TABLE III
NUMERICAL RESULTS

CONTAINERSHIP AT 32 KNOTS			
Solutions of the MQE			
$P^+ =$	0.1768×10^{-1}	-0.1282×10^{2}	0.1775×10^{-1}
	-0.1281×10^{2}	0.1236×10^{4}	0.2285×10^{2}
	0.1775×10^{-1}	0.2285×10^{2}	0.4257×10^{1}
$P^- =$	-0.1471×10^{1}	-0.2828×10^{3}	-0.5973×10^{0}
	-0.2828×10^{3}	-0.6442×10^{5}	-0.8872×10^{2}
	-0.5973×10^{0}	-0.8872×10^{2}	-0.4497×10^{1}
State Feedback Gains (Stabilizing Control)			
	-0.7115×10^{-1}	0.1292×10^{2}	0.1000×10^{0}
Eigenvalues of the Closed Loop System, (A^+)			
	-0.16777×10^{0}	-0.10081×10^{0}	-0.23708×10^{-2}
Eigenvalues of $P^+ - P^-$ Matrix:			
	0.37515×10^{0}	0.65655×10^{5}	0.8567×10^{1}
250000 DWT TANKER AT 15 KNOTS (FULL LOAD)			
Solutions of the MQE			
$P^+ =$	0.7520×10^{0}	-0.3245×10^{3}	-0.9354×10^{-2}
	-0.3245×10^{3}	0.5441×10^{5}	0.6872×10^{3}
	-0.9354×10^{-2}	0.6872×10^{3}	0.4469×10^{2}
$P^- =$	-0.1942×10^{2}	-0.5705×10^{4}	-0.1215×10^{2}
	-0.5705×10^{4}	-0.1997×10^{7}	-0.2251×10^{4}
	-0.1215×10^{2}	-0.2551×10^{4}	-0.5201×10^{2}
State Feedback Gains (Stabilizing Control):			
	-0.2019×10^{0}	0.4782×10^{2}	0.3162×10^{0}
Eigenvalues of the Closed Loop System, (A^+):			
	-0.62846×10^{-1}	-0.27199×10^{-1}	-0.23213×10^{-2}
Eigenvalues of $P^+ - P^-$ Matrix:			
	0.59082×10^{1}	0.20513×10^{7}	0.91736×10^{2}

The quadratic criterion of (7) is physically well motivated, with the weighting coefficient λ being completely defined *a priori* from the dynamics of the problem. It is possible that many other optimization problems can be successfully posed in this framework and solved in a more truly optimal manner rather than by the classical LR ($Q \geq 0$) formulation where the quadratic weighting coefficients (generally) are iteratively specified by trial-and-error by the system designer on the basis of experimental studies.

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Stabilizability of Second-Order Bilinear Systems

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***Abstract*—This note states necessary and sufficient conditions for the existence of a linear state feedback controller such that a second-order bilinear system has a globally asymptotically stable closed loop. A suitable controller is constructed for each system which satisfies the conditions.**

I. INTRODUCTION

This note concerns the stabilizability of second-order bilinear systems

$$\dot{x} = Ax + u(Dx + b) \tag{1}$$

where $A, D \in \mathbb{R}^{2\times 2}$ and $x, b \in \mathbb{R}^2$. While a great amount of literature devoted to the structural properties of such systems has developed over the past decade [1]–[3], it is fair to say that little is understood regarding the qualitative behavior of trajectories of (1). Recently, several authors have investigated the stabilizability of systems of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i(D_i x + b_i) \tag{2}$$

in $\mathbb{R}^n$ [4], [6], [7]. These papers derive sufficient conditions and construct controllers to stabilize systems which meet specific and quite restrictive requirements. In our opinion, a significant understanding of bilinear systems will not be possible until more systematic analysis has been accomplished, and this note represents a step in that direction. Specifically, we give necessary and sufficient conditions for the existence of a constant linear feedback controller to stabilize (1). Even given the limited scope of this problem, it is safe to say that the statement of necessary and sufficient conditions is deceptively simple, and is possible only because of recent results in the stability of quadratic systems developed by the authors [5]. These results depend heavily upon that work.

Problem Statement: Characterize the properties of the triple (A, b, D) such that for some $c \in \mathbb{R}^2$, for $u \triangleq c^T x$, the resulting second-order closed-loop system

$$\dot{x} = A_c x + c^T x Dx \qquad A_c \triangleq A + bc^T \tag{3}$$

is globally asymptotically stable (GAS).

This problem is completely resolved by Theorem 1, stated below. It is worth remarking that a scalar bilinear system can never be made GAS using constant linear state feedback [8]; hence, the apparently restrictive conditions of Theorem 1 should not seem surprising. In the sequel, we assume that $b \neq 0$ and $|D| \neq 0$, and we will adopt the notation and definitions used in [5]. Briefly, $|x, y|$ denotes the determinant of the array

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$[x, y]$, A_s denotes the symmetric part of the matrix A, J is the skew-symmetric matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and nodal, critical, or focal matrices have two distinct, one distinct, or no real eigenvectors, respectively.

Theorem 1: The triple (A, b, D) is stabilizable under constant linear state feedback if and only if either

i) D has complex conjugate eigenvalues and $\forall x$, $|Dx, x|$ has the same sign as $|AD^{-1}b, b|$. If $|AD^{-1}b, b| = 0$ then the special conditions given in Proposition 2, Section III hold; or,

ii) D is singular and its nonzero eigenvector is a stable eigenvector of A. If D is singular with a unique real eigenspace, then the special conditions given in Proposition 4, Section IV hold.

We present some preliminary results in Section II, then discuss condition i) of Theorem 1 in Section III, condition ii) in Section IV, and finally provide a proof of Theorem 1 by way of summary in Section V. A construction for a stabilizing linear constant controller is provided in the proof of each case, and reviewed in the summary.

II. Preliminary Results

Evidently, system (3) is an autonomous quadratic differential equation. In order to characterize the stabilizability of the triple (A, b, D) under constant linear state feedback, we must, therefore, know something about the stability of such systems.

Theorem 2 [5]: System (3) is GAS if and only if:

i) A_c has eigenvalues with a nonpositive real part;

ii) $[JD]_s$ and $[D^TJA_c]_s$ are sign definite or semidefinite with the same sign;

iii) one of the following two mutually exclusive conditions holds:

a) D is focal and $D^{-1}A_c$ is either focal or x-critical where $x \in (c_\perp)$ iff $|A_c| \neq 0$;

b) D is x-critical and singular, $|A_c| \neq 0$, and $A_c^{-1}D = \gamma D$ for some scalar γ.

The two distinct cases listed under condition iii) form a natural framework for the presentation of stabilizability conditions. In Section III, we discuss the properties of the triple (A, b, D) when D is focal, corresponding to condition iii-a), above. In Section IV, we consider the case where D is singular, corresponding to condition iii-b), above. It is an immediate consequence of Theorem 2 that we need consider no other cases.

Lemma 1: If D is not focal and not singular, then system (1) cannot be stabilized by constant linear state feedback.

Proof: If D is nodal and nonsingular, then $[JD]_s$ is indefinite, and (3) violates condition ii) of Theorem 2 for any $c \in \mathbb{R}^2$. If D is x-critical and nonsingular, then (3) violates condition iii) of Theorem 2 for any $c \in \mathbb{R}^2$. □

As a further consequence of Theorem 2, we must choose a linear control law, $u \triangleq c^Tx$, for system (1) such that $[D^TJA_c]_s$ is sign definite or semidefinite depending upon the sign of $[JD]_s$. Thus, we may naturally inquire when a vector $c \in \mathbb{R}^2$ exists such that $D^TJA_c = D^TJA + D^TJbc^T$ has a definite symmetric part. This question is resolved by the following lemma and its corollaries.

Lemma 2: For any $Q \in \mathbb{R}^{2\times 2}$ and $g \in \mathbb{R}^2$, there exists a $c \in \mathbb{R}^2$ such that $[Q + gc^T]_s \geqslant 0$ if and only if $g_\perp^T Q g_\perp \geqslant 0$, and $[Q + gc^T]_s > 0$ if and only if $g_\perp^T Q g > 0$.

Proof: i) *Necessity:* If $g_\perp^T Q g_\perp < 0$, then $g_\perp^T [Q + gc^T]_s g_\perp = g_\perp^T Q g_\perp < 0$. If $g_\perp^T Q g_\perp = 0$, then $g_\perp^T [Q + gc^T]_s g_\perp = 0$.

ii) *Sufficiency:* If $g_\perp^T Q g_\perp = 0$, then $Q = [gd^T]_s$ for some $d \in \mathbb{R}^2$. Hence, if $c \triangleq -d + \gamma g$ for $\gamma \in \mathbb{R}^+$, then $[Q + gc^T]_s = [gd^T + gc^T]_s = \gamma gg^T \geqslant 0$. Note that any other choice of c leads to an indefinite form for $[Q + gc^T]_s$.

Let $g_\perp^T Q g_\perp > 0$. Note that

$$\left|[Q + gc^T]_s\right| = |Q_s| + \left|[gc^T]_s\right| + \operatorname{tr}\{J^TQ_sJ[gc^T]_s\}$$
$$= |Q_s| - 1/4|g, c|^2 + g^TJ^TQJc.$$

Hence, if $c \triangleq \gamma g$, then

$$\left|[Q + gc^T]_s\right| = |Q_s| + \gamma g_\perp^T Q g_\perp$$

and

$$\operatorname{tr}\{[Q + gc^T]_s\} = \operatorname{tr}\{Q_s\} + \gamma g^Tg$$

and the matrix is positive definite for large enough $\gamma > 0$. □

Corollary 2.1: If D is focal, then there exists a $c \in \mathbb{R}^2$ such that $[JD]_s$ and $[D^TJA_c]_s$ agree in sign if and only if $|Dx, x| \cdot |AD^{-1}b, b| \geqslant 0$.

Proof: Assume $[JD]_s > 0$. According to Lemma 2, $[D^TJA_c]_s \geqslant 0$ iff $[D^TJb]_\perp^T D^TJA[D^TJb]_\perp \geqslant 0$. But

$$[D^TJb]_\perp^T D^TJA[D^TJb]_\perp = b^TJ^TDJ^T[D^TJA]JD^TJb = |D|^2b^TJAD^{-1}b.$$

Hence, $[D^TJA_c]_s \geqslant 0$ iff $0 \leqslant b^TJAD^{-1}b = |AD^{-1}b, b|$ since $|D|^2 > 0$. The identical proof holds for $[JD]_s < 0$. □

Corollary 2.2: If D is focal and $|Dx, x| \cdot |AD^{-1}b, b| > 0$, then if $\gamma \in \mathbb{R}$, $c \triangleq \gamma D^TJb$ implies $[JD]_s$ and $[D^TJA_c]_s$ agree in sign when $|\gamma|$ is large enough and $\operatorname{sgn}\gamma = \operatorname{sgn} x^T[JD]_s x$. In this case, $D^{-1}A_c$ is focal.

Proof: This follows directly from the construction of g in the proof of Lemma 2 when $Q \triangleq D^TJA$. □

Corollary 2.3: If D is focal and $|AD^{-1}b, b| = 0$ $(b \neq 0)$, then $[JD]_s$ and $[D^TJA_c]_s$ agree in sign iff $c \triangleq D^TJ(-d + \gamma b)$ where d is the other eigenvector of AD^{-1} and $\operatorname{sgn}\gamma = \operatorname{sgn} x^T[JD]_s x$.[1] In this case, $D^{-1}A_c$ is $D^{-1}b$-critical.

Proof: Again, this follows from the proof of Lemma 2. □

We may now proceed to consider the cases listed above in correspondence with the conditions of Theorem 2.

III. Stabilizability When D Is Focal

According to Lemma 2 and its corollaries, if D is focal, then a c exists such that conditions ii) and iii-a) of Theorem 2 hold when $|AD^{-1}b, b|$ and $[JD]_s$ have the same sign. Surprisingly enough, if $|AD^{-1}b, b| \neq 0$, the same sign condition assures the stabilizability of (A, b) in the sense of LTI pole-placement [8] and, thereby, of (A, b, D) in our sense. The following proposition exploits this coincidence, specifying stabilizability conditions which make implicit use of this fact.

Proposition 1: If D is focal and $|AD^{-1}b, b| \neq 0$, then there exists a $c \in \mathbb{R}^2$ such that (3) is GAS if and only if $|AD^{-1}b, b| \cdot |Dx, x| > 0$.

Proof:

i) *Necessity:* According to Corollary 2.1, condition ii) of Theorem 2 holds only if $|Dx, x| \cdot |AD^{-1}b, b| > 0$ under the assumptions above.

ii) *Sufficiency:* Since $|AD^{-1}b, b| \cdot |Dx, x| > 0$ implies condition ii) and iii-a) of Theorem 2 according to Corollary 2.2 when $c \triangleq \gamma D^TJb$ where $\operatorname{sgn}\gamma = \operatorname{sgn} x^TJDx = \operatorname{sgn}|Dx, x|$ and $|\gamma|$ is large enough, it remains to show that i) holds under this choice of feedback.

Since $\operatorname{tr}\{A_c\} = \operatorname{tr}\{A\} + \gamma b^TJ^TDb = \operatorname{tr}\{A\} - \gamma|Db, b|$ and $\gamma|Dx, x| > 0$ for all $x \in \mathbb{R}^2$, we have $\operatorname{tr}\{A_c\} < 0$ when $|\gamma|$ is large enough.

Since

$$|A_c| = |A| + |bc^T| + \operatorname{tr}\{J^Tcb^TJA\}$$
$$= |A| + \gamma b^TJAJ^TD^TJb$$
$$= |A| + \gamma|D|b^TJAD^{-1}b$$
$$= |A| + \gamma|D||AD^{-1}b, b|$$

we have $|A_c| > 0$ when $|\gamma|$ is large enough, since $|D| > 0$ (D is focal) and $\gamma|AD^{-1}b, b| > 0$ since $\operatorname{sgn}\gamma = \operatorname{sgn}|Dx, x| = \operatorname{sgn}|AD^{-1}b, b|$.

But $\operatorname{tr}\{A_c\} < 0, |A_c| > 0$ implies A_c stable. ◫

If D is focal, but b is an eigenvector of AD^{-1}, then according to Lemma 2 and its corollaries, $D^{-1}A_c$ cannot be made focal by arbitrary choice of c. By Corollary 2.3, there exists a unique $c \in \mathbb{R}^2$ such that conditions ii) and iii-a) of Theorem 2 hold: however, there is no guarantee that (A, b) is

[1]If AD^{-1} is b-critical, then there is no "other eigenvector" and the result follows with $d \in (b)$ as seen from the proof of Lemma 2. However, if AD^{-1} is nodal, then we require $d \notin (b)$ in order for this construction to work. Accordingly, in the sequel, the terminology *other eigenvector* shall designate $d \notin (b)$ if AD^{-1} is nodal, and $d \in (b)$ if AD^{-1} is b-critical.

stabilizable in the sense of LTI pole-placement. Hence, the conditions for stabilizability in this case are more restrictive, and are given as follows.

Proposition 2: Let D be focal, $|AD^{-1}b, b| = 0$ $(b \neq 0)$, and let d be the other eigenvector of AD^{-1} with eigenvalue δ.[2] Then there exists a $c \in \mathbb{R}^2$ such that (3) is GAS if and only if either

i) AD^{-1} is b-critical and $|A| > 0$ or

ii) AD^{-1} is nodal and $\delta|d, b| = |A|/|D|$.

Proof:

i) *Necessity:* Assume $[JD]_s > 0$ without loss of generality. By Theorem 2, it is necessary that $[D^TJA_c] \geqslant 0$, and this is true iff $c \triangleq D^TJ(-d+\gamma b)$ (where $\operatorname{sgn}\gamma = \operatorname{sgn}|Dx, x|$) according to Corollary 2.3, in which case $D^{-1}A_c$ is $D^{-1}b$-critical. Hence, by condition iii-a) of Theorem 2, it is necessary that $c_\perp \in (D^{-1}b)$ iff $|A_c| \neq 0$. By construction, $c_\perp = JD^TJ(-d + \gamma b) = |D|D^{-1}(d - \gamma b)$; hence, $c_\perp \in (D^{-1}b)$ iff $d \in (b)$. Thus, we require $|A_c| > 0$ if AD^{-1} is b-critical and $0 = |A_c| = |A| + \operatorname{tr}\{J^Tcb^TJA\}$ if AD^{-1} is nodal. Since $\operatorname{tr}\{J^Tcb^TJA\} = |D|\,|AD^{-1}(-d+\gamma b), b| = -|D|\delta|d, b|$, condition ii), above, follows.

ii) *Sufficiency:* By the foregoing construction of c, all requirements of Theorem 2 have been met except the demonstration that $\operatorname{tr}\{A_c\} < 0$. But $\operatorname{tr}\{A_c\} = \operatorname{tr}\{A\} - \operatorname{tr}\{bd^TJ^TD\} + \gamma\operatorname{tr}\{bb^TJ^TD\}$, and the last term is equal to $-\gamma|Db, b| < 0$ since $\operatorname{sgn}\gamma = \operatorname{sgn}|Dx, x|$ for all x. Hence, $\operatorname{tr}\{A_c\} < 0$ when $|\gamma|$ is large enough. ∎

IV. Stabilizability When D Is Singular

If D is nonsingular and not focal, then system (3) violates Theorem 2 as shown by Lemma 1. However, if $|D| = 0$, then for some $d, e \in \mathbb{R}^2$, $D = de^T$. Hence, $c^Tx\,Dx = e^TxD'x$, and by choosing $c \in (d_\perp)$, D' in system (2) is d-critical and singular (if $e \in (d_\perp)$, then D is d-critical and singular to begin with). Therefore, when D is singular, it is possible to stabilize (1) in some cases. Before presenting these cases, we state the following useful result.

Lemma 3: If D is singular and d-critical and $|A| > 0$, then condition ii) of Theorem 2 holds iff $Ad = \alpha d$ and $\alpha < 0$.

Proof: Since $x^TD^TJAx = |Ax, Dx|$ and $x^TJDx = |Dx, x|$, condition ii) is equivalent to

$$0 \leqslant |Ax, Dx|\cdot|Dx, x| = |A|\cdot|x, A^{-1}dd_\perp^Tx|\cdot|dd_\perp^Tx, x|$$
$$= (d_\perp^Tx)^2|A|\cdot|x, A^{-1}d|\cdot|d, x|$$

which is true, assuming $|A| > 0$, if and only if $0 \leqslant |x, A^{-1}d|\cdot|d, x|$. The latter is possible if and only if $A^{-1}d = \alpha d$, $\alpha < 0$. ∎

In general, when $D = de^T$, $d \notin (e_\perp)$, hence, D is nodal as well as singular. In this case, stabilizability conditions are quite simple to state.

Proposition 3: If $D = de^T$ is nodal then there exists a $c \in \mathbb{R}^2$ such that (3) is GAS iff $Ad = \alpha d$, $\alpha < 0$.

Proof:

i) *Necessity:* According to condition ii) of Theorem 2 we require $c \in (d_\perp)$ or $c \triangleq \beta Jd$ for D' to not be nodal. In this case, condition iii-b) applies, and we require $|A_c| \neq 0$ which necessitates $|A_c| > 0$ according to condition i) of Theorem 2. Hence, condition ii) holds iff $A_cd = \alpha d$, $\alpha < 0$ according to Lemma 3. But $A_cd = Ad$ by construction of c, hence $Ad = \alpha d$, $\alpha < 0$.

ii) *Sufficiency:* Since $c \triangleq \beta Jd$ satisfies conditions ii) and iii) of Theorem 2, it suffices to show that i) holds for suitable β. Since $\operatorname{tr}\{A_c\} = \operatorname{tr}\{A\} + \beta|d, b|$ and $|A_c| = |A| + \beta b^TJAd = |A| + \alpha\beta|d, b|$, $|\beta|$ large and $\operatorname{sgn}\beta = -\operatorname{sgn}|d, b|$ implies A_c stable. □

If, however, $d \in (e_\perp)$, then a much greater choice of c is available and stabilizability conditions are more complex.

Proposition 4: If $D = dd_\perp^T$, then there exists a $c \in \mathbb{R}^2$ such that (3) is GAS iff either

a) b is an eigenvector of A in the null space of D and (A, b) is a stabilizable pair (in the sense of LTI theory) or

b) there exists a $\gamma \in \mathbb{R}$ such that

$$\operatorname{tr}\{A\} - \frac{|Ab, d|}{|b, d|} + \gamma < 0$$
$$|A| - \gamma\frac{|Ab, d|}{|b, d|} > 0.$$

Proof: According to conditions i) and iii-b) of Theorem 2, $|A_c| > 0$ and $A_c^{-1}D = \gamma D$, hence, $A_cd = \gamma d$, or $0 = |A_cd, d| = |Ad, d| + c^Td|b, d|$. If $|d, b| = 0$, then we require $0 = |Ad, d| = |Ab, b|$, hence, (A, b) is not controllable, and we must have LTI stabilizability, in which case any c such that A_c is stable meets the conditions of Theorem 2. This accounts for case a), above.

If $|d, b| \neq 0$, then $|A_cd, d| = 0$ iff $c \triangleq (\gamma d_\perp - A^Td_\perp)1/|d, b|$ (for some $\gamma \in \mathbb{R}$). Hence,

$$\operatorname{tr}\{A_c\} = \operatorname{tr}\{A\} - \frac{d_\perp^TAb}{d_\perp^Tb} + \gamma = \operatorname{tr}\{A\} - \frac{|Ab, d|}{|b, d|} + \gamma$$

and

$$|A_c| = |A| - \gamma\frac{d_\perp^TAb}{d_\perp^Tb} = |A| - \gamma\frac{|Ab, d|}{|b, d|}$$

giving rise to the condition in b), above □

V. Summary and Conclusion

The central result of this paper is the statement of necessary and sufficient conditions for the stabilizability of (1) under constant linear state feedback as given by Theorem 1 in the Introduction. As a formal proof of that theorem we may summarize the results of Sections II–IV.

If D is focal and $|AD^{-1}b, b| \neq 0$, then (3) is GAS iff $\operatorname{sgn}|AD^{-1}b, b| = \operatorname{sgn}|Dx, x|$, according to Proposition 1 (Section III). In this case, a stabilizing controller is given by $c \triangleq \gamma D^TJb$, $\operatorname{sgn}\gamma = \operatorname{sgn}|Dx, x|$, and $|\gamma|$ suitably large. If $|AD^{-1}b, b| = 0$ and d is the other eigenvector of AD^{-1}, then a stabilizing controller given by $c \triangleq D^TJ(-d+\gamma b)$, $\operatorname{sgn}\gamma = \operatorname{sgn}|Dx, x|$ may be chosen iff the conditions of Proposition 2 (Section III) hold. Thus, if D is focal, condition i) of Theorem 1 is necessary and sufficient for stabilizability.

If D is singular and nodal then (3) is GAS iff d, its nonzero eigenvector, is a stable eigenvector of A, according to Proposition 3 (Section IV). In this case, $c \triangleq \beta Jd$, $\operatorname{sgn}\beta = -\operatorname{sgn}|d, b|$, and $|\beta|$ suitably large is a stabilizing controller. If D is singular and critical, then (3) is GAS iff the conditions of Proposition 4 (Section IV) hold. If D is b-critical and those conditions are met, then any c which stabilizes (A, b) in the sense of LTI pole-placement, stabilizes (1). If D is d-critical, $b \notin (d)$, and the conditions are met, then $c \triangleq (\gamma d_\perp - A^Td_\perp)1/|d, b|$ is a stabilizing controller. Thus, if D is singular, condition ii) of Theorem 1 is necessary and sufficient for stabilizability.

If D is neither focal nor singular, then (3) is never GAS, according to Lemma 1 (Section II). Thus, Theorem 1 lists complete necessary and sufficient conditions for stabilizability, as claimed.

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[2] See Footnote 1.