On Fractional Paradigm and Intermediate Zones in Electromagnetism: I. Planar Observation†

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Abstract

In this Letter the kernel of the integral transform that relates the field quantities over an observation flat plane to the corresponding quantities on another observation plane parallel with the first one is fractionalized for the two-dimensional (2-D) monochromatic wave propagation. It is shown that such fractionalized kernels, with fractionalization parameter ν between zero and unity, are the kernels of the integral transforms that provide the field quantities over the parallel planes *between* the two original planes. With proper choice of the first two planes, these fractional kernels can provide us with a natural way of interpreting the fields in the intermediate zones (i.e., the region between the near and the far zones) in certain electromagnetic problems. The evolution of these fractional kernels into the Fresnel and Fraunhofer diffraction kernels is addressed. The limit of these fractional kernels for the static case is also mentioned.

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Introduction

In the past few years, we have been interested in developing and studying the *fractional paradigm in electromagnetic theory* [1-7]. In such a paradigm, the goal is to bring the tools of fractional calculus (see e.g., [8])--the topic that deals with fractional differentiation and integration-- and in general fractionalization of various operators and the electromagnetic theory together and to find applications of such fractionalized operators in electromagnetic problems. The basic idea behind this paradigm is to find the fractional or "interpolated" responses that are effectively intermediate situations between the canonical solutions to a given problem [1]. If the canonical cases are labeled Case 1 and Case 2 and if there exists an operator $\underline{\underline{\boldsymbol{L}}}$ that maps Case 1 onto Case 2, then one would inquire whether the fractionalization of such an operator would provide us with the "fractional" or interpolated solutions that are considered *intermediate* between the Cases 1 and 2. Some of the detail behind this idea and a general recipe for the fractionalization of operator $\underline{\underline{\boldsymbol{L}}}$ are described in [1,2]. Here, we present another case study that addresses the issue of the fractionalization of operators and its roles in intermediate zones in electromagnetism.

The idea is this: since fractional derivatives/integrals effectively address the "intermediate behavior" for the differentiation/integration operators, can fractionalization of some appropriate operators provide us with proper tools to tackle certain electromagnetic problems involving intermediate zones? It is well known that for a given charge and/or current distributions, whether it is a static or dynamic case, the quantities of interest such as electric fields or potentials resulted from such a source may be expressed in terms of integrals involving the appropriate Green functions and the given source. Furthermore, it is also well known that in order to find some simpler expressions for the quantity of interest, usually more attention has been paid to the analysis of the far fields and the near fields while less attention has been aimed at the intermediate zones. In the present work we show that if the appropriate kernels of integral transforms, which relate quantities of interest in the near zone to the quantities in the far-zone region, are considered and then are properly "fractionalized", such "fractionalized kernels" can act as kernels of integral transforms that "link" the near-zone quantities to the quantities at the intermediate zones. This would provide another logical interpretation for the intermediate zones as the regions where the integral transform with fractionalized kernels can give the fields and potentials. The detail of this problem is given in the following section.

Geometry and Formulation of the Problem

Consider a Cartesian coordinate system (x, y, z) and a monochromatic source represented as the volume current density $\boldsymbol{J}$ in free space. The time dependence of $e^{-i\omega t}$ is assumed throughout this work. Without loss of generality for presenting the idea and for the sake of simplicity in the mathematical formulation here, we assume the problem to be a two-dimensional (2-D) one in which all quantities of interest are independent of *y*-coordinate, the source $\boldsymbol{J} = \boldsymbol{J}(x, z)$ is a function of the *x*- and *z*-coordinates, and its transverse cross section in the *x-z* plane is confined to a limited region. Let us denote the potential (or a Cartesian component of the fields) of interest with the symbol $\psi(x, z)$. In the region outside the source, this function satisfies the Helmholtz equation $\nabla^2\psi + k_o{}^2\psi = 0$ where $k_o \equiv \omega\sqrt{\mu_o\varepsilon_o}$ with μ_o and ε_o being the permeability and permittivity of free space, respectively. We choose two observation planes parallel with the x-y plane, one located at $z = z_o > 0$ and the other at $z = z_1 > z_o$ where z_o and z_1 are both outside the source region, and the region between these two planes is source-free. (See Fig. 1.) These two values of z can be at any locations along the z-axis (as long as the space $z_o \le z \le z_1$ is source free), and if desired one can assume that z_o is in the near zone of the source while z_1 is in the far-zone region. The potential distributions on these two planes are denoted by $\psi(x, z_o)$ and $\psi(x, z_1)$, respectively. Applying standard Green's theorem (see e.g. [9, page 149]) to the half space $z \ge z_o$, one can express $\psi(x, z_1)$ in terms of $\psi(x, z_o)$ using the following integral transform[1]

$$\psi(x, z_1) = \int_{-\infty}^{+\infty} K(x, z_1; x', z_o)\, \psi(x', z_o) dx' \tag{1}$$

where, after proper mathematical steps, the kernel $K(x, z_1; x', z_o)$ can be explicitly written as

$$K(x, z_1; x', z_o) = K(x - x', z_1 - z_o) = \frac{i k_o (z_1 - z_o)}{2\sqrt{(x - x')^2 + (z_1 - z_o)^2}} H_1^{(1)}\left(k_o \sqrt{(x - x')^2 + (z_1 - z_o)^2}\right) \tag{2}$$

with $H_1^{(1)}(\cdot)$ being the first-order Hankel function of the first kind. The spatial Fourier transform (with respect to variable x) of the two functions $\psi(x, z_o)$ and $\psi(x, z_1)$ are written as

$$\tilde{\psi}(k, z_o) \equiv \int_{-\infty}^{+\infty} \psi(x, z_o) e^{-ikx} dx$$

$$\tilde{\psi}(k, z_1) \equiv \int_{-\infty}^{+\infty} \psi(x, z_1) e^{-ikx} dx \tag{3}$$

[1] Here it is assumed that in this physical situation, in the half space $z \ge z_o$, $\psi(x, z)$ approaches zero when $|x| \to \infty$.

where k is the spatial angular frequency (Fourier variable). Considering the space-invariance property of the kernel $K(x-x', z_1-z_o)$ given in Eq. (2), Eq. (1) can be written in the spatial Fourier domain as

$$\tilde{\psi}(k,z_1)=\tilde{K}(k,z_1-z_o)\,\tilde{\psi}(k,z_o) \tag{4}$$

where $\tilde{K}(k,z_1-z_o)$ is the spatial Fourier transform of the kernel $K(x,z_1-z_o)$. Considering a suitable integral representation of the Hankel functions (see e.g., [10, page 89]) and the expression for the kernel $K(x-x', z_1-z_o)$ given in Eq. (2), one can find $\tilde{K}(k,z_1-z_o)$ to be

$$\tilde{K}(k,z_1-z_o)=e^{i\sqrt{k_o^2-k^2}\,(z_1-z_o)} \tag{5}$$

which is consistent with the propagator in the positive z direction for the Helmholtz operator $\nabla^2+k_o^2$. (See e.g., [11, p. 50])

Returning to the notion of canonical cases denoted as Case 1 and Case 2 as specific solutions to a given problem [1], here for the problem at hand, which is a 2-D wave propagation, we can label the function $\psi(x,z_o)$ on the plane $z=z_o$ as the Case 1 and the function $\psi(x,z_1)$ on the plane $z=z_1$ as the Case 2, both of which are obviously solutions to the Helmholtz equation. Then searching for a mapping that links the Case 1 to Case 2, one finds Eq. (1) as such mapping. Therefore, the linear operator $\underline{\underline{\boldsymbol{L}}}$ that maps $\psi(x,z_o)$ into $\psi(x,z_1)$ is indeed the integral transform with the kernel $K(x-x', z_1-z_o)$. So we can write

$$\underline{\underline{\boldsymbol{L}}}\equiv\int_{-\infty}^{+\infty}K(x-x', z_1-z_o)\cdots dx' \quad . \tag{6}$$

Now we pose the following question: If we properly "fractionalize" this operator $\underline{\underline{\boldsymbol{L}}}$ and symbolically show such a fractionalized operator as $\underline{\underline{\boldsymbol{L}}}^{\nu}$ where ν is a fractionalization parameter with values between zero and unity, would applying of $\underline{\underline{\boldsymbol{L}}}^{\nu}$ on the function $\psi(x, z_o)$ provide us with a function that is the solution to our Helmholtz equation for observation points on a plane located somewhere *between* z_o and z_1? In other words, would this fractional operator act as an integral transform to find the solutions in the intermediate zone? To find answers to these questions, we first need to fractionalize the operator $\underline{\underline{\boldsymbol{L}}}$ in Eq. (6). We have presented elsewhere a recipe for fractionalization of some class of linear operators [2,1]. The detail is not repeated here, but a brief mention of the relevant parts of that recipe is given here. The first step in fractionalization of an operator is to find the eigenfunctions and eigenvalues of the operator. To that end, we write

$$\underline{\underline{\boldsymbol{L}}}\, f_m(x) = a_m\, f_m(x) \tag{7}$$

where $f_m(x)$ and a_m are the eigenfunctions and eigenvalues of the operator $\underline{\underline{\boldsymbol{L}}}$. Here $m = 1,2,3,\cdots,n$ with n being the dimension of the space of the domain (and the range) of the operator $\underline{\underline{\boldsymbol{L}}}$. For the operator $\underline{\underline{\boldsymbol{L}}}$ given in Eq. (6), we then have

$$\int_{-\infty}^{+\infty} K(x - x', z_1 - z_o)\, f_m(x')\, dx' = a_m f_m(x). \tag{8}$$

To describe the eigenfunctions and eigenvalues of this operator, we take the spatial Fourier transform of Eq. (8) (with respect to variable x) and obtain

$$\widetilde{K}(k, z_1 - z_o)\, \widetilde{f}_m(k) = a_m\, \widetilde{f}_m(k), \tag{9}$$

where $\widetilde{f}_m(k)$ represents the spatial Fourier transform of $f_m(x)$. Substituting Eq. (5) into Eq. (9), one gets the following equation for $\widetilde{f}_m(k)$

$$\left[e^{i\sqrt{k_o^2 - k^2}\,(z_1 - z_o)} - a_m \right] \widetilde{f}_m(k) = 0. \tag{10}$$

From this equation, a set of solutions for $\tilde{f}_m(k)$ and their corresponding eigenvalues a_m is found to be

$$\tilde{f}_m(k) = \delta(k - k_m) \tag{11}$$

with $a_m = e^{i\sqrt{k_o^2 - k_m^2}\,(z_1 - z_o)}$ where $\delta(\cdot)$ is the Dirac delta function, and k_m is any given value in the spatial Fourier domain. Taking the inverse spatial Fourier transform of Eq. (11) results in

$$f_m(x) = \tfrac{1}{2\pi} e^{ik_m x} \tag{12}$$

which is in total agreement with the well-known fact in the system theory that for a space-invariant linear system the eigenfunctions are complex exponential functions as $e^{ik_m x}$.

Since k_m is any given point along the spatial Fourier variable k, we choose the notation h instead of k_m to show the continuous nature of k_m, and therefore use the symbols $f_h(x)$ and a_h instead of $f_m(x)$ and a_m, respectively. So we have the eigenfunctions and eigenvalues of our operator $\underline{\underline{\boldsymbol{L}}}$ as

$$f_h(x) = e^{ihx}, \tag{13a}$$

$$a_h = e^{i\sqrt{k_o^2 - h^2}\,(z_1 - z_o)}. \tag{13b}$$

The eigenvalues here indeed equal to the Fourier transform of the kernel K, i.e., $a_h = \tilde{K}(h, z_1 - z_o)$. Having expressed the eigenfunctions and eigenvalues of the operator $\underline{\underline{\boldsymbol{L}}}$, the next step is to define the fractionalization of the operator $\underline{\underline{\boldsymbol{L}}}$ as the new operator $\underline{\underline{\boldsymbol{L}}}^{\nu}$ whose eigenfunctions are the same as those of $\underline{\underline{\boldsymbol{L}}}$, but whose eigenvalues are $a_h{}^{\nu}$ where ν the fractionalization parameter here is between zero and unity. That is

$$\underline{\underline{\boldsymbol{L}}}^{\nu} f_h = a_h{}^{\nu} f_h. \tag{14}$$

If we write the new "fractional" operator $\underline{\underline{\boldsymbol{L}}}^{\nu}$ as the following integral operator

$$\underline{\underline{L}}^{\nu} \equiv \int_{-\infty}^{+\infty} K_{\nu}(x-x',z_1-z_o)\cdots dx' \tag{15}$$

we need to find the kernel $K_{\nu}(x-x',z_1-z_o)$, which we name *fractional kernel* here. Having the expression for the eigenvalues of the fractional operator $\underline{\underline{L}}^{\nu}$, we can find the Fourier transform of this fractional kernel as

$$\tilde{K}_{\nu}(h,z_1-z_o) = a_h^{\ \nu} = e^{i\nu\sqrt{k_o^{\,2}-h^2}\,(z_1-z_o)}. \tag{16}$$

The expression for $K_{\nu}(x-x',z_1-z_o)$ can then be obtained by inverse spatial Fourier transforming Eq. (16). That results in

$$K_{\nu}(x-x',z_1-z_o) = \frac{i\,\nu k_o\,(z_1-z_o)}{2\sqrt{(x-x')^2+\nu^2(z_1-z_o)^2}}\,H_1^{(1)}\Big(k_o\sqrt{(x-x')^2+\nu^2(z_1-z_o)^2}\Big), \tag{17}$$

with $0 \le \nu \le 1$. [2] In the next section, we will discuss some of the notable features of this fractional kernel.

Physical Remarks

When one compares Eqs. (2) and (17), one notices that

$$K_{\nu}\big(x-x',z_1-z_o\big) = K\big(x-x',\nu(z_1-z_o)\big). \tag{18}$$

It can be easily seen that the integral transform with such fractional kernel satisfies the general features of fractionalized operators discussed in our previous work [1,2]. Specifically it can be shown that when the fractionalization parameter ν approaches unity, the fractional kernel $K_{\nu}\big(x-x',z_1-z_o\big)$ becomes our original kernel $K\big(x-x',z_1-z_o\big)$; When the parameter ν goes to zero, the fractional kernel in Eq. (17) can be shown to approach the Dirac delta function $\delta(x-x')$, as expected, since $K_0\big(x-x',z_1-z_o\big) = K(x-x',0)$ becomes a kernel that "maps" the function $\psi(x,z_o)$ at the plane $z = z_o$ onto itself; and finally it can be shown that the integral transform with this fractional kernel (i.e., Eq. (15)) satisfies the additivity properties in fractional parameter ν, i.e., $\underline{\underline{\boldsymbol{L}}}^{\nu_1} \circ \underline{\underline{\boldsymbol{L}}}^{\nu_2} = \underline{\underline{\boldsymbol{L}}}^{\nu_1+\nu_2}$ where $\underline{\underline{\boldsymbol{L}}}^{\nu}$ is described in Eq. (15). This is because we can show that $K_{\nu_1} * K_{\nu_2} = K_{\nu_1+\nu_2}$ where the symbol $*$ here denotes the convolution. From Eq. (18), we can observe that when $0 < \nu < 1$, the argument $\nu(z_1 - z_o)$ can be effectively interpreted as the distance between the original plane at z_o and another plane *between* z_o and z_1. If the z-coordinate of this "intermediate" plane is denoted by z_ν, we can then write $z_\nu - z_o = \nu(z_1 - z_o)$ from which z_ν can be explicitly given as

$$z_\nu = z_o + \nu(z_1 - z_o). \tag{19}$$

and thus the fractional kernel K_ν can be explicitly written as

$$K_\nu\left(x - x', z_1 - z_o\right) = K\left(x - x', z_\nu - z_o\right) = \frac{i\,k_o\,(z_\nu - z_o)}{2\sqrt{(x - x')^2 + (z_\nu - z_o)^2}}\,H_1^{(1)}\left(k_o\sqrt{(x - x')^2 + (z_\nu - z_o)^2}\right) \tag{20}$$

We notice again that when $\nu \to 0$, the intermediate plane would approach z_o, and when $\nu \to 1$, it will go to the original plane at z_1. Therefore, the fractional kernel K_ν given in Eq. (20), which is the result of fractionalization of the original kernel K in Eq. (17) connecting the function at the plane z_o to the corresponding function at the plane z_1, is a kernel that "links" the function at the plane z_o to the function at an intermediate plane at z_ν. As the fractionalization parameters varies between zero and unity, the location of intermediate plane changes from z_o to z_1. If the original kernel is the mapping between the near-zone and the far-zone fields, the fractional kernel with parameter ν between zero and unity would provide another way to interpret the fields in the intermediate zones.

[2] For now, we are still taking ν to be between zero and unity. Later in our discussion, this restriction will be removed.

Therefore, for observation points on parallel flat planes (i.e., planar observation) in the Cartesian coordinate system as described here, the fractionalization of the near-zone to far-zone mapping provides us with the kernel for the intermediate zones.

In the general expression for the fractional kernel, when the parameter ν is specified at a non-integer value between zero and unity, the value of z_ν is then determined from Eq. (19). Conversely, if we are interested in determining the fractional kernel for a specified plane of observation at z_ν, then we can find the value of ν from the expression

$$\nu = \frac{z_\nu - z_o}{z_1 - z_o}. \tag{21}$$

Knowing ν, we can then determine the appropriate fractional kernel from Eq. (17) (or needless to say, alternatively, this kernel can be obtained directly from Eq. (20) when z_ν is specified.)

The fractionalization of such kernels and the relationship between the fractionalization parameter ν and the location of the observation plane z_ν become even more evident, if the z-coordinate is chosen such that the original plane is at $z_o = 0$. [3] In this case, $z_\nu = \nu z_1$, and it can be easily seen that as ν varies from zero to unity, z_ν evolves from zero to z_1.

It must be noted that although the fractionalization parameter ν is taken to be between zero and unity in the aforementioned discussion, it does not have to be limited to this range and it can take values larger than unity in the expression for the fractional kernel in Eq. (20) (or Eq. (17) or (18)). For $\nu > 1$, the value of z_ν in Eq. (19) becomes

[3] In this case, the source $\boldsymbol{J}$ shown in Fig. 1 should be moved to the left (or the Cartesian coordinate should be moved to the right) so that the region $z \geq 0$ should still be source-free.

greater than z_1, which implies that Eq. (20) can also obviously provide us with the kernels appropriate for the planes beyond the region $z_o \le z \le z_1$. It is worth noting that when $\nu k_o (z_1 - z_o) >> 1$ the fractional kernel in Eq. (20) can be approximately written as

$$K_\nu\left(x - x' , z_1 - z_o\right) = K\left(x - x' , z_\nu - z_o\right) \cong \sqrt{\tfrac{k_o}{2\pi}} \frac{i\,(z_\nu - z_o)\, e^{-i\frac{3\pi}{4}}}{\left[(x - x')^2 + (z_\nu - z_o)^2\right]^{\frac{3}{4}}} e^{ik_o \sqrt{(x-x')^2 + (z_\nu - z_o)^2}} . \quad (22)$$

where $\nu(z_1 - z_o)$ is substituted by $z_\nu - z_o$. Additionally, if we now use the standard Fresnel approximation (see, e.g., [11, p. 59]), for which here as a sufficient condition one would assume that $\nu^3 (z_1 - z_o)^3 >> \frac{k_o}{8} (x - x')^4_{\max}$, we obtain

$$K_\nu\left(x - x' , z_1 - z_o\right) = K\left(x - x' , z_\nu - z_o\right) \cong \frac{e^{ik_o (z_\nu - z_o)}}{\sqrt{i\,\lambda_o\,(z_\nu - z_o)}} e^{i\frac{k_o}{2(z_\nu - z_o)}(x-x')^2} \quad (23)$$

which is the usual kernel in the region of Fresnel diffraction for the 2-D propagation when $\nu(z_1 - z_o)$ is substituted by $z_\nu - z_o$, and $\lambda_o = \frac{2\pi}{k_o}$. In the Fraunhofer diffraction region (see e.g., [11, p. 61]), where we can assume $2\nu(z_1 - z_o) >> k_o (x - x')^2_{\max}$, the fractional kernel can be further simplified as

$$K_\nu\left(x - x' , z_1 - z_o\right) = K\left(x - x' , z_\nu - z_o\right) \cong \frac{e^{ik_o (z_\nu - z_o)} e^{i\frac{k_o x^2}{2(z_\nu - z_o)}}}{\sqrt{i\,\lambda_o\,(z_\nu - z_o)}} e^{i\frac{k_o x x'}{z_\nu - z_o}} , \quad (24)$$

which is the conventional Fraunhofer diffraction kernel for the 2-D propagation problem. So as the parameter ν increases from zero and gets increasing larger, the fractional kernel given in Eq. (20) evolves from the Dirac delta function $\delta(x - x')$ for $\nu = 0$, and becomes the kernels for the intermediate zone, and as ν keeps increasing, this fractional kernel evolves into the kernel which, for a certain constraint on x, x', z_ν, and z_o, becomes the kernel for the Fresnel diffraction region, and eventually when ν increases even further it will get to the kernel that again for some specific constraint on x, x', z_ν, and z_o becomes the kernel for the Fraunhofer region. As is well known, aside from the term $\frac{e^{ik_o(z_\nu - z_o)} e^{i\frac{k_o x^2}{2(z_\nu - z_o)}}}{\sqrt{i\lambda_o (z_\nu - z_o)}}$ in the kernel for the Fraunhofer diffraction region, this kernel represents the Fourier transform kernel. In the optics literature, it has been shown that while Fraunhofer diffraction produces the Fourier transform of the object plane, the Fresnel diffraction may result in Fractional Fourier transform of the object [12-14]. The reader interested in the concept of fractional Fourier transformed is referred to the excellent work of Ozaktas, Mendlovic, Lohmann and their co-workers in the literature (see, e.g., [13-18].) Our fractional kernel given in Eq. (20) (or Eq. (17) or (18)) expresses the *exact* kernels for all possible intermediate planes starting from the source/aperture (at z_o) evolving all the way towards the observation planes in the far zone when the fractional parameter ν varies from zero to ∞. So when the fractionalization parameter ν in Eq. (20) (or Eq. (17)) becomes large such that the Fresnel approximation may be applicable, then the fractional kernel would approach the kernel of the Fractional Fourier transform. However, it must be noted that when ν is taken a small value and the term $\nu(z_1 - z_o) \equiv z_\nu - z_o$ is no longer satisfies the conditions $\nu k_o (z_1 - z_o) >> 1$ and the Fresnel approximation, the kernel for the Fresnel diffraction region cannot be used, and instead the exact form of the fractional kernel given in Eq. (20) (or Eq. (17)) should be utilized. So for the fractional kernel given in this report, no constraint on x, x', z_ν, and z_o are necessary. In fact, the concept of fractional kernel can even be extended to the static problems. If one is interested in the fractional kernels for the 2-D electrostatic problem, one can find such kernels by evaluating the limit of Eq. (17) (or Eq. (20)) when $k_o \to 0$. These fractional kernels for the static case can then be explicitly written as

$$K_{\nu}(x-x',z_1-z_o)_{k_o\to 0} = \frac{\nu(z_1-z_o)}{\pi\left[(x-x')^2+\nu^2(z_1-z_o)^2\right]} = \frac{z_\nu - z_o}{\pi\left[(x-x')^2+(z_\nu-z_o)^2\right]} \qquad (25)$$

The fractional kernels presented in this Letter have been for the case of monochromatic 2-D wave propagation, and for observation points on parallel flat planes (i.e., planar observation) in the Cartesian coordinate system. The fractional kernels for the cases of observations over the cylindrical and spherical boundaries, which encounter somewhat different mathematical features, have also been studied [19], and will be reported in a future publication. The analysis reported here can be easily and straightforwardly extended to the three-dimensional (3-D) case for planar observations.

Among the problems of interest following the analysis reported in this Letter is the effect of complexification of the fractionalization parameter ν. Exploring potential applications of the fractionalization of kernels in treating and processing the fields in the intermediate zones of electromagnetic radiation and scattering problems is another problem of interest to pursue. These are currently under study by the author.

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References

[1] N. Engheta, "Fractional Paradigm in Electromagnetic Theory," a chapter in the book entitled *Frontiers of Electromagnetics*, D. H. Werner and R. Mittra (eds.), IEEE Press, to appear in 1999.

[2] N. Engheta, “Fractional Curl Operator in Electromagnetics,” *Microwave and Optical Technology Letters*, Vol. 17, No. 2, pp. 86-91, February 5, 1998.

[3] N. Engheta, "On the Role of Fractional Calculus in Electromagnetic Theory," in *IEEE Antennas and Propagation Magazine*, Vol. 39, No. 4, pp. 35-46, August 1997.

[4] N. Engheta, "Electrostatic "Fractional" Image Methods for Perfectly Conducting Wedges and Cones," *IEEE Trans. Antennas & Propagation*, Vol. 44, No. 12, pp. 1565-1574, Dec. 1996.

[5] N. Engheta, "On Fractional Calculus and Fractional Multipoles in Electromagnetism," *IEEE Trans. Antennas & Propagation*, Vol. 44, No. 4, pp. 554-566, Apr. 1996. Erratum: Vol. 44, No. 9, p. 1307, September 1996.

[6] N. Engheta, "Use of Fractional Integration to Propose Some "Fractional" Solutions for the Scalar Helmholtz Equation," a chapter in *Progress in Electromagnetics Research (PIER)* monograph Series *Vol. 12* , Jin A. Kong (ed.), EMW Pub., Cambridge, MA, pp. 107-132, ch. 5, 1996.

[7] N. Engheta, “A Note on Fractional Calculus and the Image Method for Dielectric Spheres," *J. of Electromagnetic Waves and Applications*, Vol. 9, No. 9, pp. 1179-1188, September 1995.

[8] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[9] A. Ishimaru, *Electromagnetic Wave Propagation, Radiation, and Scattering*, Prentice-Hall, Inc., New Jersey, 1991.

[10] A. Sommerfeld, *Partial Differential Equations in Physics*, (Lectures on Theoretical Physics, Vol. VI), Academic Press, New York, 1949.

[11] J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill Pub., New York, 1968.

[12] P. Pellat-Finet, "Fresnel Diffraction and the Fractional Fourier Transform," *Optics Letters*, Vol. 19, No. 18, pp. 1388-1390, September 15, 1994.

[13][4] H. M. Ozaktas, M. A. Kutay, and D. Mendlovic "Introduction to the Fractional Fourier Transform and Its Applications," in *Advances in Imaging and Electron Physics, Vol. 106*, Peter W. Hawkes (ed.), Academic Press, California, 1999, pp. 239-291.

[14] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier Transform," *Journal of Optical Society of America A*, Vol. 12, No. 4, pp. 743-751, April 1995.

[15] H. M. Ozaktas and M. F. Erden, "Relationship among Ray Optical, Gaussian Beam, and Fractional Fourier Transform Descriptions of First-Order Optical Systems," *Optics Communications*, Vol. 143, pp. 75-86, November 1, 1997.

[16] A. W. Lohmann, "Image Rotation, Wigner Rotation, and the Fractional Fourier Transform," *J. Opt. Soc. Am. A*, Vol. 10, No. 10, pp. 2181-2186, 1993.

[17] D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, "Graded-Index Fibers, Wigner-Distribution Functions, and the Fractional Fourier Transform," *Applied Optics*, Vol. 33, No. 26, pp. 6188-6193, 10 Sept. 1994.

[18] H. M. Ozaktas, B. Barshan, D. Mendlovic, and L. Onural, "Convolution, Filtering, and Multiplexing in Fractional Fourier Domains and Their Relation to Chirp and Wavelet Transforms," *Journal of the Optical Society of America, A.*, Vol. 11, No. 2, pp. 547-559, February 1994.

[4] We thank Professor H. M. Ozaktas of Bilkent University for sharing with us Ref. [13] and some of his other publications on the topic of Fractional Fourier Transform.

[19] N. Engheta, "Fractionalization of Kernels for Electromagnetic Intermediate-Zone Fields in Cylindrical and Spherical Geometries," a talk presented in the *Progress in Electromagnetic Research Symposium (PIERS'98)*, Nantes, France, July 13-17, 1998. The one-page abstract appeared in Vol. 33 of the Proceedings of this conference, page 1144.

Figure Caption

Fig. 1. Geometry of the Problem for the two-dimensional (2-D) wave propagation. A monochromatic current source $\boldsymbol{J}(x, y)$ is located in a Cartesian coordinate system (x, y, z). All quantities of interest are independent of *y*-coordinate. The original two observation flat planes are at $z = z_o$ and $z = z_1$.

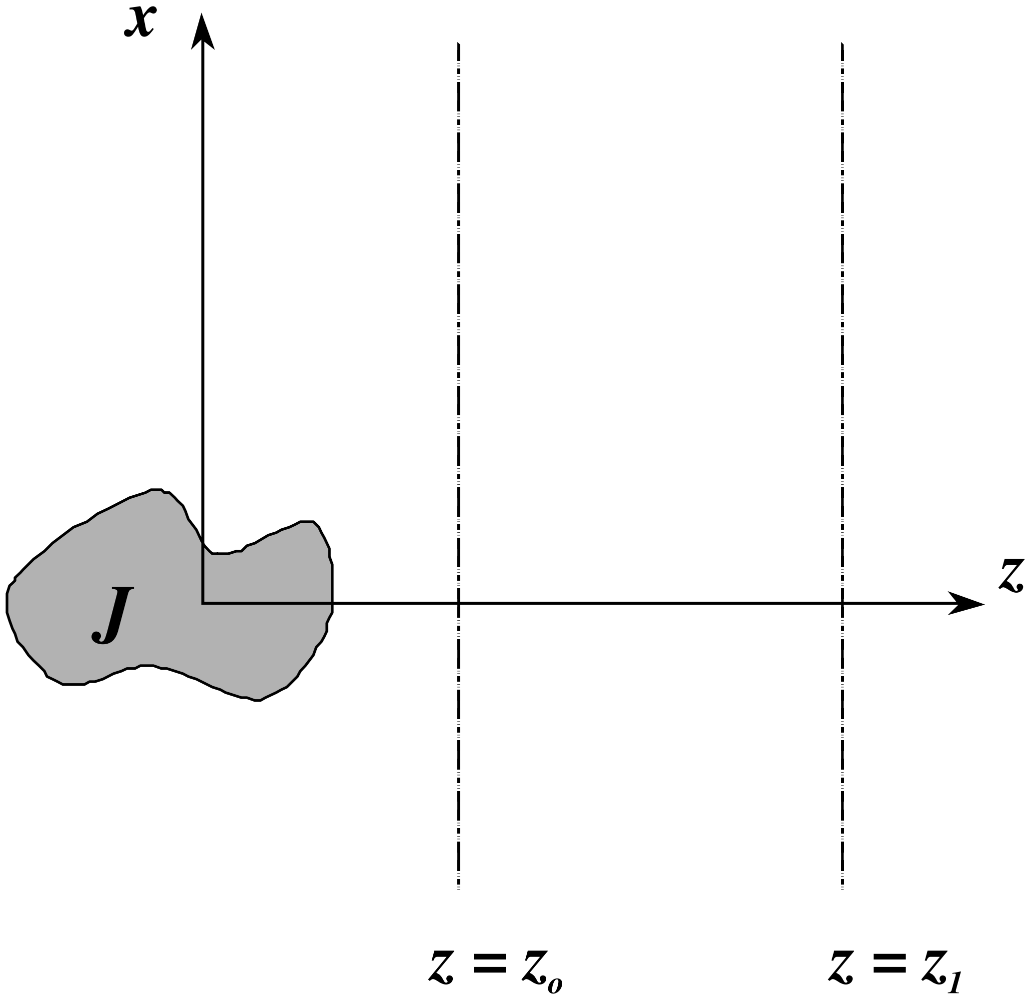

FIGURE 1