i A}}\,\textsf{init}}{!^j A \Rightarrow !^i A}\,!^j L
\qquad\qquad\qquad\qquad
i \preceq j
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\mathsf{init} \qquad \dfrac{}{B \Rightarrow B}\,\mathsf{init}}{A, B \Rightarrow A \otimes B}\,\otimes R}{!^i A, !^i B \Rightarrow !^i (A \otimes B)}\,!^i R\mathsf{K}}{!^i A \otimes !^i B \Rightarrow !^i (A \otimes B)}\,\otimes R \qquad\qquad \mathsf{K} \in m(i)$$

$$\dfrac{\dfrac{}{A \Rightarrow A}\,\mathsf{init}}{!^i A \Rightarrow A}\,!^i L \qquad\qquad \mathsf{T} \in m(i)$$

$$\dfrac{\genfrac{}{}{1.5pt}{}{}{!^i A \Rightarrow !^i A}\,\mathsf{init}}{!^i A \Rightarrow !^i !^i A}\,!^i R4 \qquad\qquad 4 \in m(i)$$

$$\dfrac{\genfrac{}{}{1.5pt}{}{\genfrac{}{}{1.5pt}{}{}{!^i A \Rightarrow !^i A}\,\mathsf{init} \qquad \genfrac{}{}{1.5pt}{}{}{!^i A \Rightarrow !^i A}\,\mathsf{init}}{!^i A, !^i A \Rightarrow !^i A \otimes !^i A}\,\otimes R}{!^i A \Rightarrow !^i A \otimes !^i A}\,!^i\mathsf{C} \qquad\qquad \mathsf{C} \in s(i)$$

$$\dfrac{\genfrac{}{}{1.5pt}{}{}{\Rightarrow 1}\,1R}{!^i A \Rightarrow 1}\,!^i\mathsf{W} \qquad\qquad \mathsf{W} \in s(i)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,\mathsf{init} \qquad \genfrac{}{}{1.5pt}{}{}{!^i A \Rightarrow !^i A}\,\mathsf{init}}{B, !^i A \Rightarrow B \otimes !^i A}\,\otimes R}{!^i A, B \Rightarrow B \otimes !^i A}\,!^i\mathsf{E1}}{!^i A \otimes B \Rightarrow B \otimes !^i A}\,\otimes L \qquad\qquad \mathsf{E1} \in s(i)$$

□

Note that none of the ‘normal’ structural rules, e.g. (CK), (E1K) were used in the proof of completeness. Recall that the reason for their inclusion in the set of rules is for the admission of cut. Each individual axiom is provable using the more restricted forms of the structural rules, but a proof involving multiple of these axioms and cut may not have a cut-free proof with the single-formula subexponential structural rules (C), (E), etc. For example the previously considered $!a \otimes !b \Rightarrow !(a \otimes b) \otimes !(a \otimes b)$

2.1. Equivalent Systems

The main goal of such a definition is to encompass a large class of pre-existing substructural logics. It is clear that such a system exists, the advantages are not in the presentation of the system, but rather in

- giving uniform notation for the discussion of a broad class of logics with very different motivations, and
- enabling related results in these systems to be proven uniformly under one roof.

We include a table of some interesting logics and sketch corresponding specification in MSL_Σ by summarizing the global and modal rules.

Acronym	System	Global Rules	Modal Rules
IPL	intuitionistic propositional logic	A1, A2, E, C, W	
L	Lambek calculus	A1, A2	
LL	(propositional) linear logic	A1, A2, E	
iMALL	multiplicative-additive intuitionistic LL	A1, A2, E	
iLL	intuitionistic LL	A1, A2, E	K, T, 4, C, W
FL	full (multiplicative-additive) L	A1, A2	
cLL	non-commutative iMALL	A1, A2	
acLL	non-commutative, non-associative iMALL		
acLL_Σ	acLL with subexponentials		Σ
NL	non-associative L		
FNL	full (multiplicative-additive) NL		
SLL	soft LL	A1, A2, E	K, T, C, W
SELL	subexponential LL	A1, A2, E	K, T, 4, Σ
AL	Affine logic	A1, A2, E, W	K, T, 4, C
SDML	simply dependent multimodal linear logics		
SMALC	FL with subexponentials	A1, A2	

Table 2.1: Acronyms of systems described by MSL_Σ.

2.2. Cut Elimination

Further, we present a unified syntactic proof of cut elimination in MSL_Σ.

Lemma 2.2.1. *If* $\Delta \Rightarrow A$ *and* $\Gamma\{A\} \Rightarrow C$ *have (cut-free) proofs in* MSL_Σ *with* $\mathsf{C} \notin S$*, then* $\Gamma\{\Delta\} \Rightarrow C$ *has a (cut-free) proof in* MSL_Σ*.*

Proof. We prove this by double induction on

- κ, the complexity of A, and
- δ, the number of rule applications in the proofs of $\Delta \Rightarrow A$ and $\Gamma\{A\} \Rightarrow C$.

We proceed casewise by the final rules in the proofs of $\Delta \Rightarrow A$ and $\Gamma\{A\} \Rightarrow C$ if the top level connective of A is not the exponential, and then handle the exponential case more carefully.

Axioms

If either proof is (init) alone, then the other proof suffices directly.

$$\dfrac{\Delta \Rightarrow A \quad \dfrac{}{A \Rightarrow A}\,\mathsf{init}}{\Delta \Rightarrow A}\,\mathsf{cut} \qquad \dfrac{\dfrac{}{A \Rightarrow A}\,\mathsf{init} \quad \Gamma\{A\} \Rightarrow C}{\Gamma\{A\} \Rightarrow C}\,\mathsf{cut}$$

Left Nonprincipal

If a left rule $\mathcal{R}$ is the final rule in the proof of $\Delta \Rightarrow A$, then we have the following general form, potentially with an extra assumption $\mathcal{A}$.

$$\dfrac{\dfrac{\mathcal{A}^? \quad \Delta' \Rightarrow A}{\Delta \Rightarrow A}\,\mathcal{R} \quad \Gamma\{A\} \Rightarrow C}{\Gamma\{\Delta\} \Rightarrow C}\,\mathsf{cut}$$

This can be rewritten to the following.

$$\dfrac{\mathcal{A}^? \quad \dfrac{\Delta' \Rightarrow A \quad \Gamma\{A\} \Rightarrow C}{\Gamma\{\Delta'\} \Rightarrow C}\,\mathsf{cut}}{\Gamma\{\Delta\} \Rightarrow C}\,\mathcal{R}$$

Here the formula complexity is maintained, but the number of rule applications is one fewer.

Right Nonprincipal

The case of nonprincipal rules in $\Gamma\{A\} \Rightarrow C$ requires more granularity.

For a rule $\mathcal{R}$ that is $(\otimes L)$, $(!^i L)$, (C), (CK), (W), or $(1L)$ applied nonprincipally, or $(\rightarrow L)$ or $(\leftarrow L)$ applied independently of A, potentially with another assumption $\mathcal{A}$, we have the following setup.

$$\dfrac{\Delta \Rightarrow A \quad \dfrac{\mathcal{A}^? \quad \Gamma'\{A\} \Rightarrow C}{\Gamma\{A\} \Rightarrow C}\,\mathcal{R}}{\Gamma\{\Delta\} \Rightarrow C}\,\mathsf{cut}$$

This can be rewritten to the following.

$$\dfrac{\mathcal{A}^? \quad \dfrac{\Delta \Rightarrow A \quad \Gamma'\{A\} \Rightarrow C}{\Gamma'\{\Delta\} \Rightarrow C}\,\mathsf{cut}}{\Gamma\{\Delta\} \Rightarrow C}\,\mathcal{R}$$

Here, κ is same in all cases, but δ decreases by one, so we apply the inductive hypothesis.

We need also to consider the applications of $(\rightarrow L)$ and $(\leftarrow L)$ where A moves to the left assumption; up to symmetry this is the following.

$$\dfrac{\Delta \Rightarrow A \quad \dfrac{\Pi\{A\} \Rightarrow D \quad \Gamma\{E\} \Rightarrow C}{\Gamma\{(\Pi\{A\}, D \rightarrow E)\} \Rightarrow C}\rightarrow L}{\Gamma\{(\Pi\{\Delta\}, D \rightarrow E)\} \Rightarrow C}\,\mathsf{cut}$$

This can be rewritten to the following.

$$\dfrac{\dfrac{\Delta \Rightarrow A \quad \Pi\{A\} \Rightarrow D}{\Pi\{\Delta\} \Rightarrow D}\,\mathsf{cut} \quad \Gamma\{E\} \Rightarrow C}{\Gamma\{(\Pi\{\Delta\}, D \rightarrow E)\} \Rightarrow C}\rightarrow L$$

Once more, κ is maintained while δ decreases by one.

Up to symmetry, we have the following rewrite for $(\otimes R)$.

$$\dfrac{\Delta \Rightarrow A \quad \dfrac{\Gamma\{A\} \Rightarrow B \quad \Pi \Rightarrow C}{(\Gamma\{A\}, \Pi) \Rightarrow B \otimes C}\otimes R}{(\Gamma\{\Delta\}, \Pi) \Rightarrow B \otimes C}\,\mathsf{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta \Rightarrow A \quad \Gamma\{A\} \Rightarrow B}{\Gamma\{\Delta\} \Rightarrow B}\,\mathsf{cut} \quad \Pi \Rightarrow C}{(\Gamma\{\Delta\}, \Pi) \Rightarrow B \otimes C}\otimes R$$

Additionally, we have the following rewrite for $(\rightarrow R)$.

$$\dfrac{\Delta \Rightarrow A \quad \dfrac{(B, \Gamma\{A\}) \Rightarrow C}{\Gamma\{A\} \Rightarrow B \rightarrow C}\rightarrow R}{\Gamma\{\Delta\} \Rightarrow B \rightarrow C}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Rightarrow A \quad (B, \Gamma\{A\}) \Rightarrow C}{(B, \Gamma\{\Delta\}) \Rightarrow C}\,\mathsf{cut}}{\Gamma\{\Delta\} \Rightarrow B \rightarrow C}\rightarrow R$$

$\square$

The case of $(\leftarrow R)$ is symmetric. In all of these cases, κ is maintained, but δ is decreased.

This exhausts all applications of rules to $\Gamma\{A\} \Rightarrow C$ where A is nonprincipal.

Nonexponential Principal Pairs

All remaining cases have A principal in the final rule of the proofs of $\Delta \Rightarrow A$ and $\Gamma\{A\} \Rightarrow C$. As usual, we consider the top level connective of A and show how to rewrite the proof by appealing to the inductive hypothesis.

In the case of tensor, we have the following

$$\dfrac{\dfrac{\Delta \Rightarrow A \quad \Pi \Rightarrow B}{(\Delta, \Pi) \Rightarrow A \otimes B}\otimes R \quad \dfrac{\Gamma\{(A, B)\} \Rightarrow C}{\Gamma\{A \otimes B\} \Rightarrow C}\otimes L}{\Gamma\{(\Delta, \Pi)\} \Rightarrow C}\,\mathsf{cut} \quad \rightsquigarrow$$

$$\dfrac{\Delta \Rightarrow A \qquad \dfrac{\Pi \Rightarrow B \quad \Gamma\{(A,B)\} \Rightarrow C}{\Gamma\{(A,\Pi)\} \Rightarrow C}\,\mathsf{cut}}{\Gamma\{(\Delta,\Pi)\} \Rightarrow C}\,\mathsf{cut}$$

Both cut formulas have lower complexity, so we appeal twice to the inductive hypothesis.

Next we consider if the top level connective is $\rightarrow$. Here we have the following rewrite.

$$\dfrac{\dfrac{(A,\Pi) \Rightarrow B}{\Pi \Rightarrow A \rightarrow B}\rightarrow R \qquad \dfrac{\Delta \Rightarrow A \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{(\Delta, A \rightarrow B)\} \Rightarrow C}\rightarrow L}{\Gamma\{(\Delta,\Pi)\} \Rightarrow C}\,\mathsf{cut} \quad \rightsquigarrow$$

$$\dfrac{\Delta \Rightarrow A \qquad \dfrac{(A,\Pi) \Rightarrow B \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{(A,\Pi)\} \Rightarrow C}\,\mathsf{cut}}{\Gamma\{(\Delta,\Pi)\} \Rightarrow C}\,\mathsf{cut}$$

Again, both applications of cut have smaller κ, so we appeal twice to the inductive hypothesis.

For the constant 1 we have

$$\dfrac{\dfrac{}{\Rightarrow 1}\,1R \qquad \dfrac{\Gamma\{\,\} \Rightarrow C}{\Gamma\{1\} \Rightarrow C}\,1L}{\Gamma\{\,\} \Rightarrow C}\,\mathsf{cut}$$

and the top proof of $\Gamma\{\,\} \Rightarrow C$ suffices, removing the (cut).

Subexponentials

Lastly, we need only consider if the cut formula is of the form $!^i A$ and is active in the final rules of both proofs. The only rules in which a banged succedent is active are (init), ($!^i R$), ($!^i R\mathsf{4}$), ($!^i R\mathsf{K}$), and ($!^i R\mathsf{K4}$); for convenience call this class of rules, excluding (init), $\mathcal{RI}$. We exclude (init) because we have already considered the case where the first rule above the left proof is (init). The subset of the rules in $\mathcal{RI}$ that are in MSL_Σ depends on Σ.

Following the deep cut elimination of Braüner and dePaiva Braüner et al. (1996); Braüner and de Paiva

(1998), applied to noncommutative logics by Kanovich et. al. Kanovich et al. (2019), the track all of the occurrences of $!^i A$ in the proof of $\Gamma\{!^i A\} \Rightarrow C$ upwards to the introduction of their 'bang', i.e. $(!^i)$. The rules that can introduce bang on an antecedent are (init), (W), $(!^i L)$, $(!^i R)$, $(!^i R\mathsf{K})$, and $(!^i R\mathsf{K4})$; for convenience label this class of rules $\mathcal{LI}$.

Hence, we are in the following scenario.

$$\dfrac{\dfrac{\Delta' \Rightarrow A}{!^i\Delta \Rightarrow !^i A}\ \mathcal{RI} \qquad \begin{matrix}\mathcal{LI}\text{ applications}\\ \vdots \\ \Gamma\{!^i A\} \Rightarrow C\end{matrix}}{\Gamma\{!^i\Delta\} \Rightarrow C}\ \mathsf{cut}$$

Because of (C) or (CK), there may be multiple applications of $\mathcal{LI}$ rules. Note that these applications may be on the same or different branches of a proof.

Additionally, the form of Δ and Δ' both depend on the $\mathcal{RI}$ rule being applied, which itself depends on the labels in M for the considered $!^S_M\mathsf{MacLL}$. For example, if $k \notin M$, then Δ is one formula. If the rule is $(!^i R\mathsf{K})$ then $\Delta' \equiv \Delta$, but if it is $(!^i R\mathsf{K4})$, then $\Delta' \equiv !^{i*}\Delta$.

We modify the proof of $\Gamma\{!^i A\} \Rightarrow C$ by replacing all occurrences of $!^i A$ with $!^i\Delta$, and then cutting in A and/or adding certain rules elsewhere as necessary at each $\mathcal{LI}$ application. This proof will only have cuts of lower complexity, and we then appeal to the inductive hypothesis several times.

It is not immediate for this replacement of a formula $!^i A$ by a structure $!^i\Delta$ to be valid generally. If $k \notin M$, then Δ must be a single formula, and we are fine. If $k \in M$, then we use the fact that (CK) allows for the contraction of entire substructures rather than only individual formulas, and the replacement is valid.

Consider each $\mathcal{LI}$ rule individually.

If the rule is (init), then in the modified proof, the sequent $!^i\Delta \Rightarrow !^i A$ appears, and we replace its proof with the proof ending in the $\mathcal{RI}$ application. Symbolically,

$$\dfrac{}{!^i A \Rightarrow !^i A}\,\textsf{init} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdots}{\Delta' \Rightarrow A}}{!^i \Delta \Rightarrow !^i A}\,\mathcal{RI}$$

If the rule is (W), then $w \in S$, and we instead apply (W) individually to each formula in $!^i\Delta$.

$$\dfrac{\Pi\{\ \} \Rightarrow D}{\Pi\{!^i A\} \Rightarrow D}\,\textsf{W} \quad\rightsquigarrow\quad \dfrac{\Pi\{\ \} \Rightarrow D}{\Pi\{!^i \Delta\} \Rightarrow D}\,\textsf{W}$$

Depending on the $\mathcal{RI}$ rule applied, $\Delta' \equiv \Delta$ or $\Delta' \equiv !^{i*}\Delta$. In either case, we can achieve Δ' from $!^i\Delta$ by some, potentially zero, number of applications of ($!^i L$). Therefore, for applications of ($!^i L\textsf{T}$) to $!^i A$, we can make the following replacement.

$$\dfrac{\Pi\{A\} \Rightarrow D}{\Pi\{!^i A\} \Rightarrow D}\,!^i L\textsf{T} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdots}{\Delta' \Rightarrow A}}{!^i\Delta \Rightarrow A}\,!^i L \qquad \Pi\{A\} \Rightarrow C}{\Pi\{!^i \Delta\} \Rightarrow D}\,\textsf{cut}$$

If the $\mathcal{LI}$ application is ($!^i R$), then by the definition of $!^S_M\textsf{MacLL}$, we can conclude that the $\mathcal{RI}$ rule must be ($!^i R$) or ($!^i R4$), and in particular $\Delta \equiv B$ for some B. In both cases we have the following respective replacements.

$$\dfrac{A \Rightarrow D}{!^i A \Rightarrow !^i D}\,!^i R \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdots}{B \Rightarrow A} \qquad A \Rightarrow D}{B \Rightarrow D}\,\textsf{cut}}{!^i B \Rightarrow !^i D}\,!^i R$$

$$\dfrac{A \Rightarrow D}{!^i A \Rightarrow !^i D}\,!^i R \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdots}{!^i B \Rightarrow A} \qquad A \Rightarrow D}{!^i B \Rightarrow D}\,\textsf{cut}}{!^i B \Rightarrow !^i D}\,!^i R4$$

The case of ($!^i R\mathsf{K}$) is much the same. The $\mathcal{RI}$ rule must have been ($!^i R\mathsf{K}$). We have the following rewrite.

$$\dfrac{\Pi\{A\} \Rightarrow D}{!^i\Pi\{!^i A\} \Rightarrow !^i D}\ !^i R\mathsf{K} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots}{\Delta \Rightarrow A} \quad \Pi\{A\} \Rightarrow D}{\Pi\{\Delta\} \Rightarrow D}\ \mathsf{cut}}{!^i\Pi\{!^i\Delta\} \Rightarrow !^i D}\ !^i R\mathsf{K}$$

Lastly, we need to consider applications of ($!^i R\mathsf{K4}$) where the bang on $!^i A$ is introduced. Here, the $\mathcal{RI}$ rule must have also been ($!^i R\mathsf{K4}$). In this case we have the following rewrite.

$$\dfrac{!^{i*}\Pi\{A\} \Rightarrow D}{!^i\Pi\{!^i A\} \Rightarrow !^i D}\ !^i R\mathsf{K} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots}{!^{i*}\Delta \Rightarrow A} \quad !^{i*}\Pi\{A\} \Rightarrow D}{!^{i*}\Pi\left\{!^{i*}\Delta\right\} \Rightarrow D}\ \mathsf{cut}}{!^i\Pi\{!^i\Delta\} \Rightarrow !^i D}\ !^i R\mathsf{K}$$

This exhausts all cases, proving the lemma. As a consequence, we have the following.

Theorem 2.2.1 (Cut admissibility)**.** *If a sequent is provable in* $\mathsf{MSL}_\Sigma + (\mathsf{cut})$ *with* $\mathsf{C} \notin S$*, then it has a proof in* MSL_Σ.

2.3. Subexponential Analyticity

Cut admissibility yields a subformula property for MSL_Σ. However, this subformula property is with respect to Gentzen subformulas where $!^j A$ is a subformula of $!^i A$ if $i \preceq j$. In particular, if Σ contains infinitely many subexponential labels, then a formula may have infinitely many subformulas.

We would prefer for formulas to have finitely many subformulas that can appear in a proof. Any sequent will only contain finitely many bangs and thus only a finite subset of the subexponential labels. Thus, it is sufficient to show that there is a proof where only labels appearing in the conclusion are used.

Theorem 2.3.1. *If* $\Gamma \Rightarrow C$ *is provable in* MSL_Σ*, there is a proof using only labels in* Γ *or* C*.*

Proof. We proceed by induction on the length of the proof of $\Gamma \Rightarrow C$.

Now consider the last rule. If the last rule is anything other than $(!^i L)$, the labels in each assumption are a subset of the labels in the conclusion, and we are done by direct appeal to the induction hypothesis.

Therefore, consider if the last rule is $(!^i L)$. Note that it commutes upward part every rule except another $(!^i L)$ with the same active formula, in which case both can be merged by the transitivity of $\preceq$. □

CHAPTER 3

CLASSICAL SYSTEM

Linear logic has two presentations, classical and intuitionistic. The classical system displays some of the symmetries latent in the intuitionistic system.

Schellinx (1991) observed that, unlike in structural propositional logic, classical linear logic (without additive constants) is a conservative extension of intuitionistic linear logic.

Later, Pentus (1998) extended this to noncommutative linear logic. This presented proof nets, and introduced a convenient ‘counter’ on formulas helpful in proving the conservativity of classical sublinear logics over their intuitionistic counterparts.

Then **?** proved the analogous result for nonassociative noncommutative linear logic. This notably included a normal form of nonassociative structures that serve as proof nets.

On the multimodal front, Kanovich et al. (2019) extended the noncommutative result of Pentus (1998) to include subexponentials. This adapts Pentus's counter, which is at first surprising, as a vital precondition for Pentus's counter, and these linear conservativity results in general is the lack of nonlinear structural rules like contraction and weakening, which the subexponentials of Kanovich et al. (2019) possess.

Finally, Blaisdell et al. (2023) completes this square, merging the results of **?** and Kanovich et al. (2019) by showing the conservativity of classical nonassociative noncommutative linear logic with subexponentials over its intuitionistic counterpart.

The content of Blaisdell et al. (2023) constitutes the majority of this chapter. The main distinction is that it uniformizes the above proofs by parametrizing the linear structural rules.

3.1. Nonassociative Classical Contexts

We start by defining the one-sided variant of structured sequents.

Definition 4. *A* classical structured sequent *has the form* $\Rightarrow \Gamma$ *where* Γ *is a non-empty structure.*

The rules for the structured system for classical non-associative non-commutative linear logic **?** are depicted in Figure 3.2.

3.2. The Rules

With the classical sequents defined, we present a large class of one-sided sequent rules and then we divide these rules into logical systems in direct analogy with the intuitionistic case. This is followed immediately by a justification and explanation of the curious 'top-level' rules.

Propositional rules

$$\frac{\Rightarrow \Gamma, B \quad \Rightarrow \Delta, A}{\Rightarrow ((\Gamma, \Delta), A \otimes B)} \otimes \qquad \frac{\Gamma\{(A, B)\}}{\Rightarrow \Gamma\{A \parr B\}} \parr$$

$$\frac{\Rightarrow \Gamma\{A_i\}}{\Rightarrow \Gamma\{A_1 \oplus A_2\}} \oplus_i \qquad \frac{\Rightarrow \Gamma\{A_1\} \quad \Rightarrow \Gamma\{A_2\}}{\Rightarrow \Gamma\{A_1 \,\&\, A_2\}} \,\&$$

$$\frac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \Gamma\{\bot\}} \bot \qquad \frac{}{\Rightarrow 1} 1 \qquad \frac{}{\Rightarrow \Gamma\{\top\}} \top$$

Structural Rules

$$\frac{\Rightarrow \Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\}}{\Rightarrow \Gamma\{((\Delta_1, \Delta_2), \Delta_3)\}} \mathsf{A1} \qquad \frac{\Rightarrow \Gamma\{((\Delta_1, \Delta_2), \Delta_3)\}}{\Rightarrow \Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\}} \mathsf{A2}$$

$$\frac{\Rightarrow \Gamma\{(\Delta_2, \Delta_1)\}}{\Rightarrow \Gamma\{(\Delta_1, \Delta_2)\}} \mathsf{E} \qquad \frac{\Rightarrow \Gamma\{(\Delta, \Delta)\}}{\Rightarrow \Gamma\{\Delta\}} \mathsf{C} \qquad \frac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \Gamma\{\Delta\}} \mathsf{W}$$

Top-Level Structural Rules

$$\frac{\Rightarrow (\Delta, \Gamma)}{\Rightarrow (\Gamma, \Delta)} \mathsf{E}_t \qquad \frac{\Rightarrow (\Gamma, (\Delta, \Pi))}{\Rightarrow ((\Gamma, \Delta), \Pi)} \mathsf{A1}_t \qquad \frac{\Rightarrow ((\Gamma, \Delta), \Pi)}{\Rightarrow (\Gamma, (\Delta, \Pi))} \mathsf{A2}_t$$

Initial and cut Rules

$$\frac{}{\Rightarrow (A, A^\perp)} \mathsf{init} \qquad \frac{\Rightarrow (\Gamma, A) \quad \Rightarrow (A^\perp, \Delta)}{\Rightarrow (\Gamma, \Delta)} \mathsf{cut}$$

Figure 3.1: Propositional, structural, initial, and cut rules for classical multimodal substructural logic.

Subexponential Modal Rules

$$\frac{\Rightarrow \Gamma\{?^{\succeq i} A\}}{\Rightarrow \Gamma\{?^i A\}}\ ?^i \qquad \frac{\Rightarrow \Gamma\{A\}}{\Rightarrow \Gamma\{?^i A\}}\ ?^i\mathsf{T}$$

$$\frac{\Rightarrow (A, C)}{\Rightarrow (?^i A, !^i C)}\ !^i \qquad \frac{\Rightarrow (\Gamma, C)}{\Rightarrow (?^i \Gamma, !^i C)}\ !^i\mathsf{K}$$

$$\frac{\Rightarrow (?^i A, C)}{\Rightarrow (?^i A, !^i C)}\ !^i 4 \qquad \frac{\Rightarrow (?^{i*} \Gamma, C)}{\Rightarrow (?^i \Gamma, !^i C)}\ !^i\mathsf{K4}$$

Subexponential Structural Rules

$$\frac{\Rightarrow ((\Delta_1, \Delta_2), ?^i A)}{\Rightarrow ((\Delta_2, \Delta_1), ?^i A)}\ ?^i\mathsf{E} \qquad \frac{\Rightarrow ((\Delta_2, \Delta_1), ?^i \Gamma)}{\Rightarrow ((\Delta_1, \Delta_2), ?^i \Gamma)}\ ?^i\mathsf{EK}$$

$$\frac{\Rightarrow (((\Delta_1, \Delta_2), \Delta_3), ?^i A)}{\Rightarrow ((\Delta_1, (\Delta_2, \Delta_3)), ?^i A)}\ ?^i\mathsf{A1} \qquad \frac{\Rightarrow (((\Delta_1, \Delta_2), \Delta_3), ?^i \Gamma)}{\Rightarrow ((\Delta_1, (\Delta_2, \Delta_3)), ?^i \Gamma)}\ ?^i\mathsf{A1K}$$

$$\frac{\Rightarrow ((\Delta_1, (\Delta_2, \Delta_3)), ?^i A)}{\Rightarrow (((\Delta_1, \Delta_2), \Delta_3), ?^i A)}\ ?^i\mathsf{A2} \qquad \frac{\Rightarrow ((\Delta_1, (\Delta_2, \Delta_3)), ?^i \Gamma)}{\Rightarrow (((\Delta_1, \Delta_2), \Delta_3), ?^i \Gamma)}\ ?^i\mathsf{A2K}$$

$$\frac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \Gamma\{?^w A\}}\ ?^i\mathsf{W} \qquad \frac{\Rightarrow \Gamma\{?^i \Delta\}\{?^i \Delta\}}{\Rightarrow \Gamma\{?^i \Delta\}\{\,\}}\ ?^i\mathsf{nC}$$

Figure 3.2: Subexponential rules for classical multimodal substructural logic.

3.2.1. The Systems

Definition 5. *The* classical multimodal substructural logic associated to a multimodal substructural specification $\Sigma = (I, \preceq, m, s, S)$ *written* MSL_Σ, *is the logic on classical structured sequents with the following rules.*

- *All propositional rules and* (init),
- *The rule* $(\mathcal{A})$ *for* $\mathcal{A} \in \{\mathsf{A1}, \mathsf{A2}, \mathsf{E}, \mathsf{W}, \mathsf{C}, \mathsf{nC}\}$ *if* $\mathcal{A} \in S$,
- *The rule* $(?^i\mathcal{A})$ *for* $i \in I$ *if* $\mathcal{A} \in s(i)$,
- *The rule* $(?^i\mathcal{A}\mathsf{K})$ *for* $i \in I$ *if* $\mathcal{A} \in s(i)$, $\mathsf{K} \in m(i)$, *and such a rule exists,*
- *The rule* $(?^i)$ *for* $i \in I$,

- *The rule* $(?^i L\mathsf{T})$ *if* $\mathsf{T} \in m(i)$ *for* $i \in I$,

- *The rule* $(!^i)$ *if* $\mathsf{K} \notin m(i)$ *for* $i \in I$,

- *The rule* $(!^i 4)$ *if* $\mathsf{K} \notin m(i), 4 \in m(i)$ *for* $i \in I$,

- *The rule* $(!^i \mathsf{K})$ *if* $\mathsf{K} \in m(i), 4 \notin m(i)$ *for* $i \in I$, *and*

- *The rule* $(!^i \mathsf{K}4)$ *if* $\mathsf{K}, 4 \in m(i)$ *for* $i \in I$.

We again expand upon this definition and describe the relationship between the modal axioms and subexponential rules.

For each label i, the rule $(?^i)$ is always included as this determines the relationship between the subexponentials of differing labels.

The rule $(?^i L\mathsf{T})$ is included for a label i if and only if that label has $\mathsf{T} \in m(i)$.

The included $?^i$ rule(s) depends on the inclusion or exclusion of both K and 4.

	$\mathsf{K} \notin m(i)$	$\mathsf{K} \in m(i)$
$4 \notin m(i)$	$(?^i)$	$(?^i \mathsf{K})$
$4 \in m(i)$	$(?^i), (?^i 4)$	$(?^i \mathsf{K}4)$

3.2.2. Top-Level Structural Rules

The structural rules of MSL_Σ make structures very flexible. By considering more sophisticated syntax, we can represent classes of equivalent structures by a unique normal form. In the associative case, this normal form is a circular ordering of the a priori linearly ordered formulas Yetter (1990). The non-associative case is more complex. In **?**, they represent (non-empty) structures as acyclic graphs (unrooted trees) where each vertex has degree 1 or 3. Vertices of degree 1 are leaves labelled by formulae and vertices of degree 3 are inner nodes. Further, the cyclic ordering of every node's neighbors is maintained.

Such graphs with cyclic order on inner nodes are called *unrooted cyclically-ordered-neigbor 3-regular trees with leaves* Groote and Lamarche (2002).

These objects are syntactically cumbersome, so we choose not to use them in the system or the proofs, though it should be noted that this 'normal form' of classical structures is an incredible tool for intuition, and is the underlying basis for most of the proofs below.

3.3. The Designator

These top-level rules give the structures of the sequents interesting dynamics. In our proofs we follow the 'syntactic' handling of classical sequents adapting Buszkowski (2016).

Although there are many structures equivalent to any given one, designated a subtree and a location in the tree yields a unique representation. In this thesis, we opt to give all proofs using sequents rather than cyclically ordered acyclic graphs and choose the first right branch of the structure as the designated location.

This is possible because of the following definitions and technical lemmas.

Definition 6. *We define* structural equivalence*, written* $\sim$*, between two structures to be the reflexive, symmetric, transitive closure of*

$$(\Gamma, \Delta) \sim (\Delta, \Gamma)$$

$$(\Gamma, (\Delta, \Pi)) \sim ((\Gamma, \Delta), \Pi)$$

Here, $\Theta \sim \Xi$ *if and only if* Ξ *can be transformed to* Θ *using only the structural rules* $(\mathsf{E}_t), (\mathsf{A1}_t)$, *and* $(\mathsf{A2}_t)$.

Note that for the tree representation in **?**, this corresponds to choosing a particular edge in the graph. We formalize this by defining the following.

Definition 7. *Following a construction in Buszkowski (2016), or a context with a hole* $\Gamma\{\,\}$*, we define the* designated *structure* $\widetilde{\Gamma\{*\}}$ *inductively by the following:*

$$\widetilde{(\Gamma\{*\},\Delta)} :\equiv \widetilde{(\Delta,\Gamma\{*\})}$$
$$\widetilde{(\Gamma,(\Delta,\Pi\{*\}))} :\equiv \widetilde{((\Gamma,\Delta),\Pi\{*\})}$$
$$\widetilde{(\Gamma,(\Delta\{*\},\Pi))} :\equiv \widetilde{((\Pi,\Gamma),\Delta\{*\})}$$
$$\widetilde{(\Gamma,*)} :\equiv \Gamma$$

To ensure that this definition is complete, we also specify the empty case $\widetilde{*} :\equiv \cdot$. *We call the overline in the notation* $\widetilde{\Gamma\{*\}}$ *the* designator.

Observe that the designator is well defined. Indeed, first note that the left hand sides are cumulatively exhaustive. If $*$ is on the left, it is handled by the first case. If it is on the right, it is either immediately to the right or it is on the right side's left or right branch. These are handled by the fourth, second, and third cases respectively. Note also that the left hand sides are mutually exclusive. Finally, note that recursive application of this definition terminates, because each case places $*$ on the right branch and the depth of $*$ on the right decreases in every subsequent case.

The following lemma shows that this definition indeed gives us an equivalent structure that puts the designated subtree on the right.

Lemma 3.3.1 (Correctness of Designator)**.** *For any structure* $\Theta\{\Xi\}$ *with distinguished subtree, we have*

$$(\widetilde{\Theta\{*\}},\Xi) \sim \Theta\{\Xi\}.$$

Proof. We prove this by induction on the depth of Ξ. We consider $\Theta\{\,\}$ casewise.

If $\Theta\{\,\} \equiv (\Gamma\{\,\},\Delta)$, then using the induction hypothesis we have

$$
\begin{aligned}
(\widetilde{(\Gamma\{*\},\Delta)},\Xi) &:\equiv (\widetilde{(\Delta,\Gamma\{*\})},\Xi)\\
&\sim (\Delta,\Gamma\{\Xi\})\\
&\sim (\Gamma\{\Xi\},\Delta)
\end{aligned}
$$

Further, if $\Theta\{\,\} \equiv (\Gamma,(\Delta,\Pi\{\,\}))$, then

$$
\begin{aligned}
(\widetilde{(\Gamma,(\Delta,\Pi\{*\}))},\Xi) &:\equiv (\widetilde{((\Gamma,\Delta),\Pi\{*\})},\Xi)\\
&\sim ((\Gamma,\Delta),\Pi\{\Xi\})\\
&\sim (\Gamma,(\Delta,\Pi\{\Xi\}))
\end{aligned}
$$

Most interestingly, if $\Theta\{\,\} \equiv (\Gamma,(\Delta\{\,\},\Pi))$, then

$$
\begin{aligned}
(\widetilde{(\Gamma,(\Delta\{*\},\Pi))},\Xi) &:\equiv (\widetilde{((\Pi,\Gamma),\Delta\{*\})},\Xi)\\
&\sim ((\Pi,\Gamma),\Delta\{\Xi\})\\
&\sim (\Pi,(\Gamma,\Delta\{\Xi\}))\\
&\sim ((\Gamma,\Delta\{\Xi\}),\Pi)\\
&\sim (\Gamma,(\Delta\{\Xi\},\Pi))
\end{aligned}
$$

Finally, as the base case, if $\Theta\{*\} :\equiv (\Gamma,*)$, then

$$\widetilde{((\Gamma, *), \Xi)} :\equiv (\Gamma, \Xi)$$

Also note that this holds in the empty case. □

While this correctness lemma says that we can designate any substructure as the one that should appear on the right, the following says that this happens uniquely.

Lemma 3.3.2. *If* $\Theta\{*\} \sim \Xi\{*\}$*, then* $\widetilde{\Theta\{*\}} \equiv \widetilde{\Xi\{*\}}$.

Proof. It is sufficient to prove this when $\sim$ is a single forward step, as $\equiv$ is reflexive, symmetric, and transitive. We consider all possible single structural steps casewise.

First, consider exchange, i.e. $\Theta\{*\} = (\Gamma\{*\}, \Delta)$ and $\Xi\{*\} :\equiv (\Delta, \Gamma\{*\})$ or vice-versa. Then,

$$\widetilde{(\Gamma\{*\}, \Delta)} :\equiv \widetilde{(\Delta, \Gamma\{*\})}$$

Associativity requires more cases. We need to consider three places $*$ could appear.

First, if $*$ is on the left, i.e. $\Theta\{*\} = (\Gamma\{*\}, (\Delta, \Pi))$ and $\Xi\{*\} \equiv ((\Gamma\{*\}, \Delta), \Pi)$ then

$$\begin{aligned}
\widetilde{(\Gamma\{*\}, (\Delta, \Pi))} &:\equiv \widetilde{((\Delta, \Pi), \Gamma\{*\})} \\
&\equiv: \widetilde{(\Pi, (\Gamma\{*\}, \Delta))} \\
&\equiv: \widetilde{((\Gamma\{*\}, \Delta), \Pi)}
\end{aligned}$$

If $*$ appears in the middle, then

$$\widetilde{(\Gamma,(\Delta\{*\},\Pi))} :\equiv \widetilde{((\Pi,\Gamma),\Delta\{*\})}$$
$$\equiv: \widetilde{(\Pi,(\Gamma,\Delta\{*\}))}$$
$$\equiv: \widetilde{((\Gamma,\Delta\{*\}),\Pi)}$$

Finally, if $*$ appears rightmost, then

$$\widetilde{(\Gamma,(\Delta,\Pi\{*\}))} :\equiv \widetilde{((\Gamma,\Delta),\Pi\{*\})}$$

□

Corollary 3.3.1 (Uniqueness)**.** *If* $(\Gamma,\Pi) \sim (\Delta,\Pi)$ *both contain a distinguished occurrence of* Π*, then* $\Gamma \equiv \Delta$ *(and thus* $(\Gamma,\Pi) \equiv (\Delta,\Pi)$ *as well).*

Proof. Since the occurrence of Π is distinguished, this tells us that $(\Gamma,*) \sim (\Delta,*)$. Therefore, by the preceding lemma,

$$\Gamma \equiv: \widetilde{(\Gamma,*)} \equiv \widetilde{(\Delta,*)} :\equiv \Delta,$$

proving the claim. □

Finally we present another helpful technical lemma which's proof is a straightforward induction on the definition of the designator.

Lemma 3.3.3 (Independent Substructure Preservation)**.** *If* Δ *is a substructure of* $\Gamma\{\Delta\}\{*\}$ *(that does not contain* $*$*), then* Δ *is a substructure of* $\widetilde{\Gamma\{\Delta\}\{*\}}$*, and further replacing* Δ *by* Π *in* $\widetilde{\Gamma\{\Delta\}\{*\}}$

yields $\widetilde{\Gamma\{\Pi\}\{*\}}$.

Finally, from now on derivations will be considered *modulo* designator, in the sense that the operations for determining the designator are not really performed: they should be seen simply as a handy representation of formulas, and not as syntactic manipulations over them. Under this view, we write

$$\frac{\Rightarrow \Delta}{\Rightarrow \Gamma} \sim \quad \text{for} \quad \Gamma \sim \Delta.$$

only as a convenient representation, not a formal inference rule.

3.4. Cut Admissibility

Theorem 3.4.1. *If sequent is provable in* $\mathsf{MSL}_\Sigma + (\mathsf{cut})$ *with* $\mathsf{C} \notin S$*, then it is provable in* MSL_Σ*.*

Proof. We simultaneously eliminate (cut) and (mix)

$$\frac{\Rightarrow (\Gamma, !^c A^\perp) \quad \Rightarrow (?^c A, \Delta\{?^c A\} \cdots \{?^c A\})}{\Rightarrow (\Gamma, \Delta\{\,\} \cdots \{\,\})} \ \mathsf{mix}$$

where $\mathsf{nC} \in s(c)$.

We proceed by nested induction on the complexity of the cut formula and then on the depth of the (cut) or (mix).

In the case of (cut), we consider case-wise the first non-structural rules above (cut).

We first consider if the principal formula of this rule on one of the two premises is not the cut formula. By symmetry we consider this only on the left.

If this rule is $(\otimes)$, then we consider

$$\frac{\dfrac{\Rightarrow ((\widetilde{\Gamma\{*\}, C}), A \otimes B)}{\Rightarrow (\Gamma\{A \otimes B\}, C)} \sim \qquad \Rightarrow (C^\perp, \Delta)}{\Rightarrow (\Gamma\{A \otimes B\}, \Delta)} \ \mathsf{cut}$$

We set $(\Pi, \Xi) :\equiv \widetilde{(\Gamma\{*\}, C)}$, and consider two subcases depending on which side contains C. By the independent substructure preservation lemma we have the following for these cases.

$$
\dfrac{\dfrac{\dfrac{\Rightarrow (\Xi, A) \quad \Rightarrow (\Pi\{C\}, B)}{\Rightarrow ((\Pi\{C\}, \Xi), A \otimes B) \equiv \Rightarrow ((\Gamma\{*\}, C), A \otimes B)}\otimes}{\Rightarrow (\Gamma\{A \otimes B\}, C)}\sim \qquad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\Gamma\{A \otimes B\}, \Delta)}\mathsf{cut} \qquad \rightsquigarrow
$$

$$
\dfrac{\dfrac{\Rightarrow (\Xi, A) \qquad \dfrac{\dfrac{\dfrac{\Rightarrow (\Pi\{C\}, B)}{\Rightarrow (\widetilde{(\Pi\{*\}, B)}, C)}\sim \qquad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{(\Pi\{*\}, B)}, \Delta)}\mathsf{cut}}{\Rightarrow (\Pi\{\Delta\}, B)}\sim}{\Rightarrow ((\Pi\{\Delta\}, \Xi), A \otimes B) \equiv \Rightarrow ((\Gamma\{*\}, \Delta), A \otimes B)}\otimes}{\Rightarrow (\Gamma\{A \otimes B\}, \Delta)}\sim
$$

$$
\dfrac{\dfrac{\dfrac{\Rightarrow (\Xi\{C\}, A) \quad \Rightarrow (\Pi, B)}{\Rightarrow ((\Pi, \Xi\{C\}), A \otimes B) \equiv \Rightarrow ((\Gamma\{*\}, C), A \otimes B)}\otimes}{\Rightarrow (\Gamma\{A \otimes B\}, C)}\sim \qquad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\Gamma\{A \otimes B\}, \Delta)}\mathsf{cut} \qquad \rightsquigarrow
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow (\Pi\{C\}, A)}{\Rightarrow (\widetilde{(\Xi\{*\}, A)}, C)}\sim \qquad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{(\Xi\{*\}, A)}, \Delta)}\mathsf{cut}}{\Rightarrow (\Xi\{\Delta\}, A)}\sim \qquad \Rightarrow (\Pi, B)}{\Rightarrow ((\Pi, \Xi\{\Delta\}), A \otimes B) \equiv \Rightarrow ((\Gamma\{*\}, \Delta), A \otimes B)}\otimes}{\Rightarrow (\Gamma\{A \otimes B\}, \Delta)}\sim
$$

More simply, if the first nonstructural rule is ($\&$), then we have

$$
\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{A\}\{C\} \quad \Rightarrow \Gamma\{B\}\{C\}}{\Rightarrow \Gamma\{A \,\&\, B\}\{C\}}\,\&}{\Rightarrow (\widetilde{\Gamma\{A \,\&\, B\}\{*\}}, C)}\,\sim \qquad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{\Gamma\{A \,\&\, B\}\{*\}}, \Delta)}\,\mathsf{cut} \qquad \rightsquigarrow
$$

$$
\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{A\}\{C\}}{\Rightarrow (\widetilde{\Gamma\{A\}\{*\}}, C)}\,\sim \quad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{\Gamma\{A\}\{*\}}, \Delta)}\,\mathsf{cut} \qquad \dfrac{\dfrac{\Rightarrow \Gamma\{B\}\{C\}}{\Rightarrow (\widetilde{\Gamma\{B\}\{*\}}, C)}\,\sim \quad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{\Gamma\{B\}\{*\}}, \Delta)}\,\mathsf{cut}}{\Rightarrow (\widetilde{\Gamma\{A \,\&\, B\}\{*\}}, \Delta)}\,\&
$$

For the one-premise rules $(\parr), (\oplus_i), (\perp), (?^i), (?^i\mathsf{T})$ and the subexponential structural rules (applied independently of the cut formula), we have

$$
\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{\Pi'\}\{C\}}{\Rightarrow \Gamma\{\Pi\}\{C\}}\,\mathsf{R}}{\Rightarrow (\widetilde{\Gamma\{\Pi\}\{*\}}, C)}\,\sim \quad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{\Gamma\{\Pi\}\{*\}}, \Delta)}\,\mathsf{cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{\Pi'\}\{C\}}{\Rightarrow (\widetilde{\Gamma\{\Pi'\}\{*\}}, C)}\,\sim \quad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{\Gamma\{\Pi'\}\{*\}}, \Delta)}\,\mathsf{cut}}{\Rightarrow (\widetilde{\Gamma\{\Pi\}\{*\}}, \Delta)}\,\mathsf{R}
$$

For $(\top)$, the (cut) disappears.

$$
\dfrac{\dfrac{\dfrac{}{\Rightarrow \Gamma\{\top\}\{C\}}\,\top}{\Rightarrow (\widetilde{\Gamma\{\top\}\{*\}}, C)}\,\sim \quad \Rightarrow (C^{\perp}, \Delta)}{\Rightarrow (\widetilde{\Gamma\{\top\}\{*\}}, \Delta)}\,\mathsf{cut} \qquad \rightsquigarrow \qquad \dfrac{}{\Rightarrow (\widetilde{\Gamma\{\top\}\{*\}}, \Delta)}\,\top
$$

The rules (1) and $(!^i), (!^i\mathsf{K}), (!^i4)$ and $(!^i\mathsf{K}4)$ cannot be applied nonprincipally.

If (init) is applied above (cut), the (cut) disappears as usual. Up to symmetry we have the following.

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow (A, A^{\perp})}\,\mathsf{init}}{\Rightarrow (A^{\perp}, A)}\,\sim \qquad \Rightarrow (A^{\perp}, \Delta)}{\Rightarrow (A^{\perp}, \Delta)}\,\mathsf{cut} \qquad \rightsquigarrow \quad \Rightarrow (A^{\perp}, \Delta)$$

Thus, all that remains to be checked is if the cut formula is principal on both sides. So we consider casewise each dual pair of connectives.

Firstly, we consider the multiplicative binary connectives.

$$\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{(A, B)\}}{\Rightarrow \Gamma\{A \parr B\}}\,\parr}{\Rightarrow (\widetilde{\Gamma\{*\}}, A \parr B)}\,\sim \qquad \dfrac{\dfrac{\Rightarrow (\Pi, B^{\perp}) \quad \Rightarrow (\Delta, A^{\perp})}{\Rightarrow ((\Delta, \Pi), B^{\perp} \otimes A^{\perp})}\,\otimes}{\Rightarrow (B^{\perp} \otimes A^{\perp}, (\Delta, \Pi))}\,\sim}{\Rightarrow (\widetilde{\Gamma\{*\}}, (\Delta, \Pi))}\,\mathsf{cut} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{(A, B)\}}{\Rightarrow (\widetilde{\Gamma\{*\}}, (A, B))}\,\sim}{\Rightarrow ((B, \widetilde{\Gamma\{*\}}), A)}\,\mathsf{E, A1, E} \qquad \dfrac{\Rightarrow (\Delta, A^{\perp})}{\Rightarrow (A^{\perp}, \Delta)}\,\mathsf{E}}{\Rightarrow ((B, \widetilde{\Gamma\{*\}}), \Delta)}\,\mathsf{cut}}{\Rightarrow ((\widetilde{\Gamma\{*\}}, \Delta), B}\,\mathsf{E, A1} \qquad \dfrac{\Rightarrow (\Pi, B^{\perp})}{\Rightarrow (B^{\perp}, \Pi)}\,\mathsf{E}}{\Rightarrow ((\widetilde{\Gamma\{*\}}, \Delta), \Pi)}\,\mathsf{cut}}{\Rightarrow (\widetilde{\Gamma\{*\}}, (\Delta, \Pi))}\,\mathsf{A1}$$

Next, we have the additive binary connectives.

$$\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{A_i\}}{\Rightarrow \Gamma\{A_1 \oplus A_2\}}\,\oplus_i}{\Rightarrow (\widetilde{\Gamma\{*\}}, A_1 \oplus A_2)}\,\sim \qquad \dfrac{\dfrac{\Rightarrow \Delta\{A_1^{\perp}\} \quad \Rightarrow \Delta\{A_2^{\perp}\}}{\Rightarrow \Delta\{A_1^{\perp} \,\&\, A_2^{\perp}\}}\,\&}{\Rightarrow (A_1^{\perp} \,\&\, A_2^{\perp}, \widetilde{\Delta\{*\}})}\,\sim}{\Rightarrow (\widetilde{\Gamma\{*\}}, \widetilde{\Delta\{*\}})}\,\mathsf{cut} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\Rightarrow \Gamma\{A_i\}}{\Rightarrow (\widetilde{\Gamma\{*\}}, A_i)}\,\sim \qquad \dfrac{\Rightarrow \Delta\{A_i^{\perp}\}}{\Rightarrow (A_i^{\perp}, \widetilde{\Delta\{*\}})}\,\sim}{\Rightarrow (\widetilde{\Gamma\{*\}}, \widetilde{\Delta\{*\}})}\,\mathsf{cut}$$

The multiplicative units are straightforward.

$$
\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \Gamma\{\bot\}}\;\bot}{\Rightarrow (\widetilde{\Gamma\{*\}}, \bot)}\;\sim \qquad \dfrac{\dfrac{}{\Rightarrow 1}\;1}{\Rightarrow 1}\;\sim}{\Rightarrow \widetilde{\Gamma\{*\}}}\;\mathsf{cut} \qquad \rightsquigarrow \qquad \dfrac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \widetilde{\Gamma\{*\}}}\;\sim
$$

There is no rule for 0, so neither 0 nor $\top$ can be the cut formula when both sides are principal.

Thus, all that remains is when the cut formula has a subexponential as its top level connective. By symmetry, say that the cut formula in the left premise has a bang. Thus, the first nonstructural rule on the left is $(!^i), (!^i\mathsf{K}), (!^i 4)$, or $(!^i\mathsf{K}4)$. However, on the right we need to consider the $?^i$ and $!^i$ rules as well as the subexponential structural rules, so we consider these interactions individually.

We start with the all possible interactions between bang rules.

If $m(i)$ includes neither K nor 4.

$$
\dfrac{\dfrac{\dfrac{\Rightarrow (C, A)}{\Rightarrow (?^i C, !^i A)}\;!^i}{\Rightarrow (?^i C, !^i A)}\;\sim \qquad \dfrac{\dfrac{\Rightarrow (A^\bot, B)}{\Rightarrow (?^i A^\bot, !^i B)}\;!^i}{\Rightarrow (?^i A^\bot, !^i B)}\;\sim}{\Rightarrow (?^i C, !^i B)}\;\mathsf{cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Rightarrow (C, A) \quad \Rightarrow (A^\bot, B)}{\Rightarrow (C, B)}\;\mathsf{cut}}{\Rightarrow (?^i C, !^i B)}\;!^i
$$

If $m(i)$ includes 4 but not K, we have four subcases. One is covered by the previous reduction for $(!^i)$, the other three involve at least one $(!^i 4)$.

$$
\dfrac{\dfrac{\dfrac{\Rightarrow (?^i C, A)}{\Rightarrow (?^i C, !^i A)}\;!^i 4}{\Rightarrow (?^i C, !^i A)}\;\sim \qquad \dfrac{\dfrac{\Rightarrow (A^\bot, B)}{\Rightarrow (?^i A^\bot, !^i B)}\;!^i}{\Rightarrow (?^i A^\bot, !^i B)}\;\sim}{\Rightarrow (?^i C, !^i B)}\;\mathsf{cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Rightarrow (?^i C, A) \quad \Rightarrow (A^\bot, B)}{\Rightarrow (?^i C, B)}\;\mathsf{cut}}{\Rightarrow (?^i C, !^i B)}\;!^i 4
$$

$$\dfrac{\dfrac{\dfrac{\Rightarrow (C, A)}{\Rightarrow (?^iC, !^iA)}\, !^i}{\Rightarrow (?^iC, !^iA)}\sim \qquad \dfrac{\dfrac{\Rightarrow (?^iA^\perp, B)}{\Rightarrow (?^iA^\perp, !^iB)}\, !^i4}{\Rightarrow (?^iA^\perp, !^iB)}\sim}{\Rightarrow (?^iC, !^iB)}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Rightarrow (C, A)}{\Rightarrow (?^iC, !^iA^\perp)}\, !^i \qquad \Rightarrow (?^iA^\perp, B)}{\Rightarrow (?^iC, B)}\,\mathsf{cut}}{\Rightarrow (?^iC, !^iB)}\, !^i4$$

$$\dfrac{\dfrac{\dfrac{\Rightarrow (?^iC, A)}{\Rightarrow (?^iC, !^iA)}\, !^i4}{\Rightarrow (?^iC, !^iA)}\sim \qquad \dfrac{\dfrac{\Rightarrow (?^iA^\perp, B)}{\Rightarrow (?^iA^\perp, !^iB)}\, !^i4}{\Rightarrow (?^iA^\perp, !^iB)}\sim}{\Rightarrow (?^iC, !^iB)}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Rightarrow (?^iC, A)}{\Rightarrow (?^iC, !^iA^\perp)}\, !^i4 \qquad \Rightarrow (?^iA^\perp, B)}{\Rightarrow (?^iC, B)}\,\mathsf{cut}}{\Rightarrow (?^iC, !^iB)}\, !^i4$$

If $m(i)$ includes K but not 4 we are back to only one case.

$$\dfrac{\dfrac{\dfrac{\Rightarrow (\Gamma, A)}{\Rightarrow (?^i\Gamma, !^iA)}\, !^i\mathsf{K}}{\Rightarrow (?^i\Gamma, !^iA)}\sim \qquad \dfrac{\dfrac{\Rightarrow (\Delta\{A^\perp\}, B)}{\Rightarrow (?^i\Delta\{?^iA^\perp\}, !^iB)}\, !^i\mathsf{K}}{\Rightarrow (?^iA^\perp, \widetilde{(?^i\Delta\{*\}, !^iB)})}\sim}{\Rightarrow (?^i\Gamma, \widetilde{(?^i\Delta\{*\}, !^iB)})}\,\mathsf{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow (?^i\Gamma, A) \qquad \dfrac{\Rightarrow (\Delta\{A^\perp\}, B)}{\Rightarrow (A^\perp, \widetilde{(\Delta\{*\}, B)})}\sim}{\Rightarrow (\Gamma, \widetilde{(\Delta\{*\}, B)})}\,\mathsf{cut}}{\Rightarrow (\Delta\{\Gamma\}, B)}\sim}{\Rightarrow (?^i\Delta\{?^i\Gamma\}, !^iB)}\, !^i\mathsf{K}}{\Rightarrow (?^i\Gamma, \widetilde{(?^i\Delta\{*\}, !^iB)})}\sim$$

If $m(i)$ includes K and 4.

$$\dfrac{\dfrac{\dfrac{\Rightarrow (?^{i*}\Gamma, A)}{\Rightarrow (?^i\Gamma, !^iA)}\, !^i\mathsf{K}4}{\Rightarrow (?^i\Gamma, !^iA)}\sim \qquad \dfrac{\dfrac{\Rightarrow (?^i\Delta\{?^iA^\perp\}, B)}{\Rightarrow (?^i\Delta\{?^iA^\perp\}, !^iB)}\, !^i\mathsf{K}4}{\Rightarrow (?^iA^\perp, \widetilde{(?^i\Delta\{*\}, !^iB)})}\sim}{\Rightarrow (?^i\Gamma, \widetilde{(\Delta\{*\}, !^iB)})}\,\mathsf{cut} \quad \rightsquigarrow$$

$$
\dfrac{
\dfrac{\dfrac{\Rightarrow (?^i\Gamma, A)}{\Rightarrow (?^i\Gamma, !^i A)}\;!^i\mathsf{K4}}{\Rightarrow (?^i\Gamma, !^i A)}\;\sim
\qquad
\dfrac{\Rightarrow (?^i\Delta\{?^i A^\perp\}, B)}{\Rightarrow (?^i A^\perp, \widetilde{(?^i\Delta\{*\}, B)})}\;\sim
}{
\dfrac{\dfrac{\dfrac{\Rightarrow (?^i\Gamma, \widetilde{(?^i\Delta\{*\}, B)})}{\Rightarrow (?^i\Delta\{?^i\Gamma\}, B)}\;\sim}{\Rightarrow (?^i\Delta\{?^i\Gamma\}, !^i B)}\;!^i\mathsf{K4}}{\Rightarrow (?^i\Gamma, \widetilde{(?^i\Delta\{*\}, !^i B)})}\;\sim
}\;\mathsf{cut}
$$

This concludes the interaction of all $!^i$ rules, so we now consider the interaction of a principal $!^i$ rule and a $?^i$ rule, giving as an example the case where $m(i) \supseteq \{\mathsf{K}, 4\}$.

$$
\dfrac{
\dfrac{\dfrac{\Rightarrow (?^{i*}\Gamma, A)}{\Rightarrow (?^i\Gamma, !^i A)}\;!^i\mathsf{K4}}{\Rightarrow (?^i\Gamma, !^i A)}\;\sim
\qquad
\dfrac{\dfrac{\Rightarrow (?^i\Delta\{?^i A^\perp\}, B)}{\Rightarrow (?^i\Delta\{?^i A^\perp\}, !^i B)}\;!^i\mathsf{K4}}{\Rightarrow (?^i A^\perp, \widetilde{(?^i\Delta\{*\}, !^i B)})}\;\sim
}{
\Rightarrow (?^i\Gamma, \widetilde{(\Delta\{*\}, !^i B)})
}\;\mathsf{cut}
\qquad \rightsquigarrow
$$

$$
\dfrac{
\dfrac{\dfrac{\Rightarrow (?^i\Gamma, A)}{\Rightarrow (?^i\Gamma, !^i A)}\;!^i}{\Rightarrow (?^i\Gamma, !^i A)}\;\sim
\qquad
\dfrac{\Rightarrow (?^i\Delta\{?^i A^\perp\}, B)}{\Rightarrow (?^i A^\perp, \widetilde{(?^i\Delta\{*\}, B)})}\;\sim
}{
\dfrac{\dfrac{\dfrac{\Rightarrow (?^i\Gamma, \widetilde{(?^i\Delta\{*\}, B)})}{\Rightarrow (?^i\Delta\{?^i\Gamma\}, B)}\;\sim}{\Rightarrow (?^i\Delta\{?^i\Gamma\}, !^i B)}\;!^i\mathsf{K4}}{\Rightarrow (?^i\Gamma, \widetilde{(?^i\Delta\{*\}, !^i B)})}\;\sim
}\;\mathsf{cut}
$$

In the case of a contraction subexponential rule, (cut) becomes (mix).

$$
\dfrac{
\dfrac{\dfrac{\Rightarrow (?^{c*}\Gamma, A)}{\Rightarrow (?^c\Gamma, !^c A)}\;!^i\mathsf{K4}}{\Rightarrow (?^c\Gamma, !^c A)}\;\sim
\qquad
\dfrac{\dfrac{\Rightarrow \Delta\{?^c\Pi\{?^i A^\perp\}\}\{?^c\Pi\{?^i A^\perp\}\}}{\Rightarrow \Delta\{?^c\Pi\{?^i A^\perp\}\}\{\,\}}\;\mathsf{nCK}}{\Rightarrow (?^i A^\perp, ?^c\Pi\{*\})}\;\sim
}{
\Rightarrow (?^c\Gamma, \widetilde{\Delta\{?^c\Pi\{*\}\}}\{\,\})
}\;\mathsf{cut}
\qquad \rightsquigarrow
$$

$$\dfrac{\dfrac{\Rightarrow (?^{c*}\Gamma, A)}{\Rightarrow (?^{c}\Gamma, !^{i}A)}\ !^{i} \quad \Rightarrow \Delta\{?^{c}\Pi\{?^{i}A^{\perp}\}\}\{?^{c}\Pi\{?^{i}A^{\perp}\}\}}{\dfrac{\Rightarrow (?^{c}\Gamma, \widehat{\Delta\{?^{c}\Pi\{*\}\}\{?^{c}\Pi\{\,\}\}})}{\Rightarrow (?^{c}\Gamma, \widehat{\Delta\{?^{c}\Pi\{*\}\}\{\,\}})}\ \mathsf{nCK}}\ \mathsf{mix}$$

The case of (mix) elimination is similar to that of (cut). □

3.5. Embedding

In this section, we will show an embedding of the intuitionistic system MSL_Σ into MSL_Σ. This embedding is standard

Consider the translation $\widehat{\cdot}$ on formulas defined below.

$$\begin{array}{rcl@{\qquad}rcl}
\widehat{p} & :\equiv & p & \widehat{A \otimes B} & :\equiv & \widehat{A} \otimes \widehat{B} \\
\widehat{A \to B} & :\equiv & \widehat{A}^{\perp} \parr \widehat{B} & \widehat{B \leftarrow A} & :\equiv & \widehat{B} \parr \widehat{A}^{\perp} \\
\widehat{A \,\&\, B} & :\equiv & \widehat{A} \,\&\, \widehat{B} & \widehat{A \oplus B} & :\equiv & \widehat{A} \oplus \widehat{B} \\
\widehat{!^{i}A} & :\equiv & !^{i}\widehat{A} & \widehat{1} & :\equiv & 1 \\
\widehat{\top} & :\equiv & \top & & &
\end{array}$$

This translation is extended this to structures by the following:

$$\widehat{(\Gamma, \Delta)}^{\perp} :\equiv (\widehat{\Delta}^{\perp}, \widehat{\Gamma}^{\perp})$$

Note both that the order is reversed by the tight negation and also that we will only every need the negative translation for structures.

We will show this embedding is faithful if no subexponentials license associativity.

Theorem 3.5.1. *If in the specification* Σ *we have* $S \subseteq \{\mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$ *and for all labels* i *we have* $s(i) \subseteq \{\mathsf{nC}, \mathsf{C}, \mathsf{W}, \mathsf{E}\}$*, then a* MSL_Σ *sequent* $\Gamma \Rightarrow A$ *is provable iff* $\Rightarrow (\widehat{\Gamma}^{\perp}, \widehat{A})$ *is provable in* MSL_Σ*.*

We start with the easier direction, showing that this embedding is sound. This embedding is sound even with the inclusion of associativity.

Lemma 3.5.1 (Soundness)**.** *If an* MSL_Σ *sequent* $\Gamma \Rightarrow A$ *is provable, then* $\Rightarrow (\widehat{\Gamma}^{\perp}, \widehat{A})$ *is provable in* MSL_Σ.

Proof. We prove this directly be induction on proofs by showing that the translations of each MSL_Σ rule is a valid MSL_Σ partial proof. Consider the bottom rule of a proof. We will show some key cases.

$$\dfrac{\Delta \Rightarrow A \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{(B \leftarrow A, \Delta)\} \Rightarrow C} \leftarrow L \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Rightarrow (\widehat{\Delta}^{\perp}, \widehat{A}) \quad \dfrac{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{B}^{\perp}\right\}, C)}{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, C)}, \widehat{B}^{\perp})} \sim}{\Rightarrow ((\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, C)}, \widehat{\Delta}^{\perp}), \widehat{A} \otimes \widehat{B}^{\perp})} \otimes}{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, C)}, (\widehat{\Delta}^{\perp}, \widehat{A} \otimes \widehat{B}^{\perp}))} \mathsf{A1}}{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{(\widehat{\Delta}^{\perp}, \widehat{A} \otimes \widehat{B}^{\perp})\right\}, C)} \sim$$

$$\dfrac{\Gamma\{(\Pi, !^{e}\Delta)\} \Rightarrow C}{\Gamma\{(!^{e}\Delta, \Pi)\} \Rightarrow C} \mathsf{E1} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{(?^{e}\widehat{\Delta}^{\perp}, \widehat{\Pi}^{\perp})\right\}, \widehat{C})}{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, (?^{e}\widehat{\Delta}^{\perp}, \widehat{\Pi}^{\perp}))} \sim}{\Rightarrow ((\widehat{\Pi}^{\perp}, \widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}), ?^{e}\widehat{\Delta}^{\perp})} \mathsf{A1}, \mathsf{E}, \mathsf{A1}}{\Rightarrow ((\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{\Pi}^{\perp}), ?^{e}\widehat{\Delta}^{\perp})} ?\mathsf{E}}{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, (\widehat{\Pi}^{\perp}, ?^{e}\widehat{\Delta}^{\perp}))} \mathsf{A2}}{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{(\widehat{\Pi}^{\perp}, ?^{e}\widehat{\Delta}^{\perp})\right\}, \widehat{C})} \sim$$

□

We now prove the more surprising direction, that the embedding of the intuitionistic system into the classical system is complete.

We start by proposing a counter on formulas, which is an extension of the counter defined in Kanovich et al. (2017), in its turn an extension of the counter in Pentus (1998).

Definition 8. *For a* MSL_Σ *formula* A*, we define the integer number* $\natural(A)$ *by induction as follows.*

$$
\begin{array}{rcl@{\qquad}rcl}
\natural(p) & := & 0 & \natural(A \parr B) & := & \natural(A) + \natural(B) - 1 \\
\natural(\bar{p}) & := & 1 & \natural(A \otimes B) & := & \natural(A) + \natural(B) \\
\natural(1) & := & 0 & \natural(A \oplus B) = \natural(A \,\&\, B) & := & \natural(A) \\
\natural(\bot) & := & 1 & \natural(?^i A) = \natural(!^i A) & := & \natural(A)
\end{array}
$$

and extend to structures by

$$\natural((\Gamma, \Delta)) := \natural(\Gamma) + \natural(\Delta)$$

We need this counter for the following technical lemmas, which are easily proven by straightforward induction.

Lemma 3.5.2. *For any* MSL_Σ *formula* C*, we have* $\natural(\widehat{C}) = 0$ *and* $\natural(\widehat{C}^\perp) = 1$.

This follows directly by induction on MSL_Σ formulas.

Corollary 3.5.1. *A formula cannot be both of the form* $\widehat{A}$ *and* $\widehat{B}^\perp$.

Lemma 3.5.3. *Let* $\Rightarrow \Gamma$ *be a sequent with* n *formulas where every formula is of the form* $\widehat{C}$ *or* $\widehat{C}^\perp$ *that is provable in* MSL_Σ *with* $S \subseteq \{\mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$*. Then* $\sum_{A \in \Gamma} \natural(A) = n - 1$.

This follows from an easy induction on MSL_Σ proofs, noting that the global structural rules (nC), (C), and (W) are not in the considered MSL_Σ.

Definition 9. *We say that a sequent with exactly one formula of the form* $\widehat{C}$ *and the rest of the form* $\widehat{C}^\perp$ *is* intuitionistically polarizable*. We call the formula of the form* $\widehat{C}$ *the* positive formula.

Lemma 3.5.4 (Intuitionistic Polarization)**.** *If a sequent with all formulas are of the form* $\widehat{C}$ *or* $\widehat{C}^\perp$ *is provable, then exactly one of the formulas is of the form* $\widehat{C}$*, i.e. it is intuitionistically polarizable.*

Proof. Let n be the number of formulas in Γ. Since by previous lemmas $\natural(\widehat{C}) = 0$, $\natural(\widehat{C}^\perp) = 1$, and $\sum_{A \in \Gamma} \natural(A) = n - 1$, there are exactly $n - 1$ formulas of the form $\widehat{C}^\perp$. □

Remark 3.5.1. *Note that any intuitionistically polarizable sequent is structurally equivalent to a unique sequent of the form* $(\widehat{\Gamma}^{\perp}, \widehat{C})$.

We now sketch the proof of completeness.

Lemma 3.5.5 (Completeness)**.** *Let* Σ *be a specification with* $S \subseteq \{\mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$ *where all labels* i *have* $s(i) \subseteq \{\mathsf{C}, \mathsf{W}, \mathsf{E}\}$ *and let* $\Gamma \Rightarrow A$ *be an* MSL_{Σ} *sequent. If* $\Rightarrow (\widehat{\Gamma}^{\perp}, \widehat{A})$ *is provable in* MSL_{Σ}*, then* $\Gamma \Rightarrow A$ *is provable in* MSL_{Σ}.

Proof. We prove the theorem by induction on the length of MSL_{Σ} proofs.

If the first nonstructural rule is $(\otimes)$, we must consider the following subcases.

$$\dfrac{\dfrac{\Rightarrow (\widehat{\Gamma}^{\perp}, \widehat{A}) \quad \Rightarrow (\widehat{\Delta}^{\perp}, \widehat{B})}{\Rightarrow ((\widehat{\Delta}^{\perp}, \widehat{\Gamma}^{\perp}), \widehat{A} \otimes \widehat{B})}\otimes}{\Rightarrow ((\widehat{\Delta}^{\perp}, \widehat{\Gamma}^{\perp}), \widehat{A} \otimes \widehat{B})}\sim \quad \rightsquigarrow \quad \dfrac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\otimes R$$

$$\dfrac{\dfrac{\Rightarrow (\widehat{\Delta}^{\perp}, \widehat{A}) \quad \dfrac{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{B}^{\perp}\right\}, \widehat{C})}{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{B}^{\perp})}\sim}{\Rightarrow ((\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{\Delta}^{\perp}), \widehat{A} \otimes \widehat{B}^{\perp})}\otimes}{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{\Delta}^{\perp}, \widehat{A} \otimes \widehat{B}^{\perp}\right\}, \widehat{C})}\sim \quad \rightsquigarrow \quad \dfrac{\Delta \Rightarrow A \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{B \leftarrow A, \Delta\} \Rightarrow C}\leftarrow L$$

$$\dfrac{\dfrac{\dfrac{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{B}^{\perp}\right\}, \widehat{C})}{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{B}^{\perp})}\sim \quad \Rightarrow (\widehat{\Delta}^{\perp}, \widehat{A})}{\Rightarrow ((\widehat{\Delta}^{\perp}, \widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}), \widehat{B}^{\perp} \otimes \widehat{A})}\otimes}{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{B}^{\perp} \otimes \widehat{A}, \widehat{\Delta}^{\perp}\right\}, \widehat{C})}\sim \quad \rightsquigarrow \quad \dfrac{\Delta \Rightarrow A \quad \Gamma\{B\} \Rightarrow C}{\Gamma\{\Delta, A \to B\} \Rightarrow C}\leftarrow L$$

There are two remaining ways that $\otimes$ can appear in the translation of a sequent and be principal.

$$\dfrac{\dfrac{\Rightarrow (\widehat{\Delta}^{\perp}, \widehat{B}^{\perp}) \quad \Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{A})}{\Rightarrow ((\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{\Delta}^{\perp}), \widehat{B}^{\perp} \otimes \widehat{A})}\otimes}{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{\Delta}^{\perp}, \widehat{B}^{\perp} \otimes \widehat{A}\right\}, \widehat{C})}\sim \qquad \dfrac{\dfrac{\Rightarrow (\widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}, \widehat{A}) \quad \Rightarrow (\widehat{\Delta}^{\perp}, \widehat{B}^{\perp})}{\Rightarrow ((\widehat{\Delta}^{\perp}, \widetilde{(\widehat{\Gamma}^{\perp}\{*\}, \widehat{C})}), \widehat{A} \otimes \widehat{B}^{\perp})}\otimes}{\Rightarrow (\widehat{\Gamma}^{\perp}\left\{\widehat{A} \otimes \widehat{B}^{\perp}, \widehat{\Delta}^{\perp}\right\}, \widehat{C})}\sim$$

However, the premises are not intuitionistically polarizable, and therefore cannot be provable; in other words, these cases are impossible.

Most interestingly we have subexponentially licensed structural rules.The positive formula in the sequent cannot be of the form $?^i A$, and is thus not part of the active substructure of any subexponential structural rules. Hence, by the independent substructure lemma, we can make the following transformations, where $\widetilde{\Gamma\{*\}}^r$ indicates reversing $\Gamma\{*\}$, designating, and reversing back.

$$
\begin{array}{c}
\Rightarrow (\widetilde{\widehat{\Gamma}^\perp\left\{?^c\widehat{\Delta}^\perp\right\}\left\{?^c\widehat{\Delta}^\perp\right\}\{*\}}, \widehat{C}) \sim \\
\dfrac{\Rightarrow (\widehat{\Gamma}^\perp\left\{?^c\widehat{\Delta}^\perp\right\}\left\{?^c\widehat{\Delta}^\perp\right\}\left\{\widehat{C}\right\})}{\Rightarrow (\widehat{\Gamma}^\perp\left\{?^c\widehat{\Delta}^\perp\right\}\{\,\}\left\{\widehat{C}\right\})}\ \mathsf{C} \\
\hline\hline
\Rightarrow (\widetilde{\widehat{\Gamma}^\perp\left\{?^c\widehat{\Delta}^\perp\right\}\{\,\}\{*\}}, \widehat{C})
\end{array}
\sim
\qquad \rightsquigarrow \qquad
\dfrac{\widetilde{\Gamma\{!^c\Delta\}\{!^c\Delta\}\{*\}}^r \Rightarrow C}{\widetilde{\Gamma\{!^c\Delta\}\{*\}}^r \Rightarrow C}\ \mathsf{C}
$$

□

CHAPTER 4

UPPER BOUNDS

We show the decidability of various nonassociative noncommutative logics. In logics without local or global contraction, the standard proof system is already bounded, so we have decidability for free. Therefore, we consider logics with contraction.

In the case of local contraction, we adapt a construction of Groote et al. (2004), itself inspired by an unpublished construction of Kanovich which encodes exponentials as nonlogical axioms. We then appeal to a result of Bulińska (2009) extending a result of Buszkowski (2005) which shows the decidability of deducibility in nonassociative noncommutative linear logic.

We present two main results via this method.

- the PTIME decidability of multiplicative subexponential logics without K whose only subexponential structural rules are C, W, and
- the $\mathsf{EXPTIME}$ decidability of multiplicative subexponential logics with exactly the axioms K, C, and W.

In both cases, we encode provability into deducibility in nonassociative Lambek calculus with unit, $\mathsf{NL1}$, which we define here, without Lambek's restriction, to be the fragment of MSL_Σ where Σ has no subexponential labels, $\mathcal{S} = \emptyset$, and only multplicative connectives are included.

Looking at the list of considered axioms, we note that only the structural $!^i\mathsf{C}$ and $!^i\mathsf{W}$ strictly quantify over formulas appearing under bangs. This will allow us, in two different ways to restrict to a finite number of instances of these axioms, where we then appeal to the result of Bulińska (2009) showing PTIME decidability of deducibility in $\mathsf{NL1}$.

In the case of logics without K, we can use the analyticity of MSL_Σ to directly restrict our attention to instances of $!^i\mathsf{C}$ and $!^i\mathsf{W}$ where $!^iA$ is a subformula of some formula we started with. Intuitively this is because without K, we need only contract or weaken individual formulas. Even without K we still cannot directly encode the other structural subexponential rules as finitely many axioms; consider for example ($!^i\mathsf{E}$), where there is no way to bound the substructures that a single banged formula may have to commute with.

In the case of logics with K, we may need to contract entire substructures. There is no immediate way to limit the number of required instances of $!^i\mathsf{C}$ via analyticity. However, in the addition presence of $!^i\mathsf{W}$, we have that a substructure wrapped entirely in bangs is equiprovable with any substructure containing the same set of formulas. Thus, in this case, we informally require an instance of the axiom $!^i\mathsf{C}$ for each of the finitely many subsets of the set of subformulas of our original sequent.

4.1. Encoding Local Contraction and Weakening as Axioms

Fix some multimodal substructural specification Σ with countably many labels.

We start by encoding finite sets of multiplicative subexponential formulas, and then use this to define axioms equivalent to the subexponentials' sturctural rules.

4.1.1. Encoding Formulas

Let $\mathcal{F}$ be the set of formulas built from denumerably many propositional variables $p_0, p_1, ...$, and the connectives $\otimes, \rightarrow \leftarrow$, and 1.

Similarly, let $!\mathcal{F}$ be the set of formulas built from denumerably many propositional variables $p_0, p_1, ...$, $\otimes, \rightarrow \leftarrow$, and 1 as well as all $!^i$.

Note that $!\mathcal{F}$ is countable. Further, there is an efficient encoding $I \times !\mathcal{F} \hookrightarrow \mathbb{N}$.

Definition 10. *Fix some efficient encoding* $f : I \times !\mathcal{F} \hookrightarrow \mathbb{N}$.

We use this to define the following translation similar to Groote et al. (2004).

$$
\begin{aligned}
p_i^* &:\equiv p_{2i+1} \\
(A \circledast B)^* &:\equiv A^* \circledast B^* \textit{ where } \circledast \in \{\otimes, \rightarrow \leftarrow\} \\
1^* &:\equiv 1 \\
(!^i A)^* &:\equiv p_{2f(i,A)}
\end{aligned}
$$

We extend this to structures via

$$((\Delta, \Pi))^* :\equiv (\Delta^*, \Pi^*)$$

4.1.2. The Axioms

The intention is to find a set of axioms such that deduction from these axioms is equivalent to provability using the subexponential rules. Loosely, each substructural rule corresponds to exactly one axiom schema, and analyticity of MSL_Σ allows us to consider only finitely many instances of a schema at a time.

The only snag is the right subexponential rules. These cannot be neatly represented as a single axiom schema, essentially because the assumptions must be proven in a context completely distinct from the conclusion.

Now we define a set of axioms $\Phi_0(X)$ containing the axiom schemas that can represent instances of rules for formulas appearing in X. Recall that every $(!^i A)^*$ is just the propositional variable $p_{2f(i,A)}$; we choose to write $(!^i A)^*$ below rather than $p_{2f(i,A)}$ for ease of reading.

Definition 11. *We say a structure is distinct if no two of it's formulas are identical.*

$$
\begin{aligned}
\Phi_0(X) :=\{ & (!^i A)^* \Rightarrow A^*, && \text{if } \mathsf{T} \in m(i) \\
& (!^i A)^* \Rightarrow (!^i A)^* \otimes (!^i A)^*, && \text{if } \mathsf{C} \in s(i) \\
& \bigotimes (!^i \Gamma)^* \Rightarrow \bigotimes (!^i \Gamma)^* \otimes \bigotimes (!^i \Gamma)^*, && \text{if } \mathsf{K} \in m(i), \mathsf{C} \in s(i) \\
& (!^i A)^* \Rightarrow 1, && \text{if } \mathsf{W} \in s(i) \\
& (!^j A)^* \Rightarrow (!^i A)^*, \\
& : i \preceq j \in I, !^i A \in X, !^i \Gamma \subseteq X, \Gamma \text{ a distinct structure}\}
\end{aligned}
$$

These axioms cover all but the right subexponential rules. For the right subexponential rules, when $\mathsf{K} \notin m(i)$, the conclusion is $!^i A \Rightarrow !^i C$, other it is $!^i \Gamma \Rightarrow !^i C$. Thus, to complete the set of axioms we add these in as axioms, but only when the premise of the right rule is implied by previous axioms. Specifically, we define $\Phi_n(X)$ inductively as follows:

$$
\begin{aligned}
\Phi_{n+1}(X) := \{ & (!^i A)^* \Rightarrow (!^i C)^* : \\
& \mathsf{NL1} + \Phi_n(X) \vdash A^* \Rightarrow C^* && \text{or} \\
& \mathsf{NL1} + \Phi_n(X) \vdash (!^i A)^* && \text{if } 4 \in m(i) \\
& \text{for } i \in I, A, C \in X\} && \text{if } \mathsf{K} \notin m(i)
\end{aligned}
$$

or

$$
\begin{aligned}
\Phi_{n+1}(X) := \{ & \bigotimes (!^i \Gamma)^* \Rightarrow (!^i C)^* : \\
& \mathsf{NL1} + \Phi_n(X) \vdash \bigotimes (!^i \Gamma)^* \Rightarrow C^* \\
& \text{for } i \in I, \Gamma \subseteq X, C \in X\} && \text{if } \mathsf{K}, 4 \in m(i)
\end{aligned}
$$

Then we close up all of these axioms as such,

$$\Phi_\infty(X) := \bigcup_{n\in\mathbb{N}} \Phi_n(X).$$

This is sufficient to prove the desired soundness and completeness results.

4.1.3. Soundness

We take the subexponential logic MSL_Σ as the ground truth and consider the soundness and completeness of deducibility of the translation with respect to $\mathsf{NL1}$ with nonlogical axioms.

If X is infinite, then $\Phi_\infty(X)$ is infinite, which is undesirable from a complexity standpoint. The intent of adding the set X is to restrict the necessary set of axioms when a sufficient set of instances is known. Thus, X will be more salient when considering completeness.

Theorem 4.1.1 (Soundness)**.** *Let Σ be a specification where $S = \emptyset$ and $s(i) \subseteq \{\mathsf{C}, \mathsf{W}\}$ for all labels i. If $\Gamma \Rightarrow C$ is a sequent containing only $\otimes, \to, \leftarrow, 1$ and $!^i$, then for any X, if*

$$\mathsf{NL1} + \Phi_\infty(X) \vdash \Gamma^* \Rightarrow C^* \quad \textit{then} \quad \mathsf{MSL}_\Sigma \vdash \Gamma \Rightarrow C$$

Proof. Since any proof uses only finitely many axioms, it is sufficient to prove the claim for all $\Phi_n(X)$ rather than $\Phi_\infty(X)$ directly. We do this by induction on n.

All of the non-axiom rules in $\mathsf{NL1} + \Phi_n(X)$ translate directly under $*$, so in each case we simply need to prove the axioms.

In the case of $\Phi_0(X)$ the schemas all follow from straightforward deductions, listed here.

$$\frac{}{(!^i A)^* \Rightarrow A^*}\ \Phi_0(X) \qquad \rightsquigarrow \qquad \dfrac{\dfrac{}{A \Rightarrow A}\ \mathsf{init}}{!^i A \Rightarrow A}\ !^i L\mathsf{T} \qquad \text{if } \mathsf{T} \in m(i)$$

$$\frac{}{(!^i A)^* \Rightarrow (!^i A)^* \otimes (!^i A)^*}\ \Phi_0(X) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{!^i A \Rightarrow\ !^i A}\ \mathsf{init} \qquad \dfrac{}{!^i A \Rightarrow\ !^i A}\ \mathsf{init}}{!^i A, !^i A \Rightarrow\ !^i A \otimes !^i A}\ \otimes R}{!^i A \Rightarrow\ !^i A \otimes !^i A}\ \mathsf{C/CK} \qquad \text{if } \mathsf{C} \in s(i)$$

$$\frac{}{\bigotimes (!^i \Gamma)^* \Rightarrow \bigotimes (!^i \Gamma)^* \otimes \bigotimes (!^i \Gamma)^*}\ \Phi_0(X) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{}{!^i \Gamma \Rightarrow \bigotimes !^i \Gamma}\ \otimes R, \mathsf{init} \qquad \dfrac{}{!^i \Gamma \Rightarrow \bigotimes !^i \Gamma}\ \otimes R, \mathsf{init}}{!^i \Gamma, !^i \Gamma \Rightarrow \bigotimes !^i \Gamma \otimes \bigotimes !^i \Gamma}\ \otimes R}{!^i \Gamma \Rightarrow \bigotimes !^i \Gamma \otimes \bigotimes !^i \Gamma}\ \mathsf{CK}}{\bigotimes !^i \Gamma \Rightarrow \bigotimes !^i \Gamma \otimes \bigotimes !^i \Gamma}\ \otimes L$$

if $\mathsf{K} \in m(i), \mathsf{C} \in s(i)$

$$\frac{}{(!^i A)^* \Rightarrow 1}\ \Phi_0(X) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\Rightarrow 1}\ 1R}{!^i A \Rightarrow 1}\ \mathsf{W} \qquad \text{if } \mathsf{W} \in s(i)$$

$$\frac{}{(!^j A)^* \Rightarrow (!^i A)^*}\ \Phi_0(X) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{!^i A \Rightarrow\ !^i A}\ \mathsf{init}}{!^j A \Rightarrow\ !^i A}\ !^j L \qquad \text{where } i \preceq j$$

In the inductive case, that is $\Phi_{n+1}(X)$, if $\mathsf{K} \notin m(i)$ and $(!^i A)^* \Rightarrow (!^i C)^*$ is added as an axiom, then $\mathsf{NL1} + \Phi_n(X) \vdash A^* \Rightarrow C^*$ or $\mathsf{NL1} + \Phi_n(X) \vdash (!^i A)^* \Rightarrow C^*$ and $4 \in m(i)$. By the inductive hypothesis, this tells us that either $\mathsf{MSL}_\Sigma \vdash A \Rightarrow C$ or $\mathsf{MSL}_\Sigma \vdash\ !^i A \Rightarrow C$ and $4 \in m(i)$. In the first case, we see via

$$\dfrac{A \Rightarrow C}{!^i A \Rightarrow\ !^i C}\ !^i R$$

that $\mathsf{MSL}_\Sigma \vdash\ !^i A \Rightarrow\ !^i C$. In the latter case, we see the same via the $(!^i R4)$ rule:

$$\dfrac{!^i A \Rightarrow C}{!^i A \Rightarrow\ !^i C}\ !^i R4$$

Alternatively, if $\mathsf{K}, 4 \in m(i)$ and $\bigotimes (!^i \Gamma)^* \Rightarrow (!^i C)^*$ is added as an axiom, then $\mathsf{NL1} + \Phi_n(X) \vdash \bigotimes (!^i \Gamma)^* \Rightarrow C^*$. By the inductive hypothesis, we have $\mathsf{MSL}_\Sigma \vdash \bigotimes !^i \Gamma \Rightarrow C$. Thus, we have the MSL_Σ provability of $\bigotimes !^i \Gamma \Rightarrow\ !^i C$ from the deduction

$$
\dfrac{\dfrac{\dfrac{}{!^i\Gamma \Rightarrow \bigotimes !^i\Gamma}\ \otimes R, \mathsf{init} \qquad \bigotimes !^i\Gamma \Rightarrow C}{\dfrac{!^i\Gamma \Rightarrow C}{!^i\Gamma \Rightarrow !^iC}\ !^iR\mathsf{K4}}\ \mathsf{cut}}{\bigotimes !^i\Gamma \Rightarrow !^iC}\ \otimes L
$$

Thus, all of the axioms are provable in MSL_Σ, and all other rules of $\mathsf{NL} + \Phi_\infty(X)$, that is those of $\mathsf{NL1}$ are also rules of MSL_Σ, so we are done. □

4.1.4. Completeness

Completeness is more intricate.

Theorem 4.1.2 (Completeness)**.** *Let* Σ *be a specification where* $S = \emptyset$*,* $s(i) \subseteq \{\mathsf{C}, \mathsf{W}\}$*, and if* $\mathsf{K} \in m(i)$ *then* $m(i) = \{\mathsf{K}, \mathsf{T}, 4\}$ *and* $s(i) = \{\mathsf{C}, \mathsf{W}\}$ *for all labels* i*, and let* $\Gamma \Rightarrow C$ *is a sequent containing only* $\otimes, \rightarrow, \leftarrow, 1,$ *and* $!^i$.

Say X *is a set containing at least the Gentzen subformulas in* $\Gamma \Rightarrow C$ *with labels restricted to the labels in* $\Gamma \Rightarrow C$*. Then if*

$$
\mathsf{MSL}_\Sigma \vdash \Gamma \Rightarrow C \quad \textit{then} \quad \mathsf{NL1} + \Phi_\infty(X) \vdash \Gamma^* \Rightarrow C^*.
$$

Proof. We induct on cut-free analytic MSL_Σ proofs which exist via cut-elimination and subexponential label analyticity.

We consider the last rule casewise, and translate to $\mathsf{NL} + \Phi_\infty(X)$.

All of the non-subexponential rules translate directly. Said differently, all of the propositional rules and the (init) rule that appear in this fragment are also rules in $\mathsf{NL1}$. Thus, it is sufficient to explicitly consider the exponential rules of MSL_Σ.

Each of the subexponential rules can be simulated by appropriate use of cut with the additional nonlogical axioms. We consider these individually.

$$\frac{\Gamma\{!^j A\} \Rightarrow C}{\Gamma\{!^i A\} \Rightarrow C}\ !^i L \quad \rightsquigarrow \quad \frac{\overline{!^i A \Rightarrow !^j A}\ \Phi_\infty(X) \qquad \Gamma\{!^j A\} \Rightarrow C}{\Gamma\{!^i A\} \Rightarrow C}\ \mathsf{cut} \qquad \text{if } j \preceq i$$

$$\frac{\Gamma\{A\} \Rightarrow C}{\Gamma\{!^i A\} \Rightarrow C}\ !^i L\mathsf{T} \quad \rightsquigarrow \quad \frac{\overline{!^i A \Rightarrow !^j A}\ \Phi_\infty(X) \qquad \Gamma\{!^j A\} \Rightarrow C}{\Gamma\{!^i A\} \Rightarrow C}\ \mathsf{cut} \qquad \text{if } \mathsf{T} \in m(i)$$

For the right subexponential rules, the induction hypothesis applied to the assumption gives that $\mathsf{NL1} + \Phi_\infty(X)$ proves the translation, and thus $\mathsf{NL1} + \Phi_n(X)$ proves the translation for some n, by the finiteness of proofs. Thus, there will be an axiom in $\Phi_{n+1}(X)$ corresponding exactly to the conclusion.

$$\frac{A \Rightarrow C}{!^i A \Rightarrow !^i C}\ !^i R \quad \rightsquigarrow \quad \overline{(!^i A)^* \Rightarrow (!^i C)^*}\ \Phi_\infty(X) \qquad \text{if } \mathsf{K} \notin m(i)$$

$$\frac{!^i A \Rightarrow C}{!^i A \Rightarrow !^i C}\ !^i R \quad \rightsquigarrow \quad \overline{(!^i A)^* \Rightarrow (!^i C)^*}\ \Phi_\infty(X) \qquad \text{if } \mathsf{K} \notin m(i), 4 \in m(i)$$

The most interesting right case is if $m(i) = \{\mathsf{K}, \mathsf{T}, 4\}$ and $s(i) = \{\mathsf{C}, \mathsf{W}\}$ (the only case considered where $\mathsf{K} \in m(i)$). In this case, we need to be careful with structural rules. We are looking for a translation of

$$\frac{!^i \Gamma \Rightarrow C}{!^i \Gamma \Rightarrow !^i C}\ !^i R\mathsf{K}4$$

By the assumption of the rule and the induction hypothesis, we have that $(!^i\Gamma)^* \Rightarrow C^*$ is provable in $\mathsf{NL1} + \Phi_\infty(X)$. If Γ contains duplicate formulas, by removing some formulas in-place a la weakening, we can obtain a distinct structure containing the same set of formulas; call it $\Delta \equiv (D_k)_k$.

Each D_i is still a subformula and Δ is distinct, so the set $\Phi_\infty(X)$ includes $\bigotimes(!^i\Delta)^* \Rightarrow \bigotimes(!^i\Delta)^* \otimes \bigotimes(!^i\Delta)^*$. By cutting this in several times, we essentially contract $(!^i\Delta)^*$ to have the shape of Γ, we then cut in many instances of weakening $!^i D_k \Rightarrow 1$ to obtain $!^i\Gamma$ itself. This deduciton is below.

$$\dfrac{\dfrac{}{(!^i\Delta)^* \Rightarrow \bigotimes(!^i\Delta)^*}\ \otimes R, \mathsf{init} \qquad \dfrac{}{\bigotimes(!^i\Delta)^* \Rightarrow \bigotimes(!^i\Delta)^* \otimes \bigotimes(!!^i\Delta)^*} \qquad \dfrac{\dfrac{\dfrac{}{(!^i D_k)^* \Rightarrow 1} \quad \begin{matrix}(!^i\Gamma)^* \Rightarrow C^* \equiv \\ (!^i D_k)^*_k \Rightarrow C^*\end{matrix}}{(!^i\Delta)^*_k \Rightarrow C^*}\ \mathsf{cut}, 1L}{\bigotimes(!^i\Delta)^*_k \Rightarrow C^*}\ \otimes L}{(!^i\Delta)^* \Rightarrow C^*}\ \mathsf{cut}, \otimes L$$

Thus, $(!^i\Delta)^* \Rightarrow C^*$ is also provable in $\mathsf{NL1} + \Phi_\infty(X)$. Thus, it is provable in some $\mathsf{NL1} + \Phi_n(X)$,. Since Δ is distinct, this means that

$$((!^i\Delta)^* \Rightarrow (!^i C)^*) \in \Phi_{n+1}(X)$$

We then apply a similar trick, and again strategically cut in weakening to prove our original goal.

$$\dfrac{\dfrac{}{(!^i D_k)^* \Rightarrow 1} \quad \dfrac{}{(!^i\Delta)^* \Rightarrow (!^i C)^*}\ \Phi_{n+1}(X)}{(!^i\Gamma)^* \Rightarrow (!^i C)^*}\ \mathsf{cut}, 1L$$

Proving the inductive claim in this final case. □

4.2. Local Contraction in Multiplicative Logics without K

Decidability is an easy corollary of Bulińska (2009) with these soundness and completeness results.

Theorem 4.2.1. *Let Σ be a specification where $S = \emptyset$ and $s(i) \subseteq \{\mathsf{C}, \mathsf{W}\}$ and $\mathsf{K} \notin m(i)$ for all labels i. Provability for sequents consisting of $\otimes, \rightarrow, \leftarrow, 1$, and $!^i$ is PTIME decidable.*

Proof. The previous section reduces provability with $\otimes, \rightarrow, \leftarrow, 1$, and $!^i$ to deducability from $\Phi_\infty(X)$ with $\otimes, \rightarrow, \leftarrow$, and 1 where X is the set of Gentzen subformulas in the considered sequent.

Since $\Phi_\infty(X)$ is finite, it closes after finitely many computable steps and is thus computable.

Thus, decidability of the considered provability follows from the decidability of deducability of nonassociative noncommutative linear logic with $\otimes, \to, \leftarrow$, and 1, given in Bulińska (2009).

More concretely, $\Phi_\infty(X)$ has size at most $|X|^2 + (|I|^2 + 3)|X|$, so the construction of $\Phi_\infty(X)$ closes in at most this many steps. Explicitly,

$$\Phi_\infty(X) = \Phi_{|X|^2+(|I|^2+3)|X|}(X).$$

Thus, $\Phi_\infty(X)$ can be constructed in polynomial time, and Bulińska (2009) showed that deducibility in NL with the multiplicative unit 1 is PTIME in the size of the axioms and original sequent. $\square$

It is worth noting that this covers the case where contraction is included but not weakening. This relevant case often yields undecidability in related logics.

4.3. Local Contraction in Multiplicative Logics with K

In the presence of K, we have much stronger requirements on the subexponentials.

Theorem 4.3.1. *Let Σ be a specification where $S = \emptyset$, $m(i) = \{\mathsf{K}, \mathsf{T}, 4\}$, and $s(i) = \{\mathsf{C}, \mathsf{W}\}$ for all labels i. Provability for sequents consisting of $\otimes, \to, \leftarrow, 1$, and $!^i$ is $\mathsf{EXPTIME}$ decidable.*

Proof. Again, by soundness and completeness above we can reduce to deducibility in NL with 1.

However, this time, the size of $\Phi_\infty(X)$ is bounded only exponentially in $|X|$, and thus the PTIME decidability of deducibility of $\mathsf{NL1}$ yields $\mathsf{EXPTIME}$ decidability of this logic. $\square$

CHAPTER 5

LOWER BOUNDS

In this chapter, we reduce previously explored decision problems in logic to provability in particular fragments of MSL_Σ.

We consider various classes of multimodal substructural specification and embed other logics faithfully into them. These embedding yield linear time reductions, giving hardness results for provability in MSL_Σ.

5.1. PSPACE-Hardness of Logics Simulating Full Structurality

Among serving other functions, the exponential of linear logic serves as an avenue for encoding fully structural intuitionistic propositional logic into linear logic. Since intuitionistic propositional logic, IPL, and in fact even its implicational fragment $\mathsf{IPL}_\to$, are PSPACE complete, we obtain a strong lower bound for any logic in which we can carry out the analogous embedding.

With this motivation, we first define a formal notion of a sufficiently structural subexponential. Specifications implying such structure with therefore be called *structurally subexponential.*

Definition 12. *A multimodal substructural specification* $\Sigma = (I, \preceq, m, s, S)$ *is called* structurally subexponential *if there is a label* $x \in I$*, called the exponential label, such that* $\mathsf{T} \in m(x)$ *and for every pair of structures* Π, Δ *where* $set(\Pi) \subseteq set(\Delta)$ *there is a deduction*

$$\dfrac{!^x\Pi \Rightarrow C}{!^x\Delta \Rightarrow C}$$

in MSL_Σ *using only subexponential structural rules.*

When discussing a MSL_Σ *with a distinguished exponential label, we call* $!^x$ *the* exponential *and write it as simply* $!$.

Now we move on to the encoding.

5.1.1. The Encoding

For our encoding of fully structural implicational intuitionistic logic $\mathsf{IPL}_{\to}$ into MSL_{Σ}, we consider the presentation of $\mathsf{IPL}_{\to}$ with the following three rules, where Γ is a set of formulas.

$$\frac{}{\Gamma, p \Rightarrow p}\ \mathsf{init} \qquad \frac{\Gamma, A \to B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \to L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \to R$$

Further, note that the following weakening rule is admissible.

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\ \mathsf{W}$$

We start with the following translation from $\mathsf{IPL}_{\to}$ formulas to MSL_{Σ} formulas with ! and $\to$.

Definition 13. *Let $\Sigma = (I, \preceq, m, s, S)$ be a structurally exponential multimodal substructural specification.*

Now define the translations $\tau_{\pm}$ from $\mathsf{IPL}_{\to}$ formulas to MSL_{Σ} formulas inductively similarly to the embedding of $\mathsf{IPL}_{\to}$ into linear logic.

$$\tau_+(p) :\equiv p$$
$$\tau_+(A \to B) :\equiv \tau_-(A) \to \tau_+(B)$$
$$\tau_-(p) :\equiv !p$$
$$\tau_-(A \to B) :\equiv !(\tau_+(A) \to \tau_-(B))$$

For a set Γ, let $\overline{\Gamma}$ be some choice of tree containing all of the formulas in a set Γ, and let $\tau_-(\Gamma) :=$

$\{\tau_-(A) : A \in \Gamma\}$ *be the elementwise application of* τ_- *to a set* Γ.

Definition 14. *For convenience, we also define an inverse translation* σ *from implicational subexponential* MSL_Σ *formulas to* $\mathsf{IPL}_\to$ *formulas inductively on formulas by*

$$
\begin{aligned}
\sigma(p) &:\equiv p \\
\sigma(A \to B) &:\equiv \sigma(A) \to \sigma(B) \\
\sigma(!A) &:\equiv \sigma(A)
\end{aligned}
$$

We also define the entrywise application of σ *to a structure* Γ *inductively by*

$$\sigma((\Delta, \Pi)) :\equiv (\sigma(\Delta), \sigma(\Pi))$$

This is a one-sided inverse as formalized in the following lemma.

Lemma 5.1.1. *For all* $\mathsf{IPL}_\to$ *formulas* A*, we have* $\sigma(\tau_\pm(A)) \equiv A$.

With all of this in place, we can formalize and prove soundness and completeness.

5.1.2. Soundness

We start by showing the soundness of exponentially specified MSL_Σ with respect to $\mathsf{IPL}_\to$.

Lemma 5.1.2 (Soundness)**.** *If* Σ *is an exponential multimodal substructural specification and*

$$\mathsf{MSL}_\Sigma \vdash \overline{\tau_-(\Gamma)} \Rightarrow \tau_+(C)$$

for some tree $\overline{\tau_-(\Gamma)}$*, then*

$$\mathsf{IPL}_\to \vdash \Gamma \Rightarrow C.$$

Proof. By cut elimination for MSL_Σ, it is sufficient to consider only the rules $(\mathsf{init}), (\to L), (\to R)$, the subexponential rules, and the structural rules, since we are only considering formulas with $\to$ and !. Further, ! is never the top level connective of $\tau_+(C)$, so we need not consider $(!R), (!R\mathsf{K}), (!R4)$, or $(!R\mathsf{K}4)$.

The σ-translation of each of the propositional rules are directly deducible in $\mathsf{IPL}_\to + (\mathsf{W})$. Both $(!L)$ and $(!L\mathsf{T})$ act identically under σ. Similarly, the rules $(!\mathsf{nC}), (!\mathsf{C}), (!\mathsf{E}), (!\mathsf{A1}), (!\mathsf{A2}), (\mathsf{nC}), (\mathsf{C}), (\mathsf{E}), (\mathsf{A1})$, and $(!\mathsf{A2})$ act identically on sets, as they contain the same set of formulas in the context of their premises and conclusions. The rules $(!\mathsf{W})$ and (W) either act identically if the weakened formula appears more than once in the conclusion, or it removes the formula from the premise. In the latter case, the rule, under σ, can be replicated by (W) in $\mathsf{IPL}_\to + (\mathsf{W})$.

Thus, we have a direct translation of the original proof, in fact one of at most the same length, in $\mathsf{IPL}_\to + (\mathsf{W})$

Thus, if $\mathsf{MSL}_\Sigma \vdash \overline{\tau_-(\Gamma)} \Rightarrow \tau_+(C)$, then

$$\mathsf{IPL}_\to + (\mathsf{W}) \vdash set(\sigma(\overline{\tau_-(\Gamma)})) \Rightarrow \sigma(\tau_+(C)).$$

Since (W) is admissible in $\mathsf{IPL}_\to$, we also have

$$\mathsf{IPL}_\to \vdash set(\sigma(\overline{\tau_-(\Gamma)})) \Rightarrow \sigma(\tau_+(C)).$$

However, σ is a left inverse of σ, so the above statement is syntactically just that

$$\mathsf{IPL}_\to \vdash \Gamma \Rightarrow C,$$

proving the claim. □

5.1.3. Completeness

When showing completeness, the rules cannot be directly simulated by fixed sized proofs. We need to appeal to the structural subexponentiality of the specification to properly position substructures.

Lemma 5.1.3 (Completeness)**.** *Let Σ be a structurally subexponential multimodal substructural specification. If $\mathsf{IPL}_{\rightarrow} \vdash \Gamma \Rightarrow C$, then for any choice of tree $\overline{!\tau(\Gamma)}$, we have $\mathsf{IPL}_{\rightarrow} \vdash \overline{!\tau(\Gamma)} \Rightarrow \tau(C)$.*

Proof. We proceed by induction on $\mathsf{IPL}_{\rightarrow}$ proofs, then casewise on the final rule.

If the final rule is (init), then we have the following MSL_{Σ} proof of the translation, noting that the top-level connective of all formulas on the left is the exponential and $set(p) \subseteq set(\Gamma, p)$.

$$\dfrac{\dfrac{\overline{p \Rightarrow p}\ \mathsf{init}}{!p \Rightarrow p}\ !L\mathsf{T}}{\overline{\overline{\tau(\Gamma), !p} \Rightarrow p}}$$

Note that our use of ($!L\mathsf{T}$) here is justified by the fact that $\mathsf{T} \in m(*)$ for exponentials by definition.

If the final rule is ($\rightarrow R$), then the following one rule translation is sufficient to apply the inductive hypothesis.

$$\dfrac{(!\tau_{-}(A), \overline{\tau_{-}(\Gamma)}) \Rightarrow \tau_{+}(B)}{\overline{\tau_{-}(\Gamma)} \Rightarrow \tau_{-}(A) \rightarrow \tau + (B)} \rightarrow R$$

Finally and most interestingly, we have the case of ($\rightarrow L$). Here, we apply the rearrangement given by the definition of the exponential to copy $\overline{\tau(\Gamma)}$ and position one of the copies directly to the left of the active implication.

$$
\begin{array}{c}
\overline{\tau_-(\Gamma)} \Rightarrow \tau_+(A) \quad \overline{\tau_-(\Gamma)}, \tau_-(B) \Rightarrow \tau_+(C) \\
\hline
(\overline{\tau_-(\Gamma)}, (\overline{\tau_-(\Gamma)}, \tau_+(A) \to \tau_-(B))) \Rightarrow \tau_+(C) \\
\hline
(\overline{\tau_-(\Gamma)}, (\overline{\tau_-(\Gamma)}, !(\tau_+(A) \to \tau_-(B)))) \Rightarrow \tau_+(C) \\
\hline\hline
\overline{\tau(\Gamma), !(\tau(A) \to \tau_-(B))} \Rightarrow \tau_+(C)
\end{array}
\begin{array}{l} \to L \\ !L \end{array}
$$

This exhausts the cases, proving our claim. □

5.1.4. PSPACE-hardness

With soundness and completeness, PSPACE-hardness is an easy corollary.

Lemma 5.1.4. *For an exponential multimodal substructural specification Σ, provability for* MSL_Σ *is* PSPACE*-hard.*

Proof. By the above soundness and completeness, we can reduce provability in $\mathsf{IPL}_\to$ to provability in MSL_Σ with an exponential specification. To check if $\mathsf{IPL}_\to \vdash C$ it is sufficient to check if $\mathsf{MSL}_\Sigma \vdash \cdot \Rightarrow \tau_+(C)$, and this translation is linear time, and thus the claim follows. □

5.1.5. Local Contraction and Weakening

We now prove a corollary of the previous result that can be seen in some ways as its main motivation. To do this, we prove the following lemma.

Lemma 5.1.5 (Full Structurality of Local Contraction and Weakening)**.** *Let an* MSL_Σ *be such that Σ contains a label x with $s(x) \supseteq \{\mathsf{C}, \mathsf{W}\}$ and $\mathsf{T} \in m(x)$. Then this* MSL_Σ *is structurally subexponential with exponential label x.*

Proof. We directly exhibit the definition of structurally exponential in the presence of the subexponential structural rules $(\mathsf{C})/(\mathsf{CK})$ and (W).

Say that Π, Δ are structures with $set(\Pi) \subseteq set(\Delta)$. Define $(P_i)_i :\equiv \Pi$ and note that our assumption implies that $P_i \in \Delta$. Thus, reading bottom-up, if we contract $!\Delta$ to take the shape of $!\Pi$ and then weaken every copy of $!\Delta$ down to one formula, we have arbitrary rearrangement as in the deduction below.

$$\dfrac{\dfrac{\begin{array}{c}!\Pi \Rightarrow C \equiv \\ (!P_i)_i \Rightarrow C\end{array}}{(!\Delta)_i \Rightarrow C}\ \mathsf{W}}{!\Delta \Rightarrow C}\ !\mathsf{C}$$

□

Corollary 5.1.1. *Let an* MSL_Σ *be such that* Σ *contains a label* x *with* $s(x) \supseteq \{\mathsf{C}, \mathsf{W}\}$ *and* $\mathsf{T} \in m(x)$*. Then provability in this logic is* PSPACE*-hard.*

CHAPTER 6

CONCLUSION

6.1. Summary

We considered many substructural logics and complexity classes ranging from PTIME to PSPACE-hard to recursively enumerable.

Logics equivalent to fragments of the defined multimodal substructural logic MSL_Σ show up in varied subfields of mathematics, linguistics, computer science, and philosophy.

The exact complexity class of many of these logics is well-known. Further the proof theory, including structural cut elimination and Craig interpolation theorem is very well-explored, especially for commutative logics.

Here we

- provide a uniform framework, MSL_Σ, for the discussion of logics licensing local structural rules with multimodalities,
- provide uniform proofs of proof-theoretic theorems such as cut admissiblity and conservativity of the classical system, and
- provide new complexity bounds for several subexponential logics corresponding to fragments of MSL_Σ.

6.2. Open Problems and Future Work

The modular nature of substructural logic leads to a plethora of open problems.

6.2.1. Nonlocal Contraction

The undecidability of multiplicative additive astructural MSL_Σ with an unbounded label follows from the reduction Kanovich et al. (2019) presents to Chvalovský (2015). In fact, we don't need & for undecidability.

What remains an open problem is the decidability of fully astructural multiplicative, or even implicational, MSL_Σ with an unbounded subexponential, as mentioned in Blaisdell et al. (2022).

6.2.2. Combinatorial Array of Decision Problems

Without subexponentials, there appears to be a large set of very canonical problems. We quickly discuss a particular combinatorial way of describing interesting systems, and then discuss what is know.

It is natural to treat both associativities as one axiom. Thus, there are sixteen meaningful subsets of the global structural axioms $\{\mathsf{C}, \mathsf{W}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$.

Further, the choice of connectives is also salient. There seem to be at least three natural choices:

- the multiplicatives only,
- the additives and multiplicatives, and
- the additives and multiplicatives with additive distribution.

Here additive distribution is the axiom

$$A \,\&\, (B \oplus C) \Rightarrow (A \,\&\, B) \oplus (A \,\&\, C).$$

The distibutivity axiom is incredibly natural when investigating Craig interpolation, which is often useful for showing decidability. At a high level, this convenience comes from the fact that finitely generated boolean sublattices are finite, while this is not true in general lattices.

Further, the questions of both provability and deducibility are of interest.

In total, this raises ninety six (96) interesting and arguably canonical decision problems. There are many ways many of these problems collapse or trivialize:

- Provability for logics without contraction is trivially decidable.

- Logics with contraction and weakening admit the other structural rules.

Even so, there continues to be steady continued progress on these problems.

BIBLIOGRAPHY

K. Ajdukiewicz. Die syntaktische konnexität. *Studia Philosophica*, 1:1–27, 1935.

Y. Bar-Hillel. A quasi-arithmetical notation for syntactic description. *Language*, 29:47–58, 1953.

Eben Blaisdell, Max Kanovich, Stepan L. Kuznetsov, Elaine Pimentel, and Andre Scedrov. Non-associative, non-commutative multi-modal linear logic. page 449–467, Berlin, Heidelberg, 2022. Springer-Verlag. ISBN 978-3-031-10768-9.

Eben Blaisdell, Max Kanovich, Stepan Kuznetsov, Elaine Pimentel, and Andre Scedrov. Explorations in subexponential non-associative non-commutative linear logic. *Electronic Proceedings in Theoretical Computer Science*, 381:4–19, 08 2023. doi: 10.4204/EPTCS.381.3.

Torben Braüner and Valeria de Paiva. A formulation of linear logic based on dependency-relations. In Mogens Nielsen and Wolfgang Thomas, editors, *Computer Science Logic*, pages 129–148, Berlin, Heidelberg, 1998. Springer Berlin Heidelberg. ISBN 978-3-540-69353-6.

Torben Braüner, Ny Munkegade, and Valeria De Paiva. Cut-elimination for full intuitionistic linear logic. *BRICS Report Series*, 3, 05 1996. doi: 10.7146/brics.v3i10.19973.

Maria Bulińska. On the complexity of nonassociative Lambek calculus with unit. *Studia Logica: An International Journal for Symbolic Logic*, 93(1):1–14, 2009. ISSN 00393215, 15728730. URL http://www.jstor.org/stable/40587157.

Wojciech Buszkowski. Lambek calculus with nonlogical axioms. In *Language and Grammar*, Studies in Mathematical Linguistics and Natural Language, pages 77–93. CSLI Publications, 2005.

Wojciech Buszkowski. On classical nonassociative lambek calculus. In Maxime Amblard, Philippe de Groote, Sylvain Pogodalla, and Christian Retoré, editors, *Logical Aspects of Computational Linguistics. Celebrating 20 Years of LACL (1996-2016) - 9th International Conference, LACL 2016, Nancy, France, December 5-7, 2016, Proceedings*, volume 10054 of *Lecture Notes in Computer Science*, pages 68–84, 2016. doi: 10.1007/978-3-662-53826-5_5. URL https://doi.org/10.1007/978-3-662-53826-5_5.

Karel Chvalovský. Undecidability of consequence relation in full non-associative Lambek calculus. *J. Symb. Logic*, 80(2):567–586, 2015.

Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx. The structure of exponentials: Uncovering the dynamics of linear logic proofs. In Georg Gottlob, Alexander Leitsch, and Daniele Mundici, editors, *Kurt Gödel Colloquium*, volume 713 of *LNCS*, pages 159–171. Springer, 1993.

Uffe Engberg and Glynn Winskel. Linear logic on petri nets. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *A Decade of Concurrency Reflections and Perspectives*, pages 176–229, Berlin, Heidelberg, 1994. Springer Berlin Heidelberg. ISBN 978-3-540-48423-3.

Gerhard Gentzen. Investigations into logical deduction. In *The Collected Papers of Gerhard Gentzen*, pages 68–131. 1969.

Jean-Yves Girard. Linear logic. *Theoret. Comput. Sci.*, 50:1–102, 1987. doi: 10.1016/0304-3975(87)90045-4. URL http://dx.doi.org/10.1016/0304-3975(87)90045-4.

Jean-Yves Girard. Light linear logic. *Information and Computation*, 143(2):175–204, 1998. ISSN 0890-5401.

Philippe de Groote and F. Lamarche. Classical non-associative Lambek calculus. *Stud Logica*, 71(3): 355–388, 2002. doi: 10.1023/A:1020520915016. URL https://doi.org/10.1023/A:1020520915016.

Philippe de Groote, Bruno Guillaume, and Sylvain Salvati. Vector addition tree automata. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, LICS '04, page 64–73, USA, 2004. IEEE Computer Society. ISBN 0769521924.

Max Kanovich, Stepan Kuznetsov, Vivek Nigam, and Andre Scedrov. Soft subexponentials and multiplexing. In *Automated Reasoning: 10th International Joint Conference, IJCAR 2020, Paris, France, July 1–4, 2020, Proceedings, Part I*, page 500–517, Berlin, Heidelberg, 2020. Springer-Verlag. ISBN 978-3-030-51073-2.

Max I. Kanovich, Stepan L. Kuznetsov, Glyn Morrill, and Andre Scedrov. A polynomial-time algorithm for the lambek calculus with brackets of bounded order. In Dale Miller, editor, *2nd International Conference on Formal Structures for Computation and Deduction, FSCD 2017, September 3-9, 2017, Oxford, UK*, volume 84 of *LIPIcs*, pages 22:1–22:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017. doi: 10.4230/LIPIcs.FSCD.2017.22. URL https://doi.org/10.4230/LIPIcs.FSCD.2017.22.

Max I. Kanovich, Stepan Kuznetsov, Vivek Nigam, and Andre Scedrov. Subexponentials in non-commutative linear logic. *Math. Struct. Comput. Sci.*, 29(8):1217–1249, 2019. doi: 10.1017/S0960129518000117. URL https://doi.org/10.1017/S0960129518000117.

Natasha Kurtonina and Michael Moortgat. Structural control. 1997. URL https://api.semanticscholar.org/CorpusID:264263123.

Yves Lafont. Soft linear logic and polynomial time. *Theoretical Computer Science*, 318(1):163–180, 2004. ISSN 0304-3975. doi: https://doi.org/10.1016/j.tcs.2003.10.018. URL https://www.sciencedirect.com/science/article/pii/S0304397503005231. Implicit Computational Complexity.

J. Lambek. On the calculus of syntactic types. In R. Jakobson, editor, *Structure of Language and Its Mathematical Aspects*, pages 166–178. American Mathematical Society, 1961.

Joachim Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65(3):

154–170, 1958.

Richard Moot and Christian Retoré. Natural language semantics and computability. *J. of Logic, Lang. and Inf.*, 28(2):287–307, jun 2019. ISSN 0925-8531. doi: 10.1007/s10849-019-09290-7. URL https://doi.org/10.1007/s10849-019-09290-7.

Glyn Morrill and Oriol Valentín. Computational coverage of tlg: Nonlinearity. 06 2017.

Mati Pentus. Free monoid completeness of the lambek calculus allowing empty premises. In Lecture Notes in Logic, editor, *Proc. Logic Colloquium '96*, volume 12, pages 171–209, 1998.

V. Pratt. Linear logic for generalized quantum mechanics. In *Workshop on Physics and Computation*, pages 166–180, 1992. doi: 10.1109/PHYCMP.1992.615518.

Harold Schellinx. Some syntactical observations on linear logic. *J. Log. Comput.*, 1(4):537–559, 1991. doi: 10.1093/logcom/1.4.537. URL https://doi.org/10.1093/logcom/1.4.537.

David N. Yetter. Quantales and (noncommutative) linear logic. *J. Symb. Logic*, 55(1):41–64, 1990.