A Theory of Collusion with Partial Mutual Understanding*

Joseph E. Harrington, Jr.
Department of Business Economics & Public Policy
The Wharton School
University of Pennsylvania
3620 Locust Walk
Philadelphia, PA 19102
harrij@wharton.upenn.edu

28 July 2014

Abstract

Unlawful collusion is when firms have a mutual understanding to coordinate their behavior for the purpose of achieving a supracompetitive outcome. Given the legal focus on mutual beliefs, this paper initiates a research program to explore how much and what type of mutual beliefs among firms allows them to effectively collude. Focusing on price leadership as the collusive mechanism, it is assumed that firms commonly believe that price increases will be at least matched but lack any shared understanding about who will lead, when they will, and at what prices. Sufficient conditions are derived which ensure that supracompetitive prices emerge. However, an upper bound on price is derived which is less than the maximal equilibrium price.

*I appreciate the comments of Peter Dijkstra, George Mailath, Nikos Vettas, seminar participants at Universitat Pompeu Fabra, Universitat de València, Penn-Wharton, Antitrust Division of the U.S. Department of Justice, and Toulouse School of Economics, U. of Rochester-Simon School, and conference participants at EARIE 2011 (Stockholm), 2nd MaCCI Summer Institute in Competition Policy (Deidesheim), and XXVII Jornadas de Economía Industrial (Murcia). This research was partly conducted under a Cátedras de Excelencia at the Departamento de Economía of the Universidad Carlos III de Madrid. I want to thank Banco Santander for funding and the faculty for providing a most collegial and stimulating environment. This research has been conducted with the financial support of the National Science Foundation (SES-1148129).

1 Introduction

Collusion is when firms coordinate their behavior to suppress competition through such activities as raising prices, allocating markets, and bid rigging. In the U.S., *unlawful collusion* has come to mean that firms have an *agreement* to coordinate their behavior. In their interpretation of Section 1 of the Sherman Act, courts view an agreement as present when there is "mutual consent" among firms,[1] when firms have "a conscious commitment to a common scheme designed to achieve an unlawful objective,"[2] and when firms have a "unity of purpose or a common design and understanding, or a meeting of minds."[3] This perspective is not unique to the U.S. Article 101 of the Treaty of the European Communities (1999) declares agreements to be unlawful that "have as their object or effect the prevention, restriction or distortion of competition" and the European Union General Court defines an agreement as when firms have "joint intention" (*ACF Chemiefarma,* 1970) and a "concurrence of wills" (*Bayer v. Commission*, 2000). In sum, firms unlawfully collude when there is a *mutual understanding* to coordinate their behavior for the purpose of achieving a supracompetitive outcome.[4]

As a "meeting of minds" or "concurrence of wills" is not something that is directly observed, evidentiary standards for determining liability are based on *communications* that could produce mutual understanding and *market behavior* that is the possible consequence of mutual understanding. Some communication practices - such as firms expressly communicating a plan to coordinate prices or allocate markets - are subject to a *per se* prohibition. But then there are communication practices which are insufficient by themselves to establish liability but could deliver a guilty verdict when buttressed with corroborating market evidence that the communication was successful in generating the requisite mutual understanding. For example, there are cases in which a single firm made an announcement that was public to all firms and thereby could serve to coordinate behavior. In *Interstate Circuit, Inc. v. United States* (1939), one firm sent a letter to each of the other firms that stated a plan to coordinate on price and made clear in the letter that every firm was receiving the same letter. In *United States v. Foley* (1980), there was a dinner among realtors during which one of them announced he was raising his commission rate from six to seven percent. In both of these cases, even though there was no exchange of assurances, firms were able to successfully coordinate their behavior using indirect communication.

The primary legal challenge is determining when firms share an understanding to raise prices, allocate markets, or engage in some other form of coordinated behavior. In defining the boundaries of unlawful collusion, this challenge can be broken down to addressing two questions. First, what is the relationship between communications among firms and the level of mutual understanding that is achieved? What communication practices are effective in producing mutual beliefs among firms and with regards to what aspects of behavior is there mutual understanding? Second, what is the relationship between mutual understanding among firms and their market behavior? What types of mutual beliefs result in coordination

[1] *Esco Corp. v. United States*, 340 F.2d 1000, 1007-08 (9th Cir. 1965).

[2] *Monsanto Co. v. Spray-Rite Serv. Corp.*, 465 U.S. 752 (1984); 753.

[3] *American Tobacco Co. v. United States*, 328 U.S. 781 (1946); 810.

[4] For a critical examination of U.S. law with respect to price-fixing, the reader is referred to Kaplow (2013).

on prices and, more generally, supracompetitive outcomes?

The objective of this paper is to address the second question: How much and what type of mutual understanding results in collusive outcomes? On this question, the economic theory of collusion shows that if firms have mutual beliefs regarding a collusive strategy profile - that is, each firm has correct beliefs of this strategy profile - and incentive compatibility conditions hold (that is, the collusive strategy profile is an equilibrium) then supracompetitive outcomes will emerge. If firms were to expressly communicate with each other - as is done in hard core cartels (for examples, see Harrington 2006 and Marshall and Marx 2012) - it is reasonable to suppose that they could achieve mutual beliefs regarding the strategy profile. For example, managers could meet and express the intent to raise prices by 10% and to persist in doing so only if every firm charges 10% higher prices. If there are mutual beliefs regarding such a strategy and it is optimal for each firm to use that strategy, economic theory predicts supracompetitive outcomes will result.

While it may be reasonable to expect firms, who have engaged in direct and unconstrained communication, to have mutual beliefs regarding the collusive strategy profile, there are many episodes of collusion for which communication is limited and, as a consequence, mutual understanding may be incomplete. Firms have attempted to coordinate their behavior through a wide variety of communication practices including tacking on a few digits to a multi-million dollar bid (FCC spectrum auctions; see Cramton and Schwartz, 2000), committing to a policy of non-negotiable posted prices (turbine generators; see Harrington, 2011), announcing future intended prices (airlines; see Borenstein, 2004), and unilaterally announcing a collusive pricing strategy (free-standing newspaper inserts; see *In the Matter of Valassis Communications, Inc.,* Federal Trade Commission, File No. 051 0008, Docket No. C-4160, April 28, 2006). Presumably, the reason that firms chose not to expressly communicate is because the more direct the method of communication, the greater the chances of detection, prosecution, and conviction. Thus, in an environment for which collusion is unlawful, firms will have an incentive to engage in indirect forms of communication which is then likely to limit the amount of shared understanding. This then leads us back to our central research question: How much and what kind of mutual understanding is enough for firms to effectively collude?[5]

Epistemic game theory provides a natural approach for exploring the relationship between firms' mutual beliefs and their behavior. Rather than specify a game and an equilibrium concept - which implicitly assumes firms have mutual beliefs with regards to a strategy profile - an epistemic approach begins with players' knowledge and beliefs about the game, about players' rationality, and about other players' beliefs. Typically, epistemic game theory places no prior structure on players' beliefs regarding other players' strategies and instead derives those beliefs; while equilibrium game theory assumes mutual beliefs with regards to a particular strategy profile and characterizes those strategy profiles consistent with rationality. The approach that is being proposed here lies between these two extremes in that it assumes

[5] Even with express communication, mutual understanding regarding the strategy profile is not inevitable. For example, given firm asymmetries, there could be disagreement regarding the strategy profile and a breakdown in bargaining could occur prior to achieving mutual beliefs as to the strategy profile. Such was the case in the lysine cartel (Connor, 2001). While firms agreed to a price, they initially failed to agree to a market allocation with respect to sales quotas. While prices rose, they quickly fell. Firms then returned to bargaining, settled on a market allocation, and effective collusion ensued.

some mutual beliefs on the strategies of other players - as might be achieved through limited forms of communication - but these mutual beliefs fall short of firms having common beliefs regarding the entire strategy profile.

The research program is to specify mutual beliefs regarding the game, firms' rationality, and firms' strategies and to then derive the implications for firm behavior with a particular focus on understanding when supracompetitive outcomes will emerge. Crucial to this approach is making *relevant* and *plausible* assumptions regarding mutual beliefs on firms' strategies. Relevance means it is relevant to firms coordinating their behavior, and plausible means firms could plausibly obtain these mutual beliefs without engaging in express communication. In some instances, it could be reasonable for firms to share mutual beliefs regarding the collusive mechanism but not about the particular details associated with implementing that mechanism. For example, firms could mutually believe that they will engage in price leadership but lack mutual beliefs regarding the sequence of price increases and who will lead; or firms could share an understanding to allocate a market in terms of exclusive territories but do not have mutual beliefs as to the assignment of territories; or firms could have mutual understanding to engage in bid rotation at procurement auctions but lack a shared understanding regarding which firm is slated to win which auction. Whatever the case, it is important to motivate the mutual beliefs that are assumed.

In this paper, the focus is on price leadership which is one of the most common methods of collusion not deploying express communication.[6] Price leadership involves a firm raising price and other firms matching that price increase, with potentially multiple episodes of a firm leading and other firms following. It could have the same firm lead or firms could take turns. It could have many or few price increases; price increases could be large or small. The full strategy encompasses the timing of price increases, the size of price increases, and the identities of the firms leading the price increase. If firms do not engage in express communication, it is difficult to imagine that they could achieve mutual beliefs with respect to such details. However, suppose instead that their mutual beliefs only pertain to the property of price leadership that a price increase will be at least matched by the other firms, and failure to do so means a return to competitive prices. This level of mutual understanding could possibly be achieved through public announcements such as firms announcing that they will not undercut rival firms' prices. In that case, a firm conveys its intent to avoid aggressive competition but its announcement falls far short of expressing a plan to coordinate prices. For example, U-Haul CEO Joe Schoen conveyed during an earnings conference call with analysts that it thought rival Budget's price undercutting in the truck rental market was nonsensical and proposed moving to price leadership. While such an announcement fell short of evidentiary standards for Section 1 of the Sherman Act, the Federal Trade Commission pursued a Section 5 case (under the Federal Trade Commission Act) against U-Haul as an "invitation to collude".[7]

In this paper, it is assumed that the game and firms' rationality are common knowledge and that there is common knowledge regarding the following properties of firms' strategies: price increases will be at least matched and failure to do so results in non-collusive (static Nash equilibrium) prices. Two questions are addressed. First, are these mutual beliefs

[6] For some examples, see Markham (1951) and Scherer (1980, Chapter 6).
[7] *Matter of U-Haul Int'l Inc. and AMERCO* (FTC File No. 081-0157, July 10, 2010)

sufficient to result in supracompetitive prices? Second, if they can achieve supracompetitive prices, are those prices less than what would be achieved if there was mutual understanding of the entire strategy profile (that is, equilibrium)? To my knowledge, this is the first paper to investigate when less than full mutual understanding of a collusive strategy profile is sufficient to generate effective collusion.

There are two main results. The first finding is that if it is common knowledge that price increases will be at least matched (and failure to do so will result in competitive prices) then an upper bound on price is derived that is strictly less than the maximal price achieved under equilibrium (using the same implicit punishment). Thus, in this particular setting, less than full mutual understanding of the strategy profile does limit the extent of collusion. The reason is that without mutual beliefs regarding the collusive price, coordinating on a higher collusive price requires some firm to take the lead in raising price and this is a costly enterprise which limits the extent of price increases. While an upper bound on price is derived, no more can be said about price paths induced by this set of mutual beliefs; it could involve competitive or supracompetitive prices. Thus, in answer to the first question, mutual beliefs that price increases will be at least matched does not imply supracompetitive prices.

With only mutual beliefs that price increases will be at least matched, a firm can assign positive probability to many strategies for the other firms. For example, a firm could believe that another firm will always act as a price leader or instead that other firms believe it'll be the price leader or that another firm will initiate the price increase process and, only upon doing so, will other firms subsequently lead on price. In deriving the first result, no structure was placed on how firms update their beliefs over other firms' strategies as the history unfolds. If it is further assumed that firms engage in Bayesian updating and that prior beliefs assign positive probability to the true price path then it can be shown that firms will eventually succeed in coordinating on supracompetitive prices. Thus, supracompetitive prices are sure to emerge even though firms lack common knowledge about who will lead on price, when they will lead, and what prices they will charge. That is the second main finding of the paper.

The model is described in Section 2 - where standard assumptions are made regarding cost, demand, and firm objectives - and in Section 3 - where, consistent with the epistemic approach, assumptions are made regarding the behavior and beliefs of firms. An upper bound on price is characterized in Section 4. The implications of having firms learn about other firms' strategies is explored in Section 5. Section 6 offers a few concluding remarks. Proofs are in Appendix A.

2 Assumptions on the Market

Consider a symmetric differentiated products price game with n firms. $\pi(p_i, \mathbf{p}_{-i}) : \Re^n_+ \rightarrow \Re$ is a firm i's profit when it prices at p_i and its rivals price at $\mathbf{p}_{-i} = (p_1, ..., p_{i-1}, p_{i+1}, ..., p_n)$. Assume $\pi(p_i, \mathbf{p}_{-i})$ is bounded, twice continuously differentiable, increasing in a rival's price p_j $(j \neq i)$, and strictly concave in own price p_i. A firm's best reply function then exists:

$$\psi(\mathbf{p}_{-i}) = \arg\max_{p_i} \pi(p_i, \mathbf{p}_{-i}).$$

Further assume

$$\frac{\partial^2 \pi}{\partial p_i \partial p_j} > 0, \ \forall j \neq i$$

from which it follows that $\psi(\mathbf{p}_{-i})$ is increasing in p_j, $j \neq i$. A symmetric Nash equilibrium price, p^N, exists and is assumed to be unique,[8]

$$\psi(p, .., p) \gtreqqless p \text{ as } p \lesseqqgtr p^N,$$

and let

$$\pi^N \equiv \pi\left(p^N, ..., p^N\right) > 0.$$

Assuming $\pi(p, ..., p)$ is strictly concave in p, there exists a unique joint profit maximum p^M,

$$\sum_{j=1}^{n} \frac{\partial \pi(p, ..., p)}{\partial p_j} \gtreqqless 0 \text{ as } p \lesseqqgtr p^M,$$

and $p^M > p^N$.

Firms interact in an infinitely repeated price game with perfect monitoring. A collusive price $p' > p^N$ is sustainable with the grim trigger strategy if and only if:[9]

$$\left(\frac{1}{1-\delta}\right) \pi(p', ..., p') \geq \max_{p_i} \pi(p_i, p', ..., p') + \left(\frac{\delta}{1-\delta}\right) \pi^N, \tag{1}$$

where δ is the common discount factor. Define $\widetilde{p}$ as the best price sustainable using the grim trigger strategy:

$$\widetilde{p} \equiv \max\left\{p \in \left[p^N, p^M\right] : \left(\frac{1}{1-\delta}\right) \pi(p, ..., p) \geq \max_{p_i < p} \pi(p_i, p, ..., p) + \left(\frac{\delta}{1-\delta}\right) \pi^N\right\}.$$

Assume $\widetilde{p} > p^N$ and if $\widetilde{p} \in \left(p^N, p^M\right)$ then

$$\left(\frac{1}{1-\delta}\right) \pi(p, ..., p) \gtreqqless \pi(\psi(p, ..., p), p, ..., p) + \left(\frac{\delta}{1-\delta}\right) \pi^N \text{ as } p \lesseqqgtr \widetilde{p} \text{ for } p \in \left[p^N, p^M\right]. \tag{2}$$

$\widetilde{p}$ will prove to be a useful benchmark.

For the later analysis, consider the "price matching" objective function for a firm:

$$W(p_i, \mathbf{p}_{-i}) \equiv \pi(p_i, \mathbf{p}_{-i}) + \left(\frac{\delta}{1-\delta}\right) \pi(p_i, ..., p_i).$$

Given its rivals price at $\mathbf{p}_{-i}$ in the current period, $W(p_i, \mathbf{p}_{-i})$ is firm i's payoff from pricing at p_i *if it believed* that all firms would match that price in all ensuing periods. Consider

$$\frac{\partial W(p_i, \mathbf{p}_{-i})}{\partial p_i} = \frac{\partial \pi(p_i, \mathbf{p}_{-i})}{\partial p_i} + \left(\frac{\delta}{1-\delta}\right) \sum_{j=1}^{n} \frac{\partial \pi(p_i, ..., p_i)}{\partial p_j}.$$

[8] A sufficient condition is

$$\frac{\partial^2 \pi}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 \pi}{\partial p_i \partial p_j} < 0, \ \forall (p_i, \mathbf{p}_{-i}).$$

[9] The grim trigger strategy has any deviation from the collusive price p' result in a price of p^N forever.

If $p_i < p^M$ then the second term is positive; by raising its current price, a firm increases the future profit stream under the assumption that its price increase will be matched by its rivals. If $p_i > \psi(p_{-i})$ then the first term is negative. Evaluate $\frac{\partial W(p_i, p_{-i})}{\partial p_i}$ when firms price at a common level p:

$$\begin{aligned} \frac{\partial W(p,...,p)}{\partial p_i} &= \frac{\partial \pi(p,...,p)}{\partial p_i} + \left(\frac{\delta}{1-\delta}\right)\sum_{j=1}^{n}\frac{\partial \pi(p,...,p)}{\partial p_j} \\ &= \left(\frac{1}{1-\delta}\right)\left(\frac{\partial \pi(p,...,p)}{\partial p_i} + \delta \sum_{j\neq i}^{n}\frac{\partial \pi(p,...,p)}{\partial p_j}\right) \end{aligned}$$

Thus, when $p \in \left(p^N, p^M\right)$, raising price lowers current profit, $\frac{\partial \pi(p,...,p)}{\partial p_i} < 0$, and increases future profit, $\sum_{j=1}^{n}\frac{\partial \pi(p,...,p)}{\partial p_j} > 0$. By the preceding assumptions, $W(p_i, \mathbf{p}_{-i})$ is strictly concave in p_i since it is the weighted sum of two strictly concave functions. Hence, a unique optimal price exists,

$$\phi(\mathbf{p}_{-i}) = \arg\max_{p_i} W(p_i, \mathbf{p}_{-i}). \tag{3}$$

By the preceding assumptions, $\phi(\mathbf{p}_{-i})$ is increasing in a rival's price as

$$\frac{\partial \phi(\mathbf{p}_{-i})}{\partial p_j} = -\frac{\partial^2 W(p_i, \mathbf{p}_{-i})/\partial p_i \partial p_j}{\partial^2 W(p_i, \mathbf{p}_{-i})/\partial p_i^2} = -\frac{\frac{\partial^2 \pi(p_i, \mathbf{p}_{-i})}{\partial p_i \partial p_j}}{\frac{\partial^2 \pi(p_i, \mathbf{p}_{-i})}{\partial p_i^2} + \left(\frac{\delta}{1-\delta}\right)\left(\frac{\partial^2 \pi(p,...,p)}{\partial p^2}\right)} > 0.$$

As there is a benefit in terms of future profit from raising price (as long as it does not exceed the joint profit maximum) then the price matching best reply function results in a higher price than the standard best reply function. To show this result, consider

$$\begin{aligned} \frac{\partial W(\psi(\mathbf{p}_{-i}), \mathbf{p}_{-i})}{\partial p_i} &= \frac{\partial \pi(\psi(\mathbf{p}_{-i}), \mathbf{p}_{-i})}{\partial p_i} + \left(\frac{\delta}{1-\delta}\right)\sum_{j=1}^{n}\frac{\partial \pi(\psi(\mathbf{p}_{-i}),...,\psi(\mathbf{p}_{-i}))}{\partial p_j} \\ &= \left(\frac{\delta}{1-\delta}\right)\sum_{j=1}^{n}\frac{\partial \pi(\psi(\mathbf{p}_{-i}),...,\psi(\mathbf{p}_{-i}))}{\partial p_j} > 0, \end{aligned}$$

which is positive because $\mathbf{p}_{-i} \leq \left(p^M,...,p^M\right)$ implies $\psi(\mathbf{p}_{-i}) < p^M$.[10] By the strict concavity of W, $\phi(\mathbf{p}_{-i}) > \psi(\mathbf{p}_{-i})$.

ϕ has a fixed point p^* because it is continuous, $\phi\left(p^N,...,p^N\right) > p^N$, and

$$\frac{\partial W\left(p^M,...,p^M\right)}{\partial p_i} = \frac{\partial \pi\left(p^M,...,p^M\right)}{\partial p_i} < 0 \Rightarrow \phi\left(p^M,...,p^M\right) < p^M.$$

Further assume the fixed point is unique:

$$\phi(p,...,p) \gtreqless p \text{ as } p \lesseqgtr p^*.$$

[10] Since $\psi(p,..,p) \gtreqless p$ as $p \lesseqgtr p^N$ then $\psi\left(p^M,..,p^M\right) < p^M$. Given that ψ is increasing then $\mathbf{p}_{-i} \leq \left(p^M,...,p^M\right)$ implies $\psi(\mathbf{p}_{-i}) < \psi\left(p^M,..,p^M\right) < p^M$.

Thus, if rival firms price at p^*, a firm prefers to price at p^* rather than price differently under the assumption that its price will be matched forever. p^* will prove to be a useful benchmark.

Results are proven when the price set is finite.[11] From hereon, assume the price set is $\Omega \equiv \{0, \varepsilon, 2\varepsilon, ..., P\}$, where P is some upper bound on price and $\varepsilon > 0$ and is presumed to be small. For convenience, suppose $p^N, p^*, \widetilde{p} \in \Omega$.[12] As the finiteness of the price set could generate multiple optima, define the best reply correspondence for the price matching objective function:

$$\overline{\phi}(\mathbf{p}_{-i}) \equiv \arg\max_{p_i \in \Omega} \pi(p_i, \mathbf{p}_{-i}) + \left(\frac{\delta}{1-\delta}\right) \pi(p_i, ..., p_i).$$

The best reply correspondence is assumed to have the following property:[13]

$$\overline{\phi}(\mathbf{p}_{-i}) \begin{cases} \subseteq \{p' + \varepsilon, ..., p^*\} & \text{if } \mathbf{p}_{-i} = (p', ..., p') \text{ where } p' < p^* - \varepsilon \\ = \{p^*\} & \text{if } \mathbf{p}_{-i} = (p^*, ..., p^*) \\ \subseteq \{p^*, ..., p' - \varepsilon\} & \text{if } \mathbf{p}_{-i} = (p', ..., p') \text{ where } p' > p^* + \varepsilon \end{cases} \quad (4)$$

Recall that p^* is the unique fixed point for $\phi(\mathbf{p}_{-i})$; it is also a fixed point for $\overline{\phi}(\mathbf{p}_{-i})$. By (4), if all rival firms price at p' then firm i's best reply has it price above p' when $p' < p^* - \varepsilon$. Analogously, if $p' > p^* + \varepsilon$ then firm i's best reply has it price below p'. Note that an implication of (4) is that the set of symmetric fixed points of $\overline{\phi}(\mathbf{p}_{-i})$ is, at most, $\{p^* - \varepsilon, p^*, p^* + \varepsilon\}$.[14]

3 Assumptions on Beliefs and Behavior

Let us now turn to specifying firms' mutual beliefs regarding the game, rationality, and strategies. The game is assumed to be common knowledge. Regarding rationality, I will use extensive-form rationalizability and, in doing so, draw on some of the structure in Battigalli and Siniscalchi (2003).[15] For this purpose, define the following terms. $\mathcal{H} \equiv \{\varnothing\} \cup (\cup_{t \geq 1} \Omega^{n \times t})$ is the space of histories for the infinitely repeated price game with perfect monitoring. $s_i : \mathcal{H} \rightarrow \Omega$ is a strategy for firm i and S denotes a firm's strategy set, which is common to all firms. $\mu^i(s_{-i} | h) : \mathcal{H} \rightarrow \Delta(S^{n-1})$ is firm i's beliefs on the strategies of the other firms conditional on the history $h \in \mathcal{H}$ where $\Delta(X)$ is the set of probability distributions on set X.

Starting from the current period based on history h, let $U_i(s_i, \mu^i(\cdot | h))$ be the expected present value of profits for firm i after history h given firm i's strategy is s_i and firm i's

[11] A discussion of the case of an infinite price set is provided at the end of Section 4.

[12] If $\widetilde{p} \in \Omega$ and $\psi(\widetilde{p}, ..., \widetilde{p}) \in \Omega$ then $\widetilde{p}$ is still the best price sustainable using the grim punishment.

[13] A sufficient condition for (4) is $-\frac{\partial^2 \pi}{\partial p_i^2} \geq 2\frac{\partial^2 \pi}{\partial p_i \partial p_{-i}}$, which holds when demand and cost functions are linear.

[14] The case of linear demand and cost functions satisfies all of the assumptions made in Section 2. A proof is available on request.

[15] While their framework allows a player's private information (or type) to be both a trait (such as cost in our setting) and beliefs on other players' types, here we consider the special case in which traits are not private information.

beliefs on the other firms' strategies are $\mu^i(\cdot|h)$. Given beliefs as to other firms' strategies, a firm's strategy is sequentially rational if it is optimal for all histories.

Definition: $\widehat{s}_i$ is *sequentially rational* with respect to beliefs μ^i if, $\forall h \in \mathcal{H}$, $U_i(\widehat{s}_i, \mu^i(\cdot|h)) \geq U_i(s_i, \mu^i(\cdot|h))$ $\forall s_i \in S$.

Critical to our approach is allowing firms to have a reasonable amount of mutual understanding with respect to the strategy profile. As is laid out in the ensuing assumptions, the presumption is that firms believe that a higher price (up to some maximum level) will be at least matched and that any deviation - such as undercutting a rival firm's price - will result in a return to competitive pricing. This description of behavior is more formally stated as the *price matching plus* (PMP) property.

Definition: The strategy of firm i has the *price matching plus* (PMP) property if, for some $\overline{p}$,

$$p_i^t \begin{cases} \in \left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},...,\overline{p}\right\} & \text{if } p_j^\tau \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\overline{p}\right\} \ \forall j, \forall \tau \leq t-1 \\ & \text{and } \max\left\{p_1^{t-1},...,p_n^{t-1}\right\} < \overline{p} \\ = \overline{p} & \text{if } p_j^\tau \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\overline{p}\right\} \ \forall j, \forall \tau \leq t-1 \\ & \text{and } \max\left\{p_1^{t-1},...,p_n^{t-1}\right\} \geq \overline{p} \\ = p^N & \text{if not } p_j^\tau \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\overline{p}\right\} \\ & \forall j, \forall \tau \leq t-1 \end{cases}$$

First note that matching a price increase means setting $p_i^t = \max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$. Thus, as of period t, price increases have always been *at least* matched when $p_j^\tau \geq \max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\}$ $\forall j$, $\forall \tau \leq t-1$. Price matching plus behavior is subject to the caveat that firms are only expected to match price as high as $\overline{p}$; thus, firms are not expected to follow an excessively high price increase. Firms have then been complying with this modified price matching when $p_j^\tau \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\overline{p}\right\}$ $\forall j$, $\forall \tau \leq t-1$. In that event, the PMP property has a firm price in period t at least as high as $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$ with the caveat of not pricing in excess of $\overline{p}$. Finally, if any firm should fail to act in a manner consistent with the previously described behavior then firms revert to pricing at the non-collusive price p^N thereafter.[16]

A strategy satisfying the PMP property is referred to as PMP-compatible, and define $S^{PMP} \subset S$ as the set of PMP-compatible strategies. The key belief restriction is that there is mutual understanding among firms that they are using PMP-compatible strategies. With first level beliefs, this means that each firm's beliefs regarding other firms' strategies have support S^{PMP} or, as defined next, beliefs are PMP-consistent.

Definition: μ^i are *PMP-consistent* if $\mu^i : \mathcal{H} \to \Delta\left(\left(S^{PMP}\right)^{n-1}\right)$.

[16]The potential role of price matching here is to coordinate on a collusive outcome. Price matching has also been explored as a form of punishment; see Lu and Wright (2010) and Garrod (2012). Some papers exploring price leadership as a collusive equilibrium include Rotemberg and Saloner (1990) and Mouraviev and Rey (2011).

In defining what we mean by mutual understanding, we use the concept of mutual strong beliefs from Battigalli and Siniscalchi (2003). A firm has strong beliefs regarding some event if it is certain about that event and that certainty is maintained as long as the history is consistent with it.

Definition: A firm *strongly believes* event E if it is initially certain of E and would also be certain of E conditional on every history whose occurrence does not contradict E.

Definition: All firms have a *mutual* strong belief in E if each firm strongly believes E.

As specified as Assumption A1, the behavioral assumption is that firms are sequentially rational, and the belief assumption is that there is an infinite hierarchy of mutual strong beliefs that firms are sequentially rational and use strategies that have the PMP-property. Note that it is not assumed that firms use a PMP-compatible strategy; that will be an implication of sequential rationality and these beliefs.

Assumption A1: Assume $\cap_{q \geq 0} \mathcal{A}^q$ where

$\mathcal{A}^0$: Every firm is sequentially rational and has beliefs that are PMP-consistent.

$\mathcal{A}^1$: There is mutual strong belief in $\mathcal{A}^0$.

$\mathcal{A}^2$: There is mutual strong belief in $\mathcal{A}^0 \cap \mathcal{A}^1$.

$\vdots$

$\mathcal{A}^{m+1}$: There is mutual strong belief in $\mathcal{A}^0 \cap \cdots \cap \mathcal{A}^m$.

$\vdots$

One implication of A1 is that there is common knowledge among firms that: 1) price increases are at least matched as long as past price increases have been at least matched in the past; 2) price increases will be followed only as high as $\overline{p}$; and 3) departure from this price matching behavior results in reversion to non-collusive pricing. Let me discuss these features.

Consider the assumption that it is common knowledge that failure to at least match price increases (up to a maximum price of $\overline{p}$) results in non-collusive pricing thereafter. It is useful to break this assumption into two parts: 1) a departure from price matching results in some form of punishment; and 2) that punishment is the grim punishment. The second condition is unimportant because results are unchanged if the punishment is at least as severe as the grim punishment. The first condition seems natural in that firms are seeking to collude through the mutual understanding that price increases will be at least matched. Thus, departure from that behavior ought to induce either a breakdown of collusion or a more calculated punishment. The restrictiveness of the condition is really that it is common knowledge as to the low continuation payoff that ensues after a departure from price-matching plus behavior. Though any punishment would work for our analysis, a coordinated punishment - as opposed to simply a breakdown of collusion and a return to competition - would have to be justified in terms of how the punishment is common knowledge without express communication. I feel it is more consistent with the spirit of the enterprise to simply assume a departure results in a breakdown of collusion rather than a coordinated punishment, even though results are not dependent on that specification.[17]

The next feature of the PMP property to consider is that a firm does not price in excess of $\overline{p}$, which means that it will follow price increases only as high as $\overline{p}$ and, as a price leader, will not raise price beyond $\overline{p}$. It is surely compelling for a firm to have some upper bound to how high it will price; for example, it would be nonsensical to follow a price increase to the point that market demand is zero or even to where it is above the monopoly price. However, this assumption goes further in that the upper bound is common to firms, and is commonly known. While strong, I believe it is necessary for the ensuing results. As shown in Lemma 1, A1 implies that this upper bound on price matching cannot be less than the static Nash equilibrium price and cannot exceed the highest equilibrium price (with the grim punishment).

Lemma 1 *A1 implies* $\overline{p} \in \left\{p^N, ..., \widetilde{p}\right\}$.

The results of Section 4 require no further restriction on $\overline{p}$, while the results of Section 5 are unchanged as long as $\overline{p}$ is not too much less than $\widetilde{p}$.[18] However, it'll make for an easier presentation if it is assumed that the upper bound on price matching is the highest level consistent with A1.

Assumption A2: $\overline{p} = \widetilde{p}$.

One could imagine that $\widetilde{p}$ might be focal given that it is the *highest* feasible value of $\overline{p}$ and that higher values are more profitable for all firms. From hereon, S^{PMP} will refer to when $\overline{p} = \widetilde{p}$.

Consistent with the focus on the dynamic process by which firms come to coordinate on supracompetitive prices, it is assumed that they start out with prices at competitive levels: $(p_1^0, ..., p_n^0) = \left(p^N, ..., p^N\right)$. While it is a natural assumption, it is not necessary for the ensuing results. Lemma 2 only requires $\max\{p_1^0, ..., p_n^0\} \in \left\{p^N, ..., \widetilde{p}\right\}$, while the main results (Theorems 3 and 5) only require that the initial price vector is not too high: $(p_1^0, ..., p_n^0) \in \left\{p^N, ..., p^* + \varepsilon\right\}^n$.

Before moving on, it is useful to ask why it isn't reasonable to assume there are mutual beliefs that firms use some typical punishment strategy and price immediately at $\widetilde{p}$. So, let us consider firm i using a grim trigger strategy: i) price at p^N in periods $1, ..., t_i - 1$; ii) price at $\widetilde{p}$ in period t_i; ii) price at $\widetilde{p}$ in period $t\,(> t_i)$ if all firms priced at $\widetilde{p}$ in periods $t = t_i, ..., t-1$; and iv) price at p^N in period $t\,(> t_i)$ otherwise. Even if one could plausibly argue how firms would achieve mutual beliefs regarding this strategy, collusive prices will emerge if and only if $t_1 = \cdots = t_n$ so that there is mutual beliefs as to when to start pricing at a collusive level. There is nothing to make period 1 a focal point as that is just where the analysis begins. It is then an open question whether reasonable assumptions on mutual beliefs can be made to result in firms pricing at $\widetilde{p}$.

[17] It is argued in Appendix B that, in the event of a departure from the price matching plus property. the competitive outcome is *salient* in the sense of Lewis (1969) and Cubitt and Sugden (2003).

[18] Specifically, $\overline{p}$ must be at least as high as p^*.

4 Mutual Beliefs of Price Matching Constrains the Collusive Price

4.1 Derivation of an Upper Bound on Price

The first step in the analysis is to show that A1 is internally consistent so that if a firm believes its rivals use PMP-compatible strategies then it is optimal for that firm to use a PMP-compatible strategy. This is Lemma 2. Theorem 3 provides an upper bound on price, while Theorem 4 shows that this upper bound is strictly below a benchmark equilibrium price.

Lemma 2 shows that if a firm believes other firms use PMP-compatible strategies then it is optimal for that firm to also use a PMP-compatible strategy. In other words, the set S^{PMP} is closed under the best reply operator.

Lemma 2 *If a firm's beliefs are PMP-consistent then sequential rationality implies that the firm's strategy lies in* S^{PMP}.

Theorem 3 shows that if there is common knowledge that firms are sequentially rational and use strategies satisfying the PMP property then price is bounded above by (approximately) p^*.

Theorem 3 *If A1-A2 hold then* $\{(p_1^t, ..., p_n^t)\}_{t=1}^{\infty}$ *is weakly increasing over time and there exists finite* T *such that* $p_1^t = \cdots = p_n^t = \widehat{p}\ \forall t \geq T$ *where* $\widehat{p} \leq p^* + \varepsilon$.

In explaining the basis for Theorem 3, first note that while A1 leaves unspecified whether some firm will initiate a price increase, it is fully consistent with A1 for a firm to be a price leader. For example, if a firm believed other firms would not raise price then it would be rational for it to increase price (as long as the current price is not too high). The issue is how far would it go in raising price. If the firm expected that its price increase would only be met and never exceeded by a rival (for example, rivals are believed to only match price increases) then it would not want to raise price above (approximately) p^*.[19] Recall that p^* is the price at which a firm, if it were to raise price from p^* to any higher level (call it p'), it would lose more in current profit (because of lower demand from pricing above the level p^* set by its rivals) than it would gain in future profit (from all other firms raising price to p'). Thus, a firm that believed its rivals would never initiate price increases would not raise price beyond p^*. However, a firm might be willing to lead a price increase above p^* *if* it believed it would induce a rival to further increase price; for example, if the firm believed that firms would take turns leading price increases. Theorem 3 shows that cannot happen.

By A1 and Lemma 1, a firm will never price above $\widetilde{p}$. If $\widetilde{p} > p^*$ then it furthermore means that a firm would never raise price *to* $\widetilde{p}$ since such a price increase would only induce its rivals to match that price; it would not induce them to further raise price. Thus, if a firm is sequentially rational and believes the other firms use PMP-compatible strategies

[19] In discussing results, I will generally refer to the upper bound as p^* rather than $p^* + \varepsilon$ since ε is presumed to be small.

then it will not raise price to $\widetilde{p}$. This puts an upper bound on price of $\widetilde{p} - \varepsilon$. We next build on that result to argue that $\widetilde{p} - 2\varepsilon$ is an upper bound on price. Given mutual beliefs regarding sequential rationality and PMP-consistent beliefs, firm i then believes firm $j\ (\neq i)$ is sequentially rational and that firm j believes firm h uses a PMP-compatible strategy (for all $h \neq j$); hence, firm i knows that firm j will not raise price to $\widetilde{p}$. This means that firm i knows that if it raises price to $\widetilde{p} - \varepsilon$ that this price increase will only be matched and not exceeded, which then makes a price increase to $\widetilde{p} - \varepsilon$ unprofitable (as long as $\widetilde{p} - \varepsilon > p^*$). Given that all firms are not willing to raise price to $\widetilde{p} - \varepsilon$ then $\widetilde{p} - 2\varepsilon$ is an upper bound on price. The proof is completed by induction to end up with the conclusion that a firm would never raise price to a level exceeding p^*. Hence, price is bounded above by p^*.

In deriving this upper bound, the punishment for deviation from (at least) matching price is reversion to a stage game Nash equilibrium. If firms had mutual beliefs regarding a grim trigger strategy (so that the punishment is the same as in a PMP-compatible strategy), firms could sustain a price as high as $\widetilde{p}$. With the more limited mutual beliefs that the strategy profile lies in S^{PMP}, price can only rise as high as p^* which the next result shows is strictly less than $\widetilde{p}$.

Theorem 4 $p^* \in \left(p^N, \widetilde{p}\right)$.

Recall that p^* is the price at which the reduction in current profit from a marginal *increase* in a firm's price is exactly equal in magnitude to the *rise* in the present value of the future profit stream when that higher price is matched by all firms for the infinite future. Equivalently, p^* is the price at which the increase in current profit from a marginal *decrease* in price to $p^* - \varepsilon$ is exactly equal in magnitude to the *fall* in the present value of the future profit stream *when the firm's rivals lower price to* $p^* - \varepsilon$ (when ε is small). In comparison, $\widetilde{p}$ is the price for a firm at which the increase in current profit from a marginal decrease in price is exactly equal in magnitude to the fall in the present value of the future profit stream *when the firm's rivals lower price to* p^N.[20] Given that the punishment is more severe in the latter case, it follows that the maximal sustainable price is higher: $\widetilde{p} > p^*$.

With common knowledge that the strategy profile lies in S^{PMP}, the steady-state price is bounded above by p^* even though higher prices are sustainable. In other words, if firms started at a price of $\widetilde{p}$ then such a price would persist under A1-A2. But if firms start with prices below p^*, such as at the non-collusive price p^N, then prices will not go beyond p^*, even though higher prices are sustainable. The obstacle is that it is not in the interests of any firm to lead a price increase beyond p^*. Thus, with only mutual understanding that the strategy profile lies in S^{PMP}, what is constraining how high price can go is the trade-off a firm faces when it acts as a price leader: It foregoes current demand and profit in exchange for higher future profit from its rivals having raised their prices to match its price increase. This is to be contrasted with equilibrium where what limits how high is price is the condition that a firm finds it unprofitable to undercut it. In a sense, coordination comes for free with equilibrium. Limited mutual understanding makes the coordination on price (through price leadership) the constraining factor, not the stability of the price which is eventually

[20] That is, $\widetilde{p}$ is the highest price for which a firm incentive compatibility constraint (1), holds. For this discussion, suppose $\widetilde{p} < p^M$.

coordinated upon. Finally, note that Theorems 3 and 4 are robust to the form of the punishment. p^* is independent of the punishment and, given another punishment, $\widetilde{p}$ would just be the highest sustainable price with that punishment. In particular, if the punishment is at least as severe as the grim punishment then Theorems 3 and 4 are unchanged.[21]

In concluding, let me discuss the role of the finiteness of the price set. p^* is the highest price to which a firm will raise price if it can only anticipate that other firms will match its price. Thus, a firm is willing to take the lead and price above p^* only if, by doing so, it induces a rival to enact further price increases. Since no firm will price above $\widetilde{p}$ then raising price to $\widetilde{p}$ cannot induce rivals to lead future price increases. Thus, a firm will not raise price to a level beyond $\widetilde{p} - \varepsilon$, which means $\widetilde{p} - \varepsilon$ is an upper bound on price. This argument works iteratively to ultimately conclude that p^* is (approximately) an upper bound on price. The finiteness of price is critical in this proof strategy for it allows $\widetilde{p} - \varepsilon$ to be well-defined. However, even with an infinite price set, it is still the case that a necessary condition for a firm to lead and raise price above p^* is that it will induce a rival to enact further price increases. As that must always be true then, if the limit price were to exceed p^* when there is an infinite price set, prices cannot converge in finite time. But since it is still the case that $\widetilde{p}$ is an upper bound on price, the price increases must then get arbitrarily small; eventually, each successive price increase will bring forth a smaller future price increase by a rival. I am not arguing that this argument will prevent Theorem 3 from extending to the infinite price set but rather that it is the only argument that could possibly do so. Therefore, either Theorem 3 extends to when the price set is infinite or, if it does not, it implies a not very credible price path with never-ending price increases that eventually become arbitrarily small. The oddity of such a price path is an artifact of assuming an infinite set of prices when, in fact, the set of prices is finite.

4.2 Price Can Be Competitive or Supracompetitive

By Theorem 3, if firms have the common knowledge in A1 that firms are sequentially rational and that their strategies satisfy the PMP property then price is bounded above by p^*. But is p^* the least upper bound? And is there a lower bound on price exceeding p^N? The purpose of the current section is to show, by way of example, that it is consistent with A1-A2 for price to converge to p^* but also to fail to rise above p^N. Thus, a tighter result than Theorem 3 will require additional assumptions, which is a matter taken up in Section 5.

For the duopoly case, suppose the price set is composed of just three elements, $\{p^N, p', p^M\}$, and $p' \equiv (p^M + p^N)/2$. Assume δ is sufficiently close to one so that $p^* = \widetilde{p} = p^M$.[22] Consider

[21] The property that price falls short of the equilibrium price is similar to a finding in Lockwood and Thomas (2002). They consider a multi-player infinite horizon setting in which actions are irreversible; that is, the minimum element of player i's period t action set is the action the player selected in period $t-1$. The equilibrium path is uniformly bounded above by the first-best for basically the same reasons that price converges to a value less than $\widetilde{p}$. I want to thank Thomas Mariotti for pointing out this connection.

[22] In this example, as opposed to elsewhere in the paper, p^* is defined for when the feasible price set is $\{p^N, p', p^M\}$ rather than $\Re_+$. This, however, is a good approximation when $\delta \simeq 1$ as then $p^* \simeq p^M$ when the price set is $\Re_+$. Of course, if $\delta \simeq 1$ then $\widetilde{p} = p^M$ whether the price set is $\{p^N, p', p^M\}$ or $\Re_+$.

the following pair of functions which map from $\max\left\{p_1^{t-1}, p_2^{t-1}\right\}$ to current price p_i^t:

$$\begin{aligned} S^L\left(p^N\right) &= p', S^L\left(p'\right) = p^M, S^L\left(p^M\right) = p^M \\ S^F\left(p^N\right) &= p^N, S^F\left(p'\right) = p', S^F\left(p^M\right) = p^M \end{aligned} \quad (5)$$

S^L (where L denotes "leader") has a firm raise price to p' when the lagged maximum price is p^N, to p^M when the lagged maximum price is p', and to price at p^M when the lagged maximum price is p^M. S^F (where F denotes "follower") has a firm's price equal the lagged maximum price. When S^L (or S^F) is referred to as a strategy, it is meant that the specification in (5) applies when both firms have priced at least as high as the previous period's maximum price in all past periods, and otherwise a firm prices at p^N. Thus, these strategies satisfy the PMP property. It is shown in Appendix C that $\left(S^L, S^F\right)$ is a subgame perfect equilibrium when $\delta \simeq 1$ and

$$\pi\left(p', p^N\right) + \pi\left(p^M, p'\right) > \pi\left(p^M, p^N\right) + \pi\left(p^M, p^M\right). \quad (6)$$

(6) holds for the case of linear demand and linear cost when products are sufficiently differentiated and/or cost is sufficiently low.

Given $\left(S^L, S^F\right)$ is a subgame perfect equilibrium, it is easy to argue that both strategies satisfy A1-A2. If, for all histories, a firm's beliefs assign probability one to its rival using strategy S^F then S^L is sequentially rational for those beliefs; and if a firm's beliefs assign probability one to its rival using strategy S^L then S^F is sequentially rational for those beliefs. Therefore, S^L and S^F are consistent with $\mathcal{A}^0$. Mutual strong belief in $\mathcal{A}^0$ implies that firms' beliefs have support over strategies consistent with $\mathcal{A}^0$. Hence, S^L and S^F are consistent with $\mathcal{A}^1$; and so forth.

Any price path consistent with firms deploying a strategy pair from $\left\{S^L, S^F\right\}^2$ is then consistent with A1-A2. For example, a price path of $\left(\left(p', p^N\right), \left(p^M, p'\right) \left(p^M, p^M\right), ...\right)$, with a steady-state supracompetitive price of p^M, is consistent with A1-A2. It is achieved by firm 1 using S^L based on the belief that firm 2 use S^F, and firm 2 using S^F based upon the belief that firm 1 uses S^L. However, it is also the case that a price path of $\left(\left(p^N, p^N\right), ...\right)$ is consistent with A1-A2. It occurs when each firm's strategy is S^F which is sequentially rational if, for all histories, it believes the other firm uses S^L. Thus, common knowledge of sequential rationality and that prices increases will be at least matched does not imply supracompetitive prices; either competitive or supracompetitive prices could ensue.

5 Rational Learning and Price Matching Results in Supracompetitive Prices

5.1 Derivation of the Long-Run Supracompetitive Price

Thus far, assumptions have been made on a firm's beliefs regarding price matching - specifically, other firms will at least match price up to a maximum level of $\widetilde{p}$ - and regarding what happens when behavior is contrary to such price matching - firms revert to competitive prices. Common knowledge of those properties along with rationality - as expressed in A1 - is sufficient to place an upper bound on price of (approximately) p^*. What is not yet clear is whether price is assured of reaching p^* or whether price is mired at the competitive level. Though firms have mutual understanding about price matching, they lack mutual understanding about who will lead on price and some firm must take the the lead if prices are to rise above the competitive level. In some markets, a particular firm may be the salient leader by virtue of its size or access to information (what is referred to as barometric price leadership; see, for example, Cooper, 1997), but no such presumption is made here. Furthermore, firms prefer to be price followers than price leaders because it is costly to initiate a price hike as a firm will lose demand prior to its price being matched.[23] Hence, each firm would prefer another firm to take the lead in raising price which provides an incentive for a firm to hesitate before leading. In this section, sufficient conditions are provided for one of the firms to eventually lead and, therefore, collusion to succeed.

As described in Section 4.2, competitive pricing can occur - in spite of price matching being common knowledge - because firms' beliefs are inconsistent; each believes someone else will lead on price and each takes the role of follower with no price leader emerging. However, if firms could learn over time about rivals' strategies then perhaps a firm would learn that other firms are not likely to lead which would could induce it to raise price. This is the avenue pursued here and, in doing so, I will draw on a result of Kalai and Lehrer (1993) regarding the learning of strategies in an infinitely repeated game.

To lay the background for this analysis, it is useful to provide an alternative description of A1 by presenting it as restrictions on a firm's prior beliefs on other firms' strategies and on their posterior beliefs conditional on the observed history. Define $B(X)$ to be the set of strategies that are sequentially rational for some beliefs over other firms' strategies with support X^{n-1}. Defining $X^1 = B\left(S^{PMP}\right)$, X^1 is the set of strategies that are sequentially rational for some PMP-consistent beliefs. By Lemma 2, $X^1 \subseteq S^{PMP}$. By $\mathcal{A}^1$, each firm believes other firms use a strategy in X^1. By $\mathcal{A}^2$, each firm believes other firms believe other firms use a strategy in X^1 which, along with the belief that other firms are sequentially rational, means that each firm believes other firms use a strategy in $X^2 = B(X^1)$. Continuing in this manner, the infinite hierarchy of beliefs implies that each firm believes other firms use a strategy in X^∞ where $X^\infty = B(X^\infty)$. Thus, A1 requires that a firm has beliefs over other firms' strategies with support X^∞, and chooses a strategy (necessarily from X^∞) that is sequentially rational given those beliefs. Focusing on the restrictions placed on prior beliefs and belief revision (in response to the history), A1 requires that prior beliefs on other firms' strategies have support $(X^\infty)^{n-1}$, while the only restriction on belief revision is that posterior beliefs have support $(X^\infty)^{n-1}$.

To use the result of Kalai and Lehrer (1993), more structure needs to be placed on prior beliefs and the belief revision process. Firms start with prior beliefs on the actual strategy profile and from these beliefs are generated beliefs on infinite price paths, Ω^∞. The first

[23] Wang (2009) provides indirect evidence of the costliness of price leadership. In a retail gasoline market in Perth, Australia, Shell was the price leader over 85% of the time until a new law increased the cost of price leadership, after which the three large firms - BP, Caltex, and Shell - much more evenly shared the role of price leader. The law specified that every gasoline station was to notify the government by 2pm of its next day's retail prices, and to post prices on its price board at the start of the next day *for a duration of at least 24 hours*. Hence, a firm which led in price could not expect its rivals to match its price until the subsequent day. The difference between price being matched in an hour and in a day is actually quite significant given the high elasticity of firm demand in the retail gasoline market. For the Quebec City gasoline market, Clark and Houde (2011, p. 20) find that "a station that posts a price more than 2 cents above the minimum price in the city loses between 35% and 50% of its daily volume."

restriction is that these prior beliefs do not assign zero probability to the actual price path that is played. A3 is referred to as the "Grain of Truth" assumption.[24]

Assumption A3: Each firm's prior beliefs on infinite price paths assigns positive probability to the true price path.

Though A3 only requires positive probability on the true price path, it is well-recognized not to be a weak assumption because there is an infinite number of price paths. However, the more relevant point is that it is a far weaker assumption than the standard assumption implicit in equilibrium which is that a firm's beliefs put probability one on the true strategy profile and thus the true price path. It is also worth noting that A3 is weaker than assuming positive probability is assigned to the true strategy profile since there may be many strategy profiles that produce the same realized price path.

Thus far, minimal structure has been placed on how, in response to observed play, firms revise their beliefs as to other firms' strategies. Now, it is assumed that firms are Bayesian learners.

Assumption A4: In response to the realized price path, a firm updates its beliefs as to other firms' strategies (and, therefore, the future price path) using Bayes Rule.

What transpires is that a firm starts with prior beliefs about the strategy profile and the price path. By A1, these prior beliefs have support on those strategies consistent with mutual understanding that each firm is sequentially rational and believes other firms use PMP-compatible strategies and, by A3, assign positive probability to the set of strategy profiles that produce what will prove to be the observed price path. Given the strategy profile that firms adopted given their prior beliefs, prices will be chosen and an outcome path will emerge. Come period t, firms will have observed the first $t-1$ periods of the true underlying price path and each will update its beliefs as to other firms' strategies and the resultant future price path using Bayes Rule. By sequential rationality, it is required that each firm's strategy is optimal at that point. By the result of Kalai and Lehrer (1993), this process will entail firms eventually having beliefs over the future path of play that are "close" to the true future path of play. Theorem 5 shows that an implication of this learning of the future price path is that firms are almost sure to eventually price close to the supracompetitive price p^*.

Theorem 5 *Assume A1-A4. For all $\eta > 0$, there exists T such that, with probability of at least $1-\eta$, $p_1^t = \cdots = p_n^t = \widehat{p}\ \forall t \geq T$ where $\widehat{p} \in \{p^* - \varepsilon, p^*, p^* + \varepsilon\}$.*

Learning implies supracompetitive prices will eventually emerge even though firms only have mutual understanding over price matching and not over who will lead. To understand why prices must converge to (approximately) p^*, suppose they did not and instead converged

[24] While Kalai and Lehrer (1993) allow for strategies in which players randomize, I am assuming that the true strategy profile is a pure strategy profile. However, given that their analysis models a player's beliefs on other players' strategies as a point belief, it is important for their analysis that beliefs are over mixed (behavior) strategies. That is not the case with our analysis.

to $\widehat{p}$ where $\widehat{p} < p^* - \varepsilon$. By the result of Kalai and Lehrer (1993), firms will eventually believe with high probability that the future path has all firms price at $\widehat{p}$. But since $\widehat{p} < p^* - \varepsilon$ then $\phi(\widehat{p}, ..., \widehat{p}) > \widehat{p}$ in which case a firm would find it optimal to raise above $\widehat{p}$ given it expects other firms to respond by at least matching that price increase.[25] In other words, if a firm eventually came to believe that the other firms were extremely unlikely to raise price and the current price is less than $p^* - \varepsilon$, then a firm would find it profitable to act as a price leader. Thus, learning the future price path prevents firms from getting mired in a miscoordination in which all firms do not act as a price leader because each anticipates that other firms will do so. While such was possible when firms' prior and posterior beliefs are only required to have support X^{∞}, it is ruled out (eventually) when prior beliefs do not assign zero probability to the true price path and firms update their beliefs on other firms' strategies using Bayes Rule.

In concluding, let me make two remarks. First, A3-A4 are not sufficient to deliver supracompetitive prices, even if firms use collusive strategies. Kalai and Lehrer (1993) prove that A3-A4 (and each player knows its payoff function) is sufficient for play to converge to equilibrium play. Suppose, for example, that firms are using the collusive strategies described at the end of Section 3. Unless $t_1 = \cdots = t_n$ so that all firms know to start setting the collusive price at the same time, the long-run price is the static Nash equilibrium price p^N. Though firms use collusive strategies and eventually learn the future price path, they are in the punishment phase by the time that path is learned. Hence, supracompetitive outcomes do not occur. Second, this analysis shows how the result of Kalai and Lehrer (1993) can deliver more precise results when used in conjunction with some prior beliefs on the strategy profile.

5.2 Example of a Supracompetitive Price Path

As illustrative of Theorem 5, strategies satisfying A1-A4 will be constructed. To keep the analysis manageable, assume the price set has just two elements $\{p^N, p^M\}$ where p^N is the static Nash equilibrium price and p^M is the monopoly price where $p^N < p^M$. Firms are initially pricing at p^N. The associated profit levels are denoted:

Own firm's price	Rival firm's price	Own firm's profit
p^N	p^N	π^N
p^N	p^M	π^F
p^M	p^N	π^L
p^M	p^M	π^M

$\pi^F > \pi^M > \pi^N > \pi^L$ so that profit is highest when the other firm raises price and a firm keeps price low (and thus the latter acts as a follower, which is why its profit is referred to as π^F), and profit is lowest when the other firm keeps price low and a firm raises price (and thus the latter acts as a leader, which is why its profit is referred to as π^L). Assume $p^M = p^*$

[25] Recall that ϕ is a firm's optimal period t price when it expects other firms to price in period t at the maximum price of period $t-1$ and all future prices equal the maximum price of period t. That is, all other firms do not take the lead on raising price and only match price increases, and there will be no further price increases after the current period.

which means that a firm's payoff from raising price from p^N to p^M, given the other firm is expected to price at p^N in the current period and match p^M thereafter, exceeds that from both firms pricing at p^N forever:

$$\pi^L + \left(\frac{\delta}{1-\delta}\right)\pi^M > \frac{\pi^N}{1-\delta} \Rightarrow \delta > \frac{\pi^N - \pi^L}{\pi^M - \pi^L}. \tag{7}$$

Our attention will focus on a natural class of PMP-compatible strategies which are defined by $T \in \{1, 2, ...\} \cup \{\infty\}$ where T is the earliest period in which a firm will lead. More specifically, a T-strategy has a firm lead and raise price to p^M in period t when no firm raised price prior to period t and $t \geq T$. $T = \infty$ represents the strategy in which a firm never takes the lead in raising price. In constructing beliefs that satisfy A1, I will focus on prior beliefs for firm i that have full support on $T_j \in \{1, 2, ...\} \cup \{\infty\}$ and obey Bayes Rule so that A4 is satisfied.

Let us first show that, with the assumption of full support, $T = \infty$ is not sequentially rational. Firm 1's prior probability on $T_2 \in \{T', T'+1, ...\}$ necessarily goes to zero as $T' \to \infty$. Given that the probability of $T_2 = \infty$ is positive, Bayesian updating implies that if firm 2 has not raised price come period t then the posterior probability that firm 1 assigns to $T_2 = \infty$ goes to one as $t \to \infty$. Hence, as $t \to \infty$, the expected payoff from leading on price converges to the LHS of (7) and the expected payoff from $T_1 = \infty$ converges to the RHS of (7). Eventually, firm 1 will then prefer to lead on price which contradicts the sequential rationality of $T_1 = \infty$. Hence, if firm 1 assigns positive prior probability to $T_2 = \infty$ then Bayes Rule implies $T_1 = \infty$ is not sequentially rational.

The next step is to show that there exists beliefs such that each strategy $T \in \{1, 2, ...\}$ is sequentially rational, and that this set of strategies satisfies A1-A4. The prior beliefs of firm i are assumed to have full support on $T_j \in \{1, 2, ...\}$ but, in order to simplify the analysis, zero probability will be assigned to $T_j = \infty$. (It can be shown that the ensuing analysis is robust to allowing for a small positive probability attached to $T_j = \infty$.) For each strategy in $\{1, 2, ...\}$, prior beliefs on $\{1, 2, ...\}$ are found such that the strategy is sequentially rational. Given that all elements of $\{1, 2, ...\}$ then satisfy $\mathcal{A}^0$ using beliefs with support $\{1, 2, ...\}$, all of those strategies satisfy $\mathcal{A}^1$ as well and so forth; hence, $\{1, 2, ...\}$ satisfies A1. As these beliefs will be constructed to comply with Bayes Rule, A4 is also satisfied. Finally, A3 is satisfied given each firm uses a strategy from $\{1, 2, ...\}$, prior beliefs have full support on $\{1, 2, ...\}$, and posterior beliefs satisfy Bayes Rule.

To begin, consider the following prior beliefs of firm i on firm j's strategy:

T_j	Prior Probability
1	1/2
2	$(1/2)^2$
$\vdots$	$\vdots$
$t'_j - 1$	$(1/2)^{t'_j - 1}$
t'_j	$(1/2)^{t'_j}(1/x)$
$\vdots$	$\vdots$
$t'_j + x - 1$	$(1/2)^{t'_j}(1/x)$
$t'_j + x$	$(1/2)^{t'_j + 1}(1/x)$
$\vdots$	$\vdots$
$t'_j + 2x - 1$	$(1/2)^{t'_j + 1}(1/x)$
$\vdots$	$\vdots$

(8)

where $t'_j \in \{1, 2, ...\}$. For $T_j \in \{1, ..., t'_j - 1\}$, the probability assigned to the rival firm using a strategy that has it lead in period T_j is $(1/2)^{T_j}$ and thus is exponentially declining. Starting with period t'_j, the probability assigned over every x periods exponentially decays and that probability mass is uniformly distributed within a window of x periods. For $T_j \in \{t'_j, ..., t'_j + x - 1\}$, the probability that the rival firm's strategy has it lead in period t is $(1/2)^{t'_j}(1/x)$; and, more generally, for $T_j \in \{t'_j + \omega x, ..., t'_j + (1 + \omega)x - 1\}$, the probability that the rival firm's strategy has it lead in period t is $(1/2)^{t'_j + \omega}(1/x)$, $\omega \in \{0, 1, 2, ...\}$.

Let T_1 represent the strategy of firm 1 and assume firm 1's prior beliefs on firm 2's strategy are as specified in (8). The proof will first derive sufficient conditions for it to be sequentially rational for firm 1 to wait in period t (assuming neither firm has yet raised price) when $t < t'_2$; in other words, sequential rationality requires $T_1 \geq t'_2$. Next, sufficient conditions are derived for it to be sequentially rational for firm 1 to raise price when $t = t'_2$. Finally, it is shown that if $t > t'_2$ then it is sequentially rational for firm 1 to raise price. Thus, sequential rationality implies $T_1 = t'_2$. As this will be shown for an arbitrary t'_2 then every strategy in $\{1, 2, ...\}$ is sequentially rational for some prior beliefs.

Suppose firm 1's prior beliefs on firm 2's strategy are as specified in (8) and $t'_2 > 1$. (The case of $t'_2 = 1$ is covered when I examine $t = t'_2$.) Consider a strategy for firm 1 with $T_1 > 1$ so that firm 1 does not raise price in period 1. According to (8) with $t'_2 > 1$, firm 1 assigns probability 1/2 to $T_2 = 1$ and, in that event, firm 2 raises price in period 1 so firm 1's payoff is $\pi^F + \left(\frac{\delta}{1-\delta}\right)\pi^M$. Also with probability 1/2, firm 1 believes $T_2 > 1$ in which case firm 1's period 1 profit from $T_1 > 1$ is π^N, while a lower bound on its expected future payoff is $\pi^N/(1-\delta)$ (which firm 1 can achieve by pricing at p^N in all ensuing periods). Thus, a lower bound on firm 1's expected payoff from $T_1 > 1$ is

$$\left(\frac{1}{2}\right)\left[\pi^F + \left(\frac{\delta}{1-\delta}\right)\pi^M\right] + \left(\frac{1}{2}\right)\left(\frac{1}{1-\delta}\right)\pi^N. \tag{9}$$

In comparison, firm 1's expected payoff from $T_1 = 1$ is

$$\left[\left(\frac{1}{2}\right)\pi^L + \left(\frac{1}{2}\right)\pi^M\right] + \left(\frac{\delta}{1-\delta}\right)\pi^M. \tag{10}$$

Hence, if $t_2' > 1$ then it is sequentially rational for firm 1 not to raise price in period 1 when (9) exceeds (10):

$$\left(\frac{1}{2}\right)\left[\pi^F + \left(\frac{\delta}{1-\delta}\right)\pi^M\right] + \left(\frac{1}{2}\right)\left(\frac{1}{1-\delta}\right)\pi^N > \left(\frac{1}{2}\right)\pi^L + \left(\frac{1}{2}\right)\pi^M + \left(\frac{\delta}{1-\delta}\right)\pi^M \Rightarrow$$

$$1 - \left(\frac{\pi^M - \pi^N}{\pi^F - \pi^L}\right) > \delta. \tag{11}$$

Given the beliefs in (8) with $t_2' > 1$, if (11) holds then $T_1 = 1$ is not sequentially rational for firm 1.

Now suppose it is period 2 and neither firm raised price in period 1. Firm 1 then infers that $T_2 \geq 2$. If prior beliefs are (8) with $t_2' > 2$, Bayes Rule implies firm 1's posterior beliefs are those in (8) divided by the probability that $T_2 \geq 2$ (which is $1/2$). By the same analysis which showed that it is not sequentially rational for firm 1 to raise price in period 1 when $t_2' > 1$, it is not sequentially rational for firm 1 to raise price in period 2 when $t_2' > 2$. That this property also holds for period 2 is because the posterior probability that firm 2 raises price in period 2, given it did not raise price in period 1, equals the prior probability that firm 2 raises price in period 1. In fact, this property holds for all $t < t_2'$ so that, if neither firm has raised price come period t, it is not sequentially rational for firm 1 to raise price in period t. In sum, if (11) holds then it follow from prior beliefs (8), posterior beliefs satisfying Bayes Rule, and sequential rationality that $T_1 \geq t_2'$.

Now suppose it is period $t = t_2'$ and neither firm raised price over periods $1, ..., t_2' - 1$. The posterior beliefs of firm 1 on firm 2's strategy are

T_j	Posterior Probability as of Period t_j'
t_j'	$(1/2)\,(1/x)$
$t_j' + 1$	$(1/2)\,(1/x)$
$\vdots$	$\vdots$
$t_j' + x - 1$	$(1/2)\,(1/x)$
$t_j' + x$	$(1/2)^2\,(1/x)$
$\vdots$	$\vdots$
$t_j' + 2x - 1$	$(1/2)^2\,(1/x)$
$t_j' + 2x$	$(1/2)^3\,(1/x)$
$\vdots$	$\vdots$

(12)

It is shown in Appendix D that if (7) holds then, when x is sufficiently large, firm 1 prefers to lead at t than to wait until $t + 1$ as long as $t \geq t_j'$ because the likelihood of firm 2 leading in the near future is small.

In sum, if

$$\delta \in \left(\frac{\pi^N - \pi^L}{\pi^M - \pi^L}, 1 - \left(\frac{\pi^M - \pi^N}{\pi^F - \pi^L} \right) \right) \tag{13}$$

then, for x sufficiently large, the sequentially rational strategy for firm 1 is $T_1 = t_2'$. Given that this argument works for all $t_2' \in \{1, 2, ...\}$, every $T_1 \in \{1, 2, ...\}$ is sequentially rational. That the interval in (13) is non-empty follows from $\pi^M > \pi^N$ and $\pi^F > \pi^M$. If the discount factor is in this intermediate range then a firm is sufficiently patient that it'll initially wait to see whether its rival leads (which means waiting when $t < t_j'$) but it is sufficiently impatient that eventually it'll take the lead (which means taking the lead when $t \geq t_j'$).

Given that every $(T_1, T_2) \in \{1, 2, ...\}^2$ satisfies A1-A4, the path has one of the firms raising price at $\min\{T_1, T_2\}$ and the other firm matching price in the subsequent period. For example, suppose $1 < T_1 < T_2$. Both firms initially attach enough probability to the other firm leading sufficiently early that they don't lead themselves. However, as time progresses and the other firm has not led, each firm eventually attaches substantial probability to the other firm not leading in the near future. Firm 1 shifts enough probability to firm 2's strategy not having it take the lead in the near future that, once period T_1 is reached and no firm has led, firm 1 takes the leads and raises prices to p^M. Firm 2 matches that price increase in the subsequent period.

6 Concluding Remarks

In his classic examination of imperfect competition, Chamberlain (1948) argued that collusion would naturally emerge because each firm would recognize the incentive to maintain a collusive price, rather than undercut its rivals' prices and bring forth retaliation. We now know that it is a non-trivial matter for firms to coordinate on a collusive solution because there are so many collusive equilibria. This multiplicity poses a challenge for firms to achieve mutual understanding with respect to the collusive strategy profile, and this challenge is exacerbated should firms avoid express communication. All of this naturally raises the question of how much mutual understanding must firms have in order to effectively collude.

To my knowledge, this paper is the first to explore the relationship between mutual understanding among firms and the coordination on supracompetitive outcomes. Assuming that the game and firms' rationality is common knowledge, I investigated the implications of common knowledge that price increases will be at least matched and failure to do so results in a return to competition. It was shown there is an upper bound on price that is strictly below what equilibrium sustains (given the same underlying punishment). However, this level of mutual beliefs is not sufficient to ensure success in colluding as both competitive and supracompetitive price paths could ensue. If, in addition, firms update their beliefs on other firms' strategies using Bayes Rule and their prior beliefs assign positive probability to the true price path then supracompetitive prices will occur almost for sure. Thus, mutual understanding that price increases will be matched - but not about who will lead on price, at what time, and at what level - is sufficient for firms to effectively collude.

In this paper, the focus was on price leadership as a collusive mechanism where firms shared beliefs that price increases would be at least matched. It is worthwhile to consider other collusive mechanisms - such as various market allocation schemes and bid rigging - and to investigate how much mutual understanding must firms have if supracompetitive outcomes are to emerge. By identifying the relationship between firms' mutual understanding and their ability to effectively collude, such a research program will identify what types of mutual beliefs ought to raise concerns to competition authorities. Having identified those mutual beliefs that support collusion, communication practices likely to generate those mutual beliefs are a target for prohibition. In this manner, economic theory can contribute to identifying the boundaries of unlawful collusion.

7 Appendices

7.1 Appendix A: Proofs

Proof of Lemma 1. First note that if $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\} = \overline{p}$ then, by A1, firm i believes $\forall j \neq i$: i) firm j will price at $\overline{p}$ for the infinite future (level 1 belief); and ii) firm j believes that all other firms will price at $\overline{p}$ for the infinite future (level 2 belief). If $\overline{p} \notin \left\{p^N,...,\widetilde{p}\right\}$, we will show that (i)-(ii) are inconsistent with firm i believing firm j is sequentially rational (level 2 belief). Hence, $\overline{p} \notin \left\{p^N,...,\widetilde{p}\right\}$ is inconsistent with A1.

Suppose $\overline{p} < p^N$ and $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\} = \overline{p}$ so that (i)-(ii) hold and, therefore, firm i believes firm j will price at $\overline{p}$ for the infinite future. Consider instead firm j pricing at the static best reply function $\psi\left(\overline{p},...,\overline{p}\right)(> \overline{p})$. In that case, firm i believes firm j expects to earn the payoff on the LHS of the inequality:

$$\left(\frac{1}{1-\delta}\right)\pi\left(\psi\left(\overline{p},...,\overline{p}\right),\overline{p},...,\overline{p}\right) > \left(\frac{1}{1-\delta}\right)\pi\left(\overline{p},...,\overline{p}\right).$$

On the RHS of the inequality is the expected payoff by pricing at $\overline{p}$ forever which is clearly less and thus runs contrary to firm i believing firm j is sequentially rational.

Next suppose $\overline{p} > \widetilde{p}\left(> p^N\right)$ and $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\} = \overline{p}$ so again (i)-(ii) hold. Consider firm j pricing instead at $\psi\left(\overline{p},...,\overline{p}\right)(< \overline{p})$. Firm i believes firm j expects to earn the payoff on the LHS of the inequality:

$$\pi\left(\psi\left(\overline{p},...,\overline{p}\right),\overline{p},...,\overline{p}\right) + \left(\frac{\delta}{1-\delta}\right)\pi^N > \left(\frac{1}{1-\delta}\right)\pi\left(\overline{p},...,\overline{p}\right),$$

which again runs contrary to firm i believing firm j is sequentially rational.

In summing up, A1 implies $\overline{p} \notin \Omega - \left\{p^N,...,\widetilde{p}\right\}$. Furthermore, existence of $\overline{p}$ such that A1 is satisfied is immediate as $\overline{p} = p^N$ trivially satisfies it. Therefore, A1 implies $\overline{p} \in \left\{p^N,...,\widetilde{p}\right\}$.
■

A useful property of other firms using PMP-compatible strategies is that a lower bound on a rational firm's period t continuation payoff is the payoff associated with all firms pricing at $\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\}$ in all periods (Lemma 6). Intuitively, if the rivals to firm i are using PMP-compatible strategies then they will price at least as high as $\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\}$ in all ensuing periods, as long as firm i does not violate the PMP property and induce a shift to p^N. Hence, firm i can at least earn the profit from all firms (including i) pricing at $\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\}$.

Lemma 6 *Suppose firm i uses strategy $\widehat{s}_i$. If firm i is sequentially rational and has PMP-consistent beliefs then its expected continuation payoff, $U_i\left(\widehat{s}_i,\mu^i\left(s\cdot\left|h\right.\right)\right)$, satisfies*

$$U_i\left(\widehat{s}_i,\mu^i\left(\cdot\left|h^t\right.\right)\right) \geq \frac{\pi\left(\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\},...,\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\}\right)}{1-\delta},$$

for all h^t such that

$$p_j^{\tau} \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\widetilde{p}\right\}\ \ \forall j, \forall\tau \leq t-1.$$

Proof of Lemma 6. Consider firm i pricing at $\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\}$ in period t and then, in all ensuing periods, matching the maximum price of the other firms in the previous period:

$$p_i^t = \min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\},\ p_i^\tau = \max\left\{\mathbf{p}_{-i}^{\tau-1}\right\}\ \text{for } \tau = t+1,...$$

where

$$\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\} \equiv \max\left\{p_1^{\tau-1},...,p_{i-1}^{\tau-1},p_{i+1}^{\tau-1},...,p_n^{\tau-1}\right\}.$$

Given this behavior for firm i, consider any price path $\{(p_1^\tau,...,p_n^\tau)\}_{\tau=t}^{\infty}$ assigned positive probability by firm i (recalling that its beliefs are PMP-consistent). These are price paths in which firm i matches the maximum price of the other firms in the previous period and the other firms at least match the maximum price in the previous period though not pricing above $\overline{p}$. Firm i's payoff for any of those price paths is

$$\pi\left(p_i^t,\mathbf{p}_{-i}^t\right) + \sum_{\tau=t+1}^{\infty}\delta^{\tau-t}\pi\left(\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\},\mathbf{p}_{-i}^\tau\right).$$

Since $p_i^t \leq \max\left\{\mathbf{p}_{-i}^{t-1}\right\}$ and $\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\} \leq p_j^\tau\ \forall j \neq i,\ \forall \tau \geq t+1$, it follows from firm i's profit being increasing in the other firms' prices that

$$\begin{aligned}&\pi\left(p_i^t,\mathbf{p}_{-i}^t\right) + \sum_{\tau=t+1}^{\infty}\delta^{\tau-t}\pi\left(\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\},\mathbf{p}_{-i}^\tau\right) \\ \geq\ &\pi\left(p_i^t,...,p_i^t\right) + \sum_{\tau=t+1}^{\infty}\delta^{\tau-t}\pi\left(\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\},...,\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\}\right).\end{aligned} \tag{14}$$

Next note that $p_i^t \leq \max\left\{\mathbf{p}_{-i}^{t-1}\right\} \leq \widetilde{p} \leq p^M$ implies

$$\pi\left(\max\left\{\mathbf{p}_{-i}^{t-1}\right\},...,\max\left\{\mathbf{p}_{-i}^{t-1}\right\}\right) \geq \pi\left(p_i^t,...,p_i^t\right).$$

Therefore, the RHS of (14) is at least

$$\pi\left(p_i^t,...,p_i^t\right) + \sum_{\tau=t+1}^{\infty}\delta^{\tau-t}\pi\left(p_i^t,...,p_i^t\right) = \frac{\pi\left(p_i^t,...,p_i^t\right)}{1-\delta_i},$$

which means (14) implies

$$\pi\left(p_i^t,\mathbf{p}_{-i}^t\right) + \sum_{\tau=t+1}^{\infty}\delta^{\tau-t}\pi\left(\max\left\{\mathbf{p}_{-i}^{\tau-1}\right\},\mathbf{p}_{-i}^\tau\right) \geq \frac{\pi\left(p_i^t,...,p_i^t\right)}{1-\delta_i}. \tag{15}$$

We have then constructed a strategy for firm i that, for any price path assigned positive probability, yields a payoff of at least the RHS of (15). It then follows from sequential rationality that $U_i\left(\widehat{s}_i,\mu^i\left(\cdot\,|h^t\right)\right)$ must also exceed the RHS of (15). Given p_i^t was set equal to $\min\left\{\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right\}$, the lemma is proved. ■

Proof of Lemma 2. Suppose h^t is such that

$$p_j^\tau < \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\widetilde{p}\right\}\ \text{for some } j \text{ and some } \tau \leq t-1.$$

A PMP-compatible strategy for firm i has a firm price at p^N in the current and all future periods. Thus, if firm i's beliefs are PMP-consistent (that is, they have support in S^{PMP}) then pricing at p^N is clearly optimal as firm i believes all other firms will price at p^N forever. Hence, a PMP-compatible strategy is uniquely optimal for firm i for those histories.

For the remainder of the proof, consider h^t such that

$$p_j^\tau \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\widetilde{p}\right\}\ \forall j,\forall\tau\leq t-1.$$

To prove this lemma, it'll be shown that, for any strategy for firm i that is not in S^{PMP}, there exists a strategy in S^{PMP} which yields a strictly higher payoff and, therefore, sequential rationality implies the use of a PMP-compatible strategy. Thus, as long as a firm believes the other firms are using PMP-compatible strategies then a PMP-compatible strategy is optimal.

Given $p_j^\tau \geq \min\left\{\max\left\{p_1^{\tau-1},...,p_n^{\tau-1}\right\},\widetilde{p}\right\}\ \forall j,\forall\tau\leq t-1$, firm i's strategy can violate the PMP property (and thus not lie in S^{PMP}) either by pricing above $\widetilde{p}$ or below $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$. Let us begin by considering a PMP-incompatible strategy that has firm i price at $p' > \widetilde{p}$. When its rivals price at $\mathbf{p}_{-i}^t$, a PMP-compatible strategy that has firm i price at $\widetilde{p}$ is more profitable than pricing at p' iff:

$$\pi\left(\widetilde{p},\mathbf{p}_{-i}^t\right)+\left(\frac{\delta}{1-\delta}\right)\pi\left(\widetilde{p},...,\widetilde{p}\right) > \pi\left(p',\mathbf{p}_{-i}^t\right)+\left(\frac{\delta}{1-\delta}\right)\pi\left(\widetilde{p},...,\widetilde{p}\right), \tag{16}$$

where recall that the other firms will only follow price as high as $\widetilde{p}$. (16) holds iff

$$\pi\left(\widetilde{p},\mathbf{p}_{-i}^t\right) > \pi\left(p',\mathbf{p}_{-i}^t\right). \tag{17}$$

Given that $\mathbf{p}_{-i}^t \leq (\widetilde{p},...,\widetilde{p})$ and $p^N < \widetilde{p}$ then $\psi\left(\mathbf{p}_{-i}^t\right) \leq \psi\left(\widetilde{p},...,\widetilde{p}\right) < \widetilde{p}$. By the strict concavity of π in a firm's own price and that $\psi\left(\mathbf{p}_{-i}^t\right) < \widetilde{p} < p'$, (17) is true.

Next consider a PMP-incompatible strategy that has firm i price at $p'' < \max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$. Let us show that a PMP-compatible strategy that has firm i price at $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$ is more profitable than pricing at p'' for any $\mathbf{p}_{-i}^t \in \left[\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\widetilde{p}\right]^{n-1}$; that is, for any prices for the other firms assigned positive probability given PMP-consistent beliefs.[26] A sufficient condition for the preceding claim to be true is:

$$\begin{aligned}&\pi\left(\max\left\{p_1^{t-1},...,p_n^{t-1}\right\},\mathbf{p}_{-i}^t\right)+\left(\frac{\delta}{1-\delta}\right)\pi\left(\max\left\{\mathbf{p}_{-i}^t\right\},...,\max\left\{\mathbf{p}_{-i}^t\right\}\right) \\ > \ &\pi\left(p'',\mathbf{p}_{-i}^t\right)+\left(\frac{\delta}{1-\delta}\right)\pi^N,\end{aligned} \tag{18}$$

where the LHS of (18) is a lower bound on the payoff from pricing at $\max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$ and the RHS is the payoff from pricing at p''. In examining the LHS, note that $p_i^t = \max\left\{p_1^{t-1},...,p_n^{t-1}\right\}$ and that the other firms' strategies are PMP-compatible imply

$$\max\left\{p_1^t,...,p_n^t\right\} = \max\left\{\mathbf{p}_{-i}^t\right\}.$$

[26] Actually, it is shown to be only weakly as profitable when $\mathbf{p}_{-i}^t = (\widetilde{p},...,\widetilde{p})$.

Using Lemma 6,

$$\left(\frac{\delta}{1-\delta}\right)\pi\left(\max\left\{\mathbf{p}_{-i}^{t}\right\},...,\max\left\{\mathbf{p}_{-i}^{t}\right\}\right)$$

is a lower bound on the future payoff, which gives us the LHS of (18). When

$$\mathbf{p}_{-i}^{t}=\left(\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\},...,\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\right), \tag{19}$$

(18) is

$$\begin{aligned}
&\pi\left(\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\},...,\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\right) \\
&+\left(\frac{\delta}{1-\delta}\right)\pi\left(\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\},...,\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\right) \\
>\ &\pi\left(p'',\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\},...,\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\right)+\left(\frac{\delta}{1-\delta}\right)\pi^{N}.
\end{aligned} \tag{20}$$

Note that $\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\in\left\{p^{N},...,\widetilde{p}\right\}$ where $\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\geq p^{N}$ follows from the assumption that $\left(p_{1}^{0},...,p_{n}^{0}\right)=\left(p^{N},...,p^{N}\right)$. (20) is then true for all $p''<\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}$ as it is the equilibrium condition for a grim trigger strategy with collusive price $\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}$.[27] Thus, (18) holds for (19).

To complete the proof, it will be shown that the LHS of (18) is increasing in $\mathbf{p}_{-i}^{t}$ at a faster rate than the RHS in which case (18) holds for all

$$\mathbf{p}_{-i}^{t}\geq\left(\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\},...,\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}\right).$$

The derivative with respect to p_{j}^{t}, $j\neq i$, of the LHS of (18) is

$$\frac{\partial\pi\left(\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\},\mathbf{p}_{-i}^{t}\right)}{\partial p_{j}}+\left(\frac{\delta}{1-\delta}\right)\frac{\partial\max\left\{\mathbf{p}_{-i}^{t}\right\}}{\partial p_{j}}\sum_{k=1}^{n}\frac{\partial\pi\left(\max\left\{\mathbf{p}_{-i}^{t}\right\},...,\max\left\{\mathbf{p}_{-i}^{t}\right\}\right)}{\partial p_{k}}, \tag{21}$$

and of the RHS of (18) is

$$\frac{\partial\pi\left(p'',\mathbf{p}_{-i}^{t}\right)}{\partial p_{j}}. \tag{22}$$

(21) exceeds (22) because the second term in (21) is non-negative, given that $\mathbf{p}_{-i}^{t}\leq\left(p^{M},...,p^{M}\right)$, and the first term of (21) exceeds (22) because $\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}>p''$ and $\frac{\partial^{2}\pi}{\partial p_{i}\partial p_{j}}>0$, $j\neq i$.
■

Proof of Theorem 3. Given each firm has PMP-consistent beliefs then, by Lemma 2, sequential rationality implies each firm's strategy is PMP-compatible. As firms' strategies are PMP-compatible, it immediately follows that each firm's price is weakly increasing. Given a finite price set and the boundedness and monotonicity of prices, prices converge in finite time. Hence, there exists $\widehat{p}\in\left\{p^{N},...,\widetilde{p}\right\}$ and finite T such that $p_{1}^{t}=\cdots=p_{n}^{t}=\widehat{p}$ for all $t\geq T$. We then need to show that $\widehat{p}\leq p^{*}+\varepsilon$. If $p^{*}+\varepsilon\geq\widetilde{p}$ then, given that PMP-compatible strategies

[27] Recall that $\widetilde{p}$ is the highest price consistent with the grim trigger strategy being an equilibrium. Note that (20) holds with equality when $\max\left\{p_{1}^{t-1},...,p_{n}^{t-1}\right\}=\widetilde{p}$ and $p''=\psi\left(\widetilde{p},...,\widetilde{p}\right)$ and otherwise is a strict inequality.

do not have firms pricing above $\widetilde{p}$, it is immediate that $\widehat{p} \leq p^* + \varepsilon$. For the remainder of the proof, suppose $p^* + \varepsilon < \widetilde{p}$.

Let me provide an overview of the proof. First it is shown that if a firm is sequentially rational and has PMP-consistent beliefs then it will not price at $\widetilde{p}$. The reason is as follows. A firm would find it optimal to price above $p^* + \varepsilon$ only if it induced at least one of its rivals to enact further price increases (and not just match the firm's price); that is the defining property of p^*. However, if a firm believes its rivals will not price above $\widetilde{p}$ (which follows from having PMP-consistent beliefs and A2) then it is not optimal for a firm to raise price to $\widetilde{p}$ because it can only expect its rivals to match a price of $\widetilde{p}$, not exceed it. This argument works as well to show that each of the other firms will not raise price to $\widetilde{p}$. Hence, there is an upper bound on price of $\widetilde{p} - \varepsilon$. The proof is completed by induction using the mutual beliefs in A1. If a firm believes its rivals will not price above p' then it can be shown that a firm will find it optimal not to price above $p' - \varepsilon$, as long as $p' \geq p^* + 2\varepsilon$ which implies that an upper bound on price is $p^* + \varepsilon$, which is the desired result.

Before executing that proof strategy, the following lemma will be needed. Recall that $\overline{\phi}(\mathbf{p}_{-i})$ is the best reply correspondence for the following objective function:

$$W(p_i, \mathbf{p}_{-i}) \equiv \pi(p_i, \mathbf{p}_{-i}) + \left(\frac{\delta}{1-\delta}\right) \pi(p_i, ..., p_i).$$

Given its rivals price at $\mathbf{p}_{-i}$ in the current period, $W(p_i, \mathbf{p}_{-i})$ is firm i's payoff from pricing at p_i *if it believed* that all firms would match that price in all ensuing periods. Define $\overline{\phi}^U(\mathbf{p}_{-i})$ to be the maximal element of $\overline{\phi}(\mathbf{p}_{-i})$. Lemma 7 shows that the objective function is decreasing in p_i for all $p_i > \overline{\phi}^U(\mathbf{p}_{-i})$. This follows naturally from the strict concavity of $W(p_i, \mathbf{p}_{-i})$ but requires some care to show given the price space is finite.

Lemma 7 *If $p'' > p' \geq \overline{\phi}^U(\mathbf{p}_{-i})$ then*

$$\pi(p', \mathbf{p}_{-i}) + \left(\frac{\delta}{1-\delta}\right) \pi(p', ..., p') > \pi(p'', \mathbf{p}_{-i}) + \left(\frac{\delta}{1-\delta}\right) \pi(p'', ..., p'').$$

■

Proof of Lemma 7. To begin, let us show that $\overline{\phi}^U(\mathbf{p}_{-i})$ is non-decreasing in $\mathbf{p}_{-i}$. By the definition of $\overline{\phi}^U(\mathbf{p}_{-i})$, we know that:

$$W\left(\overline{\phi}^U(\mathbf{p}'_{-i}), \mathbf{p}'_{-i}\right) - W(p_i, \mathbf{p}'_{-i}) > 0, \ \forall p_i \in A \equiv \left\{p \in \Omega : p > \overline{\phi}^U(\mathbf{p}'_{-i})\right\}.$$

Since $\frac{\partial^2 W(p_i, \mathbf{p}_{-i})}{\partial p_i \partial p_j} = \frac{\partial^2 \pi(p_i, \mathbf{p}_{-i})}{\partial p_i \partial p_j} > 0$ then $p_i > \overline{\phi}^U(\mathbf{p}'_{-i})$ and $\mathbf{p}''_{-i} \leq \mathbf{p}'_{-i}$ imply

$$W(p_i, \mathbf{p}'_{-i}) - W(p_i, \mathbf{p}''_{-i}) \geq W\left(\overline{\phi}^U(\mathbf{p}'_{-i}), \mathbf{p}'_{-i}\right) - W\left(\overline{\phi}^U(\mathbf{p}'_{-i}), \mathbf{p}''_{-i}\right), \ \forall p_i \in A,$$

and, re-arranging, we have

$$W\left(\phi^U(\mathbf{p}'_{-i}), \mathbf{p}''_{-i}\right) - W(p_i, \mathbf{p}''_{-i}) \geq W\left(\overline{\phi}^U(\mathbf{p}'_{-i}), \mathbf{p}'_{-i}\right) - W(p_i, \mathbf{p}'_{-i}), \ \forall p_i \in A.$$

Therefore,

$$W\left(\overline{\phi}^U\left(\mathbf{p}'_{-i}\right), \mathbf{p}''_{-i}\right) - W\left(p_i, \mathbf{p}''_{-i}\right) > 0,\ \forall p_i \in A,$$

which, along with $\mathbf{p}''_{-i} \leq \mathbf{p}'_{-i}$ and the strict concavity of W in own price, imply $\overline{\phi}^U\left(\mathbf{p}''_{-i}\right) \leq \overline{\phi}^U\left(\mathbf{p}'_{-i}\right)$. Hence, $\overline{\phi}^U\left(\mathbf{p}_{-i}\right)$ is non-decreasing.

By (4), $p' > p^* + \varepsilon$ implies $\overline{\phi}^U\left(p', ..., p'\right) \leq p' - \varepsilon$. Given $\overline{\phi}^U\left(\mathbf{p}_{-i}\right)$ is non-decreasing in $\mathbf{p}_{-i}$, it follows:

$$\text{if } \mathbf{p}_{-i} \leq \left(p', ..., p'\right) \text{ and } p' > p^* + \varepsilon \text{ then } \overline{\phi}^U\left(\mathbf{p}_{-i}\right) \leq p' - \varepsilon. \tag{23}$$

From the strict concavity of W in own price, we have:

$$\text{if } p'' > p' \geq \overline{\phi}^U\left(\mathbf{p}_{-i}\right) \text{ then}$$

$$\pi\left(p', \mathbf{p}_{-i}\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p', ..., p'\right) > \pi\left(p'', \mathbf{p}_{-i}\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p'', ..., p''\right).$$

■

Let us first show that if firm i has PMP-consistent beliefs then it strictly prefers a price of $\widetilde{p} - \varepsilon$ to $\widetilde{p}$. Given PMP-consistent beliefs, firm i's beliefs on $\mathbf{p}^t_{-i}$ have support $\left[\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, \widetilde{p}\right]^{n-1}$. For any $\mathbf{p}^t_{-i} \in \left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, \widetilde{p}\right\}^{n-1}$, Lemma 6 implies that a lower bound on its payoff from $p_i^t = \widetilde{p} - \varepsilon$ is

$$\pi\left(\widetilde{p} - \varepsilon, \mathbf{p}^t_{-i}\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(\widetilde{p} - \varepsilon, ..., \widetilde{p} - \varepsilon\right). \tag{24}$$

For any prices for the other firms for period t that are assigned positive probability - $\mathbf{p}^t_{-i} \in \left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, \widetilde{p}\right\}^{n-1}$ - firm i attributes a payoff from $p_i^t = \widetilde{p}$ equal to

$$\pi\left(\widetilde{p}, \mathbf{p}^t_{-i}\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(\widetilde{p}, ..., \widetilde{p}\right). \tag{25}$$

Given that $\widetilde{p} > p^* + \varepsilon$ then $\overline{\phi}^U\left(\mathbf{p}^t_{-i}\right) \leq \widetilde{p} - \varepsilon$ for all $\mathbf{p}^t_{-i} \leq \left(\widetilde{p}, ..., \widetilde{p}\right)$ by (23). It then follows from Lemma 7 that (24) strictly exceeds (25). Therefore, for any beliefs of firm i with support $\left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, \widetilde{p}\right\}^{n-1}$, a price of $\widetilde{p} - \varepsilon$ is strictly preferred to $\widetilde{p}$. In sum, if firm i is sequentially rational and has PMP-consistent beliefs then its price will not exceed $\widetilde{p} - \varepsilon$.

Given the level 2 beliefs in Assumption A1, firm i believes firm $j\ (\neq i)$ is sequentially rational and that firm j has PMP-consistent beliefs for firm k, $\forall k \neq j$. Hence, applying the preceding argument to firm j, firm i believes firm j will not price above $\widetilde{p} - \varepsilon$. Firm i's beliefs on $\mathbf{p}^t_{-i}$ then have support $\left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, \widetilde{p} - \varepsilon\right\}^{n-1}$. If $\widetilde{p} - \varepsilon > p^* + \varepsilon$ then, by (23), $\overline{\phi}^U\left(\mathbf{p}^t_{-i}\right) \leq \widetilde{p} - 2\varepsilon\ \forall \mathbf{p}^t_{-i} \leq \left(\widetilde{p} - \varepsilon, ..., \widetilde{p} - \varepsilon\right)$.[28] By the same logic as above, a lower bound on firm i's payoff from $p_i^t = \widetilde{p} - 2\varepsilon$ is

$$\pi\left(\widetilde{p} - 2\varepsilon, \mathbf{p}^t_{-i}\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(\widetilde{p} - 2\varepsilon, ..., \widetilde{p} - 2\varepsilon\right), \tag{26}$$

[28] If instead $\widetilde{p} - \varepsilon \leq p^* + \varepsilon$ then, given that it has already been shown $\widetilde{p} - \varepsilon$ is an upper bound on the limit price, it follows that $p^* + \varepsilon$ is an upper bound and we're done.

while its payoff from $p_i^t = \widetilde{p} - \varepsilon$ is

$$\pi\left(\widetilde{p} - \varepsilon, \mathbf{p}_{-i}^t\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(\widetilde{p} - \varepsilon, ..., \widetilde{p} - \varepsilon\right). \tag{27}$$

With (27), we used the fact that firms will not price above $\widetilde{p} - \varepsilon$, which was derived in the first step. Again using Lemma 7, it is concluded that (26) strictly exceeds (27). Therefore, for any beliefs of firm i over $\mathbf{p}_{-i}^t$ with support $\left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, ..., \widetilde{p} - \varepsilon\right\}^{n-1}$, a price of $\widetilde{p} - 2\varepsilon$ is strictly preferred to $\widetilde{p} - \varepsilon$. It follows that if a firm is sequentially rational and has PMP-consistent beliefs, and believes other firms are sequentially rational and have PMP-consistent beliefs then its optimal price does not exceed $\widetilde{p} - 2\varepsilon$. Hence, all firms will not price above $\widetilde{p} - 2\varepsilon$.

The proof is completed by induction. Suppose we have shown that firm i believes that the other firms will not price above p' so firm i's beliefs on $\mathbf{p}_{-i}^t$ have support

$$\left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, ..., p'\right\}^{n-1}.$$

(That we can get to the point that a firm has those beliefs is because of the hierarchy of beliefs in A1.) If $p' > p^* + \varepsilon$ then $\overline{\phi}^U\left(\mathbf{p}_{-i}^t\right) \leq p' - \varepsilon$ for all $\mathbf{p}_{-i}^t \leq (p', ..., p')$. A lower bound on firm i's payoff from $p_i^t = p' - \varepsilon$ is

$$\pi\left(p' - \varepsilon, \mathbf{p}_{-i}^t\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p' - \varepsilon, ..., p' - \varepsilon\right), \tag{28}$$

while its payoff from $p_i^t = p'$ is

$$\pi\left(p', \mathbf{p}_{-i}^t\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p', ..., p'\right), \tag{29}$$

since all firms have an upper bound of p' on their prices. Using Lemma 7, it is concluded that (28) strictly exceeds (29). Therefore, for any beliefs of firm i over $\mathbf{p}_{-i}^t$ with support $\left\{\max\left\{p_1^{t-1}, ..., p_n^{t-1}\right\}, p'\right\}^{n-1}$, a price of $p' - \varepsilon$ is strictly preferred to p'. It follows that firms' prices are bounded above by $p' - \varepsilon$. The preceding argument is correct as long as $p' > p^* + \varepsilon$; therefore, price is bounded above by $p^* + \varepsilon$. ■

Proof of Theorem 4. p^* is defined by

$$\frac{\partial W\left(p^*, ..., p^*\right)}{\partial p_i} = \frac{\partial \pi\left(p^*, ..., p^*\right)}{\partial p_i} + \left(\frac{\delta}{1-\delta}\right)\sum_{j=1}^{n}\frac{\partial \pi\left(p^*, ..., p^*\right)}{\partial p_j} = 0$$

or

$$\frac{\partial \pi\left(p^*, ..., p^*\right)}{\partial p_i} + \delta\sum_{j\neq i}^{n}\frac{\partial \pi\left(p^*, ..., p^*\right)}{\partial p_j} = 0.$$

For all $p \geq p^M$,

$$\frac{\partial \pi\left(p, ..., p\right)}{\partial p_i} < 0 \text{ and } \sum_{j=1}^{n}\frac{\partial \pi\left(p, ..., p\right)}{\partial p_j} \leq 0,$$

which implies $p^* < p^M$ by the strict concavity of W in own price. To show $p^* > p^N$, note that $\phi(p,...,p) > \psi(p,...,p)$ and $\psi(p,...,p) \geq p$ $\forall p \leq p^N$ implies $\phi(p,...,p) > p$ $\forall p \leq p^N$. Since $\phi(p,...,p) \gtreqless p$ as $p \lesseqgtr p^*$ then $p^* > p^N$. We have then shown $p^* \in \left(p^N, p^M\right)$.

If $\widetilde{p} = p^M$ then $p^* \in \left(p^N, \widetilde{p}\right)$ and we are done. From hereon, suppose $\widetilde{p} < p^M$ in which case the incentive compatibility constraint (ICC) binds:

$$\frac{\pi(\widetilde{p},...,\widetilde{p})}{1-\delta} = \pi(\psi(\widetilde{p},...,\widetilde{p}), \widetilde{p},...,\widetilde{p}) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N,...,p^N\right) \ . \tag{30}$$

As $p \in \left(p^N, p^*\right]$ implies $\psi(p,...,p) < p \leq \phi(p,...,p)$ then, by the strict concavity of W,

$$W(p,...,p) > W(\psi(p), p,...,p),$$

which is equivalently expressed as

$$\frac{\pi(p,...,p)}{1-\delta} > \pi(\psi(p), p,...,p) + \left(\frac{\delta}{1-\delta}\right)\pi(\psi(p),...,\psi(p)). \tag{31}$$

$p > p^N$ implies $\psi(p,...,p) \in \left(p^N, p\right)$. Next note $\psi(p,...,p) < p \leq p^* < p^M$ implies $\psi(p,...,p) < p^M$. It then follows from $\psi(p,...,p) \in \left(p^N, p^M\right)$ that $\pi(\psi(p),...,\psi(p)) > \pi\left(p^N,...,p^N\right)$. Using this property in (31), we have

$$\frac{\pi(p,...,p)}{1-\delta} > \pi(\psi(p), p,...,p) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N,...,p^N\right), \ \forall p \in \left(p^N, p^*\right]. \tag{32}$$

Therefore, $p \in \left(p^N, p^*\right]$ is sustainable with the grim trigger strategy. Given (30) - where the ICC binds for $p = \widetilde{p}$ - and evaluating (32) at $p = p^*$ - so the ICC does not bind - it follows from (2) that $\widetilde{p} > p^*$. ■

Proof of Theorem 5. To draw on the result of Kalai and Lehrer (1993), I need to introduce some notation and definitions. If s is a strategy profile and $Q \subset \Lambda$ is a collection of infinite price paths for the game then define $\mu_s(Q)$ to be the probability measure on Q induced by s. By A1, a strategy profile is an element of $(X^\infty)^n \subseteq \left(S^{PMP}\right)^n$ and, by Lemma 3, the price path converges in finite time.

Suppose, contrary to the statement of Theorem 5, the true strategy profile, denoted $\widetilde{s}$, has price converge to $p' < p^* - \varepsilon$. It is then the case that there exists T' such that $\mu_{\widetilde{s}}(Q_{\widehat{p}}) = 1$ for all infinite price paths $Q_{\widehat{p}}$ with the property: $p_1^t = \cdots = p_n^t = p'$ $\forall t \geq T'$. By Theorem 1 of Kalai and Lehrer (1993), $\exists T(\eta)$ such that firm i's beliefs on the infinite price paths, denoted μ^i, have the property that $\mu_{\widetilde{s}}$ is *η-close* to μ^i for price paths starting at t, $\forall t \geq T(\eta)$.[29] Given that $\mu_{\widetilde{s}}(Q_{\widehat{p}}) = 1$ $\forall t \geq \max\{T', T(\eta)\}$ then μ^i must assign probability of at least $1-\eta$ to the future price path having the property: $p_1^t = \cdots = p_n^t = p'$ $\forall t \geq \max\{T', T(\eta)\}$.[30]

[29] Definition 1 (Kalai and Lehrer, 1993): Let $\eta > 0$ and let μ and $\widetilde{\mu}$ be two probability measures defined on the same space. μ is said to be *η-close* to $\widetilde{\mu}$ if there is a measurable set Q satisfying: (i) $\mu(Q)$ and $\widetilde{\mu}(Q)$ are greater than $1-\eta$; and (ii) for every measurable set $X \subseteq Q$: $(1-\eta)\widetilde{\mu}(X) \leq \mu(X) \leq (1+\eta)\widetilde{\mu}(X)$.

[30] I am only using property (i) in Definition 1 of Kalai and Lehrer (1993); see footnote 30

.

Given these beliefs, let us evaluate optimal play for firm i. For $t \geq \max\{T', T(\eta)\}$, firm i's expected payoff from acting according to its strategy and pricing at p' has an upper bound of

$$(1-\eta)\left(\frac{\pi(p',...,p')}{1-\delta}\right)+\eta\left[\pi(p',p^m...,p^m)+\left(\frac{\delta}{1-\delta}\right)\pi(p^m,...,p^m)\right]. \tag{33}$$

Probability $1-\eta$ is assigned to the "true" future price path in which price is always p', and the remaining probability is assigned to all rivals raising price to p^m in the current period (in order to provide an upper bound on the payoff). Now consider firm i deviating from its strategy of pricing at p' by pricing instead at $\phi(p',...,p')$. Firm i's expected payoff has a lower bound of

$$(1-\eta)\left[\pi(\phi(p',...,p'),p',...,p')+\left(\frac{\delta}{1-\delta}\right)\pi(\phi(p',...,p'),...,\phi(p',...,p'))\right]. \tag{34}$$

Probability $1-\eta$ is assigned to other firms pricing at p' in the current period and only matching firm i's price of $\phi(p',...,p')$ thereafter, which is the worst case scenario given firm i has PMP-consistent beliefs. A zero payoff is assigned to the remaining probability η to give us a lower bound. We want to show that pricing at p' is not optimal - and thus inconsistent with sequential rationality - which is the case if (34) exceeds (33):

$$\begin{aligned}&(1-\eta)\left[\pi(\phi(p',...,p'),p',...,p')+\left(\frac{\delta}{1-\delta}\right)\pi(\phi(p',...,p'),...,\phi(p',...,p'))\right]\\ > \; &(1-\eta)\left(\frac{\pi(p',...,p')}{1-\delta}\right)+\eta\left[\pi(p',p^m...,p^m)+\left(\frac{\delta}{1-\delta}\right)\pi(p^m,...,p^m)\right]\end{aligned}$$

or

$$\begin{aligned}&(1-\eta)\left[\pi(\phi(p',...,p'),p',...,p')+\left(\frac{\delta}{1-\delta}\right)\pi(\phi(p',...,p'),...,\phi(p',...,p'))-\frac{\pi(p',...,p')}{1-\delta}\right]\\ > \; &\eta\left[\pi(p',p^m...,p^m)+\left(\frac{\delta}{1-\delta}\right)\pi(p^m,...,p^m)\right].\end{aligned} \tag{35}$$

Given that $p' < p^* - \varepsilon$ then the first bracketed term in (35) is strictly positive. Hence, for η sufficiently small, (35) holds. Thus, if t is sufficiently great, firm i's beliefs are such that pricing at p' in period t is non-optimal. Therefore, I conclude that price cannot converge to a value below $p^* - \varepsilon$. Given that Theorem 3 showed that an upper bound on convergence is $p^* + \varepsilon$, it is concluded that prices converges to some value in $\{p^* - \varepsilon, p^*, p^* + \varepsilon\}$. ■

7.2 Appendix B: Competitive Outcome is Salient After a Departure from Price Matching

Suppose a firm raises price and, in the subsequent period, some other firm fails to match (or exceed) that price and thus violates the PMP property. In response to such an event, firms share the belief that all firms will return to static Nash equilibrium prices; competition replaces collusion. In specifying how firms respond to this incongruity between beliefs and behavior - they expected a price increase to be at least matched and it was not - let me provide a justification by drawing on Lewis (1969) to argue that the competitive solution is *salient*.

Lewis (1969) defines a salient outcome as "one that stands out from the rest by its uniqueness in some conspicuous respect"[31] and that precedence is one source of saliency: "We may tend to repeat the action that succeeded before if we have no strong reason to do otherwise."[32] Cubitt and Sugden (2003) stress the latter qualifier and note that "precedent allows the individual to make inductive inferences in which she has *some* confidence, but which are overridden whenever deductive analysis points clearly in a different direction."[33] With this perspective in mind, the movement from competition to collusion can be seen as a shift from inductive to deductive reasoning. Firms have been competing and, by induction, they would expect to continue to do so. However, through some other coordinating event, firms supplant inductive inferences with deductive reasoning so that a common expectation of competition is replaced with a common expectation of price matching. With this as a backdrop, my claim is that a subsequent departure in behavior from price matching implies a breakdown in the efficacy of deductive reasoning, in response to which firms revert to the original inductive analysis and therefore the competitive solution. Here I am appealing to the view that firms will "tend to pick the salient as a last resort."[34] The saliency of the competitive solution emanates from it being the most recent outcome (prior to the current episode of collusion) that was common knowledge to firms.

There are two implicit assumptions in the preceding argument that warrant discussion. First, the saliency of the competitive solution relies on it prevailing prior to this episode of tacit collusion. However, that is not essential. If some other behavior described the pre-collusion setting then that behavior can be assumed instead. What *is* critical is that how firms respond to the departure from price matching is common knowledge and the associated continuation payoff is lower than if firms had abided by the PMP property. A second assumption, which figures prominently in discussions of saliency (such as in Lewis, 1969), is that the current post-collusion situation is sufficiently similar to the pre-collusion situation so that induction on the latter is compelling. It is well-recognized that[35]

> no two interactions are exactly alike. Any two real-world interactions will differ in matters of detail, quite apart from the inescapable fact that "previous" and "current" interactions occur at different points in time. Thus, the idea of "repeating what was done in previous instances of the game" is not well-defined. Precedent has to depend on analogy: to follow precedent in the present instance is to behave in a way that is *analogous with* behaviour in past instances. ... Inductive inference is possible only because a very small subset of the set of possible patterns is privileged.

The post-collusion scenario most notably differs from the pre-collusion scenario in that the former was preceded by an episode of collusion, while the latter was (probably) not. Though

[31] Lewis, (1969), p. 35.
[32] Lewis, (1969), p. 37.
[33] Cubitt and Sugden (2003), p. 196. Also see Sugden (2011).
[34] Lewis, (1969), p. 35.
[35] Cubitt and Sugden (2003), pp. 196-7.

this difference could disrupt the saliency of the pre-collusion outcome when it comes to responding to a departure from the PMP property, it is reasonable for its saliency to remain intact which is the presumption made here.

7.3 Appendix C: $\left(S^L, S^F\right)$ is a Subgame Perfect Equilibrium

In deriving sufficient conditions for $\left(S^L, S^F\right)$ to be a subgame perfect equilibrium, let us first consider S^L and have ρ denote the lagged maximum price. If $\rho = p^N$ then $S^L\left(p^N\right) = p'$ which is optimal iff p' is at least as profitable as p^N,

$$\begin{aligned} & \pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \\ \geq\ & \pi\left(p^N, p^N\right) + \delta\pi\left(p', p^N\right) + \delta^2\pi\left(p^M, p'\right) + \left(\frac{\delta^3}{1-\delta}\right)\pi\left(p^M, p^M\right) \end{aligned} \tag{36}$$

and at least as profitable as p^M,

$$\begin{aligned} & \pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \\ \geq\ & \pi\left(p^M, p^N\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right). \end{aligned} \tag{37}$$

(36) and (37) can be simplified to:

$$\pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) + \delta^2\pi\left(p^M, p^M\right) \geq \pi\left(p^N, p^N\right) + \delta\pi\left(p', p^N\right) + \delta^2\pi\left(p^M, p'\right) \tag{38}$$

$$\pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) \geq \pi\left(p^M, p^N\right) + \delta\pi\left(p^M, p^M\right) \tag{39}$$

If $\delta \simeq 1$ then (38) is true, and (39) is true when:

$$\pi\left(p', p^N\right) + \pi\left(p^M, p'\right) > \pi\left(p^M, p^N\right) + \pi\left(p^M, p^M\right) \tag{40}$$

Now suppose $\rho = p'$. $S^L\left(p'\right) = p^M$ is optimal iff p^M is at least as profitable as p^N,

$$\pi\left(p^M, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p^N, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N, p^N\right), \tag{41}$$

and at least as profitable as p',

$$\begin{aligned} \pi\left(p^M, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \ \geq\ & \pi\left(p', p'\right) + \delta\pi\left(p^M, p'\right) \\ & + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right). \end{aligned} \tag{42}$$

If $\delta \simeq 1$ then (41) and (42) hold. Finally, if $\rho = p^M$ then $S^L\left(p^M\right) = p^M$ is optimal iff:

$$\left(\frac{1}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \max\left\{\pi\left(p^N, p^M\right), \pi\left(p', p^M\right)\right\} + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N, p^N\right), \tag{43}$$

which holds if $\delta \simeq 1$. In sum, S^L is subgame perfect if $\delta \simeq 1$ and (40) holds.

Next, let us turn to S^F. If $\rho = p^N$ then $S^F\left(p^N\right) = p^N$ is optimal iff p^N is at least as profitable as p',

$$\pi\left(p^N, p'\right) + \delta\pi\left(p', p^M\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p', p'\right) + \delta\pi\left(p', p^M\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right), \tag{44}$$

and is at least as profitable as p^M,

$$\pi\left(p^N, p'\right) + \delta\pi\left(p', p^M\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p^M, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right). \tag{45}$$

Both conditions hold for all δ.[36] If $\rho = p'$ then $S^F(p') = p'$ is optimal iff p' is at least as profitable as p^N,

$$\pi\left(p', p^M\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p^N, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N, p^N\right), \tag{46}$$

and is at least as profitable as p^M,

$$\pi\left(p', p^M\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \left(\frac{1}{1-\delta}\right)\pi\left(p^M, p^M\right). \tag{47}$$

(46) holds for $\delta \simeq 1$, and (47) holds for all δ. Finally, if $\rho = p^M$ then $S^F\left(p^M\right) = p^M$ is optimal iff (43) is true. In sum, S^F is subgame perfect if $\delta \simeq 1$.

To evaluate when (40) holds, consider:

$$\begin{aligned}
\pi\left(p', p^N\right) + \pi\left(p^M, p'\right) &> \pi\left(p^M, p^N\right) + \pi\left(p^M, p^M\right) \Leftrightarrow \\
\pi\left(\frac{p^M + p^N}{2}, p^N\right) - \pi\left(p^M, p^N\right) &> \pi\left(p^M, p^M\right) - \pi\left(p^M, \frac{p^M + p^N}{2}\right) \Leftrightarrow \\
-\int_{\frac{p^M+p^N}{2}}^{p^M}\left(\frac{\partial\pi\left(p, p^N\right)}{\partial p_1}\right)dp_1 &> \int_{\frac{p^M+p^N}{2}}^{p^M}\left(\frac{\partial\pi\left(p^M, p\right)}{\partial p_2}\right)dp_2.
\end{aligned} \tag{48}$$

Assuming linear demand and constant marginal cost,

$$\pi\left(p_i, \mathbf{p}_{-i}\right) = \left(a - bp_i + d\left(\frac{1}{n-1}\right)\sum_{j\neq i} p_j\right)\left(p_i - c\right), \text{where } a > bc > 0, b > d > 0,$$

(48) is

$$\begin{aligned}
-\int_{\frac{p^M+p^N}{2}}^{p^M}\left(a + bc - 2bp_1 + dp^N\right)dp_1 &> \int_{\frac{p^M+p^N}{2}}^{p^M} d\left(p^M - c\right)dp_2 \Leftrightarrow \\
-\left(a + bc + dp^N\right)\left(\frac{p^M - p^N}{2}\right) + b\left[\left(p^M\right)^2 - \left(\frac{p^M + p^N}{2}\right)^2\right] &> d\left(p^M - c\right)\left(\frac{p^M - p^N}{2}\right)
\end{aligned}$$

[36] Note that $\pi\left(p^N, p'\right) > \pi\left(p', p'\right)$ for if that was not the case then p' would be a static Nash equilibrium and thereby violation the assumption that p^N is the unique Nash equilibrium. Similarly, it must be true that $\pi\left(p', p^M\right) > \pi\left(p^M, p^M\right)$.

which, after some manipulations, is equivalent to

$$3bp^M + bp^N > 2a + 2bc + 2dp^N + 2dp^M - 2dc. \tag{49}$$

Substituting

$$p^N = \frac{a+bc}{2b-d},\ p^M = \frac{a+(b-d)\,c}{2\,(b-d)}$$

and again performing some manipulations, (49) is equivalent to

$$[a+(b-d)\,c]\left[(6b-4d)\,(b-d)+d^2\right]+2\,(b-2d)\,(b-d)\,dc > 0. \tag{50}$$

The first term is positive because $b > d$, while the second term is non-negative when $b \geq 2d$. Hence, if products are sufficiently differentiated then (50) is true. When instead $b < 2d$ then (50) holds when $c \simeq 0$. Hence, if cost is sufficiently small then (50) is true

7.4 Appendix D: Completion of Proof Constructing Beliefs Consistent with A1-A4

Suppose it is period $t = t_2'$ and neither firm raised price over periods $1, ..., t_2' - 1$. The posterior beliefs of firm 1 on firm 2's strategy are as specified in (12). As a first step, let us show that $T_1 = T'$ is preferable to $T_1 = T' + 1$ for all $T' \in \{t_2', ..., t_2' + x - 2\}$; that is, it is better to lead in period t than wait and lead in period $t+1$ for all $t \in \{t_2', ..., t_2' + x - 2\}$.

In comparing T' and $T'+1$, first note that they yield the same profit sequence if $T_2 < T'$ as then firm 2 raises price first. Hence, we can focus on the payoffs associated with when $T_2 \geq T'$. Next note that both strategies always yield the same profits prior to T' and the same profit of π^M starting with period $T'+2$, so we need only consider how expected profits differ in periods T' and $T'+1$. With probability $(1/2)\,(1/x)$, firm 2's strategy is T' so it raises price in period T' in which case the period T' profit to firm 1 from strategy T' is π^M and from strategy $T'+1$ is π^F; both strategies yield the same profit starting in period $T'+1$. With probability $(1/2)\,(1/x)$, firm 2's strategy is $T'+1$ in which case the period T' profit to firm 1 from strategy T' is π^L and from strategy $T'+1$ is π^M; both strategies yield the same profit starting in period $T'+1$. And with probability

$$\left(\frac{1}{2}\right)\left(\frac{x-1-(T'+1-t_2')}{x}\right)+\sum_{y=2}^{\infty}\left(\frac{1}{2}\right)^y$$

firm 2's strategy exceeds $T'+1$ in which case firm 1's payoff over periods T' and $T'+1$ from strategy T' is $\pi^L + \delta\pi^M$ and from strategy $T'+1$ is $\pi^N + \delta\pi^L$; and both strategies yield the same profit starting in period $T'+2$. Thus, the difference between the expected payoff from strategy T' and strategy $T'+1$ is

$$\begin{aligned}&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left[\left(\pi^M+\delta\pi^M\right)-\left(\pi^F+\delta\pi^M\right)\right] \qquad (51)\\ &+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left[\left(\pi^L+\delta\pi^M\right)-\left(\pi^N+\delta\pi^M\right)\right]\\ &+\left[\left(\frac{1}{2}\right)\left(\frac{x-1-(T'+1-t_2')}{x}\right)+\sum_{y=2}^{\infty}\left(\frac{1}{2}\right)^y\right]\left[\left(\pi^L+\delta\pi^M\right)-\left(\pi^N+\delta\pi^L\right)\right]\end{aligned}$$

For x sufficiently large, the sign of this expression is the same as the sign of the third term. Hence, (51) is positive if x is sufficiently large and

$$\pi^L + \delta\pi^M > \pi^N + \delta\pi^L. \tag{52}$$

(52) is equivalent to

$$\delta > \frac{\pi^N - \pi^L}{\pi^M - \pi^L}. \tag{53}$$

In sum, if x is sufficiently large and (53) holds then firm 1 prefers strategy T' to strategy $T' + 1$ for all $T' \in \{t'_2, ..., t'_2 + x - 2\}$. Note that this condition is the same as (7).

Thus far, conditions have been derived whereby if $t = t'_2$ then firm 1 prefers to lead than to wait and lead in any period in $\{t'_2 + 1, ..., t'_2 + x - 2\}$. The next step is to show that firm 1 prefers to lead in period t'_2 ($T_1 = t'_2$) than to wait and lead in period $t'_2 + x$ ($T_1 = t'_2 + x$). Note that, as of period t'_2, the posterior probability assigned by firm 1 to $T_2 = t'_2$ is $(1/2)(1/x)$ and to $T_2 = t'_2 + x$ is $(1/2)^2 (1/x)$. The expected payoff from $T_1 = t'_2$ is

$$\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{\pi^M}{1-\delta}\right) + \left[1 - \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\right]\left(\pi^L + \frac{\delta\pi^M}{1-\delta}\right). \tag{54}$$

With probability $(1/2)(1/x)$, firm 2 also raises price in period t'_2 so firm 1's current and future profit is π^M. With probability $1 - (1/2)(1/x)$, firm 2 does not raise price in period t'_2 so firm 1's current profit is π^L and its future profit stream is π^M. The expected payoff from $T_1 = t'_2 + x$ is

$$\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\pi^F + \frac{\delta\pi^M}{1-\delta}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\pi^N + \delta\pi^F + \frac{\delta^2\pi^M}{1-\delta}\right) + \\
&\cdots + \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-2}\pi^N + \delta^{x-1}\pi^F + \frac{\delta^x\pi^M}{1-\delta}\right) \\
&+ \left(\frac{1}{2}\right)^2\left(\frac{1}{x}\right)\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-1}\pi^N + \frac{\delta^x\pi^M}{1-\delta}\right) \\
&+ \left[1 - \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2\left(\frac{1}{x}\right)\right]\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-1}\pi^N + \delta^x\pi^L + \frac{\delta^{x+1}\pi^M}{1-\delta}\right).
\end{aligned}$$

The first term is the probability that $T_2 = t'_2$ multiplied by the payoff in that event, the second term is the probability that $T_2 = t'_2 + 1$ multiplied by the payoff in that event, and so forth; the penultimate term is the probability that $T_2 = t'_2 + x$ multiplied by the payoff in that event, and the final term is the probability that $T_2 > t'_2 + x$ multiplied by the payoff in that event. Collecting common profit terms,

$$\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(1+\delta+\cdots+\delta^{x-1}\right)\pi^F \\
&+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(1+\delta+\cdots+\delta^{x-1}\right)\frac{\delta\pi^M}{1-\delta} \\
&+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left[1+(1+\delta)+\cdots+\left(1+\delta+\cdots+\delta^{x-2}\right)\right]\pi^N \\
&+\left(\frac{1}{4x}\right)\left(\pi^N+\delta\pi^N+\cdots+\delta^{x-1}\pi^N+\frac{\delta^x\pi^M}{1-\delta}\right) \\
&+\left(\frac{2x-1}{4x}\right)\left(\pi^N+\delta\pi^N+\cdots+\delta^{x-1}\pi^N+\delta^x\pi^L+\frac{\delta^{x+1}\pi^M}{1-\delta}\right)
\end{aligned} \tag{55}$$

In evaluating the third term, note that

$$\begin{aligned}
&1+(1+\delta)+\cdots+\left(1+\delta+\cdots+\delta^{x-2}\right) \\
=\ &\sum_{y=1}^{x-1}\left(\frac{1-\delta^y}{1-\delta}\right)=\left(\frac{1}{1-\delta}\right)\sum_{y=1}^{x-1}\left(1-\delta^y\right) \\
=\ &\left(\frac{1}{1-\delta}\right)\left((x-1)-\sum_{y=1}^{x-1}\delta^y\right)=\left(\frac{1}{1-\delta}\right)\left((x-1)-\delta\left(\frac{1-\delta^{x-1}}{1-\delta}\right)\right)
\end{aligned} \tag{56}$$

Substituting (56) and simplifying, (55) becomes

$$\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^x}{1-\delta}\right)\pi^F+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^x}{1-\delta}\right)\frac{\delta\pi^M}{1-\delta} \\
&+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1}{1-\delta}\right)\left((x-1)-\delta\left(\frac{1-\delta^{x-1}}{1-\delta}\right)\right)\pi^N \\
&+\left(\frac{1}{4x}\right)\left(\left(\frac{1-\delta^x}{1-\delta}\right)\pi^N+\frac{\delta^x\pi^M}{1-\delta}\right)+\left(\frac{2x-1}{4x}\right)\left(\left(\frac{1-\delta^x}{1-\delta}\right)\pi^N+\delta^x\pi^L+\frac{\delta^{x+1}\pi^M}{1-\delta}\right)
\end{aligned} \tag{57}$$

Using (54) and (57), the expected payoff from $T_1=t_2'$ exceeds that from $T_1=t_2'+x$ when

$$\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{\pi^M}{1-\delta}\right)+\left[1-\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\right]\left(\pi^L+\frac{\delta\pi^M}{1-\delta}\right) \\
>\ &\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^x}{1-\delta}\right)\pi^F+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^x}{1-\delta}\right)\frac{\delta\pi^M}{1-\delta} \\
&+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1}{1-\delta}\right)\left((x-1)-\delta\left(\frac{1-\delta^{x-1}}{1-\delta}\right)\right)\pi^N \\
&+\left(\frac{1}{4x}\right)\left(\left(\frac{1-\delta^x}{1-\delta}\right)\pi^N+\frac{\delta^x\pi^M}{1-\delta}\right)+\left(\frac{2x-1}{4x}\right)\left(\left(\frac{1-\delta^x}{1-\delta}\right)\pi^N+\delta^x\pi^L+\frac{\delta^{x+1}\pi^M}{1-\delta}\right)
\end{aligned} \tag{58}$$

Letting $x\to\infty$, (58) is

$$\pi^L+\frac{\delta\pi^M}{1-\delta}>\left(\frac{1}{1-\delta}\right)\pi^N,$$

which is equivalent to (53).

In summing up the previous two steps, if (53) holds then, for x sufficiently large, at period $t = t_2' : 1)$ firm 1 prefers $T_1 = t_2'$ to $T_1 = t_2' + 1$, prefers $T_1 = t_2' + 1$ to $T_1 = t_2' + 2$, ..., prefers $T_1 = t_2' + x - 2$ to $T_1 = t_2' + x - 1$; and 2) firm 1 prefers $T_1 = t_2'$ to $T_1 = t_2' + x$. By the structure of prior (and posterior beliefs), the analysis is the same starting from period $t_2' + x$ so that: 3) firm 1 prefers $T_1 = t_2' + x$ to $T_1 = t_2' + x + 1$, prefers $T_1 = t_2' + x + 1$ to $T_1 = t_2' + x + 2$, ..., prefers $T_1 = t_2' + 2x - 2$ to $T_1 = t_2' + 2x - 1$. By transitivity and (1)-(3), firm 1 prefers $T_1 = t_2'$ to T_1 for all $T_1 \in \{t_2' + 1, ..., t_2' + 2x - 1\}$. Iterating, firm 1 prefers $T_1 = t_2'$ to T_1 for all $T_1 > t_2'$.

In sum, if (11) and (53) hold then, for x sufficiently large, the sequentially rational strategy for firm 1 is $T_1 = t_2'$. Given that this argument works for all $t_2' \in \{1, 2, ...\}$, every $T_1 \in \{1, 2, ...\}$ is sequentially rational. Combining (11) and (53), it is required that

$$\delta \in \left(\frac{\pi^N - \pi^L}{\pi^M - \pi^L}, 1 - \left(\frac{\pi^M - \pi^N}{\pi^F - \pi^L} \right) \right). \tag{59}$$

Note that there exist values for the discount factor whereby this condition holds because

$$\begin{aligned} 1 - \left(\frac{\pi^M - \pi^N}{\pi^F - \pi^L} \right) &> \frac{\pi^N - \pi^L}{\pi^M - \pi^L} \Leftrightarrow \\ \left(\pi^M - \pi^L\right)\left(\pi^F - \pi^L\right) - \left(\pi^M - \pi^L\right)\left(\pi^M - \pi^N\right) &> \left(\pi^N - \pi^L\right)\left(\pi^F - \pi^L\right) \Leftrightarrow \\ \left(\pi^M - \pi^N\right)\left(\pi^F - \pi^L\right) &> \left(\pi^M - \pi^N\right)\left(\pi^M - \pi^L\right) \Leftrightarrow \pi^F > \pi^M, \end{aligned}$$

which is true because $\pi^M > \pi^N$ and $\pi^F > \pi^M$.

In concluding, let us explain why the analysis is robust to allowing firm i's prior beliefs assign a small positive probability to $T_j = \infty$. Modify the prior beliefs in (8) so that probability $\kappa \in (0, 1)$ is assigned to $T_j = \infty$ and the probabilities for all other T_j are scaled by $1 - \kappa$. Prior beliefs are now:

T_j	Prior Probability
∞	κ
1	$(1/2)(1-\kappa)$
2	$(1/2)^2(1-\kappa)$
$\vdots$	$\vdots$
$t_j' - 1$	$(1/2)^{t_j'-1}(1-\kappa)$
t_j'	$(1/2)^{t_j'}(1/x)(1-\kappa)$
$\vdots$	$\vdots$
$t_j' + x - 1$	$(1/2)^{t_j'}(1/x)(1-\kappa)$
$t_j' + x$	$(1/2)^{t_j'+1}(1/x)(1-\kappa)$
$\vdots$	$\vdots$
$t_j' + 2x - 1$	$(1/2)^{t_j'+1}(1/x)(1-\kappa)$
$\vdots$	$\vdots$

Now that there is some prior probability that firm j will never lead ($T_j = \infty$), firm i will have a stronger incentive to lead rather than wait. However, as long as κ is small relative to $1/t'_j$ - so that the posterior probability that $T_j = \infty$ is sufficiently small for $t < t'_j$ - then firm i will continue to prefer to wait for all $t < t'_j$. Thus, the sequential rationality of not leading before t'_j is robust to $\kappa > 0$ and small. Turning to the analysis that proves it is sequentially rational for firm i to lead for $t \geq t'_j$, it is reinforced when $\kappa > 0$. A firm will be more inclined to lead when it assigns positive probability to the rival firm never leading. While the associated analysis required that x is sufficiently large, note that x does not need to be sufficiently large relative to $1/\kappa$. For the proof to go through, we just need that κ is small relative to $1/t'_j$.

References

[1] Battigalli, Pierpaolo and Marciano Siniscalchi, "Rationalization and Incomplete Information," *Advances in Theoretical Economics*, 3 (2003) (1), Article 3.

[2] Borenstein, Severin, "Rapid Price Communication and Coordination: The Airline Tariff Publishing Case (1994)," in *The Antitrust Revolution*, 4th edition, John E. Kwoka, Jr. and Lawrence J. White, eds., New York: Oxford University Press, 2004.

[3] Chamberlain, Edward, *The Theory of Monopolistic Competition*, Cambridge: Harvard University Press, 1938.

[4] Clark, Robert and Jean-François Houde, "Collusion with Asymmetric Retailers: Evidence from a Gasoline Price-Fixing Case," HEC Montréal, February 2011.

[5] Connor, John M., "'Our Customers Are Our Enemies': The Lysine Cartel of 1992-1995," *Review of Industrial Organization*, 18 (2001), 5-21.

[6] Cooper, David J., "Barometric Price Leadership," *International Journal of Industrial Organization*, 15 (1997), 301-325.

[7] Cramton, Peter and Jesse A. Schwartz, "Collusive Bidding: Lessons from the FCC Spectrum Auctions," *Journal of Regulatory Economics*, 17 (2000), 229-252.

[8] Cubitt, Robin P. and Robert Sugden, "Common Knowledge, Salience and Convention: A Reconstruction of David Lewis' Game Theory," *Economics and Philosophy*, 19 (2003), 175-210.

[9] Garrod, Luke, "Collusive Price Rigidity under Price-Matching Punishments," *International Journal of Industrial Organization*, 30 (2012), 471–482.

[10] Harrington, Joseph E. Jr., "How Do Cartels Operate?," *Foundations and Trends in Microeconomics*, Vol. 2, Issue 1, July 2006.

[11] Harrington, Joseph E. Jr., "Posted Pricing as a Plus Factor," *Journal of Competition Law and Economics*, 7 (2011), 1-35.

[12] Kalai, Ehud and Ehud Lehrer, "Rational Learning Leads to Nash Equilibrium," *Econometrica*, 561 (1993), 1019-1045.

[13] Kaplow, Louis, *Competition Policy and Price Fixing*, Princeton, N.J.: Princeton University Press, 2013.

[14] Lewis, David, *Convention: A Philosophical Study*, Cambridge: Harvard University Press, 1969.

[15] Lockwood, Ben and Jonathan P. Thomas, "Gradualism and Irreversibility," *Review of Economic Studies*, 69 (2002), 339-356.

[16] Lu, Yuanzhu and Julian Wright, "Tacit Collusion with Price-Matching Punishments," *International Journal of Industrial Organization*, 28 (2010), 298-306.

[17] Markham, Jesse W., "The Nature and Significance of Price Leadership," *American Economic Review*, 41 (1951), 891-905.

[18] Marshall, Robert C. and Leslie M. Marx, *The Economics of Collusion: Cartels and Bidding Rings*, Cambridge, Mass.: The MIT Press, 2012.

[19] Mouraviev, Igor and Patrick Rey, "Collusion and Leadership," *International Journal of Industrial Organization*, 29 (2011), 705–717.

[20] Rotemberg, Julio J. and Garth Saloner, "Price Leadership," *Journal of Industrial Economics,* 39 (1990), 93-111.

[21] Scherer, F. M., *Industrial Market Structure and Economic Performance*, 2nd Edition, Boston: Houghton Mifflin, 1980.

[22] Wang, Zhongmin, "(Mixed) Strategy in Oligopoly Pricing: Evidence from Gasoline Pricing Cycles Before and Under a Timing Regulation," *Journal of Political Economy*, 117 (2009), 987-1030.