

# Disjoint Empty Convex Pentagons in Planar Point Sets

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**Abstract.** Harborth [*Elemente der Mathematik*, Vol. 33 (5), 116–118, 1978] proved that every set of 10 points in the plane, no three on a line, contains an empty convex pentagon. From this it follows that the number of disjoint empty convex pentagons in any set of  $n$  points in the plane is least  $\lfloor \frac{n}{10} \rfloor$ . In this paper we prove that every set of 19 points in the plane, no three on a line, contains two disjoint empty convex pentagons. We also show that any set of  $2m + 9$  points in the plane, where  $m$  is a positive integer, can be subdivided into three disjoint convex regions, two of which contains  $m$  points each, and another contains a set of 9 points containing an empty convex pentagon. Combining these two results, we obtain non-trivial lower bounds on the number of disjoint empty convex pentagons in planar points sets. We show that the number of disjoint empty convex pentagons in any set of  $n$  points in the plane, no three on a line, is at least  $\lfloor \frac{5n}{47} \rfloor$ . This bound has been further improved to  $\frac{3n-1}{28}$  for infinitely many  $n$ .

**Keywords.** Convex hull, Discrete geometry, Empty convex polygons, Erdős-Szekeres theorem, Pentagons.

## 1 Introduction

The origin of the problems concerning the existence of empty convex polygons goes back to the famous theorem due to Erdős and Szekeres [10]. It states that for every positive integer  $m \geq 3$ , there exists a smallest integer  $ES(m)$ , such that any set of  $n$  points ( $n \geq ES(m)$ ) in the plane, no three on a line, contains a subset of  $m$  points which lie on the vertices of a convex polygon. Evaluating the exact value of  $ES(m)$  is a long standing open problem. A construction due to Erdős [11] shows that  $ES(m) \geq 2^{m-2} + 1$ , which is also conjectured to be sharp. It is known that  $ES(4) = 5$  and  $ES(5) = 9$  [18]. Following a long computer search, Szekeres and Peters [28] recently proved that  $ES(6) = 17$ . The value of  $ES(m)$  is unknown for all  $m > 6$ . The best known upper bound for  $m \geq 7$  is due to Tóth and Valtr [29] -  $ES(m) \leq \binom{2m-5}{m-3} + 1$ . For a more detailed description of the Erdős-Szekeres theorem and its numerous ramifications see the surveys by Bárány and Károlyi [3] and Morris and Soltan [24].

In 1978, Erdős [9] asked whether for every positive integer  $k$ , there exists a smallest integer  $H(k)$ , such that any set of at least  $H(k)$  points in the plane, no three on a line, contains  $k$  points which lie on the vertices of a convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex  $k$ -gon* or a  *$k$ -hole*. Esther Klein showed  $H(4) = 5$  and Harborth [13] proved that  $H(5) = 10$ . Horton [14] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that  $H(k)$  does not exist for  $k \geq 7$ . Recently, after a long wait, the existence of  $H(6)$  has been proved by Gerken [12] and independently by Nicolás [25]. Later Valtr [32] gave a simpler version of Gerken's proof. For results regarding the number of  $k$ -holes in planar point sets and other related problems see [2–4, 8, 27]. Existence of a hole of any fixed size in sufficiently large point sets, with some additional restrictions on the point sets, has been studied by Károlyi et al. [19, 20], Kun and Lippner [22], and Valtr [31].

Two empty convex polygons are said to be *disjoint* if their convex hulls do not intersect. For positive integers  $k \leq \ell$ , denote by  $H(k, \ell)$  the smallest integer such that any set of  $H(k, \ell)$  points in the plane, no three on a line, contains both a  $k$ -hole and a  $\ell$ -hole which are disjoint. Clearly,  $H(3, 3) = 6$  and Horton's result [14] implies that  $H(k, \ell)$  does not exist for all  $\ell \geq 7$ . Urabe [30] showed that  $H(3, 4) = 7$ , while Hosono and Urabe [17] showed that  $H(4, 4) = 9$ . Hosono and Urabe [15] also proved that  $H(3, 5) = 10$ ,  $12 \leq H(4, 5) \leq 14$ , and  $16 \leq H(5, 5) \leq 20$ . The results  $H(3, 4) = 7$  and  $H(4, 5) \leq 14$  were later reconfirmed by Wu and Ding [33]. Using the computer-aided order-type enumeration method, Aichholzer et al. [1] proved that every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole. Recently, this result was proved geometrically by Bhattacharya and Das [5, 6]. Using this Ramsey-type result, Hosono and Urabe [16] proved that  $H(4, 5) \leq 13$ , which was later tightened to  $H(4, 5) = 12$  by Bhattacharya and Das [7]. Hosono and Urabe [16] have also improved the lower bound on  $H(5, 5)$  to 17.

The problems concerning disjoint holes was, in fact, first studied by Urabe [30] while addressing the problem of partitioning of planar point sets. For any set  $S$  of points in the plane, denote by  $CH(S)$  the *convex hull* of  $S$ . Given a set  $S$  of  $n$  points in the plane, no three on a line, a *disjoint convex partition* of  $S$  is a partition of  $S$  into subsets  $S_1, S_2, \dots, S_t$ , with  $\sum_{i=1}^t |S_i| = n$ , such that for each  $i \in \{1, 2, \dots, t\}$ ,  $CH(S_i)$  forms a  $|S_i|$ -gon and  $CH(S_i) \cap CH(S_j) = \emptyset$ , for any pair of indices  $i, j$ . Observe that in any disjoint convex partition of  $S$ , the set  $S_i$  forms a  $|S_i|$ -hole and the holes formed by the sets  $S_i$  and  $S_j$  are disjoint for any pair of distinct indices  $i, j$ . If  $F(S)$  denote the minimum number of disjoint holes in any disjoint convex partition of  $S$ , then  $F(n) = \max_S F(S)$ , where the maximum is taken over all sets  $S$  of  $n$  points, is called the *disjoint convex partition number* for all sets of fixed size  $n$ . The disjoint convex partition number  $F(n)$  is bounded by  $\lceil \frac{n-1}{4} \rceil \leq F(n) \leq \lceil \frac{5n}{18} \rceil$ . The lower bound is by Urabe [30] and the upper bound by Hosono and Urabe [17]. The proof of the upper bound uses the fact that every set of 7 points in the plane contains a 3-hole and a disjoint 4-hole. Later, Xu and Ding [34] improved the lower bound to  $\lceil \frac{n+1}{4} \rceil$ . Recently, Aichholzer et al. [1] introduced the notion pseudo-convex partitioning of planar point sets, which extends the concept partitioning, in the sense, that they allow both convex polygons and pseudo-triangles in the partition.

Urabe [17] also defined the function  $F_k(n)$  as the minimum number of pairwise disjoint  $k$ -holes in any  $n$ -element point set. If  $F_k(S)$  denotes the number of  $k$ -holes in a disjoint partition of  $S$ , then  $F_k(n) = \min_S \{\max_{\pi_d} F_k(S)\}$ , where the maximum is taken over all disjoint partitions  $\pi_d$  of  $S$ , and the minimum is taken over all sets  $S$  with  $|S| = n$ . Hosono and Urabe [17] proved any set of 9 points, no three on a line, contains two disjoint 4-holes. They also showed any set of  $2m + 4$  points can be divided into three disjoint convex regions, one containing a 4-hole and the others containing  $m$  points each. Combining these two results they proved  $F_4(n) \geq \lfloor \frac{5n}{22} \rfloor$ . This bound can be improved to  $(3n - 1)/13$  for infinitely many  $n$ .

The problem, however, appears to be much more complicated in the case of disjoint 5-holes. Harborth's result [13] implies  $F_5(n) \geq \lfloor \frac{n}{10} \rfloor$ , which, to the best our knowledge, is the only known lower bound on this number. A construction by Hosono and Urabe [16] shows that  $F_5(n) \leq 1$  if  $n \leq 16$ . In general, it is known that  $F_5(n) < n/6$  [3]. Moreover, Hosono and Urabe [17] states the impossibility of an analogous result for 5-holes with  $2m + 5$  points.

In this paper, following a couple of new results for small point sets, we prove non-trivial lower bounds on  $F_5(n)$ . At first, we show that every set of 19 points in the plane, no three on a line, contains two disjoint 5-holes. In other words, this implies,  $F_5(19) \geq 2$  or  $H(5, 5) \leq 19$ . Drawing parallel from the result of Hosono and Urabe [17], we also show that any set of

$2m + 9$  points in the plane, where  $m$  is a positive integer, can be subdivided into three disjoint convex regions, two of which contains  $m$  points each, and the third one is a set of 9 points containing a 5-hole. Combining these two results, we prove  $F_5(n) \geq \lfloor \frac{5n}{47} \rfloor$ . This bound can be further improved to  $\frac{3n-1}{28}$  for infinitely many  $n$ . The proofs rely on a series of results concerning the existence of 5-holes in planar point sets having less than 10 points.

The paper is organized as follows. The results proving the existence of 5-holes in point sets having less than 10 points, and the characterization of 9-point sets not containing any 5-hole are presented in Section 3. In Section 4, we give the formal statements of our main results and use them to prove lower bounds on  $F_5(n)$ . The proofs of the 19-point result and the  $2m + 9$ -point partitioning theorem are presented in Sections 5 and 6, respectively. In Section 2 we introduce notations and definitions and in Section 7 we summarize our work and provide some directions for future work.

## 2 Notations and Definitions

We first introduce the definitions and notations required for the remainder of the paper. Let  $S$  be a finite set of points in the plane in general position, that is, no three on a line. Denote the convex hull of  $S$  by  $CH(S)$ . The boundary vertices of  $CH(S)$ , and the points of  $S$  in the interior of  $CH(S)$  are denoted by  $\mathcal{V}(CH(S))$  and  $\mathcal{I}(CH(S))$ , respectively. A region  $R$  in the plane is said to be empty in  $S$ , if  $R$  contains no elements of  $S$ . A point  $p \in S$  is said to be  $k$ -redundant in a subset  $T$  of  $S$ , if there exists a  $k$ -hole in  $T \setminus \{p\}$ .

By  $\mathcal{P} = p_1 p_2 \dots p_k$  we denote a convex  $k$ -gon with vertices  $p_1, p_2, \dots, p_k$  taken in the counter-clockwise order.  $\mathcal{V}(\mathcal{P})$  denotes the set of vertices of  $\mathcal{P}$  and  $\mathcal{I}(\mathcal{P})$  the interior of  $\mathcal{P}$ .

The  $j$ -th convex layer of  $S$ , denoted by  $L\{j, S\}$ , is the set of points that lie on the boundary of  $CH(S \setminus \{\bigcup_{i=1}^{j-1} L\{i, S\}\})$ , where  $L\{1, S\} = \mathcal{V}(CH(S))$ . If  $p, q \in S$  are such that  $pq$  is an edge of the convex hull of the  $j$ -th layer, then the open halfplane bounded by the line  $pq$  and not containing any point of  $S \setminus \{\bigcup_{i=1}^{j-1} L\{i, S\}\}$  will be referred to as the *outer* halfplane induced by the edge  $pq$ .

For any three points  $p, q, r \in S$ ,  $\mathcal{H}(pq, r)$  (respectively  $\mathcal{H}_c(pq, r)$ ) denotes the open (respectively closed) halfplane bounded by the line  $pq$  containing the point  $r$ . Similarly,  $\overline{\mathcal{H}}(pq, r)$  (respectively  $\overline{\mathcal{H}}_c(pq, r)$ ) is the open (respectively closed) halfplane bounded by  $pq$  not containing the point  $r$ .

Moreover, if  $p, q, r \in S$  is such that  $\angle rpq < \pi$ , then  $Cone(rpq)$  is the set of points in  $\mathbb{R}^2$  which lies in the interior of the angular domain  $\angle rpq$ . A point  $s \in Cone(rpq) \cap S$  is called the *nearest angular neighbor* of  $\vec{pq}$  in  $Cone(rpq)$  if  $Cone(spq)$  is empty in  $S$ . In general, whenever we have a convex region  $R$ , we think of  $R$  as the set of points in  $\mathbb{R}^2$  which lies in the region  $R$ . Thus, for any convex region  $R$  a point  $s \in R \cap S$  is called the *nearest angular neighbor* of  $\vec{pq}$  in  $R$  if  $Cone(spq) \cap R$  is empty in  $S$ . More generally, for any positive integer  $k$ , a point  $s \in S$  is called the  $k$ -th *angular neighbor* of  $\vec{pq}$  whenever  $Cone(spq) \cap R$  contains exactly  $k - 1$  points of  $S$  in its interior. Also, for any convex region  $R$ , the point  $s \in S$ , which has the shortest perpendicular distance to the line  $pq$ ,  $p, q \in S$ , is called the *nearest neighbor* of  $pq$  in  $R$ .

## 3 5-Holes With Less Than 10 Points

We begin by restating a well known result regarding the existence of 5-holes in planar point sets.

**Lemma 1.** [23] Any set of points in general position containing a convex hexagon, contains a 5-hole.

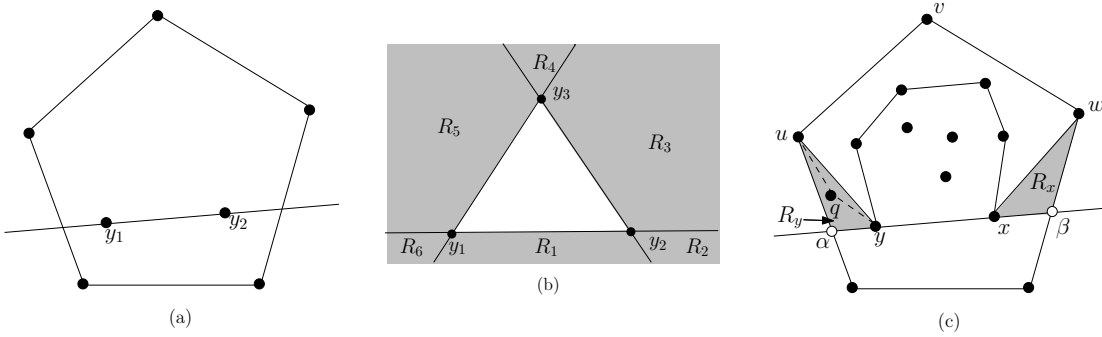
From the Erdős Szekeres theorem, we know that every sufficiently large set of points in the plane in general position, contains a convex hexagon. Lemma 1 therefore ensures that every sufficiently large set of points in the plane contains a 5-hole. Harborth [13] showed that a minimum of 10 points are required to ensure the existence of a 5-hole, that is  $H(5) = 10$ . This means, the existence of a 5-hole is not guaranteed if we have less than 10 points in the plane [13].

In the following, we prove two lemmas where we show, if the convex hull of the point set is not a triangle, a 5-hole can be obtained in less than 10 points.

**Lemma 2.** If  $Z$  is a set of points in the plane in general position, with  $|\mathcal{V}(CH(Z))| = 5$  and  $|\mathcal{I}(CH(Z))| \geq 2$ , then  $Z$  contains a 5-hole.

*Proof.* To begin with suppose there are only two points  $y_1$  and  $y_2$  in  $\mathcal{I}(CH(Z))$ . The extended straight line  $y_1y_2$  divides the plane into two halfplanes, one of which must contain at least three points of  $\mathcal{V}(CH(Z))$ . These three points along with the points  $y_1$  and  $y_2$  forms a 5-hole (Figure 1(a)).

Next suppose, there are three points  $y_1, y_2$ , and  $y_3$  in  $\mathcal{I}(CH(Z))$ . Consider the partition of the exterior of  $y_1y_2y_3$  into disjoint regions  $R_i$  as shown in Figure 1(b). Let  $|R_i|$  denote the number of points of  $\mathcal{V}(CH(Z))$  in region  $R_i$ . If  $Z$  does not contain a 5-hole, we must have:



**Fig. 1.** Illustrations for the proof of Lemma 2.

$$|R_1| \leq 1, \quad |R_3| \leq 1, \quad |R_5| \leq 1, \quad (1)$$

$$\begin{aligned} |R_6| + |R_1| + |R_2| &\leq 2, \\ |R_2| + |R_3| + |R_4| &\leq 2, \\ |R_4| + |R_5| + |R_6| &\leq 2. \end{aligned} \quad (2)$$

Adding the inequalities of (2) and using the fact  $|\mathcal{V}(CH(Z))| = 5$  we get  $|R_2| + |R_4| + |R_6| \leq 1$ . On adding this inequality with those of (1) we finally get  $\sum_{i=1}^6 |R_i| \leq 4 < 5 = |\mathcal{V}(CH(Z))|$ , which is a contradiction.

Finally, suppose  $|\mathcal{I}(CH(Z))| = k \geq 4$ . Let  $x, y \in Z$  be such that  $xy$  is an edge of  $CH(\mathcal{I}(CH(Z)))$  and  $z \in \mathcal{I}(CH(Z))$  be any other point. If  $|\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy, z)| \geq 3$ , the points  $x$  and  $y$  together with the three points of  $\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy, z)$  form a 5-hole.

When  $|\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy, z)| = 1$ , the 4 points in  $\mathcal{V}(CH(Z)) \cap \mathcal{H}(xy, z)$  along with the points  $x$  and  $y$  form a convex hexagon, which contains a 5-hole from Lemma 1. Otherwise,  $|\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy, z)| = 2$ . Denote by  $\alpha, \beta$  the points where the extended straight line passing through the points  $x$  and  $y$  intersects the boundary of  $CH(Z)$ , as shown in Figure 1(c). Let  $R_x = \mathcal{I}(wx\beta)$  and  $R_y = \mathcal{I}(uy\alpha)$  be the two triangular regions generated inside  $CH(Z)$  in the halfplane  $\mathcal{H}(xy, z)$ . If any one of  $R_x$  or  $R_y$  is non-empty in  $Z$ , the nearest neighbor  $q$  of the line  $uy$  (or  $wx$ ) in  $R_y$  (or  $R_x$ ) forms the convex hexagon  $uvwxyq$  (or  $xyuvwq$ ), which contains an 5-hole from Lemma 1. Therefore, assume that both  $R_x$  and  $R_y$  are empty in  $Z$ . Observe that the number of points of  $Z$  inside  $uvwxy$  is exactly two less than the number of points of  $Z$  inside  $CH(Z)$ . By applying this argument repeatedly on the modified pentagon we finally get a 5-hole or a convex pentagon with two or three interior points.  $\square$

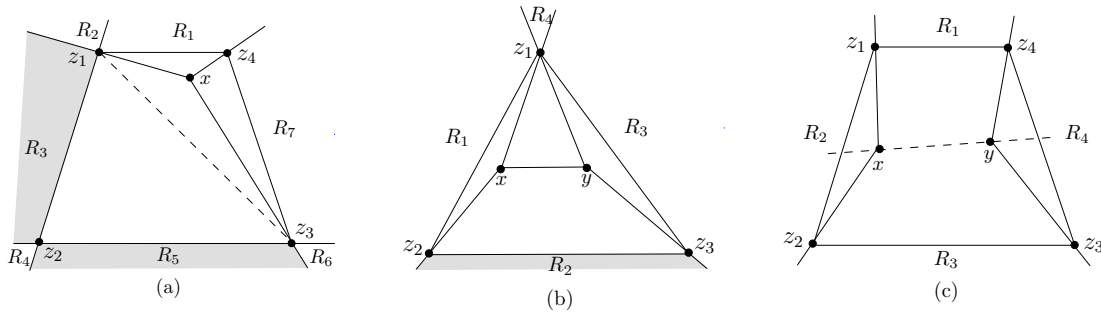
**Lemma 3.** *If  $Z$  is a set of points in the plane in general position, with  $|\mathcal{V}(CH(Z))| = 4$  and  $|\mathcal{I}(CH(Z))| \geq 5$ , then  $Z$  contains a 5-hole.*

*Proof.* Let  $CH(Z)$  be the polygon  $p_1p_2p_3p_4$ . If some outer halfplane induced by an edge of  $CH(\mathcal{I}(CH(Z)))$  contains more than two points of  $\mathcal{V}(CH(Z))$ , then  $Z$  contains a 5-hole. Therefore, we assume

**Assumption 1** *Every outer halfplane induced by the edges of  $CH(\mathcal{I}(CH(Z)))$  contains at most two points of  $\mathcal{V}(CH(Z))$ .*

To begin with suppose  $|\mathcal{I}(CH(Z))| = 5$ . If  $|\mathcal{V}(CH(\mathcal{I}(CH(Z))))| = 5$ , we are done. Thus, the convex hull of the second layer of  $Z$  is either a quadrilateral or a triangle. Let  $CH(\mathcal{I}(CH(Z)))$  be the polygon  $z_1z_2\dots z_k$ , where  $k$  is either 3 or 4. This means  $3 \leq |L\{2, Z\}| \leq 4$ , and we have the following two cases:

*Case 1:*  $|L\{2, Z\}| = 4$ . Let  $x \in L\{3, Z\}$  and w. l. o. g. assume  $x \in \mathcal{I}(z_1z_3z_4) \cap Z$ . Consider the partition of the exterior of the quadrilateral  $z_1z_2z_3z_4$  into disjoint regions  $R_i$  as shown in Figure 2(a). Let  $|R_i|$  denote the number of points of  $\mathcal{V}(CH(Z))$  in the region  $R_i$ . If there exists a point  $p_i \in R_3 \cap Z$ , then  $p_iz_2z_1z_3x$  forms a 5-hole. Therefore, assume that  $|R_3| = 0$ , and similarly,  $|R_5| = 0$ . Moreover, if  $|R_1| + |R_2| \geq 2$ ,  $((R_1 \cup R_2) \cap \mathcal{V}(CH(Z))) \cup \{z_1, z_4, x\}$  contains a 5-hole. This implies,  $|R_1| + |R_2| \leq 1$  and similarly  $|R_6| + |R_7| \leq 1$ . Therefore,  $|R_4| \geq 2$  and Assumption 1 implies that  $|R_4| = 2$ . This implies that the set of points in  $(R_4 \cap Z) \cup \{z_1, z_3, z_4\}$  forms a convex pentagon with exactly two interior points, which then contains a 5-hole from Lemma 2.



**Fig. 2.** Illustrations for the proof of Lemma 3: (a)  $|L\{2, Z\}| = 4$ , (b)  $|L\{2, Z\}| = 3$ , (c) Illustration for the proof of Theorem 1.

*Case 2:*  $|L\{2, Z\}| = 3$ . Let  $L\{3, Z\} = \{x, y\}$ . Consider the partition of the exterior of  $CH(\mathcal{I}(CH(Z)))$  as shown in Figure 2(b). Observe that  $Z$  contains a 5-hole unless  $|R_2| = 0$ ,  $|R_1| \leq 1$ , and  $|R_3| + |R_4| \leq 1$ . This implies that  $\sum_{i=1}^4 |R_i| \leq 3 < 4 = |\mathcal{V}(CH(Z))|$ , which is a contradiction.

Now, consider  $|\mathcal{I}(CH(Z))| > 5$ . W.l.o.g. assume that  $\mathcal{I}(p_1p_2p_3) \cap Z$  is non-empty. If  $|CH(Z \setminus \{p_2\})| \geq 5$ , a 5-hole in  $Z \setminus \{p_2\}$  is ensured from Lemma 1 and Lemma 2. Otherwise,  $CH(Z \setminus \{p_2\})$  is a quadrilateral with exactly one less point of  $Z$  in its interior than  $CH(Z)$ . By repeating this process we finally get a convex quadrilateral with exactly 5 points in its interior, thus reducing the problem to *Case 1* and *Case 2*.  $\square$

From the argument at the end of the proof of the previous lemma, it follows that if  $|\mathcal{I}(CH(Z))| \geq 6$ , then either  $p_1$  or  $p_3$  is 5-redundant in  $Z$ . Similarly, either  $p_2$  or  $p_4$  is 5-redundant in  $Z$ . Therefore, we have the following corollary:

**Corollary 1.** *Let  $Z$  be a set of points in the plane in general position, such that  $CH(Z)$  is the polygon  $z_1z_2z_3z_4$ , and  $|\mathcal{I}(CH(Z))| \geq 6$ . Then the following statements hold:*

- (i) *If for some  $i \in \{1, 2, 3, 4\}$ ,  $\mathcal{I}(z_{i-1}z_iz_{i+1}) \cap Z$  is non-empty, then  $z_i$  is 5-redundant in  $Z$ , where the indices are taken modulo 4.*
- (ii) *At least one of the vertices corresponding to any diagonal of  $CH(Z)$  is 5-redundant in  $Z$ .*  $\square$

Moreover, by combining Lemmas 1, 2, and 3, the following result about the existence of 5-holes is immediate.

**Corollary 2.** *Any set  $Z$  of 9 points in the plane in general position, with  $|\mathcal{V}(CH(Z))| \geq 4$ , contains a 5-hole.*  $\square$

Two sets of points,  $S_1$  and  $S_2$ , in general position, having the same number of points belong to the same *layer equivalence class* if the number of layers in both the point sets is the same and  $|L\{k, S_1\}| = |L\{k, S_2\}|$ , for all  $k$ . A set  $S$  of points with 3 different layers belongs to the layer equivalence class  $L\{a, b, c\}$  whenever  $|L\{1, S\}| = a$ ,  $|L\{2, S\}| = b$ , and  $|L\{3, S\}| = c$ , where  $a, b, c$  are positive integers.

It is known that there exist sets with 9 points without any 5-hole, belonging to the layer equivalence classes  $L\{3, 3, 3\}$  [21] and  $L\{3, 5, 1\}$  [13]. In the following theorem we show that any 9-point set not belonging to either of these two equivalent classes contains a 5-hole.

**Theorem 1.** *Any set of 9 points in the plane in general position, not containing a 5-hole either belongs to the layer equivalence class  $L\{3, 3, 3\}$  or to the layer equivalence class  $L\{3, 5, 1\}$ .*

*Proof.* Let  $S$  be a set of 9 points in general position. If  $|\mathcal{V}(CH(S))| \geq 4$ , a 5-hole is guaranteed from Corollary 2. Thus, for proving the result it suffices to show that  $S$  contains a 5-hole if  $S \in L\{3, 4, 2\}$ .

Assume  $S \in L\{3, 4, 2\}$  and suppose  $z_1, z_2, z_3, z_4$  are the vertices of the second layer. Let  $L\{3, S\} = \{x, y\}$ . The extended straight line  $xy$  divides the entire plane into two halfplanes. If one of these halfplanes contains three points of  $L\{2, S\}$ , these three points along with the points  $x$  and  $y$  form a 5-hole.

Otherwise, both halfplanes induced by the extended straight line  $xy$  contain exactly two points of  $L\{2, S\}$ . The exterior of the quadrilateral  $z_1z_2z_3z_4$  can now be partitioned into 4

disjoint regions  $R_1, R_2, R_3,$  and  $R_4,$  as shown in Figure 2(c). Let  $|R_i|$  denote the number of points of  $\mathcal{V}(CH(S))$  in the region  $R_i$ . If  $R_1$  or  $R_3$  contains any point of  $\mathcal{V}(CH(S))$ , a 5-hole is immediate. Therefore,  $|R_1| = |R_3| = 0$ , which implies that  $|R_2| + |R_4| = |\mathcal{V}(CH(S))| = 3$ . By the pigeonhole principle, either  $|R_2| \geq 2$  or  $|R_4| \geq 2$ . If  $|R_2| \geq 2$ ,  $(R_2 \cap S) \cup \{x, z_1, z_2\}$  contains a 5-hole. Otherwise,  $|R_4| \geq 2$ , and  $(R_4 \cap S) \cup \{y, z_3, z_4\}$  contains a 5-hole.

Thus, a set  $S$  of 9 points not containing a 5-hole, must either belong to  $L\{3, 3, 3\}$  or  $L\{3, 5, 1\}$ .  $\square$

## 4 Disjoint 5-Holes: Lower Bounds

In this section we present our main results concerning the existence of disjoint 5-holes in planar point sets, which leads to a non-trivial lower bound on the number of disjoint 5-holes in planar point sets. As  $H(5) = 10$ , it is clear that every set 20 points in the plane in general position, contains two disjoint 5-holes. At first, we improve upon this result by showing that any set of 19 points also contains two disjoint 5-holes.

**Theorem 2.** *Every set of 19 points in the plane in general position, contains two disjoint 5-holes.*

Drawing parallel from the  $2m + 4$ -point result for disjoint 4-holes due to Hosono and Urabe [17], we prove a partitioning theorem for disjoint 5-holes for any set of  $2m + 9$  points in the plane in general position.

**Theorem 3.** *For any set of  $2m + 9$  points in the plane in general position, it is possible to divide the plane into three disjoint convex regions such that one contains a set of 9 points which contains a 5-hole, and the others contain  $m$  points each, where  $m$  is a positive integer.*

Since  $H(5) = 10$ , the trivial lower bound on  $F_5(n)$  is  $\lfloor \frac{n}{10} \rfloor$ . Observe that any set of 47 points can be partitioned into two sets of 19 points each, and another set of 9 points containing a 5-hole, by Theorem 3. Hence, from Theorems 2 and 3, it follows that,  $F_5(47) = 5$ . Using this result, we obtain an improved lower bound on  $F_5(n)$ .

**Theorem 4.**  $F_5(n) \geq \lfloor \frac{5n}{47} \rfloor$ .

*Proof.* Let  $S$  be a set of  $n$  points in the plane, no three of which are collinear. By a horizontal sweep, we can divide the plane into  $\lceil \frac{n}{47} \rceil$  disjoint strips, of which  $\lfloor \frac{n}{47} \rfloor$  contain 47 points each and one remaining strip  $R$ , with  $|R| < 47$ . The strips having 47 points contain at least 5 disjoint 5-holes, since  $F_5(47) = 5$  (Theorems 2 and 3). If  $9k + 1 \leq |R| \leq 9k + 9$ , for  $k = 0$  or  $k = 1$ , there exist at least  $k$  disjoint 5-holes in  $R$ . If  $19 \leq |R| \leq 28$ , Theorem 2 guarantees the existence of 2 disjoint 5-holes in  $R$ . Finally, if  $9k + 2 \leq |R| \leq 9k + 10$ , for  $k = 3$  or  $4$ , at least  $k$  disjoint 5-holes exist in  $R$ . Thus, the total number of disjoint 5-holes in a set of  $n$  points is always at least  $\lfloor \frac{5n}{47} \rfloor$ .  $\square$

We can obtain a better lower bound on  $F_5(n)$  for infinitely many  $n$ , of the form  $n = 28 \cdot 2^{k-1} - 9$  with  $k \geq 1$ , by the repeated application of Theorem 3.

**Theorem 5.**  $F_5(n) \geq (3n - 1)/28$ , for  $n = 28 \cdot 2^{k-1} - 9$  and  $k \geq 1$ .

*Proof.* Let  $g(k) = 28 \cdot 2^{k-1} - 9$  and  $h(k) = 3 \cdot 2^{k-1} - 1$ . We need to show  $F_5(g(k)) \geq h(k)$ . We prove the inequality by induction on  $k$ . By Theorem 2, the inequality holds for  $k = 1$ . Suppose the result is true for  $k$ , that is,  $F_5(g(k)) \geq h(k)$ . Since,  $g(k+1) = 2g(k) + 9$ , any set of  $g(k+1)$  points can be partitioned into three disjoint convex regions, two of which contain  $g(k)$  points each, and the third a set of 9 points containing a 5-hole by Theorem 3. Hence,  $F_5(g(k+1)) = F_5(2g(k) + 9) \geq 2h(k) + 1 = h(k+1)$ . This completes the induction step, proving the result for  $n = 28 \cdot 2^{k-1} - 9$ .  $\square$

## 5 Proof of Theorem 2

Let  $S$  be a set of 19 points in the plane in general position. We say  $S$  is *admissible* if it contains two disjoint 5-holes. We prove Theorem 2 by considering the various cases based on the size of  $|\mathcal{V}(CH(S))|$ . The proof is divided into two subsections. The first section considers the cases where  $|\mathcal{V}(CH(S))| \geq 4$ , and the second section deals with the case where  $|\mathcal{V}(CH(S))| = 3$ .

### 5.1 $|\mathcal{V}(CH(S))| \geq 4$

Let  $CH(S)$  be the polygon  $s_1s_2 \dots s_k$ , where  $k = |\mathcal{V}(CH(S))|$  and  $k \geq 4$ . A diagonal  $d := s_i s_j$  of  $CH(S)$ , is called a *dividing* diagonal if

$$|\mathcal{H}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| - |\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| = c,$$

where  $c$  is 0 or 1 according as  $k$  is even or odd, and  $s_m \in \mathcal{V}(CH(S))$  is such that  $m \neq i, j$ . Consider a dividing diagonal  $d := s_i s_j$  of  $CH(S)$ . Observe that for any fixed index  $m \neq i, j$ , either  $|\mathcal{H}(s_i s_j, s_m) \cap S| \geq 9$  or  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S| \geq 9$ . Now, we have the following observation.

**Observation 1** *If for some dividing diagonal  $d = s_i s_j$  of  $CH(S)$ ,  $|\mathcal{H}(s_i s_j, s_m) \cap S| > 10$ , where  $m \neq i, j$ , then  $S$  is admissible.*

*Proof.* Let  $Z = \overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S$  and  $\beta$  and  $\gamma$  the first and the second angular neighbors of  $\overline{s_i s_j}$  in  $\mathcal{H}(s_i s_j, s_m) \cap S$ , respectively. Now,  $|\mathcal{V}(CH(Z))| \geq 3$ , since  $|\mathcal{V}(CH(S))| > 3$ . We consider different cases based on the size of  $CH(Z)$ .

**Case 1:**  $|\mathcal{V}(CH(Z))| \geq 5$ . This implies that  $|\mathcal{V}(CH(Z \cup \{\beta\}))| \geq 6$  and so  $Z \cup \{\beta\}$  contains a 5-hole by Lemma 1. This 5-hole is disjoint from the 5-hole contained in  $(\mathcal{H}(s_i s_j, s_m) \cap S) \setminus \{\beta\}$ .

**Case 2:**  $|\mathcal{V}(CH(Z))| = 4$ . If  $|\mathcal{I}(CH(Z))| \geq 2$ , then  $Z \cup \{\beta\}$  is a convex pentagon with at least two interior points. From Lemma 2,  $Z \cup \{\beta\}$  contains a 5-hole which is disjoint from the 5-hole contained in  $(\mathcal{H}(s_i s_j, s_m) \cap S) \setminus \{\beta\}$ . Otherwise,  $|\mathcal{I}(CH(Z))| \leq 1$ . Let  $Z' = Z \cup \{\beta, \gamma\}$ . It follows from Lemmas 1 and 2 that  $Z'$  always contains a 5-hole. This 5-hole is disjoint from the 5-hole contained in  $(\mathcal{H}(s_i s_j, s_m) \cap S) \setminus \{\beta, \gamma\}$ , since  $|\mathcal{H}(s_i s_j, s_m) \cap S| \setminus \{\beta, \gamma\}| \geq 12$ .

**Case 3:**  $|\mathcal{V}(CH(Z))| = 3$ . If  $|\mathcal{I}(CH(Z))| = 5$ ,  $|\mathcal{V}(CH(Z \cup \{\beta\}))| = 4$  and  $Z \cup \{\beta\}$  contains a 5-hole by Corollary 2, which is disjoint from the 5-hole contained in  $(\mathcal{H}(s_i s_j, s_m) \cap S) \setminus \{\beta\}$ . So, let  $|\mathcal{I}(CH(Z))| = b \leq 4$ , which implies,  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 16 - b$ . Let  $\eta$  be the  $(6 - b)$ -th angular neighbor of  $\overline{s_i s_j}$  in  $\mathcal{H}(s_i s_j, s_m) \cap S$ . Let  $S_1 = \mathcal{H}_c(\eta s_i, s_j) \cap S$  and  $S_2 = \overline{\mathcal{H}}(\eta s_i, s_j) \cap S$ . Now, since  $|S_1| = 9$  and  $|\mathcal{V}(CH(S_1))| \geq 4$ ,  $S_1$  contains 5-hole, by Corollary 2. This 5-hole disjoint from the 5-hole contained in  $S_2$ .  $\square$



Observation 1 implies that for any dividing diagonal  $d := s_i s_j$  and for any fixed vertex  $s_m$ , with  $m \neq i, j$ ,  $S$  is admissible unless  $|\mathcal{H}(s_i s_j, s_m) \cap S| \leq 10$  and  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S| \leq 10$ . This can now be used to show the admissibility of  $S$  whenever  $|\mathcal{V}(CH(S))| \geq 8$ .

**Lemma 4.**  $S$  is admissible whenever  $|\mathcal{V}(CH(S))| \geq 8$ .

*Proof.* Let  $d := s_i s_j$  be a dividing diagonal of  $CH(S)$ , and  $s_m \in \mathcal{V}(CH(S))$  be such that  $m \neq i, j$ . Since  $|\mathcal{V}(CH(S))| \geq 8$ , both  $|\mathcal{H}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))|$  and  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))|$  must be greater than 3. Moreover, if  $|\mathcal{H}(s_i s_j, s_m) \cap S| > 10$ , Observation 1 ensures that  $S$  is admissible. Thus, we have the following two cases:

*Case 1:*  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 10$ . Now, since  $|\mathcal{V}(CH(\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S))| \geq 4$ ,  $\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S$  contains a 5-hole which is disjoint from the 5-hole contained in  $\mathcal{H}(s_i s_j, s_m) \cap S$ .

*Case 2:*  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 9$ . As  $|\mathcal{V}(CH(S))| \geq 8$  and  $\overline{s_i s_j}$  is a dividing diagonal of  $CH(S)$ , we have  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| \geq 3$ . Let  $W = (\overline{\mathcal{H}}(s_i s_j, s_m) \cap S) \cup \{s_i\}$ . Then from Corollary 2,  $W$  contains a 5-hole, since  $|W| = 9$  and  $|\mathcal{V}(CH(W))| \geq 4$ . The 5-hole contained in  $W$  is disjoint from the 5-hole contained in  $(\mathcal{H}(s_i s_j, s_m) \cap S) \cup \{s_j\}$ . Hence  $S$  is admissible.

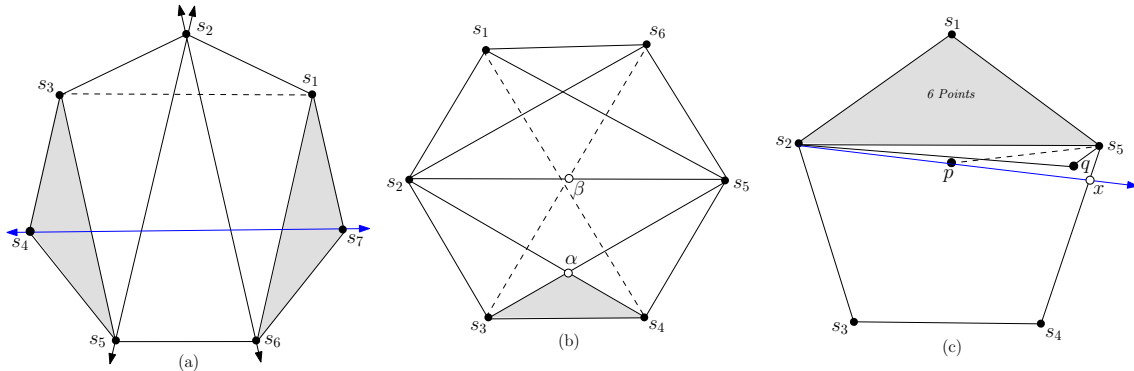
*Case 3:*  $|\mathcal{H}(s_i s_j, s_m) \cap S| \leq 8$ . In this case,  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S| \geq 9$ , and the problem reduces to the previous cases.  $\square$

Therefore, it suffices to show the admissibility of  $S$  whenever  $4 \leq |\mathcal{V}(CH(S))| \leq 7$ . Observe that  $S$  is admissible whenever  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 10$  and  $|\mathcal{V}(CH(\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S))| \geq 4$ . Moreover, *Case 2* of Lemma 4 shows that  $S$  is admissible if  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 9$  and  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| \geq 3$ . Thus, hereafter we shall assume,

**Assumption 2** For every dividing diagonal  $s_i s_j$  of  $CH(S)$ , there exists  $s_m \in \mathcal{V}(CH(S))$ , with  $m \neq i, j$ , such that either  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 10$  and  $|\mathcal{V}(CH(\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S))| = 3$ , or  $|\mathcal{H}(s_i s_j, s_m) \cap S| = 9$  and  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| \leq 2$ .

A dividing diagonal  $s_i s_j$  of  $CH(S)$  is said to be an  $(a, b)$ -splitter of  $CH(S)$ , where  $a \leq b$  are integers, if either  $|\mathcal{H}(s_i s_j, s_m) \cap S \setminus \mathcal{V}(CH(S))| = a$  and  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S \setminus \mathcal{V}(CH(S))| = b$  or  $|\mathcal{H}(s_i s_j, s_m) \cap S \setminus \mathcal{V}(CH(S))| = b$  and  $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S \setminus \mathcal{V}(CH(S))| = a$ .

The admissibility of  $S$  in the different cases which arise are now proved as follows:



**Fig. 3.** Illustrations for the proof of Lemma 5: (a)  $|\mathcal{V}(CH(S))| = 7$ , (b)  $|\mathcal{V}(CH(S))| = 6$ , (c) Illustration for the proof of Lemma 6.

**Lemma 5.**  $S$  is admissible whenever  $6 \leq |\mathcal{V}(CH(S))| \leq 7$ .

*Proof.* We consider the two cases based on the size of  $|\mathcal{V}(CH(S))|$  separately as follows:

**Case 1:**  $|\mathcal{V}(CH(S))| = 7$ . Refer to Figure 3(a). From Assumption 2 it follows that every dividing diagonal of  $CH(S)$  must be a (6, 6)-splitter of  $CH(S)$ . As both  $s_2s_5$  and  $s_2s_6$  are (6, 6)-splitters, it is clear that  $\mathcal{I}(s_2s_5s_6)$  is empty in  $S$ . Now, if  $s_2$  is 5-redundant in either  $\mathcal{H}_c(s_2s_5, s_4) \cap S$  or  $\mathcal{H}_c(s_2s_6, s_2) \cap S$ , the admissibility of  $S$  is immediate. Therefore, assume that  $s_2$  is not 5-redundant in either  $\mathcal{H}_c(s_2s_5, s_4) \cap S$  or  $\mathcal{H}_c(s_2s_6, s_2) \cap S$ . This implies that  $\mathcal{I}(s_2s_3s_4s_5) \cap S \subset \mathcal{I}(s_3s_4s_5)$  and  $\mathcal{I}(s_2s_6s_1s_7) \cap S \subset \mathcal{I}(s_1s_6s_7)$ . Therefore,  $\mathcal{I}(s_1s_2s_3)$  is empty in  $S$ . Now, since  $s_4s_7$  is also a (6, 6)-splitter of  $CH(S)$ ,  $|\mathcal{V}(CH(\mathcal{H}(s_4s_7, s_2) \cap S))| \geq 4$  (see Figure 3(a)), and Corollary 2 implies  $\mathcal{H}(s_4s_7, s_2) \cap S$  contains a 5-hole. This 5-hole disjoint from the 5-hole contained in  $\mathcal{H}_c(s_4s_7, s_5) \cap S$ .

**Case 2:**  $|\mathcal{V}(CH(S))| = 6$ . Refer to Figure 3(b). Again, Assumption 2 implies that every dividing diagonal of  $CH(S)$  must be a (6, 7)-splitter of  $CH(S)$ . W.l.o.g. assume that  $|\mathcal{I}(s_1s_2s_5s_6) \cap S| = 7$  and  $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 6$ . Let  $\alpha$  be the point of intersection of the diagonals of the quadrilateral  $s_2s_3s_4s_5$ . If  $s_2$  or  $s_5$  is 5-redundant in  $\mathcal{H}_c(s_2s_5, s_4) \cap S$ , then the admissibility of  $S$  is immediate. Therefore, assume that neither  $s_2$  nor  $s_5$  is 5-redundant in  $\mathcal{H}_c(s_2s_5, s_4) \cap S$ . This implies that  $\mathcal{I}(s_2s_3s_4s_5) \cap S \subset \mathcal{I}(s_3\alpha s_4)$ . Now, if  $|\mathcal{I}(s_1s_2s_3s_4) \cap S| = 6$ , then  $s_4$  is 5-redundant in  $\mathcal{H}_c(s_1s_4, s_2) \cap S$  and the admissibility of  $S$  follows. Similarly, if  $|\mathcal{I}(s_3s_4s_5s_6) \cap S| = 6$ , then  $S$  is admissible, as  $s_3$  is 5-redundant in  $\mathcal{H}_c(s_3s_6, s_5) \cap S$ . Hence, assume  $|\mathcal{I}(s_1s_2s_3s_4) \cap S| = |\mathcal{I}(s_3s_4s_5s_6) \cap S| = 7$ . Now, as  $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 6$ ,  $(\mathcal{I}(s_3s_4s_5s_6) \setminus \mathcal{I}(s_3s_4\alpha)) \cap S \subset \mathcal{I}(s_5s_6\beta)$ , where  $\beta$  is the point of intersection of the diagonals  $s_2s_5$  and  $s_3s_6$ . Therefore,  $|\mathcal{V}(CH(\mathcal{H}(s_3s_6, s_5) \cap S))| \geq 4$ . Therefore, the 5-hole contained in  $\mathcal{H}(s_3s_6, s_5) \cap S$  is disjoint from the 5-hole contained in  $\mathcal{H}_c(s_3s_6, s_1) \cap S$ .  $\square$

**Lemma 6.**  $S$  is admissible whenever  $|\mathcal{V}(CH(S))| = 5$ .

*Proof.* Assumption 2 implies that a dividing diagonal of  $CH(S)$  is either a (6, 8)-splitter or a (7, 7)-splitter of  $CH(S)$ . To begin with suppose, every dividing diagonal of  $CH(S)$  is a (7, 7)-splitter of  $|\mathcal{V}(CH(S))|$ . Then  $|\mathcal{I}(s_1s_2s_3) \cap S| = |\mathcal{I}(s_1s_4s_5) \cap S| = 7$ , which means that  $|\mathcal{I}(s_1s_3s_4) \cap S| = 0$ . Similarly,  $|\mathcal{I}(s_2s_4s_5) \cap S| = |\mathcal{I}(s_3s_5s_1) \cap S| = |\mathcal{I}(s_4s_2s_1) \cap S| = |\mathcal{I}(s_5s_2s_3) \cap S| = 0$ . This implies  $|\mathcal{I}(CH(S))| = 0$ , which is a contradiction.

Therefore, assume that there exists a (6, 8)-splitter of  $CH(S)$ . W.l.o.g., assume  $s_2s_5$  is a (6, 8)-splitter of  $CH(S)$ . There are two possibilities:

**Case 1:**  $|\mathcal{I}(s_1s_2s_5) \cap S| = 6$  and  $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 8$ . Refer to Figure 3(c). Let  $p$  be the nearest neighbor of  $s_2s_5$  in  $\mathcal{H}(s_2s_5, s_4) \cap S$ . W.l.o.g., assume  $\mathcal{I}(s_1s_2p) \cap S$  is non-empty. Let  $x$  be the point where  $\overrightarrow{s_2p}$  intersects the boundary of  $CH(S)$ . Then  $\mathcal{H}_c(s_2x, s_1) \cap S$  contains a 5-hole, and by Corollary 1  $s_2$  is 5-redundant in  $\mathcal{H}_c(s_2p, s_1) \cap S$ . Now, if  $\text{Cone}(s_5px) \cap S$  is empty, the 5-hole contained in  $(\mathcal{H}_c(s_2p, s_1) \cap S) \setminus \{s_2\}$  is disjoint from the 5-hole contained in  $(\overline{\mathcal{H}}(s_2p, s_1) \cap S) \cup \{s_2\}$ . Otherwise, assume  $\text{Cone}(s_5px) \cap S$  is non-empty. Let  $q$  be the first angular neighbor of  $\overrightarrow{s_2s_5}$  in  $\text{Cone}(s_5px)$ . Observe that  $\mathcal{I}(s_1s_2q) \cap S$  is non-empty, since  $\mathcal{I}(s_1s_2p) \cap S$  is assumed to be non-empty, and  $\mathcal{H}_c(s_2q, s_1) \cap S$  contains a 5-hole. Now, Corollary 1 implies that  $s_2$  is 5-redundant in  $\mathcal{H}_c(s_2q, s_1) \cap S$ , and the admissibility of  $S$  follows.

**Case 2:**  $|\mathcal{I}(s_1s_2s_5) \cap S| = 8$  and  $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 6$ . Clearly,  $\mathcal{H}_c(s_2s_5, s_3) \cap S$  contains a 5-hole. Now, if either  $s_2$  or  $s_5$  is 5-redundant in  $\mathcal{H}_c(s_2s_5, s_3) \cap S$ , then  $S$  is admissible. Therefore, assume  $\mathcal{I}(s_2s_3s_4s_5) \cap S \subset \mathcal{I}(s_3s_4\alpha)$ , where  $\alpha$  is the point where the diagonals

of the quadrilateral  $s_2s_3s_4s_5$  intersect. The problem now reduces to *Case 1* with respect to the dividing diagonal  $s_2s_4$ .  $\square$

The case  $|\mathcal{V}(CH(S))| = 4$  is dealt separately in the next section.

$|\mathcal{V}(CH(S))| = 4$  As before, let  $CH(S)$  be the polygon  $s_1s_2s_3s_4$ . From Observation 1, we have to consider the cases where a dividing diagonal of  $CH(S)$  is either a (6, 9)-splitter or a (7, 8)-splitter of  $CH(S)$ .

Firstly, suppose some dividing diagonal of  $CH(S)$ , say  $s_2s_4$ , is a (6, 9)-splitter of  $CH(S)$ . Assume that  $|\mathcal{I}(s_1s_2s_4) \cap S| = 6$  and  $|\mathcal{I}(s_2s_3s_4) \cap S| = 9$ . Begin by taking the nearest neighbor  $p$  of  $s_2s_4$  in  $\mathcal{I}(s_2s_3s_4)$ . Then choose the first angular neighbor  $q$  of either  $\overrightarrow{s_2s_4}$  or  $\overrightarrow{s_4s_2}$  in  $\mathcal{I}(s_2s_3s_4)$ , and proceed as in *Case 1* of Lemma 6 to show the admissibility of  $S$ .

Therefore, it suffices to assume that

**Assumption 3** *Both the dividing diagonals of the quadrilateral  $s_1s_2s_3s_4$  are (7, 8)-splitters of  $CH(S)$ .*

W.l.o.g., let  $|\mathcal{I}(s_1s_2s_4) \cap S| = 8$  and  $|\mathcal{I}(s_2s_3s_4) \cap S| = 7$ . Let  $\alpha$  be the point where the diagonals of  $CH(S)$  intersect. Observe, there always exists an edge of  $CH(S)$  say,  $s_2s_3$ , such that  $|\mathcal{I}(s_1s_2s_3) \cap S| = |\mathcal{I}(s_2s_3s_4) \cap S| = 7$ , and  $|\mathcal{I}(s_1s_3s_4) \cap S| = |\mathcal{I}(s_1s_2s_4) \cap S| = 8$ . This implies,  $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n$ , with  $0 \leq n \leq 7$ . We begin with the following simple observation

**Lemma 7.**  *$S$  is admissible whenever  $n = 0$ .*

*Proof.* Let  $Z = (\mathcal{H}(s_2s_4, s_1) \cap S) \cup \{s_4\}$ . Observe that  $|Z| = 10$ , which means  $Z$  contains a 5-hole. If  $|\mathcal{V}(CH(Z))| \geq 5$ ,  $s_4$  is 5-redundant in  $Z$ , and  $Z \setminus \{s_4\}$  contains a 5-hole which is disjoint from the 5-hole contained in  $\mathcal{H}_c(s_2s_4, s_3) \cap S$ . Let  $r$  be the nearest angular neighbor of  $\overrightarrow{s_1s_3}$  in  $Cone(s_4s_1s_3)$ . If  $|\mathcal{V}(CH(Z))| = 4$ , either  $r$  or  $s_4$  is 5-redundant in  $Z$  by Corollary 1, and the admissibility of  $S$  follows. Otherwise,  $|\mathcal{V}(CH(Z))| = 3$  and at least one of  $s_1$ ,  $s_4$ , or  $r$  is 5-redundant in  $Z$  and the admissibility of  $S$  follows similarly.  $\square$

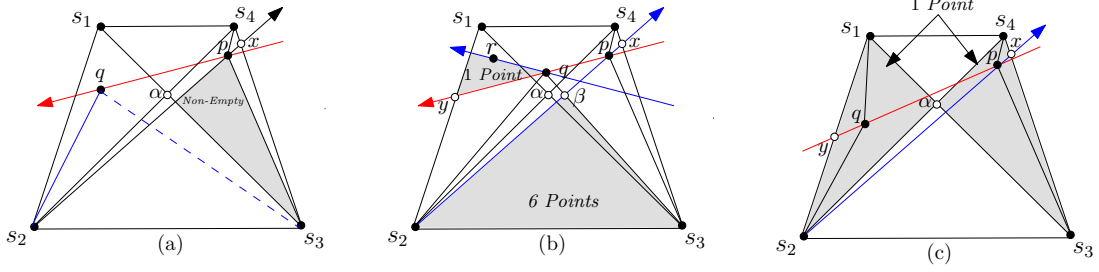
From the previous lemma, it suffices to assume  $n > 0$ . Let  $p$  be the first angular neighbor of  $\overrightarrow{s_2s_4}$  in  $Cone(s_4s_2s_3)$  and  $x$  the intersection point of  $\overrightarrow{s_2p}$  with the boundary of  $CH(S)$ . Let  $\alpha$  be the point of intersection of the diagonals of the quadrilateral  $s_1s_2s_3s_4$ . If  $Cone(s_3px) \cap S$  is non-empty,  $|\mathcal{V}(CH(\mathcal{H}_c(s_2p, s_3) \cap S))| \geq 4$ . From Corollary 2,  $\mathcal{H}_c(s_2p, s_3) \cap S$  contains a 5-hole which is disjoint from the 5-hole contained in  $(\mathcal{H}(s_2s_4, s_1) \cap S) \cup \{s_4\}$ . Therefore, we shall assume that

**Assumption 4**  *$Cone(s_3px) \cap S$  is empty.*

Assumption 4 and the fact that  $n > 0$  implies that  $p \in \mathcal{I}(s_3\alpha s_4) \cap S$  (see Figure 4(a)). Let  $q$  be the first angular neighbor of  $\overrightarrow{ps_2}$  in  $Cone(s_2ps_1)$ . The admissibility of  $S$  in the remaining cases is proved in the following two lemmas.

**Lemma 8.**  *$S$  is admissible whenever  $n \geq 2$ .*

*Proof.* To begin with suppose,  $q \in \mathcal{I}(s_2\alpha s_1) \cap S$ , as shown in Figure 4(a). By Assumption 4, there exists a point in  $\mathcal{I}(s_3s_4\alpha) \cap S$ , different from the point  $p$ , which belongs to  $\mathcal{I}(qps_3) \cap S$ . Hence, by Corollary 1,  $p$  is 5-redundant in  $\mathcal{H}_c(pq, s_2) \cap S$ , and the 5-hole contained in  $(\mathcal{H}(pq, s_2) \cap S) \cup \{q\}$  is disjoint from the 5-hole contained in  $(\mathcal{H}(pq, s_1) \cap S) \cup \{p\}$ .



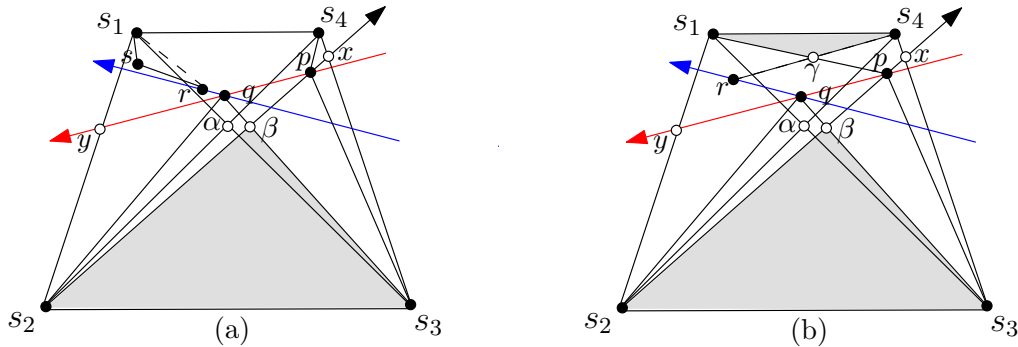
**Fig. 4.**  $|\mathcal{V}(CH(S))| = 4$ : (a)  $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n \geq 2$  and  $q \in \mathcal{I}(s_2\alpha s_1) \cap S$ , (b)  $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n \geq 2$ , and  $q \in \mathcal{I}(s_1\alpha s_4) \cap S$ , (c)  $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n = 1$ .

Otherwise, assume that  $q \in \mathcal{I}(s_1\alpha s_4) \cap S$  and refer to Figure 4(b). Observe that  $S$  is admissible if either  $p$  or  $q$  is 5-redundant in  $\mathcal{H}_c(pq, s_2) \cap S$ . Hence, assume that neither  $p$  nor  $q$  is 5-redundant in  $\mathcal{H}_c(pq, s_2) \cap S$ . This implies  $\mathcal{I}(s_2s_3pq) \cap S \subset \mathcal{I}(s_2s_3\beta)$ , where  $\beta$  is the point of intersection of the diagonals of the quadrilateral  $s_2s_3pq$ . Let  $r$  be the second angular neighbor of  $\vec{qy}$  in  $Cone(yqs_1)$ , where  $y$  is the point where  $\vec{pq}$  intersects the boundary  $CH(S)$ . Note that the point  $r$  exists because  $n \geq 2$  and  $q \in \mathcal{I}(s_1s_4\alpha) \cap S$ . Now, the 5-hole contained in  $(\mathcal{H}(qr, s_2) \cap S) \cup \{q\}$  is disjoint from the 5-hole contained in  $(\mathcal{H}(qr, s_1) \cap S) \cup \{r\}$  by Corollary 2.  $\square$

**Lemma 9.**  $S$  is admissible whenever  $n = 1$ .

*Proof.* To begin with let  $q \in \mathcal{I}(s_1\alpha s_2)$ . Refer to Figure 4(c). Assume,  $\mathcal{I}(s_4pq) \cap S$  is non-empty and let  $Z = (\mathcal{H}(pq, s_1) \cap S) \cup \{q\}$ . Observe that  $|\mathcal{V}(CH(Z))| \geq 4$ , and by Corollary 1 either  $q$  or  $s_4$  is 5-redundant in  $Z$ , and the admissibility of  $S$  follows.

Otherwise, assume that  $\mathcal{I}(s_4pq) \cap S$  is empty. If either  $q$  or  $s_4$  is 5-redundant in  $Z$ , the admissibility of  $S$  is immediate. Therefore, it suffices to assume that there exists a 5-hole in  $Z$  with  $qs_4$  as an edge. This implies that we have a 6-hole with  $ps_4$  and  $pq$  as edges. Observe that  $s_1$  cannot be a vertex of this 6-hole. Hence, there exists a 5-hole with  $ps_4$  as an edge, which does not contain  $s_1$  and  $q$  as vertices. Thus,  $s_1$  and  $q$  are 5-redundant in  $\mathcal{H}_c(s_4q, s_1) \cap S$ . This 5-hole is disjoint from the 5-hole contained in  $\mathcal{H}_c(s_1s_3, s_2) \cap S$ .



**Fig. 5.**  $|\mathcal{V}(CH(S))| = 4$  with  $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n = 1$ : (a)  $q, r \in \mathcal{I}(s_1s_4\alpha) \cap S$ , and (b)  $q \in \mathcal{I}(s_1s_4\alpha)$  and  $r \in \mathcal{I}(s_1s_2\alpha)$ .

Finally, suppose  $q \in \mathcal{I}(s_1s_4\alpha) \cap S$  (see Figure 5(a)). Observe that since  $Cone(s_3px) \cap S$  is empty by Assumption 4,  $S$  is admissible whenever either  $p$  or  $q$  is 5-redundant in  $\mathcal{H}_c(pq, s_2) \cap S$ . Hence, assume that  $\mathcal{I}(s_2s_3pq) \cap S \subset \mathcal{I}(s_2s_3\beta)$ , where  $\beta$  is the point of intersection of the diagonals of the quadrilateral  $s_2s_3pq$ . Let  $r$  be the first angular neighbor of  $\vec{qy}$  in  $Cone(yqs_1)$ ,

where  $y$  is the point where  $\overrightarrow{pq}$  intersects the boundary  $CH(S)$ . If  $r \in \mathcal{I}(s_1s_4\alpha) \cap S$ , then  $|\mathcal{V}(CH(\mathcal{H}_c(pq, s_1) \cap S))| = 6$  and both  $p$  and  $q$  are 5-redundant in  $\mathcal{H}_c(pq, s_1) \cap S$  (Figure 5(a)). Thus, the partition of  $S$  given by  $\mathcal{H}(pq, s_1) \cap S$  and  $\mathcal{H}_c(pq, s_2) \cap S$  is admissible. Otherwise, assume that  $r \in \mathcal{I}(s_1s_2\alpha) \cap S$ , as shown in Figure 5(b). Let  $\gamma$  be the point of intersection of the diagonals of the quadrilateral  $s_1rps_4$ . From Corollary 1, it is easy to see that whenever there exists a point of  $(\mathcal{H}(pq, s_1) \cap \mathcal{I}(s_1s_4\alpha)) \cap S$  outside  $\mathcal{I}(s_1s_4\gamma)$ , at least one of  $p$  or  $r$  is 5-redundant in  $(\mathcal{H}(pq, s_1)) \cap S \cup \{p\}$ , and the admissibility of  $S$  is immediate. Therefore, it suffices to assume that  $(\mathcal{H}(pq, s_1) \cap \mathcal{I}(s_1s_4\alpha)) \cap S \subset \mathcal{I}(s_1s_4\gamma)$ . Then  $|\mathcal{V}(CH(\mathcal{H}(s_2s_4, s_1) \cap S))| \geq 4$  and  $|\mathcal{H}(s_2s_4, s_1) \cap S| = 9$ . Hence, the 5-hole contained in  $\mathcal{H}(s_2s_4, s_1) \cap S$  (Corollary 2), is disjoint from the 5-hole contained in  $\mathcal{H}_c(s_2s_4, s_3) \cap S$ .  $\square$

## 5.2 $|\mathcal{V}(CH(S))| = 3$

Let  $s_1, s_2, s_3$  be the three vertices of  $CH(S)$ . Let  $\mathcal{I}(CH(S)) = \{u_1, u_2, \dots, u_{16}\}$  be such that  $u_i$  is the  $i$ -th angular neighbor of  $\overrightarrow{s_1s_2}$  in  $Cone(s_2s_1s_3)$ . For  $i \in \{1, 2, 3\}$  and  $j \in \{1, 2, \dots, 16\}$ , let  $p_{ij}$  be the point where  $\overrightarrow{s_iu_j}$  intersects the boundary of  $CH(S)$ . For example,  $p_{17}$  is the point of intersection of  $\overrightarrow{s_1u_7}$  with the boundary of  $CH(S)$ .

If  $\mathcal{I}(u_7p_{17}s_2)$  is not empty in  $S$ ,  $|\mathcal{V}(CH(\mathcal{H}_c(s_1u_7, s_2) \cap S))| \geq 4$  and by Corollary 2,  $\mathcal{H}_c(s_1u_7, s_2) \cap S$  contains a 5-hole which is disjoint from the 5-hole contained in  $\mathcal{H}(s_1u_7, s_3) \cap S$ . Therefore,  $\mathcal{I}(u_7p_{17}s_2) \cap S$  can be assumed to be empty. In fact, we can make the following more general assumption.

**Assumption 5** For all  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $Cone(p_{it}u_t s_j) \cap S$  is empty, where  $u_t$  is the seventh angular neighbor of  $\overrightarrow{s_i s_j}$  in  $Cone(s_j s_i s_k) \cap S$ .

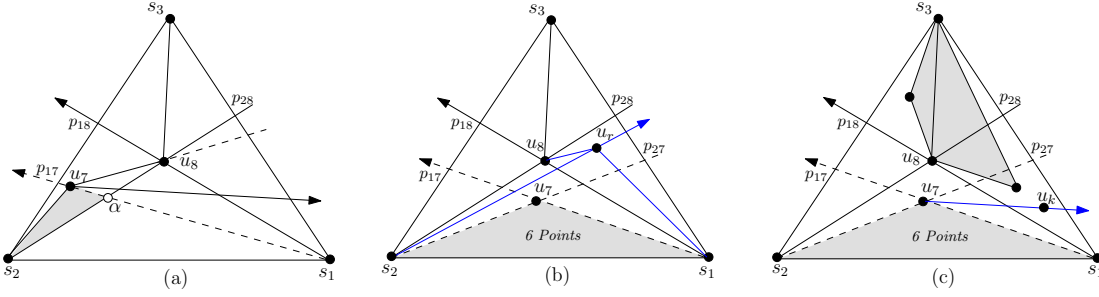
Now, we have the following observation.

**Observation 2** If for some  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $Cone(p_{it}u_t s_j) \cap S$  is non-empty, where  $u_t$  is the eighth angular neighbor of  $\overrightarrow{s_i s_j}$  in  $Cone(s_j s_i s_k)$ , then  $S$  is admissible.

*Proof.* W.l.o.g., let  $i = 1$  and  $j = 2$ , which means,  $t = 8$ . Set  $T = \mathcal{H}_c(s_1u_8, s_2) \cap S$ . Suppose, there exists a point  $u_r \in \mathcal{I}(s_2u_8p_{18}) \cap S$ . This implies that  $|\mathcal{V}(CH(T))| \geq 4$ . When  $|\mathcal{V}(CH(T))| \geq 5$ ,  $u_8$  is 5-redundant in  $T$  and  $T \setminus \{u_8\}$  contains a 5-hole which is disjoint from the 5-hole contained in  $(\mathcal{H}(s_1u_8, s_3) \cap S) \cup \{u_8\}$ .

Hence, it suffices to assume  $|\mathcal{V}(CH(T))| = 4$ . Let  $\mathcal{V}(CH(T)) = \{s_1, s_2, u_r, u_8\}$ , with  $r \leq 7$ , and  $\alpha$  the point of intersection of the diagonals of the quadrilateral  $s_1s_2u_ru_8$ . By Corollary 1, it follows that unless  $\mathcal{I}(s_1s_2u_ru_8) \cap S \subset \mathcal{I}(s_2\alpha u_r)$ , either  $s_1$  or  $u_8$  is 5-redundant in  $T$  and hence  $S$  is admissible. Therefore, assume  $\mathcal{I}(s_1s_2u_ru_8) \cap S \subset \mathcal{I}(s_2\alpha u_r)$ , which implies  $u_r = u_7$ , as shown in Figure 6(a). Suppose,  $Cone(s_1u_7u_8) \cap S$  is non-empty, and let  $u_k$  be the first angular neighbor of  $\overrightarrow{u_7s_1}$  in  $Cone(s_1u_7u_8)$ . Then  $\mathcal{I}(u_ku_7s_2) \cap S$  is non-empty, and  $u_7$  is 5-redundant in  $\mathcal{H}_c(u_7u_k, s_1) \cap S$ . Thus, the 5-hole contained in  $\mathcal{H}(u_7u_k, s_1) \cap S \cup \{u_k\}$  is disjoint from the 5-hole contained in  $(\mathcal{H}(u_7u_k, s_3) \cap S) \cup \{u_7\}$ . However, if  $Cone(s_1u_7u_8) \cap S$  is empty,  $u_7$  is 5-redundant in  $\mathcal{H}_c(u_7u_8, s_1) \cap S$  by Corollary 1, and the 5-hole contained in  $(\mathcal{H}(u_7u_8, s_1) \cap S) \cup \{u_8\}$  is disjoint from the 5-hole contained in  $(\mathcal{H}(u_7u_8, s_3) \cap S) \cup \{u_7\}$ .  $\square$

**Lemma 10.** If for some  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $Cone(p_{jt}u_t s_i) \cap S$  is empty, where  $u_t$  is the seventh angular neighbor of  $\overrightarrow{s_i s_j}$  in  $Cone(s_j s_i s_k)$ , then  $S$  is admissible.



**Fig. 6.** (a) Proof of Observation 2, (b) Proof of Lemma 10, and (c) Proof of Lemma 11.

*Proof.* W.l.o.g., let  $i = 1$  and  $j = 2$ . This means  $t = 7$  and  $Cone(s_1u_7p_{27})$  is empty in  $S$ . From Assumption 5,  $\mathcal{I}(u_7p_{17}s_2) \cap S$  is empty. Based on Observation 2 we may suppose  $Cone(s_2u_8p_{18}) \cap S$  is empty. Now, if  $Cone(p_{28}u_8s_1) \cap S$  is empty, at least one of  $s_1$ ,  $s_2$ , or  $u_8$  is 5-redundant in  $\mathcal{H}_c(s_1u_8, s_2) \cap S$ , and admissibility of  $S$  is immediate.

Therefore, assume that  $Cone(p_{28}u_8s_1) \cap S$  is non-empty, which implies that  $Cone(p_{27}s_2p_{28}) \cap S$  is non-empty, since  $Cone(s_1u_7p_{27}) \cap S$  is assumed to be empty. Let  $u_r$  be the first angular neighbor of  $\overrightarrow{s_2u_7}$  in  $Cone(p_{27}s_2p_{28}) \cap S$  (see Figure 6(b)). Now,  $S$  is admissible unless there exists a 5-hole in  $\mathcal{H}_c(s_1u_8, s_2) \cap S$  with  $s_1u_8$  as an edge. Observe that this 5-hole cannot have  $s_2$  as a vertex. Moreover, the remaining three vertices of this 5-hole, that is, the vertices apart from  $s_1$  and  $u_8$ , lie in the halfplane  $\mathcal{H}(u_r, s_2, s_1)$ . Now, this 5-hole can be extended to a convex hexagon having  $s_1$ ,  $u_8$ , and  $u_r$  as three consecutive vertices. Note that this convex hexagon may not be empty, and it does not contain  $s_2$  as a vertex. From this convex hexagon, we can get a 5-hole with  $u_r s_1$  as an edge, which does not contain  $u_8$  as a vertex and which lies in the halfplane  $\mathcal{H}(u_r, s_1, s_2)$ . Hence,  $(\mathcal{H}(s_2u_r, s_1) \cap S) \cup \{u_r\}$  contains a 5-hole which is disjoint from the 5-hole contained in  $(\mathcal{H}(s_2u_r, s_3) \cap S) \cup \{s_2\}$ .  $\square$

Hereafter, in light of the previous lemma, let us assume

**Assumption 6** For all  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $Cone(p_{jt}u_t s_i) \cap S$  is non-empty, where  $u_t$  is the seventh angular neighbor of  $\overrightarrow{s_i s_j}$  in  $Cone(s_j s_i s_k)$ .

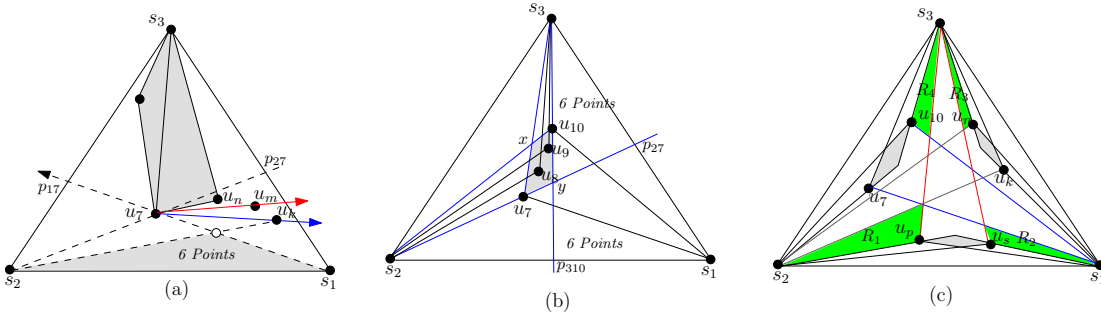
With this assumption we have the following two lemmas.

**Lemma 11.** If for some  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $Cone(s_k u_t s_j) \cap S$  is non-empty, where  $u_t$  is the eighth angular neighbor of  $\overrightarrow{s_i s_j}$  in  $Cone(s_j s_i s_k) \cap S$ , then  $S$  is admissible.

*Proof.* It suffices to prove the result for  $i = 1$  and  $j = 2$ , which means  $t = 8$ . Refer to Figure 6(c). Based on Observation 2 we may suppose  $S$  is admissible whenever  $\mathcal{I}(s_2u_8p_{18}) \cap S$  is non-empty. Therefore, assume that  $\mathcal{I}(s_2u_8p_{18}) \cap S$  is empty. Now, suppose  $\mathcal{I}(u_8s_3p_{18}) \cap S$  is non-empty, and let  $\mathcal{I}(u_8s_3p_{18}) \cap S$ . Let  $u_k$  be the first angular neighbor of  $\overrightarrow{u_7 s_1}$  in  $Cone(s_1u_7p_{27})$ , which is non-empty by Assumption 6. If  $Cone(u_k u_7 p_{27})$  is empty, from Corollary 1,  $s_2$  is 5-redundant in  $\mathcal{H}_c(u_7 u_k, s_2) \cap S$  and the admissibility of  $S$  follows. Thus, there exists some point  $u_m$  ( $m \neq k$ ) in  $Cone(u_k u_7 p_{27}) \cap S$ . Therefore,  $|\mathcal{V}(CH((\mathcal{H}(u_7 u_k, s_3) \cap S)))| \geq 4$ , and by Corollary 2,  $\mathcal{H}(u_7 u_k, s_3) \cap S$  contains a 5-hole. This 5-hole is disjoint from the 5-hole contained in  $\mathcal{H}_c(u_7 u_k, s_2) \cap S$ .  $\square$

**Lemma 12.** If for some  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $Cone(s_k u_t s_j) \cap S$  is non-empty, where  $u_t$  is the seventh angular neighbor of  $\overrightarrow{s_i s_j}$  in  $Cone(s_j s_i s_k)$ , then  $S$  is admissible.

*Proof.* W.l.o.g., let  $i = 1$  and  $j = 2$ , which means  $t = 7$ . From Assumption 5,  $\mathcal{I}(u_7 p_{17} s_2) \cap S$  is empty. Next, suppose there exists a point  $u_a$  in  $\mathcal{I}(u_7 p_{17} s_3) \cap S$ . Refer to Figure 7(a). Since  $\text{Cone}(s_1 u_7 p_{27}) \cap S$  is non-empty by Assumption 6, let  $u_k$  be the first angular neighbor of  $\overrightarrow{s_2 s_1}$  in  $\text{Cone}(s_1 u_7 p_{27})$  and  $\alpha$  the point of intersection of the diagonals of the convex quadrilateral  $u_7 s_2 s_1 u_k$ . From Corollary 1, it is easy to see that  $S$  is admissible unless  $\mathcal{I}(s_1 s_2 u_7 u_k) \cap S \subset \mathcal{I}(s_1 s_2 \alpha)$ . Now, if  $u_7$  is the eighth angular neighbor of  $\overrightarrow{s_2 s_1}$  or  $\overrightarrow{s_2 s_3}$  in  $\text{Cone}(s_1 s_2 s_3)$ , then  $S$  is admissible from Lemma 11, since  $\mathcal{I}(u_7 s_3 s_1) \cap S$  is not empty. Since the eighth angular neighbor of  $\overrightarrow{s_2 s_3}$  in  $\text{Cone}(s_1 s_2 s_3)$  is the ninth angular neighbor of  $\overrightarrow{s_2 s_1}$  in  $\text{Cone}(s_1 s_2 s_3)$ ,  $u_7$  cannot be the eighth or ninth angular neighbor  $\overrightarrow{s_2 s_1}$  in  $\text{Cone}(s_1 s_2 s_3)$ . Thus there exist at least two points,  $u_m$  and  $u_n$  in  $\text{Cone}(p_{27} u_7 u_k) \cap S$ , where  $u_m$  is the first angular neighbor of  $\overrightarrow{u_7 u_k}$  in  $\text{Cone}(p_{27} u_7 u_k)$ . Then, the 5-hole contained in  $(\mathcal{H}(u_7 u_m, s_1) \cap S) \cup \{u_m\}$  is disjoint from the 5-hole contained in  $(\mathcal{H}(u_7 u_m, s_3) \cap S) \cup \{u_7\}$ , since  $|\mathcal{V}(CH((\mathcal{H}(u_7 u_m, s_3) \cap S) \cup \{u_7\}))| \geq 4$  (see Figure 7(a)).  $\square$



**Fig. 7.** (a) Illustration for the proof of Lemma 12, (b) Diamond arrangement  $D\{u_7, u_{10}\}$ , (c) Arrangement of diamonds  $D\{u_7, u_{10}\}$ ,  $D\{u_k, u_n\}$ , and  $D\{u_p, u_s\}$  in  $\mathcal{I}(s_1 s_2 s_3)$ .

The following lemma proves the admissibility of  $S$  in the remaining cases.

**Lemma 13.** *If for all  $i \neq j \neq k \in \{1, 2, 3\}$ ,  $\text{Cone}(s_k u_\alpha s_j) \cap S$  and  $\text{Cone}(s_k u_\beta s_j) \cap S$  are empty, where  $u_\alpha, u_\beta$  are the seventh and eighth angular neighbors of  $\overrightarrow{s_i s_j}$  in  $\text{Cone}(s_j s_i s_k)$ , respectively, then  $S$  is admissible.*

*Proof.* Lemmas 11 and 12 imply that  $S$  is admissible unless the interiors of  $s_2 u_7 s_3$ ,  $s_2 u_8 s_3$ ,  $s_2 u_9 s_3$ , and  $s_2 u_{10} s_3$  are empty in  $S$ . Thus, points  $u_7, u_8, u_9, u_{10}$  must be arranged inside  $CH(S)$  as shown in Figure 7(b). We call such a set of 4 points a *diamond* and denote it by  $D\{u_7, u_{10}\}$ . Note that,  $|\mathcal{I}(s_1 s_2 u_7) \cap S| = |\mathcal{I}(s_1 s_3 u_{10}) \cap S| = 6$ .

Since  $\text{Cone}(s_1 u_7 p_{27}) \cap S$  is non-empty by Assumption 6,  $u_7$  cannot be the seventh, eighth, ninth, and tenth angular neighbors of  $\overrightarrow{s_2 s_1}$  in  $\text{Cone}(s_1 s_2 s_3)$ . Let  $u_k$  be the seventh angular neighbor of  $\overrightarrow{s_2 s_1}$  in  $\text{Cone}(s_1 s_2 s_3)$ . Suppose that  $u_k \in \mathcal{I}(u_7 s_2 s_1)$ . Then we have  $|\mathcal{I}(s_1 u_k p_{2k}) \cap S| \geq 1$ , as  $|\mathcal{I}(u_7 s_1 s_2) \cap S| = 6$ . Hence,  $|\mathcal{V}(CH(\mathcal{H}_c(s_2 u_k, s_1) \cap S))| \geq 4$ , and since  $|\mathcal{H}_c(s_2 u_k, s_1) \cap S| = 9$ , the admissibility of  $S$ , in this case, follows from Corollary 2.

Therefore, it can be assumed that the seventh angular neighbor of  $\overrightarrow{s_2 s_1}$ , that is,  $u_k$  lies in  $\mathcal{I}(p_{27} u_7 s_1) \cap S$ . Then Lemmas 11 and 12 imply that the eighth, ninth, and tenth angular neighbors of  $\overrightarrow{s_2 s_1}$  are in  $\text{Cone}(s_1 u_7 p_{27})$ . Let  $u_i, u_m$ , and  $u_n$  denote the eighth, ninth, and tenth angular neighbors of  $\overrightarrow{s_2 s_1}$  in  $\text{Cone}(s_1 s_2 s_3)$ , respectively. From similar arguments as before, these three points along with the point  $u_k$  form a diamond,  $D\{u_k, u_n\}$ , which is disjoint from diamond  $D\{u_7, u_{10}\}$  (see Figure 7(c)).

Let  $u_s$  be the seventh angular neighbor of  $\overline{s_3s_1}$  in  $Cone(s_1s_3u_{10})$  as shown in Figure 7(c). Again, Assumption 6 and the same logic as before implies  $S$  is admissible if  $u_{10}$  is the eighth, ninth or tenth angular neighbor of  $\overline{s_3s_1}$  in  $Cone(s_1s_3u_{10})$ . Let  $u_r, u_q,$  and  $u_p$  be the eighth, ninth, and tenth angular neighbors of  $\overline{s_3s_1}$  in  $Cone(s_1s_3u_{10})$ , respectively. As before, these three points along with the point  $u_s$ , form another diamond  $D\{u_p, u_s\}$ , which is disjoint from both  $D\{u_7, u_{10}\}$  and  $D\{u_k, u_n\}$ .

Let  $R_1, R_2, R_3, R_4$  be the shaded regions inside  $CH(S)$ , as shown in Figure 7(c). To begin with suppose that  $|R_1 \cap S| \geq 1$ . Let  $u_z$  be the first angular neighbor of  $\overline{u_p s_3}$  in  $Cone(p_2p_u p_s s_3)$ . Note that  $|\mathcal{H}_c(u_p u_z, s_3) \cap S| = 10$  and  $\mathcal{I}(s_2 u_z u_p) \cap S$  is non-empty, as  $|R_1 \cap S| \geq 1$ . This implies that  $u_p$  is 5-redundant in  $\mathcal{H}_c(u_p u_z, s_3) \cap S$ . Therefore, the 5-hole contained in  $(\mathcal{H}(u_p u_z, s_3) \cap S) \cup \{u_z\}$  is disjoint from the 5-hole contained in  $(\mathcal{H}(u_p u_z, s_1) \cap S) \cap \{u_p\}$ . Therefore, assume that  $|R_1 \cap S| = 0$ . This implies that  $|R_4 \cap S| = 2$ , as  $|\mathcal{I}(s_2 s_3 u_p) \cap S| = 6$ . The admissibility of  $S$  now follows from exactly similar arguments by taking the nearest angular neighbor of  $\overline{u_{10} s_1}$  in  $Cone(s_1 u_{10} p_{310})$ .  $\square$

Since all the different cases have been considered, the proof of the case  $|\mathcal{V}(CH(S))| = 3$ , and hence the theorem is finally completed.

## 6 Proof of Theorem 3

Let  $S$  be any set of  $2m + 9$  points in the plane in general position, and  $u_1, u_2,$  and  $w_m$  be vertices of  $CH(S)$  such that  $u_1 u_2$  and  $u_1 w_m$  are edges of  $CH(S)$ . We label the points in the set  $S$  inductively as follows.

- (i)  $u_i$  be the  $(i - 2)$ -th angular neighbor of  $\overline{u_1 u_2}$  in  $Cone(w_m u_1 u_2)$ , where  $i \in \{3, 4, \dots, m\}$ .
- (ii)  $v_i$  be the  $i$ -th angular neighbor of  $\overline{u_1 u_m}$  in  $Cone(w_m u_1 u_m)$ , where  $i \in \{1, 2, \dots, 9\}$ .
- (iii)  $w_i$  be the  $i$ -th angular neighbor of  $\overline{u_1 v_9}$  in  $Cone(w_m u_1 v_9)$ , where  $i \in \{1, 2, \dots, m\}$ .

Therefore,  $S = U \cup V \cup W$ , where  $U = \{u_1, u_2, \dots, u_m\}$ ,  $V = \{v_1, v_2, \dots, v_9\}$ , and  $W = \{w_1, w_2, \dots, w_m\}$ .

A disjoint convex partition of  $S$  into three subsets  $S_1, S_2, S_3$  is said to be a *separable* partition of  $S$  (or *separable* for  $S$ ) if  $|S_1| = |S_3| = m$  and the set of 9 points  $S_2$  contains a 5-hole. The set  $S$  is said to be *separable* if there exists a partition which is separable for  $S$ . For proving Theorem 3 we have to identify a separable partition for every set of  $2m + 9$  points in the plane in general position. It is clear, from Corollary 2, that  $S$  is separable whenever  $|\mathcal{V}(CH(V))| \geq 4$ .

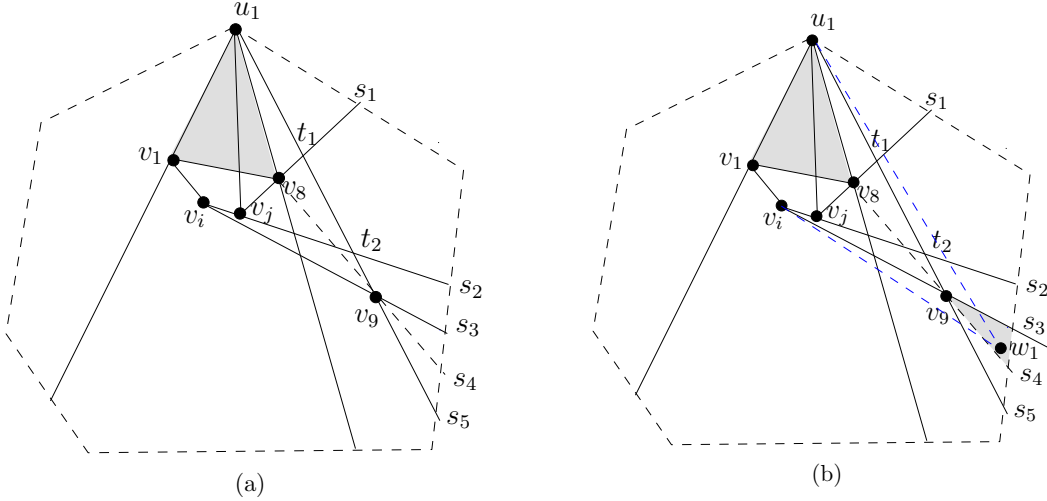
Let  $T = V \setminus \{v_9\} \cup \{u_1\}$ . If  $|\mathcal{V}(CH(T))| \geq 6$ ,  $u_1$  is 5-redundant in  $T$  and  $S_1 = U$ ,  $S_2 = V$ , and  $S_3 = W$  is a separable partition of  $S$ .

Therefore, assume that  $|\mathcal{V}(CH(T))| \leq 5$ . The three cases based on the size of  $|\mathcal{V}(CH(T))|$  are considered separately in the following lemmas.

**Lemma 14.**  *$S$  is separable whenever  $|\mathcal{V}(CH(T))| = 5$ .*

*Proof.* Let  $\{u_1, v_1, v_i, v_j, v_8\}$  be the vertices of the convex hull of  $T$ . It suffices to assume that  $\mathcal{I}(u_1 v_1 v_i)$  and  $\mathcal{I}(u_1 v_1 v_8)$  are empty in  $S$ , otherwise either  $v_1$  or  $u_1$  is, respectively, 5-redundant and  $S$  is separable. Let the lines  $\overline{v_j v_8}$  and  $\overline{v_i v_j}$  intersect  $\overline{u_1 v_9}$  at the points  $t_1, t_2$ , and  $CH(S)$  at the points  $s_1, s_2$ , respectively (Figure 8(a)). Now, we consider the following cases based on the location of the point  $v_9$  on the line segment  $u_1 s_5$ , where  $s_5$  is the point where  $\overline{u_1 v_9}$  intersects the boundary of  $CH(S)$ .





**Fig. 8.** Illustrations for the proof of Lemma 14.

*Case 1:*  $v_9$  lies on the line segment  $u_1t_2$ . This implies,  $|\mathcal{V}(CH(V))| \geq 4$  and by Corollary 2,  $S_1 = U$ ,  $S_2 = V$ , and  $S_3 = W$  is a separable partition of  $S$ .

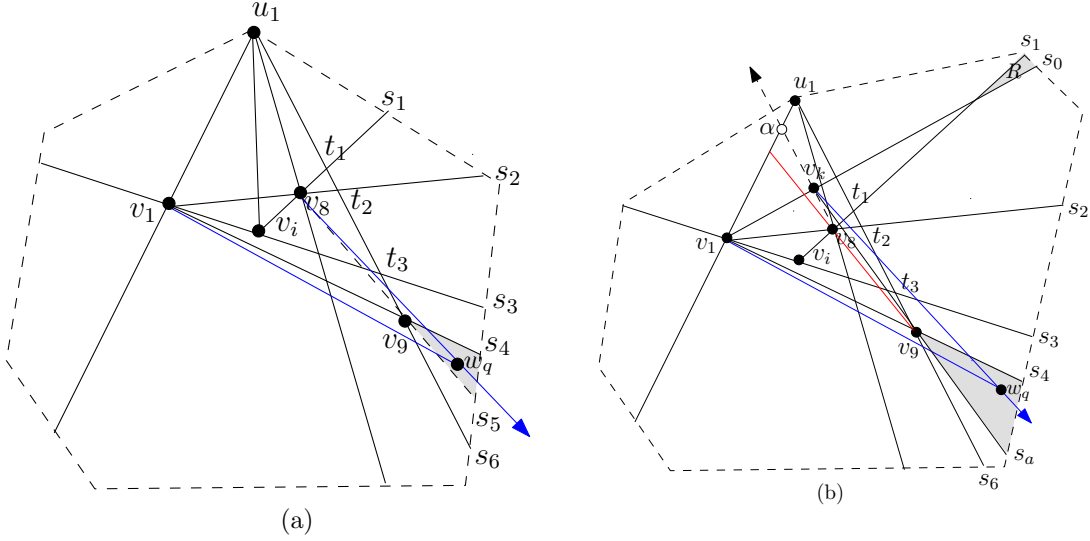
*Case 2:*  $v_9$  lies on the line segment  $t_2s_5$ . Let  $s_3$  and  $s_4$  be the points where the lines  $\overrightarrow{v_i v_9}$  and  $\overrightarrow{v_8 v_9}$  intersects  $CH(S)$ , respectively. (Note that if  $v_9 = s_5$ , then the points  $s_3$  and  $s_4$  coincide with the point  $v_9$ .) If  $Cone(u_1t_1s_1) \cap S$  is non-empty, let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_8 u_1}$  in  $Cone(u_1t_1s_1)$ . This implies,  $|\mathcal{V}(CH(V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 5$  and by Corollary 2  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is a separable partition of  $S$ . So, assume that  $Cone(u_1t_1s_1) \cap S$  empty.

*Case 2.1:*  $Cone(s_1v_j s_2) \cap W$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_j s_1}$  in  $Cone(s_1v_j s_2)$ . Then,  $|\mathcal{V}(CH(V \setminus \{v_9\} \cup \{w_q\}))| \geq 4$ , and the partition,  $S_1 = U$ ,  $S_2 = V \setminus \{v_9\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is separable for  $S$ .

*Case 2.2:*  $Cone(s_1v_j s_2) \cap W$  is empty and  $Cone(s_5v_9 s_4) \cap W$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_9 s_5}$  in  $Cone(s_5v_9 s_4)$ . Observe that  $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 4$  and  $\mathcal{I}(v_8 v_9 w_q) \cap S$  is empty. Now, if  $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 5$ , then  $v_1$  is clearly 5-redundant in  $V \cup \{w_q\}$ . Otherwise, Corollary 1 now implies that  $v_1$  is 5-redundant in  $V \cup \{w_q\}$ . Therefore, the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{u_1\}$  is separable for  $S$ .

*Case 2.3:*  $Cone(s_1v_j s_2) \cap W$  and  $Cone(s_5v_9 s_4) \cap W$  are both empty. If  $w_1$ , the nearest angular neighbor of  $\overrightarrow{u_1 s_5}$  in  $W$ , lies in  $Cone(s_2v_i s_3)$ ,  $|\mathcal{V}(CH(V \setminus \{v_1\} \cup \{u_1, w_1\}))| = 4$  and  $u_1$  is 5-redundant in  $V \setminus \{v_1\} \cup \{u_1, w_1\}$  by Corollary 1. Therefore,  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_1\}$ , and  $S_3 = W \setminus \{w_1\} \cup \{u_1\}$  is separable for  $S$ . Finally, consider that  $w_1 \in Cone(s_4v_9 s_3)$  and let  $Z = V \setminus \{v_1\} \cup \{u_1, w_1\}$ . Observe that  $|\mathcal{V}(CH(Z))| = 3$  (Figure 8(b)). Now, since  $|Z| = 10$ ,  $Z$  must contain a 5-hole. Note that since  $\mathcal{I}(u_1v_1v_8)$  is assumed to be empty in  $S$ , it follows that all the four vertices of the 4-hole  $u_1v_8v_9w_1$  cannot be a part of any 5-hole in  $Z$ . Moreover, there cannot be a 5-hole in  $Z$  with the points  $u_1, v_9, w_1$  or the points  $u_1, v_8, v_9$  as vertices, since  $Cone(s_5u_1w_1)$  and  $Cone(u_1w_1v_8)$  are empty in  $Z$ . Emptiness of  $Cone(s_5u_1w_1) \cap Z$  and  $Cone(u_1w_1v_8) \cap Z$  also implies that there cannot be a 5-hole in  $Z$  with both the points  $u_1$  and  $w_1$  as vertices. Thus, either  $u_1$  or  $w_1$  is 5-redundant in  $Z$ , and separability of  $S$  follows.  $\square$

**Lemma 15.**  $S$  is separable whenever  $|\mathcal{V}(CH(T))| = 4$ .



**Fig. 9.** Illustrations for the proof of Lemma 15: *Case 1* and *Case 2*.

*Proof.* Suppose  $\{u_1, v_1, v_i, v_8\}$  are the vertices of the convex hull of  $T$ . Let the lines  $\overrightarrow{v_i v_8}$ ,  $\overrightarrow{v_1 v_8}$ , and  $\overrightarrow{v_1 v_i}$  intersect  $\overrightarrow{u_1 v_9}$  at the points  $t_1, t_2, t_3$ , and  $CH(S)$  at the points  $s_1, s_2, s_3$ , respectively (see Figure 9(a)). If  $v_9$  lies on the line segment  $u_1 t_1$  or  $t_2 t_3$ , then  $|\mathcal{V}(CH(V))| \geq 4$  and  $S_1 = U$ ,  $S_2 = V$ , and  $S_3 = W$  is separable for  $S$ . So, assume that  $v_9$  lies on the line segment  $t_1 t_2$ , or on the line segment  $t_3 s_6$ , where  $s_6$  is the point of intersection of  $\overrightarrow{u_1 v_9}$  and  $CH(S)$ . Now, we consider the following cases.

*Case 1:*  $v_9$  lies on the line segment  $t_3 s_6$ , and  $\mathcal{I}(u_1 v_1 v_8) \cap S$  is empty. Let  $s_4$  and  $s_5$  be the points where  $\overrightarrow{v_1 v_9}$  and  $\overrightarrow{v_8 v_9}$  intersect the boundary of  $CH(S)$ , respectively.

*Case 1.1:*  $Cone(u_1 v_8 s_1) \cap W$  is non-empty. If  $w_q$  be the first angular neighbor of  $\overrightarrow{v_8 u_1}$  in  $Cone(u_1 v_8 s_1)$ , then  $|\mathcal{V}(CH(V \setminus \{v_9\} \cup \{w_q\}))| = 4$ . Hence,  $S_1 = U$ ,  $S_2 = V \setminus \{v_9\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is a separable partition.

*Case 1.2:*  $Cone(u_1 v_8 s_1) \cap W$  is empty, and  $Cone(s_6 v_9 s_5) \cap W$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_9 s_6}$  in  $Cone(s_6 v_9 s_5)$ . Note that  $CH(V \cup \{w_q\})$  is a quadrilateral and  $\mathcal{I}(v_8 v_9 w_q) \cap S$  is empty. This implies that  $v_1$  is 5-redundant in  $V \cup \{w_q\}$  by Corollary 1. Therefore,  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{u_1\}$  is separable for  $S$ .

*Case 1.3:* Both  $Cone(u_1 v_8 s_1) \cap W$  and  $Cone(s_6 v_9 s_5) \cap W$  are empty, but  $Cone(s_5 v_8 s_2) \cap W$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_8 v_9}$  in  $Cone(s_5 v_8 s_2)$ . To begin with, assume  $w_q \in Cone(s_5 v_8 s_2) \setminus Cone(s_5 v_9 s_4)$ . Then  $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 4$  and  $V \cup \{w_q\}$  contains a 5-hole. Now, by Corollary 1, either  $v_1$  or  $w_q$  is 5-redundant in  $V \cup \{w_q\}$ , and the separability of  $S$  is immediate. Otherwise,  $w_q \in Cone(s_5 v_9 s_4)$ , and  $|\mathcal{V}(CH(V \cup \{w_q\}))| = 3$  (Figure 9(a)). Now,  $V \cup \{w_q\}$  contains a 5-hole and at least one of  $v_1, v_8$ , and  $w_q$  is 5-redundant in  $V \cup \{w_q\}$ . If  $w_q$  is 5-redundant, the separability of  $S$  is immediate. If  $v_1$  is 5-redundant, the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{u_1\}$  is a separable partition of  $S$ . Finally, if  $v_8$  is 5-redundant, then the partition  $S_1 = U$ ,  $S_2 = V \setminus \{v_8\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_8\}$  is a separable partition of  $S$ .

*Case 1.4:*  $W \subset Cone(s_1 v_8 s_2)$ . Let  $w_q$  be the nearest angular neighbor of  $\overrightarrow{v_i s_1}$  in  $Cone(s_1 v_i s_3)$ . If  $\mathcal{I}(u_1 v_1 v_i) \cap S$  is non-empty, then  $|\mathcal{V}(CH(V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 4$  and the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is

separable for  $S$ . Otherwise, assume  $\mathcal{I}(u_1v_1v_i) \cap S$  is empty. Let  $w_1$  be the first angular neighbor of  $\overrightarrow{u_1s_6}$  in  $W$ . Then,  $|\mathcal{V}(CH(V \setminus \{v_1\} \cup \{w_1\}))| \geq 4$ , and the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_1\}$ , and  $S_3 = W \setminus \{w_1\} \cup \{u_1\}$  is separable for  $S$ .

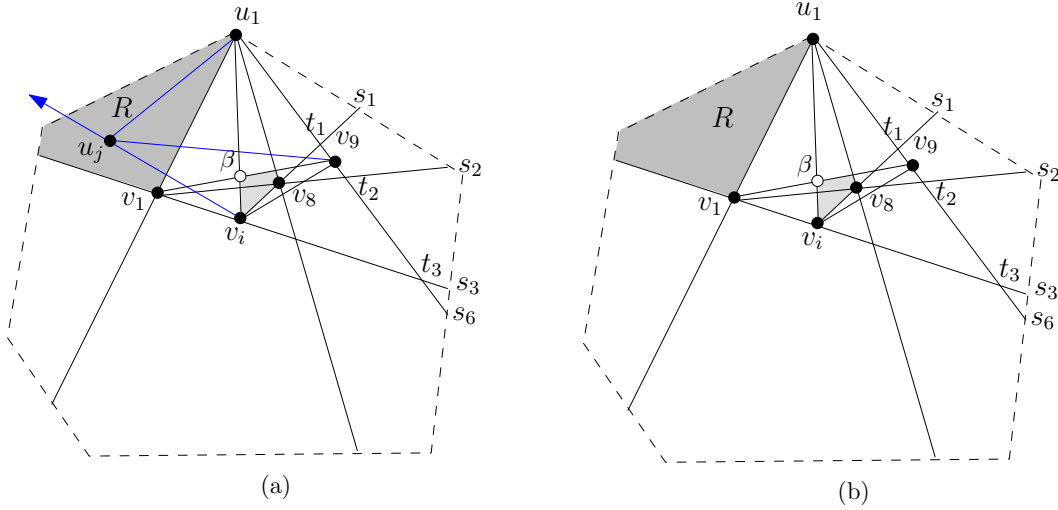


Fig. 10. Illustrations for the proof of Lemma 15: Case 3.

**Case 2:**  $v_9$  lies on the line segment  $t_3s_6$ , and  $\mathcal{I}(u_1v_1v_8) \cap S$  is non-empty. Let  $v_k$  be the first angular neighbor of  $\overrightarrow{v_8u_1}$  in  $Cone(u_1v_8v_1)$ , and let  $s_0, s_4$  and  $s_a$  be the points where  $\overrightarrow{v_1v_k}$ ,  $\overrightarrow{v_1v_9}$  and  $\overrightarrow{v_kv_9}$  intersect  $CH(S)$ , respectively. Note that if  $v_k \in \overline{\mathcal{H}}(v_9v_8, u_1) \cap V$ , then  $|\mathcal{V}(CH(V))| \geq 4$  and the separability of  $S$  is immediate. Therefore, assume that  $v_k \in \mathcal{H}(v_9v_8, u_1) \cap V$  (see Figure 9(b)). Let  $\alpha$  be the point where  $\overrightarrow{v_8v_k}$  intersects  $\overrightarrow{u_1v_1}$ . If  $\mathcal{I}(v_1v_k\alpha) \cap V$  is non-empty, then  $|\mathcal{V}(CH(V))| \geq 5$ , and the separability of  $S$  is immediate. Therefore, assume that  $\mathcal{I}(v_1v_k\alpha) \cap V$  is empty, that is,  $\mathcal{I}(v_1v_ku_1) \cap V$  is empty.

**Case 2.1:**  $Cone(s_6v_9s_a) \cap W$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_9s_6}$  in  $Cone(s_6v_9s_a)$ . Then  $|\mathcal{V}(CH(V \cup \{w_q\}))| = 4$  and by Corollary 1 either  $v_1$  or  $v_9$  is 5-redundant in  $V \cup \{w_q\}$ . The separability of  $S$  now follows easily.

**Case 2.2:**  $Cone(s_6v_9s_a) \cap W$  is empty and  $Cone(s_0v_k s_a) \cap W$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_k v_9}$  in  $Cone(s_0v_k s_a)$ . If  $w_q \in Cone(s_0v_k s_a) \setminus Cone(s_4v_9 s_a)$  then  $|\mathcal{V}(CH(V \setminus \{v_1\} \cup \{w_q\}))| \geq 4$ , and the separability of  $S$  is immediate. Otherwise, assume  $w_q \in Cone(s_4v_9 s_a)$ . Then  $|\mathcal{V}(CH(V \cup \{w_q\}))| = 3$  and either  $v_1, v_k$ , or  $w_q$  is 5-redundant in  $V \cup \{w_q\}$ , and the separability of  $S$  is immediate.

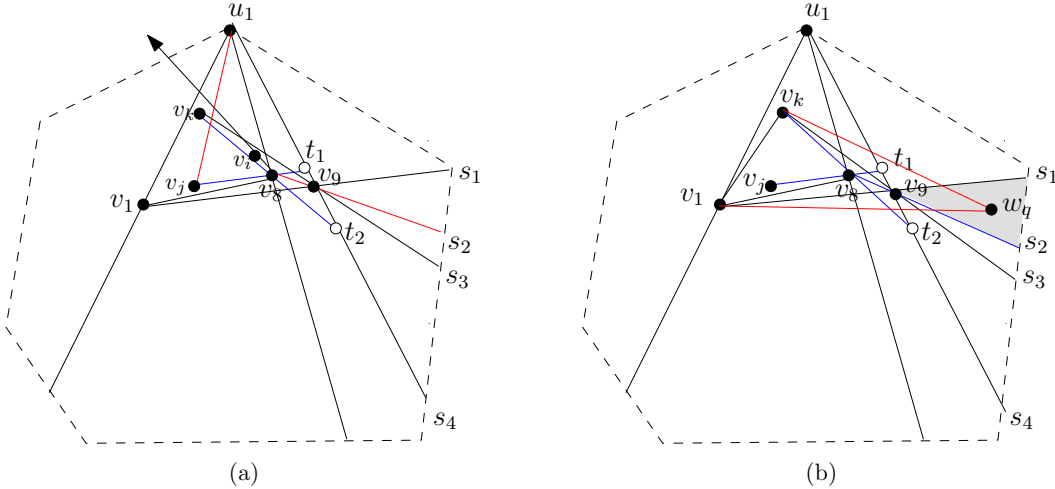
**Case 2.3:** Both  $Cone(s_6v_9s_a) \cap W$  and  $Cone(s_0v_k s_a) \cap W$  are empty, but  $Cone(u_1v_k s_0) \cap W$  is non-empty. Now, if  $Cone(u_1v_8 s_1) \cap W$  is non-empty, the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1, v_9\} \cup \{u_1, w_i\}$ , and  $S_3 = W \setminus \{w_i\} \cup \{v_9\}$  is separable for  $S$ , where  $w_i$  is the first angular neighbor of  $\overrightarrow{v_8u_1}$  in  $Cone(u_1v_8 s_1) \cap W$ . Therefore, assume that  $Cone(u_1v_8 s_1) \cap W$  is empty. This implies,  $W \subset R \cap S$ , where  $R$  is the shaded region as shown in Figure 9(b). Let  $w_q$  be the nearest angular neighbor of  $\overrightarrow{v_i v_8}$  in  $Cone(s_1v_i s_3)$ . If  $\mathcal{I}(u_1v_1v_i) \cap S$  is non-empty, then  $|\mathcal{V}(CH(V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 4$  and the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is separable for  $S$ . Otherwise, assume  $\mathcal{I}(u_1v_1v_i) \cap S$  is empty. Let  $w_1$  be the first angular neighbor of  $\overrightarrow{u_1s_6}$  in  $W$ . Then,  $|\mathcal{V}(CH(V \setminus \{v_1\} \cup \{w_1\}))| \geq 4$ , and the partition

$S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_1\}$ , and  $S_3 = W \setminus \{w_1\} \cup \{u_1\}$  is separable for  $S$ .

*Case 3:*  $v_9$  lies on the line segment  $t_1t_2$ . Observe that if either  $u_1$  or  $v_1$  is 5-redundant in  $V \cup \{u_1\}$ , then the separability of  $S$  is immediate. Therefore, from Corollary 1, it suffices to assume that all the points inside  $CH(V \cup \{u_1\})$  must lie in  $\mathcal{I}(v_9v_i\beta)$ , where  $\beta$  is the point of intersection of the diagonals of the quadrilateral  $u_1v_1v_iv_9$ . Next, suppose that  $R \cap S$  is non-empty, where  $R$  is the shaded region inside  $CH(S)$  as shown in Figure 10(a). Let  $u_j \in R \cap S$  be the first angular neighbor of  $\overrightarrow{v_iu_1}$  in  $R$ . Then  $|\mathcal{V}(CH(V \setminus \{v_1\} \cup \{u_1, u_j\}))| = 4$  and  $v_i$  is 5-redundant in  $V \setminus \{v_1\} \cup \{u_1, u_j\}$ , since  $\mathcal{I}(u_jv_iv_9) \cap S$  is non-empty (Corollary 1). Hence, the partition of  $S$  given by  $S_1 = U \setminus \{u_1, u_j\} \cup \{v_1, v_i\}$ ,  $S_2 = V \setminus \{v_1, v_i\} \cup \{u_1, u_j\}$ ,  $S_3 = W$  is separable. On the other hand, if  $R \cap S$  is empty, then the partition  $S_1 = U \setminus \{u_1\} \cup \{v_i\}$ ,  $S_2 = V \setminus \{v_i\} \cup \{u_1\}$ , and  $S_3 = W$  is separable, since  $v_i$  is 5-redundant in  $V \cup \{u_1\}$  by Corollary 1 (see Figure 10(b)).

**Lemma 16.**  $S$  is separable whenever  $|\mathcal{V}(CH(T))| = 3$ .

*Proof.* Let  $\mathcal{V}(CH(T)) = \{u_1, v_1, v_8\}$ . Let  $v_i$  and  $v_j$  be the first angular neighbors of  $\overrightarrow{v_8u_1}$  and  $\overrightarrow{v_8v_1}$  respectively in  $Cone(u_1v_8v_1)$ . Let  $\overrightarrow{v_jv_8}$  and  $\overrightarrow{v_iv_8}$  intersect  $\overrightarrow{u_1v_9}$  at  $t_1$  and  $t_2$ , respectively (Figure 11(a)). If  $v_9$  lies on the line segment  $u_1t_1$ ,  $|\mathcal{V}(CH(V \setminus \{v_1\} \cup \{u_1\}))| \geq 4$  and by Corollary 2,  $V \setminus \{v_1\} \cup \{u_1\}$  contains a 5-hole. Thus,  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{u_1\}$ , and  $S_3 = W$  is a separable partition of  $S$ . Similarly, if  $v_9$  lies on the line segment  $t_2s_4$ , where  $s_4$  is the point where  $\overrightarrow{u_1v_9}$  intersects the boundary of  $CH(S)$ , then  $|\mathcal{V}(CH(V))| \geq 4$ , and  $S_1 = U$ ,  $S_2 = V$ , and  $S_3 = W$  is separable for  $S$ .



**Fig. 11.** Illustrations for the proof of Lemma 16.

Therefore,  $v_9$  lies on the line segment  $t_1t_2$ . Clearly,  $S$  is separable unless  $|\mathcal{V}(CH(V))| = 3$ . Let  $\mathcal{V}(CH(V)) = \{v_1, v_k, v_9\}$ . (Note that  $v_k$  need not be the point  $v_i$  as shown in Figure 11(a)). Let  $s_1$ ,  $s_2$ , and  $s_3$  be the points where  $\overrightarrow{v_1v_9}$ ,  $\overrightarrow{v_8v_9}$ , and  $\overrightarrow{v_kv_9}$  intersect  $CH(S)$ , respectively. Now, we have the following cases:

*Case 1:*  $Cone(u_1v_8t_1) \cap S$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overrightarrow{v_8u_1}$ , in  $Cone(u_1v_8t_1)$ . This implies,  $|\mathcal{V}(CH(V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 4$ , and  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1, v_9\} \cup \{u_1, w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is a separable partition of  $S$ .

*Case 2:*  $\text{Cone}(u_1v_8t_1) \cap S$  is empty and  $\text{Cone}(s_4v_9s_3) \cap S$  is non-empty. Suppose,  $w_q$  is the first angular neighbor of  $\overline{v_9s_4}$  in  $\text{Cone}(s_4v_9s_3)$ . Since  $|\mathcal{V}(\text{CH}(V \cup \{w_q\}))| \geq 4$ , either  $v_1$  or  $v_9$  is 5-redundant in  $V \cup \{w_q\}$  by Corollary 1. Thus, either  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{u_1\}$  or  $S_1 = U$ ,  $S_2 = V \setminus \{v_9\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_9\}$  is, respectively, separable for  $S$ .

*Case 3:*  $\text{Cone}(u_1v_8t_1) \cap S$  and  $\text{Cone}(s_4v_9s_3) \cap S$  are empty but  $\text{Cone}(s_3v_9s_2) \cap S$  is non-empty. If  $w_q$  is the first angular neighbor of  $\overline{v_9s_3}$  in  $\text{Cone}(s_3v_9s_2)$ , then  $v_1v_7v_8v_9w_q$  is a 5-hole, and  $S_1 = U$ ,  $S_2 = V \setminus \{v_k\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_k\}$  is separable for  $S$ .

*Case 4:* The three sets  $\text{Cone}(u_1v_8t_1) \cap S$ ,  $\text{Cone}(s_4v_9s_3) \cap S$ , and  $\text{Cone}(s_3v_9s_2) \cap S$  are all empty, but  $\text{Cone}(t_1v_8s_2) \cap S$  is non-empty. Let  $w_q$  be the first angular neighbor of  $\overline{v_8v_9}$  in  $\text{Cone}(u_1v_8v_9)$ . Clearly,  $w_q \in \text{Cone}(t_1v_8s_2)$ .

*Case 4.1:*  $w_q \in \text{Cone}(t_1v_8s_2) \setminus \text{Cone}(s_2v_9s_1)$ . In this case,  $|\mathcal{V}(\text{CH}(V \cup \{w_q\}))| = 4$  and  $v_1$  is 5-redundant in  $V \cup \{w_q\}$  by Corollary 1. Then the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{u_1\}$  is separable for  $S$ .

*Case 4.2:*  $w_q \in \text{Cone}(s_2v_9s_1)$  (see Figure 11(b)). Let  $Z = V \cup \{w_q\}$ . Observe,  $|\mathcal{V}(\text{CH}(Z))| = 3$  and  $Z$  must contain a 5-hole, since  $|Z| = 10$ . Now, either  $v_1$ ,  $v_k$ , or  $w_q$  is 5-redundant in  $Z$ . If  $w_q$  is 5-redundant, the separability of  $S$  is immediate. If  $v_1$  is 5-redundant, the partition  $S_1 = U \setminus \{u_1\} \cup \{v_1\}$ ,  $S_2 = V \setminus \{v_1\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{u_1\}$  is a separable partition of  $S$ . Finally, if  $v_k$  is 5-redundant, then the partition  $S_1 = U$ ,  $S_2 = V \setminus \{v_k\} \cup \{w_q\}$ , and  $S_3 = W \setminus \{w_q\} \cup \{v_k\}$  is a separable partition of  $S$ .  $\square$

This finishes the analysis of all the different cases, and completes the proof of Theorem 3.

## 7 Conclusion

In this paper we address problems concerning the existence of disjoint 5-holes in planar point sets. We prove that every set of 19 points in the plane, in general position, contains two disjoint 5-holes. Next, we show that any set of  $2m + 9$  points in the plane can be subdivided into three disjoint convex regions such that one contains a set of 9 points which contains a 5-hole, and the others contain  $m$  points each, where  $m$  is a positive integer. Combining these two results we show that the number of disjoint empty convex pentagons in any set of  $n$  points in the plane in general position, is at least  $\lfloor \frac{5n}{47} \rfloor$ . This bound has been further improved to  $\frac{3n-1}{28}$  for infinitely many  $n$ .

In other words, we have shown that  $H(5, 5) \leq 19$ . This improves upon the results of Hosono and Urabe [15, 16], where they showed  $17 \leq H(5, 5) \leq 20$ . There is still a gap between the upper and lower bounds of  $H(5, 5)$ , which probably requires a more complicated and detailed argument to be settled.

However, we are still quite far from establishing non-trivial bounds on  $F_6(n)$  and  $H(6, \ell)$ , for  $0 \leq \ell \leq 6$ , since the exact value of  $H(6) = H(6, 0)$  is still unknown. The best known bounds are  $H(6) \leq ES(9) \leq 1717$  and  $H(6) \geq 30$  by Gerken [12] and Overmars [26], respectively.

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