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The Dilaton Equation in Semirigid String Theory

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We show how to obtain explicit integration measures on ordinary moduli space corresponding to the correlation functions of pure 2-dimensional topological gravity. In particular our prescription tells how to remove the zero modes of the $\beta\gamma$ system. We then use our formula to derive the "dilaton equation" introduced by E. Verlinde and H. Verlinde, a relation between the N-point and $(N-1)$-point correlations of this theory. Just as in critical string theory we use the fact that certain BRST-exact states fail to decouple. Instead they build up Čech classes, in this instance the Euler class of an N-times punctured surface. Throughout we use the "semirigid" formulation of topological gravity. Thus the Liouville sector of other approaches never enters.

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1. Introduction

The recent development of two-dimensional quantum gravity has seen an alarming proliferation of apparently equivalent theories, some of which seem at first to be utterly dissimilar. Matrix models, continuum gravity with conformal matter, and topological gravity all seem to give the same answers for scaling exponents and indeed for correlation functions. Since it seems hard to attempt a direct correspondence among the elementary dynamical variables of these theories,[1] the general approach to proving their equivalence has been to prove simple relations among their n-point amplitudes. In particular, *recursion* relations can determine part or all of the amplitudes of topological gravity in terms of a few elementary ones [2][3][4][5].

Moreover, such recursion formulæ can teach us more about the common structure presumably underlying each of the incarnations of 2d gravity. For example, Witten has suggested that the compatibility of recursion relations be regarded as an integrability, or "cocycle," condition defining the topological phase [2]. But a full understanding of the recursion formulæ, let alone their deep meaning, is still elusive. For example, a direct derivation of the KdV formula from intersection theory is still lacking.

Each of the various recursion formulæ for topological gravity extant today has a slightly different status. The clearest ones have a direct derivation from the fundamental definition of the theory in terms of intersection theory. These include the special relations for genus zero and one [2], but they also include the remarkable "puncture equation" of ref. [3], as well as [6][7] the "dilaton equation" of ref. [4]. The two latter relations are valid in every genus; adopting the normalization of [2], one has

$$\mathrm{PE}: \qquad \langle\sigma_0 \prod_{i=1}^{N} \sigma_{n_i}\rangle_g = \sum_{j=1}^{N} n_j \langle \prod_{i=1}^{N} \sigma_{n_i-\delta_i^j}\rangle_g \tag{1.1}$$

$$\mathrm{DE}: \qquad \langle\sigma_1 \prod_{i=1}^{N} \sigma_{n_i}\rangle_g = (2g-2+N)\langle \prod_{i=1}^{N} \sigma_{n_i}\rangle_g \quad . \tag{1.2}$$

In addition to the intersection-theory formulation, 2d topological gravity is supposed to admit another instantiation as a free conformal quantum field theory [8][9][10]. In such a framework E. Verlinde and H. Verlinde found a whole series of recursion relations generalizing (1.1)–(1.2) [4]. The first two of these are in fact the puncture and dilaton

[1] But see the recent paper of Kontsevich [1].

equations. The higher recursion formulæ are more complicated; they involve terms on the rhs with surfaces of lower genus. Again these relations were all derived in the context of a field-theoretic formulation of topological gravity, using a somewhat indirect combination of field theory and self-consistency arguments. It would be nice to have a direct derivation from a simple field theory.

In this paper we will give the derivation requested above for the dilaton equation, (1.2), and much more. First we will need to define our local field theory precisely. This is more than a matter of choosing a lagrangian [11]. One must also settle upon a class of observables. This is a nontrivial step; indeed as we will see the topological theory will emerge as a *truncation* of a larger, nontopological theory. (Similar phenomena are well known in topological matter theories [12].) Even after a class of observables has been chosen there remains the problem of computing the correlations. For this one needs a way to construct a volume form on moduli space, which can then be integrated to get answers. It is not in general obvious how to construct such a volume form; later on we will mention a plausible alternative which is simply wrong.

What is needed is clearly some organizing principle determining the lagrangian *and* the physical states *and* the correct volume form, in a mutually consistent way. The "semirigid" geometry recently introduced in ref. [13] will provide such a lodestone for us. It constructs our theory as a truncation of local $N = 2$ supergravity by imposing a self-consistent constraint. The physical states are those of $N = 2$ subject to the constraint. The usual operator formalism techniques then yield a volume form on a constrained subspace of the moduli space of $N = 2$ surfaces. As shown in [13] this volume form then projects to the desired density on ordinary moduli space $\mathcal{M}$.

Remarkably [4] all the physical observables of the constrained theory are BRST-exact. They fail to decouple, however, because while each satisfies the weak physical state condition [14], still each is the BRST-variation of something which does *not*. Thus each computes a Čech cohomology class, just like the dilaton of critical string theory [14][15][16].

In the following sections we will find an explicit formula for the integration measure of topological gravity, including the disposition of the $\beta\gamma$ zero modes. We will see how the semirigid prescription leads to some unexpected terms in the measure, just as it did in the supercurrent [13], and how both sets of unexpected terms are crucial to get the desired answers. We also recall [13] how our geometrical principle gives a precise description for how our surfaces may degenerate.

Applying this point of view to $\mathcal{O}_1$ we will then at once get the dilaton equation, just as in the bosonic string [16]. A similar analysis can be carried out for the puncture equation [17]. Our derivation makes no use of the "Liouville sector" of [4]. Indeed this sector does not arise at all in the analysis of [18], nor in the semirigid framework; nor for that matter did it appear in the critical bosonic string, where the dilaton equation is again valid. Our arguments can be used to analyze the higher $\mathcal{O}_n$ as well, though we will not attempt this here.

Throughout this paper we will consider pure topological gravity. Thus all coupling constants are zero and we work at the first critical point of the matrix model.

2. General remarks

There is by now an extensive literature on topological quantum field theory (TQFT).[2] All such theories share some key features. All begin as gauge-invariant quantum field theories with a *scalar* supersymmetry generated by Q_S. The supersymmetry ensures the near-perfect cancellation of bosonic and fermionic contributions to all amplitudes, and in particular the absence of propagating physical modes. Similarly, after BRST gauge-fixing one finds the BRST cohomology to be trivial.

Nevertheless these theories are not empty. On one hand gauge-fixing does not completely eliminate the original degrees of freedom; a moduli space $\mathcal{M}$ of inequivalent configurations remains. On the other hand, as explained in [20][21][22], the appropriate space of physical states consists of the "equivariant" cohomology with respect to the gauge group: an operator is deemed trivial only if it is the BRST variation of a *gauge-invariant* operator. There are indeed some states nontrivial in this sense. Since they are all trivial in the broader sense, however, they all nearly decouple; indeed they would decouple completely were $\mathcal{M}$ topologically trivial. Instead their correlations pick up topological properties of $\mathcal{M}$. This phenomenon appears quite explicitly in bosonic string theory as well [14]; here the dilaton is almost trivial. In a TQFT *all* physical observables are of this type.

Thus the essence of topological field theory is that it has a BRST-like complex of states with an action of the gauge group, no ordinary cohomology, but some nontrivial equivariant cohomology.

[2] For a review with references see [19]. In this paper we use "TQFT" synonymously with "cohomological QFT."

Such a field theory can be written down for 2d gravity [8][9][10][18][4]. Let us briefly recall the formulation given in [13]. Begin with local $N = 2$ supergravity on a space with coordinates z, θ, ξ, with derivatives $D_\theta \equiv \partial_\theta + \xi\partial_z$, $\widetilde{D}_\xi \equiv \partial_\xi + \theta\partial_z$. We now gauge-fix as usual to get a free $B\,C$ ghost system. We then expand about a symmetry-breaking vev for one of the ghost field components. More precisely we impose the *constraint* $\widetilde{D}_\xi C^{\theta\xi} \equiv$ const. The meaning of this constraint is that it breaks the local superconformal symmetry down to an anomaly-free subalgebra, including one isolated generator which plays the role of Q_S. But of course we cannot stop here. Whenever we impose a first-class constraint we get the problem that time evolution becomes undetermined; any amount of the constraint may be added to the hamiltonian. Alternately in quantum mechanics the constraint on C is incompatible with the canonical commutation relations (or free propagator) of the BC system. The cure for all these problems is of course to impose canonically-conjugate conditions on the *observables* of the theory. For example, in Yang-Mills theory if we choose to impose Gauss's law before quantization we find we may only observe gauge-invariant quantities, since the Gauss constraint generates gauge transformations under Poisson bracket. Similarly in our case we must require of all observables that they not depend on two of the components of B. In the language of [13] this allows dependence on b, β only, not on $\breve{b}, \breve{\beta}$. The unbroken symmetry generators L_n, G_n, and Q_S all have this property automatically. For example,

$$G = -2\beta\partial c - (\partial\beta)c - b. \tag{2.1}$$

The $N = 2$ BRST charge also descends to our constrained theory. We find

$$Q_T \equiv Q^{N=2}_{BRST} = -\tfrac{1}{2}\oint \mathrm{d}\mathbf{z}\, \widetilde{D}B\, C\, DC \quad ,$$

where $\mathrm{d}\mathbf{z}$ denotes $[\mathrm{d}z|\mathrm{d}\theta\mathrm{d}\xi]$. Q_T is seen to be nilpotent, even though the original $N = 2$ ghost system was anomalous. It is also invariant under L_n, G_n and so defines the required scalar BRST charge. Moreover one has $Q_T = Q_S + Q_V$ where Q_V is the BRST charge associated to the Virasoro algebra, so Q_T is the operator of [4]. Finally one has $\{Q_T, b_n\} = L_n$ and $[Q_T, \beta_n] = -G_n$. Adapting the argument of [23], these commutation relations tell us that Q_T plays the role of the exterior derivative on an appropriate supermoduli space $\widehat{\mathcal{M}}$. Namely, $\widehat{\mathcal{M}}$ is the space of supermanifolds built from pieces of the $z|\theta\xi$ plane patched together by maps corresponding to the unbroken generators L_n, G_n:

$$\begin{aligned} z' &= f(z) + \theta\rho(z) \\ \theta' &= \theta \\ \xi' &= \rho(z) + \xi\partial f(z) - \theta\xi\rho(z) \quad . \end{aligned} \tag{2.2}$$

Thus θ has spin zero. We will call supermanifolds of the form (2.2) “semirigid” Riemann surfaces, or SSRS.

Physical vertex operators $\mathcal{O}$ must now be annihilated by $Q_T + \bar{Q}_T$ and subject to the above constraint on their B-dependence. Any collection $\{\mathcal{O}_i\}$ of such operators yields a form of some sort on the space $\widehat{\mathcal{P}}_{g,N}$ of SSRS with g handles, N marked points, and local superconformal coordinates at those points, just as in the fermionic string [23]. Note that in this form B potentially enters both via the explicit operators $\mathcal{O}$, and via the insertions $B[v] \equiv \oint \mathrm{d}\mathbf{z}\, Bv$ needed to soak up the background charge of the BC system. These insertions are not manifestly functions of $\widetilde{D}B$ only, but they are nonetheless compatible with the constraint if v is a *chiral* tensor, $\widetilde{D}v = 0$. Indeed such v do generate the infinitesimal form of (2.2).[3] Demanding that all external states be independent of $\breve{b}, \breve{\beta}$ thus gives a consistent truncation of our theory. Henceforth we will drop the fields $\breve{b}, \breve{\beta}$ altogether.

What we need, however, is a volume form on $\mathcal{M}_{g,N}$, the *ordinary* moduli space *without* local coordinates, for it is here that the intersection-theory definition of topological gravity lives [2]. We can get to $\mathcal{M}_{g,N}$ in two steps. First, as explained in [23] we can reduce from $\widehat{\mathcal{P}}_{g,N}$ to $\widehat{\mathcal{M}}_{g,N}$ if in addition to

$$(Q_T + \bar{Q}_T)|\mathcal{O}\rangle = 0 \tag{2.3}$$

we require the “strong physical state condition:”

$$b_n|\mathcal{O}\rangle = \beta_n|\mathcal{O}\rangle = L_n|\mathcal{O}\rangle = G_n|\mathcal{O}\rangle = 0 \quad , \qquad n \geq 0. \tag{2.4}$$

Of course to get nonzero answers we must also require that $|\mathcal{O}\rangle$ have appropriate ghost charges. It turns out that the present theory has only one nontrivial operator of this type, the “puncture” operator $|\mathcal{O}_0\rangle = c_1\bar{c}_1\delta(\gamma_1)\delta(\bar{\gamma}_1)|0\rangle$. But a weaker condition than (2.4) suffices, as explained in [14][15][16]. If (2.4) is replaced by the “weak physical state condition, ” or WPSC,

$$(b_0 - \bar{b}_0)|\mathcal{O}\rangle = 0 \quad , \tag{2.5}$$

then we can get a volume form on $\widehat{\mathcal{M}}_{g,N}$ if we first choose a “slice,” a choice $\hat{\sigma} : \widehat{\mathcal{M}} \to \widehat{\mathcal{P}}$ of local coordinates near each puncture.

Before we ask how our answers depend on the choice of $\hat{\sigma}$, we still must pass from the supermoduli space $\widehat{\mathcal{M}}$ to the ordinary space $\mathcal{M}$. Fortunately this is easy. There is a

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[3] Note that the generalization of (2.2) with $\theta' = \theta +$const. is *not* generated by chiral v, which is why we did not allow it.

natural projection [13] $\pi : \widehat{\mathcal{M}} \to \mathcal{M}$; we integrate over the fibers of π to obtain at last a volume form Ω on $\mathcal{M}_{g,N}$.

The analysis of [14][15] now shows that changing the slice $\hat{\sigma}$ changes Ω by a total derivative. It is well known that such total derivative ambiguities can be important in string theory. Indeed we will recall later how they enter in the bosonic string calculation of [16]. Remarkably, however, we will prove that in topological gravity this residual dependence vanishes, at least in the collision of $\mathcal{O}_1$ with any $\mathcal{O}_n$, yielding contact interactions which are completely independent of choices. In fact they are just winding numbers in the complex plane.

As mentioned above the insertion of $|\mathcal{O}\rangle = (Q_T + \bar{Q}_T)|\mathcal{O}'\rangle$ where $|\mathcal{O}'\rangle$ obeys (2.5) also yields a total derivative on $\widehat{\mathcal{M}}$. The states obeying (2.5) and not trivial in this "equivariant" sense are

$$|\mathcal{O}_n\rangle = (\gamma_0)^n c_1 \bar{c}_1 |-1\rangle \ , \qquad |-1\rangle \ \equiv \ \delta(\gamma_1)\delta(\bar{\gamma}_1)|0\rangle \quad . \tag{2.6}$$

We write $|n\rangle$ to denote the Fock vacuum at Bose sea level n. The states (2.6) are not quite the same as the states discussed in [4], but they differ by terms which are truly trivial. The difference is nevertheless important as we shall see in due course. We would like to identify $\mathcal{O}_n$ with σ_n in (1.1), (1.2) (up to some constant) and in particular $\mathcal{O}_1/2\pi i$ with the dilaton, σ_1. Indeed an analysis along the lines of [15] shows [17] that the contribution to an $(N+1)$-point function including an $\mathcal{O}_1$ is $2\pi i(2g-2)$ times the n-point function without the dilaton, plus collision terms which we must compute.

Our goal is thus to establish well-defined delta-function contact terms in the correlations of $\mathcal{O}_1$ with other $\mathcal{O}_n$ and hence complete the derivation of (1.2). In order to say that we really understand such terms we need to find a way to smooth them out, recovering their singular form in some limit. We did this in [16] for the case of the bosonic string dilaton. There the crucial observation was that the states $|\mathcal{O}_n\rangle$, $n > 0$, since they fail the strong physical state condition, are sensitive to their normal ordering.[4] The normal ordering which is nonsingular as a surface pinches is however incompatible with the choice which can be made globally. We must interpolate between these and in the process pick up the desired smoothed contact terms.

[4] Originally it seemed that the Liouville sector could be used to covariantize vertex operators so that this normal-ordering analysis would not be needed, but this no longer seems to be the case [24].

3. The measure

In this section we will make the above program very concrete. Namely we will begin with a collection of physical states $|\mathcal{O}_{n_i}\rangle$ obeying the conditions (2.3), (2.5) and the choice of a slice $\sigma : \mathcal{M} \to \mathcal{P}$ on an open set of $\mathcal{M}$. We will promote σ to a slice $\hat{\sigma}$ for $\widehat{\mathcal{M}}$, compute the volume form $\widehat{\Omega}$ on $\widehat{\mathcal{M}}$, and integrate it over the odd directions to get Ω on $\mathcal{M}$.

We need to make some additional choices before we can write down formulas. One can check that these choices in fact drop out of our final formula for the measure, or in other words that our formalism is covariant. First we choose coordinates $m^1, \ldots, m^K$ for $\mathcal{M}_{g,N}$, where

$$K \equiv 3g - 3 + N \quad .$$

In addition we realize the map $\sigma : \mathcal{M} \to \mathcal{P}$ by giving a family of Riemann surfaces depending on $\vec{m}$. Thus we give a collection of patching functions $u_\alpha = f_{\alpha\beta}(u_\beta; \vec{m}\,)$, with the understanding that at most one puncture P_i lies on any patch $\mathcal{U}_\alpha$, and that in that case $u_\alpha(P_i) \equiv 0$.

To keep the notation simple we will suppose our family to be of a special form. Namely we will choose one puncture $P_0 \in \mathcal{U}_0$ and suppose $\mathcal{U}_0$ to be completely surrounded by another patch $\mathcal{U}_1$ with annular overlap. Moreover all $f_{\alpha\beta}$ are assumed independent of $\vec{m}$ except for $u_0 = f_{01}(u_1; \vec{m}\,)$. In a small enough region of $\mathcal{M}$ we can always represent σ this way. (It is easy to generalize to other situations.)

We can now promote our family of Riemann surfaces to a family of semirigid Riemann surfaces following [13]. As shown in [13], to study surfaces patched from maps of the form (2.2) it suffices to consider those constructed from the $z|\theta$ plane via[5]

$$\begin{aligned} z' &= f(z) + \theta\rho(z) \\ \theta' &= \theta \quad . \end{aligned} \tag{3.1}$$

Again θ is a globally-defined coordinate. In fact we can say still more. Given any family of Riemann surfaces built from patches $\mathcal{U}_\alpha \subset \mathbf{C}$ with transition functions

$$u_\alpha = f_{\alpha\beta}(u_\beta; \vec{m})$$

[5] As explained in [13] we could have equally used the description in ref. [24] of this geometry; however, our choice will lead to easier algebra.

depending on parameters $\vec{m}$ we can always introduce an equal number of Grassmann parameters $\vec{\zeta}$ and construct

$$
\begin{aligned}
z_\alpha &= F_{\alpha\beta}(z_\beta, \theta_\beta; \vec{m}, \vec{\zeta}\,) \;\equiv\; f_{\alpha\beta}(z_\beta; \vec{m} + \theta\vec{\zeta}\,) \\
\theta_\alpha &= \theta_\beta
\end{aligned}
\tag{3.2}
$$

which is of the form (3.1). Changing to a different set of coordinates $\vec{m}'$ induces a split coordinate transformation on $(\vec{m}, \vec{\zeta}\,)$. In our case we have

$$
z_0 \;\equiv\; F_{01}(z_1, \theta; \vec{m}, \vec{\zeta}\,) = f_{01}(z_1; \vec{m}\,) + \theta\zeta^a \frac{\partial f_{01}(z_1; \vec{m}\,)}{\partial m^a}
\tag{3.3}
$$

and we regard all the z_α as fixed except for z_0.

Our choice of slice (3.3) has two key features. First the coordinates $(\vec{m}, \vec{\zeta}\,)$ defined by (3.2) define a split structure for $\widehat{\mathcal{M}}$, and hence in particular a projection $\pi : \widehat{\mathcal{M}} \to \mathcal{M}$. That is, the $\vec{m}$ are defined to be lifted from the corresponding coordinates on $\mathcal{M}$. One can check that starting from a different set of coordinates $\vec{m}'$ induces a split transformation of $(\vec{m}, \vec{\zeta}\,)$, and that choosing a different presentation of the same family of surfaces likewise changes nothing. Since the coordinates $(\vec{m}, \vec{\zeta}\,)$ are adapted to the projection, to integrate $\widehat{\Omega}$ along the fibers of π we simply insert $\frac{\partial}{\partial \zeta^a}$ into the odd slots of $\widehat{\Omega}$ and integrate the fiber coordinates $\vec{\zeta}$.

The second key feature of (3.3) is that everywhere it obeys

$$
\frac{\partial z_0}{\partial \bar{\zeta}^i} = 0 \;, \qquad \frac{\partial z_0}{\partial \zeta^a} = \theta \frac{\partial z_0}{\partial m^a} \quad .
\tag{3.4}
$$

Eqns. (3.4) are more general than our special family (3.3); they apply to any family promoted via (3.2) from a family of ordinary Riemann surfaces. They make global sense because different coordinate systems $(\vec{m}', \vec{\zeta}')$ are related by split, holomorphic transformations, and because θ is a global coordinate. The virtue of (3.4) is that it will make the measure $\widehat{\Omega}$ constructed using (3.3) very simple.

To find this measure we need the ghost insertions appropriate to the slice (3.3). As usual [25][14][16] this means we must differentiate z_0 with respect to the moduli, then reexpress the answer in terms of z_0 itself. Thus in the bosonic string we would need

$$
v_a(z_0, \vec{m}\,) \;\equiv\; \frac{\partial f_{01}}{\partial m^a} \circ f_{01}^{-1} \;, \quad \widetilde{v}_a(z_0, \vec{m}\,) \;\equiv\; \frac{\partial f_{01}}{\partial \bar{m}^a} \circ f_{01}^{-1} \quad ,
\tag{3.5}
$$

where the composition and inverse refer to the z-dependence. We now need a corresponding expression for F_{01}. First we note that

$$z_1 = f_{01}^{-1}(z_0, \vec{m}\,) - \theta\zeta^a \Big(\frac{1}{f'_{01} \circ f_{01}^{-1}}\Big) v_a \quad ,$$

where prime denotes derivative with respect to the first argument. Next we define $V_a(z_1, \theta; \vec{m}, \vec{\zeta}\,)$ by

$$F_{01}(z_1, \theta; \vec{m} + \vec{\Delta}, \vec{\zeta}\,) = \Big(\mathbb{I} + \Delta \cdot V + \bar{\Delta} \cdot \widetilde{V}\Big) \circ F_{01}(z_1, \theta; \vec{m}, \vec{\zeta}\,) + \mathcal{O}(\Delta^2) \quad ,$$

analogously to (3.5). Using the identities

$$\begin{aligned}
&\delta(A \circ B^{-1}) = (\delta A) \circ B^{-1} - (A' \circ B^{-1}) \cdot \frac{1}{B' \circ B^{-1}} \cdot (\delta B \circ B^{-1}) \\
&(A \circ B^{-1})' = A' \circ B^{-1} \cdot \frac{1}{B' \circ B^{-1}} \quad ,
\end{aligned}$$

we compute $\frac{\partial v_a}{\partial m^b}$ and v'_a to obtain

$$V_a = v_a + \theta\zeta^b \frac{\partial v_a}{\partial m^b} \;, \quad \widetilde{V}_a = \widetilde{v}_a + \theta\zeta^b \frac{\partial \widetilde{v}_a}{\partial m^b} \quad . \tag{3.6}$$

Similarly varying $\vec{\zeta}$ instead of $\vec{m}$ yields

$$\Upsilon_a = \theta v_a \;, \qquad\qquad \widetilde{\Upsilon}_a = 0 \quad .$$

These imply that the desired ghost insertions are

$$\hat{B}\Big(\frac{\partial}{\partial m^a}\Big) \;\equiv\; b_a - \zeta^b \beta_{ab} + \overline{b_{\tilde{a}}} - \bar{\zeta}^b \overline{\beta_{\tilde{a}b}} \tag{3.7}$$

$$\delta\Big(\hat{B}\Big(\frac{\partial}{\partial m^a}\Big)\Big) \;\equiv\; \delta(\beta_a) \tag{3.8}$$

where

$$\begin{aligned}
&b_a = b[v_a] \;\equiv\; \oint b_{zz}(z) v_a^z(z)\,\mathrm{d}z \;; \quad b_{\tilde{a}} = b[\widetilde{v}_a] \\
&\beta_a = \beta[v_a] \\
&\beta_{ab} = \beta[\partial v_a / \partial m^b] \;; \quad \beta_{\tilde{a}b} = \beta[\partial \widetilde{v}_a / \partial m^b] \quad .
\end{aligned} \tag{3.9}$$

As mentioned earlier the simplicity of (3.7) is a consequence of (3.4).

To integrate $\vec{\zeta}$ we will need the explicit $\vec{\zeta}$-dependence of the state $\langle\Sigma(\vec{m},\vec{\zeta})|$ associated to our surface by the operator formalism. Using (3.3) we get

$$\begin{aligned}\langle\Sigma(\vec{m},\vec{\zeta})| &= \langle\Sigma, F_{01}| = \langle\Sigma, (\mathbf{1}+\theta\zeta^a v_a)\circ f_{01}| \\ &= \langle\Sigma, f_{01}|(\mathbf{1}-\zeta^a G_a - \bar{\zeta}^a\bar{G}_a) \quad .\end{aligned} \tag{3.10}$$

Let us now assemble the ingredients we have found. To evaluate the volume form $\widehat{\Omega}$ on the tangent vectors $\frac{\partial}{\partial m^1},\ldots,\frac{\partial}{\partial\zeta^1},\ldots,\frac{\partial}{\partial\bar{m}^1},\ldots,\frac{\partial}{\partial\bar{\zeta}^1},\ldots,\frac{\partial}{\partial\bar{\zeta}^K}$ we must compute a certain correlation function, in which we insert the desired states $\mathcal{O}_{n_1},\ldots,\mathcal{O}_{n_N}$ and the ghost insertions (3.7)–(3.8). The states (2.6) have ghost charges

$$\mathcal{O}_n \; : \qquad (U_{bc}, U_{\beta\gamma}; U_{\bar{b}\bar{c}}, U_{\bar{\beta}\bar{\gamma}}) = (1, n-1, 1, -1) \quad .$$

Let us assume for now that the inserted states obey the condition for topological amplitudes [18][2]:

$$\sum n_i = K \ , \qquad K \;\equiv\; 3g-3+N \quad . \tag{3.11}$$

Then $\otimes_{i=1}^N|\mathcal{O}_{n_i}\rangle$ has charges $(N, 3g-3, N, -N)$. The anomaly on a surface of genus g equals $(3g-3, -3g+3, 3g-3, -3g+3)$. Also we have seen that the ghost insertions corresponding to odd tangent vectors (3.8) are very simple, contributing $(0, K, 0, K)$. Thus the various other insertions on the surface must contribute $(-K, -K, -K, 0)$.

There are K insertions of type $\hat{B}(\frac{\partial}{\partial m^a})$, and K of type $\hat{B}(\frac{\partial}{\partial\bar{m}^a})$. From (3.7) each can have four types of terms:

(i) b terms of charge (-1,0,0,0) , $\quad$ (ii) $\bar{b}$ terms of charge (0,0,-1,0) ,

(iii) $\zeta\beta$ terms of charge (0,-1,0,0) , $\quad$ (iv) $\bar{\zeta}\bar{\beta}$ terms of charge (0,0,0,-1) .

The Grassmann integral over $\mathrm{d}\vec{\zeta}\mathrm{d}\vec{\bar{\zeta}}$ will in general bring down some factors of the supercurrent. Again using (3.2), (3.4) all such terms involve $\zeta^a G_a$ (never $\zeta^a\zeta^b L_n$, *etc.*) and we get four more types of insertion (see (2.1)):

(α) $\zeta c\beta$ terms of charge (1,-1,0,0) , $\quad$ (β) $\bar{\zeta}\bar{c}\bar{\beta}$ terms of charge (0,0,1,-1) ,

(γ) ζb terms of charge (-1,0,0,0) , $\quad$ (δ) $\bar{\zeta}\bar{b}$ terms of charge (0,0,-1,0) .

Imposing now the anomalous conservation of $U_{\bar{\beta}\bar{\gamma}}$ we find that terms (iv), (β) cannot contribute. Integrating $\mathrm{d}\vec{\bar{\zeta}}$ then shows we must use *all* K of the (δ) terms. Thus the inhomogeneous terms of $\bar{G}$ are necessary to get nonzero answers. Imposing conservation of $U_{\bar{b}\bar{c}}$ we then find that terms (*ii*) cannot contribute. Finally charge conservation in the holomorphic sector shows that terms (γ) cannot contribute either.

Thus our integrand vanishes unless every $\hat{B}\big(\frac{\partial}{\partial \bar{m}^a}\big)$ gives rise to a *holomorphic* insertion, *i.e.* all $2K$ terms of type (*i*), (*iii*) must be used. This in turn requires our slice to be maximally nonholomorphic, *i.e.* all of the $\widetilde{v}_a$ in (3.5) must be nonzero in order to get a nonzero answer. This useful property depends crucially on taking the holomorphic form of the states $|\mathcal{O}_n\rangle$, eqn. (2.6).

Having done the integral over $\mathrm{d}\vec{\bar{\zeta}}$, we finally integrate $\vec{\zeta}$ to get the desired volume form Ω on $\mathcal{M}_{g,N}$:

$$\Omega\Big(\frac{\partial}{\partial m^1}, \ldots, \frac{\partial}{\partial \bar{m}^K}\Big) = \int \mathrm{d}^K\vec{\zeta}\ \langle \Sigma(\vec{m})\mid \prod_{c=1}^{K}(\mathbf{1}-\zeta^c G_c)\prod_{a=1}^{K}\big[(b_a-\zeta^b\beta_{ab})(b_{\tilde{a}}-\zeta^b\beta_{\tilde{a}b})\big]$$
$$\times \prod_{b=1}^{K}\big[\overline{b_b}\delta(\beta_b)\overline{\delta(\beta_b)}\big]\mid \mathcal{O}_{n_1}\rangle\otimes\cdots\otimes|\mathcal{O}_{n_N}\rangle \quad . \tag{3.12}$$

Here $\langle\Sigma(\vec{m})|$ is the state in the $bc\beta\gamma$ Fock space associated to the ordinary Riemann surface at $\vec{\zeta}=0$ and

$$G^{(0)} = -2\beta\partial c - (\partial\beta)c \tag{3.13}$$

is the quadratic bit of G.

Eqn. (3.12) is our desired explicit formula for the measure on $\mathcal{M}$. We have written it entirely in terms of ordinary CFT on ordinary Riemann surfaces. It depends on the chosen slice σ via (3.9), (3.6), and (3.5). One can show however that changing our family f_{01} of patching functions without changing σ modifies the v_a by Borel vectors and leaves (3.12) unchanged. Notice that we have managed to conserve all four charges *separately*, and not just the linear combination $U_{bc}+2U_{\beta\gamma}$. Note also that in the formula (3.12) we can drop the antiholomorphic fields altogether: the left-moving part of the correlation function is just a constant. This follows from the form (2.6) of the states; the insertions $b_a\delta(\beta_a)$ convert all of them into the SL_2-invariant $|0\rangle$ in the barred sector.

The striking thing about (3.12) is that only the first of the three terms in parentheses looks familiar. In this term there are ordinary b-ghost insertions for moduli, as is familiar from the bosonic string, and enough picture-changing operators $G_b^{(0)}\delta(\beta_b)$ to satisfy the Bose sea anomaly. The other terms seem superfluous. Nevertheless they are a definite consequence of the semirigid approach, coming from the extra terms in (3.7) and of the inhomogeneous term $-b$ of G.[6] They are also definitely necessary to get the desired answers. Without them, for instance, the dilaton-dilaton contact term comes out incorrectly normalized relative to dilaton-puncture.

Eqn. (3.12) defines the measure in general. We now turn to an application to see how contact terms can arise in this theory.

4. The dilaton equation

We begin with an extremely simple rederivation of the result of ref. [16] for the contact interaction in the bosonic string. A key conclusion of [16] was that in the bosonic string the contact terms thus defined are not quite fixed by geometry. We had to *choose* a standard fixture, a 3-punctured sphere with coordinates which we tied to the rest of the surface by the usual plumbing construction. Placing the punctures at ∞, 0, and 1, we can write this fixture as

$$\Big(\mathbf{P}^1,\, z^{-1},\, z+a_1z^2+a_2z^3+\cdots,\, z-1+\tilde{a}_1(z-1)^2+\tilde{a}_2(z-1)^3+\cdots\Big) \quad . \tag{4.1}$$

One then finds that while $a_2,\ \tilde{a}_2\ldots$ drop out of the calculation, the strength of the contact interaction depends on $a_1,\ \tilde{a}_1$. We gave a physical motivation for one choice of (4.1)[7] but this residual dependence is annoying. Thus in the following derivation we will separate the contact term into a piece which is obviously independent of all choices, plus a correction. This separation is equally valid for topological gravity. We will then examine the correction and show that it *vanishes* for topological gravity but not for the bosonic string. Briefly the answer lies in the nature of the semirigid plumbing construction [13]. Finally we will comment on the derivation along the lines of [16], which yields the same answers in a rather surprising way.

For the bosonic string we use ordinary geometry. We insert a dilaton at P and an ordinary vertex operator $|\psi\rangle$ at Q. To describe a degenerating Riemann surface we take two *fixed* surfaces with coordinates $(\Sigma_L,\sigma_X),\ (\Sigma_R,\sigma_Y)$ and join them via the plumbing $\sigma_X=q/\sigma_Y$. Taking the only q dependence to be in the neck as $q\to 0$ is the essence of the stable-curve compactification. The key to the analysis of [16] was to generalize this prescription to curves with marked points *and* coordinates centered at those points, so that

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[6] For another approach to these terms using multi-valued Beltrami differentials see [24].

[7] It amounts to $a_1=\tilde{a}_1=\frac{1}{2}$, *etc.*

we can insert arbitrary states not obeying (2.4). Thus now Σ_L, Σ_R have coordinates on various other points $\{P_i\}$ besides X, Y, and again these are independent of q; the only q dependence is again in the neck.

For the case at hand we tie (4.1) onto any (Σ_L, σ_X) to get $(\Sigma_L, \check{\sigma}_P, \check{\sigma}_Q)$ where

$$\check{\sigma}_P = \frac{\sigma_X}{q} - 1 + a_1(\frac{\sigma_X}{q} - 1)^2 + \cdots \qquad |q| \ll \epsilon \tag{4.2}$$

and similarly at Q, where ϵ is some small fixed number. For our construction to make sense the value of ϵ should drop out in the end. When $|q| \gg \epsilon$, however, we should use a coordinate like

$$\check{\sigma}'_P = \sigma_X - q \qquad |q| \gg \epsilon \quad , \tag{4.3}$$

as it is (4.3), not (4.2), which makes global sense as P, Q move over the surface. Recall that both $\check{\sigma}, \check{\sigma}'$ depend on two moduli, namely q and the location r of the attachment point X where σ_X is centered. We see from (4.2) that different choices of fixture (4.1) amount to different slices and hence potentially different answers.

Since (4.2) and (4.3) disagree, to get a smooth slice we must interpolate between them in the region $|q| \sim \epsilon$. Outside this region, say at $|q| \sim 2\epsilon$, we may take

$$\check{\sigma}_P = \frac{|q|}{q}(\sigma_X - q) = \frac{|q|}{q}\check{\sigma}'_P \qquad |q| \sim 2\epsilon \quad , \tag{4.4}$$

but we *cannot* smoothly remove the overall phase because it *winds* as we move around $q = 0$.

The analog of (2.6) for the bosonic string is the dilaton state $|D\rangle = 2c_1 c_{-1}|0\rangle$. Suppose we insert this using the slice (4.3). To find the appropriate ghost insertions we follow sect. three, differentiating the slice and expressing the answer in power series in $\check{\sigma}'$. This gives $\hat{b}(\frac{\partial}{\partial q})\hat{b}(\frac{\partial}{\partial \bar{q}}) = b_{-1}\bar{b}_{-1}$, which kills $|D\rangle$. Something similar happens with (4.2). To get a nonzero answer we need our slice to depend *non*holomorphically on q. But this is precisely what is happening in the interpolation region, $\epsilon < |q| < 2\epsilon$, as we see from (4.4). Thus we have smoothed the delta function into this annular region as desired. To compute it we now observe that $|D\rangle = Qc_0|0\rangle$, so that the form

$$\Omega = \mathrm{d}\nu$$

is a total derivative on $\epsilon < |q| < 2\epsilon$. On the boundary $|q| = 2\epsilon$ we find that

$$\nu\Big(\cdots, \frac{\partial}{\partial q}\Big) = \langle \Sigma_L, \check{\sigma}_p, \check{\sigma}_Q \mid \cdots \hat{b}\big(\frac{\partial}{\partial q}\big)\, c_0|0\rangle_P \otimes \mid \psi\rangle_Q \tag{4.5}$$

and similarly when inserting $\frac{\partial}{\partial\bar{q}}$. On the left the ellipsis denotes other tangent vectors besides $\frac{\partial}{\partial q}$, $\frac{\partial}{\partial\bar{q}}$; on the right it denotes the corresponding ghost insertions. Differentiating (4.4) shows that

$$\hat{b}(\frac{\partial}{\partial q})\Big|_{|q|\sim 2\epsilon} = \frac{|q|}{q} b^P_{-1} + \frac{1}{2q}(b^P_0 + b^Q_0 - \bar{b}^P_0 - \bar{b}^Q_0) \quad .$$

Since $|\psi\rangle$ obeys (2.4) we get

$$\begin{aligned}\oint_{|q|=2\epsilon} \nu =& \oint \Big(\frac{\mathrm{d}q}{2q} - \frac{\mathrm{d}\bar{q}}{2\bar{q}}\Big)\langle\Sigma_L, \check{\sigma}_Q \mid \cdots \mid \psi\rangle_Q \\ =& 2\pi i \Omega_\psi\end{aligned} \tag{4.6}$$

where Ω_ψ is the measure with no dilaton inserted. As promised this is a topological invariant — it is precisely the *winding number* of the phase $\frac{|q|}{q}$ relating (4.3) to (4.4).

We are not done, however. On $|q| = \epsilon$ we know $\Omega = 0$ because (4.2) is holomorphic in q. Still it does not follow that $\nu = 0$, since in (4.5) we dropped one of the tangents $\frac{\partial}{\partial q}$, $\frac{\partial}{\partial\bar{q}}$. In fact differentiating (4.2) and the similar expression at P gives

$$\hat{b}(\frac{\partial}{\partial q})\Big|_{|q|\sim\epsilon} = b^P_{-1} + (1+2a_1)b^P_0 + b^Q_0 + (b_{\geq 1} \text{ terms}) \quad .$$

Again substituting into (4.5) gives (taking into account the orientation of the boundary)

$$-\oint_{|q|=\epsilon} \nu = -\oint \frac{\mathrm{d}q}{q}(1+2a_1)\langle\Sigma, \check{\sigma}_Q \mid \cdots \mid \psi\rangle_Q = -2\pi i(1+2a_1)\Omega_\psi \quad . \tag{4.7}$$

Combining (4.6) with (4.7) gives the full answer, which does depend on the slice via a_1 as anticipated. Now choosing $a_1 = -\frac{1}{2}$ eliminates the inner piece, giving us the answer of [16].

We should pause to comment on a confusing point. If $|\psi\rangle$ above is *itself* a dilaton then we cannot drop the terms involving $b^Q_{\geq 1}$ above. Such terms modify the strength of the contact term, but they do *not* lead to divergent answers, in apparent contradiction with [16]. Mathematically the source of this difference is clear: by switching from $(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle$ to $2c_1 c_{-1}|0\rangle$ we discarded a genuine, but divergent, total derivative. Physically the meaning of this step is simply that the fusion of two of the present dilatons can never yield the tachyon $c_1\bar{c}_1|0\rangle$, while in [16] this did happen. For similar reasons we chose the holomorphic form of the observables of topological gravity in (2.6).

We are now ready to repeat our contact term analysis, this time in topological gravity. In our analysis of the bosonic string we began by observing that the measure $\Omega = 0$ close to $q \to 0$. In sect. three however we proved a stronger statement in topological gravity: $\Omega = 0$ everywhere unless the slice σ is as far as possible from being holomorphic. That is, if any $\frac{\partial \sigma_i}{\partial \bar{m}^a} = 0$ then $\Omega = 0$. In particular (4.2) is holomorphic in q, so once again $\Omega \equiv 0$ near $q \to 0$, and hence we can use the same strategy as before to split the contact term into two bits.

Now we use (2.6) at both P, Q, with $n = 1$ at P. With the notion of puncture given in [13] we see that again (4.1) is the most general fixture in $\widehat{\mathcal{P}}_{0,3}$.[8] Now however we must change the plumbing from $\sigma_X = q/\sigma_Y$ to the one appropriate to semirigid geometry [13]:

$$\sigma_X = (q + \theta\delta)/\sigma_Y \quad . \tag{4.8}$$

Here θ is the same on both sides of the pinch and δ is an anticommuting modulus, the partner of q. Eqn. (4.8) obeys (3.4); it gives (*cf.* (4.2))

$$\check{\sigma} = q^{-1}\big[\sigma_X - Eq - \theta\delta q^{-1}\sigma_X\big] + a_1 q^{-2}\big[\sigma_X - Eq - \theta\delta q^{-1}\sigma_X\big]^2 + \cdots , \qquad |q| \ll \epsilon \quad , \tag{4.9}$$

which is to be compared to (*cf.* (4.3))

$$\check{\sigma}' = \sigma_X - Eq - E\theta\delta , \qquad |q| > 2\epsilon \quad . \tag{4.10}$$

To make the notation compact we have introduced a constant E which is $= 0$ for $\check{\sigma}_Q$, $= 1$ for $\check{\sigma}_P$. Again both forms depend on two even moduli q, r and two odd δ, ρ; the attachment point X is located at r, ρ in some fixed set of coordinates. Again we can smoothly interpolate $\check{\sigma}$ from its asymptotic form (4.9) as $|q| \to 0$ to $\frac{|q|}{q}\check{\sigma}'$ at $|q| \sim 2\epsilon$; again the phase is unremovable. Then exactly the same argument as before yields the residue (4.6), since again b_0^Q annihilates $|\mathcal{O}_n\rangle_Q$. We need only to make two minor adaptations to (3.12). First of course not one, but two local coordinates $\check{\sigma}_P, \check{\sigma}_Q$ depend on q, δ. Secondly we are not inserting $|\mathcal{O}_1\rangle_P$ to get a top form, but rather $(c_0 - \bar{c}_0)c_1\bar{c}_1| - 1\rangle_P$ to get a $(2K-1)$-form. One finds that the integral $\int \mathrm{d}\delta \mathrm{d}\bar{\delta}$ brings in the inhomogeneous bits of both G_{-1}^P and $\bar{G}_{-1}^P$; together with $b_0\delta(\beta_{-1})\delta(\bar{\beta}_{-1})$ from the ghost insertions this converts the inserted state to $|0\rangle_P$, which we again can erase.

[8] For example we cannot add $\lambda\theta$ to the local coordinate because λ would have to be anticommuting, and hence a function of the odd moduli. But the 3-punctured sphere has no moduli at all [13].

Once again we have the answer we want, but we are not yet finished. As in the bosonic string we must now turn to the other boundary $|q| = \epsilon$ of the annular patch. Here the slice is given by (4.9). This time one finds that this contribution is always zero! The algebra is slightly tedious, so we have relegated it to the Appendix. The implication, however, is simple: since the measure is zero for $|q| < \epsilon$, and the contribution from the annular patch is just the winding number (4.6), we at once find the dilaton equation, eqn. (1.2), with $\sigma_1 = \mathcal{O}_1/2\pi i$. The answer is completely independent of the choice of the degeneration fixture (4.1), as promised.

Instead of using Stokes' theorem, one can go through all the calculations of [16] in semirigid geometry. We have recovered the same answer at least for $\mathcal{O}_0$ or $\mathcal{O}_1$ at Q. In particular the independence of $a_1,\ \tilde{a}_1$ follows, though now it looks much more surprising. A key feature of this derivation is that as we noted at the end of sect. three, the integrals over the odd moduli ρ, δ, do *not* all bring down factors of the supercurrent G. Rather, some of these integrals differentiate the explicit moduli dependence of the normal ordering (see (3.7)), a phenomenon first seen in the heterotic string [15]. For this reason, just naively picture-changing all the operators $\mathcal{O}_n$ n times and then proceeding on ordinary moduli space does not yield the correct dilaton equation.

5. Conclusion

Conformal and superconformal geometry tell us all we need to know to construct string and superstring amplitudes. Namely, the geometry dictates the class of surfaces on which correlation functions may be taken to live, the dynamics of the geometric fields on those surfaces, the possible couplings to matter, and the space of physical states. What we have seen here is how semirigid geometry dictates all these ingredients for pure topological gravity. In particular the geometry unambiguously requires certain unexpected elements in the measure, and these elements are necessary to yield the required contact terms in the dilaton equation.

The semirigid approach also provides a common superspace home for both ghost and matter fields; it tells us that we can couple pure topological gravity to matter systems obtained by appropriate truncation of locally $N = 2$ supersymmetric matter. In particular it leads us to expect that higher matrix models correspond to higher $N = 2$ minimal models coupled to topological gravity, as seems to be the case [26][27], rather than to topological $SL(n, \mathbf{R})$ gauge theory. It has been suggested that these higher topological gravities generate the intersection numbers of some exotic new moduli spaces [24]. Whether the semirigid approach can shed light on the construction of the latter spaces is an interesting open question.

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Appendix A. A calculation

We want to compute the analog of (4.7) for the topological theory. To simplify the equations we will set to zero all the arbitrary coefficients a_i, $\widetilde{a}_i$ beyond the dangerous a_1 and take $\widetilde{a}_1 = a_1 \equiv a$. We will give only the main points of the calculation.

We want to construct the differential form ν whose exterior derivative is the measure Ω. We are not computing the *difference* of ν across patch boundaries, but rather the contribution to $\int_{\epsilon<|q|<2\epsilon} \mathrm{d}\nu$ from the inner boundary. Since for $|q| \sim \epsilon$ the slice has already reached its asymptotic form (4.9), the calculation is fairly easy. This of course is why we split the calculation up in this way.

From (4.9) we have $f_{01}(z_1) = \frac{z_1}{q} - E + a\big(\frac{z_1}{q} - E\big)^2$, so

$$\begin{aligned} v_q &= -\frac{1}{q}\left[E + (1+2aE)z_0 + (a-2a^2E){z_0}^2 + \mathcal{O}({z_0}^3)\right] \;, \quad \widetilde{v}_q = 0 \\ \frac{\partial v_q}{\partial q} &= -\frac{1}{q} v_q \quad . \end{aligned}$$

From (3.7)–(3.10) we thus get (suppressing all moduli except $q, \overline{q}$)

$$\nu = \langle \Sigma, f_{01} \mid \overline{G_q} G_q (\mathrm{d}q\, b_q + \mathrm{d}\bar{q}\, \overline{b_q}) |\delta(\beta_q)|^2 \cdot c_0^P c_1^P \bar{c}_1^P (\gamma_0^Q)^n c_1^Q \bar{c}_1^Q \mid -1\rangle_P \otimes \mid -1\rangle_Q \quad . \tag{A.1}$$

Note that the term β_{qq} of (3.7) is annihilated by $\delta(\beta_q)$. Consider the first term of (A.1). We may replace G_q by $G_q^{(0)}$ since $(b_q)^2 = 0$ (see (2.1), (3.13)). Commuting $G_q^{(0)}$ to the right we only pick up modes of c^P, c^Q which either kill $|0\rangle$ or are already present. Thus this term vanishes. The second term is even easier: $\delta(\bar{\beta}_q)$ converts the antiholomorphic part of $|-1\rangle_P$ into the SL_2-invariant vacuum, which $\overline{G_q^{(0)}}$ then kills.

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