How to Integrate the Fermionic String Measure*

HoSeong La†
Physics Department
Boston University
Boston, MA 02215 USA

Philip Nelson
Physics Department
University of Pennsylvania
Philadelphia, PA 19104 USA

ABSTRACT

We show how to treat boundary divergences in heterotic string theory covariantly and unambiguously. The method applies even to theories with nonvanishing tadpoles; in this case the Fischler-Susskind mechanism suffices to ensure unambiguous amplitudes. No splitting or projection of supermoduli space is needed.

One way to quantize a field theory is to quantize small fluctuations about a classical ground state configuration. At tree level the vanishing vev of the fluctuations at a classical vacuum identifies an extremum of the classical potential. At higher loop level, if the vev of the fluctuation of any massless state at the classical vacuum does not vanish, then physical amplitudes will diverge due to processes in which a state propagates for a very long time before it disappears into the vacuum.

In particular, such divergences signify the instability of the naive tree-level vacuum, making it unsuitable as a starting point for a perturbation expansion. As in any perturbative dynamical symmetry breaking problem, one needs to determine appropriate background shifts order by order in the coupling constants. To do this it is necessary to cut off diagrams which in real space have long on-shell lines, and to introduce compensating shifts in the background as a counterterm. The

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† Address after June 1st, 1989: Physics Department, University of Pennsylvania, Philadelphia, PA 19104.

good limit to lead to finite, unambiguous amplitudes can be successfully achieved when the background shift obeys a finite, loop corrected equation of motion as the cutoff is removed. Then we can obtain the effective action with the true vacuum configurations as well as a well-defined perturbation theory.

We claim that string theory works same way. If there is any divergence due to the massless modes, with a good cutoff choice we can renormalize the amplitudes by shifting the vacuum properly. There is a subtlety not found in field theory. In string theory, given a surface, it is impossible to say what precisely is the length of any internal tube. There are no precise interaction points on a surface whose separation we could measure because we do not have any preferred metric choice[1].

The fundamental observation of Fischler and Susskind is that the two troubles we have in string theory, namely, the nonexistence of natural IR cutoff and the broken world-sheet conformal invariance due to the divergences, cancel each other[1], leaving us with a well-defined prescription to get the loop corrected string equation of motion.

The mechanism proposed in [1] and refined in [2][3][4] (and elsewhere) works as follows. We must impose a single cutoff on both worldsheet conformal field theory and on our moduli integrals. Then we introduce proper counterterms, requiring a good limit as the cutoff is removed. These counterterms can be interpreted as changes in the space-time background fields and thus we find modified field equations. This cutoff requires us to choose some additional data not contained in the given Riemann surface. In [1][3][5] and elsewhere this choice was made implicitly via a coordinate choice. Analogous constructions were given for fermionic strings in [6][7] and elsewhere.

It is desirable to disclose the geometrical data needed to specify a cut off covariantly. Formalisms in which all chosen geometrical objects are explicitly visible are called "covariant"; thus for example special relativity is not covariant because the choice of a particular metric has been implicitly made. One normally works in

[1] In fact this has been considered as a merit of string theory because the ultraviolet divergences in quantum field theory are precisely due to the overlapping of such interaction points, and string theory can avoid it. While the ultraviolet finiteness in string theory can be ensured by the modular invariance, still many interesting string theories suffer from infrared divergences.

special "inertial" coordinates in which this metric is a constant; by acknowledging the metric and assigning its appropriate transformation properties we can promote the formulas of special relativity (*e.g.* Maxwell's equations) to coordinate-invariant status.

In the bosonic string case Polchinski used the choice of world sheet metric, simultaneously cutting off narrow necks and defining the normal ordering needed to insert background corrections off the usual mass shell[4]. In this talk we will make a different choice, one better suited to fermionic string, but the principle is similar[8]. In any case, having brought this choice of extra data into view, one can investigate the circumstances under which it drops out. This turns out to determine the background shifts, much as the tree-level condition of conformal invariance determines the tree-level effective action[9]. Remarkably one finds that the resulting *consistency* condition coincides with the *stability* (no-tadpole) condition[1], a fact which was less surprising at tree level [9]. The condition also arises from enforcing BRST decoupling, as conjectured by Mansfield [10] and shown in a simple case in [6][7].

All of our considerations can be extended to the super case [11] [12]. These will provide a proper way to handle the divergences of string amplitudes at the boundary of the moduli space. There has been some confusion in the literature on the super case (see [11]), much of which can be traced to the use of explicit coordinates for supermoduli space. As discussed above, coordinate-dependent formalisms have hidden an implicit choice of cutoff prescription [13][14][15].

To put the issues in focus, suppose we are given a volume form μ on the complex plane of the form $\mu = |q|^{-2}\ \mathrm{d}q \wedge \mathrm{d}\bar{q}+$ (regular), where q is the standard complex coordinate. The integral of μ over the unit disk diverges. We can try to define $I_\delta = \int_{|q|>\delta} \mu$, but we must keep in mind that this depends not only on δ and the form μ, but also on the choice of coordinate q used to exclude δ-ball. If $q' = aq + bq^2 + \cdots$ is another coordinate centered at zero then we can define I'_δ by excluding $\{|q'| < \delta\}$; then

$$I'(\delta) = I(\delta) - 4\pi i \log |a| + \mathcal{O}(\delta). \tag{1}$$

Thus if we propose to find a "counterterm" to add to $I(\delta)$ so as to get a good limit as $\delta \to 0$, the former must also depend on the same choice of q in a way which cancels the dependence in $I(\delta)$. Note that as the cutoff is removed it is only the magnitude of a, the first Taylor coefficient, which matters; were μ more divergent we would have needed more coefficients.

In this talk we shall describe the two ingredients used in [11][12]. We use the extended moduli space [16] to introduce proper cutoffs and factorizations of string amplitudes. The necessary geometrical facts can be obtained by the the Beilinson-Kontsevich action of Virasoro on that space [17]. The boundary can be analyzed by the operator formalism from [18][19](while a general reference is [20]), and the generalization of the "conformal normal ordering"(CNO) prescription of Polchinski[21] is used to describe the inserted states which generate the BRST anomaly.

Let $\mathcal{M}_{g,n}$ be the moduli space of smooth Riemann surfaces with g handles and n labeled points. We also denote by $\mathcal{P}_{g,n}$ the extended moduli space of pointed surfaces, which consists of triples $\widetilde{X} \equiv (X, P_i, z_i)$ for $1 \leq i \leq n$. z_i are local complex coordinates defined on some open neighborhood of the marked points P_i such that $z_i(P_i) = 0$. Of course, the explicit mention of P_i is redundant; once z_i is given we have $P_i = \{z_i = 0\}$, so sometimes we will write $\widetilde{X} = (X, z_i)$. Given $\widetilde{X}$ we can always throw away z_i to obtain $(X, P_i) \in \mathcal{M}_{g,n}$, i.e. we always have a projection $\pi : \mathcal{P}_{g,n} \to \mathcal{M}_{g,n}$. Another way, which will be more suitable to fermionic strings, to think about π is that $\pi(X, z) = (X, [z])$, where $[z]$ is z taken modulo multiplication by any local holomorphic function not vanishing at P.

Note that for a given puncture different coordinates preserving the origin always project to the same point in $\mathcal{M}_{g,n}$. For convenience consider the one puncture case and let G be some analytic map on the plane C which preserves the origin. Infinitesimally

$$G(w) = w - \sum_{n=0}^{\infty} \epsilon_n w^{n+1}. \tag{2}$$

Then the composition $z' = G \circ z$ is a new local coordinate centered at the same point P as before. Since we can compose the maps G, we see that $\pi : \mathcal{P}_{g,1} \to \mathcal{M}_{g,1}$ is a principal fiber bundle. The Lie algebra of its group we will call the positive Virasoro algebra, Vir_+.

Even if G moves the origin we can still define the above action, but now it moves the point P. For example if $G(w) = w - \epsilon_{-1}$, then $z' = z - \epsilon_{-1}$ vanishes at

$P' \neq P$, where P' is defined precisely by $z(P') = \epsilon_{-1}$. In fact we can even define an action on $\mathcal{P}_{g,1}$ of any $G : \mathbf{C} \to \mathbf{C}$ which is holomorphic and single-valued on some neighborhood of the origin (excluding 0 itself) and meromorphic at 0. The Lie algebra of such G we call Vir. To define the action suppose first that z is a well-defined map to the unit disk of C and G is well-defined on this disk, except at 0. Excise the unit z-disk from X and reattach it using the map G restricted to an annulus. This gives a new surface X' with a new local coordinate z'. If we like, we can instead begin with a smaller disk and rescale everything to arrive at the same action; then if the original z or G is defined on some arbitrary domain, we just cut out a small enough disk around P and apply the above prescription[2].

Our convention for generators for Vir is that ℓ_n generates the transformation $G \circ z = z - \epsilon z^{n+1}$. This diffeomorphism is generated by a vector field $v = v^z \frac{\partial}{\partial z}$ on C:

$$\ell_n \leftrightarrow -z^{n+1} \frac{\partial}{\partial z}. \tag{3}$$

Strictly speaking only *real* vector fields generate diffeomorphisms; thus only generators of the form $\epsilon_n \ell_n + \bar{\epsilon}_n \bar{\ell}_n$ make sense, where the numbers $\bar{\epsilon}_n$ are complex conjugate to ϵ_n. But having said that we see the sense in which $\epsilon_n \ell_n$ by itself gives a vector in the complexified tangent of $\mathcal{P}_{g,1}$. In fact it is a holomorphic tangent.

We will denote the action of Vir on $\mathcal{P}_{g,1}$ by $i_{\widetilde{X}}(v)$, a tangent to $\mathcal{P}_{g,1}$ at $\widetilde{X}$ corresponding to $v \in$ Vir. We have seen that for $v \in \mathrm{Vir}_+$ this deformation of $\widetilde{X}$ is vertical; it does not affect $\pi(\widetilde{X}) = (X, P)$. More generally $i_{\widetilde{X}}(v)$ will project to some tangent on $\mathcal{M}$, denoted by $\pi_* i_{\widetilde{X}}(v)$. We stress that Vir does *not* act directly on $\mathcal{M}_{g,1}$, however; if $\widetilde{X} = (X, P, z)$ and $\widetilde{X}' = (X, P, z')$ then $i_{\widetilde{X}}(v)$ and $i_{\widetilde{X}'}(v)$ will in general project to *different* vectors on $\mathcal{M}_{g,1}$.

To choose a local coordinate z is to find a section of $\mathcal{P}_{g,1}$. Let σ be such a choice: $\sigma(X, P) \equiv (X, P, z_{(X,P)})$. Note that $z_{(X,P)}(Q)$ for fixed X is a function of *two* points on X, with the property that $z_{(X,P)}(P) = 0$. Also we require that it is holomorphic in Q, but *not* necessarily in P or X. We certainly cannot always choose σ globally. If we could, then $\mathrm{d}z_{(X,P)}|_P$ would be a nowhere-vanishing cotangent field, but this does not exist unless X is a torus. If we allow σ to

[2] Strictly speaking what we get in this way is not an action of the group of G's but rather of its Lie algebra Vir.

jump by a phase, though, then there is no further obstruction. That is, we take σ to be defined on patches; across patch boundaries we permit jumps of the form $z'_{(X,P)}(Q) = e^{i\alpha(X,P)} z_{(X,P)}(Q)$ where α is a real function independent of Q. Infinitesimally, then, z and z' are related by the action of a single generator, namely $\ell_0 - \bar{\ell}_0$. We will see that a coordinate family defined up to $U(1)$ will be all we need to cut off string amplitudes [4].

Given a point (X, P) of $\mathcal{M}_{g,1}$ we have no preferred $\widetilde{X}$ in $\mathcal{P}_{g,1}$ which projects to (X, P). Even if $\widetilde{X}$ is given, we still don't know how to lift a tangent $V \in T_{(X,P)}\mathcal{M}_{g,1}$ to $\widetilde{V} \in T_{\widetilde{X}}\mathcal{P}_{g,1}$; any $\widetilde{V}$ with $\pi_* \widetilde{V} = V$ will do. If we have a *slice* σ near (X, P), though, then we can simply require that $\widetilde{V}$ be parallel to the slice [8].[3] Suppose for example that near Q our coordinate family obeys

$$z_P(\cdot) = z_Q(\cdot) - \Delta_{PQ}, \tag{4}$$

where $\Delta_{PQ} = z_Q(P)$ is a small number and X is fixed. Certainly $z_P(\cdot)$ is centered at P and holomorphic in both P and Q. Keeping X fixed, a coordinate for the modulus corresponding to the P in $\mathcal{M}_{g,1}$ is then just $u(X, P) = z_{Q_0}(P)$, the location of P in one *fixed* coordinate system.

The most general coordinate family is given to first order in Δ_{PQ} by

$$z_P(\cdot) = z_Q(\cdot) - \Delta_{PQ} + \sum_{n=0}^{\infty} (E_n \Delta_{PQ} + F_n \bar{\Delta}_{PQ}) z_Q(\cdot)^{n+1} + \mathcal{O}(\Delta_{PQ}^2). \tag{5}$$

for some coefficients E_n, F_n depending on P. The extra terms are just a family of transformations (2), not necessarily holomorphic in P, reducing to the identity at $P = Q$. We then read off

$$\sigma_* \left(\frac{\partial}{\partial u} \right) = i_{\widetilde{X}} \left(\ell_{-1} - E_n \ell_n - \bar{F}_n \bar{\ell}_n \right) \quad , \text{ general coord family.} \tag{6}$$

The last term comes from the $\mathcal{O}(\Delta_{PQ})$ term of $\bar{z}_P(\cdot)$. If $E_n = F_n = 0$, (6) reduces to CNO family given in [21] as (5) reduces to (4).

After we have the necessary data, we would like a way to exclude a tube around the divisor Δ of once-pinched curves, and hence cut off string integrals, which

[3] Note that this use of the word "slice" has nothing to do with the slices of [14][15].

typically diverge there. We can do this locally as follows. Choose a coordinate q on $\mathcal{M}_g$ such that $q \equiv 0$ on Δ but $\mathrm{d}q$ is nowhere zero on Δ. Then let $\mathcal{M}_{g,\delta} = \{|q| > \delta\}$. The first condition means that $\mathrm{d}q$ annihilates the tangent to Δ, or in other words that $\mathrm{d}q$ is in the "conormal" bundle to Δ. The second condition says that $\mathrm{d}q$ "trivializes" the conormal bundle. Since in general this bundle is not trivial, q cannot be globally defined. Even if it were, however, we would still need to investigate the effects of different choices of cutoff (choices of transverse coordinate q) on our answers.

Given any local choice of q as above, and in particular a holomorphic choice, we can write any other choice as

$$q' = F(\vec{m}_{(1)}, \vec{m}_{(2)}, q) \cdot q \quad , \tag{7}$$

where F is a function on $\mathcal{M}_g$ which never vanishes on Δ. In general F is some arbitrary function. As $q \to 0$, we can find a more special F.

Consider a Riemann surface X_1 of genus g_1 with one puncture P_1, and another X_2 of genus g_2 with puncture P_2. Given a complex number $q \neq 0$, we would like to construct a nearly-pinched surface X. Choose local coordinates z_1, z_2 near P_1, P_2. We can now excise the disks $\{|z_i| < |q|\}$ and glue $\{|q| < |z_1| < 1\}$ to $\{|q| < |z_2| < 1\}$ via the map $z_2 = q/z_1$; as $q \to 0$ we recover the pinched surface. What we have defined is not a unique isomorphism $\mathcal{M}_{g_1,1} \times \mathbf{C} \times \mathcal{M}_{g_2,1} \simeq \mathcal{M}_g$ but rather a map [22]

$$\infty : \mathcal{P}_{g_1,1} \times \mathbf{C} \times \mathcal{P}_{g_2,1} \to \mathcal{M}_g. \tag{8}$$

It is certainly many-to-one, since $\mathcal{P}$ is infinite-dimensional.

To get *coordinates* for $\mathcal{M}_g$ we begin with a *family* of X_1 parameterized by $m^i_{(1)}$, $i = 1, \cdots, 3g_1 - 2$. (One modulus describes the location of P_1) and repeat for side two. Now choose families $\sigma_{1,2}$ of local coordinates. Composing $\sigma_{1,2}$ with the above map ∞ gives us the desired map $A : \mathcal{M}_{g_1,1} \times \mathbf{C} \times \mathcal{M}_{g_2,1} \to \mathcal{M}_g$, and with its coordinates $\vec{m}_{(1)}, \vec{m}_{(2)}, q$ for $\mathcal{M}_g$ near Δ, but now A depends on σ_1 and σ_2. If we change σ_i to σ'_i then the *same* values of $\vec{m}_{(1)}, \vec{m}_{(2)}, q$ will correspond to a *different* surface X'. Phrased differently, changing σ_i to σ'_i induces a coordinate change

$$\vec{m}_{(i)} \mapsto \vec{m}'_{(i)}(\vec{m}_{(i)}, q) \tag{9}$$

$$q \mapsto q'(\vec{m}_{(i)}, q) \quad . \tag{10}$$

It is of course the change in q which interests us most.

We claim that under this transformation (7) takes the form

$$q' = F_1(\vec{m}_{(1)}) F_2(\vec{m}_{(2)}) q + \mathcal{O}(q^2) \quad . \tag{11}$$

The coefficient of q on the right side is the transition function for the conormal bundle on Δ, since q^2 can't contribute to $\mathrm{d}q|_{q=0}$. We know that Δ has the product structure $\Delta \simeq \mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$. What (11) says is that the conormal bundle *also* has a natural product structure, which let us introduce the counterterms for each side independently. It is this "factorization" that makes the Fischler-Susskind mechanism work; in the fermionic case it will again hold, and it will eliminate the integration problem[11][12].

The Polyakov measure has an elegant formulation [23][18]. Suppose we want to insert one external state ψ on X. Given tangent vectors $V_1, \cdots, \bar{V}_{3g-2}$ to $\mathcal{M}_{g,1}$ at some X we seek an alternating form $\mu_\psi(V_1, \cdots \bar{V}_{3g-2})$. Choose any P, z and compute $|\widetilde{X}\rangle$. Next choose $v_i \in \mathrm{Vir}$ such that $i_{\widetilde{X}}(v_i)$ projects down to V_i. Thus each v_i is ambiguous by the addition of $\ell_n, n \geq 0$.

Let us define a form on $\mathcal{P}_{g,1}$

$$\widetilde{\mu}_\psi(v_1, \cdots, \bar{v}_{3g-2}) := \Big(b(v_1^T) \cdots \bar{b}(\bar{v}_{3g-2}^T) |\widetilde{X}\rangle, \psi \Big) \, .$$

Now suppose ψ is BRST-exact, $\psi = (Q + \overline{Q})\lambda$. We find

$$\widetilde{\mu}_{(Q+\overline{Q})\lambda} = \mathrm{d}\widetilde{\mu}_\lambda . \tag{12}$$

Sometimes $\widetilde{\mu}_\psi$ is actually the lift of a form μ_ψ on $\mathcal{M}_{g,1}$, just as with the Polyakov measure [18][8]. This happens when ψ satisfies:

$$L_n \psi = b_n \psi = 0, \quad n \geq 0 \quad . \tag{13}$$

If (13) is not satisfied we can obtain a form μ_ψ by brute force, as follows: locally choose any section σ of $\mathcal{P}_{g,1} \to \mathcal{M}_{g,1}$, and let

$$\mu_{\psi,\sigma} \equiv \sigma^* \widetilde{\mu}_\psi \quad ,$$

the pullback. Thus at $(X,P)\in\mathcal{M}_{g,1}$

$$\mu_{\psi,\sigma}(V_1,\cdots,\bar{V}_{3g-2})=\widetilde{\mu}_\psi(\sigma_*V_1,\cdots,\sigma_*\bar{V}_{3g-2}) \quad , \tag{14}$$

where the RHS is evaluated at $\sigma(X,P)$. As we explained in [8], (14) reproduces the "$\hat{b}$" prescription of [4]; the corrections to b_{-1} are clearly visible in (6) and the following paragraph. When ψ satisfies the physical conditions (13), the existence of the global σ does not matter, since in any case $\mu_{\psi,\sigma}$ is independent of σ.

As pointed out in [4], however, states which violate (13) can enter into the factorization formula. Still no global choice of σ is necessary. The relevant factorizing states do satisfy a weaker form of (13), namely

$$(L_0-\bar{L}_0)\psi=(b_0-\bar{b}_0)\psi=0. \tag{15}$$

In this case $\mu_{\psi,\sigma}$ is at least insensitive to $U(1)$ changes in σ, and we *can* choose a global coordinate family defined modulo $U(1)$. Now we need to use (6) in (14) explicitly. In particular an insertion of ψ will be accompanied by $\hat{b}_{-1}\hat{\bar{b}}_{-1}$ times $\mathrm{d}u\wedge\mathrm{d}\bar{u}$, where again u is the location of the insertion point and

$$\hat{b}_{-1}=b_{-1}-E_nb_n-\bar{F}_n\bar{b}_n. \tag{16}$$

Substituting the special family with $\bar{F}_1=-\frac{1}{8}R$, etc, we recover the prescription in [4].

In fact the BRST anomaly due to the divergences of string amplitudes at the boundary of the moduli space which can be analyzed in terms of the factorization formula are precisely due to the massless states satisfying (15), which can be compensated by a background shift also satisfying (15). Consider the Polyakov measure and factorize

$$\mu=\int\frac{d^2q}{|q|^2}\sum_a q^{h_a}\bar{q}^{\bar{h}_a}\left\langle b_0\bar{b}_0\phi_{a(P_1;z_1)}\right\rangle_{X_1}\left\langle\phi^a_{(P_2;z_2)}\right\rangle_{X_2}, \tag{17}$$

where $\langle\ \rangle_X$ is defined with $\hat{b}$ zero mode insertions and $\{\phi^a\}$ is a basis of the Hilbert space at the marked point. Thus we can get the tadpole BRST anomaly: insert a BRST exact state $Q\chi$ on X_1 to get total derivative (12)and integrate the boundary integral over $\{|q|>\delta\}$ with a cutoff δ to get

$$\text{anomaly}=\sum_{h_a=\bar{h}_a}\delta^{-h_a-\bar{h}_a}\left\langle\chi\bar{b}_0\phi_{a(P_1;z_1)}\right\rangle_{X_1}\cdot 2\pi i\cdot i\mathcal{Z}^a_{(g_2)}, \tag{18}$$

where $i\mathcal{Z}^a_{(g_2)}$ is the vev of ϕ^a on X_2.

Now we want to have a background shift $\delta^{(g_2)}\phi_{b.g.}$ such that

$$\left\langle [Q\delta^{(g_2)}\phi_{b.g.}]_{(P_1;\widetilde{z_1})} \right\rangle_{X_1} = -\text{anomaly} \ , \tag{19}$$

where $\widetilde{z_1} = z_1/\delta$. From (15) we must choose $\delta^{(g_2)}\phi_{b.g.}$ satisfying $(b_0 - \bar{b}_0)\delta^{(g_2)}\phi_{b.g.} = 0$. Requiring that (18) cancel (19) gives a condition for the required massless background shift

$$Q\delta^{(g_2)}\phi_{b.g.} = \sum_{h_a=\bar{h}_a} [\delta]^{-L_0-\bar{L}_0}\bar{b}_0\phi_a \cdot 2\pi\mathcal{Z}^a_{(g_2)} \ , \tag{20}$$

where as $\delta \to 0$ only $h_a = \bar{h}_a = 0$ survives. (20) leads to the loop corrected effective field equations. When a background satisfies these equations, BRST spurious states decouple.

Now we can return to the ambiguity schematically given by (1). When we use the coordinates discussed below (8), we see from (11) that the log in (1) splits into two pieces. It turns out that (20) ensures that one of these terms vanishes; a similar term from the insertion of $Q\chi$ on X_2 takes care of the other[12].

Next, let us consider the fermionic string case. Supermoduli space $\hat{\mathcal{M}}_g$ has a compactification by stable super curves[24]. These are built from smooth super Riemann surfaces (SRS) joined by separated, universal degenerations. Now however, there are *two* distinct degenerations [24][25], the super (or Neveu-Schwarz) pinches and the spin (or Ramond) pinches. In this talk we will deal only with super pinches, since in the heterotic string in flat spacetime only spacetime bosons can disappear into the (tree-level) vacuum.

A density of compact support has a well-defined integral (see *e.g.* [26][20]). Thus to integrate μ we can take any open cover $\{\mathcal{U}_\alpha\}$ of the compactified $\hat{\mathcal{M}}_g$ and any associated partition of unity $\{\rho_\alpha\}$. If a given $\mathcal{U}_\alpha$ does not intersect Δ then we simply integrate $\int_{\mathcal{U}_\alpha}\rho_\alpha\mu$ without further difficulty. Thus we see that in *principle* there is interesting physics only at the boundary Δ.[4] In *practice* it will almost certainly be helpful to employ the methods of [14][27][28][15][29][30] to compute integrals on the interior of $\hat{\mathcal{M}}_g$, but these methods are not our present concern.

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[4] The preceding argument is due to E. Witten.

We *do* need to understand the boundary. There is a well-defined integral over a supermanifold-with-boundary [26], so we now turn to the super version of the remarks made before on how to exclude a tube surrounding Δ.

The space $\hat{\mathcal{P}}_{g,1}$ consists of SRS with a chosen "point" P and local superconformal coordinate $\mathbf{z} = (z, \theta)$ centered at P. We put "point" in quotes because in general it is a problematic construction in superspace; in our case however, P is again wholly redundant once $\mathbf{z}$ is given. In fact we can again project $\hat{\mathcal{P}}_{g,1}$ to $\hat{\mathcal{M}}_{g,1}$ by sending (X, z, θ) to $(X, [z])$ where $[z]$ is z taken modulo multiplication by invertible local functions. Given such a z we recover P as "the point where $z = Dz = 0$," and clearly $f \cdot z$ defines the same P for invertible f.

We now get an action of the Neveu-Schwarz algebra $\mathit{Diff}\, S^{1|1}$ on $\hat{\mathcal{P}}_{g,1}$. The algebra consists of superconformal vector fields meromorphic near the origin of the z, θ plane. Once given a fixed local coordinate z, θ we can obtain a coordinate family analogous to (6), which we will call the generalization of the "superconformal normal ordering" (SCNO) family. But let us first consider the SCNO family analogous to (4). Suppose that Q is a point located at $(z, \theta) = (u, \zeta)$. (More precisely, $Q = [z - u + \zeta\theta]$.) Then to Q we associate

$$(z_Q, \theta_Q) = (z - u + \zeta\theta, \theta - \zeta), \tag{21}$$

a superconformal coordinate centered at Q. This association is not natural, as one finds by starting with another (z', θ'). For now, however, (z, θ) will be *fixed* while (z_Q, θ_Q) vary with Q.

The infinitesimal transformation $\epsilon\ell_{-1}$ takes $(z - u + \zeta\theta, \theta - \zeta)$ to $(z - u + \zeta\theta - \epsilon, \theta - \zeta)$. But the same thing can be accomplished by sending $u \mapsto u + \epsilon$. We thus have

$$\sigma_* \left(\frac{\partial}{\partial u}\bigg|_Q \right) = i_{\sigma(Q)}(\ell_{-1}), \qquad \text{SCNO family.} \tag{22}$$

A similar argument shows that

$$\sigma_* \left(\frac{\partial}{\partial \zeta}\bigg|_Q + \zeta \frac{\partial}{\partial u}\bigg|_Q \right) = -2 i_{\sigma(Q)}(g_{-1/2}), \qquad \text{SCNO family.} \tag{23}$$

It will be convenient to define a function on $\mathcal{M}_{g,1}$ with values in the algebra:

$$k_{-1/2} \equiv g_{-1/2} + \tfrac{1}{2}\zeta\ell_{-1} \quad . \tag{24}$$

Then

$$\sigma_* \left(\frac{\partial}{\partial \zeta}\bigg|_Q \right) = -2 i_{\sigma(Q)}(k_{-1/2}) \quad . \tag{25}$$

Just as before, a more general coordinate family differs from (21) by a family of superconformal transformations; for such a σ the RHS of (22), (23), (25) receive corrections involving the derivatives of σ, similar to (6).

Once again we can make a plumbing construction: given SRS with punctures and local superconformal coordinates we sew them to get a pinching family parametriz even and odd moduli $\vec{m}_i,\ \vec{\tau}_i,\ i=1,2$, and an even coordinate t[25]. (t is the square root of our previous q.) Again a modification of $\mathbf{z}_1, \mathbf{z}_2$ by rescaling (ℓ_0 transformation) rescales t, while other changes add only $\mathcal{O}(t^2)$ terms:

$$t' = F_1(\vec{m}_1, \vec{\tau}_1) F_2(\vec{m}_2, \vec{\tau}_2) t + \mathcal{O}(t^2), \tag{26}$$

analogous to (11). Thus we once again find a product structure both at Δ and *near* Δ, as expressed by the preferred family of pinching coordinates related by (26).

For heterotic string the Polyakov density now takes a form very parallel to the bosonic case [31][32][19][33]. If $V_1, \cdots, V_{3g-3}, \Upsilon_1, \cdots, \Upsilon_{2g-2}$ are a basis of tangents to $\hat{\mathcal{M}}_g$ with V_i even and Υ_α odd, define corresponding $v_1, \cdots \nu_{2g-2}$ in the Neveu-Schwarz algebra and let

$$\mu(V_1, \ldots, \bar{V}_1, \ldots, \Upsilon_1, \ldots) \equiv \Big\langle \big[\bar{B}(\bar{v}_{3g-3}) \cdots \delta[B(\nu_{2g-2})] \cdots B(v_{3g-3}) \cdots \big]_{(P;\mathbf{z})} \Big\rangle_X \quad . \tag{27}$$

The operators $\delta[B(\nu)]$ are discussed extensively in [32][19][33]. With this definition μ transforms as a density. Similarly one can define $\widetilde{\mu}_\psi$ for an inserted state ψ, and relation (12) again holds. Given a family σ of superconformal coordinates, one again defines a density $\mu_{\psi,\sigma}$ on $\hat{\mathcal{M}}_g$ using (14). For the SCNO coordinate family described before we see that an insertion of ψ should be accompanied by $B(\ell_{-1})\bar{B}(\bar{\ell}_{-1})\delta[B(-2k_{-1/2})]$, times $[\mathrm{d}u\mathrm{d}\bar{u}|\mathrm{d}\zeta]$, where $k_{-1/2}$ is defined in (24). The insertion equals $-\frac{1}{2}b_{-1}\bar{b}_{-1}\delta[\beta_{-1/2} + \frac{1}{2}\zeta\ell_{-1}]$, which in turn is $-\frac{1}{2}b_{-1}\bar{b}_{-1}\delta(\beta_{-1/2})$. Performing the integral over ζ then differentiates $|\widetilde{X}\rangle$, since ζ enters nowhere else; thus we recover the usual picture-changing formalism.

As in the bosonic case for the factorization of the fermionic string amplitudes on the extended moduli space we need to insert the states with the generalization of the SCNO coordinate family. This generalization accompanies the corrections corresponding to the vector fields extending to holomorphically to the pinching point.

Now again with the necessary background shifts to compensate the BRST anomaly, we can define unambiguous, finite integral by requiring the cancellation of choice dependence, which leads to the loop corrected equations of motion, ensuring the BRST decoupling[11][12].

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