EVOLUTION OF THE SECONDARY SPECTRUM MARKET

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Dedicated to my parents

ABSTRACT

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The secondary spectrum market where primaries (license holders) lease the secondaries (unlicensed users) in lieu of the financial remuneration can eliminate the inefficiencies of the "static" spectrum allocation policy. We redress some of the challenges that have inhibited the wide scale deployment of the secondary spectrum market.

We first consider a secondary spectrum market where the primaries quote their prices for their available channels at a single location. The transmission rates offered by the channels of primaries evolve randomly because of the fading and noise. The secondaries decide to buy among the channels based on the transmission rate and the prices. We formulate the problem as a non cooperative game with the primaries as players. Each primary selects a price based on its own channel state only, as it is unaware of the channel states of the other primaries. We show that under the unique NE strategy profile a primary prices its channel to render the channel which provides high transmission rate more preferable; *this negates the perception that prices ought to be selected to render channels equally preferable to the secondary regardless of their transmission rates.*

Next, we consider the setting where the secondary spectrum market operates over multiple locations. Each primary needs to select an independent set in a conflict graph and the price at each location. We consider two scenarios–i) the number of locations is

small, and ii) the number of locations is large. We show that when the number of locations is small, in a symmetric NE strategy, each primary sells its channel to an independent set whose cardinality exceeds a certain threshold. The threshold also decreases as the transmission rate offered by the channel decreases. The symmetric NE is unique in a widely seen conflict graph-the linear conflict graph. In contrast, when the number of locations is large, a primary only sells its channel in the maximum independent set and the symmetric NE in not unique in the linear conflict graph.

Subsequently, we consider the setting where a primary owns a channel at a single location and can acquire the competitor's channel state information (C-CSI) by incurring a cost. Each primary now needs to decide whether to acquire the C-CSI or not and a price based on the information it has. We formulate the problem as a non cooperative game with two primaries as players and characterize the NE strategies. We first characterize the Nash Equilibrium (NE) of this game for a symmetric model where the C-CSI is perfect. We show that the payoff of a primary is independent of the C-CSI acquisition cost. We then generalize our analysis to allow for imperfect estimation and cases where the two primaries have different C-CSI costs or different channel availabilities. Our results show interestingly that the payoff of a primary increases when there is estimation error. We also show that surprisingly, the expected payoff of a primary may decrease when the C-CSI acquisition cost decreases when primaries have different availabilities.

Finally, we consider the setting where a primary allows multiple secondaries use the channel of a primary at a location. The interference must be limited at each primary-user terminal (primary-UT) in order to maintain a quality of service for each primary-UT. The secondary-base stations (secondary-BSs) are self-interested entities and only maximize their own utilities which makes it difficult to obtain a simple interference mitigation policy. We formulate the problem as a non cooperative coupled constrained concave game. We use the concept of the normalized Nash equilibrium (NNE) since it caters to the distributed setting. We develop a distributed algorithm which converges to the unique NNE for a large class of utility functions. In the distributed algorithm, the secondary-BSs do not need to exchange information among themselves, and the minimal cooperation from the primary-UTs. When the NNE is not unique or difficult to compute, we introduce the concept of WNNE which retains most of the properties of the NNE, but it can be computed easily compared to the NNE.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Motivation

The demand for mobile broadband is increasing due to the proliferation of wireless devices (e.g. smartphones, tablets, kindles etc.). According to a recent study, almost 30 billion devices will be connected via wireless by 2020. As a result, there is a widespread belief that the wireless spectrum is becoming increasingly crowded. However, recent studies suggest that the licensed spectrum is largely underutilized [1]. This is because, traditionally, a spectrum regulator (e.g. FCC in USA) allocates a fixed frequency band to a service provider (primary[1]) for its " exclusive use" (which is known as a 'static' spectrum allocation policy), so, the frequency band can not be used by others even if it is not utilized by the licensee [2].

To eliminate the above inefficient usage of the licensed spectrum, researchers propose

[1] We use the word primary service provider and primary interchangeably.

[2] This is also known as "the tragedy of anticommons"[37]

that the primary providers should allow unlicensed users (secondaries[3]) to access the channel. Towards that end, both the technical and legal hurdles have already been removed. The advent of the cognitive radio technology has enabled the users to intelligently select the idle frequency bands and switch across the bands as per the necessity. FCC has legalized the TV "white space usage" in 2004.

However, the secondary spectrum access will only proliferate if it is rendered profitable to the primaries. Realizing the above, TV white space trading where primaries lease their spectrum in lieu of financial remuneration has already been initiated. As of now[4], company such as 'Spectrum Bridge' acts as a spectrum broker and acquires the information of the unused spectrum from the primaries and advertises them in its database. The secondaries can buy them through an auction mechanism or direct negotiation with the primaries. The spectrum brokers retain a portion of the price paid by the secondary and forward the rest to the primary. *The primaries lease their spectrum on the long term basis (e.g. yearly, or monthly).*

However, the spectrum trading is yet to be widely deployed. This is because [2]–

- Primary can use the channel only in a limited manner during the period of the lease. Since the lease is on a long term basis, the primary is reluctant to lease the spectrum as it may need it in near future.

[3]The word 'secondary' may denote to either the local wireless service provider which does not have license of the spectrum or the end-users which can buy the Cognitive Radio devices and can access the spectrum of the primary.

[4]There are also some spectrum trading based on the one-to-one negotiation between the secondary and the primary, however it has some serious drawbacks; such as the secondary can not choose among all the TV white spaces, and compare the prices from other primaries.

- It is difficult for a primary to decide a price on the long term basis as the valuation of its spectrum may change in future depending on the availability of the other spectrum.

- Primaries feel that the spectrum brokers (e.g. ‘Spectrum Bridge’, ‘M2M Spectrum Networks’) may acquire most of the profit reducing the profits of the primaries.

- Spectrum Brokers have to maintain a database of unused spectrum over a large number of locations which is a challenging task. A large amount of information has to be exchanged between the primary and the spectrum broker, and the secondary and the spectrum broker for a transaction.

- The transaction costs are high. There is also a delay in approval of such lease (FCC generally takes 45-60 days to approve such lease).

Failure of the above markets have motivated the researchers to consider the alternatives such as short term leasing of the spectrum. The short term leasing has some potential advantages. For example, the primaries are no longer required to adapt their own usage in order to honor the secondary usage on a long term basis. It will also enable the primary to decide price efficiently as the primary only has to know the valuation of the spectrum for a small time period. Spectrum brokers are also unnecessary, thus all the complications with the spectrum brokers do not arise. Though the peak usage of the primary’s spectrum is high, the usage of a primary’s spectrum fluctuates drastically over short time [8]. Thus, if the secondary spectrum is operated on a short term, more spectrum can be made available to the secondaries. The transaction cost in such a short term market will also be low. The real time secondary spectrum market also has some additional benefits. Apart from the local service providers (which mainly act as the secondaries in the long term lease) the end-users can also act as secondaries and use the spectrum on the short term basis by paying the primaries, thus, the demand will also increase. Apart from the traditional primaries such as the wireless service providers, and the TV broadcasters, owners of various WiFi Access points (e.g. hotel owners, Airport authorities, hotspot owners) can also lease their channels to the secondaries on a short term basis.

Real time secondary spectrum market where the primaries lease their spectrum on a short term basis has already been proposed in the wireless economy literature [65, 8]. Based on the above observations and from [8] the criteria for the success of the secondary spectrum market are–i) the leases are made in the short term basis (e.g. hourly, or daily basis), ii) the trading should be well understood by the participants and the exchange of information should be limited, iii) primaries have to be compensated adequately– multiple primaries can directly quote their prices to the secondaries i.e. the market should be devoid of spectrum brokers, iv) the trading should facilitate efficient allocation, and v) primaries can lease the channels to the secondary without affecting the performance degradation of the primary-UTs in order to avoid "the tragedy of commons"[5].

However, there exist major challenges before the deploying of the secondary spectrum market on the short term basis. There are two broad categories of challenges. First, there are some policy issues which have to be circumvented through new regulations. For example, FCC has to reduce the transaction costs, and the current delays in transactions. FCC is already working in this direction [77]. Second, there are several technical issues

[5]FCC has also defined "interference temperature" as a specific way to constrain interference at the primary-user terminals.

which have so far inhibited the deployment of the secondary spectrum market. This dissertation seeks to redress some of those challenges enabling the large scale proliferation of the secondary spectrum market.

We consider a secondary spectrum market which is easy to operate in the real time. In this market, a primary chooses a price for its available channel in a database without coordinating with the other primaries. The secondaries then decide on the channels to buy based on the qualities. The market is readily scalable. Note that it eliminates the need of spectrum brokers. Yet, the primaries need a framework to decide their prices which we focus on this dissertation. The selection of the prices will be different depending on various associated factors such as– the variation of the channel states over time, the information that the primaries have, the various stages of the deployment of the secondary spectrum market, and the cost of obtaining an additional information. In the following, we discuss the challenges involved in designing the frameworks for selecting the prices in each of the above in detail. We also discuss the challenges involved in designing an interference mitigation policy when multiple self-interested secondaries use the channel of a primary at a location.

1.2 Quality Sensitive Price Competition in the Secondary Spectrum Market: Single Location

The quality of the wireless spectrum evolves randomly over time because of the fading, the noise and the usage statistics of the subscribers of the primaries. Such time dependent factors do not arise in the long term leasing. At a certain time the transmission rate offered by the channel of a primary may be very low which may be of low value to a secondary, but the same channel can offer a very high transmission rate at some other time which may be of high value to a secondary. In general, the primary is aware of its own channel state, however, *it is not aware of the channel state of its competitors.* This is because–

- A primary has to estimate the channel state of its competitor by sensing and analyzing the traffic pattern which may be costly. Thus, the primary may decide against it.
- Even if the primary estimates the channel state of the competitor, the estimation error may be large because of the *fast fading*, noise and measurement error. Thus, acquiring the channel state information (CSI) of the competitor may not be beneficial to a primary.
- Note that if a primary knows that the channel state of its competitor is poor, then it can select a very high price as it will have a *monopoly* power. The regulator may dictate against acquiring the CSI of the competitor in order to avoid a primary having the *monopoly* power.

We expect that initially secondary market on the short term basis is likely to be introduced in geographically dispersed locations which are unlikely to interfere with each other. In Chapter 2 we consider a secondary spectrum market where each primary selects a price based on its own channel state to the secondary at a location[6]. The secondaries select channels depending on the prices and the transmission rates. However, since a primary is

[6]Since there is only one secondary, the primary can specify the interference limit to the secondary [65].

unaware of transmission rates offered by channels of its competitors; this special feature known as *uncertainty in competition*[7] complicates the analysis. For example, if a primary quotes a high price, it will earn a large profit if it sells its channel, but may not be able to sell at all; on the other hand a low price will enhance the probability of a sale but may also fetch lower profits in the event of a sale. Further, the impact of the transmission rate on the pricing strategy of a primary is also not apriori clear. *In Chapter 2 we characterize the pricing strategies, analyze their properties and the payoffs of the primaries.*

1.3 Multiple Locations: Spatial Reuse

The market must eventually operate on a region consisting of multiple locations. In Chapter 3 we consider the setting where a primary owns a channel over multiple locations and seeks to lease its channel to at most one secondary at each location. Radio spectrum possess a special property known as *spatial reuse*: The same spectrum band can be utilized simultaneously at geographically dispersed locations without interference; but the same band can not be utilized simultaneously at interfering locations. This special feature adds another dimension in the strategic interaction as now a primary has to cull a set of non-interfering locations, which is denoted as an *independent set* in the conflict graph representation of the region[80]; at which to offer its channel apart from selecting a price at every node of that set. Intuitively, a primary would like to make its channel available at an independent set of the maximum size (cardinality). However, a primary is not

[7]Note that a secondary only buys a channel which offers a transmission rate above a certain threshold. Thus, if the transmission rate offered by the channel of a primary is very low, the primary can not sell its channel. Thus, a primary is also not aware of the number of competitors with whom it is competing.

aware of the channel states of its competitors. Thus, at a maximum independent set a primary may face a stiff competition from the other primaries which have higher quality channels at the maximum independent sets. Thus, the primary may achieve higher payoff by setting high price at small independent sets (where the competition is not so intense). It is also not apriori clear what are the independent sets that a primary will select based on the transmission rate offered by the channel.

In Chapter 3, we consider two settings–i) the channel state is the same throughout the region, ii) the channel state can vary across the locations. The first setting is likely to arise in initial deployment of the secondary market where the market will consist of a small number of locations. When the number of locations is large (which occurs in the later stage of deployment of the secondary market) the second setting is likely to arise.

1.4 To acquire the CSI of the competitor or not?

In Chapter 4 we extend the setting discussed in section 1.2 to consider the scenario where a primary can acquire the CSI of the other primaries. However, a primary incurs a cost to acquire the CSI. For example, a primary needs to sense the channel of other primaries and analyze their traffic patterns. The primary, then, needs to estimate the channel state based on that information. All these processes require power which is costly to procure. Each primary needs to decide i) whether to acquire the CSI of other primaries, and ii) the price depending on whether it has acquired the CSI of the competitors or not. The channel is available (unavailable, resp.) if the transmission rate offered is higher (lower, resp.) compared to a threshold.

We now illustrate the challenges involved in analyzing the setting. A primary needs to select whether to acquire the CSI of its competitors and a price when its own channel is available. However, while taking its own decision a primary does not know whether its competitors decide to acquire the CSI or not. Knowledge of the CSI of the other primary has potential advantage. For example, if the primary knows that the competitors' channels are not available, then, it can select the highest possible price and still can sell its channel because of the lack of competition. Thus, conventional wisdom suggests that a primary should acquire the CSI of the competitors and gain a higher payoff. However, conventional wisdom is not definitive. For example, if the other primaries also acquire the CSI, then they may select lower prices if the number of available channels are high which also may reduce the expected payoff of the primary as the primary also needs to select a lower price to sell its channel. Additionally, a primary incurs a cost to acquire the CSI of the competitors. Thus, it is not apriori clear whether a primary acquires the CSI of the other primaries.

The inherent uncertainty in the competitor's decision also complicates the pricing strategy of the primary. The pricing strategy not only depends on the information the primary has, but also the information its competitors' have. For example, suppose that there are two primaries. If the primary (A) knows that the channel of the competitor (B) is unavailable, then it will select a higher price, however, if the channel of the primary B is available, then whether primary A will be able to sell its channel inherently depends on the price selected by primary B. On the other hand, the price of primary B inherently depends on whether it acquires the CSI of the primary A or not. Primary B may select different prices depending on whether it acquires the CSI of primary A or not. This in turn complicates the decision of primary A as primary A is unaware of the decision of the primary B.

If a primary acquires the CSI of its competitors' it selects a price depending on how many competitors' channel states are known and how many channels are available. Note that in Section 1.2 we consider the setting where the primaries *do not know* the channel states of their competitors. Thus, a primary only needs to select a price depending on only its own channel state in the setting considered in Section 1.2. As explained in the previous paragraph, in this setting, the price of the primary also depends on the information its competitors have. However, a primary (A, say) is also unaware whether the other primaries have acquired the CSI of their competitors' and how many channels are available among the acquired CSIs (if it does so), while in the setting considered in Section 1.2 the primary A knows that the CSIs of the competitors are also unknown to other primaries. Thus, there is also additional uncertainty regarding the information that the competitors of primary A have in this setting. The error in estimating the channel state is also very common because of the noise in the environment, the lack of co-ordination among the primaries, and the fast fading of the channel. Thus, even if a primary estimates the CSI of the competitor, the actual channel state of the competitor can be different from the estimated state. To simplify the analysis, we consider that there are two primaries and the channel is either available or unavailable i.e. the available channels are statistically identical. It also resembles the competition setting in practice as in many countries the wireless market is mostly shared by only two primary service providers. For example, in the USA, Verizon and AT&T have a combined market share of almost 70%.

We characterize the impact of the estimation errors, the cost of acquiring the CSI of the competitors and the channel availability probabilities on the decision, and the payoffs of the primaries in Chapter 4.

1.5 Co-existence of Multiple secondaries in the channel of a Primary

When multiple secondaries use the spectrum of a primary at a location, the interference mitigation policy is required to limit the interference at each primary-UT. However, the secondaries are independent entities which only want to maximize their own utilities. A secondary is also unaware of the channel parameters of the other users, thus, the interference mitigation in such a setting is an uphill task [8].

In Chapter 5, we consider the setting where a primary allows multiple secondaries to use the channel simultaneously at a given location along with the primary users[8]. Specifically, we consider a setting where multiple pairs of secondary-Base stations (secondary-BSs) and secondary-user terminals (secondary-UTs) co-exist with multiple primary-user terminals (primary-UTs). A secondary-BS serves the secondary-UT. A secondary-BS must select its transmission power using cognitive radio technology such that *the total interference from secondary-BSs at each primary-UT is below an acceptable threshold.*

Since each secondary-BS is an independent entity and selects its transmission power level in order to maximize only its own utility, a non cooperative game theoretic setting

[8]The above setting differs from the "spectrum commons model" since we consider that secondaries must transmit with powers such that the interference must be limited at the primary-users.

is a preferable model in this context. A distributed algorithm is also preferable to obtain an equilibrium in order to keep the exchange of information among the secondary-BSs limited. Despite the minimal exchange of information, the secondary-BSs must satisfy the interference constraint at each primary-UT. Since the secondary-BSs are not operated by the primary-BS owners and primary-UTs are oblivious of the number of active secondary-BSs, the exchange of information between primary- and secondary-BSs should also be limited. To summarize, our goal is to obtain an equilibrium power allocation strategy profile using a distributed approach based on a non cooperative game theoretic setting with secondary-BSs as players. Additionally, the primary-UTs should be oblivious of the secondary-BSs and the interference constraint should be satisfied at each primary-UT. In Chapter 5 we provide an equilibrium strategy profile among the secondaries which satisfy the above characteristics.

1.6 Our Contributions and Related Literature

1.6.1 Uncertainty in Competition: Single Location Game

In Chapter 2 we consider the setting where the primary owns a channel only at a single location. As discussed in Section 1.2 the primary is only aware of the channel state of its own channel, however, it is not aware of the channel state of its competitors. We formulate the price selection as a game in which each primary selects a price depending on the transmission rate its channel provides. We consider that the preference of the secondaries can be captured by a penalty function which associates a penalty value to each channel that is available for sale depending on its transmission rate and the price quoted. Secondaries prefer the channels which induce lower penalty values. Since there is an one-to-one relationship between price and penalty we consider penalty selection strategies instead of price selection strategies. We characterize the Nash equilibrium (NE) penalty selection strategies. We show that for a large class of penalty functions, there exists a *unique* NE strategy profile, which we explicitly compute.

We show that the unique NE strategy profile is symmetric i.e. price selection strategy of all primaries are statistically identical. We show that at channel state i, the primary selects a penalty from the interval $[L_i, U_i]$ where $U_i \leq L_j$ if the channel quality is low when the channel state is in state j compared to state i . Thus, our analysis reveals that primaries select price in a manner such that the preference order of transmission rates is retained. *This negates the intuition that prices ought to be selected so as to render all transmission rates equally preferable to a secondary.* The analysis also reveals that the unique NE strategy profile consists of "nice" cumulative distributions in that they are continuous and strictly increasing; the former rules out pure strategy NEs and the latter ensures that the support sets are contiguous. We also numerically show that as the number of states goes to infinity each primary selects a pure pricing strategy at a given channel state.

Subsequently, utilizing the explicit computation algorithm for the symmetric NE strategies, we analytically investigate the reduction in expected profit suffered under the unique symmetric NE pricing strategies as compared to the maximum possible value allowing for collusion among primaries. Finally, we extend our one shot game at single location, to a repeated game where primaries interact with each other multiple number of times and compute a subgame perfect Nash equilibrium (SPNE) in which a primary attains a payoff which is arbitrarily close to the payoff that a primary would have obtained if primaries select prices jointly; thus, price competition does not lower payoff in a repeated game.

Related Literature

Price selection in oligopolies has been extensively investigated in economics as a non co-operative Bertrand Game [59] and its modifications [64, 51]. The price competition among wireless service providers have also been explored to a great extent ([41, 58, 57, 83, 62, 63, 91, 47, 76, 13, 89, 43, 87, 73, 49]). We divide this genre of works in two parts: i) Papers which model price competition as Auction where a central auctioneer or spectrum broker collect the bids and allocates the spectrum to the secodnaries ([73, 85]), and ii) Papers which model the price competition as a non co-operative game ([41, 58, 57, 83, 62, 63, 47, 43, 87, 76, 89, 55, 13, 49]). We now distinguish our work with respect to these papers.

As compared to the genre of work in the first category our model has some advantages. First, in our model, a central auctioneer or spectrum broker is not required. Second, since the channel states of the primaries evolve randomly, the prices also fluctuate in the auction framework considered in [73, 85]. For example, when the number of secondaries is higher than the number of available channels, the price is set at the highest possible value which reduce the utilities of the secondaries; on the other hand, when the number of high quality channels are higher compared to the number of secondaries, the price is set at the lowest possible value which reduce the payoffs of the primaries to 0. Thus, the primaries and secondaries may not participate in such a market. However, in our setting since a primary is not aware of the channel states of the other primaries, it neither selects too high price nor too low price. Thus, primaries and secondaries always achieve positive payoffs which provide incentives to the primaries and secondaries to participate in the secondary market. Third, it is not clear that players will bid truthfully in the auction framework. In [73, 85] a VCG type auction has been proposed which is proved to be truthful, however, in the VCG auction the payment made to the primaries is not straightforward, the payment to a primary depends on the " externality" added by a primary on the other competitors. A computationally intensive algorithm is required to find the price to be paid to the primary. On the contrary, in our setting the primaries quote their prices, and the secondary pays the primary whatever it quotes if the secondary buys the channel. Hence, our setting is readily scalable and implementable in practice. Fourth, despite the non cooperative setting, our result shows that a *socially efficient outcome*[9] is achieved in our setting. Specifically, we show that the high quality channels are rendered more profitable compared to the low quality channels.

Some papers in the second category [43, 87, 76, 89, 13, 49] considered the quality of primaries as a factor while selecting the price. But all of these papers mentioned above, ignore *uncertainty in competition* which distinguish secondary spectrum oligopoly from standard oligopolies: a primary selects a price knowing only the transmission rate of its own channel; it is unaware of transmission rates offered by channels of its competitors. We have considered the uncertainty in competition.

Some recent works ([45],[42],[50], [46]) that consider uncertainty in competition assume that the commodity on sale can be in one of two states: available or otherwise. This

[9] In Chapter, we also provide one more advantage of our approach compared to the auction mechanism. Prices have more variability when it is set in an auction framework compared to our approach.

assumption does not capture different transmission rates offered by available channels. A primary may now need to employ different pricing strategies and different independent set selection strategies for different transmission rates, while in the former case a single pricing and independent set selection strategy will suffice as a price needs not be quoted for an unavailable commodity. Our investigation in Chapter 2 seeks to contribute in this space.

1.6.2 Spatial Reuse

Subsequently in Chapter 3, we consider the setting where a primary owns a channel over multiple locations which we briefly introduced in Section 1.3. We devise the problem as a game in which each primary's strategy space consists of independent set selection strategy and the pricing strategy at each node of the independent set when the channel is available for sale. The channel is assumed to be in either one of the states $0, 1 \ldots, n$ where lower channel state represents lower transmission quality. When the channel is in state 0, the transmission rate is very low and thus, we consider the channel is not available for sale.

Scenario 1

We consider two possible scenarios. First , we consider the setting when the region is small consisting of few locations. Therefore, the usage statistics and the propagation condition of a channel do not vary substantially over the region. Thus, we assume that the channel state is identical at each location in this setting. In the initial stages of deployment of the secondary market, it is expected that the secondary market will be introduced in small regions consisting of a few locations. Hence, the price competition in this setting reduces to a price selection problem where the transmission quality of each primary remains the same throughout the region.

We first show that there may exist multiple asymmetric NEs. Asymmetric NEs are difficult to implement in the symmetric game that we consider. We, therefore, focus only on finding symmetric NEs subsequently. We prove a *separation theorem* which entails that the NE pricing strategy at each location can be uniquely computed if the independent set selection strategy is known. By virtue of our work in Chapter 2 which characterizes pricing strategies of primaries for different transmission rates when the region has only one location (i.e. no spatial reuse), we then focus only on the independent set selection strategy.

In this setting, we focus on a particular class of conflict graphs, introduced as *mean valid graph* [46] since most of the small graphs observed in practice are mean valid graphs. In a mean valid graph, nodes can be partitioned in d disjoint maximal independent sets namely $I_1, \ldots, I_d$ [46]. But the total number of independent sets in such a graph may be substantially large; generally, the number of independent sets grows exponentially with the number of nodes. We show that there exists a symmetric NE strategy which selects independent sets only amongst $I_1, \ldots, I_d$ which characterize the mean valid graph; we explicitly compute the strategy. Such a strategy profile can be stored using a d dimensional vector. Thus, the space required to store strategy profile scales with d rather than increasing exponentially with nodes. Primaries also need to know only $I_1, \ldots, I_d$ rather than the entire graph in order to compute a symmetric NE.

The characterization of the symmetric NE strategy profile reveals that a primary only selects an independent set whose cardinality is greater than or equal to a certain threshold. This threshold turns out to be a non-decreasing function of channel quality. Thus, when the channel quality is high, a primary restricts itself only to independent sets of large cardinalities; when the channel quality is poor, the primary diversifies among independent sets of different sizes. We show using an example that arises in practice that primaries only offer their poor quality channels at independent sets of lower cardinalities. Thus, a social planner may have to provide some incentives to primaries so as to ensure that users of those locations can get access to higher quality channels.

Next, we examine the uniqueness among symmetric NE strategy profiles in mean valid graphs. Nodes in such a graph can be partitioned into different collections of maximal independent sets (Fig. 3.6). A primary in general would not know the partition other primaries are selecting. Our result reveals that each such partition leads to a unique symmetric NE; yet primaries need not co-ordinate with each other regarding the partition one is selecting. Hence the symmetric NE strategy profile is easy to implement. We also show that all these symmetric NEs lead to the same node selection probabilities. The NE pricing strategy at a node depends only on the probability with which it is selected. Thus, all these symmetric NEs are functionally unique. Finally, we focus on a special class of mean valid graphs known as *linear graphs* (Fig. 3.1) which frequently arises in practice such as in the modeling of communication nodes over a highway or a row of shops. We prove that the symmetric NE strategy is unique (is not merely functionally unique) in linear graphs.

Scenario 2

Subsequently, we consider the scenario when the secondary spectrum market is operated on a large region consisting of several locations. In this setting the transmission quality of a channel may be different at different locations in the region. Thus, a primary needs to specify a strategy for each possible channel state across the network. The number of channel states and thus, the strategy space increases exponentially with number of nodes. The conflict graph representation of the region depends on the channel state across each location since a primary must select an independent set of nodes only among those nodes where the channel is available for sale. A primary is not aware of the conflict graph from which other primaries are selecting their independent sets let alone their channel states. The characterization of a symmetric NE strategy profile in the above setting is thus, more challenging. We simplify the model by assuming that the channel is either available or not (i.e. $n = 1$), but the availability can differ across the nodes.

We focus on node symmetric or node transitive graphs [70] such as finite cyclic graph, infinite lattice graphs (e.g. infinite linear graph (infinite in both directions), infinite square graph, infinite grid graph, infinite triangular graphs)[79] which arise in practice when the region becomes large. We allow some statistical correlations which arise naturally among the channel states at different locations when the number of locations is large. We show that there exists a symmetric NE strategy profile (SP_{sym}) for those graphs. In the symmetric NE strategy profile, a primary randomizes uniformly among the maximum independent sets (the independent set of the highest cardinality). A primary thus only need to enumerate the maximum independent sets in order to determine SP_{sym}. In contrast to the setting where the channel state remains the same through the network, in $\mathrm{SP}_{\mathrm{sym}}$ the channel is offered at every node with equal probability. We also show that $\mathrm{SP}_{\mathrm{sym}}$ may not be an NE in a finite linear graph which *is not a node symmetric graph.* We show that the symmetric NE may not be unique for a linear graph unlike the setting where the channel state remains the same throughout the network.

In $\mathrm{SP}_{\mathrm{sym}}$ each primary needs to enumerate the maximum independent sets. The number of independent sets grow exponentially with the nodes. However, at a given channel state vector over the region, the conflict graph may consist of several components. A primary can find maximum independent sets and SP_{sym} in each component in parallel. However, the number of maximum independent sets in a component grows exponentially with the number of nodes in the component. We, thus, investigate the size of the expected component size both analytically and empirically. Empirical result shows that the average size of components is often moderate and the upper bound computed analytically is often loose. However, empirical and analytical results show that the component size can be substantially large when the channel availability probability is large. In order to control the component size, we consider the setting where each primary decides to estimate the channel state at a node with a certain probability (p). A primary then sells its channel at nodes only amongst the nodes where it estimates the channel. We show that $\mathrm{SP}_{\mathrm{sym}}$ is a NE strategy in this setting as well. However, if p is small, then a primary can only sell its channel at few locations which will potentially reduce the payoff. A primary thus needs to select p judiciously in order to attain a required trade-off between the computation cost and the expected payoff.

Finally, we numerically compare the expected profit obtained by the primaries using our NE strategy profile in both of the settings to the maximum possible profit allowing for collusion among primaries.

Related Literature

Price selection in oligopolies has been extensively investigated in economics which we have analyzed in the previous section. However, most of the papers did not consider the *spatial reuse* property of the secondary spectrum market. Only papers [31, 91, 46] considered the spatial reuse property. [91, 31] considered an Auction framework where a central auctioneer runs the auction to determine the winners. In our framework, we do not need any auctioneer. The computational complexity is also higher to determine the winner with the spatial reuse in an auction framework. We have provided computationally efficient NE strategies. The different channel states across the network is also not considered in the above framework.

We now distinguish our contributions compared to [46] which is the closest to our work. First, [46] considered that the channel state remains the same throughout the region and the state of the channel can be either 0 (not available for sale) or 1 (available); this assumption does not capture the different transmission qualities offered by the available channels. When we consider that the channel state remains the same throughout the network we consider that the available channel can be in one of the n states depending on the transmission qualities. Thus, in our setting a primary now needs to employ different pricing strategies and different independent set selection strategies for different channel states while in the former case a single pricing and independent set selection strategy would suffice as the price need not be quoted for an unavailable commodity. Second, we also consider the setting where the channel state need not be the same unlike in [46]. In our setting a primary does not know the conflict graph of other primaries from which they will select their independent sets. Thus, the collection of independent sets from which a primary selects an independent set may be different for different primaries at a given time slot since the channel state vector may be different for different primaries. Whereas in [46] the channel is either available at all locations or unavailable at any location. Thus in [46], a primary knows the conflict graph from which other primaries will select their independent sets when their channels are available. Thus, the characterization of an NE becomes significantly challenging in our setting compared to [46]. The result we obtain also significantly differs from [46]. For example, in [46] a primary can select an independent set of lower cardinalities, however, in our setting, a primary only selects the maximum independent set. Additionally, the symmetric NE is unique in a finite linear graph in [46], whereas there are infinitely many symmetric NEs in our setting.

1.6.3 Provision of acquiring the CSI of the competitor

In Chapter 4 we consider the setting where a primary can acquire the CSI of its competitor by incurring the cost which we briefly introduced in Section 1.4. Note from Section 1.4 we only consider the setting where there are two primaries and the channel is either available or unavailable. We model the setting as a non-cooperative game with two primaries as players. When the channel of a primary is available, each primary decides i) whether to acquire the competitor's CSI or not, and ii) a price. Selection of the price depends on the information the primary has. Specifically, if the primary acquires the competitor's CSI it may select different prices depending on whether its competitor's channel is available or not. On the other hand, when the primary does not acquire the competitor's CSI, it has to select a price irrespective of the competitor's channel state. We characterize the Nash equilibrium (NE) strategies.

Basic Model: We first consider a *basic model* where each primary accurately estimates the channel state of its competitor by acquiring its CSI. The channel availability probability and the cost of acquiring the competitor's CSI (C-CSI) are the same for both the primaries. We introduce a $[T, p]$ class of strategy and show that there exists NE strategy profile which is of the above form. In the $[T, p]$ type strategy–i) a primary acquires the C-CSI with probability (w.p.) p when the cost is below T and ii) does not acquire the C-CSI when the cost is above T. We, additionally, show that p increases as the cost of acquiring the C-CSI decreases. It is apparent that as the cost of acquiring the C-CSI decreases, a primary should acquire the C-CSI with a higher probability. However, we show that a primary never acquires the C-CSI with probability 1 irrespective of the cost. Additionally, we also show that T depends on the availability probability of the competitor and it increases as the uncertainty of the availability of the competitor's channel increases.

We find that the expected payoff of a primary is independent of the cost of acquiring the C-CSI and is the same as in the setting where acquiring the C-CSI is not possible. Thus, we have the following counter-intuitive result: *the ability to acquire the competitor's CSI does not increase the expected payoff of the primary.*

Intuitively, when a primary knows (does not know, resp.) the channel of its competitor is available, a primary should select prices more conservatively (aggressively, resp.). Our results validate the above intuition and go beyond. We show that when the primary knows that its competitor's channel is available then it selects its price from an interval $[L, \tilde{p}_1]$, on the other hand when the primary does not know the channel state of its competitor then it selects its channel from the interval $[\tilde{p}_1, v]$ where v is the highest possible price. We have fully characterized $\tilde{p}_1$ and L. In the NE pricing strategy, both the primaries select their prices from an interval using a continuous distribution.

Impact of the Estimation Error: We, subsequently, investigate the impact of the channel estimation error on the decision, payoff and the pricing strategy of the primary. Specifically, we consider the setting where each primary accurately estimates the channel state of its competitor with a probability q_s. Conventional wisdom suggests that decrease in the error in estimating the channel state should increase the payoff. However, conventional wisdom is not definitive because of the following. A primary selects a higher price even when it estimates that the channel of the competitor is available as the channel may be unavailable with a positive probability if there is an error in estimation. The pricing strategy also inherently depends on the estimation error. We characterize the impact of the estimation error on the strategy and the payoff of a primary in Chapter 4.

We show that there exists a $[T, p]$ type NE strategy where the threshold T decreases as the error in estimation increases. Intuitively, increasing estimation error leads to more uncertainty about the C-CSI, making acquiring the C-CSI less attractive for larger costs. We show that the probability p with which a primary acquires the C-CSI increases as the acquisition cost decreases. *Interestingly*, we show that the expected payoff of a primary is *higher* when there is *an error* in estimating the C-CSI. Thus, *it negates conventional wisdom that the payoff of the primary should decrease as the error increases.* In contrast to the basic model, the expected payoff also increases as the cost of acquiring the C-CSI decreases.

We show that when the primary estimates that the competitor's channel is available (unavailable, resp.) then it randomizes among the prices in the interval $[\tilde{p}_1, L_N]$ ($[L_0, v]$, resp.) where $L_0 > L_N$ and v is the highest possible price. When the primary does not estimate the C-CSI, then, it neither selects a too high price nor a too low price; specifically, it selects its price from the interval $[L_N, L_0]$. Apparently, when a primary knows that the channel of its competitor is available (unavailable, resp.) with a higher probability, a primary selects a lower (higher, resp.) price. In the basic model, when the primary accurately estimates that the channel of the competitor is unavailable, it selects the highest possible price v w.p. 1. However, when there is an error in estimating the channel state, the channel of the competitor may be available even when it estimates that the channel is unavailable. Thus, a primary also selects a lower price. We also show that the variance of the price selected by a primary increases as the estimation error (the acquisition cost, resp.) decreases.

Impact of Unequal cost of acquiring the CSI: We, subsequently, investigate the setting where different primaries may have different costs of acquiring the C-CSI. The impact of different C-CSI acquisition costs on the frequency with which each primary acquires the C-CSI is not apriori clear. For example, the primary (say, 1) who incurs a lower cost, can acquire the CSI of the competitor (say 2) more often. However, primary 2 may also acquire the C-CSI and select a lower price when the channel of primary 1 is available. Thus, primary 1 has to select a lower price which reduces the possibility of acquiring the C-CSI. The pricing strategy also inherently depends on the frequency with which each primary acquires the CSI of its competitor. We characterize the NE strategies, and the payoffs of the primaries.

We show that the NE strategy is of the form $[T, p_i]$ for primary i. Though the thresholds are the same p_is are different for different primaries. We show that the primary 1 acquires the C-CSI with a higher probability compared to the primary 2. The expected payoff of primary 2 is the same the payoff it obtained when it did not acquire the C-CSI, thus, the provision of the acquiring the C-CSI does not impact the payoff of the primary which has higher acquisition cost. However, in contrast to the basic model, the expected payoff of primary 1 is higher compared to the expected payoff of the primary 2 when primary 1 acquires the C-CSI . Additionally, the expected payoff of the primary 1 decreases as the difference between the costs of acquiring the CSI decreases.

We show that the primary i selects its price from the interval $[L, \tilde{p}_i]$ ($[\tilde{p}_i, v]$, resp.) when primary i acquires (does not acquire, resp.) the C-CSI and the channel of that primary is available. However, we show that in contrast to the basic model, $\tilde{p}_1 > \tilde{p}_2$. We, also, show that primary 2 selects its price from a distribution which has a discontinuity at the highest possible price when it does not acquire the C-CSI. Thus, the primary which has higher cost of acquiring the C-CSI, selects higher prices with higher probabilities when it does not acquire the C-CSI.

Impact of Unequal availability probabilities: We, subsequently, investigate the impact of primaries having different availability probabilities on the competition. The frequency with which each primary acquires the C-CSI can not be readily concluded because of the following. Acquiring the C-CSI depends on the availability probability of the competitor. If the availability probability is high or low, a primary may not acquire the C-CSI. On the other hand if the uncertainty is high regarding the availability of the competitor's channel, a primary will more likely to acquire the C-CSI. Since primaries have different availability probabilities, thus, it is not apriori clear whether the primary which has higher availability probability (say, 1) will acquire the C-CSI compared to the other primary (say, 2). Conventional wisdom suggests that the payoff of the primary should not decrease with the decrease in the cost of acquiring the C-CSI. However, it is not definitive because of the following. If primary 1 acquires the C-CSI with a higher probability, it selects a lower price when the channel of primary 2 is available. Since the channel of primary 1 is available with a higher probability, the channel state of primary 2 will be known to primary 1 with a higher probability. Thus, primary 2 selects a lower price in order to sell its available channel which may reduce the payoff of primary 2. The pricing strategy of each primary also inherently depends on the frequencies with which each primary acquires the CSI of its competitor. We resolve all these quandaries.

We show a NE strategy which is of the form $[T_i, p_i]$ for primary i. We show that $T_1 > T_2$ and $p_1 > p_2$. Thus, interestingly, irrespective of the availability probabilities primary 1 acquires the C-CSI more frequently compared to the primary 2 when the cost is below T_1. The expected payoff of primary 1 is also higher compared to primary 2. Moreover, the expected payoff of primary 2 decreases as the cost of acquiring the C-CSI of its competitor decreases which negates *conventional wisdom that the payoff of a primary should not decrease as the cost of acquiring the C-CSI decreases.*

We show that primary i selects its price from the interval $[L, \tilde{p}_i]$ ($[\tilde{p}_i, v]$,resp.) when it acquires (does not acquire, resp.) the C-CSI and the channel of the competitor is available. We, show that $\tilde{p}_1 > \tilde{p}_2$, thus, even when primary 1 acquires the C-CSI it selects a higher price compared to primary 2 when the channel of the competitor is available. We also show that in contrast to the basic model, primary 1 selects its price from a function which has a discontinuity at the highest possible price when it does not acquire the C-CSI. Apparently, since primary 1 has a higher availability probability, it selects higher prices with higher probabilities when it does not acquire the C-CSI.

Related Literature

Some recent papers [45, 42] imposed the constraint where the players *can not know* the channel states of their competitors. However, in our setting a primary (A, say) is also unaware whether the other primary has an acquired the CSI of A, while in [45, 42] the primary A knows that its CSI is unknown to other primary. Thus, a primary now needs to judiciously decide whether to acquire the CSI of its competitor or not and selects a price based on the acquired CSI. None of the other papers discussed in Section 1.6.1 considered the impact of the CSI on the decision of the player which we consider. Naturally, the impact of the cost of acquiring the CSI, error in estimating the channel state and different channel availability probabilities on the decision of the primaries have not been considered in the above papers. We contribute in this space.

1.6.4 Co-existence of Multiple secondaries

In Chapter 5 we consider the setting where a primary allows multiple secondaries to share the spectrum at a location. Specifically, we consider a setting with multiple secondary-BSs and multiple primary-UTs. Each secondary-BS serves only one secondary-UT. We formulate the power allocation problem among secondary-BSs as a coupled constrained concave game [67] with secondary-BSs as *players*. Each secondary-BS selects its transmission power to maximize its own utility subject to the constraint that the total interference at each primary-UT must be below a threshold. Hence, the strategy of a player as well as its utility also depends on the strategy of other players. There are multiple Nash equilibria in the coupled constrained concave game in general. In order to design distributed algorithms converging to a well defined and unique equilibrium we resort to the concept of normalized Nash equilibrium (NNE) introduced by Rosen in [67].

We consider two scenarios, in the first one the interference at a secondary-UT from other secondary-BSs, i.e., the inter-secondary-network interference is negligible. This scenario is likely to arise when the secondary-BSs have small coverage area and the number of secondary networks is small. In the second scenario, the inter secondary network interference is non negligible and the utilities of secondary-BSs *directly* depend on the policies selected by the other secondary-BSs. This situation is likely to arise when the coverage of secondary-BS is large and/or the number of secondary networks is large. The analysis of the former scenario is ancillary to the analysis of the latter one and motivates the introduction of the concept of WNNE to extend the appealing properties of the NNE in the former scenario to systems with inter-secondary network interference.

We show that the computation of an NNE reduces to solving a convex optimization problem when the game admits a concave potential function [61]. We propose a distributed algorithm which converges to the NNE when the game admits a strictly concave potential function. In the algorithm (Algorithm DIST) the secondary-BSs do not need to exchange information among them and the primary-UTs only need to track the total interference. *Primary-UTs select prices for the total interference caused by the secondary-BSs. Thus, this mechanism also provides an incentive to primary-UTs to participate in the secondary* *access.* We show that in the setting with negligible interference among secondary-UTs, the NNE is always unique and the game admits a strictly concave potential function. Thus, the algorithm DIST can be used in this setting to obtain the *unique* NNE.

In the setting with inter-secondary-network interference, the NNE may not be unique and the game does not necessarily admit a potential function. Nevertheless, we identify a class of utility functions which admits unique NNEs and strictly concave potential functions even in presence of co-channel interference among secondary-BSs. Thus, algorithm DIST can be used to obtain the unique NNE. We introduce the concept of Weakly Normalized Nash equilibrium (WNNE) as an equilibrium selection concept when it is difficult to compute an NNE in presence of inter-secondary-network interference.

We illustrate the significance of the WNNE by analyzing a specific game with a function that provides an achievable rate, commonly referred to as the Shannon function, as a utility function and inter-secondary network interference. We show that this game admits a unique NNE only under certain conditions depending on the parameters of the channels and does not admit a potential function. The standard algorithm to compute standard NNE increases exponentially with the number of secondary-BSs. We provide an algorithm whose complexity only scales linearly with the number of secondary-BSs. The implementation of this algorithm requires that a secondary-BS knows all the channel coefficients from each secondary-BS to all secondary-UTs and the primary-UTs. On the contrary, the WNNE can be obtained with lower complexity using the distributed algorithm DIST. When algorithm DIST is applied, each secondary-BS only needs to know the coefficients of the local channels from the BS itself and the primary-UTs. A secondary-BS does not need to know the coefficients of the channels involving other secondary-BSs and/or secondary-UTs. Moreover, WNNEs can be obtained for all possible realizations of the channel parameters.

Finally, we numerically evaluate various properties of NNE and WNNE solution for some well known utility functions.

Related Literature

Game theoretic approaches have been widely applied to wireless communication problems (see e.g. [35]). However, only those works which are related to resource allocation in heterogeneous or cognitive radio network are of particular interest to us. Algorithms for power allocation at the secondary-UTs relying on the cooperation of primary-UTs have been studied in [5, 38, 88, 48]. Power allocation in cognitive radio using Stackelberg game is studied [4, 66] with primary-UTs as leaders and secondary-BSs as followers. Power allocation for heterogeneous networks (HetNets) using Stackelberg game is studied in [34, 75]. The setting of HetNet is analogous to the setting considered in this chapter with macro- and femto-cells playing the role of primary and secondary networks, respectively. In contrast to the above mentioned works, in the game theoretical framework proposed in this chapter, primary-UTs are oblivious of the number of secondary-BSs. Primary-UTs are almost passive entities that only select prices depending on the total interference: no knowledge of each secondary-BS's utility or the channel state information is required. Thus, our model is readily scalable compared to previous models. In [40], the authors obtained an equilibrium power allocation strategy using a non cooperative game theoretic distributed algorithm. However, no interference constraints are enforced at the primary-UTs, i.e., the total interference at each primary-UT is not constrained to be below a certain threshold.

Optimal power allocation in HetNet with interference mitigation techniques has also been studied using *evolutionary game theoretic approaches*[72, 6]. Both these papers assume an identical discrete finite strategy space for each player whereas the strategy space of each player in our setting is continuous, uncountably infinite and the strategy of a player inherently depends on the strategy of other players since we consider a coupled constrained game. Additionally, in [72] and [6] a player must know the average utility of the players in order to obtain a stable equilibrium and thus, it requires either communications among players [6] or a central controller [72]. In contrast, in our setting a player, i.e., a secondary-BS, does not need to know the utility functions of other secondary-BSs in the distributed algorithm DIST to attain an NNE (or, WNNE). In [84], the authors also studied resource allocation among secondary-UTs using a stochastic learning method. Unlike the DIST algorithm, the method proposed in [84] needs a central controller and coordination among players. Thus, our approach is more practically viable and readily implementable compared to the above mentioned works.

Distributed power allocation in cognitive radio network in a non cooperative game theoretic setting has also been studied in [82, 53, 78]. Our approach differs from those works in the following aspects. Previous works focused on the characteristics of the equilibriums for very specific utility functions. In contrast, we characterize the uniqueness of the NNE for a wide class of utility functions. Moreover, for the cases when it is difficult to compute the NNE, we introduce the WNNE as an equilibrium selection approach since it retains most of the favorable properties of NNE. Thus, our equilibrium selection methods can be applied to more generalized and challenging settings. Additionally, in the above mentioned works, secondary-BSs need to exchange information among them for the distributed algorithm to converge. In the present work, algorithm DIST, which provides NNE or WNNE, does not require exchange of information among secondary-BSs. Finally, only in [82] the Shannon capacity function is adopted as a utility function. The algorithm proposed in [82] to compute an NE scales exponentially with the number of secondary-BSs. In contrast, for the Shannon capacity function in presence of inter-secondary-network interference, we identify the conditions under which an NNE is unique and we propose an algorithm to obtain the unique NNE which scales only linearly with the number of secondary-BSs.

The closest work to ours is [17] where the NNE is adopted as equilibrium concept for optimal power allocation among femto-BSs in a HetNet. In [17], a single macro-UT with interference free femto-cells is considered. The setting studied there corresponds to a setting with a single primary-UT and negligible inter-secondary-network interference. We relax both these assumptions in this chapter. We contribute in this space. The presence of multiple primary-UTs in the system raises a problem of computational complexity in determining the unique NNE: the NNE computation does not boil down to an ordinary water-filling problem as in [17]. By applying standard techniques it turns out that the problem has exponential complexity in the number of primary-UTs and secondary-BSs and we are not aware of the techniques that solve the problem with lower complexity. Along with increasing complexity, the consideration of co-channel interference at secondary-UTs in our setting has the effect of destroying the property of uniqueness of NNE always satisfied in the setting considered in [17].

1.7 Publications

- Chapter 2 is based on [26]. The shorter version has been published in [21].
- Chapter 3 is based on [24] (Archived version[28]). The shorter versions have been published in [22, 25].
- Chapter 4 is based on [27] (Archived version [29]). The shorter version will be published in [30].
- Chapter 5 is based on [19]. The shorter versions are published in [20, 18].
- Other published papers are [23], [81].

Chapter 2

Uncertainty in Competition: Single Location Game

We investigate a spectrum oligopoly market where each primary seeks to sell its channel to a secondary at a location. Transmission rate of a channel evolves randomly. Each primary needs to select a price depending on the transmission rate of its channel. However, the primary is unaware of the channel states of the other primaries while taking its decision. Each secondary selects a channel depending on the price and the transmission rate of the channel. We formulate the above problem as a non-cooperative game with primaries as the players. We show that there exists a unique Nash Equilibrium (NE) and explicitly compute it. Under the NE strategy profile a primary prices its channel to render the channel which provides higher transmission rate more preferable; this negates the perception that prices ought to be selected to render channels equally preferable to the secondary regardless of their transmission rates. We show that the non cooperation of the primaries may lead to a loss in the revenue in the asymptotic limit. In the repeated version of the game, we characterize a subgame perfect NE where a primary can attain a payoff arbitrarily close to the payoff it would obtain when primaries cooperate.

The chapter is organized as follows: In Section 2.1 we describe the system model and the strategies of the players. In Section 2.2 we characterize the structure of an NE, and show the existence and uniqueness. We also provide an algorithm to explicitly compute the NE. In Section 2.3 we compute the ratio of the payoffs of the primaries in the competitive setting and the collusive setting in an asymptotic limit. In Section 2.4 we consider the setting where the primaries interact with each other multiple times. In Section 2.5 we briefly characterize the results when some of the assumptions made in this chapter are relaxed. We finally conclude and discuss some future works in Section 2.6. We prove the results in Section 2.7.

2.1 System Model

We consider a spectrum market with $l (l \geq 2)$ primaries. Each primary owns a channel at a single location. Different channels leased by primaries to secondaries constitute disjoint frequency bands. A primary only allows at most one secondary to use it. There are m secondaries. We initially consider the case when primaries know m, later generalize our results for random, apriori unknown m (Section 2.2.3).

2.1.1 Transmission Rate

The channel of a primary provides a certain transmission rate to a secondary who is granted access. Transmission rate (i.e. Shanon Capacity) depends on 1) the number of subscribers of a primary that are using the channel[1] and 2) the propagation condition of the radio signal. The transmission rate evolves randomly over time owing to the random fluctuations of the usage of subscribers of primaries and the propagation condition[2]. We assume that at every time slot, the channel of a primary belongs to one of the states[3] $0, 1, \ldots, n$. State i provides a lower transmission rate to a secondary than state j if $i < j$ and state 0 arises when the channel is not available for sale i.e. secondaries can not use the channel when it is in state[4] 0. A channel is in state $i \geq 1$ w.p. $q_i > 0$ and in state

[1]Shanon Capacity [12] for user i at a channel is equal to $\log\left(1 + \frac{p_i h_i}{\sum_{j \neq i} p_j h_j + \sigma^2}\right)$ where p_k is the power with which user k is transmitting, σ^2 is the power of white noise, h_k is the channel gain between transmitter and receiver which depends on the propagation condition. If a secondary is using the channel then p_i, h_i of the numerator are the attributes associated with the secondary while $p_j, h_j j \neq i$ are those of the subscribers of the primaries. In general, the power p_j for subscriber of primaries is constant for subscriber j of primary, but the number of subscribers vary randomly over time. The power p_i with which a secondary will transmit may be a constant or may decrease with the number of subscribers of primaries in order to limit the interference caused to each subscriber. The above factors contributes to the random fluctuation in the capacity of a channel offered to a secondary. In our setting p_i, h_is are assumed to be the same across the secondaries for a channel which we justify later. However, these values can be different for different channels.

[2]Referring to footnote 2, h_k and σ^2 evolve randomly owing to the random scattering of the particles in the ionosphere and troposphere; this phenomenon is also known as *fading*.

[3]We discretize the available transmission rates into a fixed number of states n. This is a standard approximation to discretize the continuous function[16, 56]. The corresponding inaccuracy becomes negligible with increase in n.

[4]Generally a minimum transmission rate is required to send data. State 0 indicates that the transmission rate is below that threshold due to either the excessive usage of subscribers of primaries or the transmission condition.

0 w.p. $1-q$ where $q=\sum_{i=1}^{n} q_i$, independent of the channel states of other primaries [5]. Thus, the state of the channel of each primary is independent and identically distributed. We do not make any assumption on the relationship between q_i and l or q_i and[6] m. We assume

$$q < 1. \tag{2.1}$$

We assume that the transmission rate offered by the channel of a primary is the same to all secondaries. We justify the above assumption in the following. We consider the setting where the secondaries are one of the following types: i) Service provider who does not lease spectrum from the FCC and serves the end-users through secondary access, ii) end-users who directly buy a channel from primaries. In initial stages of deployment of the secondary market, secondaries will be of the first type. When the secondaries are of the first type, then a primary would not know the transmission rate to the end-users who are subscribers of the service provider. A primary measures the channel qualities across different positions in the locality (e.g. a cell) and considers the average as the channel quality that an end-user subscribed to a secondary service provider will get at the location (e.g. a cell). This average will be identical across different end-users subscribed to different secondary service providers and hence, the channel quality is identical across

[5] We have shown that at a given slot, the channel state differs across the primaries mainly because of the differences of i) the number of subscribers that are using the channel and ii) the propagation conditions. Since different primaries have different subscriber bases, thus, their usage behaviors are largely uncorrelated. Also, channels of different primaries operate on different frequency bands and have different noise levels, thus, the propagation conditions are also uncorrelated across the channels.

[6] Since each primary sells its channel to only one secondary, thus, referring to footnotes 2 and 3 transmission rate (or q_i) at a channel does not depend on m (secondary demand) in practice.

the secondaries.

If the secondaries are of the second type, then the primary may know the transmission rate that an end-user will attain which may be different at different positions. However, if a primary needs to select a price for each position of the location, then it needs to compute the transmission rate at each possible position at that location (e.g. a cell). Thus, the computation and storage requirement for the primary would be large. Such position based pricing scheme will also not be attractive to the end-users since they may perceive it discriminatory as the price changes when its position changes within a location (e.g. a cell). Thus, such a position based pricing scheme may not be practically implementable. Hence, a primary estimates the channel quality and decides the price for the estimated channel quality by considering that the channel quality will not significantly vary across the location. This is because end-users who are interested to buy the channel from a primary at a location most likely have similar propagation paths: for example, secondary users who buy the channels are most often present in buildings (e.g. shopping complex, an office or residential area). The distance from the base station of the primary to the end-users is also similar because the end-users are close to each other in a location. Thus, the path loss component will also similar. Hence, the channel quality is considered to be identical across the secondaries.

Though the quality of a channel is identical for secondaries, the quality can vary across the channels. A primary can get an estimate of the transmission rate by sending a pilot test signal at different positions with the location and then, applying some standard estimation techniques[7].

2.1.2 Penalty functions of Secondaries and Strategy of Primaries

Each primary selects a price for its channel if it is available for sale. We formulate the decision problem of primaries as a non-cooperative game. A primary selects a price with the knowledge of the state of its channel, but without knowing the states of the other channels; a primary however knows $l, m, n, q_1, \ldots, q_n$.

Secondaries are passive entities. They select channels depending on the price and the transmission rate a channel offers. We assume that the preference of secondaries can be represented by a penalty function. If a primary selects a price p at channel state i, then the channel incurs a penalty $g_i(p)$ for all secondaries. As the name suggests, a secondary prefers a channel with a lower penalty. Since lower prices should induce lower penalty, thus, we assume that each $g_i(\cdot)$ is strictly increasing; therefore, $g_i(\cdot)$ is invertible. A primary selects a price for its available channel, but, since there is an one-to-one relation between the price and penalty at each state, we can equivalently consider that primaries select penalties instead. For a given price, a channel of higher transmission rate must induce lower penalty, thus, $g_i(p) < g_j(p)$ if $i > j$. We can also consider that desirability of a channel for a secondary is the negative of penalty. A secondary will not buy any channel whose desirability falls below a certain threshold, equivalently, whose penalty exceeds a certain threshold. We consider that such threshold is the same for each secondary and we denote it as v i.e. no secondary will buy any channel whose penalty exceeds v. Secondaries have the same penalty function and the same upper bound for penalty value (v), thus, secondaries are statistically identical.

Primary i chooses its penalty using an arbitrary probability distribution function (d.f.)

$\psi_{i,j}(\cdot)$ [7] when its channel is in state $j \geq 1$. If $j = 0$ (i.e., the channel is unavailable), primary i chooses a penalty of $v + 1$: this is equivalent to considering that such a channel is not offered for sale as no secondary buys a channel whose penalty exceeds v.

Definition 2.1. A strategy of a primary i for state $j \geq 1$, $\psi_{i,j}(\cdot)$ provides the penalty distribution it uses at each node, when the channel state is $j \geq 1$. $S_i = (\psi_{i,1}, ..., \psi_{i,n})$ denotes the strategy of primary i, and $(S_1, ..., S_l)$ denotes the strategy profile of all primaries (players). S_{-i} denotes the strategy profile of primaries other than i.

2.1.3 Payoff of Primaries

We denote $f_i(\cdot)$ as the inverse of $g_i(\cdot)$. Thus, $f_i(x)$ denotes the price when the penalty is x at channel state i. We assume that $g_i(\cdot)$ is continuous, thus $f_i(\cdot)$ is continuous and strictly increasing. Also, $f_i(x) < f_j(x)$ for each x and $i < j$. Each primary incurs a transaction cost $c > 0$ when the primary leases its channel to a secondary, and therefore never selects a price lower than c.

If primary i selects a penalty x when its channel state is j, then its payoff is

$$\begin{cases} f_j(x) - c & \text{if the primary sells its channel} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if Y is the number of channels offered for sale, for which the penalties are upper bounded by v, then those with $\min(Y, m)$ lowest penalties are sold since secondaries select

[7] Probability distribution refers cumulative distribution function (c.d.f.). C.d.f. of a random variable X is the function $G(x), G(x) = P(X \leq x) \quad \forall x \in \Re$ [14].

channels in the increasing order of penalties. The ties among channels with identical penalties are broken randomly and symmetrically among the primaries.

Definition 2.2. $u_{i,j}(\psi_{i,j}, S_{-i})$ is the expected payoff when primary i's channel is at state j and selects strategy $\psi_{i,j}(\cdot)$ and other primaries use strategy S_{-i}.

2.1.4 Nash Equilibrium

We use Nash Equilibrium (NE) as a solution concept which we define below

Definition 2.3. [59] A *Nash equilibrium* $(S_1, \ldots, S_n)$ is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy. So, with $S_i = (\psi_{i,1}, ..., \psi_{i,n})$, $(S_1, \ldots, S_n)$, is an NE if for each primary i and channel state j

$$u_{i,j}(\psi_{i,j}, S_{-i}) \geq u_{i,j}(\tilde{\psi}_{i,j}, S_{-i}) \ \forall \ \tilde{\psi}_{i,j}. \tag{2.2}$$

An NE $(S_1, \ldots, S_n)$ is a *symmetric NE* if $S_i = S_j$ for all i, j.

An NE is asymmetric if $S_i \neq S_j$ for some $i, j \in \{1, \ldots, l\}$.

Note that if $m \geq l$, then primaries select the highest penalty v with probability 1. This is because, when $m \geq l$, then, the channel of a primary will always be sold. Henceforth, we will consider that $m < l$.

2.1.5 A Class of Penalty Functions

Since $g_i(\cdot)$ is strictly increasing in p and $g_i(p) > g_j(p)$ for $i < j$, we focus on penalty functions of the form $g_i(p) = h_1(p) - h_2(i)$, where $h_1(\cdot)$ and $h_2(\cdot)$ are strictly increasing in their respective arguments. Note that $-g_i(p)$ may be considered as a utility that a secondary would get at channel state i when the price is set at p. Such utility functions are commonly assumed to be concave [74]; which is possible only if $g_i(\cdot)$ is convex in p i.e. $h_1(\cdot)$ is convex. It is easy to show that when $g_i(p) = h_1(p) - h_2(i)$ and $h_1(\cdot)$ is convex, satisfies the following property:

Assumption 1

$$\frac{f_i(y) - c}{f_j(y) - c} < \frac{f_i(x) - c}{f_j(x) - c} \text{ for all } x > y > g_i(c), i < j. \tag{2.3}$$

Moreover, when $g_i(p) = h_1(p)/h_2(i)$, then, the inequality in (2.3) is satisfied for some certain convex functions $h_1(\cdot)$ like $h_1(p) = p^r(r \geq 1), \exp(p)$. In addition, there is also a large set of functions that satisfy (2.3), such as: $g_i(p) = \zeta\left(p - h_2(i)\right), g_i(p) = \zeta\left(p/h_2(i)\right)$ where $\zeta(\cdot)$ is continuous and strictly increasing. Moreover, $g_i(\cdot)$ are such that the inverses are of the form $f_i(x) = h(x) + h_2(i), f_i(x) = h(x) * h_2(i)$, where $h(\cdot)$ is *any* strictly increasing function, satisfy Assumption 1. Henceforth, we only consider penalty functions which satisfy (2.3).

We mostly consider $g_i(\cdot)$s which do not depend on $l, m, n, q_1, \ldots, q_n$ (e.g. Fig. 2.1, 2.2, 2.3). But in some case we also consider that $g_i(\cdot)$ is a function of n (e.g. Fig. 2.4).

2.2 Structure, Computation, Uniqueness and Existence of NE

First, we identify key structural properties of a NE (should it exist). Next we show that the above properties lead to a unique strategy profile which we explicitly compute - thus the symmetric NE is unique should it exist. We finally prove that the strategy profile resulting from the structural properties above is indeed a NE thereby establishing the existence.

2.2.1 Structure of an NE

We provide some important properties that any NE $(S_1, \ldots, S_l)$ $(S_i = \{\psi_{i,1}, \ldots, \psi_{i,n}\})$ must satisfy. First, we show that each primary must use the same strategy profile (Theorem 2.1). In the next chapter, we show that this is no longer the case when there are multiple locations. In fact we show that there may be multiple asymmetric NEs when there are multiple locations. We show that under the NE strategy profile a primary selects lower values of the penalties when the channel quality is high (Theorem 2.3). In Remark 2.1 we also show that the result attained in our setting is equivalent to the socially optimum solution where the sum of the utilities of the secondaries and the payoffs of the primaries are maximized. We have also shown that $\psi_{i,j}(\cdot)$ are continuous and contiguous (Theorem 2.2 and 2.4).

Theorem 2.1. *Each primary must use the same strategy i.e.* $\psi_{i,j}(\cdot) = \psi_{k,j}(\cdot)\ \forall i, k \in \{1, \ldots, l\}$ *and* $j \in \{1, \ldots, n\}$.

Theorem 2.1 implies that an NE strategy profile can not be asymmetric. Since channel statistics are identical and payoff functions are identical to each primary, thus this game is symmetric. Given that the game is symmetric, apparently there should only be symmetric NE strategies. Although there are symmetric games where asymmetric NEs do exist [15], we are able to rule that out in our setting using the assumptions that naturally arise in practice namely those which are stated in Sections 2.1.1, 2.1.2 and Assumption 1 which is satisfied by a large class of functions that are likely to arise in practice (Section 2.1.5). Thus, a significance of Theorem 2.1 is that Theorem 1 holds for a large class of penalty functions which are likely to arise in practice. However, we show that there may exist asymmetric NE in absence of Assumption 1 (Section 2.5.2).

Now, we point out another significance of the above theorem. In a symmetric game it is difficult to implement an asymmetric NE. For example, for two players if (S_1, S_2) is an NE with $S_1 \neq S_2$, then (S_2, S_1) is also an NE due to the symmetry of the game. The realization of such an NE is only possible when one player knows whether the other is using S_1 or S_2. But, coordination among players is infeasible apriori as the game is non co-operative. Thus, Theorem 2.1 entails that we can avoid such complications in this game. We show that this game has a unique symmetric NE through Lemma 2.2 and Theorem 2.5. Thus, Theorem 2.1, Lemma 2.2 and Theorem 2.5 together entail that there exists a unique NE strategy profile.

Since every primary uses the same strategy, thus, **we drop the indices corresponding to primaries and represent the strategy of any primary as $S = (\psi_1(\cdot), \psi_2(\cdot), ..., \psi_n(\cdot))$, where $\psi_i(\cdot)$ denotes the strategy at channel state i.** Thus, we can represent a strategy profile in terms of only one primary.

Definition 2.4. $\phi_j(x)$ is the expected profit of a primary whose channel is in state j and selects a penalty x [8].

Theorem 2.2. $\psi_i(.), i \in \{1, .., n\}$ *is a continuous probability distribution. Function* $\phi_j(\cdot)$ *is continuous.*

The above theorem implies that $\psi_i(\cdot)$ does not have any jump at any penalty value. i.e. no penalty value is chosen with positive probability. We now intuitively justify the

[8] Note that $\phi_j(x)$ depend on strategies of other primaries , to keep notational simplicity, we do not make it explicit

property. There are uncountably infinite number of penalty values and thus, clearly $\psi_i(\cdot)$ can only have jump at some of those values. Intuitively, there is no inherent asymmetry amongst the penalty values within the interval $(g_i(c), v)$ i.e. at the penalty values except the end points of the interval $[g_i(c), v]$. Thus, a primary does not prioritize any of those penalty values. Now, we intuitively explain why $\psi_i(\cdot)$ does not have jump at the end points. First, at penalty $g_i(c)$, a primary gets a payoff of 0 when the channel state is i; but the payoff at any penalty value greater that $g_i(c)$ is positive, thus $\psi_i(\cdot)$ does not prioritize the penalty value $g_i(c)$. On the other hand, intuitively if a primary selects penalty v with positive probability, then the rest would select slightly lower penalty in order to enhance their sales and thus, the probability that the primary would sell its channel decreases. Thus, $\psi_i(\cdot)$ also does not have a jump at v.

Note that in a deterministic N.E. strategy at channel state i, then $\psi_i(\cdot)$ must have a jump from 0 to 1 at the above penalty value. Such $\psi_i(\cdot)$ is *not* continuous. Thus, the above theorem rules out any deterministic N.E. strategy. The fact that $\phi_j(\cdot)$ is continuous has an important technical consequence; this guarantees the existence of the best response penalty in Definition 2.6 stated in Section 2.2.2.

Definition 2.5. We denote the lower and upper endpoints of the support set[9] of $\psi_i(.)$ as L_i and U_i respectively i.e.

$$L_i = \inf\{x : \psi_i(x) > 0\}.$$

$$U_i = \inf\{x : \psi_i(x) = 1\}.$$

[9] The support set of a probability distribution is the smallest closed set such that the probability of its complement is 0.[14]

We next show that primaries select higher penalty when the transmission rate is low. More specifically, we show that upper endpoint of the support set of $\psi_i(\cdot)$ is upper bounded by the lower endpoint of $\psi_{i-1}(\cdot)$.

Theorem 2.3. *$U_i \leq L_{i-1}$, if $j < i$.*

Theorem 2.3 is apparently counter intuitive. We prove it using the assumptions stated in Section 2.1. In particular, we rely on Assumption 1 which is satisfied by a large class of penalty functions (Section 2.1.5). Thus, the significance of Theorem 2.3 is that the counter intuitive structure holds for a large class of penalty functions. However, in Section 2.5.3 we show that Theorem 2.3 needs not to hold in absence of Assumption 1.

Remark 2.1. Theorem 2.3 shows in our setting *socially desirable outcome* can be achieved i.e. there can not be any unsold high quality channel if the low quality channel is sold[10]. To illustrate the fact consider the penalty function $g_i(x) = x - h(i)$ where $h(\cdot)$ is a strictly increasing function. The above penalty function satisfies the Assumption 1. Now, suppose a social planner wants to allocate the available channels to the secondaries in order to maximize the social welfare i.e. it wants to maximize the sum of the utilities of the secondaries and the payoffs of the primaries. The utility of the secondary is $-g_i(x)$ and the payoff of the primary is $x - c$, thus, if the channel which in state i is alloted to a secondary at price x then the sum of the utility of the secondary and the payoff of the primary is $h(i)$. Since $h(\cdot)$ is strictly increasing, thus, the socially efficient outcome is to sort the channels in the descending order of the channel states and then allocate channels to the secondaries until all the demand is met or the number of available channels are

[10]Note that if the high quality channels are larger than the number of secondaries, then there can be unsold high quality channels.

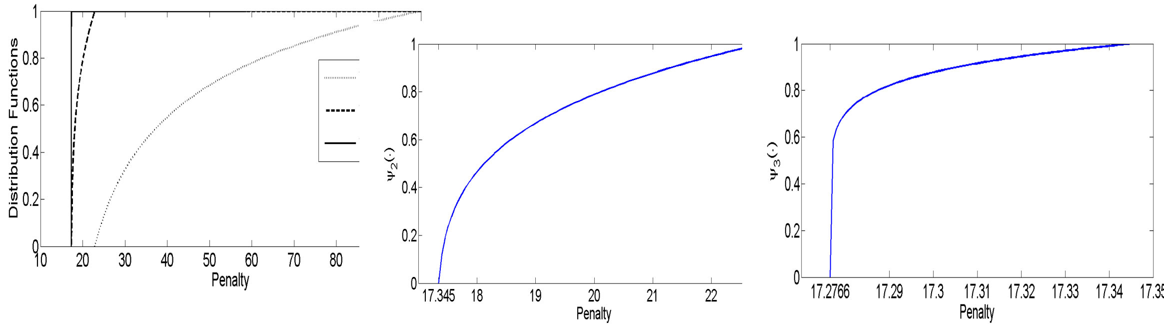

Figure 2.1: Figure in the left hand side shows the d.f. $\psi_i(\cdot), i = 1, \ldots, 3$ as a function of penalty for an example setting: $v = 100, c = 1, l = 21, m = 10, n = 3, q_1 = q_2 = q_3 = 0.2$ and $g_i(x) = x - i^3$. Note that support sets of $\psi_i(\cdot)$s are disjoint with $L_3 = 17.2766$, $U_3 = 17.345 = L_2$, $U_2 = 22.864 = L_1$, and $U_1 = 100 = v$. Figures in the center and the right hand side show d.f. $\psi_2(\cdot)$ and $\psi_3(\cdot)$ respectively, using different scales compared to the left hand figure.

exhausted. Thus, we show that though the primaries are selfish entities they select prices such that a socially desirable outcome is achieved.

Fig. 2.1 illustrates L_is and U_is in an example scenario (The distribution $\psi_i(\cdot)$ in Fig. 2.1 is plotted using (2.9)).

In general, a continuous NE penalty selection distribution may not be contiguous i.e. support set may be a union of multiple number of disjoint closed intervals. Thus, the support set of $\psi_i(\cdot)$ may be of the following form $[a_1, b_1] \cup \ldots \cup [a_d, b_d]$ with $b_k < a_{k+1}, k \in \{1, \ldots, d-1\}, a_1 = L_i$ and $b_d = U_i$. In this case, $\psi_i(\cdot)$ is strictly increasing in each of $[a_k, b_k] k \in \{1, \ldots, d\}$, but it is constant in the "gap" between the intervals i.e. $\psi_i(\cdot)$ is constant in the interval $[a_{k-1}, b_k]$ $k \in \{2, \ldots, d\}$. We rule out the above possibility in the following theorem i.e. the support set of $\psi_i(\cdot)$ consists of only one closed interval $[L_i, U_i]$ which also establishes that $\psi_i(\cdot)$ is strictly increasing from L_i to U_i. In the following theorem we also rule out any "gap" between support sets for different $\psi_i(\cdot)$, $i = 1, \ldots, n$.

Theorem 2.4. *The support set of* $\psi_i(\cdot), i = 1, .., n$ *is* $[L_i, U_i]$ *and* $U_i = L_{i-1}$ *for* $i = 2, .., n$, $U_1 = v$.

Theorem 2.3 states that $L_{i-1} \geq U_i$. Theorem 2.4 further confirms that $L_{i-1} = U_i$ i.e. there is no "gap" between the support sets. Theorem 2.4 also implies that there is no "gap" within a support set. We now explain the intuition that leads to the reason. If there are a and b which are in the support sets such that the primaries do not select any penalty in the interval (a, b), then, a primary can get strictly a higher payoff at any penalty in the interval (a, b) compared to penalty at a or just below a. Thus, a primary would select penalties at or just below a with probability 0 which implies that a can not be in the support set of an NE strategy profile. We prove Theorem 2.4 using the above insights and Theorem 2.3.

Figure 2.1 illustrates d.f. $\psi_i(\cdot)$ for an example scenario.

2.2.2 Computation, Uniqueness and Existence

We now show that the structural properties of an NE identified in Theorems 2.1-2.4 are satisfied by a unique strategy profile, which we explicitly compute (Lemma 2.2). This proves the uniqueness of a NE subject to the existence. We show that the strategy profile is indeed an NE in Theorem 2.5. We start with the following definitions.

Definition 2.6. A *best response* penalty for a channel in state $j \geq 1$ is x if and only if

$$\phi_j(x) = \sup_{y \in \Re} \phi_j(y). \tag{2.4}$$

Let $u_{j,max} = \phi_j(x)$ for a best response x[11] for state j, $j \geq 1$ i.e., $u_{j,max}$ is the maximum expected profit that a primary earns when its channel is in state j, $j \geq 1$.

[11]Since $\phi_j(\cdot)$ is continuous by Theorem 2.2 and penalty must be selected within the interval $[g_j(c), v]$,

Remark 2.2. In an NE strategy profile a primary only selects a best response penalty with a positive probability. Thus, a primary selects x with positive probability at channel state i, then expected payoff to the primary must be $u_{i,max}$ at x.

Definition 2.7. Let $w(x)$ (w_i, respectively) be the probability of at least m successes out of $l-1$ independent Bernoulli trials, each of which occurs with probability x ($\sum_{k=i}^{n} q_k$, respectively). Thus,

$$w(x) \quad = \quad \sum_{i=m}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-i-1}. \tag{2.5}$$

$$w_i \quad = \quad w\left(\sum_{j=i}^{n} q_j\right) \quad \text{for } i = 1, ..., n \quad \& w_{n+1} = 0. \tag{2.6}$$

The following lemma provides the explicit expression for the maximum expected payoff that a primary can get under an NE strategy profile.

Lemma 2.1. *For* $1 \leq i \leq n$*,*

$$u_{i,max} \quad = \quad p_i - c.$$

$$\textit{where, } p_i \quad = \quad c + (f_i(L_{i-1}) - c)(1 - w_i). \tag{2.7}$$

$$\textit{and } L_i \quad = \quad g_i\left(\frac{p_i - c}{1 - w_{i+1}} + c\right), L_0 = v. \tag{2.8}$$

Remark 2.3. Expected payoff obtained by a primary under an NE strategy profile at channel state i is given by $p_i - c$.

Remark 2.4. Starting from $L_0 = v$, we obtain p_1 (from (2.7)) and using p_1 we obtain L_1 by (2.8) which we use to obtain p_2 from (2.7). Thus, recursively we obtain p_i and L_i for $i = 1, \ldots, n$.

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thus the maximum exists in (2.4). This maximum is equal to $u_{j,max}$ and $\phi_j(x)$ is equal to $u_{j,max}$ for some x.

We now obtain expressions for $\psi_i(\cdot)$ using expression of L_i and p_i. We use the fact that $w(\cdot)$ is strictly increasing and continuous in $[0, 1]$.

Lemma 2.2. *An NE strategy profile (if it exists) $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$ must comprise of:*

$$\begin{aligned}\psi_i(x) =& 0, \textit{ if } x < L_i \\ & \frac{1}{q_i}(w^{-1}(\frac{f_i(x) - p_i}{f_i(x) - c}) - \sum_{j=i+1}^{n} q_j), \textit{ if } L_{i-1} \geq x \geq L_i \\ & 1, \textit{ if } x > L_{i-1}.\end{aligned} \tag{2.9}$$

where $p_i, L_i, i = 0, .., n$ are defined in (2.7).

Remark 2.5. Using (2.9) we can easily compute the strategy profile $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$. Fig. 2.1 illustrates d.f. $\psi_i(\cdot)$ for an example scenario.

The following lemma ensures that $\psi_i(\cdot)$ as defined in Lemma 2.2 is indeed a strategy profile.

Lemma 2.3. *$\psi_i(\cdot)$ as defined in Lemma 2.2 is a strictly increasing and continuous probability distribution function.*

Fig. 2.1 illustrates continuous and strictly increasing $\psi_i(\cdot)$ for $i = 1, \ldots, 3$ for an example setting.

Explicit computation in Lemma 2.2 shows that the NE strategy profile is unique, if it exists. There is a plethora of symmetric games [59] where NE strategy profile does not exist. However, we establish that any strategy profile of the form (2.9) is an NE.

Theorem 2.5. *Strategy profile as defined in Lemma 2.2 is an NE.*

Hence, we have shown that

Theorem 2.6. *The strategy profile, in which each primary randomizes over the penalties in the range $[L_i, L_{i-1}]$ using the continuous probability distribution function $\psi_i(\cdot)$ (defined in Lemma 2.2) when the channel state is i, is the unique NE strategy profile.*

2.2.3 Random Demand

Note that all our results readily generalize to allow for random number of secondaries (M) with probability mass functions (p.m.f.) $\Pr(M = m) = \gamma_m$. A primary does not have the exact realization of number of secondaries, but it knows the p.m.f. . We only have to redefine $w(x)$ as-

$$\sum_{k=0}^{\max(M)} \gamma_k \sum_{i=k}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-1-i}$$

and $w_{n+1} = \gamma_0$.

2.3 Performance Evaluations of NE Strategy Profile in Asymptotic Limit

In this section, we analyze the reduction in payoff under NE strategy profile due to the competition. First, we study the expected payoff that a primary obtains under the unique NE strategy profile in the asymptotic limit (Lemma 2.4). Subsequently, we compare the expected payoff of primaries under the NE strategy profile with the payoff that primaries get when they collude (Lemma 2.5). Subsequently, we investigate the asymptotic variation of strategy profiles of primaries with n in an example setting (Fig. 2.4).

Recall from Remark 2.3 that expected payoff obtained by a primary under the unique NE strategy profile at channel state i is given by $p_i - c$. Next,

Definition 2.8. Let R_{NE} denote the ex-ante expected profit of a primary at the Nash equilibrium. Then,

$$R_{NE} = \sum_{i=1}^{n} (q_i.(p_i - c)). \tag{2.10}$$

Note that $l \cdot R_{NE}$ denotes the total ex-ante expected payoff obtained by primaries at the NE strategy profile. We obtain

Lemma 2.4. *Let $c_j = g_j(c), j = 1, .., n$. When $l \to \infty$, then*

$$p_i - c \to \begin{cases} f_i(v) - c & \textit{if } (l-1)\sum_{j=1}^{n} q_j < m \\ f_i(c_k) - c \quad \textit{if} & (l-1)\sum_{j=k+1}^{n} q_j < m \\ & < (l-1)\sum_{j=k}^{n} q_j \quad 1 \leq k < i \\ 0 \quad \textit{if} & m < (l-1)\sum_{j=i}^{n} q_j. \end{cases}$$

We illustrate Lemma 2.4 using an example in Figure 2.2. Intuitively, competition increases with the decrease in m. Primaries choose prices progressively closer to the lower limit c. Thus, the expected payoff $p_i - c, i = 1 \ldots, n$ decreases as m decreases. The above lemma reveals that, as m becomes smaller only those primaries whose channels provide higher transmission rate can have strictly positive payoff i.e. $p_i - c$ is positive (Fig. 2.2).

From Lemma 2.4 and (2.10) we readily obtain

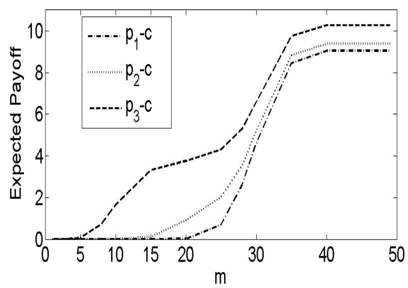


Figure 2.2: This figure illustrates the variation of $p_i - c, i = 1, \ldots, n$ with m in an example setting: $l = 51, n = 3, v = 100, c = 1, q_1 = q_2 = q_3 = 0.2$ and $g_i(x) = x^2 - i^3$. For $m \leq 5$, $p_i - c \approx 0$ for all i. For $5 \leq 5 \leq 15$, $p_i - c \approx 0$ for $i = 1, 2.$. For $15 \leq m \leq 20$, $p_1 - c \approx 0$. When m exceeds 40, $p_i - c, i = 1, 2, 3$ closely match the highest possible expected value as Lemma 2.4 indicates.

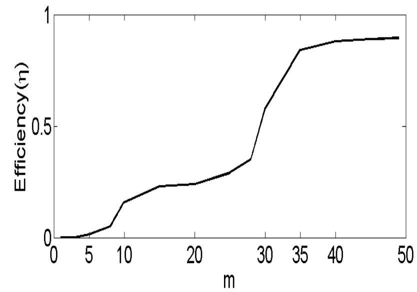


Figure 2.3: Variation of efficiency (η) with m in an example setting: $g_i(x) = x^2 - i^3$, $l = 51, n = 3, q_1 = q_2 = q_3 = 0.2, v = 100$ and $c = 1$. When $m \leq 5$, $\eta \approx 0$. When $m \geq 35$, $\eta \approx 1$.

Corollary 2.1. *When $l \to \infty$, then,*

$$R_{NE} \to \begin{cases} \sum_{j=1}^{n} q_j.(f_j(v) - c) & \textit{If } (l-1)\sum_{j=1}^{n} q_j < m \\ \sum_{j=i+1}^{n} q_j.(f_j(c_i) - c) & \textit{If } (l-1)\sum_{j=i+1}^{n} q_j < m \\ & < (l-1)\sum_{j=i}^{n} q_j, \quad i \in \{1,..,n-1\} \\ 0 & \textit{If } m < (l-1)q_n. \end{cases}$$

Thus, asymptotically R_{NE} decreases as m decreases (Fig. 2.2).

Definition 2.9. Let R_{OPT} be the maximum expected profit earned through collusive selection of prices by the primaries. *Efficiency* η is defined as $\dfrac{l \cdot R_{NE}}{R_{OPT}}$.

Efficiency is a measure of the reduction in the expected profit owing to competition. The asymptotic behavior of η is characterized by the following lemma.

Lemma 2.5. *When $l \to \infty$, then*

$$\eta \to \begin{cases} 1 & \textit{If } (l-1)\sum_{j=1}^{n} q_j < m \\ 0 & \textit{If } m < (l-1)q_n. \end{cases}$$

We illustrate the variation of efficiency with m using an example in Figure 2.3. Intuitively, the competition decreases with increase in m; thus primaries set their penalties close to the highest possible value for all states. This leads to high efficiency. On the other hand, competition becomes intense when m decreases, thus, R_{NE} becomes very small as Corollary 2.1 reveals. But, if primaries collude, primaries can maximize the aggregate payoff by offering only the channels of highest possible states by selecting highest penalties. Thus, efficiency becomes very small when m is very small (Fig. 2.3).

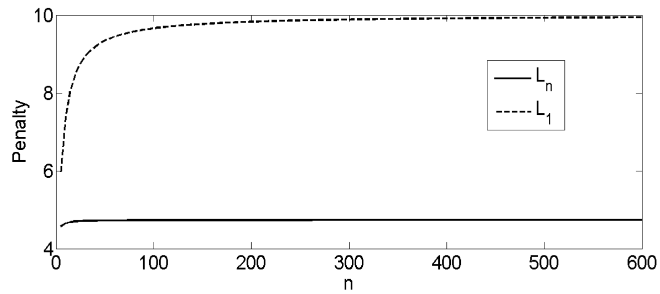


Figure 2.4: Figure shows the plot of L_1 and L_n with n. We consider $v = 10, c = 0, l = 21, m = 10, q_1 = \ldots = q_n = 0.5/n$ and the following penalty function: $g_i(x) = x - (r_{max} - r_{min}) * i/n$, where r_{max} denotes the maximum possible transmission rate which a secondary user can transmit and r_{min} denotes the minimum transmission rate required to transmit the signal. Note that the penalty function is of the form $g_i(x) = h_1(x) - h_2(i)$, where, $h_1(x) = x$ is strictly increasing concave function in x, and $h_2(i) = (r_{max} - r_{min}) * i/n$ is strictly decreasing in i. Thus, $g_i(\cdot)$ satisfies Assumption 1. We have equally divided the available rate region into the number of states n. We consider that $h_2(\cdot)$ is the representative rate at state i. We consider $r_{max} = 3.5$ and $r_{min} = 0.5$.

The transmission rates of an available channel constitute a continuum in practice. We have discretized the transmission rates of an available channel in multiple states for the ease of analysis. However, the theory allows us to investigate numerically how the penalty distribution strategies behave in the asymptotic limit (Fig. 2.4).

Fig. 2.4 reveals that L_n increases with n and eventually converges to a point which is strictly less than v. On the other hand, L_1 converges to v as n becomes large (Fig. 2.4). Thus, the lower endpoints (and thus, the upper endpoints (since $U_j = L_{j-1}$)) of penalty selection strategies converge at different points when n becomes large.

2.4 Repeated Game

We have so far considered the setting where primaries and secondaries interact only once. In practice, however, primaries and secondaries interact repeatedly. To analyze repeated interactions we consider a scenario where the one shot game is repeated an infinite number of times. We characterize the subgame perfect Nash Equilibrium where the expected payoff of primaries can be made arbitrarily close to the payoff that primaries would obtain if they would collude.

The one shot game is played at time slots $t = 0, 1, 2, \ldots,$. Let, $\phi_{i,t}$ denote the expected payoff at stage t, when the channel state is i. Hence, the payoff of a primary, when its channel state is i, is given by

$$\phi_i = (1-\delta)\sum_{t=0}^{\infty} \delta^t \phi_{i,t} \tag{2.11}$$

where, $\delta \in (0, 1)$ is the discount factor.

Since NE constitutes a weak notion in repeated game [59], we will focus on Subgame Perfect Nash Equilibrium (SPNE).

Definition 2.10. [59] Strategy profile $(S_1, \ldots, S_n)$ constitutes a SPNE if the strategy profile prescribes an NE at every subgame.

The one shot unique NE that we have characterized, is also a SPNE in repeated game. Total expected payoff that a primary can get, is R_{NE} (Definition 2.8) under one-shot game. We have already shown that this payoff is not always efficient (Lemma 2.5) i.e. $\eta \nrightarrow 1$. Here, we present an efficient SPNE (Theorem 2.7), provided that δ is sufficiently high.

Fix a sequence of $\{\epsilon_i\}_1^n$ such that

$$0 = \epsilon_1 < \epsilon_2 < \ldots < \epsilon_n. \tag{2.12}$$

$$\epsilon_i \leq v - L_{i-1}, \epsilon_i \leq v - g_i((f_i(v) - c)(1 - w_i) + c). \tag{2.13}$$

We provide a *Nash Reversion Strategy* such that a primary will get a payoff arbitrarily close to the payoff that it would obtain when all primaries collude.

Strategy Profile (SP_R): *Each primary selects penalty $v - \epsilon_i$, (where ϵ_i satisfies (2.12) and (2.13)), when state of the channel is i, at $t = 0$ and also at time slot $\tau = 1, 2, \ldots$ as long as all other primaries have chosen $v - \epsilon_j$, $j \in \{1, \ldots, n\}$, when their channel state is j at all time slots $0, 1, \ldots, \tau - 1$. Otherwise, play the unique one shot game NE strategy $\psi_i(\cdot)$ (Lemma 2.2).*

Remark 2.6. Note that if everyone sticks to the strategy, then each primary selects a penalty of $v - \epsilon_i$ at every time slot, when the channel state is i. Under the collusive setting, each primary selects penalty v. Thus, for every $\gamma > 0$, we can choose sufficiently small $\epsilon_i, i = 1, \ldots, n$ and sufficiently high δ such that the efficiency (definition 2.9) is at least $1 - \gamma$.

Now we are ready to state the main result of this section.

Theorem 2.7. *Suppose $\{\epsilon_i\}_{i=1}^n$ are such that they satisfy (2.12) and (2.13). Then, there exists $\delta_{min} \in (0, 1)$ such that for any discount factor $\delta \geq \delta_{min}$ the strategy profile SP_R is a SPNE.*

Remark 2.7. Thus, there exists a SPNE strategy profile (for sufficiently high δ) where each primary obtains an expected payoff arbitrarily close to the payoff it would have obtained if all primaries collude.

2.5 Pending Question: What happens when Assumption 1 is relaxed?

We show that if penalty functions do not satisfy Assumption 1, then the system may have multiple NEs (section 2.5.1), asymmetric NE (section 2.5.2) and the strategy profile that we have characterized in (2.9) may not be an NE (section 2.5.3).

2.5.1 Multiple NEs

We first give a set of penalty functions which do not satisfy Assumption 1 and then we state a strategy profile which is an NE for this set of penalties along with the strategy profile that we have characterized in (2.9).

Let $f_i(\cdot)$ be such that

$$\frac{f_i(x)-c}{f_j(x)-c} = \frac{f_i(y)-c}{f_j(y)-c} \quad (x > y > g_i(c), i < j) \tag{2.14}$$

Examples of such kind of functions are $g_i(x) = (x-c)^p/i$.

It can be easily verified that strategy profile, described as in (2.9), is still an NE strategy profile under the above setting. We will provide another NE strategy profile.

First, we will introduce some notations which will be used throughout this section.

$$\bar{p}_i = (f_i(v)-c)(1-w_1)+c \tag{2.15}$$

$$\bar{L} = g_1(\bar{p}_1) \tag{2.16}$$

Now, we show that there exists a symmetric NE strategy profile where a primary selects the same strategy for its each state of the channel. This establishes that the the system has multiple NEs.

Let's consider the following symmetric strategy profile where at channel state i a primary's strategy profile is $\bar{\psi}_i(\cdot) = \bar{\psi}(\cdot)$ for $i = 1, \ldots, n$, where

$$\begin{aligned}\bar{\psi}(x) =& 0 \quad (\text{if } x < \bar{L})\\ & \frac{1}{\sum_{j=1}^{n} q_j} w^{-1}\left(1 - \frac{\bar{p}_1 - c}{f_1(x) - c}\right) \quad (\text{if } v \geq x \geq \bar{L})\\ & 1 \quad (\text{if } x > v)\end{aligned} \tag{2.17}$$

First, we show that $\bar{\psi}(\cdot)$ is a probability d.f.

Lemma 2.6. *$\bar{\psi}(\cdot)$ as defined in (2.17) is a probability distribution function.*

Note that in this strategy profile each primary selects the same strategy irrespective of the channel state. Next, we show that strategy profile as described in (2.17) is an NE strategy profile.

Theorem 2.8. *Consider the strategy profile where $\bar{\psi}_i(\cdot) = \bar{\psi}(\cdot)$, for $i = 1, \ldots, n$. This strategy profile constitute an NE.*

2.5.2 Asymmetric NE

If Assumption 1 is not satisfied, then there may exist asymmetric NEs. Note that when Assumption 1 is satisfied the unique NE is symmetric. We again consider the penalty functions are of the type given in (2.14). We consider $n = 2, l = 2, m = 1$ and $q_1 = q_2$. Here, we denote $\psi_{i,j}(\cdot)$ as the strategy profile for primary $i, i = 1, 2$ at channel state j, $j = 1, 2$. Let

$$\hat{L} = g_2((f_2(v) - c)(1 - q_1 - q_2)/(1 - q_2) + c) \tag{2.18}$$

Note from (2.18) that

$$
\begin{aligned}
(f_2(\hat{L}) - c)(1 - q_2) &= (f_2(v) - c)(1 - q_1 - q_2) \\
(f_1(\hat{L}) - c)(1 - q_2) &= (f_1(v) - c)(1 - q_1 - q_2) \quad \text{(from (2.14))} \\
(f_1(\hat{L}) - c)(1 - q_1) &= (f_1(v) - c)(1 - q_1 - q_2) \quad \text{(since } q_1 = q_2\text{)}
\end{aligned}
\tag{2.19}
$$

Next

$$
\hat{L}_{low} = g_1((f_1(\hat{L}) - c)(1 - q_2) + c) \tag{2.20}
$$

Again using (2.14) and the fact that $q_1 = q_2$ we also obtain

$$
f_2(\hat{L}_{low}) - c = (f_2(\hat{L}) - c)(1 - q_1) \tag{2.21}
$$

Consider the following strategy profile

$$
\begin{aligned}
\psi_{1,1}(x) &= 1, \quad (x \geq v), \\
&= \frac{1}{q_1}(1 - q_2 - \frac{(f_2(v) - c)(1 - q_1 - q_2)}{f_2(x) - c}) \quad (v > x > \hat{L}) \\
&= 0, \quad x \leq \hat{L} \\
\psi_{1,2}(x) &= 1, \quad (x \geq \hat{L}), \\
&= \frac{1}{q_2}(1 - \frac{(f_1(\hat{L}) - c)(1 - q_2)}{f_1(x) - c}) \quad (\hat{L} > x > \hat{L}_{low}) \\
&= 0, \quad x \leq \hat{L}_{low}
\end{aligned}
\tag{2.22}
$$

and

$$
\begin{aligned}
\psi_{2,1}(x) &= 1, \quad (x \geq \hat{L}), \\
&= \frac{1}{q_1}(1 - \frac{(f_2(\hat{L}) - c)(1-q_1)}{f_2(x) - c}) \quad \hat{L}_{low} < x < \hat{L} \\
&= 0. \quad x \leq \hat{L}_{low} \\
\psi_{2,2}(x) &= 1, \quad x \geq v \\
&= \frac{1}{q_2}(1 - q_1 - \frac{(f_1(v) - c)(1-q_1-q_2)}{f_1(x) - c}) \quad \hat{L} > x > v \\
&= 0, \quad x \leq \hat{L}
\end{aligned} \tag{2.23}
$$

It is easy to discern that the above strategy profile is a continuous distribution function. Also note that $\psi_{1,1} \neq \psi_{2,1}$ and $\psi_{1,2} \neq \psi_{2,2}$. Hence, the strategy profile is asymmetric. The following theorem confirms that the strategy profile that we have just described is indeed an NE.

Theorem 2.9. *The strategy profile* $\psi_1 = \{\psi_{1,1}(\cdot), \psi_{1,2}(\cdot)\}$ *and* $\psi_2 = \{\psi_{2,1}(\cdot), \psi_{2,2}(\cdot)\}$ *as described in (2.22) and (2.23) respectively is an NE.*

The above theorem confirms that there may exist an asymmetric NE when the penalty functions are of the form (2.14).

2.5.3 Strategy Profile Described in (2.9) may not be an NE

We first describe a set of penalty functions that do not satisfy (2.3) and then we show that the strategy profile that we have characterized in (2.9) is not an NE. [12]

[12] Note that in theorem 2.5 we have shown that the strategy profile described in (2.9) is an NE when (2.3) is satisfied

Consider that $n = 2, c = 0$; penalty functions are as follow:

$$g_1(x) = x, f_1(x) = x \tag{2.24}$$

$$g_2(x) = \log(x), f_2(x) = e^x \tag{2.25}$$

Now, as $c = 0$, hence

$$\frac{f_1(x) - c}{f_2(x) - c} = \frac{x}{e^x} = F(x)$$
$$\frac{dF(x)}{dx} = e^{-x} - xe^{-x} \tag{2.26}$$

From (2.26), it is clear that for $x > 1$, $F(x)$ is strictly decreasing. Hence we have for $1 < x < y$

$$\frac{f_2(y) - c}{f_1(y) - c} > \frac{f_2(x) - c}{f_1(x) - c} \tag{2.27}$$

which contradicts (2.3).

Now, let $v = 5$ and $l = 20, m = 10, q_1 = 0.2, q_2 = 0.4$. Under this setting, we obtain

Theorem 2.10. *The strategy profile as defined in (2.9) is* ***not*** *an NE strategy profile.*

Remark 2.8. Thus, the condition in (2.3) is also *necessary* for a NE strategy profile to be in the form (2.9).

In the example constructed above, however, an NE strategy profile may still exist. Now, we show that in certain cases we can have a symmetric NE where $L_1 < L_2$. Thus, when channel state is 1 and 2, a primary chooses penalty from the interval $[L_1, L_2]$ and $[L_2, v]$ respectively. (Note the difference; in the strategy profile described in lemma 2.2, we have $L_1 > L_2$). Consider the following symmetric strategy profile where each primary selects strategy $\tilde{\psi}_i$ at channel state i, $i \in \{1, 2\}$-

$$\begin{aligned}
\tilde{\psi}_i(x) = & 0, \text{if } x < \tilde{L}_i \\
& \frac{1}{q_i}(w^{-1}(\frac{f_i(x) - \tilde{p}_i}{f_i(x) - c}) - \sum_{j=1}^{i-1} q_j), \text{if } \tilde{L}_{i+1} \geq x \geq \tilde{L}_i \\
& 1, \text{if } x > \tilde{L}_{i+1}
\end{aligned} \tag{2.28}$$

with $\tilde{L}_3 = v$, $\tilde{L}_1 > 1$.

and

$$\begin{aligned}
\tilde{p}_2 &= (f_2(v) - c)(1 - w(q_1 + q_2)) \\
\tilde{L}_2 &= g_2(\frac{\tilde{p}_2}{1 - w(q_1)}) \\
\tilde{L}_1 &= g_1(f_1(\tilde{L}_2)(1 - w(q_1)) \\
\tilde{p}_1 &= \frac{f_1(\tilde{L}_2)}{f_2(\tilde{L}_2)} * \tilde{p}_2
\end{aligned} \tag{2.29}$$

When $v = 5$, $l = 20, m = 10, q_1 = 0.2, q_2 = 0.4$; we obtain $\tilde{p}_2 = 27.6185, \tilde{L}_2 = 3.3201, \tilde{p}_1 = 3.3148, \tilde{L}_1 = 3.3148$.

It is easy to show that $\tilde{\psi}_i(\cdot)$ as defined in (2.28) is distribution function with $\tilde{\psi}_i(\tilde{L}_i) = 0$ and $\tilde{\psi}_i(\tilde{L}_{i+1}) = 1$. Note that under $\tilde{\psi}_i(\cdot)$, a primary selects penalty from the interval $[\tilde{L}_1, \tilde{L}_2]$ and $[\tilde{L}_2, v]$ when the channel states are 1 and 2 respectively. So, it remains to show that it is an NE strategy profile. The following theorem shows that it is indeed an NE strategy profile.

Theorem 2.11. *Strategy profile $\tilde{\psi}_i(\cdot), i = 1, \ldots, n$ as described in (2.28) is an NE .*

2.6 Conclusion and Future Work

We have analyzed a spectrum oligopoly market with primaries and secondaries where secondaries select a channel depending on the price quoted by a primary and the transmission rate a channel offers. We have shown that in the one-shot game there exists a unique NE strategy profile which we have explicitly computed. We have analyzed the expected payoff under the NE strategy profile in the asymptotic limit and compared it with the payoff that primaries would obtain when they collude. We have shown that under a repeated version of the game there exists a subgame perfect NE strategy profile where each primary obtains a payoff arbitrarily close to the payoff that it would have obtained if all primaries collude.

The characterization of an NE strategy profile under the setting i) when secondaries have different penalty functions, and ii) when demand of secondaries vary depending on the pricing strategy remains an open problem. The analytical tools and results that we have provided may provide the basis for developing a framework to solve those problems.

2.7 Proofs

2.7.1 Proof of the results of Section 2.2.1

We first state Lemma 2.7, 2.8 and 2.9 in order to prove Theorem 2.1. Theorem 2.2 readily follows from Lemma 2.7. After that we show Corollary 2.3 which we use to prove theorems 2.3 and 2.4.

Now, we introduce some notations which we use throughout this section.

Definition 2.11. Let $r_i(x)$ denote the probability of winning of primary i when it selects penalty x.

Let $t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l)$ denote the probability of at least m success out of $l-1$ (except i) independent Bernouli event with event k has success probability of x_k.

Note that $r_i(x)$ does not depend on the state of the channel since secondaries select the channels based only on the penalties. Since secondaries prefer channels which induce lower penalty thus $r_i(\cdot)$ is non-increasing function. Note that $t_i(\cdot)$ is strictly increasing in each component. Note also that

$$t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l) = w(x) \quad (\text{if } x_1 = \ldots = x_l = x). \tag{2.30}$$

If primary i selects penalty x at channel state j, then its expected payoff is

$$(f_j(x) - c)r_i(x). \tag{2.31}$$

Definition 2.12. A *Best response* penalty for primary i at channel state $j \geq 1$ is

$$x = \underset{y \in \Re}{\operatorname{argmax}} (f_j(y) - c)r_i(y)$$

Let $u_{i,j,max}$ denote the maximum expected payoff under NE strategy profile for primary i at channel state j i.e. $u_{i,j,max}$ is equal to the payoff at x at channel state j if x is a best response for primary i at channel state j.

A primary only selects penalty x with positive probability at channel state j, if x is a best penalty response at j. We state an observation that we will use throughout:

Observation 2.1. Any penalty $y \leq g_j(c)$ can not be a best response (definition 2.6) for channel state j.

Proof. Note that the profit of a primary is non-positive if the selected penalty is upper bounded by $g_j(c)$. On the other hand when a primary selects penalty x where $g_j(c) < x \leq v$, it can sell its channel at least in the event when the total number of available channels are less than m. Since $0 < \sum_{i=1}^{n} q_i < 1$ by (2.1), thus the event occurs with non-zero probability, hence the profit is strictly positive when $g_j(c) < x \leq v$. Hence, the result follows. □

We denote $f(x-) = \lim_{y \uparrow x} f(y)$ throughout this section for a function $f(\cdot)$. Now we are ready to show Lemma 2.7.

Lemma 2.7. *$\psi_{i,j}(\cdot)$ is continuous at all points, except possibly at v. Also, at most one primary can have a jump at v.*

Proof. Let $\psi_{i,j}(\cdot)$ has a jump at $x < v$ where $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, n\}$. Thus, x is a best response for primary i at channel state j. Next, we show that no player other than player i will select penalty in the interval $[x, x+\epsilon]$ with positive probability for small enough $\epsilon > 0$.

Fix a player $k \in \{1, \ldots, i-1, i+1, \ldots, l\}$ and channel state $k_1 \in \{1, \ldots, n\}$.

First, note that if $f_{k_1}(x) \leq c$, then a player can not select penalty in the interval $[x, x+\epsilon_0]$ with positive probability where $f_{k_1}(x+\epsilon_0) - c < (f_{k_1}(v) - c)(1 - w(\sum_{j=1}^{n} q_j))$ since a primary gets a payoff of at least $(f_{k_1}(v) - c)(1 - w(\sum_{j=1}^{n} q_i))$ at penalty v. Note that $\sum_{j=1}^{n} q_i < 1$, thus $\epsilon_0 > 0$. We need to consider states $k_1 \in \{1, \ldots, n\}$ such that $f_{k_1}(x) > c$.

The payoff that player k will get at a channel state k_1 at a $y \in [x, x+\epsilon_1]$ is

$$\begin{aligned}(f_{k_1}(y) - c)r_k(y) &\leq (f_{k_1}(y) - c)r_k(x) \\ &\quad (\text{since } r_k(y) \leq r_k(x)). \end{aligned} \tag{2.32}$$

For any $\delta > 0$, expected payoff for player k at $x - \delta$ at channel state k_1 is lower bounded by

$$(f_{k_1}(x-\delta) - c)r_k(x-)(\text{since } r_k(x-) \leq r_k(x-\delta)). \tag{2.33}$$

Since $\psi_{i,j}(\cdot)$ has a jump at x, thus $r_k(x-) > r_k(x)$. Hence, by continuity of $f_{k_1}(\cdot)$ there exists a $\delta > 0$ and small enough $\epsilon_1 > 0$ such that for every $y \in [x, x+\epsilon_1]$ we have

$$\begin{aligned}(f_{k_1}(x-\delta) - c)r_k(x-) &> (f_{k_1}(y) - c)r_k(x) \\ &\geq (f_{k_1}(y) - c)r_k(x) \quad (\text{from (2.32)}).\end{aligned}$$

Thus, player k has strictly higher payoff at $x - \delta$ compared to at penalty $y \in [x, x+\epsilon_1]$.

Hence, no player apart from i selects penalty in the interval $[x, x+\epsilon]$ with positive probability where $\epsilon = \min(\epsilon_0, \epsilon_1)$.

Since no player apart from player i select penalty in the interval $[x, x+\epsilon]$, thus player i will have a strictly higher payoff at $x+\epsilon$ instead of x which contradicts the fact that x is a best penalty response for player i.

If player i selects v with positive probability, then player $j, j \neq i$ will have strictly higher payoff by selecting a penalty just below v. Hence, player j will select v with 0 probability. Hence, the result follows. □

We introduce some notations which we use throughout this section:

Definition 2.13. Let $X_{m,i}$ be the mth lowest penalty selected by primaries except i.

Note that if primary i selects penalty x, then it will not be able to sell its channel if $X_{m,i} < x$. Now we show some results which directly follow from Lemma 2.7. We use these results to prove Theorem 2.1.

Observation 2.2. If $\psi_{k,j}(\cdot), k \neq i, j = 1, \ldots, n$ does not have jump at x apart from i (i may or may not have jump at x), then

$$r_i(x) = 1 - t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l)$$
$$\text{(where } x_k = \sum_{j=1}^{n} q_j \psi_{k,j}(x)). \tag{2.34}$$

Proof. Note that

$$r_i(x) = P(A|X_{m,i} = x)P(X_{m,i} = x) + P(X_{m,i} > x) \tag{2.35}$$

where $P(A|X_{m,i} = x)$ is the probability that primary i will be selected by secondaries when $X_{m,i} = x$. If $\psi_{k,j}(\cdot), k \neq i, j \in \{1, \ldots, n\}$ does not have any jump at x, then,

$$P(X_{m,i} = x) = 0. \tag{2.36}$$

On the other hand note that the probability of the event that primary k selects penalty less than or equal to x is given by $\sum_{j=1}^{n} q_j \psi_{k,j}(x)$, hence, $P(X_{m,i} > x)$ is given by

$$1 - t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l) \tag{2.37}$$

where $x_k = \sum_{j=1}^{n} q_j \psi_{k,j}(x)$. Thus, if $\psi_{k,j}(\cdot)$ does not have jump at x for all k and j, then (2.34) follows from (2.35), (2.36) and (2.37). □

By Lemma 2.7 no primary has a jump at $x < v$. Thus, $r_i(x)$ is exactly given by (2.34) when $x < v$. By Lemma 2.7, only one player can have a jump at v. Thus, if player k have a jump at v, then by Observation 2.2, $r_k(v) = r_k(v-)$. Hence, the following corollary is a direct consequence of (2.34).

Corollary 2.2. *$r_i(x) = r_k(x)$ iff $\sum_{j=1}^{n} q_j \psi_{i,j}(x) = \sum_{j=1}^{n} q_j \psi_{k,j}(x)$ and $r_i(x) > r_k(x)$ iff $\sum_{j=1}^{n} q_j \psi_{i,j}(x) > q_j \sum_{j=1}^{n} \psi_{i,j}(x)$ for $x < v$ and for $x = v$ if no primary has a jump at v. If primary k has a jump at v, then $r_k(v) = r_k(v-)$.*

Definition 2.14. Let

$$L_{i,j} = \inf\{x : \psi_{i,j}(x) > 0\}. \tag{2.38}$$

$$U_{i,j} = \inf\{x : \psi_{i,j} = 1\}. \tag{2.39}$$

$L_{i,j}, U_{i,j}$ are respectively the lowest and upper endpoint of support set of $\psi_{i,j}(\cdot)$.

Lemma 2.8. *i) $L_{i,j}$ is a best response for primary i at channel state j.*

ii) $U_{i,j}$ is a best response for primary i at channel state j, if one of the followings is true:

a) $U_{i,j} < v$.

b) $U_{i,j} = v$ and no primary has a jump at $U_{i,j}$.

c) $U_{i,j} = v$ and only primary i has a jump at v.

Proof. We prove part (i). The proof of part (ii) is similar and hence we omit.

We prove part (i) by considering the following two scenarios:

Case i: $L_{i,j} = v$: Note that $\psi_{i,j}(v) = 1$. Thus, by (2.38), $\psi_{i,j}(\cdot)$ has a jump at $L_{i,j}$; thus, $L_{i,j}$ is a best response to primary i at channel state j.

Case ii: $L_{i,j} < v$: By Lemma 2.7, no primary has a jump at $L_{i,j}$ and thus $r_i(\cdot)$ is continuous at $L_{i,j}$. Thus, by (2.38), primary i selects a penalty just above $L_{i,j}$ with positive probability when the channel state is j i.e. for every $\epsilon > 0$ there exists $y \in (L_{i,j}, L_{i,j} + \epsilon)$

such that y is a best response to primary i at channel state j. Then, we must have a sequence $z_k, k = 1, 2, \dots$ such that each z_k is a best response for primary i at channel state j and $\lim_{k\to\infty} z_k = L_{i,j}$. But profit to primary i at channel state j is $(f_j(z_k) - c)r_i(z_k)$. Now from continuity of $f_j(\cdot)$ and $r_i(\cdot)$ at $L_{i,j}$ we obtain

$$\lim_{k\to\infty} (f_j(z_k) - c)r_i(z_k) = (f_j(L_{i,j}) - c)r_i(L_{i,j}). \tag{2.40}$$

Since each $z_k, k = 1 \dots, \infty$ is a best response, thus, $L_{i,j}$ is also a best response. □

Lemma 2.9. *$U_{i,a} \leq L_{i,j}$ if $a > j$*

Proof. Fix a primary i. We first show that for any x, y such that $x < y \leq v$, if x is a best response when the state of the channel is j, then y can not be a best response when the state of the channel is a where $a > j$. If not, consider $y > x$ such that x, y are the best responses when channel states are respectively $j, a(a > j)$. Since x is a best response at channel state j, thus $f_j(x) > c$ by Observation 2.1. On the other hand, since y is a best response at channel state a, thus, $f_a(y) > c$ by Observation 2.1. Since $y > x$, thus $f_j(y) > c$. Also,

$$u_{i,a,max} = (f_a(y) - c)r_i(y). \tag{2.41}$$

Expected payoff to primary i at channel state j at y is $(f_j(y) - c)r_i(y)$. Thus, from (2.41)

$$u_{i,j,max} \geq (f_j(y) - c)r_i(y) = u_{i,a,max} \cdot \frac{f_j(y) - c}{f_a(y) - c}. \tag{2.42}$$

Since x is a best response to primary i at channel state j, thus

$$u_{i,j,max} = (f_j(x) - c)r_i(x).$$

Expected payoff of primary i at channel state a at penalty x is

$$(f_a(x) - c)r_i(x) = u_{i,j,max} \cdot \frac{f_a(x) - c}{f_j(x) - c} \tag{2.43}$$

Using (2.42) and (2.43), we obtain-

$$\begin{aligned}(f_a(x) - c)r_i(x) &\geq u_{i,a,max} \cdot \frac{(f_j(y) - c)(f_a(x) - c)}{(f_a(y) - c)(f_j(x) - c)} \\ &> u_{i,a,max} (\text{from (2.3)since } y > x, a > j, f_j(x) > c)\end{aligned} \tag{2.44}$$

which contradicts Definition 2.12.

We also obtain from the argument in the above paragraph, if $U_{i,a}$ is a best response then $U_{i,a} < L_{i,j}$.

If $U_{i,a}$ is not a best response then by Lemma 2.8, $U_{i,a} = v$ and there exists a primary other than i which has a jump at v. Thus, by Lemma 2.7, primary i does not have a jump at v. Thus, $\psi_{i,a}(\cdot)$ is continuous and thus, by the definition of $U_{i,a}$ (2.39) for every $\epsilon > 0$, there exists $y \in [v - \epsilon, v)$ such that y is a best response for primary i at channel state a. Hence, if $U_{i,a} > L_{i,j}$, then there exists a $y_1 > L_{i,j}$ such that y_1 is a best response for primary i for state a. But, we have already shown that it is not possible. Hence, the result follows. □

Proof of Theorem 2.1: Suppose the statement is not true. Thus, we must have $i, k \in \{1, \ldots, l\}$ which do not have identical strategy. Let, j be the largest index in $\{1, \ldots, n\}$ such that $\psi_{i,j}(\cdot)$ and $\psi_{k,j}(\cdot)$ differs. Thus, we must have

$$x = \inf\{y \leq v : \psi_{i,j}(y) \neq \psi_{k,j}(y)\}.$$

If $x = v$, then $\psi_{i,j}(\cdot) = \psi_{k,j}(\cdot)$ since $\psi_{i,j}(x') = 1$ for any $x' \geq v$. Hence, we must have $x < v$.

Note that by definition of j, $\psi_{i,a}(x) = \psi_{k,a}(x)$ $\forall a > j$. Since $x < v$, thus $\psi_{i,j}(x)$ and $\psi_{k,j}(\cdot)$ are continuous at x by Lemma 2.7, thus, $\psi_{i,j}(x) = \psi_{k,j}(x)$. Hence, $\psi_{i,a}(x) = \psi_{k,a}(x)$ for all $a \geq j$. By definition of x, $\psi_{i,j}(\cdot)$ and $\psi_{k,j}(\cdot)$ can not differ at a penalty less than x and thus $\psi_{i,j}(x) = \psi_{k,j}(x) \neq 1$. Thus $x < U_{i,j}$ and $x < U_{k,j}$, hence $\psi_{i,a}(x) = \psi_{k,a}(x) = 0$ $\forall a < j$ by Lemma 2.9. Since $\psi_{i,a}(x) = \psi_{k,a}(x)$ for all a and $q_a, a = 1, \ldots, n$ are exactly the same for each primary, thus, by Corollary 2.2

$$r_i(x) = r_k(x). \tag{2.45}$$

By definition of x, for every $\epsilon > 0$, there is a y such that $y \in (x, x+\epsilon)$ and $\psi_{i,j}(y) \neq \psi_{k,j}(y)$. Without loss of generality, we assume that $\psi_{i,j}(y) > \psi_{k,j}(y)$ for every y in $(x, x+\epsilon)$ for some $\epsilon > 0$. Thus, $\psi_{i,j}(x+\epsilon) > \psi_{i,j}(x)$ for every $\epsilon > 0$. We consider the following two possible scenarios:

Case i: $\psi_{k,j}(x+\epsilon) > \psi_{k,j}(x)$ for every $\epsilon > 0$.

Hence, there is a $\gamma > 0$, such that $x + \gamma \leq U_{k,j}$, $y \in (x, x+\gamma)$ is a best response for primary k at channel state j and $\psi_{i,j}(y) > \psi_{k,j}(y)$. Since $\psi_{k,j}(x+\epsilon) > \psi_{k,j}(x)$ and $\psi_{i,j}(x+\epsilon) > \psi_{i,j}(x)$ for every $\epsilon > 0$; and no primary has a jump at x, thus, x is also a best response for primary k and primary i at channel state j. But, expected payoff to primary k at x at channel state j is

$$(f_j(x) - c)r_k(x) = (f_j(x) - c)r_i(x) \quad \text{(from (2.45))}. \tag{2.46}$$

Since x is a best response for primary i at channel state j, thus,

$$(f_j(x) - c)r_i(x) \geq (f_j(y) - c)r_i(y). \tag{2.47}$$

Since $\psi_{i,j}(y) > \psi_{k,j}(y)$ and $\psi_{k,a}(y) = \psi_{k,a}(x)$ (since $L_{k,j} < y < U_{k,j}$) by Lemma 2.9 for all $a \neq j$, thus, $\sum_{a=1}^{n} q_a \psi_{i,a}(y) > \sum_{a=1}^{n} q_a \psi_{k,a}(y)$. Thus, from Corollary 2.2 $r_k(y) < r_i(y)$.

Thus, expected payoff at y is

$$
\begin{aligned}
(f_j(y) - c)r_k(y) &< (f_j(y) - c)r_i(y) \\
&\leq (f_j(x) - c)r_i(x) \quad \text{(from (2.47))} \\
&= (f_j(x) - c)r_k(x) \quad \text{(from (2.46))}.
\end{aligned} \tag{2.48}
$$

Since y and x are best response to primary k at channel state j, thus expected payoff to primary k at channel state j at x and y must be equal. But, this leads to a contradiction from (2.48) and (2.46).

Case ii: $\psi_{k,j}(x) = \psi_{k,j}(y)$ for some $y > x$:

Let $x_1 = \inf\{y > x : \psi_{k,j}(y) = \psi_{k,j}(x)\}$. Note that $\psi_{i,j}(x_1) > \psi_{k,j}(x_1)$. We consider two possible scenarios:

Case ii a: $x_1 < v$:

Since no primary has a jump at x_1 by Lemma 2.7, thus, by definition of x_1, it is a best response for primary k at channel state j. But expected payoff to primary k at channel state j at x_1 is given by

$$
(f_j(x_1) - c)r_k(x_1). \tag{2.49}
$$

Since $\psi_{k,j}(x_1) = \psi_{k,j}(x) < 1$, thus, $x_1 < U_{k,j}$, thus, $\psi_{k,a}(x) = \psi_{k,a}(x_1)\ \forall a < j$ by Lemma 2.9. Since $\psi_{i,j}(x + \epsilon) > \psi_{i,j}(x)$ for every $\epsilon > 0$, thus, $x \geq L_{i,j}$, hence $\psi_{i,a}(x) = 1\ \forall a > j$. Since $x_1 > x$, thus, $\psi_{i,a}(x_1) = \psi_{i,a}(x) = 1$ for all $a > j$. If $\psi_{k,a}(x) < \psi_{k,a}(x_1) \leq 1$ for some $a > j$, then $\psi_{i,a}(\cdot)$ differs from $\psi_{k,a}(\cdot)$ (since $1 = \psi_{i,a}(x) = \psi_{i,a}(x_1)$ for all $a > j$) at least at x, but this is against the definition of j. Hence, $\psi_{k,a}(x) = \psi_{k,a}(x_1)\ \forall a$. Since $\psi_{i,j}(x_1) > \psi_{k,j}(x_1)$ and $\psi_{i,a}(x_1) \geq \psi_{i,a}(x)$ for all $a < j$,thus, $\sum_{a=1}^{n} q_a\psi_{i,a}(x) > \sum_{a=1}^{n} q_a\psi_{k,a}(x_1)$.

Thus, by Corollary 2.2, $r_i(x_1) > r_k(x_1)$. Since x is a best response for player i at channel state j, thus

$$(f_j(x) - c)r_i(x) \geq (f_j(x_1) - c)r_i(x_1)(\text{since } x_1 > x). \tag{2.50}$$

Hence,

$$\begin{aligned}(f_j(x) - c)r_k(x) &= (f_j(x) - c)r_i(x) \quad (\text{from } (2.45)) \\ &\geq (f_j(x_1) - c)r_i(x_1) \quad (\text{from } (2.50)) \\ &> (f_j(x_1) - c)r_k(x_1).\end{aligned}$$

which contradicts the fact that x_1 is a best response for player k when the channel state is j.

case ii b: $x_1 = v$:

Since $\psi_{k,j}(v)$ must be 1 and $\psi_{k,j}(v-) = \psi_{k,j}(x) < 1$. Hence, $\psi_{k,j}(\cdot)$ must have a jump at v and thus v is a best response to primary k when the channel state is j. Thus, by Corollary 2.2, $r_k(v) = r_k(v-)$. Thus, by the continuity of $f_j(\cdot)$, penalties close to v is also a best response for primary k at channel state j i.e.

$$(f_j(v) - c)r_k(v) = (f_j(v-) - c)r_k(v-). \tag{2.51}$$

Since $\psi_{k,j}(v-) = \psi_{k,j}(x)$ by the definition of x_1, thus, similar to argument in case ii a, we obtain $r_k(y) < r_i(y)$ for $x < y < v$, i.e. there exists an $\epsilon > 0$, $r_i(v - \epsilon) > r_k(v - \epsilon)$ but $(f_j(v) - c)r_k(v) = (f_j(v - \epsilon) - c)r_k(v - \epsilon)$ (by (2.51)). Thus, there exists an $\epsilon > 0$, such that

$$\begin{aligned}(f_j(v - \epsilon) - c)r_k(v) &= (f_j(v - \epsilon) - c)r_k(v - \epsilon) \\ &< (f_j(v - \epsilon) - c)r_i(v - \epsilon). \end{aligned} \tag{2.52}$$

Note that the right hand side is the expression for the expected payoff of primary i at channel state j at penalty $v - \epsilon$. Since x is a best response to primary i at channel state j, thus,

$$\begin{aligned} (f_j(x) - c)r_k(x) &< (f_j(x) - c)r_i(x) \quad \text{(from (2.45))} \\ &\geq (f_j(v - \epsilon) - c)r_i(v - \epsilon) \\ &> (f_j(v) - c)r_k(v) \quad \text{(from (2.52))} \end{aligned}$$

which contradicts that v is a best response for primary k at channel state j. Hence, $x_1 \neq v$. Hence, this case does not arise.

Thus from case i, case ii.a, and case ii.b we obtain the desired result. □

Henceforth, we denote $\psi_{i,j}$ and $r_i(\cdot)$ as $\psi_j(\cdot)$ and $r(\cdot)$ respectively by dropping the index corresponding to primary i. Note from Definition 2.4 that

$$\phi_j(x) = (f_j(x) - c)r(x). \tag{2.53}$$

Also note that $L_{i,j} = L_j$ and $U_{i,j} = U_j$ for all $i \in \{1, \ldots, l\}$. Since strategy profiles of primaries are identical, thus, we can consider strategy profile in terms of only one primary (say, primary 1).

Proof of Theorem 2.2: By Lemma 2.7 $\psi_i(\cdot)$ does not have a jump at $x < v$. If a primary has a jump at v, then by symmetric property other primaries also have a jump at v, which is not possible by Lemma 2.7 since $l \geq 2$. Thus, $\psi_i(\cdot)$ does not have a jump for any $i \in \{1, \ldots, n\}$.

Now, we show that $\phi_j(\cdot)$ is continuous. Now, we provide a closed form expression for $\phi_j(x)$ using (2.53). Since $\psi_{i,j} = \psi_j(x)$, thus, from Observation 2.2, (2.30) and Theorem 2.2

the expected payoff for primary i at x at channel state j is given by

$$\phi_j(x) = (f_j(x) - c)(1 - w(\sum_{k=1}^{n} q_j \psi_k(x))).$$

The continuity follows from the equation due to the continuity of $w(\cdot)$ (Definition 2.7) and $\psi_k, k = 1, \ldots, n$. □

We obtain

$$\phi_j(x) = (f_j(x) - c)(1 - w(\sum_{k=1}^{n} q_j \psi_k(x))). \tag{2.54}$$

Now, we show a corollary which is a direct consequence of Theorem 2.2. We use this result to prove Theorems 2.3 and 2.4.

Corollary 2.3. *Every element in the support set of $\psi_i(\cdot)$ is a best response*[13]*; thus, so are L_i, U_i.*

Proof. Suppose that there exists a point z in the support set of $\psi_i(\cdot)$, which is not a best response. Therefore, primary 1 plays at z with probability 0 when the channel state is i. Now, one of the following two cases must arise.

Case I: $\exists$ a neighborhood [69] of radius $\delta > 0$ around z, such that no point in this neighborhood is a best response. Neighborhood of radius $\delta > 0$ of z is an open set (Theorem

[13]Note that in general every element of a support set need not be a best response. To illustrate the fact consider a 3 player non co-operative game and the following NE strategy profile: players 1 and 2 have identical strategy profile which is a uniform distribution from $[L, v]$ and player 3 selects v with probability 1. Since the support set is closed, thus v is in the support set for players 1 and 2. But, players 1 and 2 will attain strictly higher payoff at just below v compared to at v. Thus, v is not a best penalty response for players 1 and2. We show that this is not the case here because of the continuity of strategy profile. Specifically, a primary attains the highest possible expected payoff at every penalty in the support set in our setting.

2.19 of [69]). Hence, we can eliminate that neighborhood and can attain a smaller closed set, such that its complement has probability zero under $\psi_i(\cdot)$, which contradicts the fact that z is in the support set of $\psi_i(\cdot)$.

Case II: For every $\epsilon > 0$, $\exists y \in (z-\epsilon, z+\epsilon)$, such that y is a best response. Then, we must have a sequence $z_k, k = 1, 2, \ldots$ such that each z_k is a best response, and $\lim_{k\to\infty} z_k = z$ [69]. But profit to primary 1 for channel state i at each of z_k is $(f_i(z_k)-c)(1-w(\sum_{j=1}^{n} q_j\psi_j(z_k)))$ by (2.54). Thus, we can show that z is also a best response from the continuity of $w(\cdot)$ ad $f_j(\cdot)$. We can conclude the result by noting that U_i, L_i (Definition 2.5) are in the support set of $\psi_i(\cdot)$. □

Remark 2.9. By corollary 2.3, at any channel state i, a primary attains the same expected payoff ($u_{i,max}$) at every point in the support set.

Now we are ready to prove theorems 2.3 and 2.4.

Proof of Theorem 2.3: Theorem 2.3 is the direct consequence of Lemma 2.9 since $L_{i,j} = L_j$ and $U_{i,j} = U_j$ $\forall i$. □

Proof Of Theorem 2.4: Suppose the statement is not true. But, it follows that there exists an interval $(x, y) \subseteq [L_n, v]$, such that no primary offers penalty in the interval (x, y) with positive probability. So, we must have $\tilde{a}$ such that

$$\tilde{a} = \inf\{b \leq x : \psi_j(b) = \psi_j(x), \forall j\}.$$

By definition of $\tilde{a}$, $\tilde{a}$ is a best response for at least one state i. But, as no primary offers penalty in the range $(\tilde{a}, y)$, so from (2.54), $\phi_i(z) > \phi_i(\tilde{a})$ for each $z \in (\tilde{a}, y)$. This is because $w(\sum_{j=1}^{n} q_j\psi_j(z)) = w(\sum_{j=1}^{n} q_j\psi_j(\tilde{a}))$ and $f_i(\tilde{a}) < f_i(z)$. Thus, $\tilde{a}$ can not be a best response for state i. □

2.7.2 Proof of Results of Section 2.2.2

We prove Lemma 2.1, 2.2, 2.3. Then, we state and prove Observation 2.3 which we use to prove Theorem 2.5.

Proof of Lemma 2.1: We first outline a recursive computation strategy that leads to the expressions in (2.7) and (2.8).

Using Theorem 2.4, we have $U_1 = v$ and thus v is a best response at channel state 1 (Corollary 2.3). If a primary chooses penalty v then it sells only when $X_m > v$, this allows us to compute $u_{1,max}$. By Theorem 2.4, primaries with channel states $2, 3, \ldots, n$ choose penalty below L_1 and primaries with channel state 1 select penalty greater than L_1 with probability 1. This allows us to calculate the payoff at L_1 which must be equal to $u_{1,max}$. The above equality allows us to compute the expression for L_1.

Since $L_1 = U_2$ (Theorem 2.4) and U_2 is a best response at channel state 2, which enables us to obtain the expression for $u_{2,max}$. By Theorem 2.4 primaries with channel states $3, \ldots, n$ choose penalty below L_2 and primaries with channel state 1 and 2 select penalty greater than L_2 with probability 1. This allows us to calculate the payoff at L_2 which must be equal to $u_{2,max}$. The above equality allows us to compute the expression for L_2. Using recursion, we can get the values of $u_{i,max}, L_i$, $i = 1, \ldots, n$. The detailed argument follows:

We first prove (2.7) using induction, (2.8) follows from (2.7).

From theorem 2.4, $\psi_i(\cdot)$'s support set is $[L_i, L_{i-1}]$ for $i = 2, ..., n$ and $[L_1, v]$ for $i = 1$. Thus, v is a best response for channel state 1 (by Corollary 2.3), hence

$$u_{1,max} = (f_1(v) - c)(1 - w(\sum_{i=1}^{n} q_i)) = p_1 - c.$$

Thus, (2.7) holds for $i = 1$ with $L_0 = v$. Let, (2.7) be true for $i = t < n$. We have to show that (2.7) is satisfied for $i = t + 1$ assuming that it is true for $i = t$. Thus,by induction hypothesis,

$$u_{t,max} = p_t - c = (f_t(L_{t-1}) - c)(1 - w_t). \tag{2.55}$$

Now, L_t is a best response for state t, and thus,

$$\phi_t(L_t) = (f_t(L_t) - c)(1 - w_{t+1}) = p_t - c.$$

Now, as L_t is also a best response for state $t + 1$ by Corollary 2.3, thus

$$\phi_{t+1}(L_t) = (f_{t+1}(L_t) - c)(1 - w_{t+1}) = u_{t+1,max}.$$

Thus, $u_{t+1,max} = p_{t+1} - c$ and it satisfies (2.7). Thus, (2.7) follows from mathematical induction.

(2.8) follows since $(f_i(L_i) - c)(1 - w_{i+1}) = p_i - c$ and $g_i(\cdot)$ is the inverse of $f_i(\cdot)$. $\square$

Proof of Lemma 2.2: By Theorem 2.4, L_i, L_{i-1} are respectively the lower end-point and the upper end-point of the support set of $\psi_i(\cdot)$. We should have for $x < L_i$, $\psi_i(x) = 0$ and for $x > L_{i-1}$,$\psi_i(x) = 1$. From Corollary 2.3, every point $x \in [L_i, L_{i-1}]$ is a best response for state i, and hence,

$$(f_i(x) - c)(1 - w(\sum_{j=i+1}^{n} q_j + q_i.\psi_i(x))) = u_{i,max} = p_i - c.$$

Thus, the expression for $\psi_i(\cdot)$ follows. We conclude the proof by noting that the domain and range of $w(.)$ is $[0, 1]$, and $\frac{p_i - c}{f_i(x) - c} < 1$ for $x \in [L_i, L_{i-1}]$: so $w^{-1}(.)$ is defined at $1 - \frac{p_i - c}{f_i(x) - c}$. $\square$

Proof of Lemma 2.3: Note that

$$\psi_i(L_i) = \frac{1}{q_i}(w^{-1}(1 - \frac{p_i - c}{f_i(L_i) - c}) - \sum_{j=i+1}^{n} q_j)$$

$$= \frac{1}{q_i}(w^{-1}(w_{i+1}) - \sum_{j=i+1}^{n} q_j) \quad \text{from(2.8)}$$

$$= 0 \quad \text{(by(2.6))}.$$

From (2.9) and (2.7), we obtain

$$\psi_i(L_{i-1}) = \frac{1}{q_i}(w^{-1}(1 - \frac{p_i - c}{f_i(L_{i-1}) - c}) - \sum_{j=i+1}^{n} q_j)$$

$$= \frac{1}{q_i}(w^{-1}(w_i) - \sum_{j=i+1}^{n} q_j)$$

$$= \frac{1}{q_i}.q_i = 1 \quad (\text{as} w_i = w(\sum_{j=i}^{n} q_j)).$$

$w(.)$ is continuous, strictly increasing on compact set $[0, \sum_{j=1}^{n} q_j]$, so w^{-1} is also continuous (theorem 4.17 in [69]). Also, $\frac{p_i - c}{f_i(x) - c}$ is continuous for $x \geq L_i$ as $f_i(x) > c$, so $\psi_i(.)$ is continuous as it is a composition of two continuous functions. Again, $w^{-1}(.)$ is strictly increasing (as $w(\cdot)$ is strictly increasing), $1 - \frac{p_i - c}{f_i(x) - c}$ is strictly increasing (as $f_i(\cdot)$ is strictly increasing), so $\psi_i(.)$ is strictly increasing on $[L_i, L_{i-1}]$ (as it is a composition of two strictly increasing functions (Theorem 4.7 in [69])) □.

Now, we state and prove a result (Observation 2.3). Subsequently we prove Theorem 2.5.

First, note that as $1 - w_i > 0, \forall i \in \{1, \ldots, n\}$, thus, $p_i - c > 0$. Hence, from (2.8) it is evident that

$$f_k(L_k) > c. \tag{2.56}$$

Observation 2.3. For $t > s, t, s \in \{1, \ldots, n\}$

$$p_t - c = (p_s - c) \prod_{i=s}^{t-1} \frac{f_{i+1}(L_i) - c}{f_i(L_i) - c}. \tag{2.57}$$

Proof. Since $f_i^{-1}(\cdot) = g_i$, thus from (2.8) we obtain for $i - 1$

$$p_{i-1} - c = (f_{i-1}(L_{i-1}) - c)(1 - w_i). \tag{2.58}$$

Hence, from (2.7), (2.56), and (2.58)

$$p_i - c = (p_{i-1} - c) \frac{f_i(L_{i-1}) - c}{f_{i-1}(L_{i-1}) - c}. \tag{2.59}$$

We obtain the result using recursion. □

Proof of Theorem 2.5: Fix a state $j \in \{1, \ldots, n\}$. First, we show that if a primary follows its strategy profile then it would attain a payoff of $p_j - c$ at channel state j. Next, we will show that if a primary unilaterally deviates from its strategy profile, then it would obtain a payoff of at most of $p_j - c$ (Case i and Case ii) when the channel state is j.

If the state of the channel of primary 1 is $i \geq 1$ and it selects penalty x, then its expected profit is-

$$\begin{aligned} \phi_i(x) &= (f_i(x) - c) r(x) \\ &= (f_i(x) - c)(1 - w(\sum_{k=1}^{n} q_k . \psi_k(x))). \end{aligned} \tag{2.60}$$

First, suppose $x \in [L_j, L_{j-1}]$. From (2.60) and (2.9), we obtain

$$\begin{aligned}
\phi_j(x) &= (f_j(x) - c)(1 - w(\sum_{i=1}^{n} q_i \psi_i(x))) \\
&= (f_j(x) - c)(1 - w(\sum_{k=j+1}^{n} q_k + q_j \psi_j(x))) \\
&= (f_j(x) - c)(1 - w(w^{-1}(1 - \frac{p_j - c}{f_j(x) - c}))) \quad \text{(from (2.9))} \\
&= p_j - c. \qquad (2.61)
\end{aligned}$$

Since $\psi_i(L_n) = 0 \; \forall i$, we have

$$\phi_j(L_n) = (f_j(L_n) - c)(1 - w(0)) = f_j(L_n) - c. \qquad (2.62)$$

From (2.62) expected payoff to a primary at state j at L_n is $f_j(L_n) - c$. At any $y < L_n$ expected payoff to a primary at state j will be strictly less than $f_j(L_n) - c$. Hence, it suffices to show that for $x \in [L_k, L_{k-1}], k \neq j, k \in \{1, .., n\}$, profit to primary 1 is at most $p_j - c$, when the channel state is j.

Now, let $x \in [L_k, L_{k-1}]$. From (2.60) and (2.9), expected payoff at x

$$\begin{aligned}
\phi_j(x) &= (f_j(x) - c)(1 - w(\sum_{i=k+1}^{n} q_i + q_k \psi_k(x))) \\
&= (f_j(x) - c)(1 - w(w^{-1}(1 - \frac{p_k - c}{f_k(x) - c})))\text{(from (2.9))} \\
&= \frac{(p_k - c)(f_j(x) - c)}{f_k(x) - c}. \qquad (2.63)
\end{aligned}$$

We will show that $\phi_j(x) - (p_j - c)$ is non-positive. As, $k \neq j$, so only the following two cases are possible.

Case i: $k < j$: From (2.3), (2.56) and for $i < j$, we have-

$$\frac{f_i(L_{i-1}) - c}{f_i(L_i) - c} > \frac{f_j(L_{i-1}) - c}{f_j(L_i) - c} \text{(as } L_i < L_{i-1}\text{)}. \qquad (2.64)$$

From Observation 2.3 we obtain-

$$p_j - c = \frac{(p_k - c)(f_j(L_{j-1}) - c)}{f_k(L_k) - c} \prod_{i=k+1}^{j-1} \frac{f_i(L_{i-1}) - c}{f_i(L_i) - c}.$$

Using (2.64) the above expression becomes

$$\begin{aligned} p_j - c &\geq \frac{(p_k - c)(f_j(L_{j-1}) - c)}{f_k(L_k) - c} \prod_{i=k+1}^{j-1} \frac{f_j(L_{i-1}) - c}{f_j(L_i) - c} \\ &= \frac{(p_k - c)(f_j(L_k) - c)}{f_k(L_k) - c}. \end{aligned} \tag{2.65}$$

Hence, from (2.63) and (2.65), we obtain-

$$\begin{aligned} &\phi_j(x) - (p_j - c) \\ &\leq (p_k - c)\left(\frac{f_j(x) - c}{f_k(x) - c} - \frac{f_j(L_k) - c}{f_k(L_k) - c}\right). \end{aligned} \tag{2.66}$$

Since $x \in [L_k, L_{k-1}]$, $j > k$ and $f_k(L_k) > c$ (by (2.56)); hence, from (2.66) and Assumption 1, we have-

$$\phi_j(x) \leq p_j - c. \tag{2.67}$$

Case ii: $j < k$: If $f_j(x) \leq c$ then a primary gets a non-positive payoff at channel state j, which is strictly below $p_j - c$. Hence we consider the case when $f_j(x) > c$. Since $x \leq L_{k-1}$ thus $f_j(L_{k-1}) > c$. Now, if $i > j$ and $f_j(L_i) > c$, we have from (2.3) and (2.56)-

$$\frac{f_i(L_{i-1}) - c}{f_j(L_{i-1}) - c} < \frac{f_i(L_i) - c}{f_j(L_i) - c} \quad (\text{as } L_i < L_{i-1}) \tag{2.68}$$

Since $f_j(L_{k-1}) > c$, thus

$$f_j(L_i) > c \quad (\text{for } j \leq i < k, \text{as } L_i \geq L_{k-1}). \tag{2.69}$$

Now, from Observation 2.3 we obtain-

$$
\begin{aligned}
p_k - c &= (p_j - c)\prod_{i=j}^{k-1}\frac{f_{i+1}(L_i) - c}{f_i(L_i) - c} \\
&= (p_j - c).\frac{f_k(L_{k-1}) - c}{f_j(L_j) - c}\prod_{i=j+1}^{k-1}\frac{f_i(L_{i-1}) - c}{f_i(L_i) - c} \\
&\leq (p_j - c).\frac{f_k(L_{k-1}) - c}{f_j(L_j) - c}\prod_{i=j+1}^{k-1}\frac{f_j(L_{i-1}) - c}{f_j(L_i) - c} \\
&\text{(from (2.68), \&(2.69))} \\
&= (p_j - c).\frac{f_k(L_{k-1}) - c}{f_j(L_{k-1}) - c}. \qquad (2.70)
\end{aligned}
$$

Thus, from (2.63) and (2.70), we obtain-

$$
\begin{aligned}
&\phi_j(x) - (p_j - c) \\
&\leq (p_j - c)(\frac{f_k(L_{k-1}) - c}{f_j(L_{k-1}) - c}.\frac{f_j(x) - c}{f_k(x) - c} - 1) \\
&\leq 0(\text{as } x \leq L_{k-1}, j < k \quad \text{and from Assumption 1}). \qquad (2.71)
\end{aligned}
$$

Hence, from (2.71), (2.67), and (2.61), every $x \in [L_j, L_{j-1}]$ is a best response to primary 1 when channel state is j. Since j is arbitrary, it is true for any $j \in \{1, \ldots, n\}$ and thus (2.9) constitutes a Nash Equilibrium strategy profile. □

2.7.3 Proof of results of Section 2.3

We first establish Lemma 2.4. Subsequently, we prove Lemma 2.5.

Proof of Lemma 2.4: We divide the proof in three parts:

- First, we prove that when $m \geq (l-1)(\sum_{j=1}^{n} q_j + \epsilon)$ for some $\epsilon > 0$, then $p_i - c \to f_i(v) - c$ as $l \to \infty$ (Part I).

- Next we show that if $(l-1)\sum_{j=k}^{n}(q_j+\epsilon) \geq m \geq (l-1)\sum_{j=k+1}^{n}(q_j-\epsilon)$ for some $\epsilon > 0$, then $p_i - c \to f_i(c_k) - c$ if $i > k$ and $p_i - c \to 0$ if $i \leq k$ (Part II).

- Finally, we show if $m \leq (l-1)(q_n+\epsilon)$ for some $\epsilon > 0$, then $p_i - c \to 0$ as $l \to \infty$ for any $i \in \{1, \dots, n\}$ (Part III).

Part I: Suppose $m \geq (l-1)(\sum_{j=1}^{n} q_j + \epsilon)$ for some $\epsilon > 0$.

Since $L_{i-1} \leq v$, thus, from (2.7)-

$$p_i - c \leq (f_i(v) - c) \quad i = 1, \dots, n. \tag{2.72}$$

When primary 1 selects penalty v at channel state $i \geq 1$, then its expected profit is $\phi_i(v) = (f_i(v) - c)(1 - w_1)$. Now, from Theorem 2.6 under the NE strategy profile,

$$p_i - c \geq \phi_i(v) = (f_i(v) - c)(1 - w_1). \tag{2.73}$$

Let $Z_i, i = 1, .., l-1$ be Bernoulli trials with success probabilities $\sum_{j=1}^{n} q_i$ and $Z = \sum_{i=1}^{l-1} Z_i$; so $P(Z \geq m)$ is equal to w_1 by (2.6). Since $m \geq (l-1)(\sum_{i=1}^{n} q_i + \epsilon)$ for some $\epsilon > 0$ and $E(Z) = (l-1)\sum_{i=1}^{n} q_i$, by weak law of large numbers [68], $w_1 \to 0$ as $l \to \infty$. Hence, $p_i - c \to f_i(v) - c$ as $l \to \infty$ by (2.72) and (2.73).Thus, the result follows. $\square$.

Part II: We show the result by evaluating the expressions for $p_j - c, j = 1, \dots, n$ in the asymptotic limit. Towards this end, we first evaluate the expressions for w_j and L_j in the asymptotic limit. We obtain the expression for $p_j - c$ when we combine those two values.

Suppose $(l-1)\sum_{j=k}^{n}(q_j+\epsilon) \geq m \geq (l-1)\sum_{j=k+1}^{n}(q_j-\epsilon)$ for some $\epsilon > 0$. Since w_{k+1} is the probability of at least m successes out of $l-1$ independent Bernoulli trials, each of which occurs with probability $\sum_{j=k+1}^{n} q_j$ (by (2.6)). Hence from the weak law of large numbers [68]

$$1 - w_{k+1} \to 1 \quad \text{as } l \to \infty. \tag{2.74}$$

Since $w_j < w_i$, for any $j > i$ (from (2.6)), we have from (2.74) for $j \geq k+1$

$$1 - w_j \to 1 \quad \text{as } l \to \infty. \tag{2.75}$$

Again, as $m \leq (l-1)(\sum_{j=k}^{n} q_j - \epsilon)$, so, from weak law of large numbers[68], for every $\epsilon > 0$, $\exists L$, such that $1 - w_k < \epsilon$, whenever $l \geq L$. Hence,

$$\begin{aligned} &1 - w_k \underset{l\to\infty}{\to} 0 \\ &1 - w_j \underset{l\to\infty}{\to} 0 \quad (\text{for } j \leq k, w_j \geq w_k). \end{aligned} \tag{2.76}$$

Thus, it is evident from (2.7) and (2.76) that if $i \leq k$, then

$$p_i - c \underset{l\to\infty}{\to} 0. \tag{2.77}$$

Thus, from (2.8), (2.75), and (2.77)

$$L_k \underset{l\to\infty}{\to} g_k(c) = c_k. \tag{2.78}$$

We obtain for $j > k$ from (2.7) and (2.75)

$$p_j \underset{l\to\infty}{\to} f_j(L_{j-1}). \quad (\text{from (2.75)}). \tag{2.79}$$

Again, using (2.8) and (2.75), we obtain for $j > k$

$$p_j \underset{l\to\infty}{\to} f_j(L_j) \quad (\text{from (2.75)}). \tag{2.80}$$

$f_j(\cdot)$ is strictly increasing, thus from (2.79) and (2.80), $L_j \rightarrow L_{j-1}$ (for $j > k$). Hence, for $j > k$,

$$
\begin{aligned}
& L_j \underset{l \to \infty}{\rightarrow} L_k \\
& L_j \underset{l \to \infty}{\rightarrow} c_k \quad \text{(from(2.78))}. \qquad (2.81)
\end{aligned}
$$

Thus, from (2.81), and (2.80), we obtain for any $i > k$

$$
p_i - c \underset{l \to \infty}{\rightarrow} (f_i(c_k) - c). \qquad (2.82)
$$

Thus, from (2.77) $p_i - c \rightarrow 0$ as $l \rightarrow \infty$ if $i \leq k$. From (2.82) we obtain $p_i - c \rightarrow f_i(c_k) - c$ as $l \rightarrow \infty$ if $i > k$. Hence, the result follows. □

Part III: Suppose that $m \leq (l-1)(q_n - \epsilon)$, for some $\epsilon > 0$. Let, $Z_i, i = 1, ..., l-1$ be the Bernoulli trials with success probabilities q_n and $Z = \sum_{i=1}^{l-1} Z_i$, $E(Z) = (l-1)q_n$. Hence,

$$
\begin{aligned}
1 - w_n \quad \leq \quad & P(Z \leq m) \\
& \leq P(Z \leq (l-1)(q_n - \epsilon)) \\
& \leq P(|Z - (l-1)q_n| \geq (l-1)\epsilon) \\
& \leq 2\exp(-\frac{2(l-1)^2\epsilon^2}{l-1}) \\
& \text{(from Hoeffding's Inequality [39])} \\
& = 2\exp(-2(l-1)\epsilon^2). \qquad (2.83)
\end{aligned}
$$

Note that $1 - w_i < 1 - w_j$ (if $j > i$), $f_k(L_{k-1}) > f_{k-1}(L_{k-1})$. Hence, it can be readily seen from (2.7) that

$$
p_i - c \leq (f_i(L_{i-1}) - c)(1 - w_n). \qquad (2.84)
$$

Thus, the result follows from (2.83) and (2.84). □

When $m \leq (l-1)(q_n - \epsilon)$ for some $\epsilon > 0$, then the upper bound for R_{NE} (see (2.10)) from (2.84)is

$$R_{NE} \leq (1 - w_n)(\sum_{j=1}^{n} q_j.(f_j(L_{j-1}) - c)). \tag{2.85}$$

Thus, for $m \leq (l-1)(q_n - \epsilon)$, $\epsilon > 0$, from (2.83) and (2.85), we obtain.

$$R_{NE} \leq \gamma \cdot \exp(-2\epsilon^2.(l-1)) \tag{2.86}$$

where $\gamma = 2(1-w_n)(\sum_{j=1}^{n} q_j.(f_j(L_{j-1})-c))$. We will use this bound in proving Lemma 2.5.

From, the definition of η, it should be clear that

$$\eta \leq 1. \tag{2.87}$$

Now, we show Lemma 2.5

Proof of Lemma 2.5: We divide the proof in the following two parts

- First, we show that if $m \geq (l-1)(\sum_{i=1}^{n} q_i + \epsilon)$ for some $\epsilon > 0$, then $\eta \to 1$ as $l \to \infty$ (Part I).
- Next, we show that if $m \leq (l-1)(q_n - \epsilon)$, for some $\epsilon > 0$, then $\eta \to 0$ as $l \to \infty$ (Part II).

Part I: First suppose that $m \geq (l-1)(\sum_{i=1}^{n} q_i + \epsilon)$ for some $\epsilon > 0$.

From, definition of R_{OPT}, it is obvious that

$$R_{OPT} \leq l \cdot (\sum_{i=1}^{n}(q_i.(f_i(v) - c))). \tag{2.88}$$

Hence the result follows from Corollary 2.1, (2.88) and (2.87). □

Part II: Now, suppose that $m \leq (l-1)(q_n - \epsilon)$, for some $\epsilon > 0$. We prove that $\eta \to 0$ as $l \to 0$.

We prove the result by showing that R_{NE} decreases at fast rate to 0 compared to R_{OPT} when $l \to \infty$.

Let, Z be the number of primaries, whose channel is in state n. Hence,

$$R_{OPT} \geq E(\min(Z, m))(f_n(v) - c)$$
$$\frac{R_{OPT}}{f_n(v) - c} \geq E(\min(Z, m)). \tag{2.89}$$

Note that $E(Z) = l \cdot q_n$, $Var(Z) = l \cdot q_n(1 - q_n)$.

We introduce a new random variable Y as follows-

$$Y = \begin{cases} m, & \text{if} Z \geq m \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} E(\min(Z, m)) &\geq E(Y) \\ &= m.P(Z \geq m) \\ &\geq m.(1 - P(Z \leq (l-1)(q_n - \epsilon))) \\ &\geq m.(1 - P(|Z - l.q_n| \geq (l-1)\epsilon) \\ &\geq m.(1 - \frac{l.q_n.(1-q_n)}{(l-1)^2.\epsilon^2}) \\ &\quad \text{(From Chebyshev's Inequality).} \end{aligned} \tag{2.90}$$

Hence, from (2.86), (2.89) and (2.90), we obtain-

$$\eta \leq \frac{l.\gamma.\exp(-2(l-1)\epsilon^2)}{m.(1 - \frac{l.q_n.(1-q_n)}{(l-1)^2.\epsilon^2}).(f_n(v) - c)}.$$

Thus, η tends to zero for $m \leq (l-1)(q_n - \epsilon)$,as l tends to infinity (as $m \neq 0$). □

2.7.4 Proof of Results of Section 2.4

Here, we prove Theorem 2.7. Towards this end, we state and prove Observation 2.4.

First, we evaluate the total expected payoff that a primary will get under the strategy profile (SP_R). Note that the strategy SP_R is symmetric, thus, the expected payoff of primaries would be identical and thus, we only evaluate the expected payoff of primary 1. Now we introduce some notations which we use throughout this section:

Definition 2.15. Let X_m be the mth smallest offered penalty offered by primaries $i = 2, \ldots, l$.

Let, A_i denote the event that at a time slot, primary 1's channel will be bought, when its channel state is i and selects penalty $v - \epsilon_i$ and primary $2, \ldots, l$ selects penalty $v - \epsilon_j$, when its channel state is j, $j \in \{1, \ldots, n\}$. Let's recall the definition of X_m (definition 2.15). From the law of total probability,

$$\begin{aligned}\Pr(A_i) = &\Pr(A_i|X_m > v - \epsilon_i)\Pr(X_m > v - \epsilon_i)\\ &+ \Pr(A_i|X_m = v - \epsilon_i)\Pr(X_m = v - \epsilon_i)\\ &+ \Pr(A_i|X_m < v - \epsilon_i)\Pr(X_m < v - \epsilon_i)\end{aligned} \tag{2.91}$$

Now, note that $\Pr(A_i|X_m) = 1$ if $X_m > v - \epsilon_i$ and $\Pr(A_i|X_m) = 0$ if $X_m < v - \epsilon_i$.

Note from (2.12) that

$$\begin{aligned}\Pr(X_m > v - \epsilon_i) &= \sum_{j=0}^{m-1} \binom{l-1}{j} (\sum_{k=i}^{n} q_k)^j (1 - \sum_{k=i}^{n} q_k)^{l-1-j}\\ &= 1 - w_i\end{aligned} \tag{2.92}$$

Thus, the first term of right hand side (r.h.s.) of(2.91) is $1 - w_i$. We will denote the second term of the r.h.s. of (2.91) as β_i. Since the third term of the r.h.s. of (2.91) is zero, hence,

$$\Pr(A_i) = 1 - w_i + \beta_i \tag{2.93}$$

Thus, if primary follows the strategy profile as described, then its total expected payoff at any stage of the game will be

$$R_{SNE} = \sum_{j=1}^{n} q_j(f_j(v - \epsilon_j))(1 - w_j + \beta_j) \tag{2.94}$$

Next observation will be used in proving Theorem 2.7.

Observation 2.4. If a primary selects a penalty which is strictly greater than $v - \epsilon_i$, then the probability of winning is $\leq 1 - w_i$.

Proof. Consider that a primary selects penalty $x > v - \epsilon_i$. Note that only if $X_m \geq x$, then the channel of the primary may be bought[14]. Hence,

$$\Pr(\text{the channel of the primary is bought}) \leq \Pr(X_m \geq x) \tag{2.95}$$

Since, $x > v - \epsilon_i$, thus

$$\Pr(X_m > v - \epsilon_i) = \Pr(X_m \geq x) + \Pr(x > X_m > v - \epsilon_i)$$

$$=> \Pr(X_m > v - \epsilon_i) \geq \Pr(X_m \geq x) \tag{2.96}$$

Hence, using (2.96) in (2.95), we obtain

$$\Pr(\text{the channel of the primary is bought}) \leq \Pr(X_m > v - \epsilon_i) \tag{2.97}$$

But from (2.92), $1 - w_i = \Pr(X_m > v - \epsilon_i)$, hence the result follows from (2.97). □

[14] when $X_m = x$ then there is a nonzero probability that the channel may not be bought

Now, we are ready to prove Theorem 2.7.

proof of Theorem 2.7: Fix any state i. We prove the theorem in two part. In Part 1, we show that when the game is at a stage where other primaries select penalty $v-\epsilon_j, j = 1,\ldots,n$ at channel state j, then primary 1 does not have any unilateral deviation by selecting penalty different from $v-\epsilon_i$ for sufficiently high δ. In part 2, we show that if the game is in a stage where all the other primaries play the unique NE strategy profile, then primary 1 also does not have any profitable unilateral deviation. This will ensure that SP_R is a subgame perfect NE.

Proof of Part 1: First, we show that, deviating to lower penalty compared to $v-\epsilon_i$ is not profitable (case 1) and then we show that deviating to a higher penalty compared to $v-\epsilon_i$, is also not profitable (case 2) when other primaries select penalty $v-\epsilon_j, j = 1,\ldots,n$ at channel state j.

Case 1: First, suppose that primary 1 offers penalty, which is strictly less than $v-\epsilon_i$.

A primary can attain at most a payoff of $f_i(v-\epsilon_i)$ at this stage. After this deviation, all the primaries play the unique N.E. strategy. The payoff is given by R_{NE}. Now,

$$\begin{aligned} R_{NE} &= \sum_{j=1}^{n} q_j(p_j - c) \\ &= \sum_{j=1}^{n} q_j(f_j(L_{j-1}) - c)(1-w_j) \quad \text{(from (2.7))} \\ &\leq \sum_{j=1}^{n} q_j(f_j(v-\epsilon_j) - c)(1-w_j) \quad \text{(from(2.13))} \end{aligned} \tag{2.98}$$

If a primary deviates at stage T, then its expected payoff starting from stage T would be at most

$$(1-\delta)[\delta^T(f_i(v-\epsilon_i)-c)+\sum_{t=T+1}^{\infty}\delta^t R_{NE}]$$
$$=\delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)+\delta R_{NE}] \tag{2.99}$$

If a primary would not have deviated, then its expected payoff would have been

$$(1-\delta)[\delta^T(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)+\sum_{t=T+1}^{\infty}\delta^t R_{SNE}]$$
$$=\delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)+\delta R_{SNE}] \tag{2.100}$$

Hence, from (2.99) and (2.100), the following condition must be satisfied for sub game perfect equilibrium

$$\delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)+\delta R_{NE}] \leq$$
$$\delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)+\delta R_{SNE}] \tag{2.101}$$

From (2.94) and (2.98), it is enough to satisfy the following inequality in order to satisfy inequality (2.101)

$$(1-\delta)(f_i(v-\epsilon_i)-c)+\delta\sum_{j=1}^{n}q_j(f_j(v-\epsilon_j)-c)(1-w_j) \leq$$
$$(1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)$$
$$+\delta\sum_{j=1}^{n}q_j(f_j(v-\epsilon_j)-c)(1-w_j+\beta_j) \tag{2.102}$$

By simple algebraic manipulation in (2.102), we obtain

$$\delta \geq \frac{(f_i(v-\epsilon_i)-c)(w_i-\beta_i)}{(f_i(v-\epsilon_i)-c)(w_i-\beta_i)+\sum_{j=1}^{n}q_j(f_j(v-\epsilon_j)-c)\beta_j} \tag{2.103}$$

The proof is complete by observing that the right hand side of (2.103) is strictly less than 1. Hence, if δ is greater than the following expression

$$\max_{i \in 1,\ldots,n} \frac{(f_i(v-\epsilon_i)-c)(w_i-\beta_i)}{(f_i(v-\epsilon_i)-c)(w_i-\beta_i)+\sum_{j=1}^{n} q_j(f_j(v-\epsilon_j)-c)\beta_j}$$

then, a primary will not have any profitable one shot deviation.

Case 2: Now, suppose that primary 1 offers penalty which is strictly greater than $v-\epsilon_i$.

Since primary 1 offers penalty strictly greater than $v-\epsilon_i$ and $v-\epsilon_i \leq v$, thus, from observation 2.4, primary 1 can at most attain a payoff of $(f_i(v)-c)(1-w_i)$ by offering penalty higher than $v-\epsilon_i$. After the deviation, all the primaries play the one-shot NE strategy profile. If a primary deviates at stage T, then its expected payoff starting from stage T would be

$$(1-\delta)[\delta^T(f_i(v)-c)(1-w_i)+\sum_{t=T+1}^{\infty}\delta^t R_{NE}]$$
$$= \delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)+\delta R_{NE}] \tag{2.104}$$

If a primary would not have deviated, then its expected payoff would be

$$(1-\delta)[\delta^T(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)+\sum_{t=T+1}^{\infty}\delta^t R_{SNE}]$$
$$= \delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)+\delta R_{SNE}] \tag{2.105}$$

Hence, from (2.104) and (2.105), the following condition must be satisfied for sub game perfect equilibrium

$$\delta^T[(1-\delta)(f_i(v)-c)(1-w_i)+\delta R_{NE}] \leq$$
$$\delta^T[(1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i)+\delta R_{SNE}] \tag{2.106}$$

Hence, from (2.94) satisfying the following condition will be enough for the strategy profile to be a SPNE.

$$\begin{aligned} & (1-\delta)(f_i(v)-c)(1-w_i)+\delta R_{NE} \\ & \leq (1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i) \\ & + \delta \sum_{j=1}^{n} q_j(f_j(v-\epsilon_j)-c)(1-w_j+\beta_j) \end{aligned} \tag{2.107}$$

But, from (2.13)

$$(f_i(v)-c)(1-w_i) \leq (f_i(v-\epsilon_i)-c)$$

Hence, it is enough to satisfy the following inequality in order to satisfy inequality (2.107)

$$\begin{aligned} & (1-\delta)(f_i(v-\epsilon_i)-c)+\delta \sum_{j=1}^{n} q_j(f_j(v-\epsilon_j)-c)(1-w_j) \leq \\ & (1-\delta)(f_i(v-\epsilon_i)-c)(1-w_i+\beta_i) \\ & + \delta \sum_{j=1}^{n} q_j(f_j(v-\epsilon_j)-c)(1-w_j+\beta_j) \end{aligned} \tag{2.108}$$

which is exactly similar to (2.102). Hence, the rest of the proof will be similar to case 1.

proof of part 2: If other primaries play the unique one-shot NE strategy, selecting the unique one-shot NE strategy is a best response for a primary. Hence, no primary has any profitable unilateral deviation.

2.7.5 Proof of Results of Section 2.5

First, we prove Lemma 2.6. Subsequently, we state and prove Observation 2.5 which we use to show Theorem 2.8. The proof of Theorem 2.9 is similar and hence we omit it. Finally, we show Theorems 2.10 and 2.11.

Proof of Lemma 2.6: First , it is evident from (2.15) and (2.16)

$$0 \leq \frac{\bar{p}_1 - c}{f_1(x) - c} \quad (\text{if} x \geq \bar{L})$$

Hence, $w^{-1}(\cdot)$ is defined at $x \geq \bar{L}$. Note that

$$\bar{\psi}(\bar{L}) = \frac{1}{\sum_{j=1}^{n} q_j} w^{-1}(1 - \frac{\bar{p}_1 - c}{f_1(g_1(\bar{p}_1)) - c}) \quad (\text{from}(2.16))$$

$$= 0$$

Note that

$$\begin{aligned}
\bar{\psi}(v) &= \frac{1}{\sum_{j=1}^{n} q_j} w^{-1}(1 - \frac{\bar{p}_1 - c}{f_1(v) - c}) \\
&= \frac{1}{\sum_{j=1}^{n} q_j} w^{-1}(1 - \frac{(f_1(v) - c)(1 - w_1)}{f_1(v) - c}) \\
&= \frac{1}{\sum_{j=1}^{n} q_j} w^{-1}(w_1) \\
&= 1
\end{aligned}$$

We already know that $w^{-1}(\cdot)$ is continuous and strictly increasing. Since $1 - \dfrac{\bar{p}_1 - c}{f_1(x) - c}$ is strictly increasing and continuous for $x \geq \bar{L}$. Hence, $\bar{\psi}(\cdot)$ is continuous and strictly increasing on $[\bar{L}, v]$. $\square$.

Observation 2.5.

$$f_i(\bar{L}) = \bar{p}_i \quad (i = 1, \ldots, n) \tag{2.109}$$

Proof. The result is trivially true for $i = 1$ by definition (2.16) as $g_1(\cdot) = f_1^{-1}(\cdot)$. We will show the statement for $i \geq 2$. Since $f_i(\bar{L}) > f_1(\bar{L}) > c$, we have from (2.14)

$$\frac{f_1(\bar{L}) - c}{f_i(\bar{L}) - c} = \frac{f_1(v) - c}{f_i(v) - c}$$

$$\frac{\bar{p}_1 - c}{f_i(\bar{L}) - c} = \frac{f_1(v) - c}{f_i(v) - c}$$

$$\frac{(f_1(v) - c)(1 - w_1)}{f_i(\bar{L}) - c} = \frac{f_1(v) - c}{f_i(v) - c} \quad \text{(from(2.15))}$$

$$f_i(\bar{L}) - c = \bar{p}_i - c \quad \text{(from(2.15))}$$

Hence, the result follows. □

Proof of Theorem 2.8: We show that for any $x \in [\bar{L}, v]$, a primary attains a payoff of $\bar{p}_i - c$ at channel state i. Then, we will show that if a primary selects a penalty outside the interval a primary's payoff is strictly less than $\bar{p}_i - c$ at channel state i.

Suppose, $x \in [\bar{L}, v]$. Now, fix any channel state $i \in \{1, \ldots, n\}$. If primary 1 selects penalty x at channels state i, then its expected payoff is

$$\begin{aligned}
\phi_i(x) &= (f_i(x) - c)(1 - w(\sum_{j=1}^{n} q_j \bar{\psi}_j(x))) \\
&= (f_i(x) - c)(1 - w(\bar{\psi}(x) \sum_{j=1}^{n} q_j)) \\
&= (f_i(x) - c)(1 - w(w^{-1}(1 - \frac{\bar{p}_1 - c}{f_1(x) - c}))) \\
&= \frac{(\bar{p}_1 - c)(f_i(x) - c)}{f_1(x) - c} \\
&= \frac{(f_1(v) - c)(f_i(x) - c)(1 - w_1)}{f_1(x) - c} \\
&\text{(Using(2.15)}, f_1(\bar{L}) > c) \\
&= (f_i(v) - c)(1 - w_1) \quad \text{(using(2.14))} \\
&= \bar{p}_i - c \quad \text{(from(2.15))}
\end{aligned} \tag{2.110}$$

Now, at any $x < \bar{L}$, expected payoff will be strictly less than $f_i(\bar{L}) - c$. But, from Observation 2.5, $f_i(\bar{L}) - c = \bar{p}_i - c$.

Thus, from (2.110), when channel state is $i \geq 1$, every point in the interval $[\bar{L}, v]$ is a best response to primary 1. Hence, the result follows. □

Proof of Theorem 2.10: By simple calculation, we obtain the following values

$$p_1 = 0.9305, L_1 = 1.1432, L_2 = 0.9372$$

Now, consider the following unilateral deviation for primary 1: primary 1 will choose a penalty $x \in (0.9305, 0.9372)$ with probability 1, when the channel state is 1. Since, no primary selects penalty lower than 0.9372 under the strategy profile (2.9), thus expected payoff that primary 1 will obtain is $f_1(x) - c = x - c$, which is strictly larger than $p_1 - c$ (since $x > 0.9305 = p_1$), the expected payoff that primary 1 gets by Theorem 2.6 when it selects strategy according to (2.9). Thus, the strategy profile as defined in (2.9) is not an NE. □

Proof of Theorem 2.11: We show that when the channel state is $i = 2$ a primary does not have any profitable unilateral deviation from the strategy profile. The proof for $i = 1$ is similar and thus we omit it.

First, we show that under the strategy profile a primary attains a payoff of $\tilde{p}_2 - c$ at channel state $i = 2$. Next, we show that if a primary deviates at channel state 2, then its expected payoff is upper bounded by $\tilde{p}_2 - c$.

Note that when channel state is 2 and primary chooses penalty $x \in [\tilde{L}_2, v]$, expected payoff to primary 1 is-

$$
\begin{aligned}
&(f_2(x)-c)(1-w(\sum_{i=1}^{2} q_i * \tilde{\psi}_i(x))) \\
&= (f_2(x)-c)(1-w((w^{-1}(\frac{f_2(x)-\tilde{p}_2}{f_2(x)-c})+q_1-q_1))) \\
&= \tilde{p}_2 - c \qquad (2.111)
\end{aligned}
$$

Now, suppose $x \in [\tilde{L}_1, \tilde{L}_2]$. From (2.111), expected payoff to a primary when it selects penalty x at channel state 2, is-

$$
\begin{aligned}
&(f_2(x)-c)(1-w(q_1 * \tilde{\psi}_1(x))) \\
&= (f_2(x)-c)(1-w(w^{-1}(\frac{f_1(x)-\tilde{p}_1}{f_1(x)-c}))) \\
&= (f_2(x)-c)\frac{\tilde{p}_1-c}{f_1(x)-c} \\
&= \frac{(f_2(x)-c)(f_1(\tilde{L}_2)-c)}{(f_1(x)-c)(f_2(\tilde{L}_2)-c)} * (\tilde{p}_2-c) \\
&\text{(from (2.29) and } c=0) \\
&< \tilde{p}_2 - c \quad (\text{from (2.27) as} \tilde{L}_2 \geq x \geq \tilde{L}_1 > 1) \qquad (2.112)
\end{aligned}
$$

Note that at $\tilde{L}_1$, $\tilde{\psi}_i(x) = 0, i = 1, 2$. Hence, the expected payoff to a primary when it selects penalty $\tilde{L}_1$ at channel state 2 is given by $f_2(\tilde{L}_1) - c$. From (2.112) we obtain $\tilde{p}_2 > f_2(\tilde{L}_1)$, hence any penalty $x < \tilde{L}_1$ will induce payoff of strictly lower than $\tilde{p}_2$ when channel state is 2. Hence, the result follows. □

Chapter 3

Multiple Locations

We investigate a spectrum oligopoly market where each primary seeks to sell secondary access to its channel at multiple locations. Transmission qualities of a channel evolve randomly. Each primary needs to select a price and a set of non-interfering locations (which is the independent set in the conflict graph of the region) at which to offer its channel without knowing the transmission qualities of the channels of its competitors. At each location each secondary selects a channel depending on the price and the quality of the channels. We formulate the above problem as a non-cooperative game with primaries as players. We consider two scenarios-i) when the region is small, ii) when the region is large. In the first setting, we focus on a class of conflict graphs, known as mean valid graphs which commonly arise when the region is small. We explicitly characterize a computationally efficient symmetric Nash equilibrium (NE). The NE is threshold type in that primaries only choose independent set whose cardinality is greater than a certain threshold. The threshold on the cardinality increases with increase in quality of the channel. We show the functional uniqueness of the above NE. We show that the symmetric NE strategy profile is unique (not merely functionally unique) in a special class of conflict graphs (linear graph) which commonly arises in practice. When the region is large, we consider node symmetric conflict graphs as such conflict graphs commonly arise when the number of locations is large. We explicitly compute a symmetric NE that randomizes equally among the maximum independent sets at a given channel state vector. We show that the two symmetric NEs computed in two settings exhibit important structural differences. For example, when the number of locations is large, a primary only selects a maximum independent set, however when the number of locations is small, the primary also selects an independent set which may not be a maximum. The symmetric NE is also not unique in a linear conflict graph when the number of locations is large.

The chapter is organized as follows: In Section 3.1 we describe the system model and the strategies of the primaries. In Section 3.2 we show that there can be multiple asymmetric NEs unlike in the single location game. Asymmetric NEs are difficult to implement in the symmetric game which we consider. We, thus, focus on symmetric NE. In Lemma 3.1 we show that the pricing strategy can be computed once the independent set selection strategy is known (using the single location pricing strategy which has been characterized in the previous chapter). We, thus, focus on finding the independent set selection strategy. In Section 3.3 we characterize the structure of a computationally efficient symmetric NE strategy, show the existence and also compute the same in the setting where the channel state remains the same. In Section 3.4 we consider the setting where the channel state may be different at different locations. We characterize a symmetric NE and explicitly compute it. In Section 3.6 we numerically analyze the ratio of the payoffs of the primaries in non cooperative setting compared to the setting where the primaries collude with each other.

3.1 System Model

Unless otherwise stated, we consider that there are l number of primaries and m number of secondaries at each location throughout this chapter. We, however, generalize our result for random *apriori* unknown m in Section 3.5. Different channels constitute disjoint frequency bands. Each primary only allows at most secondary to transmit at a given location.

3.1.1 Transmission Rate and Channel State

Similar to the Chapter 2 the transmission rate offered by the channel at a location depends on the state. The channel can be in one of the states ranging from $0, 1, \ldots, n$ at each location (Table 3.1).

Let J denote the channel state vector which indicates the channel state at each node. For example, when the number of nodes are 3, then $J = (1, 1, 0)$ is a channel state vector which indicates that the channel is in state 1, 1, and 0 at nodes $1, 2$, and 3 respectively. We assume that the channels are statistically identical, specifically the probability that the channel state vector of a primary is J is q_J. We also assume that the probability of the event where the channel state is 0 at every location is non-zero i.e.

$$q_J > 0 \quad \text{when } J = \{0, 0, \ldots, 0\} \tag{3.1}$$

Notation	Connotation	Significance & Assumption
$0, 1 \ldots, n$	States of each channel.	Transmission rate at state i is higher compared to state j if $i > j$. The channel is said to be in state 0 if it is not available for sale.
$g_i(\cdot)$	Penalty function for all secondaries at channel state i. It is a function of price and transmission rate at channel state i.	$g_i(\cdot)$ is strictly increasing in price and $g_i(x) < g_j(x)$ if $i > j$.
$f_i(\cdot)$	Inverse of $g_i(\cdot)$	$f_i(\cdot)$ is strictly increasing in penalty and $f_i(x) > f_j(x)$ if $i > j$.
v	Upper bound for penalty for all secondaries	Secondaries do not buy a channel whose penalty exceeds v.
c	Transaction cost incurred by primary at each location	$c > 0$

Table 3.1: Symbols defined in Chapter 2

3.1.2 Penalty functions and Assumptions

Notations $l, m, n, c, v, g_i(\cdot), f_i(\cdot), i = 1, \ldots, n$ have the same connotation as in the Chapter 2. For completeness, we summarize all these in Table 3.1.

As justified in Chapter 2, we assume that the inverse of penalty functions satisfy the following assumption:

Assumption 3.1.

$$\frac{f_i(y) - c}{f_j(y) - c} < \frac{f_i(x) - c}{f_j(x) - c} \quad \text{for all } x > y > g_i(c), i < j. \tag{3.2}$$

In the special class, when $n = 1$ i.e. the channel is either available or not, then the available channels offer the same transmission rates. Hence, we do not need the penalty functions to capture the preference order of secondaries for available channels having different transmission rates. Thus, the penalty functions are redundant when $n = 1$. But to be consistent with the notations, we still use the penalty function $g_1(\cdot)$ and the inverse penalty function $f_1(\cdot)$ when $n = 1$. We do not need Assumption 1 when $n = 1$ and we only assume that penalty function $g_1(\cdot)$ is strictly increasing.

3.1.3 Conflict Graph

Each primary owns a channel over a broad region consisting of several *locations*. Typically, secondary users can not transmit simultaneously using the same channel at adjacent locations due to interference. In order to sell its channel a primary needs to find a set of locations which do not interfere with each other. Wireless networks have been traditionally modeled as *conflict graphs* (Figures 3.1, 3.2, 3.3) in most of the existing literature including in several seminal papers [33, 44, 32]. Let $G = (V, E)$ be the overall conflict graph of the region where V is the set of nodes and E is the set of edges; an edge exists between two nodes iff transmission at the corresponding locations interfere. In a conflict graph, the set of nodes in which no edge exists between any pair of nodes is called an *independent set* (Fig. 3.1, 3.3). Thus, secondaries at all nodes in an independent set, can transmit simultaneously using the same channel without any interference.

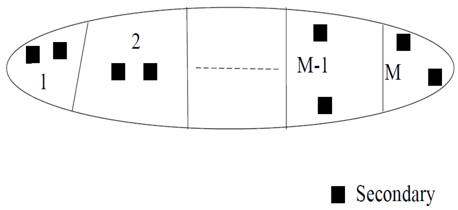
Note that when the channel of a primary is at state 0 at a node, then the primary can not sell its channel at that node. Thus, a primary ought to offer its channel at a set of non interfering locations among the locations where the channel is available for sale (i.e. the state of the channel is not 0). Let $G_J = (V_J, E_J)$ be the conflict graph representation of the channel state vector J: V_J is the set of nodes (locations) where the channel is available for sale at channel state vector J of a primary and E_J is the set of edges in G between the nodes of V_J. G_J is obtained by removing nodes and the edges corresponding to those nodes from G where the channel is not available i.e. the channel is at state 0. Thus, G_J is a subgraph of G. Figure 3.4 represents a conflict graph G of a region and the conflict graph G_J when the channel state vector is J. A primary needs to select an independent set from G_J when the channel state vector is J.

3.1.4 Strategy and Payoff of Each Primary

Let $\mathcal{P}$ denote the set of all possible channel state vectors except when the channel state is 0 across all the locations. Note that $|\mathcal{P}| = (n+1)^{|V|} - 1$.

For each channel state vector $J \in \mathcal{P}$ a primary selects[1]: a) an independent set of the conflict graph G_J where it will sell its channel; b) a price at every node of that

[1] A primary does not need to select a strategy when the channel state is 0 at all locations.



(a)

(b)

Figure 3.1: Figure in (a) shows a wireless network with M number of locations. There are $m = 2$ secondaries at each location. Signals at locations 1 and 2 and 2 and 3 interfere with each other, but signals at locations 1 and 3 do not interfere. Linear Graph in figure (b) models the conflict graph of the network in (a). Note that there is an edge between nodes 1 and 2, but not between nodes 1 and 3. $I_1 = \{1, 3, 5, \ldots, M_o\}$ and $I_2 = \{2, 4, \ldots, M_e\}$ constitute independent sets, where M_o (M_e, respectively) is the greatest odd (even, respectively) less than or equal to M. There are other independent sets too e.g. {1,4,6}. Also {1,2,4} is not an independent set since there is an edge between nodes 1 and 2.

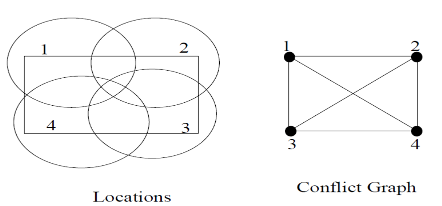


Figure 3.2: The rectangle represents a shop in a shopping complex or a department in a university campus. Circles $1, 2, 3, 4$ are the ranges of Wireless access points. Each circle corresponds to a node in the conflict graph. Since ranges of Wireless access points intersect with each other, thus there exists an edge between every pair of nodes.

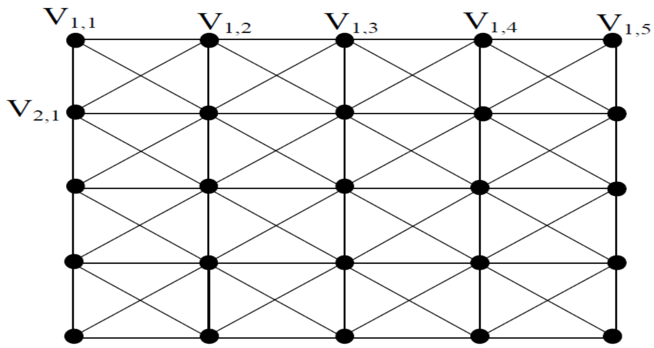


Figure 3.3: The above graph is the conflict graph representation of a larger region consisting of several networks depicted in Fig. 3.2. It is a grid conflict graph with k rows and columns (here $k = 5$). Nodes correspond to the Wireless access points. $\{V_{1,1}, V_{1,3}\}$ is an independent set and users at these two nodes can transmit simultaneously. But $\{V_{1,1}, V_{1,2}\}$ or $\{V_{1,1}, V_{2,1}\}$ are not independent sets.

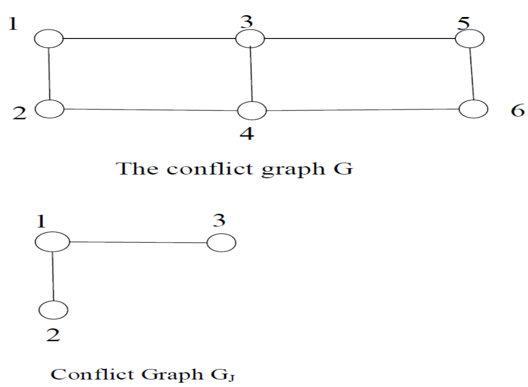


Figure 3.4: The conflict graph for the overall region is G which corresponds to the situation where the channel is available at all nodes in the region, G_J is the conflict graph when the channel state vector is $J = (j_1, j_2, j_3, 0, 0, 0)$ where $j_i \geq 1, i = 1, 2, 3$. Since the channel states are 0 at nodes $4, 5$, and 6, thus, G_J is obtained by removing those nodes and the edges corresponding to those nodes.

independent set. A primary arrives at its decision with the knowledge of its own channel state vector q_J but without knowing the *channel state vector of other primaries.* A primary however knows $l, m, n, G, g_1, \ldots, g_n, f_1, \ldots, f_n$, and $q_J, J \in \mathcal{P}$. As discussed in Section 2.1.2 secondaries strictly prefer a channel which induces lower penalty compared to the higher penalty one. Since there is a one-to-one correspondence between the price and the penalty at a given channel state, similar to Chapter 2 we consider that primaries select penalties instead of prices. The ties among channels with identical penalties are broken randomly and symmetrically among the primaries. We formulate the decision problem of primaries as a non-cooperative game with primaries as players.

Definition 3.1. A strategy of a primary i $\psi_{i,J}$ provides the probability mass function (p.m.f) for selection among the independent sets (I.S.s) and the penalty distribution it uses at each node, when its channel state vector is J . $S_i = (\psi_{i,1}, ..., \psi_{i,|\mathcal{P}|})$ denotes the strategy of primary i, and $(S_1, ..., S_l)$ denotes the strategy profile of all primaries (players). S_{-i} denotes the strategy profile of primaries other than i.

Note from Table 3.1 that each primary incurs a transaction cost c at each location where it is able to sell its channel. If primary i selects a penalty x at node s when the channel state is j, then its payoff at node s is[2]

[2]Note that if Y_s is the number of channels offered for sale at a node s, for which the penalties are upper bounded by v, then those with $\min(Y_s, m)$ lowest penalties are sold since secondaries select channels in the increasing order of penalties.

$$\begin{cases} f_j(x) - c & \text{if the primary sells its channel} \\ 0 & \text{otherwise.} \end{cases}$$

The payoff of a primary over an independent set is the sum of payoff that it gets at each node of that independent set. Thus, if a primary is unable to sell at any node of an independent set, then its payoff is 0 over that independent set.

Definition 3.2. $u_{i,J}(\psi_{i,J}, S_{-i})$ is the expected payoff when primary i's channel state vector is J and selects strategy $\psi_{i,J}(\cdot)$ and other primaries use strategy S_{-i}.

3.1.5 Solution Concept

We seek to obtain a Nash Equilibrium (NE) strategy profile which we define below using $u_{i,J}$ (Definition 3.2), $\psi_{i,J}$ and S_{-i} (Definition 3.1):

Definition 3.3. [59] A *Nash equilibrium* $(S_1, \ldots, S_l)$ is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy. So, with $S_i = (\psi_{i,1}, ..., \psi_{i,|\mathcal{P}|})$, $(S_1, \ldots, S_l)$, is a Nash equilibrium (NE) if for each primary i and channel state vector J

$$u_{i,J}(\psi_{i,J}, S_{-i}) \geq u_{i,J}(\tilde{\psi}_{i,J}, S_{-i}) \ \forall \ \tilde{\psi}_{i,J}. \tag{3.3}$$

An NE $(S_1, \ldots, S_l)$ is a *symmetric NE* if $S_i = S_j$ for all i, j.

If $S_i \neq S_k$ for some $i, k \in \{1, \ldots, l\}$ in an NE strategy profile, then the strategy profile is an asymmetric NE.

In a symmetric game, as the one we consider, it is difficult to implement an asymmetric NE. For example, if there are two players and (S_1, S_2) is an asymmetric NE i.e. $S_1 \neq S_2$, then (S_2, S_1) is also an NE due to the symmetry of the game. The realization of such an NE is only possible when one player knows whether the other is using S_1 or S_2. But, apriori coordination among players is infeasible as the game is non co-operative.

Note that if $m \geq l$, then primaries select the highest penalty v at each node and will select one of the maximum independent sets of G_J at channel state vector J with probability 1. This is because, when $m \geq l$, then, the channel of a primary will always be sold at a location. Hence, a primary will be always be able to sell its channel at the highest possible penalty. Henceforth, we will consider that $m < l$.

3.1.6 Two Different Settings

We consider two different settings: i) First, we consider that the region is small and consists of a few (but, multiple) locations (Section 3.3). Initially, it is expected that the secondary market will be introduced in a small region consisting of few locations. In a small region, the usage statistic and the propagation condition of a channel will be similar at each location, thus, in an analytical abstraction we consider that the transmission rate offered by a channel is the same at each location. In this setting, the interference relations amongst the locations may not be symmetric in general which we accommodate in our model. Since we only consider that the channel state is the same across the nodes, thus, $q_J = 0$ for all those channel state vectors where the channel state is not identical at each location.

ii) Second, we consider the region consists of large number of locations (Section 3.4). This is likely to happen in later stages of deployment of the secondary market. Since the geographical region is large, the transmission rate offered by a channel at different locations may be different. Thus, we consider that the channel state of a primary can be different at different locations. Given the large region, there will be an inherent symmetry in the interference relations among the locations which we characterize and exploit.

3.2 Multiple NEs, and A Separation Result

3.2.1 Multiple Asymmetric NEs

We first show that there can be multiple NEs in this game unlike in the single location game (Chapter 2). Consider the linear conflict graph (Fig. 3.1) with 2 nodes, 2 primaries and 1 secondary.

We show multiple asymmetric NEs for two different settings which we have discussed in Section 3.1.6. First, we consider the setting where the channel state is the same across the network. Thus, a primary needs to select a strategy when the channel state is not 0 across the network. Note that if primaries selects different nodes, then each primary can attain a maximum profit of $(f_i(v)-c)$ at the channel state i which corresponds to selecting penalty v. Thus, both the following strategy profiles are asymmetric NE: 1) primary 1 (2, respectively) selects V_1 (V_2, respectively) w.p. 1 and selects penalty v irrespective of the channel state; 2) primary 1 (2, respectively) selects V_2 (V_1, respectively) w.p. 1 and selects penalty v w.p. 1 irrespective of the channel state across the network. The realization of one of the above NEs is possible only when a primary knows other's strategy apriori; this is ruled out due to non-cooperation. Thus, asymmetric NE can not be realized in this game.

Now, we will provide multiple asymmetric NE strategies for the above linear conflict graph when the channel state can be different at different locations using the NE penalty strategy for single location as presented in Chapter 2. We need to specify strategy at each possible channel state vector. We consider $n = 1$ i.e. at any given node the channel is either available (state 1) or not (state 0). We also consider that the channel state of a primary is 1 at a given location w.p. q_1 independent of the channel state at other location. The following strategy profiles are NE strategy profiles: i) When the channel state vector is $(0, 1)$ ($(1, 0)$ respv.) then a primary selects node 2 (1 respv.) w.p. 1 and selects the single location penalty strategy stated in Theorem 2.6 with $q_1 q_0$ in place of q_1[3]. When the channel state vector is $(1, 1)$ then primary 1 (primary 2 respv.) selects node 1 (node 2 respv.) w.p. 1 and selects penalty v w.p. 1.

ii) When the channel state vector is either $(0, 1)$ or $(1, 0)$ then the strategy profile is the same as before. When channel state vector is $(1, 1)$ then primary 1 (primary 2 respv.) selects node 2 (node 1 respv.) w.p. 1 and selects penalty v w.p. 1.

Note that NE strategy profiles cited above are asymmetric. The game is a symmetric one since primaries have the same action sets, payoff functions and their channels are statistically identical. In a symmetric game, we have already discussed in Section 3.1.4 that implementing an asymmetric NE is difficult. We therefore focus on finding a symmetric NE and investigate whether it is unique. Clearly, for any symmetric NE, we can represent the strategy of any primary as $S = (\psi_1(.), \psi_2(.), ..., \psi_{|\mathcal{P}|}(.))$ where we drop the index corresponding to the primary.

[3] $q_1 q_0$ is the probability that the channel state vector is either $(0, 1)$ or $(1, 0)$.

3.2.2 A Separation Result

We now observe that the NE penalty selection at a node in an independent set can be uniquely computed using the single location NE penalty selection strategy characterized in Chapter 2. First, we introduce some notations– Let $\mathcal{I}_J$ be the set of independent sets of the graph G_J. Let $\mathcal{P}_{a,j}$ be the set of channel state vectors where the channel state is j at node a.

Definition 3.4. Let $\beta_J(I)$ be the probability with which the independent set $I \in \mathcal{I}_J$ is selected by a primary, under a symmetric NE strategy when the channel state vector is J.

Note that though $\beta_J(I)$ depends on the symmetric NE strategy, we do not make it explicit in the notation in order to keep the notational simplicity. Thus,

$$\alpha_{a,j} = \sum_{I \in \mathcal{I}_J : a \in I} \sum_{J : J \in \mathcal{P}_{a,j}} q_J \beta_J(I) \tag{3.4}$$

Let for node s and channel state $j \in \{1, \ldots, n\}$

$$\begin{aligned} L_{s,0} &= U_{s,1} = v, \\ p_{s,j} - c &= (f_j(U_{s,j}) - c)(1 - w(\sum_{k=j}^{n} \alpha_{s,k})) \qquad (3.5) \\ \text{and } L_{s,j} &= g_j(\frac{p_j - c}{1 - w(\sum_{k=j+1}^{n} \alpha_{s,k})} + c), U_{s,j} = L_{s,j-1} \qquad (3.6) \end{aligned}$$

where $w(\cdot)$ is given in (2.5). Since $U_{s,1} = v$, thus we obtain $p_{s,j}, L_{s,j}$ (which in turn gives U_{j+1}) recursively starting from $j = 1$ using (3.5) and (3.6). Let for node s and channel state $j \in \{1, \ldots, n\}$

$$\phi_{s,j}(x) = 0, \text{if } x < L_{s,j}$$
$$\frac{1}{q_j}(w^{-1}(\frac{f_j(x) - p_{s,j}}{f_j(x) - c}) - \sum_{k=j+1}^{n} \alpha_{s,k}), \text{if } L_{s,j-1} \geq x \geq L_{a,j}$$
$$1, \text{if } x > L_{s,j-1}. \tag{3.7}$$

Note that $\phi_{s,i}(\cdot)$ is the same as $\psi_i(\cdot)$ (introduced in Lemma 2.2) with $\alpha_{s,j}q_j$ in place of q_j.

Thus from Theorem 2.6

Lemma 3.1. *Suppose, under a symmetric NE, each primary offers its channel which is at state j at node s for sale at node s w.p. $\alpha_{s,j}$. Then, the unique NE penalty distribution of each primary is the d.f. $\phi_{s,j}(\cdot)$ as described in (3.7). The payoff of the primary at channel state j is $p_{s,j} - c$ and it is attained at every penalty in the interval $[L_{s,j}, L_{s,j-1}]$.*

Since the penalty selection strategy of a primary is unique given the independent set selection strategy $\{\beta_J(I)\}$ (by Lemma 3.1), henceforth, we only focus on independent set selection probability which provides the node selection probability as defined in (3.4).

3.3 Same Channel State Across the region

We first consider the setting where the channel state is the same across the network. Recall from Section 3.1.6 that this setting occurs when the region is of moderate size. We first introduce some notations specific to this setting (Section 3.3.1). We focus on symmetric NEs on a specific class of conflict graphs known as mean valid graph since conflict graphs of most of the commonly observed wireless networks of moderate sizes belong to this category (Section 3.3.2). We subsequently focus on a policy which provides a storage and computation efficient NE strategy (if it exists) (Section 3.3.3). We identify certain key properties that any NE strategy profile of the above policy (should it exist) must satisfy (Section 3.3.4). Then, we show that the identified structure is a unique and there exists a strategy profile which satisfies the identified structure (Theorem 3.2). We show that the strategy profile which satisfies the identified structure is an NE (Theorem 3.3). Finally, we investigate the uniqueness and implementation issues of the symmetric NE profile (Section 3.3.7).

3.3.1 Modifications of Notations

Since the channel state is the same across the region, we denote the channel state vector J as the scalar j in this setting when the channel state is j at each location. For example, if the channel state is 3 everywhere, we denote the channel state at the network as 3. $q_J = 0$ for all J where the channel state is not identical at each location and we denote the probability that the channel state is j over the region as q_j with slight abuse of notation. Note that in this setting, when the channel state is $j \geq 1$, then the channel is available at each node, hence, a primary always selects an independent set from the conflict graph G when the channel is available.

We replace $\beta_J(I)$ in Definition 3.4 with $\beta_j(I)$ which denotes the probability with which a primary selects independent set I under a symmetric NE strategy. Note that $\mathcal{P}_{a,j}$ is now simply j. $\alpha_{a,j}$ is thus,

$$\alpha_{a,j} = \sum_{I:a\in I} q_j \beta_j(I) \tag{3.8}$$

Also note from (3.1) that the channel state is 0 over the network with some non zero probability i.e.

$$\sum_{j=1}^{n} q_j < 1 \tag{3.9}$$

The cardinality of the strategy space $\mathcal{P}$ in this setting is n. The NE strategy profile is thus represented as $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$ in this setting. Note that though the state of a channel is the same across the nodes, the propagation condition and the usage level of different channels can be different, thus, a primary is still not aware of the states of the channel of other primaries.

3.3.2 Mean Valid Graphs

In practice most of the finite size wireless networks are of the following types:

- Wireless network of roadside shops.
- Wireless network of buildings.
- Cellular networks with hexagonal or square cells.

Conflict graphs of all the above wireless networks belong to a category, introduced as *mean valid graphs*[46].

Definition 3.5. [46] A graph $G = (V, E)$ is said to be a mean valid graph if and only if

1. Its vertex set can be partitioned into d disjoint maximal[4] I.S. for some integer $d \geq 2 : V = I_1 \cup I_2 \cup \ldots \cup I_d$[5] where $I_s, s \in \{1, \ldots, d\}$, is a maximal independent set

[4] An independent set I is said to be maximal if for each $a \notin I, a \in V$, $I \cup \{a\}$ is not an independent set [80].

[5] For example, linear conflict graph (Fig. 3.1) is mean valid graph with $d = 2$, with I_1 being the set

and $I_s \cap I_r = \emptyset, s \neq r$. Let, $|I_s| = M_s$,

$$M_1 \geq M_2 \geq \ldots \geq M_d. \tag{3.10}$$

and $I_s = \{a_{s,k} : k = 1, \ldots, M_s\}$.

2. Suppose $I \in \mathcal{I}$contains $m_s(I)$ nodes from $I_s, s = 1, \ldots, d$, then ,

$$\sum_{s=1}^{d} \frac{m_s(I)}{M_s} \leq 1 \quad \forall I \in \mathcal{I}. \tag{3.11}$$

$I_1, \ldots, I_d$ are said to characterize the mean valid graph. The following graphs are mean valid graphs[46].

- Linear Graph constitutes a conflict graph for locations along a highway or a row of shops (Fig. 3.1). It is a mean valid graph with $d = 2$.
- Grid Graph constitutes a conflict graph for a building (Fig. 3.3) or cellular network with square cells. It is a mean valid graph with $d = 4$. Three dimensional grid graph is also a mean valid graph with $d = 8$.
- Conflict graph of a cellular network with hexagonal cells is also a mean valid graph with $d = 3$, if it has an even number of rows and all rows have the same number of nodes which should be a multiple of 3.

Henceforth, we focus on mean valid graphs in this setting.

of odd numbered nodes and I_2 being the set of even numbered nodes. In Fig. 3.3 $d = 4$, with $I_1 = \{V_{1,1}, V_{1,3}, \ldots, V_{1,k_o}, V_{3,1}, V_{3,3}, \ldots, V_{3,k_o}, \ldots\}, I_2 = \{V_{1,2}, V_{1,4}, \ldots, V_{1,k_e}, V_{3,2}, V_{3,4}, \ldots, V_{3,k_e}, \ldots\}, I_3 = \{V_{2,1}, V_{2,3}, \ldots, V_{2,k_o}, V_{4,1}, V_{4,3}, \ldots, V_{4,k_o}, \ldots\}, I_4 = \{V_{2,2}, V_{2,4}, \ldots, V_{2,k_e}, V_{4,2}, V_{4,4}, \ldots, V_{4,k_e}, \ldots\}$, where k_o (respectively, k_e) denote the greatest odd (respectively, even) integer less than or equal to k.

3.3.3 A storage & Computation efficient policy

As in any graph, in mean valid graphs, the number of independent sets grows exponentially with the number of nodes. We have to compute probability distribution over all independent sets in order to find an independent set selection strategy. Thus, computation and storage requirements grow exponentially as the number of nodes increases. However, mean valid graphs are characterized by maximal independent sets $I_1, \ldots, I_d$ which partition the set of nodes. So, if there exists an NE strategy profile which only selects independent sets amongst $I_1, \ldots, I_d$, then we only need to store d independent sets and the corresponding probability distribution. Thus, the storage and computation requirement only scales with d and does not increase exponentially with the number of nodes. We therefore examine if there exists an NE strategy profile under which

- Each primary selects **only** independent sets in $\{I_1, \ldots, I_d\}$. Specifically, at channel state j, independent set $I_k, k \in \{1, \ldots, d\}$ is selected with probability $t_{k,j}$.

Under the policy, thus,

$$\beta_j(I_k) = t_{k,j} \quad \forall k \in \{1, \ldots d\} \quad \text{such that} \sum_{k=1}^{d} \beta_j(I_k) = 1. \tag{3.12}$$

Thus, from (3.8) and (3.12) for any two nodes $s, r \in I_k, k \in \{1, \ldots, d\}, j \in \{1, \ldots, n\}$:

$$\alpha_{s,j} = \alpha_{r,j} = q_j t_{k,j} \quad \sum_{k=1}^{d} t_{k,j} = 1. \tag{3.13}$$

In the next section, we show that there exists a unique symmetric NE strategy which satisfies (3.13).

3.3.4 Characterization of Symmetric NE

We first, characterize the properties that any symmetric NE strategy profile of the form (3.13) must satisfy.

By virtue of Lemma 3.1, we know the penalty selection strategy for each state at a given node for a given NE independent set selection strategy. The support sets of penalty distributions are contiguous (3.7). However, the end-points of the support sets are not necessarily the same across the location. Surprisingly, we show that the upper endpoints of the penalty selection strategy at a particular channel state i, $i = 1, \ldots, n$ are identical across different locations regardless of the choice of independent sets (Lemma 3.2). We show that there exists a threshold such that only those independent sets, whose cardinalities are equal to or greater than that threshold, are selected with positive probabilities (Lemma 3.3). Drawing from the above lemmas we characterize the structure that any NE strategy profile of the form (3.13) (if it exists) has to satisfy (Theorem 3.1). The proofs of the results have been provided at the end of this subsection.

We start with some notations which we use throughout. Recall from (2.5) that $w(x)$ is the probability that there is at least m success out of $n-1$ Bernouli events where each event has success probability of x.

Definition 3.6. Let,

$$W(x) = 1 - w(x). \tag{3.14}$$

Since $w(\cdot)$ is continuous and strictly increasing (by (2.5)),thus, $W(\cdot)$ is a continuous and a strictly decreasing function with $W(0) = 1$.

Definition 3.7. Let $\gamma_{s,j}$ denote the probability that a channel of state j or higher is offered at a node of I_s. Thus,

$$\gamma_{s,j} = \sum_{k=j}^{n} t_{s,k} q_k = \sum_{k=j}^{n} \alpha_{a,k}. \tag{3.15}$$

From (3.15), we obtain a recursive method to calculate $\gamma_{s,j}$.

$$\gamma_{s,j-1} = \sum_{k=j-1}^{n} t_{s,k} q_k = t_{s,j-1} q_{j-1} + \gamma_{s,j}. \tag{3.16}$$

In the class of policies of the form (3.13), $\alpha_{a,j}$ is equal to $q_j t_{s,j}$ for every node a in independent set $I_s, s \in \{1, \ldots, d\}$. Thus, by Lemma 3.1 the penalty selection strategy at any node of I_s is given by (3.7) with $q_j t_{s,j}$ in place of α_j. Thus, by (3.5), (3.6), and Lemma 3.1, expected payoff obtained by a primary at every node of I_s at channel state j is–

$$\begin{aligned} p_{s,j} - c &= (f_j(U_{s,j}) - c)(1 - w(\sum_{i=j}^{n} t_{s,i} q_i)) \\ &= (f_j(U_{s,j}) - c) W(\gamma_{s,j}) \end{aligned} \tag{3.17}$$

where

$$U_{s,j} = g_j\left(\frac{p_{s,j} - c}{W(\gamma_{s,j})} + c\right) \quad U_{s,1} = v, U_{s,j} = L_{s,j-1} \tag{3.18}$$

$$L_{s,j} = g_j\left(\frac{p_{s,j} - c}{W(\gamma_{s,j+1})} + c\right) \quad L_{s,0} = v. \tag{3.19}$$

Remark 3.1. Starting from $U_{s,1} = v$, we can find $p_{s,1}$ using (3.17) which we use to find $L_{s,1}$ (from (3.19)). Since $L_{s,1} = U_{s,2}$, thus utilizing $U_{s,2}$ we obtain $p_{s,2}$ (from (3.17)) which in turn gives $L_{s,2}$ (from (3.19)). Thus, recursively we obtain $U_{s,j}, p_{s,j}, L_{s,j}$ for all $s \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$. Hence, we can easily compute a penalty selection strategy at each node of I_s for a given $t_{s,j}$.

Remark 3.2. Note from Lemmas 2.2 and 3.1 that each primary selects penalty only from the interval $[L_{s,j}, U_{s,j}]$ at channel state j at every node of I_s when $t_{s,j} > 0$.

Since $p_{s,j} - c$ is the expected payoff that a primary gets at any node in I_s at channel state j when primaries select I_s with probability $t_{s,j} > 0$, thus, the expected payoff to a primary at channel state j over independent set I_s when $t_{s,j} > 0$ is

$$M_s(p_{s,j} - c) = M_s(f_j(U_{s,j}) - c)W(\gamma_{s,j}) \quad \text{(from(3.17))}. \tag{3.20}$$

Now, we introduce some notations that we use throughout.

Definition 3.8. Let $P_j(I_k)$ denote the maximum expected payoff that a primary can get at independent set I_k at channel state j when other primaries select a symmetric NE strategy profile which is of the form (3.13) . Let P_j^* be the maximum among $P_j(I_r)$ $r \in \{1, \ldots, d\}$ i.e.

$$P_j^* = \max_{r \in \{1,\ldots,d\}} P_j(I_r).$$

Let B_j denote the set of indices out of $I_1, \ldots, I_d$ which are selected with positive probability under a symmetric NE strategy profile at channel state j.

At channel state j an independent set is selected with positive probability in an NE strategy profile only if the expected payoff at that independent set is[6] P_j^* ; hence when

[6]Consider that in an NE strategy profile I_s is selected w.p. $t_{s,j} > 0$, but expected payoff is strictly less than P_j^* which it obtains at I_r (say). Let in the NE strategy profile I_r is selected w.p. $t_{r,j}$. Note that the expected payoff of a primary at an independent set only depends on the strategy of other primaries. Thus, a primary can unilaterally deviate by selecting I_r w.p. $t_{s,j} + t_{r,j}$ and I_s w.p. 0; but under the new strategy profile its expected payoff is strictly higher. Hence, the original strategy profile can not be an NE.

the channel state is j, then

$$M_s(f_j(U_{s,j}) - c)W(\gamma_{s,j}) = P_j^* \quad \text{if } s \in B_j(\text{from}(3.20)). \tag{3.21}$$

Now, we are ready to state the results.

Lemma 3.2. *If $t_{s,j} > 0, t_{r,j} > 0$, then $U_{s,j} = U_{r,j}$.*

The above lemma shows that upper end points of penalty selection strategy is the same across the nodes of the independent sets which are chosen with positive probability.[7]

Remark 3.3. From lemma 3.2 we can write $U_{s,j}$ as U_j $\forall s \in B_j$. So, for any $s, r \in B_j$, we must have from (3.21)

$$M_s(f_j(U_j) - c)W(\gamma_{s,j}) = M_r(f_j(U_j) - c)W(\gamma_{r,j}) = P_j^*.$$

$$M_s W(\gamma_{s,j}) = M_r W(\gamma_{r,j}). \tag{3.22}$$

Next lemma characterizes the best response set B_j.

Lemma 3.3. *There exists an integer $d_j \in \{1, \ldots, d\}$, such that $I_1, \ldots, I_{d_j}$ are selected with positive probability and $I_{d_j+1}, \ldots, I_d$ are selected with zero probability at channel state j.*

Thus, from (3.10), only those independent sets whose cardinalities are greater than or equal to M_{d_j} are selected with positive probabilities at channels state j . We show in Lemma 3.7 that this above threshold M_{d_j} is a non-decreasing function in channel state j.

[7]Note that we have not shown any relation between $L_{s,j}$ and $L_{r,j}$. Thus, even though $U_{s,j} = U_{r,j}$, it is possible that $L_{s,j} \neq L_{r,j}$. But if $t_{s,j+1} > 0, t_{r,j+1} > 0$, then from Lemma 3.2 we obtain $U_{s,j+1} = U_{r,j+1}$; since $L_{s,j} = U_{s,j+1}, L_{r,j} = U_{r,j+1}$, thus we have $L_{s,j} = L_{r,j}$. Hence, lower endpoint of penalty selection strategy at every node of independent sets I_s, I_r is also the same if both I_s, I_r are selected with positive probabilities for both the states j and $j+1$.

In an NE strategy only those independent sets are selected with positive probabilities which give an expected payoff of P_j^*, thus we can evaluate the expected payoff under the NE strategy using Lemma 3.3. Since we know from Lemma 3.3 that NE strategy profile only selects those independent sets whose indices are less than or equal to d_j, thus, under NE strategy expected payoff of a primary at channel state j is given by

$$P_j^* = M_s(f_j(U_j) - c)W(\gamma_{s,j}) \quad s \leq d_j. \tag{3.23}$$

We will also show that $P_j^* \geq M_r(f_j(U_j) - c)W(\gamma_{r,j})$ for $r > d_j$ to prove Lemma 3.3 (Lemma 3.6 in Section 3.3.4). Drawing from the above it readily follows that

Theorem 3.1. *The structure of a symmetric NE strategy profile which satisfies (3.13) (if it exists), is of the following form $\forall a \in I_s$ for $j \in \{1, \ldots, n\}$*

$$\alpha_{a,j} = q_j t_{s,j}, \sum_{s=1}^{d} t_{s,j} = 1, t_{s,j} > 0, s \leq d_j, t_{s,j} = 0, s > d_j \tag{3.24}$$

such that

$$\begin{aligned} M_1 W(\gamma_{1,j}) = \ldots = M_{d_j} W(\gamma_{d_j,j}) \geq M_{d_j+1} W(\gamma_{d_j+1,j}) \\ \geq M_{d_j+2} W(\gamma_{d_j+2,j}) \geq \ldots \geq M_d W(\gamma_{d,j}). \end{aligned} \tag{3.25}$$

Note that the number of equations increases linearly with the number of states n.

Theorem 3.1 provides an iterative way to compute $t_{s,j}$ for all s, j. Noting that $\gamma_{s,n} = t_{s,n} q_n$, (3.25) has only one variable $t_{s,n}$ at $j = n$ for $s \in \{1, \ldots, d\}$. Thus, we first compute $t_{s,n}$ for all s using (3.24) and (3.25) for $j = n$. From (3.16), $\gamma_{s,n-1}$ depends on $\gamma_{s,n}$ and $t_{s,j-1}$. Since we have already computed $t_{s,n}$ or $\gamma_{s,n}$, thus we solve for $t_{s,n-1}$ from (3.24) and (3.25). Thus, recursively we obtain $t_{s,j}$ for all s and j. *A primary only needs to know*

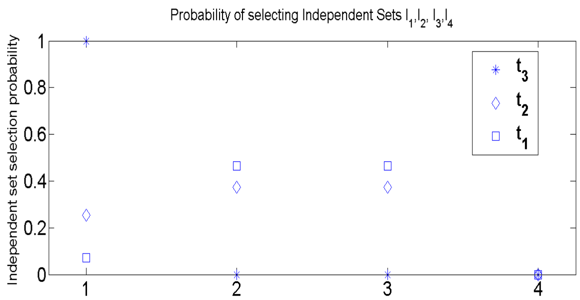


Figure 3.5: This figure shows $t_j = (t_{1,j}, \ldots, t_{d,j})$ at channel state $j = 1, 2, 3$ for Example 3.1.

$I_1, \ldots, I_d$ *to compute the independent set selection strategy and does not need to know the information regarding the network (e.g. edges).*

Example 3.1. We consider a grid graph with $d = 4$ and $k = 5$ (Fig. 3.3) . Here, $M_1 = 9, M_2 = M_3 = 6, M_4 = 4$. We consider $l = 20, m = 6, n = 3, q_1 = q_2 = q_3 = 0.2$. We first calculate $t_{s,3}$ for all s. We obtain $M_1W(\gamma_{1,3}) = 7.5324$, $M_2W(\gamma_{2,3}) = 6, M_3W(\gamma_{3,3}) = 6$, $M_4W(\gamma_{4,3}) = 4$. Thus, $d_3 = 1$ and the solution of (3.24) and (3.25) is: $t_3 = (1, 0, 0, 0)$, where $t_j = (t_{1,j}, \ldots, t_{d,j})$ for $j = 1, \ldots, 3$. Next, we compute $t_{s,2}$ following the recursive algorithm we stated. We obtain $d_2 = 3$ and $t_2 = (0.2532, 0.3734, 0.3734, 0)$. Finally, we calculate $t_{s,1}$. We obtain $d_1 = 3$ and $t_1 = (0.071, 0.4645, 0.4645, 0)$ Fig. 3.5 shows plots of $t_{s,j}$ for all s and j.

Proof of Lemma 3.2

We first deduce some results which we use throughout.

Since $\gamma_{s,j} \leq \sum_{i=1}^{n} q_i < 1$, thus $p_{s,j} - c > 0$. Hence,

$$f_j(U_{s,j}) > c, f_j(L_{s,j}) > c. \tag{3.26}$$

Now, we provide the expression for expected payoff that a primary attains at $L_{s,i}$ $i = 1, \ldots, n$ at any node in I_s at channel state j. Note that players with channel state higher than i select a penalty lower than or equal to $L_{s,i}$ with probability 1 and players with channel state lower than or equal to i select a penalty lower than or equal to $L_{s,i}$ with probability 0 at every node of I_s. Thus, the expected payoff to a primary when it selects penalty $L_{s,i}$ at channel state $j \in \{1, \dots, n\}$ at any node of I_s is

$$(f_j(L_{s,i}) - c)W(\sum_{k=i+1}^{n} q_k t_{s,k}) = (f_j(L_{s,i}) - c)W(\gamma_{s,i+1}). \tag{3.27}$$

Now, we state and prove Observations 3.1 and 3.2 which we use throughout.

Observation 3.1. $\gamma_{s,k} = \gamma_{s,k_1} + \sum_{i=k}^{k_1-1} t_{s,i} q_i$ for $s \in \{1, \dots, d\}$, $n \geq k_1 > k$.

The observation readily follows from (3.15). Since from (3.15)

$$\gamma_{s,k} = \sum_{i=k}^{k_1-1} t_{s,i} q_i + \sum_{i=k_1}^{n} t_{s,i} q_i = \sum_{i=k}^{k_1-1} t_{s,i} q_i + \gamma_{s,k_1}.$$

Observation 3.2. $U_{s,j} = L_{s,j}$ for $j \in \{1, \dots, n\}$ if and only if (iff) $t_{s,j} = 0$. $U_{s,j} = L_{s,k}$ iff $t_{s,i} = 0 \ \forall k < i < j$. Hence, $U_{s,j} = v$ iff $t_{s,k} = 0 \ \forall k < j$.

Proof. $L_{s,j} = U_{s,j}$ implies from (3.18) and (3.19) that $\gamma_{s,j+1} = \gamma_{s,j}$; thus by Observation 3.1 we have $t_{s,j} = 0$. On the other hand if $t_{s,j} = 0$ then by Observation 3.1 $\gamma_{s,j} = \gamma_{s,j+1}$. Thus, it follows that $U_{s,j} = L_{s,j}$ iff $t_s, j = 0$.

Since $L_{s,k} = U_{s,k+1}$, hence $U_{s,j} = L_{s,k}$ iff $t_{s,i} = 0 \ \forall k < i < j$. Hence, $U_{s,j} = L_{s,1} = U_{s,1}$ iff $t_{s,i} = 0 \ \forall i < j$. On the other hand by (3.18) $U_{s,1} = v$. Thus, the result follows. □

Now we are ready to show Lemma 3.2.

Proof of Lemma 3.2:

Since both $s, r \in B_j$, hence from (3.21)

$$M_s(f_j(U_{s,j}) - c)W(\gamma_{s,j}) = M_r(f_j(U_{r,j}) - c)W(\gamma_{r,j}) = P_j^*. \tag{3.28}$$

Suppose, the statement is false, i.e. $U_{s,j} \neq U_{r,j}$ when $s, r \in B_j$. Without loss of generality, we can assume that $U_{s,j} > U_{r,j}$. So, $U_{r,j} < v$. Thus, by Observation 3.2, there exists k such that $t_{r,k} > 0$, $k < j$ and $L_{r,k} = U_{r,j}$. Thus, from (3.21)

$$\begin{aligned} P_k^* &= M_r(f_k(U_{r,k}) - c)W(\gamma_{r,k}) \\ &= M_r(f_k(L_{r,k}) - c)W(\gamma_{r,k+1})(\text{from } (3.18)\&(3.19)). \end{aligned} \tag{3.29}$$

If a primary selects penalty $U_{s,j}(= L_{s,j-1})$ at a node of I_s when its channel state is k, then from (3.27) its expected payoff would be

$$(f_k(U_{s,j}) - c)W(\gamma_{s,j}) = \frac{(f_k(U_{s,j}) - c)}{(f_j(U_{s,j}) - c)} \frac{P_j^*}{M_s} \quad (\text{from } (3.28)).$$

Thus a primary obtains an expected payoff of at least

$$M_s(f_k(U_{s,j}) - c)W(\gamma_{s,j}) = P_j^* \frac{(f_k(U_{s,j}) - c)}{(f_j(U_{s,j}) - c)}.$$

at independent set I_s at channel state k. By definition of P_k^*,

$$P_j^* \frac{(f_k(U_{s,j}) - c)}{(f_j(U_{s,j}) - c)} \leq P_k^*. \tag{3.30}$$

Since $U_{r,j} = L_{r,k}$ and thus $f_k(U_{r,j}) > c$ (by (3.26)). Thus expected payoff at I_r at channel state j is at least $M_r(f_j(U_{r,j}) - c)W(\gamma_{r,k+1})$ which is

$$\begin{aligned} &= \frac{(f_j(U_{r,j}) - c)}{f_k(U_{r,j}) - c} P_k^* \quad (\text{from } (3.29)) \\ &\geq P_j^* \frac{(f_j(U_{r,j}) - c)(f_k(U_{s,j}) - c)}{(f_k(U_{r,j}) - c)(f_j(U_{s,j}) - c)} \quad (\text{from } (3.30)) \\ &> P_j^* \quad (\text{from } (3.2), j > k, U_{s,j} > U_{r,j}) \end{aligned}$$

which is not possible by Definition 3.8. □

Proof of Lemma 3.3

We state and prove Lemmas 3.4, 3.5, and 3.6 which we use to prove Lemma 3.3.

Lemma 3.4. *$L_{s,k} \geq U_j$ if $s \in B_k, s \notin B_j, k < j, j \geq 2$.*

Remark 3.4. Note that if $s \in B_k \cap B_j$ and $k < j$, then from (3.18), $L_{s,k} \geq U_j$. But, it is not apriori clear the relationship between $L_{s,k}$ and U_j when $s \in B_k$ but $s \notin B_j$ for $k < j$. The above lemma provides the answer.

Since $s \in B_k$, thus expected payoff obtained at I_s at channel state k is P_k^* (by (3.21)). If $U_j > L_{s,k}$ for some $k < j$ and $s \notin B_j$, then it can be shown that by selecting independent set I_r (where $r \in B_j$) a primary can attain a strictly higher payoff compared to P_k^* at channel state k which is not possible by Definition 3.8. The argument will be similar to the proof of Lemma 3.2. Thus, we omit it.

It is not clear that $P_j^* \geq M_r(f_j(U_j)-c)W(\gamma_{r,j})$ if $r \notin B_j$. Since $r \notin B_j$, thus, a primary will not employ any penalty selection strategy at any node of I_r when the channel state is j. Thus, at any given node in I_r, the expected payoff at U_j is still unknown. The following lemma provides the answer.

Lemma 3.5. *If $r \notin B_j$, then $P_j^* \geq M_r(f_j(U_j) - c)W(\gamma_{r,j})$.*

Proof. Since $r \notin B_j$, hence we must have $t_{r,j} = 0$. Suppose the statement is false, then for some $r \notin B_j$, we must have

$$P_j^* < M_r(f_j(U_j) - c)W(\gamma_{r,j}). \tag{3.31}$$

Now we show that a primary will attain an expected payoff which is strictly higher than P_j^* at channel state j at independent set I_r.

Let, $k = \max\{i \in \{1, \ldots, j-1\} : r \in B_i\}$, if $r \notin B_i, \forall i < j$, then set $k = 0$. By definition of k, $t_{r,i} = 0 \ \forall k < i < j$. Thus by Observation 3.1, $\gamma_{r,k+1} = \gamma_{r,j}$. Thus, from (3.27) the expected payoff at $L_{r,k}$ (if $k = 0$, then $L_{r,0} = v$) at channel state j is

$$(f_j(L_{r,k}) - c)W(\gamma_{r,k+1}) = (f_j(L_{r,k}) - c)W(\gamma_{r,j}). \tag{3.32}$$

Now from Lemma 3.4 $L_{r,k} \geq U_j$ when $k > 0$. If $k = 0$, then $L_{r,k} = v$ by Observation 3.2. Thus, $L_{r,k} \geq U_j \ \forall k$. Hence, from (3.32) total expected payoff at I_r is at least

$$\begin{aligned} M_r(f_j(L_{r,k}) - c)W(\gamma_{r,j}) &\geq M_r(f_j(U_j) - c)W(\gamma_{r,j}) \\ &> P_j^* \quad \text{(from (3.31)} \end{aligned} \tag{3.33}$$

which contradicts P_j^* from Definition 3.8. □

Thus, if $s, s_1 \in B_j$ and $s_2 \notin B_j$, then we have

$$\begin{aligned} P_j^* &= M_s(f_j(U_j) - c)W(\gamma_{s,j}) = M_{s_1}(f_j(U_j) - c)W(\gamma_{s_1,j}) \\ &\geq M_{s_2}(f_j(U_j) - c)W(\gamma_{s_2,j}) \quad \text{(from Lemma 3.5)} \end{aligned}$$

$$M_s W(\gamma_{s,j}) = M_{s_1} W(\gamma_{s_1,j}) \geq M_{s_2} W(\gamma_{s_2,j}). \tag{3.34}$$

We now state and prove Lemma 3.6.

Lemma 3.6. *$M_r W(\gamma_{r,j}) \geq M_s W(\gamma_{s,j})$ if $r < s$ for all $j \in \{1, \ldots, n\}$.*

Proof. Suppose the statement is false i.e. $M_r W(\gamma_{r,j}) < M_s W(\gamma_{s,j})$ for some $r < s$ and $j \in \{1, \ldots, n\}$. Since $M_r \geq M_s$ (by (3.10)), thus there must exist a $k \in \{j, \ldots, n\}$ such that $M_r W(\gamma_{r,k+1}) \geq M_s W(\gamma_{s,k+1})$ but $M_r W(\gamma_{r,k}) < M_s W(\gamma_{s,k})$ with $\gamma_{r,n+1} = \gamma_{s,n+1} = 0$.

Since $\gamma_{s_1,k} \geq \gamma_{s_1,k+1}$ (by Observation 3.1) $\forall s_1 \in \{1, \ldots, d\}$ and $W(\cdot)$ is strictly decreasing function, thus, we have

$$\begin{aligned} M_r W(\gamma_{r,k}) &< M_s W(\gamma_{s,k}) \\ &\leq M_s W(\gamma_{s,k+1}) \leq M_r W(\gamma_{r,k+1}). \end{aligned} \tag{3.35}$$

Since $W(\cdot)$ is strictly decreasing function and $\gamma_{r,k} = t_{r,k} q_k + \gamma_{r,k+1}$ (from Observation 3.1), thus $t_{r,k} > 0$ from (3.35); which implies that $r \in B_k$. But this contradicts (3.34). Hence, the result follows. □

Now, we are ready to show Lemma 3.3.

proof of Lemma 3.3: Suppose that $r < s$, but $r \notin B_k, s \in B_k$ for some $k \in \{1, \ldots, n\}$. Note from Observation 3.1 that $\gamma_{s,k} > \gamma_{s,k+1}$ since $t_{s,k} > 0$. Since $W(\cdot)$ is strictly decreasing thus $W(\gamma_{s,k}) < W(\gamma_{s,k+1})$. On the other hand, since $r \notin B_k$, thus $t_{r,k} = 0$. Thus, from Observation 3.1 $\gamma_{r,k+1} = \gamma_{r,k}$ and therefore, we obtain $W(\gamma_{r,k+1}) = W(\gamma_{r,k})$. Thus we obtain from Lemma 3.6–

$$M_r W(\gamma_{r,k}) = M_r W(\gamma_{r,k+1}) \geq M_s W(\gamma_{s,k+1}) > M_s W(\gamma_{s,k}).$$

But $s \in B_k, r \notin B_k$, thus the above inequality contradicts (3.34). □

3.3.5 Existence

Theorem 3.1 characterizes the structure of independent set selection strategy which is of the form (3.13). We have not yet shown whether there exists such a distribution and whether such a distribution is unique. We resolve both the issues in the following theorem which we have proved in Appendix 3.A.1:

Theorem 3.2. *There exists a unique probability distribution* $t_j = (t_{1,j}, \ldots, t_{d,j}), j = 1, \ldots, n$ *which satisfies (3.24) & (3.25).*

We now show that independent set selection strategy profile described in (3.24) and (3.25) is an NE.

Theorem 3.3. *At channel state* $j \in \{1, \ldots, n\}$*, consider the following strategy profile. The unique independent set selection strategy profile is given by (3.24) and (3.25) and at every node of* $I_s, s \in \{1, \ldots, d\}$*, penalty selection strategy is* $\psi_j(\cdot)$ *with* $q_j t_{s,j}$ *in place of* q_j *as described in Lemma 2.2. Such a strategy profile constitutes an NE in the class of mean valid graphs.*

Thus, there exists a symmetric NE which selects an independent set among $I_1, \ldots, I_d$. Such a selection strategy is storage and computationally efficient as explained in the first paragraph of Section 3.3.3. By virtue of Theorem 3.1 we also know how to compute the probabilities of these independent sets by solving n equations.

Outline of Proof of Theorem 3.3

We first show that a primary at channel state j attains an expected payoff of P_j^* at each independent set I_s, $s \leq d_j$. Subsequently, we show that at any independent set $I_s, s > d_j$, the maximum attainable payoff of a primary at channel state j is less than P_j^* when other primaries select strategies according to (3.24) and (3.25). Finally, we show that if a primary selects an independent set which does not belong to the partition, then, its maximum expected payoff is also less than P_j^*. Thus, it shows that a primary attains maximum expected payoff only at independent sets $I_s, s \leq d_j$, hence, a primary does not have any incentive to deviate unilaterally from the strategy profile which proves the theorem. The detail of the proof is given in Appendix 3.A.2.

3.3.6 Properties of Threshold

Recall from Lemma 3.3 and Theorem 3.1 that a primary only selects those independent sets which have cardinalities greater than or equal to M_{d_j} with positive probabilities at channel state j. In this section, we discuss some important properties of $d_j, j = 1, \ldots, n$.

Lemma 3.7. *Threshold is a non-decreasing function of transmission rate i.e.* $d_j \geq d_{j+1}$

From (3.10) and Lemma 3.7 we have $M_{d_j} \leq M_{d_{j+1}}$. From Example 3.1, we obtain $d_3 < d_2 = d_1$ which validates the above lemma. In Example 3.1 only I_1 is selected when the channel state is the highest i.e. 3. Thus, a primary never selects I_2, I_3 and I_4 when its channel has the highest transmission rate.

This tells that in practice, secondary users in some locations can never get access to a channel of higher quality. In Example 3.1, users in the locations belonging to independent sets I_2, I_3 and I_4 will never get access to the highest quality channel. To avoid such socially unacceptable situation a social planner may have to provide some incentives to primaries so that they offer their high quality channels in independent sets of lower cardinalities. Designing such an incentive constitutes an important problem for future research.

Since $t_{s,j} > 0$ for $s \leq d_j$ the following result is immediate from Lemma 3.7.

Corollary 3.1. $t_{s,k} > 0$ *implies that* $t_{s,j} > 0$ *where* $j < k$; $t_{1,j} > 0\ \forall j \in \{1, \ldots, n\}$.

Thus, independent set I_1 is always selected with positive probability at every channel state (Fig. 3.5). Corollary 3.1 implies that if a given primary offers its channel at an independent set $I_s, s \in \{1, \ldots, d\}$ with positive probability when the channel provides higher transmission rate, then the primary also offers its channel at I_s with positive probability when its channel provides lower transmission rate. But note that the converse is not always true.

Proof of Lemma 3.7

Suppose, the statement is false, i.e. $d_j < d_{j+1}$ for some j.

From (3.25) we obtain for state $j+1$

$$M_1 W(\gamma_{1,j+1}) = M_{d_j} W(\gamma_{d_j,j+1}) = M_{d_{j+1}} W(\gamma_{d_{j+1},j+1}). \tag{3.36}$$

Since $t_{d_j,j} > 0$ thus $\gamma_{d_j,j} > \gamma_{d_j,j+1}$ by Observation 3.1. Since $W(\cdot)$ is strictly decreasing, thus we have

$$W(\gamma_{d_j,j}) < W(\gamma_{d_j,j+1}). \tag{3.37}$$

Since $d_j < d_{j+1}$, thus $t_{d_{j+1},j} = 0$. Thus, from Observation 3.1, $\gamma_{d_{j+1},j+1} = \gamma_{d_{j+1},j}$. Thus from (3.36) and (3.37), we obtain

$$\begin{aligned} M_{d_j} W(\gamma_{d_j,j}) &< M_{d_{j+1}} W(\gamma_{d_{j+1},j+1}) \\ &= M_{d_{j+1}} W(\gamma_{d_{j+1},j}). \end{aligned} \tag{3.38}$$

Since $d_j < d_{j+1}$ thus (3.38) contradicts (3.25). □

3.3.7 Uniqueness of Symmetric NE & Implementation Issues

Till now we have shown that when primaries select among maximal independent sets characterizing the mean valid graphs, then there exists a unique symmetric NE (Theorems 3.2

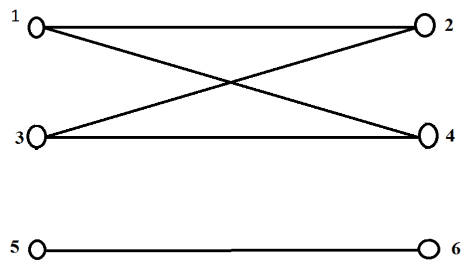


Figure 3.6: The above mean valid graph has two different sets of partitions: 1) $I_1 = \{1,3,5\}, I_2 = \{2,4,6\}$ and 2) $\bar{I}_1 = \{1,3,6\}, \bar{I}_2 = \{2,4,5\}$. If $\alpha_{a,j}$ ($\bar{\alpha}_{a,j}$, respectively) is the node selection probability at node a under NE strategy profile where primaries select I_1, I_2 ($\bar{I}_1, \bar{I}_2$, respectively), then according to Theorem 3.4, we obtain $\alpha_{a,j} = \bar{\alpha}_{a,j}$ for all channel states j. There exists independent sets which are different from I_1, I_2 and $\bar{I}_1, \bar{I}_2$ e.g. $\{1,3\}$, $\{2,4\}$.

and 3.3). Figure 3.6 reveals that partition of nodes amongst maximal independent sets need not be unique. We have shown that each such partition leads to a unique symmetric NE (Theorems 3.2 and 3.3). Thus, symmetric NE is not unique.

A primary would not know the partitions other primaries are selecting since the co-ordination among the primaries is infeasible in a non co-operative game. Theorem 3.4 entails that co-ordination among the players is not required when the independent set selection strategy is of the form (3.24) and (3.25). We obtain an even stronger result in a special case: we show that there is a unique symmetric NE in a linear conflict graph (Theorem 3.5).

Theorem 3.4. *Consider that nodes in a mean valid graph can be partitioned into two different sets of maximal independent sets: i)* $I_1, \ldots, I_d$, *and ii)* $\bar{I}_1, \ldots, \bar{I}_{\bar{d}}$. *Suppose at channel state* $j = 1, \ldots, n$, $0 \leq n_j \leq l$ *number of primaries select independent sets among* $I_1, \ldots, I_d$ *and* $l - n_j$ *number of primaries select independent sets among* $\bar{I}_1, \ldots, \bar{I}_{\bar{d}}$

according to (3.24) and (3.25). Then the strategy profile constitutes an NE.

Additionally, let $\alpha_{a,j}$ ($\bar{\alpha}_{a,j}$ *respv.) be the probability with which primary* i *offers its channel at node* a *at channel state* j *when it selects independent sets among* $I_1, \ldots, I_d$ ($\bar{I}_1, \ldots, \bar{I}_{\bar{d}}$ *reps.) such that (3.24) and (3.25) are satisfied, then* $\alpha_{a,j} = \bar{\alpha}_{a,j}$.

The first part of the above theorem implies that regardless of the partition other primaries select, a primary can attain its NE strategy profile by selecting independent sets using one of the partition. Hence, a primary needs not co-ordinate with other primaries in order to decide which partition it will choose. Thus, the strategy profile of the form (3.24) and (3.25) is easy to implement.

The second part of the theorem implies that regardless of the partition primary i selects, the node selection probability will be identical. Thus, the independent set selection strategies are *functionally unique*. Note that when different primaries select independent set selection strategies using different partitions, then the strategy profile is not symmetric, however, the node selection probabilities will be identical.

In Theorem 3.4 we show that when primaries select independent sets which belong to a partition, then the symmetric NE will lead to the same node selection probability. But there are independent sets which do not belong to a partition characterizing the mean valid graph (Fig. 3.6). We have not ruled out a symmetric NE which selects an independent set which is outside of a partition characterizing the mean valid graph. We rule this out in the special class of linear conflict graphs. Linear conflict graphs frequently arise in practice: e.g. in the modeling of wireless access point across a highway or along a row of shops.

We show[8] in Appendix 3.A.4–

Theorem 3.5. *There exists a unique (not merely functionally unique) symmetric NE strategy profile in a linear conflict graph. In the symmetric NE each primary selects only independent sets I_1 and I_2, where I_1 (I_2, respectively) consists of odd (even, respectively) numbered nodes (Fig. 3.1).*

Proof of Theorem 3.4

First, we provide an outline of the proof.

Suppose both the partitions 1) $\bar{I}_1, \ldots, \bar{I}_{\bar{d}}$ and 2) $I_1, \ldots, I_d$ characterize a mean valid graph G (i.e. they satisfy conditions 1 and 2 of Definition 3.5). Let $|\bar{I}_s| = \bar{M}_s$ for $s \in \{1, \ldots, \bar{d}\}$ with

$$\bar{M}_1 \geq \bar{M}_2 \geq \ldots \geq \bar{M}_{\bar{d}}.$$

We show in Appendix 3.A.3

Lemma 3.8. *$M_s = \bar{M}_s$, thus $d = \bar{d}$.*

Thus, $|I_s| = |\bar{I}_s|$, $s \in \{1, \ldots, d\}$. Since the solution of (3.24) and (3.25) only depend on the cardinalities of I_s, hence if a primary selects the partition $\bar{I}_1, \ldots, \bar{I}_d$ then a primary selects independent sets by solving (3.24) and (3.25). Since the solution of (3.24) and (3.25) is unique by Theorem 3.2, hence, if $|I_s| = |\bar{I}_k|$, they are selected with identical probability. Thus, if $a \in I_s$, and $a \in \bar{I}_k$ such that $|I_s| = |\bar{I}_k|$ then, the node selection

[8] In a linear conflict graph, the number of independent sets grows exponentially with M. Since I_1, I_2 are not the only independent sets (Fig. 3.1), thus, it is not apriori clear whether every NE strategy profile only selects independent sets among I_1, I_2 with positive probability.

probability at node a at any channel state will be identical. However, if $a \in \bar{I}_k$ and $|\bar{I}_k| \neq |I_s|$, then the node selection probability may be different. We eliminate the above possibility in the following which we also show in Appendix 3.A.3.

Lemma 3.9. *If $|I_j| \neq |\bar{I}_k|$, then $I_j \cap \bar{I}_k = \Phi$.*

We have explained the relationship between $I_1, \ldots, I_d$ and $\bar{I}_1, \ldots, \bar{I}_d$ in Fig. 3.7. The proof of Theorem 3.4 readily follows from the fact that the node selection probability is identical irrespective of the partitions selected by primaries. The detailed proof is given below:

Proof of Theorem 3.4: First, we show the following: if $\alpha_{a,j}$ ($\bar{\alpha}_{a,j}$ resp.) is the node selection probability when a primary selects among independent sets among $I_1, \ldots, I_d$ ($\bar{I}_1, \ldots, \bar{I}_{\bar{d}}$ resp.) such that (3.24) and (3.25) are satisfied, then $\alpha_{a,j} = \bar{\alpha}_{a,j}$. It will essentially prove the second part of the theorem.

Fix a node a. Let $a \in I_s$ and $a \in \bar{I}_k$. By theorem 3.2 there exists a unique solution $t_j = (t_{1,j}, \ldots, t_{d,j})$ of (3.24) and (3.25). Since $a \in I_s$, thus,

$$\alpha_{a,j} = q_j t_{s,j}. \tag{3.39}$$

Since $\bar{d} = d$ and $|\bar{I}_s| = |I_s|$ for all $s \in \{1, \ldots, d\}$ by Lemma 3.8, thus, (3.24) and (3.25) are identical irrespective of whether a primary selects independent sets among $I_1, \ldots, I_d$ or $\bar{I}_1, \ldots, \bar{I}_d$. Since there exists unique solution of (3.25) and (3.24) (by Theorem 3.2) , thus t_j is the only solution of (3.24) and (3.25). Hence, probability with which the independent set $\bar{I}_i$ is selected at channel state j is $t_{i,j}$. Since node $a \in \bar{I}_k$, thus, from (3.8)

$$\bar{\alpha}_{a,j} = q_j t_{k,j}. \tag{3.40}$$

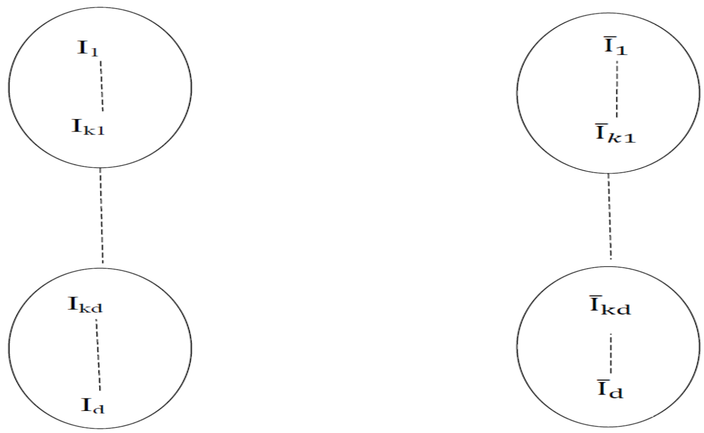


Figure 3.7: Independent sets of same cardinality are grouped together. Thus, $|I_1| = \ldots = |I_{k1}|$. $I_1 \cup \ldots \cup I_{k1} = \bar{I}_1 \cup \ldots \cup \bar{I}_{k1}$. If node a belongs to I_1, then it must belong to $\bar{I}_1 \cup \ldots \cup \bar{I}_{k1}$, but it can not belong to in $\bar{I}_s$, $s > k1$.

So, it is clear that if $s = k$, then $\alpha_{a,j}$ and $\bar{\alpha}_{a,j}$ are identical (by (3.39) and (3.40)). Thus, we are only left to show when $s \neq k$ then (3.39) and (3.40) are equal which we show in the following.

By Lemmas 3.9 and 3.8, we must have $|I_k| = |\bar{I}_k| = |I_s|$ since $a \in I_s$ and $a \in \bar{I}_k$. Since the solution of (3.24) and (3.25) is the unique (by Theorem 3.2), thus,

$$t_{k,j} = t_{s,j}.$$

Thus, $\alpha_{a,j}$ and $\bar{\alpha}_{a,j}$ are also identical (by (3.39) and (3.40)) when $s \neq k$. Hence, we show that $\alpha_{a,j} = \bar{\alpha}_{a,j}$.

Now, we show that if a primary selects independent sets among $I_1, \ldots, I_d$ irrespective of partition the other primaries select such that (3.24) and (3.25) are satisfied, then the strategy profile is an NE. This will conclude the proof since by symmetry, it will follow that if a primary selects independent sets among $\bar{I}_1, \ldots, \bar{I}_d$ irrespective of the partition other primaries select then the strategy profile is an NE.

We have so far showed that every node in I_s is selected with identical probability by each primary irrespective of the partition selected by them when the independent set selection strategy is of the form (3.24) and (3.25). Thus, at every node $a \in I_s$, each primary offers its channel at node a when the channel state is j or higher w.p. $\sum_{k=j}^{n} \alpha_{a,k} = \sum_{k=j}^{n} q_j t_{s,j}$ which is equal to $\gamma_{s,j}$ (recall from (3.15)) irrespective of the partition selected by the primaries. In proving that a primary does not have any incentive to deviate unilaterally from the strategy profile which is of the form (3.24) and (3.25) (Theorem 3.3), we only use the properties of $\gamma_{s,j}$. Hence, if a primary selects independent sets among $I_1, \ldots, I_d$ according to (3.24) and (3.25) irrespective of the partitions selected by other primaries, then it is an NE. Hence, the result follows. □

3.4 Different channel states at different locations

At later stages of deployment, the secondary market will operate at a region consisting of a large number of locations. The channel states will be different at different locations in this large region which we consider in this section. We first present specific assumptions that we have made in this scenario (Section 3.4.1). For example, nodes of commonly observed large conflict graphs exhibit an inherent symmetry in the interference relations, we therefore consider a class of conflict graphs, known as *node symmetric graphs* in the literature [70]. We subsequently obtain a symmetric NE strategy profile SP_{sym} in a node symmetric graph (Section 3.4.2). We show some important structural properties of SP_{sym} which are significantly different from the symmetric NE strategy profile obtained in the scenario where the channel state is identical across the network (Theorem 3.6, Lemmas 3.11, and 3.12). We show that SP_{sym} may not be a NE when the conflict graph is not a node symmetric (Lemma 3.13). Finally, we analytically and empirically evaluate the computational issues of computing the strategy SP_{sym} and how a primary can attain a desired trade-off between the expected payoff and the computational cost by selective estimation of channel states at randomly selected subset of nodes (Section 3.4.3).

3.4.1 Specific Assumptions

We revert to the notations introduced in Sections 3.1 and 3.2. Specifically, we do not need simplifications of the notations used for the first setting which have been introduced in Section 3.3.1.

n=1

In the previous setting (Section 3.3) we consider that the channel state is the same across the locations, thus, a primary always selects an independent set from the conflict graph G whenever the channel is available (i.e. the channel is not in state 0). Thus, a primary knows that its competitors always select independent sets from G (a primary does not select any independent set when the channel state is 0). In the current setting, the conflict graph representation of the region depends on the channel state vectors. Since the conflict graph representation can be different for different channel state vectors (G_J may not be equal to G_K when $J \neq K$), thus, a primary does not know the conflict graphs from which its competitors are selecting their independent sets. Thus, the collection of independent sets from which a primary selects its independent set may be different for different primaries. Additionally, the strategy space $\mathcal{P}$ ($|\mathcal{P}| = (n+1)^{|V|} - 1$) increase exponentially with the number of nodes. Thus, obtaining an NE in this setting is challenging. In order to simplify the setting, we consider

Assumption 3.2. $n = 1$ i.e. the channel is either available (i.e. at state 1) or not available (i.e. at state 0) at each node, but still the channel state can be different at different nodes.

Note that even though $n = 1$, the cardinality of strategy space $\mathcal{P}$ is $2^V - 1$ which is still exponential in the number of nodes and the conflict graph representation will be different for different channel state vectors.

Definition 3.9. Since $n = 1$, we drop the index j from $\alpha_{a,j}$ and $\mathcal{P}_{a,j}$ in (3.4) corresponding to the channel state at a given location. We denote α_a as the probability with which an available channel at node a is offered under a symmetric strategy profile and $\mathcal{P}_a$ as the set of channel state vectors where the channel state is 1 at node a.

Note that from Lemma 3.1 and (3.5), the upper endpoint of the penalty selection strategy is v at all nodes. The maximum expected payoff of a primary at node a under a symmetric NE strategy is

$$p_a - c = (f_1(v) - c)(1 - w(\alpha_a)) \tag{3.41}$$

Node Symmetric Graphs

We consider large size wireless networks. As an analytical abstraction, we mainly consider infinite size conflict graphs. In large conflict graphs, there is an inherent symmetry in the interference relations among the nodes in the network. We, therefore consider node symmetric graphs, which in the literature is also known as *node transitive graphs* [70].

Figure 3.8: Left hand figure shows a linear graph with 4 nodes. Right hand figure an automorphism where $F(1) = 4, F(2) = 3, F(3) = 2, F(4) = 1$. However, there is no automorphism between nodes 2 and 1. If there is an automorphism such that $F(2) = 1$, then by the property of automorphism nodes $F(2)$ and $F(3)$ should be adjacent and nodes $F(2)$ and $F(1)$ also should be adjacent, but since $F(2) = 1$, thus, either $F(3)$ or $F(1)$ will not be adjacent to node $F(2)$ since node 1 only has one degree in G.

First, we provide a formal definition of node symmetric graph. Towards that end, we first define an *automorphism* in a conflict graph G. We denote $V(G)$ as the set of nodes of G.

Definition 3.10. An automorphism is a bijective mapping $F : V(G) \to V(G)$ such that nodes $F(a)$ and $F(b)$ are adjacent[9] if and only if nodes a, b are adjacent in G.

In an automorphism, the nodes are renumbered such that it maintains the adjacency between the nodes. For example, consider a linear graph consisting of 4 nodes. Fig. 3.8 shows an automorphism on this graph. Now we are ready to define the node symmetric graph.

Definition 3.11. [70] In a node symmetric graph, for every pair of vertices a and b of G, there is some automorphism $F : V(G) \to V(G)$ such that $F(a) = b$.

For a graph to be node symmetric every node should be mapped to every other node through an automorphism. Informally, in a node symmetric graph the graphs looks the same from each node.

[9]In an undirected graph, two nodes are adjacent iff there is an edge between them.

For example cyclic graph is a node symmetric graph. But linear graph with 4 nodes is not a node symmetric graph since there is no automorphism between nodes 1 and 2 (Fig. 3.8).

Now, we provide some examples of infinite node symmetric graphs which resemble the conflict graphs of large wireless networks.

- Infinite linear graph with no end points (Fig. 3.10): This is an abstraction of the conflict graph of the network of a large number of wireless access points arranged in a linear fashion.

- Infinite square graphs (Fig. 3.11): This is an abstraction of the conflict graph of wireless networks in a large region with square cells.

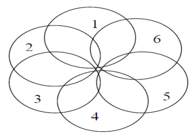
- Infinite grid graphs (Fig. 3.12): This is an abstraction of the conflict graph of a large shopping mall.

- Infinite triangular graphs (Fig. 3.13): This is an abstraction of the conflict graph representation of large number of hexagonal cells [60].

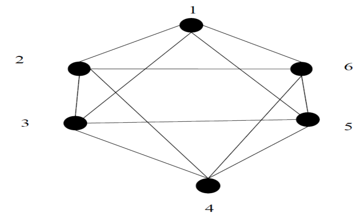
There are also several commonly observed node symmetric conflict graphs which are finite. For example, cyclic graph of any size is a node symmetric graph [10]. Cyclic conflict graph represents a collection of wireless access points arranged in a circular fashion, possibly circumambulating a city or ring size road. Figure 3.9 also shows a finite node symmetric graph and the corresponding wireless network. The complete graphs [11] are also node

[10]Note that cyclic graph is not a mean valid graph if $|V| > 3$ and $|V|$ is odd, thus, node symmetric graphs may not be mean valid graphs.

[11]In a complete graph a node has edge with every other node.



(a)



(b)

Figure 3.9: Circles in Figure (a) shows the ranges of the wireless access points located at the center of the circle. Figure (b) shows the corresponding conflict graph with each circle is represented as a node. Each circle intersects with 1 hop and 2 neighbors, thus, in the conflict graph each node has edges with 1 hop and 2 hop neighbors. The conflict graph is a node symmetric graph.

Figure 3.10: Infinite linear graph with no end-points: each node has degree 2.

symmetric graphs. We find a symmetric NE in a node symmetric graph irrespective of whether it is finite or infinite (Theorem 3.6).

Note from Section 3.3.2 that the commonly observed conflict graphs of small networks are mean valid graphs which we analyze in the previous setting. *These graphs may not be node symmetric graphs.*

Probability Distribution of Channel State Vectors

In the previous setting, we consider an extreme case where the channel state is identical across each location. In a large network, the channel states will be different. However, the channel states are often spatially proximal. Since the graph is large, like the interference relationship we expect that the statistical correlation pattern would also exhibit some

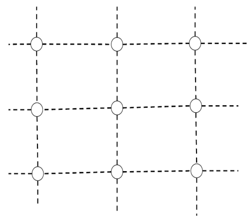


Figure 3.11: Infinite square graph: each node has degree 4.

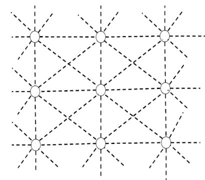


Figure 3.12: Infinite Grid graph: each node has degree 8.

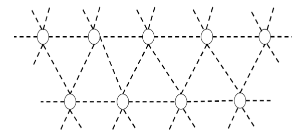


Figure 3.13: Infinite Triangular Graph: each node has degree 6.

symmetry. We consider one such symmetric relationship among the channel states across the network which arise naturally.

First, we define an isomorphism between two graphs:

Definition 3.12. Two graphs G and H are isomorphic to each other if there is a bijective mapping $F : V(G) \to V(H)$ such that any two vertices $F(a), F(b)$ are adjacent in H if and only if a, b are adjacent in G.

Informally, if two graphs look alike subject to renumbering of nodes, then they are isomorphic to each other. Note that automorphism is a special case of isomorphism which occurs when $H = G$ (Definition 3.10).

We assume that

Assumption 3.3. q_J and q_K are identical whenever the G_J and G_K are isomorphic to each other.

Intuitively, since G_J and G_K are alike subject to the renumbering of nodes, we therefore expect $q_J = q_K$. We show in Section 3.4.1 that the above assumption implies that

Lemma 3.10. *The probability that a channel of a primary is in state* 1 *at a given location is the same across the network.*

However, the converse of the above result not true in general.

We now provide some examples of joint probability distributions which arise in practice and satisfy Assumption 3.3.

Independent and identically distributed channel states: The state of the channel is $i = 1$ w.p. q at a given location independent of the channel states at other locations. At a given channel state vector J, if the channel is available at n_j number of nodes, then $q_J = q^{n_j}(1-q)^{|V|-n_j}$. When G_J and G_K are isomorphic, then both contain the same number of nodes, thus, the number of locations where the channel state is 1 (0, respv.) are the same in channel state vectors J and K. Hence, the probability distributions q_J and q_K are identical whenever G_J and G_K are isomorphic.

Correlated Channel states: We now show that Assumption 3.3 can accommodate statistical correlations across the channel states at different nodes. We provide an example in a small node symmetric graph. Consider a linear graph with 2 nodes such that $q_{(1,0)} = q_{(0,1)}$. Since $G_{(0,1)}$ and $G_{(1,0)}$ are the only possible isomorphic graphs in this case, thus, the above joint probability distribution satisfies Assumption 3.3. Now, if $q_{(1,1)} > q_{(1,0)} = q_{(0,1)}$ and $q_{(0,0)} > q_{(1,0)} = q_{(0,1)}$, then, the channel states are not independent[12]. Thus, Assumption 3.3 allows correlation among the channel states across the locations. Also note that the above probability distributions commonly arise in practice.

[12]Suppose that the channel is in state 1 at node i w.p. q_i independent of the channel state at other location, then, $q_{1,0} = q_{0,1}$ implies that $q_1 = q_2$. Now, $q_1(1-q_1)$ can not be less than both $q_1^2 (= q_{1,1})$ & $(1-q_1)^2 (= q_{0,0})$, hence, independent channel states can not satisfy the above joint distribution.

This is because when the channel is in state 1 (0 respv.) at one location, then there is a higher probability that the channel is in state 1 (0 respv.) compared to state 0 (1 respv.) at other location.

The joint probability distributions of random variables associated with spatial locations and exhibiting correlations are often represented as Markov Random Field. We provide a formal definition of Markov random field in Appendix 3.A.6 and show that the Markov random field modeling of channel states where the channel states in neighboring locations are correlated, satisfy Assumption 3.3 under some additional assumptions which naturally arise (Lemma 3.25 in Appendix 3.A.6).

Proof of Lemma 3.10

We first show Observation 3.3. Subsequently, we show Lemma 3.10.

Observation 3.3. Consider any pair of nodes a and b. For distinct channel state vectors $J, J_1 \in \mathcal{P}_a$ (Definition 3.9), there exists distinct channel state vectors $K, K_1 \in \mathcal{P}_b$ such that G_K and G_{K_1} are isomorphic to G_J and G_{J_1} respectively, such that the in the isomorphic function $F(a) = b$ (Definition 3.12).

Proof. We first show that for a channel state vector $J \in \mathcal{P}_a$ there exists a channel state vector $K \in \mathcal{P}_b$ such that G_K is isomorphic to G_J and in the isomorphic function $F(a) = b$. Since the graph is node symmetric, thus, there exists an automorphism $F(\cdot)$ (Definition 3.10) such that $F(a) = b$. Now consider the channel state vector K where the channel is available only at nodes $F(a_1)$ if $a_1 \in V(G_J)$. In the conflict graph representation of G_K, the set of edges are the edges incident on $F(a_1)$ where $a_1 \in G_J$. Since $F(\cdot)$ is itself is an automorphism on G, thus $F(a_1)$ and $F(a_2)$ are adjacent in G_K if and only if a_1 and a_2 are adjacent in G_J. Hence, $F(\cdot)$ is an isomorphic mapping from $V(G_J)$ to $V(G_K)$ such that $F(a) = b$.

Note that since J is arbitrary, thus, if $J_1 \in \mathcal{P}_a$, then, following the above procedure we obtain an isomorphic graph G_{K_1} such that $F(a) = b$ in G_{K_1}. Since $F(\cdot)$ is automorphism and thus, $F(\cdot)$ is bijective. Thus, if $J_1 \neq J$, then, using the function $F(\cdot)$ we obtain a channel state vector K_1 which is different from K. Also note that $b \in V(G_{K_1})$ since $F(a) = b$ and $a \in V(G_{J_1})$. Hence, the result follows. □

Now, we show Lemma 3.10.

Proof. Consider any two nodes a and b. Recall the definition of $\mathcal{P}_a$ (Definition 3.9). First, note that $|\mathcal{P}_a| = |\mathcal{P}_b| = 2^{|V|-1}$ since the channel state must be 1 at node a (node b, respv.) for every channel state vector in $\mathcal{P}_a$ ($\mathcal{P}_b$ respv.). Now, the probability that the channel state is 1 at node a is

$$\beta_a = \sum_{J:J\in\mathcal{P}_a} q_J$$

and the probability that the channel state is 1 at node b is

$$\beta_b = \sum_{K:K\in\mathcal{P}_b} q_K$$

Note that by Observation 3.3, for distinct channel state vectors $J, J_1 \in \mathcal{P}_a$ there exist distinct channel state vectors $K, K_1 \in \mathcal{P}_b$ such that G_K, G_{K_1} are isomorphic to G_J and G_{J_1} respectively. Also note that cardinalities of $\mathcal{P}_a$ and $\mathcal{P}_b$ are the same. Since $q_J = q_K$ whenever G_J and G_K are isomorphic to each other, thus, we obtain

$$\beta_a = \beta_b \tag{3.42}$$

Hence, the result follows. □

3.4.2 Symmetric NE strategy Profile

We, first, obtain a symmetric NE strategy profile (Theorem 3.6). We then show that the NE strategy has an important structural difference compared to the NE strategy in the previous setting (Section 3.3) where the channel state is the same across the network (Lemmas 3.11, 3.12).

We first start with introducing a notation. Let $I_{max,J}$ be the set of maximum independent sets (i.e. independent sets of highest cardinalities) of the graph G_J .

Strategy Profile (SP_{sym}*)*: A primary selects each of the independent set within the set $I_{max,J}$ with probability $\dfrac{1}{|I_{max,J}|}$ and select other independent sets with probability 0 at channel state vector J.

Theorem 3.6. *The Strategy profile* SP_{sym} *is an NE strategy profile.*

A primary only needs to find the maximum independent sets in order to find the NE strategy profile SP_{sym}. In contrast to SP_{sym}, a primary may select an independent set which is not a maximum independent set in the scenario where the channel state is identical across the locations (Theorems 3.1 and 3.3). Note that in SP_{sym} a primary puts equal weight on each of the maximum independent sets in G_J. Hence, a primary *needs not communicate* with other primaries to obtain its strategy. Hence, SP_{sym} is easy to implement.

Lemma 3.11. *Expected payoff at every node is the same under* SP_{sym}*.*

Intuitively, since the graph is node symmetric, each node belongs to the same number of maximum independent sets, thus a channel is offered with the same probability at every node under SP_{sym}; thus, the expected payoff is the same at every node. *Since each node* *is selected with the same probability, hence there is an equity in secondary access of the channel amongst different nodes as opposed to that scenario where the channel state is identical across the network (Example 1).*

We show that unlike in the scenario where the channel state is the same across the network (Theorem 3.5), the symmetric NE may not be unique in linear conflict graph in this setting.

Lemma 3.12. *There may exist infinitely many symmetric NEs in the linear conflict graph.*

The proof of the above lemma is algebraic and we relegate it to Appendix 3.A.5.

We also show in Appendix 3.A.5 that symmetry in interference relations among the nodes is required for SP_{sym} to be an NE.

Lemma 3.13. *SP_{sym} may not be an NE for a finite linear graph which is not a node symmetric graph.*

Proof of Theorem 3.6

We use Observation 3.3 stated in previous subsection (Section 3.4.1). Since the strategy profile is symmetric, it is enough to show that primary 1 does not have any incentive to deviate from SP_{sym} when other primaries also select SP_{sym}.

We first give an outline of the proof. First, we show that the maximum expected payoff attainable by primary 1 is identical across the nodes using the node symmetric property and Assumption 3.3. Thus, it directly implies that primary 1 will attain the maximum expected payoff by selecting a maximum independent set. Since SP_{sym} only randomizes among the maximum independent sets, thus, primary 1 will not have any incentive to deviate from SP_{sym} which in turn proves Theorem 3.4. The details of the proof is given below.

In order to show Theorem 3.6 we show the following:

i) First, we show that the node selection probability α_a for a primary is identical when Assumption 3.3 is satisfied for each node under SP_{sym} using Node symmetric graph and Observation 3.3.

ii) Subsequently, we show that when all primaries other than primary 1 select SP_{sym}, then the maximum expected payoff obtained by primary 1 is identical across the nodes.

iii) Finally, we show that primary 1 does not have any incentive to deviate unilaterally from SP_{sym} which shows that SP_{sym} is indeed an NE.

Part i): First, we introduce some notations. Let $I^a_{max,J}$ be the set of maximum independent sets of G_J which contains node a. Note that the node a can only be selected at a channel state vector J if $J \in \mathcal{P}_a$ (Definition 3.9).

Thus, under the strategy profile SP_{sym} the node selection probability at node a i.e. α_a is

$$\alpha_a = \sum_{J \in \mathcal{P}_a} \frac{|I^a_{max,J}|}{|I_{max,J}|} q_J \tag{3.43}$$

Now, we show that $\alpha_a = \alpha_b$ where $b \neq a$. By Observation 3.3 for every G_J, there exists a distinct G_K which is isomorphic to G_J such that in the isomorphic mapping $F(a) = b$. Thus, $|I^b_{max,K}| = |I^a_{max,J}|$. Since G_J and G_K are isomorphic to each other thus $|I_{max,J}| = |I_{max,K}|$. Also note that, $q_J = q_K$ since G_J is isomorphic to G_K by Assumption 3.3. Finally, note that the cardinalities of $\mathcal{P}_a$ and $\mathcal{P}_b$ are the same. Hence,

$\alpha_a = \alpha_b$ for any two nodes $a, b \in V$ by (3.43). Hence, the node selection probability is the same at every node.

Part ii): When all the other primaries apart from primary 1 selects SP_{sym}, then at node a , the channel is offered for sale at node a w.p. α_a by other primaries apart from primary 1. Thus, by Lemma 3.1, when all the other primaries select SP_{sym}, then the maximum expected payoff obtained by primary 1 at node a is $(f_1(v) - c)(1 - w(\alpha_a))$ (from (3.41)). Moreover, by Lemma 3.1 the payoff is obtained by selecting any penalty within $[L_1, v]$. Since α_a's are identical, hence, the maximum attainable expected payoff by primary 1 is identical at each node.

Part iii): Consider a channel state vector J. Since the maximum attainable expected payoff at every node is identical, hence, primary 1 can attain the total maximum expected payoff only by selecting a maximum independent set of G_J when other primaries select the strategy SP_{sym}. Under SP_{sym}, primary 1 randomizes among the maximum sets of G_J. Hence, the total expected payoff of primary 1 is equal to the maximum expected payoff. Hence, primary 1 does not have any incentive to deviate from SP_{sym} when other primaries select SP_{sym}. Thus, SP_{sym} is an NE. □

Proof of Lemma 3.11

Note that the proof of this result directly follows from part (ii) of the Theorem 3.6 where we have shown that primary 1 will attain the same expected payoff at every node of the conflict graph if primary 1 selects SP_{sym} when the other primaries also select strategy SP_{sym}. □

3.4.3 Computational Complexity

In SP_{sym} a primary needs to enumerate the maximum independent sets at a given channel state vector. In general, the number of maximum independent sets scales exponentially with the number of nodes. But if a graph consists of disjoint components, then a primary can compute the maximum independent sets of each component and compute the strategy profile in each component in parallel. Hence, the size of the component will govern the computation time.

The conflict graph of a primary depends on the channel state vector which evolves randomly. Hence, the conflict graphs are random graphs. Thus, it is important to find the average size of a component in a conflict graph which will govern the average computation time of maximum independent sets. In the following, we provide a bound on the expected size of a component for some node symmetric graphs that arise in practice. We also discuss how primaries can govern the component size using random sampling technique (selecting each node w.p. p) . Throughout this section, we consider that the channel states are I.I.D. where the channel state is 1 at a given location w.p. q.

Let Δ be the degree of a node. We consider those node symmetric graphs where Δ is finite. Nodes in most of the conflict graphs that we have discussed in Section 3.4.1 have finite degrees. For example, in cyclic graph (of any size) $\Delta = 2$, in infinite linear graph $\Delta = 2$, in infinite square graph (Fig. 3.10), $\Delta = 4$ (Fig. 3.11), in infinite grid graph $\Delta = 8$ (Fig. 3.12), in infinite triangular graph $\Delta = 6$ (Fig. 3.13).

We find out the expected size of a component C originating from node a in a conflict graph G_J. Since the graph is a node symmetric graph, hence the expected size of a component originating from any other node will be the same. Each node has an expected degree of $q\Delta$. The component C grows when G_J contains neighbors of node a, the neighbors of the neighbors of node a and so on. Thus, the growth of C can be compared to the Galton-Watson branching process [36] where each individual gives birth to $q\Delta$ number of children on average. The only difference in our approach to the Galton-Watson process is that the number of nodes added each step may be smaller as some of the neighbors of a node may already be in C, thus, reducing the number of neighbors that can be added in C. Thus, the expected size of C can be upper bounded by the expected umber of total descendants in Galton-Watson process [36]. Hence, the upper bound of expected size of C is obtained from [36]

Lemma 3.14. $E(C) \leq \dfrac{1}{1-q\Delta}$ *if* $q\Delta < 1$.

A primary can not control q, hence, the component size (and thus, the computational complexity) can be large for higher q. Thus, a primary may estimate its channel quality only at a subset of the locations of the region instead of the whole region and sell its channel only among the locations where it knows the transmission quality in order to minimize the computation cost. Equivalently, a primary will consider that the channel state is 0 at locations where it does not estimate its channel quality. In one simplistic setting which we consider, each primary computes the transmission quality at a node w.p. p independent of the other nodes. A primary does not know the nodes where its competitors are estimating their channel states. But , it knows p. Thus, a primary is aware of the fact that the channel state is 1 at any given node of its competitor w.p. pq independent of the channel states at other locations. Thus, the probability distribution satisfies Assumption 3.3. *Hence, the strategy profile SP_{sym} will be a symmetric NE strategy profile in this setting where the channel states are I.I.D. and the channel is in state* 1 *at a given location w.p.* pq *instead of* q. Thus, from Lemma 3.14 the expected size of component is now upper bounded by

$$E(C) \leq \frac{1}{1 - pq\Delta} \quad \text{if } pq\Delta < 1 \tag{3.44}$$

Note that the above procedure also decreases the measurement and estimation cost, since a primary only estimates the channel states at a randomly selected subset of locations.

Note that the right hand side in (3.44) increases as $pq\Delta$ increases. If a primary selects lower (higher, respv.) p the expected component size will decrease (increase, respv.), and thus, the computation complexity will decrease (increase, respv.); The bound in (3.44) also decreases (increases, respv.). However, the expected payoff of a primary will also decrease (increase, respv.) with decrease in p (increase, respv.) since the number of nodes where a primary can potentially sell its channel also decreases (increases, respv). Hence, a primary has to judiciously select p in order to achieve a desired trade-off between the computation complexity, and the expected payoff.

We now empirically investigate the variation of the mean size of the largest component with the number of nodes and the parameter pq. For each value of pq, we generate a certain number of random graphs. We compute the average of the largest component over 25 samples. The number 25 has been chosen since the average converges in 25 samples. Figure 3.14 shows the variation of the mean size of the largest component as the number of nodes increases. Figure 3.14 reveals that the growth of the average size of the largest component in a square graph is linear (not exponential) with the number of nodes whereas

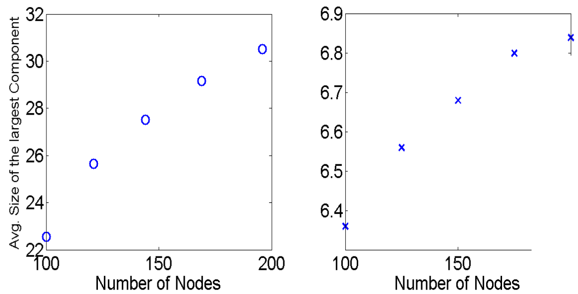


Figure 3.14: Mean size of the largest component in a Square Graph and Linear graph for $pq = 0.5$. The square graph consists of a j rows and columns. We vary j while we increase the number of nodes.

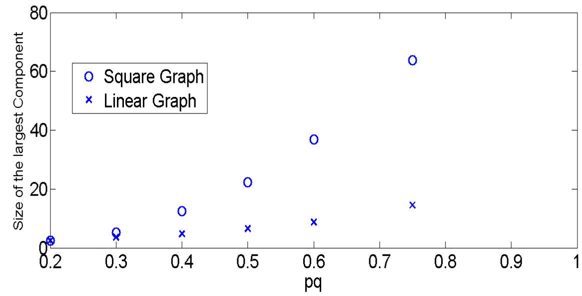


Figure 3.15: Mean size of the largest component in a Square Graph and Linear graph with 100 nodes.

the growth of the mean size of the largest component in a linear graph is very slow with the number of nodes. Additionally, when $pq = 0.5$, the upper bound in (3.44) is infinite both for square and linear graph, however, Fig. 3.14 shows that the expected size of the largest component is moderate in the square graph as well as in the linear graph even when the number of nodes are large. Fig. 3.15 reveals that when pq is exceeds a threshold the mean size of a largest component increases substantially in both linear conflict graph and square conflict graph. However, Fig. 3.15 reveals that the upper bound computed in (3.44) is loose. For example, when $0.25 \leq pq \leq 0.6$, the upper bound in (3.44) is infinite for square graph, however, Fig. 3.15 shows that the average size of the largest component is moderate. In the linear graph, the mean size of the largest component is small even when $0.5 \leq pq \leq 0.75$ whereas the upper bound computed in (3.44) is infinite when $0.5 \leq pq \leq 0.75$.

3.5 Random Demand

Till now we have assumed that the number of secondaries (m) is constant at each node. But our analysis will readily generalize to the scenario where the number of secondaries at a given location is $Z \geq 1$ where Z is a random variable independent of the number of secondaries at other locations with an additional assumption the p.m.f. $\Pr(Z = m) = \kappa_m$ must satisfy the condition $\sum_{m=1}^{l-1} \kappa_m > 0$ (i.e. the total number of primaries exceeds the total number of secondaries with positive probability but not w.p. 1). A primary does not know Z apriori, however, it knows the p.m.f of Z. The analysis will go through with the following modifications in (2.5)

$$w(x) = \sum_{m=1}^{\min\{\max(Z),l-1\}} \kappa_m \sum_{i=m}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-i-1} \tag{3.45}$$

3.6 Numerical Evaluations

We numerically study the impact of competition on the payoffs of the primaries in the scenarios which we consider. Towards that end, we compare the payoff under the symmetric NE strategy, $R_{M,NE}$, with the maximum possible value of social welfare, R_{OPT} which is obtained when all the primaries collude.

$$R_{M,NE} = \text{Number of Primaries } \times \text{Expected payoff of each primary}$$

Definition 3.13. The efficiency of NE is the ratio of the total expected payoff of primaries and the optimal value of social welfare (R_{OPT}).

In other words, efficiency (η_M)= $\dfrac{l \cdot R_{M,NE}}{R_{M,OPT}}$.

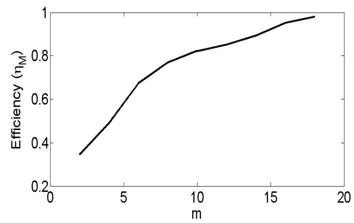


Figure 3.16: This figure shows the variation of efficiency with m. We consider a 5×5 grid graph (see Fig. 3.3). This is a mean valid graph with $d = 4$ and $|I_1| = 9, |I_2| = |I_3| = 6, |I_4| = 4$. We use the following parameter values, $l = 20, n = 3, v = 100, c = 1$, $g_i(x) = x^2 - i^3$, $q_1 = q_2 = q_3 = 0.2$.

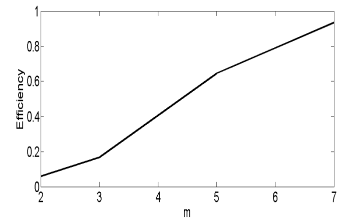


Figure 3.17: This figure shows the variation of efficiency with m when the channel states are independent and identically distributed at each location with $q_i = 0.5, i = 0, 1$. We consider a cyclic graph of 100 We use the following parameter values: $l = 10, v = 11, c = 1$.

Fig. 3.16 shows the variation of efficiency with the number of secondaries (m) in the scenario where the channel state remains the same throughout the network. Fig. 3.17 shows the variation of efficiency with m in the scenario where the channel state can be different at different locations. Both the figures reveal that η_M increases with increase in m. This is because when m is low, competition becomes intense and primaries select lower penalties. Primaries also select independent sets of lower cardinalities when the channel state is the same at each location in Fig. 3.16. But if they collude with each other, they still can offer highest penalty and only select the independent sets of the largest cardinalities in both of the settings, which lead to high payoff.

3.7 Conclusions and Future Work

We have studied a price competition model with the spatial reuse property where each primary selects a price and a set of non-interfering locations depending on the quality of its channel. We have considered two settings. In the first setting, we consider that the channel state is the same across the networks. We have shown that there exists a symmetric NE strategy profile in the class of mean valid graphs and we have computed a storage and computational efficient NE. The NE strategy profile can be readily implemented as primaries need not communicate with each other even when the NE strategy is not unique. We show that the symmetric NE strategy profile is unique in a linear conflict graph.

In the second setting, we allow that the channel state can be different at different locations. The above consideration significantly complicates the analysis as we have discussed in Section 3.4.1. We, therefore, consider that the channel is either available or unavailable at each node. We have shown that there exists a symmetric NE strategy profile in the class of node symmetric graphs. In order to obtain the symmetric NE strategy, a primary only needs to enumerate the maximum independent sets. The NE strategy is also easy to implement. We have shown that symmetric NE strategy may not be *unique* in a linear conflict graph in contrast to the first setting.

The characterization of an NE in the second setting where the available channel may belong to multiple states remains open. The characterization of an NE when the demand at different locations are correlated is also a work for the future. The analytical results and tools that we have provided in this chapter may provide the basis for developing a framework for this problem.

3.A Appendix

We prove Theorem 3.2 (Section 3.3.5) in Appendix 3.A.1. Subsequently, we show Theorem 3.3 (Section 3.3.5) in Appendix 3.A.2. We show Lemmas 3.8 and 3.9 used in Section 3.3.7 to prove Theorem 3.4 in Appendix 3.A.3. We show Theorem 3.5 (Section 3.3.7) in Appendix 3.A.4. We prove Lemmas 3.12 and 3.13 (Section 3.4.2) in Section 3.A.5. Finally, in Appendix 3.A.6 we provide a formal definition of Markov Random Field and show that Markov random field modeling of correlated channel states satisfy Assumption 3.3 (Section 3.4.1) under some additional assumptions which naturally arise in practice.

3.A.1 Proof of Theorem 3.2(Section 3.3.5)

We proceed in two parts. First, we will prove that there exists a distribution $t_j = (t_{1,j}, \ldots, t_{d,j})$ which satisfies (3.24) and (3.25). Subsequently, we will prove that such a distribution is the unique one.

Existence: First, we will show that the statement is true for $j = n$. Now, let $x \in [M_1 W(q_n), M_1]$ and $s \in \{1, \ldots, d\}$. We will show that if $x \leq M_s$ then

$$M_s W(rq_n) = x \tag{3.46}$$

has a unique solution in r, which we will denote as $t_{s,n}(x)$. Let, $h(t_{s,n}) = M_s W(t_{s,n} q_n)$, then

$$\begin{aligned} h(1) &= M_s W(q_n) \\ &\leq M_1 W(q_n) \leq x \end{aligned} \tag{3.47}$$

and

$$h(0) = M_s \geq x. \tag{3.48}$$

As $W(\cdot)$ is strictly decreasing and continuous, so is $h(\cdot)$, thus from (3.47) and (3.48), there exists a unique solution $t_{s,n}(x)$ between 0 and 1, such that $h(t_{s,n}) = x$. Note that

$$t_{s,n}(x) = 0 \quad (\text{if} \quad x = M_s). \tag{3.49}$$

$h(t_{s,n})$ is strictly decreasing in $0 \leq t_{s,n} \leq 1$. Hence, inverse exists and h^{-1} is also continuous as h is continuous. But, $x = h(t_{s,n}(x))$. Hence, $t_{s,n}(x) = h^{-1}(x)$. Thus, $t_{s,n}(x)$ is continuous for $x \leq M_s$. For $x > M_s$, define $t_{s,n}(x) = 0$. With the above definition and (3.49) we obtain $t_{s,n}(x)$ is continuous function on $[M_1W(q_n), M_1]$ and thus

$$t_{s,n}(x) = 0 \quad (\text{if } x \geq M_s). \tag{3.50}$$

Now, let

$$T_n(x) = \sum_{s=1}^{d} t_{s,n}(x). \tag{3.51}$$

As $h(t_{s,n})$ is strictly decreasing on $0 \leq t_{s,n} \leq 1$ for $s \in \{1, \ldots, d\}$, $t_{s,n}(x)$ is strictly decreasing for $x \leq M_s$. Hence, $T_n(x)$ is strictly decreasing for $x \in [M_1W(q_n), M_1]$. Also, note that, $t_{s,n}(x) = 0$ for $M_s < x \leq M_1$. Thus, for $x = M_1$, $t_{s,n}(x) = 0\ \forall s$, as $M_s \leq M_1$, hence for $x = M_1$,

$$T_n(x) = 0 \tag{3.52}$$

Now, for $x = M_1W(q_n)$, $t_{1,n}(x) = 1$, $t_{s,n}(x) \geq 0\ s \in \{2, \ldots, d\}$, thus

$$T_n(x) \geq 1. \tag{3.53}$$

As $t_{s,n}(x)$ are continuous, so is $T_n(x)$. Thus, from(3.52), (3.53) and intermediate value property, there exists a $x^* \in [M_1W(q_n), M_1]$ such that $T_n(x^*) = 1$ and this is unique as $T_n(\cdot)$ is strictly decreasing. Let, $d'_n = \max\{s : M_s > x^*\}$. By definition of $t_{s,n}$, for $s = 1, \ldots, d'_n$, $M_sW(t_{s,n}(x^*)q_n) = x^*\ t_{s,n}(x^*) > 0$ and for $s > d'_n$, $t_{s,n}(x^*) = 0$ (by (3.50)). Since $\gamma_{s,n} = t_{s,n}q_n$ (from (3.15)) and $W(0) = 1$, thus $M_sW(\gamma_{s,n}) = x^*$ for $s \leq d'{}_n$ and for $s > d'{}_n$ $M_sW(\gamma_{s,n}) \leq x^*$. Hence, $\{t_{1,n}(x^*), \ldots, t_{d,n}(x^*)\}$ constitute a probability distribution and satisfy the equations (3.24) and (3.25), with $d_n = d'_n$ and $\gamma_{s,n+1} = 0$. Thus, the result is true for n.

Let, the statement be true for $k+1$, we have to show that the statement is indeed true for k. As the statement is true for $k+1$, thus, there exists unique distribution $t_{k+1} = (t_{1,k+1}, \ldots, t_{d,k+1})$ such that (3.24) and (3.25) holds for $j = k+1$. The argument will be similar to the case when $i = n$ with finding unique solution to the equation $M_sW(t_{s,k}(x)q_k + \gamma_{s,k+1}) = x$ for $x \in [M_1W(q_k + \gamma_{1,k+1}), M_1W(\gamma_{1,k+1})]$ for $x > M_sW(\gamma_{s,k+1})$ and making $t_{s,k} = 0$ for $x \geq M_sW(\gamma_{s,k+1})$. Hence we omit the proof. Thus, the result is true by the principle of mathematical induction. □

Uniqueness: We will prove the uniqueness by Induction hypothesis. First, consider the state n.

To reach a contradiction, assume that there exists $e, f \in \{1, \ldots, d\}$ such that $t'_n = (t'_{1,n}, \ldots, t'_{d,n})$, $\bar{t}_n = (\bar{t}_{1,n}, \ldots, \bar{t}_{d,n})$, $t'_{s,n} = 0$ (respectively, $\bar{t}_{s,n} = 0$) for $s > e$ (respectively $s > f$) and for some y and z:

$$y = M_1W(t'_{1,n}q_n) = \ldots = M_eW(t'_{e,n}q_n) \geq M_{e+1}W(t'_{e+1,n}q_n) \tag{3.54}$$

$$z = M_1W(\bar{t}_{1,n}q_n) = \ldots = M_fW(\bar{t}_{f,n}q_n) \geq M_{f+1}W(\bar{t}_{f+1q_n}). \tag{3.55}$$

First, suppose $e = f$, if $y = z$, then $M_s W(t'_{s,n} q_n) = M_s W(\bar{t}_{s,n} q_n)$ for $s \in \{1, \ldots, e\}$. But, $W(\cdot)$ is a strictly decreasing and one-to-one mapping, thus $t'_{s,n} = \bar{t}_{s,n}$ for $s \in \{1, \ldots, e\}$ and $t'_{s,n} = 0 = \bar{t}_{s,n}$ for $s > e$, which leads to a contradiction.

If $e = f$, but $y > z$, then $M_s W(t'_{s,n} q_n) > M_s W(\bar{t}_{s,n} q_n)$ for $s \in \{1, \ldots, e\}$. As $W(\cdot)$ is strictly decreasing function, hence we must have $t'_{s,n} < \bar{t}_{s,n}$. Now, $t'_{s,n} = 0$ for $s > e$. Thus,

$$\sum_{s=1}^{d} t'_{s,n} = \sum_{s=1}^{e} t'_{s,n} < \sum_{s=1}^{d} \bar{t}_{s,n} = 1.$$

The above inequality lea leads to a contradiction. Thus $y > z$ is not possible, by symmetry, $z > y$ is not possible.

Now, suppose $e > f$, thus ${t'}_{f+1,n} > 0$. Since $W(\cdot)$ is strictly decreasing function, thus

$$M_{f+1} W({t'}_{f+1,n} q_n) < M_{f+1}. \tag{3.56}$$

Since $\bar{t}_{f+1,n} = 0$, thus

$$M_{f+1} W(\gamma_{f+1,n}) = M_{f+1}. \tag{3.57}$$

Thus from (3.54), (3.55), (3.57) and (3.56), $y = M_{f+1} W(t'_{f+1,n} q_n) < M_{f+1} \leq z$. So, for $s \in \{1, \ldots, f\}$:

$$M_s W(t'_{s,n} q_n) < M_s W(\bar{t}_{s,n} q_n).$$

Hence, $t'_{s,n} > \bar{t}_{s,n}$, thus, $\sum_{s=1}^{f} t'_{s,n} > \sum_{s=1}^{f} \bar{t}_{s,n} = 1$, which leads to a contradiction. Hence, $e > f$ is not possible, by symmetry, $e < f$ is not possible.

Thus, the result is true for n.

Now, assume that the statement is true for states $k+1, \ldots, n$. Since, the statement is true for states $k+1, \ldots, n$, thus ${t'}_{s,j} = \bar{t}_{s,j}$ $\forall s$, $\forall j \geq k+1$. Hence,${\gamma'}_{s,j} = \bar{\gamma}_{s,j}$ $\forall s$,$\forall j \geq k+1$.

From (3.15), $\gamma_{s,k} = \gamma_{s,k+1} + t_{s,k}q_k$. As ${\gamma'}_{s,k+1} = \bar{\gamma}_{s,k+1}$, the proof will be similar to the case when state is n.

The result follows from the induction hypothesis. □

3.A.2 Proof of Theorem 3.3(Section 3.3.5)

In order to prove Theorem 3.3 we have to show that any independent set selection strategy of the form (3.24) and (3.25) is an NE. We can not use results derived in Section 3.3.4 which we derive assuming that the strategy profile is an NE. However, we use Lemma 3.7, and Corollary 3.1 that any independent set selection strategy profile of the form (3.24) and (3.25) satisfies regardless of whether it is an NE or not. We also use the following result which can be easily seen from from Lemma 3.7. Since $d_j \geq d_{j+1}$ (by Lemma 3.7), hence from (3.24), $t_{s,k} = 0 \ \forall s > d_j, k \geq j$. Thus, from (3.15) we obtain

$$\gamma_{s,j} = 0 \quad \text{for } s > d_j. \tag{3.58}$$

Hence, we can write (3.25) as

$$M_1 W(\gamma_{1,j}) = \ldots = M_{d_j} W(\gamma_{d_j,j}) \geq M_{d_j+1} \geq M_{d_j+2} \geq \ldots \geq M_d. \tag{3.59}$$

Now we state and prove Lemmas 3.15 and 3.16 which are satisfied by any strategy profile of the form (3.24) and (3.25). We use these results to prove Theorem 3.3.

Lemma 3.15. *$U_{s,j} = U_{k,j}$ if $t_{s,j}, t_{k,j} > 0$*

Proof. We prove the statement using induction argument.

The statement is trivially true for $j = 1$ because $U_{s,1} = v \ \forall s$ by (3.18).

Now, suppose the statement is true for $j = i$. Then, for any $s, k \in \{1, \dots, d\}$, $t_{s,i} > 0, t_{k,i} > 0$, we have

$$U_{s,i} = U_{k,i} \tag{3.60}$$

Now, let $t_{r,i+1} > 0, t_{r_1,i+1} > 0$ for $r, r_1 \in \{1, \dots, d\}$. Note that $L_{r,i} = U_{r,i+1}$ and $L_{r_1,i} = U_{r_1,i+1}$ from (3.18). Thus, from (3.19),

$$U_{r,i+1} = L_{r,i} = g_i(\frac{p_{r,i} - c}{W(\gamma_{r,i+1})} + c) \tag{3.61}$$

$$U_{r_1,i+1} = L_{r_1,i} = g_i(\frac{p_{r_1,i} - c}{W(\gamma_{r_1,i+1})} + c) \tag{3.62}$$

From corollary 3.1 $t_{r,i} > 0, t_{r_1,i} > 0$ since $t_{r,i+1}, t_{r_1,i+1} > 0$. Using (3.60) for r, r_1 we have $U_{r,i} = U_{r_1,i}$, hence from (3.18)

$$\frac{p_{r,i} - c}{W(\gamma_{r,i})} = \frac{p_{r_1,i} - c}{W(\gamma_{r_1,i})} \tag{3.63}$$

Now, since $t_{r,i+1} > 0, t_{r_1,i+1} > 0, t_{r,i} > 0, t_{r_1,i} > 0$ thus $r, r_1 \leq d_{i+1} \leq d_i$ (the last inequality follows from lemma 3.7). Hence,using (3.25) for r, r_1 we obtain

$$M_r W(\gamma_{r,i}) = M_{r_1} W(\gamma_{r,i}) \tag{3.64}$$

$$M_r W(\gamma_{r,i+1}) = M_{r_1} W(\gamma_{r_1,i+1}) \tag{3.65}$$

Thus, from (3.63), (3.64) and (3.65), we obtain

$$\frac{p_{r,i} - c}{W(\gamma_{r,i+1})} = \frac{p_{r_1,i} - c}{W(\gamma_{r_1,i+1})}$$

$$U_{r,i+1} = U_{r_1,i+1} \quad \text{(from (3.61) and (3.62))}$$

Hence, $U_{r,i+1} = U_{r_1,i+1}$. The result follows from the induction hypothesis. □

Remark 3.5. **Henceforth we denote $U_{s,j}$ as U_j when $t_{s,j} > 0$ i.e.** $s \leq d_j$. Note that we have obtained similar result (Lemma 3.2) for any NE strategy profile of the form (3.13). But Lemma 3.15 is valid for any strategy profile of the form (3.24) and (3.25) satisfies regardless of whether it is an NE or not.

Lemma 3.16. *If $j \geq 2$, then for $d_k \geq s > d_j$, $L_{s,k} \geq U_j$, where $k < j$.*

Proof. Let $i = \max\{y \in \{1, \dots, j-1\} : t_{s,y} > 0\}$. Thus, $t_{s,i} > 0$, $t_{s,i+1} = 0$, and $s > d_{i+1}$. Since $t_{1,j} > 0 \; \forall j$ by corollary 3.1, thus $U_k > U_i$, (or $U_{1,k} > U_{1,i}$) if $k < i$. So, it is enough to show that $L_{s,i} \geq U_{i+1}$ because $i + 1 \leq j$ and thus $U_{i+1} \geq U_j$.

Since $s > d_{i+1}$, thus $s > d_k \; \forall k > i$ by Lemma 3.7. Thus, $t_{s,k} = 0 \; \forall k > i$, thus $\gamma_{s,i+1} = 0$ (from observation 3.1) and thus $W(\gamma_{s,i+1}) = 1$. Thus, from (3.19)

$$\begin{aligned} L_{s,i} &= g_i(p_{s,i} - c + c) \\ &= g_i((f_i(U_i) - c)W(\gamma_{s,i}) + c) \quad \text{(from(3.17))} \end{aligned} \tag{3.66}$$

Since $L_{1,i} = U_{i+1}$ and $p_{1,i} - c = (f_i(U_i) - c)W(\gamma_{1,i})$ from (3.17); hence, from (3.19)

$$L_{1,i} = U_{i+1} = g_i(\frac{(f_i(U_i) - c)W(\gamma_{1,i})}{W(\gamma_{1,i+1})} + c) \tag{3.67}$$

Since $s > d_{i+1}$, hence using (3.59) for state $i + 1$, we obtain

$$M_1 W(\gamma_{1,i+1}) \geq M_s \tag{3.68}$$

Again using (3.59) for state i and noting that $1, s \leq d_i$, we obtain

$$\begin{aligned} M_1 W(\gamma_{1,i}) &= M_s W(\gamma_{s,i}) \\ \frac{W(\gamma_{1,i})}{W(\gamma_{s,i})} &= \frac{M_s}{M_1} \leq W(\gamma_{1,i+1}) \quad \text{(from(3.68))} \end{aligned} \tag{3.69}$$

Since $g_i(\cdot)$ is strictly increasing, in order to show that $U_i \leq L_{s,i}$, from (3.66) and (3.67) it is sufficient to show the following

$$\frac{(f_i(U_i) - c)W(\gamma_{1,i})}{W(\gamma_{1,i+1})} \leq (f_i(U_i) - c)W(\gamma_{s,i}) \tag{3.70}$$

Since $f_i(U_i) > c$, (3.70) readily follows from (3.69). □

Now we are ready to show Theorem 3.3.

Proof of Theorem 3.3: We will show that for channel state $j \in \{1, \ldots, n\}$, probability distribution $t_{s,j}$ as described in (3.24) and (3.25) for $s \in \{1, \ldots, d\}$ is a best response.

1. First, we will show that under the strategy profile for $s \leq d_j$, at any independent set I_s,maximum expected payoff is given by P_j^* (equation (3.23)) and the maximum value is obtained at each penalty value in the interval $[L_{s,j}, U_j]$ at every node of I_s. (Case i)

2. Next, we will show that for any choice of penalty a primary can only attain at most an expected payoff of P_j^* at any I_s, $s > d_j$ (Case ii and Case iii).

3. Finally, we will show that if a primary selects any other independent set i.e. apart from $I_1, \ldots, I_d$ then its expected payoff is upper bounded by P_j^* for any choice of penalty (Case iv).

Case i:At independent set $I_s, s \leq d_j$.

In this case $t_{s,j} > 0$. From Lemma 3.1 and (3.17) at a node $D \in I_s$, $s \leq d_j$, a primary gets a maximum payoff of $p_{s,j} - c$ when the channel state is j. Since $s \leq d_j$, hence $U_{s,j} = U_j$. Thus,

$$p_{s,j} - c = (f_j(U_j) - c)W(\gamma_{s,j}) \quad \text{(from (3.17))}.$$

Thus, the expected payoff that a primary obtains when channel state is j, at independent set I_s,

$$M_s(f_j(U_j) - c)W(\gamma_{s,j}) = P_j^* \quad \text{(from(3.23))}.$$

Hence, payoff at each node of independent set $I_s, s \leq d_j$ is

$$(f_j(U_j) - c)W(\gamma_{s,j}) = p_{s,j} - c = \frac{P_j^*}{M_s}. \tag{3.71}$$

From Lemma 3.1, the best response penalty set is $[L_{s,j}, U_j]$ under $t_{s,k}$ $k = 1, \ldots, n$ at node D. Thus, for any $x \notin [U_j, L_{s,j}]$, payoff is atmost equal to (3.71). This completes case i.

Case ii: At independent set I_s, where$d_1 \geq s > d_j$.

Note from Lemma 3.7 that $d_i \geq d_{i+1}$. Thus, in this case we must have $k = \max\{i \in \{1, \ldots, j-1\} : s \leq d_i\}$. As $s \leq d_1$, hence this case arises only when $j \geq 2$. So, we have $L_{s,k} \geq U_j, j > k$ from Lemma 3.16.

Now, $\gamma_{s,k+1} = 0$ as $s > d_{k+1}$ from (3.58) and thus, $W(\gamma_{s,k+1}) = 1$. Thus, the expected payoff to a primary at $L_{s,k}$, when the channel state is j, is $(f_j(L_{s,k}) - c)$. So, any penalty less than $L_{s,k}$ will fetch a strictly lower payoff compared to penalty $L_{s,k}$ at any node at I_s. Hence, it is enough to show that if a primary chooses penalty in the interval $[L_{s,k}, v]$ at a node of I_s, then its payoff will be strictly less than P_j^*/M_s.

If $x \in [L_{s,k}, v]$ then x must belong to interval $[L_{s,r}, L_{s,r-1}]$ for some $r \leq k$, with $L_{s,0} = v$. Without loss of generality, we can assume that $x \in [L_{s,i}, L_{s,i-1}]$ where $i \leq k$. From Corollary 3.1 $t_{s,i} > 0$, since $t_{s,k} > 0$; thus x is a best penalty response for channel state i by Lemma 3.1. The expected payoff to a primary, when it selects penalty x at channel state i at a node $D \in I_s$, is given by

$$(f_i(x) - c)P(A) = p_{s,i} - c = (f_i(U_i) - c)W(\gamma_{s,i}) \tag{3.72}$$

where, $P(A)$ is the probability of winning, when a primary selects penalty x at channel state i, at any node $D \in I_s$.

Since $s \leq d_i$, thus from (3.71) we obtain for state i

$$P_i^* = M_s(f_i(U_i) - c)W(\gamma_{s,i}) = M_s(f_i(x) - c)P(A). \tag{3.73}$$

Since $x \geq L_{s,i}$ and $f_i(L_{s,i}) > c$ from (3.26), thus $f_i(x) > c$. Since probability of winning only depends on the penalty selected by a primary, thus, when a primary selects penalty x at node $D \in I_s$, at channel state j, its expected payoff is

$$(f_j(x) - c)P(A) = \frac{P_i^*}{M_s}\frac{f_j(x) - c}{f_i(x) - c} \quad \text{(from(3.73))}. \tag{3.74}$$

Since $1 \leq d_i$, thus expected payoff at any node of I_1 at channel state i is given by (3.19) and (3.71)

$$p_{1,i} - c = (f_i(L_{1,i}) - c)W(\gamma_{1,i+1}) = \frac{P_i^*}{M_1}. \tag{3.75}$$

Again since $1 \leq d_j$, thus, at any node of independent set I_1, maximum expected payoff obtained by a primary at channel state j is P_j^*/M_1 as given in (3.71). Expected payoff at $L_{1,i}$ at channel state j is

$$(f_j(L_{1,i}) - c)W(\gamma_{1,i+1}) \leq \frac{P_j^*}{M_1}. \tag{3.76}$$

If $s \leq d_{i+1}$, then $L_{s,i} = U_{i+1}$; on the other hand if $s > d_{i+1}$, then by Lemma 3.16, $L_{s,i} \geq U_{i+1}$. Hence, $x \geq L_{s,i} \geq U_{i+1}$. Also, note that $i < j$ by definition of i. Since

$L_{1,i} = U_{i+1}$ (as $1 \le d_{i+1}$); hence, at penalty x, at channel state j and at a node $D \in I_s$, expected payoff to a primary is

$$\le \frac{P_j^*}{M_s} \frac{(f_j(x) - c)(f_i(U_{i+1}) - c)}{(f_j(U_{i+1}) - c)(f_i(x) - c)} \quad \text{(from (3.75)\&(3.76))}$$
$$\le \frac{P_j^*}{M_s} \quad \text{(by (3.2) as } x \ge U_{i+1}, i < j, f_i(U_{i+1}) > c). \tag{3.77}$$

Case iii: $s > d_1$.

We have from (3.59)

$$M_1 W(\gamma_{1,1}) = \ldots = M_{d_1} W(\gamma_{d_1,1}) \ge M_s \quad s > d_1. \tag{3.78}$$

Since $1 \le d_j$ thus the maximum expected payoff at v is upper bounded by P_j^*/M_1 by (3.71). But, expected payoff to a primary at channel state j at v at any node of I_1 is

$$(f_j(v) - c)W(\gamma_{1,1}) \le \frac{P_j^*}{M_1}. \tag{3.79}$$

Since the expected payoff a primary can attain at a node is at most $f_j(v) - c$ at channel state j. Thus, a primary's expected payoff at any node $D \in I_s$ is always upper bounded by

$$f_j(v) - c \le \frac{M_1}{M_s}(f_j(v) - c)W(\gamma_{1,1}) \quad \text{(from(3.78))}$$
$$\le \frac{P_j^*}{M_s} \quad \text{(from (3.79))}. \tag{3.80}$$

Case iv: At any independent set other than $I_1, \ldots, I_d$:

From (3.71), (3.77) and (3.80), at any node at independent set $I_s, s > d_j$, we obtain that maximum expected payoff a primary can obtain for state j-

$$\le \frac{P_j^*}{M_s}. \tag{3.81}$$

The graph we have considered, is a d-partite graph (Section 3.3.2) . Now, consider an independent set I which contains $m_s(I)$ number of nodes from $I_s, s = 1, \dots, d$. Then at channel state j, expected payoff at independent set I is sum of all payoffs at all the nodes contained in I. Hence, from (3.81)

$$\begin{aligned}\text{Expected Payoff at } I &\leq \sum_{s=1}^{d} \frac{P_j^*}{M_s} m_s(I) \\ &= P_j^* \sum_{s=1}^{d} \frac{m_s(I)}{M_s} \\ &\leq P_j^* \quad (\text{from}(3.11)).\end{aligned}$$

Thus, at any independent set I, expected payoff to a primary at channel state j is at most P_j^* for any selection of penalty. From case (i) a primary attains P_j^* at $I_s, s \leq d_j$ following the strategy profile. Hence, the result follows. □

3.A.3 Proof of Lemmas 3.8 and 3.9

Throughout this section we use $|I_i|$ and $M_i, i \in \{1, \dots, d\}$ ($|\bar{I}_i|$ and $\bar{M}_i, i = 1, \dots, \bar{d}$ respv.) interchangeably.

Proof of Lemma 3.8

First we show that $M_1 = \bar{M}_1$.

Let $M_1 \neq \bar{M}_1$. Without loss of generality assume that $M_1 > \bar{M}_1$. Let $\bar{I}_1$ consists of $m_s(\bar{I}_1)$ number of nodes from I_s. Then

$$\sum_{s=1}^{d} \frac{m_s(\bar{I}_1)}{M_s} \geq \sum_{s=1}^{d} \frac{m_s(\bar{I}_1)}{M_1} = \frac{\bar{M}_1}{M_1} > 1. \tag{3.82}$$

which contradicts (3.11).

Suppose that $M_j \neq \bar{M}_j$ for some smallest index $j \in \{2, \ldots, d\}$. Without loss of generality, we assume that $M_j < \bar{M}_j$. By the definition of j, $M_k = \bar{M}_k$ for $k < j$, thus $\sum_{k=1}^{j-1} M_k = \sum_{k=1}^{j-1} \bar{M}_k$. Note that

$$M_{j-1} = \bar{M}_{j-1} \geq \bar{M}_j > M_j. \tag{3.83}$$

We consider two possible scenarios:

Case i: $\bar{I}_k, k \in \{1, \ldots, j-1\}$ does not contain node from I_s $s \geq j$.

Since $\sum_{k=1}^{j-1} |\bar{I}_k| = \sum_{k=1}^{j=1} \bar{M}_k = \sum_{k=1}^{j-1} M_k$, thus, $\bar{I}_j$ must consist of nodes of only $I_s, s \geq j$. Let $\bar{I}_j$ consist of $m_s(\bar{I}_j)$ nodes of I_s. Then,

$$\sum_{k=j}^{d} \frac{m_k(\bar{I}_j)}{M_k} \geq \sum_{k=j}^{d} \frac{m_k(\bar{I}_j)}{M_j} = \frac{|\bar{I}_j|}{|I_j|} > 1 \tag{3.84}$$

which is not possible by (3.11).

Case ii:$\bar{I}_k$ contains at least one node from I_s $s \geq j$ for some $k \in \{1, \ldots, j-1\}$.

Let $\bar{I}_k$ consist of $m_i(\bar{I}_k)$ number of nodes from I_i. Since $M_{j-1} > M_j$, thus, $\frac{m_s(\bar{I}_k)}{M_i} < \frac{m_s(\bar{I}_k)}{M_s}$ for any $s \leq j$ and $i > j$. By (3.11) for each $k \in \{1, \ldots, j-1\}$ we have

$$\sum_{i=1}^{d} \frac{m_i(\bar{I}_k)}{M_i} \leq 1$$
$$\sum_{k=1}^{j-1} \sum_{i=1}^{d} \frac{m_i(\bar{I}_k)}{M_i} \leq j-1. \tag{3.85}$$

Since $\bar{I}_k$ s are disjoint, thus, $|\bar{I}_1 \cup \ldots \bar{I}_{j-1}| = \sum_{k=1}^{j-1} \bar{M}_k = \sum_{k=1}^{j-1} M_k$. Thus, $\sum_{i=1}^{j-1} M_i = \sum_{k=1}^{j-1} \sum_{i=1}^{d} m_i(\bar{I}_k)$. Hence,

$$\sum_{i=j}^{d} \sum_{k=1}^{j-1} m_i(\bar{I}_k) = \sum_{i=1}^{j-1} (M_i - \sum_{k=1}^{j-1} m_i(\bar{I}_k)). \tag{3.86}$$

Since $\bar{I}_k$ contains at least one node from $I_s, s \geq j$, thus, the above expression is strictly positive. Note that

$$\sum_{k=1}^{j-1}\sum_{i=1}^{d}\frac{m_i(\bar{I}_k)}{M_i} \geq \sum_{k=1}^{j-1}(\sum_{i=1}^{j-1}\frac{m_i(\bar{I}_k)}{M_i} + \sum_{i=j}^{d}\frac{m_i(\bar{I}_k)}{M_j})$$

$$= \sum_{i=1}^{j-1}\sum_{k=1}^{j-1}\frac{m_i(\bar{I}_k)}{M_i} + \sum_{i=j}^{d}\sum_{k=1}^{j-1}\frac{m_i(\bar{I}_k)}{M_j}$$

$$> \sum_{i=1}^{j-1}\sum_{k=1}^{j-1}\frac{m_i(\bar{I}_k)}{M_i} + \sum_{i=1}^{j-1}\frac{M_i - \sum_{k=1}^{j-1} m_i(\bar{I}_k)}{M_i}$$

(from (3.86)and $M_i > M_j, i < j$)

$$= \sum_{i=1}^{j-1}\frac{M_i}{M_i} = j - 1. \tag{3.87}$$

which contradicts (3.85), hence this case can not arise. Hence, the result follows. □

Proof of Lemma 3.9

Let the lowest index be j such that $I_j \cap \bar{I}_k \neq \Phi$ but $|I_j| \neq \bar{I}_k$. Thus, I_j contains at least one node from I_k. Without loss of generality we can assume that $|I_j| < |\bar{I}_k|$.

Since $|I_k| = |\bar{I}_k|$ for all k by Lemma 3.8, thus, $M_k > M_j$. Let $k_1 = \max\{i \in \{1, \ldots, j-1\} : M_i > M_j\}$. Let I_i consists of $m_s(I_i)$ number of nodes from $\bar{I}_s$. Thus,

$$\sum_{i=1}^{k_1}|I_i| = \sum_{s=1}^{d}\sum_{i=1}^{k_1} m_s(I_i)$$

$$\sum_{i=1}^{k_1}\sum_{s=k_1+1}^{d} m_s(I_i) = \sum_{i=1}^{k_1} M_i - \sum_{i=1}^{k_1}\sum_{s=1}^{k_1} m_s(I_i) \tag{3.88}$$

Note that LHS of (3.88) is always non-negative. Now, we will show that (3.88) is strictly positive. If it is not strictly positive then we must have

$$\sum_{i=1}^{k_1} M_i = \sum_{i=1}^{k_1}\sum_{s=1}^{k_1} m_s(I_i). \tag{3.89}$$

But RHS of (3.89) is equal to

$$|(\bar{I}_1 \cup \bar{I}_2 \ldots \cup \bar{I}_{k_1}) \cap (I_1 \cup I_2 \ldots \cup I_{k_1})|. \tag{3.90}$$

and LHS of (3.89) is equal to

$$|\bar{I}_1 \cup \ldots \cup \bar{I}_{k_1}| = |I_1 \cup \ldots \cup I_{k_1}|. \tag{3.91}$$

Thus,

$$I_1 \cup \ldots \cup I_{k_1} = \bar{I}_1 \cup \ldots \cup \bar{I}_{k_1}. \tag{3.92}$$

But I_j contains at least one node from $\bar{I}_l$ and $k \leq k_1 < j$. Thus, I_j contains at least one node in common with $I_1 \cup \ldots \cup I_{k_1}$ which is not possible since I_js are disjoint. Thus (3.88) is strictly positive. Thus, there must exist a $i \in \{1, \ldots, k_1\}$ such that I_i contains at least one node from $\bar{I}_s$ $s > k_1$. Since $i \leq k_1$ and $s > k_1$, thus, $|\bar{I}_s| = |I_s| < |I_i|$. Hence, we have found a $i < j$ such that I_i contains at least one node from $\bar{I}_s$ such that $|\bar{I}_s| < |I_i|$ which contradicts the definition of j. Hence, the result follows. □

3.A.4 Proof of Theorem 3.5 (Section 3.3.7)

In order to prove Theorem 3.5 we must consider all symmetric NE strategy profiles which need not be of the form (3.13); this precludes the use of the results in section 3.3.4. First, we characterize some properties that any symmetric NE strategy must follow (Lemmas 3.18, 3.20) in Appendix 3.A.4. Then we deduce some important properties (Lemma 3.22 and 3.23) that any NE strategy profile must satisfy in a linear graph in Appendix 3.A.4. We then use those properties to prove Theorem 3.5.

Properties of any symmetric NE strategy profile (Lemmas 3.18 and 3.20)

In order to prove Lemma 3.18 we state and prove Observations 3.4, and 3.5 and Lemma3.17. We subsequently state and prove Lemma 3.19 to prove Lemma 3.20.

We start with some notations, which we use throughout.

Definition 3.14. $u_{s,i,max}$ denotes the maximum expected payoff under an NE strategy for state i at node s [13] .

Recall from (3.8) that the channel is offered at node a when the state is j with probability $\alpha_{a,j}$. With slight abuse of notation, we define $\gamma_{a,i}$ for node a in the following manner:

$$\gamma_{a,i} = \sum_{j=i}^{n} \alpha_{a,j} \tag{3.93}$$

Thus, $\gamma_{a,i}$ denotes the probability that the channel is offered at node a when the state is higher or equal to i.

Since we have to consider all NE strategy profiles which may not be of the form (3.13), thus, the payoff, upper and lower endpoint need not be the identical at each node of I_s at a given channel state. By Lemma 3.1 if $\alpha_{a,j}$ is known then the above parameters can be obtained using Lemma 2.2 with $\alpha_{a,j}$ in place of q_j. With slight abuse of notation we denote $p_{a,i}, L_{a,i}$ and $U_{a,i}$ for node a i.e. for $i = 1, \ldots, n$

$$p_{a,i} = c + (f_i(U_{a,i}) - c)W(\gamma_{a,i})(\text{from (3.93)\&(3.14)}) \tag{3.94}$$

$$L_{a,i} = g_i(\frac{p_{a,i} - c}{W(\gamma_{a,i+1})} + c), U_{a,i} = L_{a,i-1}, L_{a,0} = v \tag{3.95}$$

[13]Even if node a is selected with probability 0 when the channel state is i, we can still defined $u_{a,i,max}$ as the maximum expected payoff that a primary would have obtained if it would select node a

By Lemma 3.1 $p_{a,i} - c$ is the expected payoff at node a when the channel state is i if node a is selected with positive probability. By Lemma 3.1 a primary selected penalty from the interval $[L_{s,j}, U_{a,j}]$ when the channel state is j using the distribution (2.5) with $\alpha_{a,j}$ in place of q_j.

Now we state some observations which we use throughout.

Observation 3.4. At node a, $\gamma_{a,k} = \gamma_{a,k_1} + \sum_{i=k}^{k_1-1} \alpha_{a,i}$ where $n \geq k_1 > k$.

Observation 3.4 readily follows from (3.93). Since from (3.93)

$$\gamma_{a,k} = \sum_{i=k}^{k_1-1} \alpha_{a,i} + \sum_{i=k_1}^{n} \alpha_{a,i} = \sum_{i=k}^{k_1-1} \alpha_{a,i} + \gamma_{a,k_1} \quad \text{(from (3.93))}$$

Similar to observation 3.2, using observation 3.4, (3.95) and (3.94) we obtain

Observation 3.5. At node a, $U_{a,j} = L_{a,j}$ for $j \in \{1, \ldots, n\}$ iff $t_{a,j} = 0$. $U_{a,j} = L_{a,k}$ iff $t_{a,i} = 0 \ \forall k < i < j$. Hence, $U_{a,j} = v$ iff $t_{a,k} = 0 \ \forall k < j$.

Lemma 3.17. *Maximum expected payoff under the NE strategy profile at a node s is obtained at $L_{s,i}$ when channel states are i and $i+1$. When the channel state is 1, primary attains its maximum expected payoff at v at any node.*

Note that if $\alpha_{s,i} > 0$ then by Lemma 3.1 $L_{s,i}$ is a best penalty response at channel state i. Here we show that even if $\alpha_{s,i} = 0$, then the maximum expected payoff is obtained at $L_{s,i}$ at node s under any NE strategy profile. The above proof readily follows from Assumption 3.1. Hence, we omit it here.

Now, we provide expressions for $u_{s,i,max}, u_{s,i+1,max}$ for node s, $i \in \{1, \ldots, n-1\}$ in terms of $L_{s,i}$which we use to prove Lemmas 3.18 and 3.20.

Since v is a best response at channel state 1 at any node in the network by Lemma 3.17, thus,

$$u_{s,1,max} = (f_1(v) - c)W(\gamma_{s,1}) \tag{3.96}$$

By (3.95) expected payoff at $L_{s,i}$ is

$$(f_i(L_{s,i}) - c)W(\gamma_{s,i+1}) = u_{s,i,max} \tag{3.97}$$

$$u_{s,i+1,max} = (f_{i+1}(L_{s,i}) - c)W(\gamma_{s,i+1}) \tag{3.98}$$

Lemma 3.18. *i) For, $i \in \{1, \ldots, n-1\}$, if $u_{s,i,max} \geq u_{r,i,max}$ and $\gamma_{s,i} \leq \gamma_{r,i}, \alpha_{r,i} < \alpha_{s,i}$, then $u_{s,i+1,max} > u_{r,i+1,max}$.*

ii) If $u_{s,i,max} \geq u_{r,i,max}$ and $\gamma_{s,i} < \gamma_{r,i}$, $\alpha_{s,i} \geq \alpha_{r,i}$, then $u_{s,i+1,max} > u_{r,i+1,max}$.

Proof. First we show part (i). Proof of part (ii) follows by simple modification of the proof of part (i).

Suppose, the statement is false, i.e. $u_{s,i+1,max} \leq u_{r,i+1,max}$ for some s and r. As $\gamma_{s,i} \leq \gamma_{r,i}$ thus,

$$\begin{aligned} &\gamma_{s,i+1} + \alpha_{s,i} \leq \gamma_{r,i+1} + \alpha_{r,i} \quad \text{(by observation 3.4)} \\ &\gamma_{s,i+1} < \gamma_{r,i+1} \quad (\text{since } \alpha_{s,i} > \alpha_{r,i}) \end{aligned} \tag{3.99}$$

Now, as $u_{s,i+1,max} \leq u_{r,i+1,max}$, hence from (3.98)

$$\begin{aligned} (f_{i+1}(L_{s,i}) - c)W(\gamma_{s,i+1}) &\leq (f_{i+1}(L_{r,i}) - c)W(\gamma_{r,i+1}) \\ \frac{W(\gamma_{s,i+1})}{W(\gamma_{r,i+1})} &\leq \frac{f_{i+1}(L_{r,i}) - c}{f_{i+1}(L_{s,i}) - c} \end{aligned} \tag{3.100}$$

Since $\gamma_{r,i+1} > \gamma_{s,i+1}$ (from (3.99)) $W(\cdot)$ is strictly decreasing, thus $W(\gamma_{r,i+1}) < W(\gamma_{s,i+1})$. Since $f_{i+1}(\cdot)$ is strictly increasing, thus we obtain from (3.100) $L_{s,i} < L_{r,i}$. Now, from

(3.100) and the fact that $f_i(L_{s,i}) > c$, we obtain

$$\frac{W(\gamma_{s,i+1})}{W(\gamma_{r,i+1})} < \frac{f_i(L_{r,i}) - c}{f_i(L_{s,i}) - c} \quad \text{(from (3.2) and } L_{s,i} < L_{r,i})$$

$$(f_i(L_{s,i}) - c)W(\gamma_{s,i+1}) < (f_i(L_{r,i}) - c)W(\gamma_{r,i+1})$$

$$u_{s,i,max} < u_{r,i,max} \quad \text{(from (3.97))}$$

which contradicts the fact that $u_{s,i} \geq u_{r,i}$.

Note that, if $\gamma_{s,i} < \gamma_{r,i}$ and $\alpha_{s,i} \geq \alpha_{r,i}$, then we also obtain (3.99) by simple algebraic manipulation, hence the proof of part (ii) is exactly similar to the proof of part (i). □

We use the following result in proving lemma 3.20.

Lemma 3.19. *Suppose* $u_{s,k,max} > u_{r,k,max}, \gamma_{s,k} < \gamma_{r,k}$. *Let,* $i = \min\{j \in \{k, \ldots, n\} : \alpha_{s,j} < \alpha_{r,j}\}$, *then* $\forall j$ *such that* $k \leq j \leq i$, *we must have* $u_{s,j,max} > u_{r,j,max}$.

Proof. Suppose the statement is false. So, there exists a j such that $k < j \leq i$, $u_{s,j,max} \leq u_{r,j,max}$[14]. Since $u_{s,k} > u_{r,k}$, thus, there must exist a $k_1 \in \{k, \ldots, j-1\}$, such that $u_{s,k_1,max} > u_{r,k_1,max}$ but $u_{s,k_1+1,max} \leq u_{r,k_1+1,max}$. Because otherwise we have $u_{s,j,max} > u_{r,j,max}$.

Since $\gamma_{s,k} < \gamma_{r,k}$, thus from observation 3.4

$$\gamma_{s,k_1} + \sum_{j=k}^{k_1-1} \alpha_{s,j} < \gamma_{r,k_1} + \sum_{j=k}^{k_1-1} \alpha_{r,j} \tag{3.101}$$

By definition of i, $\alpha_{s,k_2} \geq \alpha_{r,k_2}$ for $k \leq k_2 < i$, since $k_1 < j$ and $j \leq i$, thus $\alpha_{s,k_2} \geq \alpha_{r,k_2}$ $\forall k_2 \in \{k, \ldots, k_1\}$. Hence, from (3.101), we have $\gamma_{s,k_1} < \gamma_{r,k_1}$.

But $\alpha_{s,k_1} \geq \alpha_{r,k_1}$ and $u_{s,k_1,max} > u_{r,k_1,max}$, hence by lemma 3.18 we have $u_{s,k_1+1,max} > u_{r,k_1+1,max}$ which leads to a contradiction. □

[14] Note that the statement is true at state k, since $u_{s,k,max} > u_{r,k,max}$

Lemma 3.20. *Suppose, $u_{s,j,max} > u_{r,j,max}$, then there must exist a state $i \in \{1, \dots, n\}$ such that $u_{s,i,max} > u_{r,i,max}$ but $\alpha_{s,i} < \alpha_{r,i}$.*

Proof. First we show that the statement is true when $u_{s,1,max} > u_{r,1,max}$ (case i) and then we show when $u_{s,1,max} \leq u_{r,1,max}$ (case ii); which completes the proof.

Case 1: Suppose $u_{s,1,max} > u_{r,1,max}$. Since, $W(\cdot)$ is strictly decreasing, thus from (3.96) we obtain $\gamma_{s,1} < \gamma_{r,1}$. Thus, from (3.93), there must exist $k = \min\{i \in \{1, \dots, n\} : \alpha_{s,i} < \alpha_{r,i}\}$. By lemma 3.19, $u_{s,j,max} > u_{r,j,max}$ $\forall j$ such that $1 \leq j \leq k$. Since at k, $\alpha_{s,k} < \alpha_{r,k}$, $u_{s,k,max} > u_{r,k,max}$ thus, the statement is true for k.

Case 2 Now, assume that $u_{s,1,max} \leq u_{r,1,max}$. Hence, it is obvious that $j \neq 1$. So, we must have $k = \min\{i \in \{1, \dots, j-1\} : u_{s,i,max} \leq u_{r,i,max}, u_{s,i+1,max} > u_{r,i+1,max}\}$. Note that if $\gamma_{s,k+1} < \gamma_{r,k+1}$, then from (3.93) there must exist $i = \min\{j : \{k+1, \dots, n\} : \alpha_{s,j} < \alpha_{r,j}\}$. Since $u_{s,k+1,max} > u_{r,k+1,max}$, thus by lemma 3.19 at i, $u_{s,i} > u_{r,i}$ but $\alpha_{s,i} < \alpha_{r,i}$. Thus, the result is true for i if we show that $\gamma_{s,k+1} < \gamma_{r,k+1}$. Now we complete the proof by showing that $\gamma_{s,k+1} < \gamma_{r,k+1}$.

Suppose that $\gamma_{s,k+1} \geq \gamma_{r,k+1}$. By definition of k, $u_{s,k} \leq u_{r,k}$, hence we obtain from (3.97)

$$(f_k(L_{s,k}) - c)W(\gamma_{s,k+1}) \leq (f_k(L_{r,k}) - c)W(\gamma_{r,k+1}) \tag{3.102}$$

Since $u_{s,k+1,max} > u_{r,k+1,max}$, thus from (3.98)

$$(f_{k+1}(L_{s,k}) - c)W(\gamma_{s,k+1}) > (f_{k+1}(L_{r,k}) - c)W(\gamma_{r,k+1}) \tag{3.103}$$

Since$\gamma_{s,k+1} \geq \gamma_{r,k+1}$ and $W(\cdot)$ is strictly increasing, hence, $L_{r,k} < L_{s,k}$ from (3.103). Thus from (3.103)

$$\frac{W(\gamma_{s,k+1})}{W(\gamma_{r,k+1})} > \frac{f_k(L_{r,k}) - c}{f_k(L_{s,k}) - c} \tag{3.104}$$

$$\text{(from(3.2)as } c < f_k(L_{r,k}), L_{s,k} > L_{r,k})$$

But (3.104) contradicts (3.102). Hence, $\gamma_{s,k+1} < \gamma_{r,k+1}$. □

Properties of any symmetric NE strategy profile in a linear graph (Lemmas 3.22 and 3.23)

We consider a linear graph (fig. 3.1) consisting of M number of nodes. We use the properties of linear graph and a NE strategy profile to prove the results. First, we state and prove Lemma 3.21. Subsequently, we show that under an NE strategy profile the maximum expected payoff to a primary at a channel state at each node of $I_k, k \in \{1, 2\}$ must be equal (Lemma 3.22). Then, we show that under an NE strategy profile nodes of $I_k, k = \{1, 2\}$ are selected with equal probability (Lemma 3.23). Finally, we show theorem 3.5 using lemmas 3.22 and 3.23.

In order to prove Lemma 3.21 we state and prove Observations 3.6,3.7,and 3.8.

Observation 3.6. An NE independent set selection strategy profile only selects a maximal independent set with positive probability.

Proof. Suppose not; so an independent set I has been chosen with positive probability under an NE strategy profile, but it is not maximal which in turn implies that there exists a node z, such that $\bar{I} = I \cup \{z\}$ is an independent set. Since $\sum_{j=1}^{n} q_j = q < 1$ (from (3.9)), hence at node z, primary 1 will attain at least a payoff of $(f_j(v) - c)W(q) > 0$ for state j when the primary selects the highest possible penalty v. Hence, a primary can attain strictly higher payoff by choosing independent set $\bar{I}$ compared to I. Hence, the result follows. □

Observation 3.6 enables us to focus only on the maximal independent sets for an NE strategy profile.

Observation 3.7. For a maximal independent set I-

(i) If $s \in I$, but $s+2 \notin I$, then $s+3 \in I$ for some $s \in V$.

(ii) If $s+2 \in I$, but $s \notin I$, then $s-1 \in I$ for some $s \in V$.

Proof. part (i): If it is not then $I \cup \{s+2\}$ is maximal, since $s+1 \notin I$ (as $s \in I$ and I is an independent set); which contradicts that I is maximal.

part (ii): If it is not then $I \cup \{s\}$ is an independent set since $s-1 \notin I, s+1 \notin I$ which contradicts that I is maximal. □

Observation 3.8. Consider an independent set I, such that $s \in I$, but $s+2 \notin I$, for some $s \in \{1, \ldots, M-2\}$; NE independent selection strategy profile selects I with positive probability, the following condition must be satisfied for $s \leq M-3$

$$u_{s,j,max} \geq u_{s+1,j,max}, u_{s+3,j,max} \geq u_{s+2,j,max} \quad \text{for } j \in \{1, \ldots, n\} \tag{3.105}$$

Proof. Note that if $s = M-2$, then I does not contain node $M, M-1$, hence I is not maximal. Thus, an NE strategy profile can not select I by Observation 3.6. Hence, we must have $s \leq M-3$.

If $u_{s,j,max} < u_{s+1,j,max}$, then we can replace node s with node $s+1$ and we obtain an independent set $\bar{I}$ as $s+2 \notin I$. But, we can get strictly higher payoff at the independent set $\bar{I}$, as all the nodes are same except s and $u_{s,j,max} < u_{s+1,j,max}$. This contradicts that NE strategy profile selects I with positive probability.

Similarly if $u_{s+3,j,max} < u_{s+2,j,max}$ then we obtain an independent set by replacing node $s+3$ with $s+2$ in I and can get a strictly higher payoff at that independent set. □

Lemma 3.21. *i) If $u_{s,k,max} > u_{s+2,k,max}$, then $u_{1,i,max} > u_{3,i,max}$ for some $i \in \{1, \ldots, n\}$. ii)If $u_{s,k,max} < u_{s+2,k,max}$, then $u_{M,i,max} > u_{M-2,i,max}$ for some $i \in \{1, \ldots, n\}$.*

Proof. We prove (i). The proof of (ii) will be similar to the proof of part (i) by symmetry. Since $u_{s,k,max} > u_{s+2,k,max}$, hence, from Lemma 3.20, there exists $i \in \{1, \ldots, n\}$ such that $u_{s,i,max} > u_{s+2,i,max}$, but $\alpha_{s,i} < \alpha_{s+2,i}$. Hence, there must exist a maximal independent set I such that $s \notin I$, but $s+2 \in I$, which is chosen with positive probability in an NE strategy profile when the channel state is i. But, as I is maximal, thus, $s-1 \in I$ from Observation 3.7. Also from Observation 3.8, we must have

$$u_{s-1,i,max} \geq u_{s,i,max}, u_{s+2,i,max} \geq u_{s+1,i,max} \tag{3.106}$$

Since $u_{s,i,max} > u_{s+2,i,max}$, thus, from (3.106), we obtain

$$u_{s-1,i,max} > u_{s+1,i,max}$$

Hence, we obtain $u_{s-1,i,max} > u_{s+1,i,max}$ for some $i \in \{1, \ldots, n\}$ only using the fact that $u_{s,k,max} > u_{s+2,k,max}$.Thus, by recurrence on the index s we obtain the result. □

Next Lemma characterizes that under an NE strategy profile maximum expected payoff must be equal at every node of $I_k, k \in \{1, 2\}$.

Lemma 3.22. *Under NE strategy profile, we must have* $\forall j \in \{1, \ldots, n\}, \forall s, r \in I_k, k \in \{1, 2\}$

$$u_{s,j,max} = u_{r,j,max} \tag{3.107}$$

Proof. First, we prove $\alpha_{1,i} \geq \alpha_{3,i}$, $\alpha_{M,i} \geq \alpha_{M-2,i}, \forall i$.

We show that $\alpha_{1,i} \geq \alpha_{3,i}$ $\forall i$; by symmetry we get $\alpha_{M,i} \geq \alpha_{M-2,i}$. Suppose, $\alpha_{1,j} < \alpha_{3,j}$ for some $j \in \{1, \ldots, n\}$. Then, there must exist a maximal independent set I such that node $1 \notin I$, but node $3 \in I$; which is not possible (figure 3.1).

Now, we are ready to prove the lemma. Suppose the statement is false. So, we must have $u_{s,j,max} > u_{r,j,max}$ for some $j \in \{1, \ldots, n\}$ and $s, r \in I_k, k \in \{1, 2\}$. We rule out $s < r$, by symmetry it follows that $s > r$; which completes the proof.

Since $u_{s,j,max} > u_{r,j,max}$, we must have some $a \in \{s, \ldots, r-2\}$, such that $u_{a,j,max} > u_{a+2,j,max}$. Otherwise, $u_{s,j,max} \leq u_{r,j,max}$ since $r - s = 2z$ for some positive integer z. But, this entails that $u_{1,i,max} > u_{3,i,max}$ by Lemma 3.21 for some $i \in \{1, \ldots, n\}$, which in turn entails that $\alpha_{1,b} < \alpha_{3,b}$ for some $b \in \{1, \ldots, n\}$ (Lemma 3.20). But, we have already proved that $\alpha_{1,b} \geq \alpha_{3,b} \forall b \in \{1, \ldots, n\}$. Hence, the result follows. □

Next, lemma shows that under an NE strategy profile nodes in $I_k, k \in \{1, 2\}$ are selected with equal probability.

Lemma 3.23. *For state* $z = 1, \ldots, n$, $\alpha_{z,i} = \alpha_{z,j}$ *where* $i, j \in I_s, s \in \{1, 2\}$.

Proof. Let, k be the lowest channel state, for which the statement is false. Thus, there must exist node $a, b \in I_s, s \in \{1, 2\}$ such that, $\alpha_{a,k} > \alpha_{b,k}$, but $u_{a,k,max} = u_{b,k,max}$ (by Lemma 3.22). First we rule out that $k = n$ (case i) and then $k < n$ (case ii).

Case 1 Suppose, $k = n$.

By definition of k, $\alpha_{a,j} = \alpha_{b,j}\ \forall j < k$, thus from Observation 3.4, we have $\gamma_{a,1} > \gamma_{b,1}$. Since $W(\cdot)$ is strictly decreasing function, thus from (3.96) we obtain $u_{a,1,max} < u_{b,1,max}$; which contradicts (3.107).

Case 2 Now, suppose $k < n$.

Since $u_{a,1,max} = u_{b,1,max}$ by Lemma 3.22, thus from (3.96) $\gamma_{a,1} = \gamma_{b,1}$. Thus from Observation 3.4

$$\gamma_{a,k} + \sum_{j=1}^{k-1} \alpha_{a,j} = \gamma_{b,k} + \sum_{j=1}^{k-1} \alpha_{b,j} \tag{3.108}$$

By definition of k, we have $\alpha_{a,j} = \alpha_{b,j}\ \forall j \leq k-1$. Hence, from (3.108), $\gamma_{a,k} = \gamma_{b,k}$. Since $\alpha_{a,k} > \alpha_{b,k}$, $\gamma_{a,k} = \gamma_{b,k}$, and $u_{a,k,max} = u_{b,k,max}$, hence by Lemma 3.18, we obtain $u_{a,k+1,max} > u_{b,k+1,max}$. This expression again contradicts (3.107). Hence $k \neq n$. $\square$

From Lemma 3.23, we have $\alpha_{s,j} = \alpha_{r,j} = \bar{\alpha}_{k,j}$(let) where $s, r \in I_k\ k \in \{1,2\}\ j = 1,\ldots,n$. From Lemma 3.22, we have $u_{s,j,max} = u_{r,j,max} = \bar{u}_{k,j}$(let).

Proof of Theorem 3.5: First, we will show that for any NE strategy profile $\bar{\alpha}_{k,j}$ we must have $\sum_{k=1}^{2} \bar{\alpha}_{k,j} \geq 1\ \forall j$. Then, we will show that if a primary chooses a maximal independent set other than I_1 and I_2 with positive probability, then we must have $\sum_{k=1}^{2} \bar{\alpha}_{k,j} < 1$, which completes the proof.

Suppose $\sum_{k=1}^{2} \bar{\alpha}_{k,j} < 1$ but it is an NE for some j. Since I_1 and I_2 constitute a partition of V, thus, the expected payoff that any primary at channel state j will get is the following

$$\begin{aligned}
&\sum_{s\in I_1}\bar{\alpha}_{1,j}u_{s,j}+\sum_{r\in I_2}\bar{\alpha}_{2,j}u_{r,j}\\
&=\sum_{k=1}^{2}M_k\bar{\alpha}_{k,j}\bar{u}_{k,j}\quad (\text{since}|I_k|=M_k, u_{s,j}=\bar{u}_{k,j}, s\in I_k)
\end{aligned}\tag{3.109}$$

Consider the following unilateral deviation for primary 1 at channel state j: Primary 1 chooses I_1 with probability $\bar{\alpha}_{1,j}$ and I_2 with probability $1-\bar{\alpha}_{1,j}$. Since $\bar{u}_{k,j}$ remains the same, is strictly positive, and $1-\bar{\alpha}_{1,j}>\bar{\alpha}_{2,j}$, hence primary 1 gets a strictly higher payoff following the above mentioned strategy by (3.109). This contradicts that $\bar{\alpha}_{k,j}$ is an NE distribution.

Next, consider an NE strategy profile which selects a maximal independent set I, which has at least one node both from I_1 and I_2, with positive probability. Hence, there exists a node a such that $a, a+1\notin I$. Since a and $a+1$ are adjacent, hence both can not appear in any independent set $\bar{I}\in\mathcal{I}$ otherwise $\bar{I}$ can not be an independent set. Hence, by valid distribution property, we must have

$$\alpha_{a,j}+\alpha_{a+1,j}\leq 1 \tag{3.110}$$

On the other hand for independent set I, both $a, a+1\notin I$. Since I is chosen with positive probability, hence from (3.110)

$$\alpha_{a,j}+\alpha_{a+1,j}< 1 \tag{3.111}$$

Without loss of generality, we can assume that $a\in I_1$, hence $a+1\in I_2$. Thus, $\alpha_{a,j}=\bar{\alpha}_{1,j}$ and $\alpha_{a+1,j}=\bar{\alpha}_{2,j}$. We have already shown that for any NE strategy profile we must have $\sum_{k=1}^{2}\bar{\alpha}_{k,j}=1$ which contradicts (3.111). Hence, a primary can not choose an independent set which contains at least one node from I_1 and I_2 under an NE strategy; since I_1 and I_2 constitute a partition of V; thus, only subsets of either I_1 or I_2 can be selected with positive probability. Since proper subsets of either I_1 or I_2 are not maximal, they can not be chosen with positive probability under an NE strategy by Observation 3.6. Hence, the result follows. □

3.A.5 Proof of Lemmas 3.12 and 3.13 (Section 3.4.2)

Proof of Lemma 3.12

:In order to prove Lemma 3.12 first, we describe an infinite set of strategy profile SP_{l,r,r_1}. Subsequently, we show that every strategy profile in SP_{l,r,r_1} is an NE.

Note that at a channel state vector J, a linear graph consists of disjoint smaller linear graphs (Fig. 3.18). First, we introduce some notations. Let M_i be the linear graph which starts from node i i.e. the channel is not available at node $i-1$ if $i > 1$ (fig. 3.18), but it is available at node i.

In M_i the two maximal independent sets which partition the set of nodes in M_i are: $I_{1,i}$ which contains the nodes numbered $i, i+2, \ldots$ and $I_{2,i}$ which contains the nodes numbered $i+1, i+3, \ldots$. In figure 3.18, M_i and M_j constitute two disconnected linear graphs. The cardinality of M_i can be an even or odd number depending on the number of consecutive nodes where the channel is available starting from node i. To illustrate the cardinalities of $|M_i|$, consider the linear graph with 4 nodes. Here, $|M_1|$ can take any value in $\{1, 2, 3, 4\}$. When $|M_1| = 1$, then the channel is available at node 1 but not at node 2. When $|M_1| = 2$, then the channel is available at node 1 and 2, but the channel is not available at node 3. Here $I_{1,1} = \{1\}$ and $I_{2,1} = \{2\}$. When $|M_1| = 3$, then the channel is

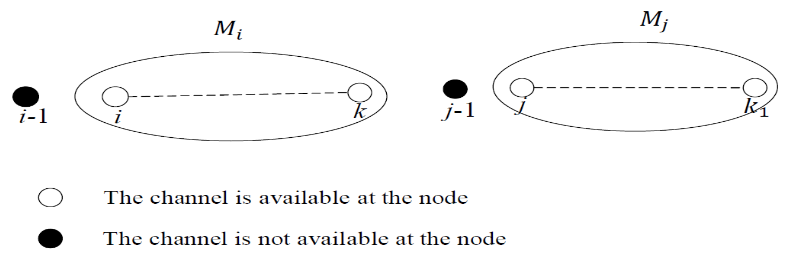


Figure 3.18: Figure shows the linear graph M_i and M_j with $|M_i| = k-i+1$ and $|M_j| = k_1-j+1$. M_i and M_j are disconnected. The maximal independent sets in M_i is $I_{1,i}$ and $I_{2,i}$ where $I_{1,i}$ contains nodes numbered $i, i+2, \ldots$ and $I_{2,i}$ contains nodes numbered $i+1, i+3, \ldots$.

available at nodes $1, 2, 3$, but the channel is unavailable at node 4. Here, $I_{1,1} = \{1, 3\}$ and $I_{2,1} = \{2\}$. When $|M_1| = 4$, the channel of the primary is available at all nodes. Thus, $I_{1,1}$ coincides with I_1 and $I_{2,1}$ coincides with I_2 where $I_1 = \{1, 3\}$ and $I_2 = \{2, 4\}$. Since for a given channel state vector J, the graph G_J can be partitioned into linear graphs M_i (fig. 3.18), thus, a primary only needs to select strategy for each such linear graph. Thus,

Lemma 3.24. *Obtaining an NE strategy profile is equivalent to obtain an NE strategy at each possible mean valid graph M_i, $i = 1, \ldots, M$.*

We use the following result to prove Theorem 3.12.

Observation 3.9. When $|M_i|$ is odd, then the only maximum independent set is $I_{1,i}$, if $|M_i|$ is even, then both $I_{1,i}$ and $I_{2,i}$ are maximum independent sets of M_i.

Note that when $|M_i|$ is even, there can be other maximum independent sets apart from $I_{1,i}$ and $I_{2,i}$[15].

Now, we consider a linear graph with 4 nodes and the channel states are I.I.D. i.e. the channel is at state 1 at a given node is w.p. $q_1 = q = 0.5$. Let $t_{i,j}$ denote the probability

[15]For example, when $|M_1| = 4$, then, the following are maximum independent sets, $\{1, 3\}$, $\{2, 4\}$, and $\{1, 4\}$ where the first two independent sets belong to $I_{1,1}$ and $I_{2,1}$ respectively

of the event that $|M_i| = j$. It is easy to show the following

$$t_{1,3} = t_{2,3}, \quad t_{4,1} = t_{1,1} \tag{3.112}$$

$$t_{3,1} = t_{2,1}, \quad t_{3,2} = t_{1,2}. \tag{3.113}$$

$$t_{2,2} = t_{1,4}, \quad t_{1,2} = 2t_{1,4}, \quad t_{3,2} = 2t_{1,4} \tag{3.114}$$

$$t_{2,1} + t_{1,2} = t_{1,1}, \quad t_{4,1} = t_{3,1} + t_{3,2} \tag{3.115}$$

Now, we describe an uncountable set of strategy profiles parameterized by parameters r and r_1.

Strategy profile SP_{l,r,r_1}: **If $|M_i|$ is odd, then $I_{1,i}$ will be selected w.p. 1. If $|M_1| = 2$, then $I_{1,1}$ will be selected w.p. r and $I_{2,1}$ will be selected w.p. $1-r$. If $|M_1| = 4$ i.e. when the channel is available at all nodes, then I_1 will be selected w.p. r_1 and I_2 will be selected w.p. $1-r_1$. If $|M_2| = 2$, then $I_{1,2}$ will be selected w.p. $\frac{1}{2}$ and $I_{2,2}$ will be selected w.p. $\frac{1}{2}$. If $|M_3| = 2$, then $I_{1,3}$ will be selected w.p. $0.75 + r$ and $I_{2,3}$ will be selected w.p. $0.25 - r$.**

where $r, r_1 \geq 0$ are such that

$$2r + r_1 = 0.75 \quad \& r \leq 0.25 \tag{3.116}$$

Since $0 \leq r \leq 0.25$ and $0 \leq r_1 \leq 0.75$, thus, it is easy to discern that the strategy profile described in SP_{l,r,r_1} constitutes a valid distribution. Note that there are uncountably infinite numbers of r, r_1 satisfying (3.116). Thus, SP_l gives rise an infinite number of strategies.

Proof of Lemma 3.12: We first prove that for every r, r_1 which satisfy (3.116) the strategy profile SP_{l,r,r_1} is an NE.

Towards this end we first show that under the strategy profile SP_{l,r,r_1} the channel is offered by a primary at every node with the same probability.

Node selection probability of node 1 i.e. α_1 is

$$\alpha_1 = t_{1,1} + t_{1,3} + t_{1,2}r + t_{1,4}r_1 \tag{3.117}$$

and node selection probability of node 2 is

$$\alpha_2 = t_{1,2}(1-r) + t_{1,4}(1-r_1) + t_{2,1} + t_{2,3} + t_{2,2}/2 \tag{3.118}$$

Node selection probability of node 3 i.e. α_3 is

$$t_{1,3} + t_{1,4}r_1 + t_{2,2}/2 + t_{3,2}(0.75 + r) + t_{3,1} \tag{3.119}$$

Node selection probability of node 4 is

$$\alpha_4 = t_{1,4} * (1-r_1) + t_{3,2} * (0.25 - r) + t_{4,1} + t_{2,3} \tag{3.120}$$

Note that

$$\begin{aligned} 2t_{1,4}r_1 + 2t_{1,2}r &= t_{1,4}(2r_1 + 4r) \quad \text{(from (3.114))} \\ &= t_{1,4} * 3/2 \quad \text{(from (3.116))} \end{aligned} \tag{3.121}$$

Thus, from (3.117) and (3.118), we obtain that $\alpha_1 - \alpha_2$ is equal to

$$t_{1,1} - t_{1,2} - t_{2,1} + t_{1,3} - t_{2,3} + 2t_{1,2}r + 2t_{1,4}r_1 - t_{1,4} - t_{2,2}/2 \tag{3.122}$$

Note that $t_{1,3} = t_{2,3}$ by (3.112) and $t_{1,1} = t_{2,1} + t_{1,2}$ by (3.115). Since $t_{2,2} = t_{1,4}$ from (3.114), thus it readily follows from (3.122) that $\alpha_1 = \alpha_2$. From (3.119) and (3.118) we obtain $\alpha_2 - \alpha_3$ is equal to

$$t_{2,1} - t_{3,1} + t_{2,3} - t_{1,3} + t_{1,4} + t_{1,2} - 0.75t_{3,2} - rt_{3,2} - rt_{1,2} - 2t_{1,4}r_1 \tag{3.123}$$

Note that $t_{1,3} = t_{2,3}$ by (3.112), $t_{1,2} = t_{3,2}$ and $t_{2,1} = t_{3,1}$ by (3.113). Thus, from (3.123)

$$\alpha_2 - \alpha_3 = t_{1,4} + t_{1,2}/4 - 2rt_{1,2} - 2r_1t_{1,4} \tag{3.124}$$

Also note from (3.114) that $t_{1,4} = t_{1,2}/2$. Thus, $t_{1,4} + t_{1,2}/4 = t_{1,4} * 3/2$. Thus, from (3.121) and (3.124) it readily follows that $\alpha_3 = \alpha_2$. From (3.119) and (3.120) we obtain that $\alpha_3 - \alpha_4$ is equal to

$$t_{1,3} - t_{2,3} + t_{3,1} + t_{3,2} - t_{4,1} + t_{2,2}/2 - t_{1,4} - t_{3,2}/2 + 2t_{1,4}r_1 + 2rt_{3,2} \tag{3.125}$$

Note that $t_{1,3} = t_{2,3}$ by (3.112). Also note that $t_{4,1} = t_{3,1} + t_{3,2}$ by (3.115). Thus, from

$$\alpha_3 - \alpha_4 = t_{2,2}/2 - t_{1,4} - t_{3,2}/2 + 2t_{1,4}r_1 + 2rt_{3,2} \tag{3.126}$$

Since $t_{3,2}/2 = t_{1,4}$ by (3.114), thus, we obtain $t_{1,4} + t_{3,2}/2 - t_{2,2}/2 = t_{1,4} * 3/2$. Since $t_{1,2} = t_{3,2}$ by (3.113), thus, from (3.121) and (3.126) it readily follows that $\alpha_4 = \alpha_3$. Hence, we obtain $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$.

Since node selection probability is identical across the nodes, thus, when all the other primaries select a strategy profile in the set SP_{l,r,r_1}, then, the maximum payoff of primary 1 at a node i is $(f_1(v) - c)(1 - w(\alpha_i))$ by Lemma 3.1 and (3.41) and this is obtained for any penalty in the interval $[L_1, v]$ with α_i in place of q_1 by Lemma 3.1. Hence, the maximum attainable expected payoff of primary 1 at each location is the same since α_i's are identical.

Now, we show that primary 1 does not have any incentive to deviate from a strategy profile for fixed r, r_1 when other primaries also select that strategy profile.

When $|M_i|$ is odd, then by Observation 3.9 $I_{1,i}$ is the only maximum independent set in M_i. Since the maximum attainable expected payoff for primary 1 is the same at every node, thus, the expected payoff at $I_{1,i}$ is the highest for primary 1 when all the other primaries select a strategy profile in SP_{l,r,r_1}. Hence, primary 1 does not have any incentive to deviate from SP_{l,r,r_1} when $|M_i|$ is odd since under SP_{l,r,r_1} primary 1 selects $I_{1,i}$ w.p. 1 when $|M_i|$ is odd.

When $|M_i|$ is even, then $|I_{1,i}| = |I_{2,i}|$. By Observation 3.9 both $I_{1,i}$, $I_{2,i}$ are the maximum independent sets. Since the maximum attainable expected payoff is the same at each node, thus, any strategy profile which randomizes between $I_{1,i}$ and $I_{2,i}$ gives the highest expected payoff to primary 1. Thus, primary 1 does not have any incentive to deviate from SP_{l,r,r_1} when $|M_i|$ is even since under SP_{l,r,r_1} primary 1 only randomizes between $I_{1,i}$ and $I_{2,i}$.

Though we only consider primary 1 since the every strategy in SP_{l,r,r_1} is symmetric, hence, no primary will have any incentive to deviate unilaterally from the strategy profile for a fixed r, r_1. Thus, we show that every r, r_1 which satisfy (3.116), the strategy set in SP_{l,r,r_1} is an NE.

Since there are uncountable number of r, r_1s which satisfy (3.116), hence there are multiple NEs in this setting. □

Proof of Lemma 3.13

We show that SP_{sym} is not a NE strategy profile in the above linear graph with 4 nodes where the channel is in state 1 at a given location w.p 0.5 irrespective of the channel states at other locations. In order to prove the result we use some of the results which we derived in the previous section to prove Lemma 3.12.

First, we point out the how SP_{sym} (described in Section 3.4.2) is different from the class of strategy profile SP_{l,r,r_1} (described in the previous section). Then, we show that SP_{sym} is not an NE in this setting.

Since $I_{1,i}$ and $I_{2,i}$ are the only maximum independent sets of M_i when $|M_i| = 2$ by Observation 3.9, thus, according to SP_{sym} (Section 3.4.2), when $|M_i| = 2$, $I_{1,i}$ and $I_{2,i}$ must be selected w.p $\frac{1}{2}$. Note that in SP_{l,r,r_1} when $|M_1| = 2$, $I_{1,1}$ is selected w.p. r, and $I_{2,1}$ is selected w.p. $1 - r$ where $r \leq 0.25$. Thus, $I_{1,1}$ and $I_{2,1}$ are not selected with equal probabilities even though they are of same sizes. Thus, SP_{sym} does not belong to SP_{l,r,r_1}. Now we show that SP_{sym} can not be an NE.

SP_{sym} puts equal weight on every maximum independent sets. When $|M_1| = 4$, then under SP_{sym}, each of the maximum independent sets $\{1,3\}, \{2,4\}$ and $\{1,4\}$ with equal probabilities. Hence, the channel is offered at node 1 w.p. $t_{1,4} * 2/3$ when $|M_1| = 4$. Thus, under SP_{sym}, the node selection probability is

$$\alpha_1 = t_{1,1} + t_{1,3} + t_{1,2}/2 + 2t_{1,4}/3 \quad (3.127)$$

and node selection probability of node 2 is

$$\alpha_2 = t_{1,2}/2 + t_{1,4}/3 + t_{2,1} + t_{2,3} + t_{2,2}/2 \quad (3.128)$$

Now, we show that $\alpha_1 > \alpha_2$. Since $t_{1,3} = t_{2,3}$ (by (3.112)) and $2t_{1,4}/3 > t_{1,4}/3$, thus, we are left to show that $t_{1,1} > t_{2,1} + t_{2,2}/2$. By simple algebraic calculation for $q_1 = q_0 = q = 0.5$, we have $t_{1,1} = 0.25, t_{2,1} = 1/8, t_{2,2} = 1/16$. Hence $t_{1,1} > t_{2,1} + t_{2,2}/2$. Thus, $\alpha_1 > \alpha_2$.

Thus, by the single location pricing strategy the maximum expected payoff attained by a primary at node 1 is $(f_1(v) - c)(1 - w(\alpha_1))$ (from (3.41)) and the expected payoff attained by a primary at node 2 is $(f_1(v) - c)(1 - w(\alpha_2))$ (by (3.41)) when the other primaries select SP_{sym}. Since $\alpha_1 > \alpha_2$ and $w(\cdot)$ is strictly increasing, thus, the expected payoff at node 2 is strictly higher compared to the node 1. Thus, when $|M_1| = 2$, if a primary selects node 2 w.p. 1, then it would attain strictly higher payoff compared to the strategy SP_{sym} where a primary selects node 2 w.p. $\frac{1}{2}$ and node 1 w.p. $\frac{1}{2}$ when $|M_1| = 2$. Hence, a primary has an incentive to deviate unilaterally from its strategy profile. Hence, SP_{sym} *is not an NE.* □

3.A.6 Markov Random Field

Background

A Markov random field is a graphical model which represents the joint probability distributions of random variables having Markov property. It is represented by an undirected graph $H = (V, E)$ in which the nodes V represents the random variables. The edges E encodes the dependencies among the random variables in the following way: if $N(A)$ is the set of neighbors of A, then in a Markov random field[54],

$$A \perp \text{other random variables}|N(A)$$

Figure 3.19 provides a cyclic Markov random field. Here, $A \perp C|B, D$. The channel states in a conflict graph are random variables whose values are either 0 or 1. Since the channel states at adjacent locations are likely to be correlated, we model the correlation amongst the adjacent locations in the conflict graph using the Markov Random field where the nodes in the Markov random field represent the channel states of the corresponding nodes of conflict graph G. Figure 3.19 represents a Markov random field when the conflict graph

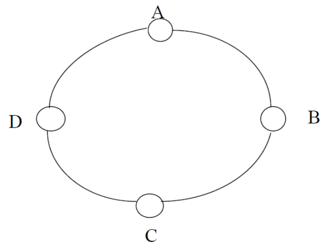


Figure 3.19: A cyclic Markov Random field.

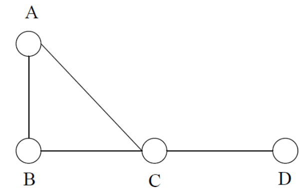


Figure 3.20: The figure shows a Markov Random Field. Here the Maximal cliques are (ABC, CD).

is a cyclic graph with 4 nodes and the values of the random variables $A, B, C, D \in \{0, 1\}$ represent the channel states at nodes A, B, C, D of conflict graph G respectively.

We now discuss the joint probability distribution in the Markov random field. Markov random fields provide a compact representation of the joint probability distribution in terms of product of *potential functions*. Potential functions are defined on the set of maximal cliques, $\mathcal{C}$, in the graphical representation of the Markov random field H. A potential function $\zeta_C(\cdot)$ represent the values of the random variable of the maximal clique $C \in \mathcal{C}$. For example, in figure 3.19 the set $\{AB\}$ is a maximal clique, thus, $\zeta_{AB}(a, b)$ denote the value of the potential function when the random variables $A = a$ and $B = b$, $a, b \in \{0, 1\}$. Note that $\zeta_C(\cdot)$ is defined on the vector $\mathbf{c}$ which represents the values of the random variables represented by nodes in the clique C. Formally, the probability of the channel state J is given by:

$$q_J = \frac{1}{Z} \prod_{C \in \mathcal{C}} \zeta_C(c_J) \tag{3.129}$$

where Z is a normalization factor and c_J denote the channel states in clique C when the overall channel state vector is J.

For example, in figure 3.19 the set of maximal cliques $\mathcal{C}$ is AB, BC, CD, DA. The joint probability distribution is given by

$$P_{A,B,C,D}(a,b,c,d) = \frac{1}{Z}\zeta_{AB}(a,b)\zeta_{BC}(b,c)\zeta_{CD}(c,d)\zeta_{DA}(d,a)$$

Since A, B, C, D only take values in $\{0, 1\}$, we can represent ζ as a matrix where $\zeta_{AB}(a, b)$ denote the value of the (a, b)th position of the matrix. For example, ζ can be the following:

$$\zeta_{AB} = \zeta_{BC} = \zeta_{CD} = \zeta_{DA} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 1 \end{bmatrix} \tag{3.130}$$

In Figure 3.20, the maximal cliques are (ABC, CD). Hence, the joint probability distributions are

$$P_{A,B,C,D}(a,b,c,d) = \frac{1}{Z}\zeta_{ABC}(a,b,c)\zeta_{CD}(c,d) \tag{3.131}$$

Definition 3.15. The Markov random field representation of random variables is symmetric if i) the maximal cliques are of identical sizes and ii) suppose $\mathbf{c}_1$ corresponds to the channel state vector of maximal clique C_1 and $\mathbf{c}_2$ corresponds to the channel state vector of maximal clique C_2, then

$$\zeta_{C_1}(\mathbf{c_1}) = \zeta_{C_2}(\mathbf{c_2}) \tag{3.132}$$

for every $\mathbf{c_1}$ and $\mathbf{c}_2$ such that $\mathbf{c}_1$ and $\mathbf{c}_2$ contain the same number of 1s (and thus, the same number of 0s since C_1, C_2 are of same sizes).

(3.130) provides an example of potential functions which are symmetric and identical. But potential functions in Fig. 3.20 can not be symmetric since the sizes of the maximal cliques are different.

Now, we are ready to provide an example which satisfies Assumption 3.3.

Result

Lemma 3.25. *The probability distributions on the channel states satisfy Assumption 3.3 if*

i) The channel states constitute a Markov random field,

ii) The graphical representation of the Markov random field H is the same as the node symmetric graph G,

iii) The Markov random field relation is symmetric[16]*, and*

iv) There are fixed integers $r_1, r_2 \ldots$ such that every clique containing $j \geq 1$ number of nodes is a subset of identical (r_j) number of maximal cliques in G.

First, it is easy to discern that the condition (iv) is satisfied by a large class of node symmetric conflict graphs including cyclic graph, infinite linear graph (Fig. 3.10), infinite square graph(Fig. 3.11), infinite grid graph (Fig. 3.12), infinite triangular graph (Fig. 3.13). For example, in the infinite triangular graph (Fig. 3.21), a clique containing 3 nodes is a maximal clique and hence, it is a part of only 1 maximal clique; any clique containing 2 nodes is a subset of 2 maximal cliques; a single node is a part of 6 maximal cliques.

In order to prove the above lemma, we first show the following for any node symmetric graph G which satisfies condition (iv):

Observation 3.10. Let n_j be the number of maximal cliques in G which contains exactly j nodes of G_J, then there are exactly n_j number of maximal cliques in G which contains exactly j nodes of G_K, when G_K is isomorphic to G_J.

[16] In a node symmetric graph, the maximal cliques are of the same size

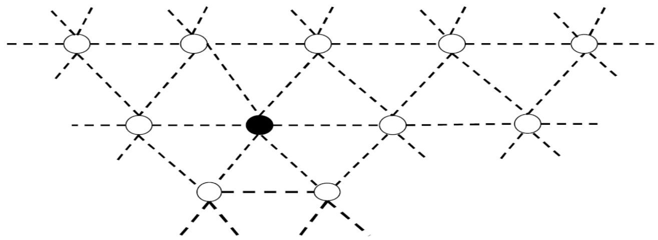


Figure 3.21: Infinite triangular graphs: the black colored node is a part of 6 maximal cliques. Any clique containing two nodes is a part of 2 maximal clique. A clique containing 3 nodes is a maximal clique in this graph.

Proof. Let G_J and G_K be isomorphic (Definition 3.12) to each other, where G_J and G_K are the conflict graphs corresponding to the channel state vectors J and K respectively. Since G_J and G_K are isomorphic, there is an isomorphic function $F_1(\cdot)$ between the nodes of G_J and G_K.

Suppose that there is a maximal clique C which contains j nodes of G_J. Thus, this set of j nodes is a subset of a maximal clique. The isomorphic function $F_1(\cdot)$ maps those j nodes into j different nodes of G_K. Also note that since these j nodes of G_J belong to a clique in the original graph G, hence they are adjacent to each other, since $F_1(\cdot)$ is isomorphic, thus, the mapped j nodes must also be adjacent to each other, hence that set of mapped j nodes is also a subset of a maximal clique in the original graph G.

Suppose the statement in the result is false. Thus, there must exist a set V_j of j nodes of G_J which is a subset of r_1 number of maximal cliques in the original graph, however the mapped set of nodes $F_1(V_j)$ of G_K is only a subset of r_2 number of maximal cliques in the original graph where $r_2 < r_1$. Thus, this violates the condition (iv). Hence, $r_2 \geq r_1$. By symmetry, we can also show that the situation where $r_2 > r_1$ can not arise. Hence, the result follows. ☐

Now, we are ready to prove Lemma 3.25.

Proof. Let G_J and G_K be isomorphic to each other, where G_J and G_K are the conflict graphs corresponding to the channel state vectors J and K respectively. We have to show that $q_J = q_K$. Let c_J be the channel state vector at the nodes of C when the channel state vector is J.

Now, at channel state vector J, the potential function value at maximal clique is $\zeta_C(c_J)$. Thus, the channel state vector $q_J$and q_K are given by

$$q_J = \prod_{C \in \mathcal{C}} \frac{1}{Z}\zeta_C(c_J)$$
$$q_K = \prod_{C \in \mathcal{C}} \frac{1}{Z}\zeta_C(c_K) \tag{3.133}$$

By Observation 3.10, the number of maximal cliques which contain j number of nodes of G_J and G_K are identical. Note that at channel state vectors J and K, the nodes where the channel state is 1 are only the nodes of G_J and G_K respectively. Hence, the number of maximal cliques which contain exactly j number of 1s are the same (and thus, the number of 0s since in the node symmetric graph, the size of maximal cliques are the same) in the channel state vectors J and K. Hence, $q_K = q_J$ from (3.132) and (3.133). ☐

Chapter 4

Provision of acquiring the CSI of the competitor

In this chapter we investigate the setting where a primary can acquire its competitor's CSI (C-CSI) by incurring a cost. As discussed in Section 1.4 we consider that there are two primaries. We formulate a non cooperative game where each primary decides whether to acquire C-CSI or not and then selects its price based on that. We introduce the class of $[T, p]$ strategies and show that there exist NE strategies which belong to this class. In a $[T, p]$ strategy a primary i) acquires the C-CSI with probability (w.p.) p if the acquisition cost is below T, and ii) does not acquire the C-CSI if the acquisition cost is above T. We show that in the NE strategy the p increases as the cost of acquiring the C-CSI decreases. We first consider a *basic model* (Section 4.2) where each primary accurately estimates the channel state of its competitor by acquiring its CSI. The channel availability probability and the acquisition costs are also the same for both the primaries. In this setting, we find that the expected payoff of a primary is independent of the cost of acquiring the C-CSI and is the same as in the setting where acquiring the C-CSI is not possible. Thus, we have the following counter-intuitive result: *the ability to acquire the C-CSI does not increase the expected payoff of the primary.* We also characterize the NE pricing strategies.

We, subsequently, investigate the impact of the estimation error on the decision, payoff and the pricing strategy of a primary (Section 4.3). We show that there exists a $[T, p]$ type NE strategy where the threshold T decreases as the estimation error increases. Interestingly, we find that the expected pay-off of a primary is *higher* when there is *an estimation error*. Thus, *it negates conventional wisdom that the payoff of the primary should decrease as the error increases.* In contrast to the basic model, the expected payoff increases as the cost of acquiring the C-CSI decreases. We characterize the NE pricing strategies and show that there exist important structural differences compared to the basic model.

We, subsequently, investigate the setting where different primaries may have different costs of acquiring the CSI of their competitors (Section 4.4); we consider that primary 1 has a lower acquisition cost compared to primary 2. We show that primary 1 acquires the C-CSI with a higher probability compared to the primary 2. The expected payoff of primary 2 is independent of the cost of acquiring the C-CSI and it is the same primary 2 would obtain when it does not acquire the C-CSI . In contrast to the basic model, the expected payoff of primary 1 is higher compared to the expected payoff of the primary 2 when primary 1 acquires the C-CSI. The expected payoff of the primary 1 decreases as the difference between the acquisition costs decreases. We characterize the NE pricing strategies and show that there exist important structural differences compared to the basic model.

We, subsequently, investigate the impact of primaries having different availability probabilities on the competition (Section 4.5); we consider that primary 1 has a higher availability probability compared to primary 2. We compute a NE strategy which is of the form $[T_i, p_i]$ for primary i. We show that $T_1 > T_2$ and $p_1 > p_2$. Thus, primary 1 acquires the C-CSI for a larger value of the acquisition cost, and also with a higher probability compared to primary 2. The expected payoff of primary 1 is also higher compared to primary 2 in contrast to the basic model. Moreover, the expected payoff of primary 2 decreases as the acquisition cost of the C-CSI decreases which negates *conventional wisdom that the payoff of a primary should not decrease as the cost of acquiring the C-CSI decreases.* We also characterize the NE pricing strategies and show that there exist some structural differences compared to the basic model.

We consider a secondary spectrum market with two primaries (players) and one or more secondaries. The choice of 2 primaries is partly motivated to simplify the analysis and it partly motivated by the fact that in most of the countries, the wireless market is mostly shared by two service providers. For example, in the USA, the market share of two primaries Verizon and AT&T is 70%.

4.1 System Model

We consider a secondary spectrum market with two primaries (players) and one secondary[1]. We first provide the basic system model in Section 4.1.1 and subsequently, we specify certain generalizations of the model in Section 4.1.2.

4.1.1 Basic Model

The transmission rate offered by a primary's channel evolves randomly because of the fading and the usage of the primary's customers. We consider that the primary's channel is available for the secondary if the transmission rate is higher than a threshold, otherwise it is rendered unavailable. For analytical exposition, we define the channel to be in state 1 if it is available, otherwise it is in state 0. Each primary's channel is available with probability (w.p.) q, where $1 > q > 0$ and q is of common knowledge.

If a primary's channel is available, the primary can sell it for the secondary use. In this

[1]If there are more that one secondary, then the decision of the primary is trivial, it will always sell its channel, thus, it will select the highest possible price and will never acquire the CSI. Our model can also accomodate the setting where the number of secondaries is not known apriori and follows a distribution function such that the probability of the event there is exactly one secondary is non-zero.

case, it decides whether to estimate (or, acquire) competitor's channel state information (C-CSI) before deciding the price for its available channel. The C-CSI estimation is accurate and thus, a primary knows the exact competitor's channel state (CCS). However, the primary incurs a cost s if it estimates the C-CSI.

The secondary is willing to pay up to v for an available channel. If the channels of both the primaries are available for sale, then, the secondary will buy the lower priced channel. If the two available channels have the same price, then a secondary will choose either of them w.p. 1/2.

4.1.2 Generalization of the Model

Estimation Error

In Section 4.3 we generalize the basic model by considering that the estimated channel state is accurate only with probability q_s. Specifically, if a primary acquires the C-CSI, then, it will estimate that the CCS is 1 (0, resp.) w.p. q_s if the original CCS is 1 (0, resp.). Without loss of generality, we assume that[2] $1/2 < q_s \leq 1$. Note that when $q_s = 1$, there is no estimation error and a primary accurately estimates the CCS, thus, the basic system model is a special case of this model.

Different Costs of Acquiring the C-CSI

In Section 4.4 we generalize the basic model to allow each primary i to incur a different cost, s_i, for acquiring the C-CSI.

[2]If $q_s = 1/2$, then there is no point of estimating the CCS as the setting becomes equivalent to the setting where a primary does not know the channel state of its competitor.

Different Channel Availability Probabilities

We generalize the basic model in Section 4.5 by allowing each primary i to have different availability probability q_i.

4.1.3 Payoff of a primary

If primary i sets its price at x and it decides to acquire the C-CSI, then, its payoff is

$$\begin{cases} x - c - s_i, & \text{if the primary is able to sell its channel,} \\ -s_i, & \text{otherwise.} \end{cases}$$

where c is a transaction cost incurred when the secondary buys the channel. Note that when both the primaries incur the same cost to acquire the C-CSI, then we have $s_i = s$.

When a primary does not acquire the C-CSI, then its payoff at price x is

$$\begin{cases} x - c, & \text{if the primary is able to sell its channel,} \\ 0, & \text{otherwise.} \end{cases}$$

4.1.4 Strategy of a Primary

If the channel of a primary is available for sale[3], it will take a decision $D \in \{Y, N\}$ where Y denotes that the primary decides to incur the cost s to estimate the C-CSI and N denotes that the primary decides not to acquire the C-CSI. Primary i also sets a price for its available channel. Note that the primaries' decisions are simultaneous so that no primary is aware of the decision of its competitor when making its own decision. If a primary selects Y, it selects a price using either a distribution $F_1(\cdot)$ or $F_0(\cdot)$ depending

[3]If the channel of the primary is unavailable, then its decision is immaterial.

on whether it estimates the CCS as 1 or 0, respectively. If a primary selects N, then it does not acquire the C-CSI, so it only selects its price using a single distribution $F(\cdot)$.

Definition 4.1. The strategy S_i of primary $i = 1, 2$ is $\sigma(D, \mathbf{F})$ where $\mathbf{F} = (F_0, F_1)$ when $D = Y$, $\mathbf{F} = (F, F)$ when $D = N$, and $\sigma(D, \mathbf{F})$ is a probability mass function over the strategies $(D, \mathbf{F})$.

The strategy of the primary other than i is denoted as S_{-i}.

Definition 4.2. $E\{u_i(S_i, S_{-i})\}$ denotes the expected payoff of primary i when its channel is available, it uses strategy S_i and the other primary uses strategy[4] S_{-i}.

4.1.5 Solution Concept

We consider a non-cooperative game where each primary only wants to maximize its own expected payoff. We use the Nash Equilibrium as a solution concept.

Definition 4.3. A *Nash equilibrium* (NE) (S_1, S_2) is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy [59]. Thus,

$$E\{u_i(S_i, S_{-i})\} \geq E\{u_i(\tilde{S}_i, S_{-i})\} \ \forall \ \tilde{S}_i. \tag{4.1}$$

A strategy profile is symmetric if $S_i = S_j$ for any pair of players i and j.

[4]Note that we consider the expected payoff of a primary as the expected payoff conditioned on the channel of the primary being available. Naturally if the channel of the primary is unavailable, it will attain a payoff of 0.

4.2 Results of the Basic Model

We, first, investigate the system model depicted in Section 4.1.1. Note that this setting is a special case of each of the more generalized settings depicted in Sections 4.1.2,4.1.2, and 4.1.2.

4.2.1 Goals

Acquiring the CSI of the competitor has potential advantages. For example, if a primary knows that the channel of its competitor is unavailable, then, the primary can select a high price because of the lack of competition. However, a primary has to incur a cost to acquire the CSI. Thus, conventional wisdom suggests that as the cost of acquiring the CSI decreases, a primary should more frequently acquire the CSI and thereby gain a higher payoff in an NE. However, conventional wisdom is not definitive because of the following. The payoff of a primary (1, say) also inherently depends on the decision of other primary (2,say). If the primary 2 decides to acquire the CSI of primary 1, then primary 2 selects a lower price when the channel of primary 1 is available, thus, in response[5], the primary 1 also selects a lower price in the NE which reduces its payoff. On the other hand, acquiring the CSI of the competitor is also not ruled out either. This is because a primary may acquire the CSI of its competitor and take advantage of the extra information. Thus, it is not apriori clear whether a primary will acquire the CSI of its competitor. It is also not clear even if a primary decides to acquire the CSI at what values of s it will do so. We resolve all the above quandaries.

The inherent uncertainty in the competitor's decision also complicates the pricing

[5]In an NE, each player selects a best response strategy in response of the strategy of the other player.

strategy of the primary. If primary 1 knows that the channel of primary 2 is available, its pricing decision still depends on if primary 2 also know that its channel is available; if not then the primary 2 may randomize among multiple prices, enabling primary 1 to charge a higher price. If primary 2 knows that the channel of primary 1 is available, primary 2 selects a lower price, in response primary 1 also selects a lower price. On the other hand, if the primary does not know the channel state of its competitor, then it may have to randomize over prices from an interval which is not known apriori. Thus, it is also not apriori clear how a primary will select its price. We also characterize NE pricing strategies.

4.2.2 Results

A class of Strategy for selecting Y

We first define a class of strategies for selecting Y.

Definition 4.4. A $[T, p]$ strategy is a strategy where a primary selects

$$\begin{cases} Y, \quad \text{w.p. } p \quad \text{when } s < T \\ N, \quad \text{w.p. } 1 \quad \text{when } s \geq T \end{cases}$$

for $0 < p \leq 1$. The probability p may be a function of s.

We show that in the basic model as well as in different generalizations, the NE strategy is a $[T, p]$ strategy where p is a strictly decreasing function of s. We also characterize T, and p in different generalizations of the basic model.

It is intuitive that in an NE, a primary will choose Y with a probability $p = f(s)$ where $f(\cdot)$ is a decreasing function. It is also intuitive that $f(s) = 0$ when $s > (v - c)$ as the maximum expected payoff that a primary attains is $v - c$. We, however, show that $f(s)$ can be 0 even for smaller values of s. We also show that p never becomes 1 for any positive value of s. We fully characterize the function $f(\cdot)$ and the value of the threshold T above which a primary does not select Y.

Main Results

Our main results are–

- Regardless of the cost s, there is no NE where both the players have full knowledge of each other's channel states w.p. 1 (Theorem 4.1). There is no NE where one primary has the complete knowledge of the channel state of its competitor, but the other does not (Theorem 4.2). Thus, a primary can only select Y, if the other primary randomizes between Y and N.

- We show that the *unique* NE strategy is a $[T, p]$ strategy where $T = q(v - c)(1 - q)$ (Theorems 4.3 and 4.4). Note that T increases when the uncertainty of the availability of the channel increases i.e. q becomes closer to 1/2. Intuitively, when either q is large or small, the uncertainty of the competitor's channel decreases, thus, a primary selects N for higher values of s. We also characterize the value of p as a function of s and show that p is a decreasing function of s.

- The expected payoff that a primary attains in any NE strategy profile is $(v-c)(1-q)$ (Theorems 4.3 and 4.4). Thus, the expected payoff of a primary is independent of the value of s. [45] shows that when a primary can not acquire the CSI of the competitor, then its payoff is $(v - c)(1 - q)$. Thus, the provision of acquiring the

CSI of the competitor does not impact the expected payoff of a primary.

- Theorem 4.3 shows that when each primary selects N, then each primary randomizes its price from the interval $[\tilde{p}, v]$. Theorem 4.4 shows that when a primary selects Y (N, resp.) and the channel of its competitor is available, then the primary selects its price from the interval $[\tilde{p}_1, \tilde{p}_2]$ ($[\tilde{p}_2, v]$,resp.). Intuitively, as the uncertainty of the availability of the competitors increases, a primary selects a higher price. We also show that $\tilde{p} < \tilde{p}_1$. Thus, a primary selects its price from a larger interval when it randomizes between Y and N.

We now describe the results in details. We first state some price distributions $\phi(\cdot)$ and $\psi(\cdot)$ which we use throughout.

$$\begin{aligned}\phi(x) =& 0 \quad \text{if } x < \tilde{p}\\ & \frac{1}{q}\left(1 - \frac{(v-c)(1-q)}{x-c}\right) \quad \text{if } \tilde{p} \leq x \leq v\\ & 1 \quad \text{if } x > v.\end{aligned} \tag{4.2}$$

$$\begin{aligned}\psi(x) =& 0 \quad \text{if } x < \tilde{p}\\ & (1 - \frac{(v-c)(1-q)}{x-c}) \quad \text{if } \tilde{p} \leq x < v\\ & 1-q, \quad \text{if } x = v\\ & 1 \quad \text{if } x > v.\end{aligned} \tag{4.3}$$

$$\text{where } \tilde{p} = (v-c)(1-q) + c. \tag{4.4}$$

4.2.3 Does there exist an NE where both primaries select Y?

Theorem 4.1. *There is no Nash equilibrium where both the primaries choose Y w.p. 1.*

Outline of the proof: Assume both players choose Y, so that they know each other's channel state. Thus, the competition becomes similar to *Bertrand Competition* [59], i.e. if the channel of its competitor is unavailable, then the primary will set its price at the v, otherwise it will set its price at the lowest value c. Now, the probability with which the channel of a primary is available is q. Thus, the expected payoff of a player is

$$(v - c - s)(1 - q) + (c - c - s)q \tag{4.5}$$

Now consider the following unilateral deviation for a primary: Primary 1 selects N and sets its price at v w.p. 1. The channel of primary 1 will be bought when the channel of primary 2 is not available for sale. Since primary 1 decides not to incur the cost s, thus, its expected payoff is

$$(v - c)(1 - q) \tag{4.6}$$

This is strictly higher than (4.5). Hence, the strategy profile can not be an NE. □

The above theorem means that there will be at least one primary which will be unaware of its competitor's channel state with a non-zero probability.

4.2.4 Does there exist an NE where one selects Y and the other selects N?

Theorem 4.2. *For positive $s > 0$, there is no NE where a primary selects Y w.p. 1 and the other selects N w.p. 1.*

First, we provide the intuition behind the result. The primary (say, 1) which selects Y tends to select lower prices with higher probability when it knows that the channel of the other primary is available. Thus, in response the primary 2 (which selects N) selects higher prices with higher probabilities in order to gain a high payoff in the event that the channel of primary 1 is unavailable since it knows that its probability of selling is very low in the event that the channel of primary 1 is available. The primary 1 can then gain a higher payoff by selecting N and higher prices as it does not have to incur the cost s. Hence, the primary 1 has an incentive to deviate from its own strategy. The detailed proof is given below.

Proof. Without loss of generality, assume that primary 1 selects Y and primary 2 selects N. First, we discuss the pricing strategies of primaries 1 and 2 and calculate the expected payoff of primary 1, subsequently, we show that primary 1 has an incentive to deviate.

When primary 1 knows that the channel of primary 2 is not available, then primary 1 will be able to sell its channel at the highest possible price, thus, it will select v w.p. 1 and its payoff if $(v - c) - s$. The above event occurs w.p. $1 - q$.

Now, we consider the case when the channel of primary 2 is available. While deciding its price, primary 2 only knows that the channel of primary 1 is available w.p. q. However, while selecting its price primary 2 knows that the primary 1 will know the channel state of primary 2 if the channel of primary 1 is available. Hence, when primary 1 knows that the channel of primary 2 is available, then the pricing decision becomes equivalent to the setting where primary 1 knows that the channel of primary 2 is available w.p. 1 and primary 2 knows that the channel of primary 1 is available w.p. q. The NE pricing strategy in the last setting has been studied in [45] and using Theorem 2 in [45] we have

Lemma 4.1. *Primary* 1 *must select its price according to* $\phi(\cdot)$ *(given in (4.2)) and pri-*

mary 2 must select its price according to $\psi(\cdot)$ (given in (4.3)).

By Lemma 4.1 when the channel of primary 2 is available for sale, then expected payoff of primary 1 at any $\tilde{p} \leq x < v$

$$(x-c)(1-\psi(x)) - s = (v-c)(1-q) - s. \tag{4.7}$$

At $x < \tilde{p}$, the payoff of primary will be strictly less than the expression in (4.7). On the other hand at v, primary 1 will get strictly a lower payoff compared to the payoff at a price just below v since $\psi(\cdot)$ has a jump at v. Hence, the maximum expected payoff to primary 1 in this case is $(v-c)(1-q) - s$.

Thus, the expected payoff of primary 1 is

$$(v-c)(1-q) + q(v-c)(1-q) - s. \tag{4.8}$$

Now, we show that if primary 1 selects N, then the primary can achieve strictly higher payoff. For $x \in [\tilde{p}, v)$, the expected payoff of primary 1 at N is

$$(x-c)(1-q\psi(x)) = (x-c)(1-q) + q(v-c)(1-q) \tag{4.9}$$

Thus, for every positive s there exists a small enough $\epsilon > 0$ such that at $x = (v-c-\epsilon)$, it will attain strictly higher payoff than (4.8). Hence, if primary 1 selects N and the price $v-\epsilon$ w.p. 1 then primary 1 attains a strictly higher payoff. The result follows. □

4.2.5 Does there exist an NE where both primaries select N?

Theorem 4.3. *Suppose that each primary selects the strategy (N, ϕ) ($\phi(\cdot)$ is given in (4.2)). The above strategy profile is the unique NE when $s \geq q(v-c)(1-q)$.*

However, the above is not an NE when $s < q(v-c)(1-q)$.

We provide an intuition behind the result. When s is high, if a primary selects Y, then it has to incur high cost compared to the potential gain it will achieve, thus, no primary has any incentive to deviate. When s is low, if a primary deviates and selects Y, then it can gain higher payoff by taking advantage of the CSI of the other primary. Thus, the strategy profile fails to be an NE when s is low. We prove that the strategy profile is an NE in Theorem 4.5 where we consider there is an error in estimating the competitor's channel state. We prove the uniqueness in Section 4.A.

Remark: The result shows that when the cost s is high, in an equilibrium both the primaries select N. It is obvious that if $s > (v - c)$, then a primary will never opt for Y. The above theorem shows that even if $s \geq (v - c)q(1 - q)$, primaries will select N.

4.2.6 Does there exist an NE when s is low?

Note from Theorems 4.1, 4.2 and 4.3 that if s is low, then there is no NE strategy where each primary selects either Y or N w.p. 1. Thus, at least one primary must randomize between Y and N when s is low.

Now, consider the following price distributions

$$\psi_1(x) = \begin{cases} 0, & \text{if } x < \tilde{p}_1, \\ \frac{1}{p}(1 - \frac{\tilde{p}_1 - c}{x - c}) & \text{if } \tilde{p}_1 \leq x \leq \tilde{p}_2, \\ 1, & \text{if } x > \tilde{p}_2. \end{cases}$$

and

$$\psi_2(x) = \begin{cases} 0, & \text{if } x < \tilde{p}_2 \\ \dfrac{1}{q(1-p)}(1 - \dfrac{(v-c)(1-q)}{x-c} - qp) & \text{if } \tilde{p}_2 \leq x \leq v \\ 1, & \text{if } x > v \end{cases}$$

where $\tilde{p}_1$ and $\tilde{p}_2$ are

$$\tilde{p}_1 = \frac{(v-c)(1-q)(1-p)}{1-qp} + c, \quad \tilde{p}_2 = \frac{(v-c)(1-q)}{1-qp} + c \tag{4.10}$$

Note that both $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are continuous. In the following, we show that a strategy profile based on these distribution is a NE when s is small enough.

Theorem 4.4. *Consider the following strategy profile: Each primary selects Y w.p. p and N w.p. $1-p$ where $p = \dfrac{q(v-c)(1-q)-s}{q(v-c)(1-q)-sq}$. When choosing Y, the primary selects its price according to $\psi_1(\cdot)$ when it knows that the channel state of the other primary is available, otherwise it selects v w.p. 1. When choosing N, the primary selects price according to $\psi_2(\cdot)$.*

The above strategy profile is the unique NE if $s < q(v-c)(1-q)$. The expected payoff that a primary attains in the NE strategy profile is $(v-c)(1-q)$.

We prove that the above straegy profile is an NE in Theorem 4.6 where we consider that there is an estimation error. We prove the uniqueness in Section 4.A. *Discussion*: Note from the above theorem that when s is low there exists an NE where *both the primaries randomize* between Y and N. It is also easy to discern that as s decreases, p increases and as $s \to 0$, $p \to 1$ (Fig. 4.1). Thus, when the cost of obtaining the competitor's CSI decreases, then the primaries will be more likely to acquire that information.

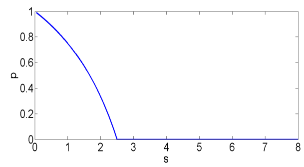


Figure 4.1: Variation of p as a function of s in an example setting: $v = 11, c = 1, q = 0.5$. When $s \geq 2.5$, $p = 0$. $p \to 1$ as $s \to 0$.

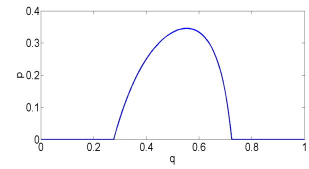


Figure 4.2: Variation of p as a function of q in an example setting: $v = 11, c = 1, s = 2$. When either $q \leq 0.28$ or $q \geq 0.72$, $p = 0$. p is maximized at $q^* = 0.55$, maximum value of p is 0.35.

Note that $q(1-q)$ is the measure of uncertainty, if the uncertainty if higher (i.e. $q = 1/2$), then the threshold is also higher. A primary never selects Y if $s \geq (v-c)/4$. By differentiating, it is easy to discern that when $s < (v-c)/4$, then p is maximized at $q^* = 1 - \sqrt{s/(v-c)}$ (Fig. 4.2). Since $s < (v-c)/4$, $q^* > 1/2$. Note also that q^* decreases as s increases. Intuitively, when s increases, primaries tend to select Y only when the uncertainty of the availability of channel increases.

The support set of $\psi_1(\cdot)$ is $[\tilde{p}_1, \tilde{p}_2]$ and $\psi_2(\cdot)$ is $[\tilde{p}_2, v]$. Thus, under Y a primary selects lower prices when the primary knows that the channel of its competitor is available compared to the setting where the primary is not aware of the channel state of its competitor. This is because in the former case the uncertainty of the appearance of the competitor is reduced.

Since p increases as s decreases, thus, from (4.10), $\tilde{p}_1$ increases as s decreases. Thus, a primary selects its price from a larger interval when s decreases. Also note that $\tilde{p}_2$ also increases as s decreases. Thus, the support set of $\psi_1(\cdot)$ increases as s decreases.

Theorems 4.3 and 4.4 imply that the expected payoff of a primary is $(v-c)(1-q)$. Note

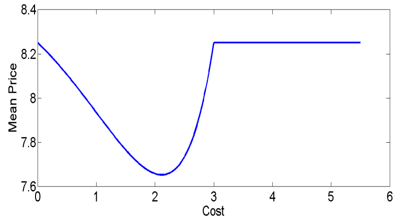


Figure 4.3: Mean price paid by the secondary

that when the primaries always know each other's channel states, the competition becomes equivalent to the Bertrand competition [59] and the expected payoff is [6] $(v-c)(1-q)$ and when the primaries are constrained to select only N, the expected payoff is again $(v-c)(1-q)$ [45, 21]. Hence, our result also builds the bridge between the two extremes. Specifically, it shows that the cost s or the availability of the competitor's CSI does not impact the expected payoff.

4.2.7 Welfare of the Secondaries

Fig. 4.3 shows the variation of the expected price paid by the secondary. Initially, the expected price decreases as the C-CSI acquisition cost s increases. The expected price reaches the minimum value, and then increases with the increase in s. When $s \geq q(v-c)(1-q)$ i.e. the primaries select N w.p. 1, the expected price is the same in the setting with $s=0$ i.e. when the primaries select Y w.p. 1. Fig. 4.3 shows that the expected price paid by the secondary is minimum at a positive cost; the minimum is not attained when $s=0$ which negates the conventional wisdom. Note that the expected payoffs of the primaries are independent of the cost s. Thus, the expected social welfare which is the sum of the expected payoffs of the primaries and the expected utility of the secondary

[6] It can also be obtained from (5).

(which is the negative of the price paid by the secondary) is in fact minimum at $s = 0$.

4.3 Impact of Error in the Estimation

We, now, investigate the impact of the estimation error. Towards this end, we consider the system model specified in Section 4.1.2. Specifically, if a primary estimates the CSI of its competitor, the estimation is accurate only with probability q_s $(1/2 < q_s \leq 1)$.

4.3.1 Goals

The impact of error in the estimation on the decision and the payoff of each primary is not apriori clear. The conventional wisdom suggests that the error in the estimation should decrease the payoff. However, the conventional wisdom is not definitive because of the following. If there is an error in estimating the channel state of the competitor, then, the primary 2 selects a higher price even when it estimates that the channel state of the primary 1 is 1, thus, in response, the primary 1 selects a higher price without reducing the winning probability, which may increase the payoff. It also remains to be seen whether the expected payoff of a primary is independent of s like in the basic model. Even if the selection of Y belongs to the class $[T, p]$ (Recall Definition 4.4), the dependence of T and p on the estimation error is also not apriori clear.

The pricing strategy also depends on the estimation error. For example, when the estimation error is 0, then a primary selects a high price when the estimated channel state of the competitor is 0 as the channel of the competitor is unavailable. However, when there is an error in estimated channel state, the actual channel state may not be 0 even when the estimated channel state is 0. Thus, a higher price may reduce the probability of winning and a lower price may reduce payoff in the event of a selling. Our goal is to characterize the pricing strategies of the primaries.

4.3.2 Main Results

We now summarize our main findings in this section here–

- We show that the NE strategy is a $[T, p]$ strategy (Definition 4.4) with $T = q(2q_s - 1)(v - c)(1 - q)$ (Theorems 4.5, 4.6). Note that in the basic model, we have also seen $[T, p]$ type strategy for selecting Y. However, due to the estimation error, the threshold is different compared to the basic model. The threshold decreases as q_s decreases i.e. primaries select N w.p. 1 for larger values of s. Intuitively, the uncertainty regarding the channel state increases as the estimation error increases, thus, the uncertainty of the channel state increases even when primary selects Y. A primary is more reluctant to select Y. Hence, primary selects N w.p. 1 for smaller values of s. We also characterize p as a function of s and show that p decreases monotonically with s.

- The expected payoff of each primary is strictly higher than $(v - c)(1 - q)$ when a primary randomizes between Y and N (i.e. $s < T$) and $q_s < 1$. In the basic model, we have shown that the expected payoff of each primary is $(v-c)(1-q)$ irrespective of the value of s. Thus, the error in estimation increases the payoff of each primary which negates the conventional wisdom that expected payoff of a primary should increase as the estimation error decreases. The payoff of each primary also increases with the decrease in s when $q_s < 1$. Hence, in contrast to the basic model, the expected payoff of each primary depends on the value of s.

- In NE pricing strategy:
  - When a primary selects Y and estimates that the channel state of its competitor is 1, then it selects its price from the interval $[\tilde{p}_1, L_N]$.
  - When a primary selects N, then it selects its price from the interval $[L_N, L_0]$.
  - When a primary selects Y and estimates that the channel state of its competitor is 0, then it selects its price from the interval $[L_0, v]$.

If $q_s = 1$, a primary always selects v w.p. 1 when the primary selects Y and the channel state of the competitor is 0 since the primary will always be able to sell its channel because of the unavailability of its competitor. However, when $q_s < 1$, there is a potential error in the estimation, thus, a primary randomizes among prices from an interval $[L_0, v]$ even when the primary estimates that the channel state of its competitor is 0. Also note that when a primary estimates that the channel state of its competitor is 1 (0,resp.), then its competitor is more likely to be available (unavailable, resp.), hence, the primary selects lower (higher, resp.) prices compared to the setting where a primary selects N.

4.3.3 High s

First, we state some results which we use throughout this section. Note that when a primary decides to estimate the CSI of its competitor, it estimates the channel state of its competitor is 1 w.p. $qq_s + (1-q)(1-q_s)$ and the primary estimates the channel state of its competitor is 0 w.p. $(1-q)q_s + q(1-q_s)$. Note that when $q_s = 1$, then the above probabilities becomes q and $1-q$ respectively. If a primary estimates that its competitor's channel state is 1, then the actual channel state is 1 w.p.

$$\frac{q_s q}{q q_s + (1-q)(1-q_s)}. \tag{4.11}$$

Similarly, if a primary estimates that its competitor's channel state is 0, then the actual channel state of its competitor is 1 w.p.

$$\frac{q(1-q_s)}{(1-q)q_s + q(1-q_s)}. \tag{4.12}$$

Note that when $q_s = 1$, then both the above probabilities become 1.

Our main result in this section shows that

Theorem 4.5. *There exists a NE where each primary selects N w.p. 1 if $s \geq (v-c)(1-q)(2qq_s - q)$. In the NE pricing strategy, each primary selects its price according to $\phi(\cdot)$ (described in (4.2)). The expected payoff of each primary is $(v-c)(1-q)$.*

Proof. We show that a primary does not have any profitable unilateral deviation when the other primary follows the strategy prescribed in the theorem. Towards this end, we, first, show that under N the maximum expected payoff of a primary is $(v-c)(1-q)$ (Step i). It is attained when the primary follows the strategy $\phi(\cdot)$ (Step ii). Subsequently, we show that if the primary selects Y, then its expected payoff is at most $(v-c)(1-q)$ which will show that the primary does not have any profitable unilateral deviation (Step iii).

Step i: At any price $x \in [\tilde{p}, v]$ the expected payoff of a primary is

$$(x-c)(1-q\phi(x)) = (v-c)(1-q) \quad \text{(from (4.2))} \tag{4.13}$$

A price strictly less than $\tilde{p}$ will fetch a payoff strictly less than $(v-c)(1-q)$ (by (4.4)). Thus, the maximum expected payoff of a primary under N is $(v-c)(1-q)$.

Step ii: Note from (4.13) that a primary attains the maximum expected payoff when it selects its price from the interval $[\tilde{p}, v]$.

Step ii: Now, we show that if primary 1 selects Y, it can not get a strictly higher payoff when $s \geq (v-c)(1-q)(2qq_s - q)$. Towards this end, we show that when a primary selects Y and estimates that the channel state of the other primary is 1, then it will attain a maximum expected payoff of $(v-c)(1-q)-s$ (Step ii.a). Subsequently, we show that if the primary selects estimates that the channel state of the competitor is 0, then it will attain a maximum expected payoff of $(v-c)(1-q)\dfrac{q_s}{(1-q_s)q+q_s(1-q)} - s$ (Step ii.b.). Finally, we show that the expected payoff of the primary is at most $(v-c)(1-q)$ when it selects Y (Step ii.c.).

Step ii.a: Suppose that the primary 1 selects Y and estimates that the channel state of primary 2 is 1. Using (4.11) the expected payoff of primary 1 at any price $x \in [\tilde{p}, v]$ is

$$
\begin{aligned}
&(x-c)\Big(1-\frac{qq_s}{qq_s+(1-q)(1-q_s)}\phi(x)\Big)-s\\
&=(x-c)\Big(1-\frac{q_s}{qq_s+(1-q)(1-q_s)}\Big(1-\frac{\tilde{p}-c}{x-c}\Big)\Big)-s\\
&=(x-c)\Big(1-\frac{q_s}{qq_s+(1-q)(1-q_s)}\Big)+(\tilde{p}-c)\frac{q_s}{qq_s+(1-q)(1-q_s)}-s \qquad (4.14)
\end{aligned}
$$

Note that $\tilde{p}-c=(v-c)(1-q)$. Since $q_s > qq_s+(1-q)(1-q_s)$ when $q_s > 1/2$, thus, the above is maximized at $\tilde{p}$, hence, the maximum expected payoff that primary 1 can attain when it estimates the channel state of its competitor is 1 is

$$
\begin{aligned}
&(v-c)(1-q)\Big(1-\frac{q_s}{qq_s+(1-q)(1-q_s)}\Big)+(v-c)(1-q)\frac{q_s}{qq_s+(1-q)(1-q_s)}-s\\
&=(v-c)(1-q)-s \qquad (4.15)
\end{aligned}
$$

Step ii.b: Now, suppose that the primary 1 estimates that the channel state of primary

2 is 0. Using (4.12) the expected payoff of primary 1 at any price $x \in [\tilde{p}, v]$ in this case is

$$
\begin{aligned}
&(x-c)(1-\frac{(1-q_s)q}{(1-q_s)q+q_s(1-q)}\phi(x))-s\\
&=(x-c)(1-\frac{1-q_s}{(1-q_s)q+q_s(1-q)}(1-\frac{\tilde{p}-c}{x-c}))-s\\
&=(x-c)(1-\frac{1-q_s}{(1-q_s)q+q_s(1-q)})+(\tilde{p}-c)\frac{1-q_s}{(1-q_s)q+q_s(1-q)}-s \qquad (4.16)
\end{aligned}
$$

The above is maximized at $x = v$. Hence, the maximum expected payoff that a primary can attain is

$$
\begin{aligned}
&(v-c)\frac{(1-q)(2q_s-1)}{(1-q_s)q+q_s(1-q)}+(v-c)(1-q)\frac{1-q_s}{(1-q_s)q+q_s(1-q)}-s\\
&=(v-c)(1-q)\frac{q_s}{(1-q_s)q+q_s(1-q)}-s \qquad (4.17)
\end{aligned}
$$

Step ii.c: Note that a primary estimates that the channel state of primary 2 is 0 w.p. $(1-q_s)q + q_s(1-q)$ and the channel state of primary 2 is 1 w.p. $qq_s + (1-q)(1-q_s)$. The primary also incurs the cost of s when it selects Y. Hence, from (4.15) and (4.17) the maximum expected payoff that primary 1 can attain by selecting Y is

$$
\begin{aligned}
&(v-c)(1-q)q_s+(v-c)(1-q)(qq_s+(1-q)(1-q_s))-s\\
&=(v-c)(1-q)(2qq_s-q+1)-s \qquad (4.18)
\end{aligned}
$$

However, since $s \geq (v-c)(1-q)(2qq_s - q)$, thus the maximum expected payoff that a primary can attain by selecting Y is $(v-c)(1-q)$. Hence, a primary does not have any profitable unilateral deviation. □

Note that the threshold $(v-c)(1-q)(2qq_s - q)$ increases as q_s increases. Intuitively, as q_s increases, the uncertainty regarding the channel state of the competitors reduces, thus, a primary tends to select N for a smaller range of the values of s.

The expected payoff of each primary is identical and equal to $(v-c)(1-q)$. Since both the players select N, thus, the expected payoff does not depend on s in this case.

4.3.4 Low s

Now, we show that there exists a NE where each primary randomizes between Y and N when $s < (v-c)(1-q)(2qq_s - q)$. Towards this end, we introduce some distribution functions parameterized by p. The significance of p is shown later.

$$\begin{aligned}\psi_{Y,1}(x) =& 0, x < \tilde{p}_1 \\ & \alpha_{1,p}(1 - \frac{\tilde{p}_1 - c}{x - c}), \tilde{p}_1 \leq x \leq L_N \\ & 1, x > L_N \end{aligned} \tag{4.19}$$

$$\begin{aligned}\psi_N(x) =& 0, \quad x < L_N \\ & \alpha_{N,p}(1 - \frac{\tilde{p}_2 - c}{x - c} - \beta_{N,p}) \quad L_N \leq x \leq L_0 \\ & 1, \quad x > L_0 \end{aligned} \tag{4.20}$$

and, when $q_s < 1$, then

$$\begin{aligned}\psi_{Y,0}(x) =& 0, \quad x < L_0 \\ & \alpha_{0,p}(1 - \frac{\tilde{p}_3 - c}{x - c} - \beta_{0,p}) \quad L_0 \leq x \leq v \\ & 1, \quad x > v \end{aligned} \tag{4.21}$$

if $q_s = 1$, then

$$\psi_{Y,0}(x) = H(x - v) \tag{4.22}$$

where $H(\cdot)$ is the heaviside step function or unit step function and

$$\tilde{p}_3 - c = \frac{(v-c)(1-q)q_s}{(1-q)q_s + q(1-q_s)}, \quad \tilde{p}_2 - c = \frac{(v-c)(1-q)q_s(1-(1-p)q - pqq_s)}{pq(1-q_s)^2 + q_s(1-q)}$$

$$L_0 - c = (\tilde{p}_2 - c)/(1-(1-p)q - pqq_s), \quad L_N - c = (\tilde{p}_2 - c)/(1 - pqq_s)$$

$$\tilde{p}_1 - c = (L_N - c)\frac{qq_s(1-pq_s) + (1-q)(1-q_s)}{qq_s + (1-q_s)(1-q)}. \tag{4.23}$$

and

$$\alpha_{1,p} = \frac{qq_s + (1-q)(1-q_s)}{pqq_s^2}, \quad \alpha_{N,p} = \frac{1}{(1-p)q}, \quad \beta_{N,p} = pqq_s$$

$$\alpha_{0,p} = \frac{q(1-q_s) + q_s(1-q)}{pq(1-q_s)^2}, \quad \beta_{0,p} = \frac{pq(1-q_s)q_s + (1-p)q(1-q_s)}{q(1-q_s) + q_s(1-q)}. \tag{4.24}$$

It is easy to discern that all the above distribution functions are continuous when $q_s < 1$. When $q_s = 1$, then only $\psi_{Y,0}(\cdot)$ is discontinuous which has a jump of 1 at v. Also note that the structures of $\psi_{Y,1}(\cdot), \psi_N(\cdot)$ and $\psi_{Y,0}(\cdot)$ are similar (i.e. variation with x is the same). However, their support sets, the scaling parameters (i.e. $\alpha_{1,p}, \alpha_{N,p}, \alpha_{0,p}$), and the constants (i.e $\beta_{0,p}$, $\beta_{N,p}$) are different.

When $q_s = 1$, the values of the parameters in (4.23) are greatly simplified which are given by–

$$\tilde{p}_3 - c = v - c, \quad \tilde{p}_2 - c = (v-c)(1-q), \quad L_0 - c = v - c$$

$$L_N - c = \frac{\tilde{p}_2 - c}{1 - pq}, \quad \tilde{p}_1 - c = (L_N - c)(1-p) \tag{4.25}$$

Thus, $L_0 - c$ and $\tilde{p}_3 - c$ are the highest when $q_s = 1$. Intuitively, when $q_s = 1$, a primary knows that the channel state of its competitor is unavailable w.p. 1 if the primary estimates that the channel state of the competitor is 0. Thus, the primary selects v w.p. 1.

Now, we are ready to state the main result of this section.

Theorem 4.6. *Consider the following strategy profile: Each primary selects Y w.p. p and N w.p. $1-p$ where p satisfies the following equality*

$$\tilde{p}_2 - c = (\tilde{p}_1 - c)(qq_s + (1-q)(1-q_s)) + (\tilde{p}_3 - c)(q(1-q_s) + q_s(1-q)) - s \qquad (4.26)$$

where $\tilde{p}_1, \tilde{p}_2$ and $\tilde{p}_3$ are given in (4.23). While selecting Y, each primary selects its price from $\psi_{Y,1}(\cdot)$ (given in (4.19)) if the estimated channel state of the other primary is 1 and each primary selects its price from $\psi_{Y,0}(\cdot)$ (given in (4.21)) if the estimated channel state of the other primary is 0. While selecting N, each primary selects its price using $\psi_N(\cdot)$ (given in (4.20)).

The above strategy profile is an NE when $s < (v-c)(1-q)(2qq_s - q)$. The above strategy profile is unique in the class of symmetric NE strategies. The expected payoff that each primary gets is $\tilde{p}_2 - c$.

Discussion: Note that when $q_s = 1$, we know from Theorem 4.4 that the strategy profile is unique one among *all* strategy profiles not only *symmetric* ones. There is no equilibrium where both the players select Y w.p. 1 even when $q_s < 1$ (we have already shown the above for $q_s = 1$ in Theorem 4.1).

Now, we show that there exists a unique solution of (4.26) in p in the interval $0 < p < 1$ when $0 < s < q(v-c)(1-q)(2q_s - 1)$.

Observation 4.1. There exists a unique solution in $p \in (0,1)$ of the equation (4.26) when $0 < s < (v-c)(1-q)(2qq_s - q)$. As s decreases p increases.

Proof. First noe that (4.26) can be written as

$$\tilde{p}_2 - c - (\tilde{p}_1 - c)(qq_s + (1-q)(1-q_s)) = (\tilde{p}_3 - c)(q(1-q_s) + q_s(1-q)) - s.mber$$

Using (4.23) we can rewrite the above as

$$(v-c)(1-q)q_s \frac{1-(1-p)q-pqq_s}{pq(1-q_s)^2+q_s(1-q)} \left(1-\frac{(qq_s(1-pq_s)+(1-q)(1-q_s))}{(1-pqq_s)}\right)$$
$$= (v-c)(1-q)q_s - s \tag{4.27}$$

First, we show that the left hand of (4.27) is strictly increasing in p (Step i). Next, we show that when $p=0$, the left hand side of (4.27) is less than the right hand side and when $p=1$, the left hand side of (4.27) is greater than the right hand side (Step ii). Since the left hand side of (4.27) is continuous in p and strictly increasing in p, there exists a unique solution $p \in (0,1)$ of (4.27). The last part easily follows as the right hand side of (4.27) decreases with s, the left hand side of (4.27) is strictly increasing in p and independent of s. Now, we show steps i and ii.

Step i: By differentiating the left hand side of (4.27) we can show that

$$1-\frac{qq_s(1-pq_s)+(1-q)(1-q_s)}{1-pqq_s} \tag{4.28}$$

is strictly increasing in p when $q_s > 1/2$. On the other hand, it is easy to discern that $(v-c)(1-q)q_s \frac{1-(1-p)q-pqq_s}{pq(1-q_s)^2+q_s(1-q)}$ is non-decreasing in p when $q_s > 1/2$. Thus, the left hand side of (4.27) is strictly increasing in p.

Step ii: When $p=0$, then the value of left hand side of the equation (4.27) is

$$(v-c)(1-q)(1-qq_s-(1-q)(1-q_s)) = (v-c)(1-q)(q_s+q-2qq_s) \tag{4.29}$$

Now, $(v-c)(1-q)q_s - (v-c)(1-q)(q_s+q-2qq_s) = (v-c)(1-q)(2qq_s-q)$. Since $s < (1-q)(v-c)(2qq_s-q)$, thus, the left hand side of (4.27) is less than the right hand side.

Now when $p = 1$, then the left hand side of (4.27) is

$$\begin{aligned}
&(v-c)(1-q)q_s[\frac{1-qq_s}{q(1-q_s)^2+(1-q)q_s} - \frac{qq_s(1-q_s)+(1-q)(1-q_s)}{q(1-q_s)^2+(1-q)q_s}] \\
&= (v-c)(1-q)q_s[\frac{qq_s^2+q+q_s-3qq_s}{q(1-q_s)^2+(1-q)q_s}] \\
&= (v-c)(1-q)q_s[\frac{q(1-q_s)^2+q_s(1-q)}{q(1-q_s)^2+(1-q)q_s}] = (v-c)(1-q)q_s \qquad (4.30)
\end{aligned}$$

Since $s > 0$, thus, the left hand side of (4.27) is greater than the right hand side.

Since the left hand side of (4.27) is continuous function of p, thus, by intermediate value theorem there exists a solution in the interval $(0, 1)$. □

Next, we show that the expected payoff of a primary is a strictly greater than $(v - c)(1-q)$ when $q_s < 1$ and the payoff increases with the decrease in s.

Lemma 4.2. *When $q_s < 1$, $\tilde{p}_2 - c$ increases with the decrease in s and $\tilde{p}_2 - c$ is strictly greater than $(v-c)(1-q)$ when $s < (v-c)(1-q)(2qq_s - q)$.*

Proof. Now, it is easy to discern that $\tilde{p}_2 - c$ is strictly increasing in p when $q_s < 1$. Now, p increases with the decrease in s (by Observation 4.1) when $s < (v-c)(1-q)(2qq_s - q)$. Hence, $\tilde{p}_2 - c$ is a strictly decreasing function in s when $q_s < 1$.

When $p = 0$, then $\tilde{p}_2 - c = (v-c)(1-q)$ (by (4.23)). Since $\tilde{p}_2 - c$ is strictly increasing in p when $q_s < 1$ and $s < q(v-c)(1-q)(2q_s - 1)$, hence $\tilde{p}_2 - c > (v-c)(1-q)$. □

Note from Theorem 4.6 that the expected payoff attained by a primary under the NE is $\tilde{p}_2 - c$. From (4.25) note that $\tilde{p}_2 - c = (v-c)(1-q)$ when $q_s = 1$. Thus, the above lemma entails that the expected payoff of each primary increases when there is an error in estimating the channel state of the competitor. This contradicts the conventional wisdom that the payoff should increase with the decrease in error in the estimation. In Section 4.3.2 we have already explained the apparent reason behind this result.

The above lemma entails that the expected payoff increases as s decreases when $q_s < 1$. Note that when $q_s = 1$, the expected payoffs of primaries are independent of s which we have already seen in the basic model (Section 4.2).

Note that when $q_s = 1$, then $\psi_{Y,0}(\cdot)$ has a jump of size 1 i.e. a primary will select v w.p. 1 as the primary will always be able to sell its channel. However, when $q_s < 1$, then a primary selects its price using a continuous distribution from the interval $[L_0, v]$ where $L_0 < v$. We have already explained the reason behind this in Section 4.3.2.

Proof of Theorem 4.6

Before digging into the details of proof, we state few more results which we use throughout. Note from (4.23) that

$$(\tilde{p}_3 - c)(q_s(1-q) + q(1-q_s))/[q_s(1-q) + pq(1-q_s)^2] = L_0 - c \tag{4.31}$$

Since $q_s > 1/2$, thus, by cross multiplication, it is easy to see that

$$\frac{q_s((1-q_s)q + (1-q)q_s)}{(1-q_s)(qq_s + (1-q_s)(1-q))} > 1 \tag{4.32}$$

We show that primary 1 can not gain higher profit by deviating from the strategy prescribed in Theorem 4.6 when primary 2 follows the strategy prescribed in Theorem 4.6. This will complete the proof. Toward this end, we first show that when primary 1 selects Y and it estimates that the channel state of its competitor is 1, then it will attain a maximum expected payoff of $\tilde{p}_1 - c - s$. The maximum expected payoff is attained when it follows the strategy $\psi_{Y,1}(\cdot)$ (Step i). Subsequently, we show that under Y, when the primary estimates the channel state as 0, then the maximum expected payoff that primary 1 can attain is $\tilde{p}_3 - c - s$ and it is attained when the primary follows the strategy $\psi_{Y,0}(\cdot)$ (Step ii). Subsequently, we show that the maximum expected payoff that primary 1 can attain under Y is $\tilde{p}_2 - c$ and it is attained when primary 1 follows the strategy (Step iii). Subsequently, we show that when primary 1 selects N, then its maximum expected payoff is $\tilde{p}_2 - c$ and it is attained when the primary follows the strategy $\psi_N(\cdot)$ (Step iv). Finally, we show that the maximum expected payoff of primary 1 is $\tilde{p}_2 - c$ and it is attained if primary 1 follows the strategy profile (Step v).

Step i: Suppose that primary 1 selects Y and estimates that the channel state of primary 2 is 1. We show that the maximum expected payoff attained by the primary 1 is $\tilde{p}_1 - c - s$ and this is attained only when the primary selects its price from the interval $[\tilde{p}_1, L_N]$. Toward this end, we first show any price in the interval $[\tilde{p}_1, L_N]$ will fetch an expected payoff of $\tilde{p}_1 - c - s$ (Step i.a.). Subsequently, we show that if primary 1 selects a price from the interval $[L_N, L_0]$ and $[L_0, v]$ it will fetch an expected payoff of less than $\tilde{p}_1 - c - s$ in Step i.b. and Step i.c. respectively. Note that at any price less than $\tilde{p}_1$ will fetch a strictly lower payoff compared to the price at $\tilde{p}_1$ as primary 2 does not select any price lower than or equal to $\tilde{p}_1$. Thus, this will complete step i.

Step i.a: Here, we are considering the scenario where primary 1 estimates that the channel state of primary 2 is 1. Under Y, when the primary 1 estimates that the channel state of primary 2 is 1, then the probability that the channel state of primary 2 is 1 is

$$\frac{q_s q}{q q_s + (1-q)(1-q_s)} \tag{4.33}$$

Suppose that primary 1 selects a price x in the interval $[\tilde{p}_1, L_N]$. When the channel state of primary 2 is 1 it will select a price less than or equal to x only if it selects Y, it estimates the channel state of primary 1 as 1 and selects a price less than or equal to x.The primary 2 selects Y w.p. p. Now, when the channel of primary 1 is available, then primary 2 estimates the channel state of primary 1 as 1 w.p. q_s and selects a price less than or equal to x w.p. $\psi_{Y,1}(x)$. The channel state of primary 2 is 1 with probability given in (4.33). Hence, the expected payoff of primary 1 when its channel is available and selects a price x in the interval $[\tilde{p}_1, L_N]$ is

$$(x-c)(1-\frac{pq_sqq_s}{qq_s+(1-q)(1-q_s)}\psi_{Y,1}(x))-s=\tilde{p}_1-c-s \quad \text{from (4.19)}. \tag{4.34}$$

Step i.b.: Now, suppose that primary 1 selects a price $x \in [L_N, L_0]$. Note that if the channel of primary 2 is available, it will select a price less than or equal to x if if one of the following occurs–i) primary 2 selects Y and estimates the channel state of primary 1 is 1, ii) primary 2 selects N and selects a price less than or equal to x. (i) occurs with probability pq_s and (ii) occurs with probability $(1-p)\psi_N(x)$. Since primary 1 estimates that the channel state of primary 2 is 1, thus, the probability that the true state of the channel of primary 2 is indeed 1 is given by (4.33). Thus, the probability that the primary 2 will select a price less than or equal to x is

$$\frac{pq_s^2q}{qq_s+(1-q)(1-q_s)}+\frac{(1-p)qq_s}{qq_s+(1-q)(1-q_s)}\psi_N(x)$$

Thus, at x, the expected payoff of primary 1 is–

$$(x-c)(1-\frac{pq_s^2q}{qq_s+(1-q)(1-q_s)}-\frac{(1-p)qq_s}{qq_s+(1-q)(1-q_s)}\psi_N(x))-s$$

$$=(x-c)(1-\frac{pq_s^2q}{qq_s+(1-q)(1-q_s)}-\frac{q_s}{qq_s+(1-q)(1-q_s)}(1-\frac{\tilde{p}_2-c}{x-c}-pqq_s))$$

$$-s \quad \text{from (4.20)}$$

$$=(x-c)(1-\frac{q_s}{qq_s+(1-q)(1-q_s)})+(\tilde{p}_2-c)\frac{q_s}{qq_s+(1-q)(1-q_s)}-s \tag{4.35}$$

Since $q_s > 1/2$, the co-efficient is negative. Thus, the above is maximized at $x = L_N$. Using (4.23) , the above expression is thus, upper bounded by

$$(L_N-c)(1-\frac{q_s}{qq_s+(1-q)(1-q_s)})+(L_N-c)\frac{q_s(1-pqq_s)}{qq_s+(1-q)(1-q_s)}-s$$

$$=(L_N-c)\frac{qq_s(1-pq_s)+(1-q)(1-q_s)}{qq_s+(1-q)(1-q_s)}-s=\tilde{p}_1-c-s \quad \text{from (4.23)} \tag{4.36}$$

Step i.c: From steps i.a. and i.b. we have already shown that when the maximum expected payoff of primary 1 is $\tilde{p}_1 - c - s$ at a price in the interval $[\tilde{p}_1, L_0]$. When $q_s = 1$, $L_0 = v$ (from (4.25)). Thus, it shows that when $q_s = 1$, the maximum expected payoff of primary 1 is indeed $\tilde{p}_1 - c - s$.

Now, we consider the case where $q_s < 1$ and primary 1 selects price $x \in [L_0, v]$. When the channel of primary 2 is available, then primary 2 selects a price less than or equal to x if one of the following occurs–i) it selects Y and estimates that the channel state of primary 1 is 1, ii) primary 2 selects N, iii) primary 2 selects Y, estimates that the channel state of primary 1 is 0 and selects a price less than or equal to x. (i) occurs with probability pq_s since the channel of primary 1 is available. (ii) occurs with probability $1-p$. (iii) occurs with probability $p(1-q_s)\psi_{Y,0}(x)$ (since the channel of primary 1 is available). On the other hand the probability that the channel of primary 2 is available is given by (4.33). Hence, the probability that primary 2 selects a price less than or equal to x is

$$\frac{pqq_s^2 + (1-p)qq_s + pqq_s(1-q_s)\psi_{Y,0}(x)}{qq_s + (1-q)(1-q_s)}$$

Thus, at any price $x \in [L_0, v]$, the expected payoff of primary 1 is

$$(x-c)(1 - \frac{pqq_s^2 + (1-p)qq_s + pqq_s(1-q_s)\psi_{Y,0}(x)}{qq_s + (1-q)(1-q_s)}) - s$$

$$= (x-c)(1 - \frac{pqq_s^2 + (1-p)qq_s}{qq_s + (1-q)(1-q_s)}) - s$$

$$- (x-c)\frac{q_s[(1-q)q_s + (1-q_s)q]}{(1-q_s)[qq_s + (1-q)(1-q_s)]}(1 - \frac{\tilde{p}_3 - c}{x-c} - \frac{pq(1-q_s)q_s + (1-p)q(1-q_s)}{(1-q)q_s + (1-q_s)q})$$

from (4.21)

$$= (x-c)(1 - \frac{[q(1-q_s) + q_s(1-q)]q_s}{(1-q_s)[qq_s + (1-q)(1-q_s)]})$$

$$+ (\tilde{p}_3 - c)\frac{[q(1-q_s) + q_s(1-q)]q_s}{(1-q_s)[qq_s + (1-q)(1-q_s)]} - s \tag{4.37}$$

By (4.32) the co-efficient of $(x-c)$ is negative, thus, the maximum of the above expression is attained at $x = L_0$. Thus, the expected payoff at x is upper bounded by expected payoff at L_0. From (4.36) (which also gives the expected payoff at L_0) we have already bounded the expected payoff at L_0 which is $\tilde{p}_1 - c - s$. Hence, the maximum expected payoff that primary 1 can attain in this case is $\tilde{p}_1 - c - s$ and it is attained at any price in the interval $[\tilde{p}_1, L_N]$.

Step ii: Suppose that primary 1 estimates that the channel state of primary 2 is 0. When $q_s = 1$, then the channel of primary 2 is unavailable with probability 1. Hence, primary 1 will attain the highest possible payoff at v and the payoff is $(v-c)-s = \tilde{p}_3 - c - s$ (by (4.25)). Thus, we consider the case when $q_s < 1$. We show that the maximum expected payoff attained by primary 1 is $\tilde{p}_3 - c - s$ and it is attained at any price in the interval

$[L_0, v]$. Towards this end, we first show that any price from the interval $[L_0, v]$ will fetch an expected payoff of $\tilde{p}_3 - c - s$ (Step ii.a.). Subsequently, we show that any price in the interval $[L_N, L_0]$ and $[\tilde{p}_1, L_N]$ will fetch an expected payoff of at most $\tilde{p}_3 - c - s$ (Steps ii.b. and ii.c. resp.). Any price less than $\tilde{p}_1$ will fetch a payoff which is strictly less than the payoff at $\tilde{p}_1$, thus, this will complete Step ii.

Step ii.a: When primary 1 estimates that the channel state of primary 2 is 0, then the probability that the channel state of primary 2 is 1 is

$$\frac{q(1-q_s)}{q(1-q_s)+q_s(1-q)} \tag{4.38}$$

Suppose that primary 1 selects a price in the interval $x \in [L_0, v]$. If the channel of primary 2 is available, then, the primary 2 will select a price less than or equal to x if one of the following occurs–i) it selects Y and estimates that the channel state of primary 1 is 1, ii) primary 2 selects N, iii) primary 2 selects Y, estimates that the channel state of primary 1 is 0 and selects a price less than or equal to x. (i) occurs with probability pq_s since the channel of primary 1 is available. (ii) occurs with probability $1-p$. (iii) occurs with probability $p(1-q_s)\psi_{Y,0}(x)$ (since the channel of primary 1 is available). On the other hand the probability that the channel of primary 2 is available is given by (4.38) as primary 1 estimates that the channel state of primary 2 is 0. Hence, the probability that primary 2 selects a price less than or equal to x is

$$\frac{pq(1-q_s)q_s+(1-p)q(1-q_s)+pq(1-q_s)^2\psi_{Y,0}(x)}{q(1-q_s)+q_s(1-q)}.$$

Hence, the expected payoff of primary 1 at x is

$$
\begin{aligned}
&(x-c)(1-\frac{pq(1-q_s)q_s+(1-p)q(1-q_s)}{q(1-q_s)+q_s(1-q)}-\frac{pq(1-q_s)^2}{q(1-q_s)+q_s(1-q)}\psi_{Y,0}(x))-s\\
&=\tilde{p}_3-c-s \quad \text{from (4.21)} \qquad (4.39)
\end{aligned}
$$

Step ii.b.: Now, suppose primary 1 selects a price x in the interval $[L_N, L_0]$. When the channel of primary 2 is available, then primary 2 selects a price less than or equal to x if one of the following occurs–i) primary 2 selects Y and estimates the channel state of primary 1 is 1, ii) primary 2 selects N and selects a price less than or equal to x. (i) occurs with probability pq_s and (ii) occurs with probability $(1-p)\psi_N(x)$. Given that the primary 1 estimates that the channel state of primary 2 is 0, the probability that channel of primary 1 is available is given by (4.38). Thus, the probability that primary 2 selects a price less than or equal to x is given by

$$\frac{(pq_s+(1-p)\psi_N(x))q(1-q_s)}{q(1-q_s)+q_s(1-q)} \qquad (4.40)$$

Hence, the expected payoff of primary 1 at x is

$$
\begin{aligned}
&(x-c)(1-\frac{pq(1-q_s)q_s}{q(1-q_s)+q_s(1-q)}-\frac{(1-p)q(1-q_s)}{q(1-q_s)+q_s(1-q)}\psi_N(x))-s\\
&=(x-c)(1-\frac{pqq_s(1-q_s)}{q(1-q_s)+q_s(1-q)})\\
&-(x-c)\frac{1-q_s}{q(1-q_s)+q_s(1-q)}(1-\frac{\tilde{p}_2-c}{x-c}-pqq_s)-s \quad \text{from (4.20)}\\
&=(x-c)(1-\frac{1-q_s}{q(1-q_s)+q_s(1-q)})+(\tilde{p}_2-c)\frac{1-q_s}{q(1-q_s)+q_s(1-q)}-s \qquad (4.41)
\end{aligned}
$$

By (4.32) the above is maximized at $x=L_0$. Hence, the maximum possible expected payoff is

$$\begin{aligned}
&(L_0-c)(1-\frac{1-q_s}{q(1-q_s)+q_s(1-q)})+(L_0-c)\frac{(1-q_s)(1-(1-p)q-pqq_s)}{q(1-q_s)+q_s(1-q)}-s\\
&=(L_0-c)(1-\frac{[(1-p)q+pqq_s](1-q_s)}{q(1-q_s)+q_s(1-q)})-s\\
&=(L_0-c)\frac{pq(1-q_s)^2+q_s(1-q)}{q(1-q_s)+q_s(1-q)}-s=\tilde{p}_3-c-s \quad \text{from (4.23)} \qquad (4.42)
\end{aligned}$$

Step ii.c.: Now, suppose that primary 1 selects a price x in the interval $[\tilde{p}_1, L_N]$. Now, if the channel of primary 2 is available, then it selects a price less than or equal to x if it selects Y, estimates that the channel state of primary 1 is 1 and selects a price less than or equal to x. The above occurs with probability $pq_s\psi_{Y,1}(x)$. The probability that the channel state of primary 2 is 1 given that the primary 1 estimates that the channel state of primary 2 is 0 is given by (4.38). Hence at x the expected payoff of primary 1 is

$$\begin{aligned}
&(x-c)(1-\frac{pq(1-q_s)q_s\psi_{Y,1}(x)}{q(1-q_s)+q_s(1-q)})-s\\
&=(x-c)(1-\frac{(1-q_s)[qq_s+(1-q)(1-q_s)]}{q_s[q(1-q_s)+q_s(1-q)]}(1-\frac{\tilde{p}_1-c}{x-c}))-s \quad \text{from (4.19)}\\
&=(x-c)(1-\frac{(1-q_s)(qq_s+(1-q)(1-q_s))}{q_s((1-q_s)q+q_s(1-q))})\\
&+(\tilde{p}_1-c)\frac{(1-q_s)(qq_s+(1-q)(1-q_s))}{q_s(q(1-q_s)+q_s(1-q))}-s
\end{aligned}$$

By (4.32) the above is maximized at $x=L_N$. Thus, the expected payoff at any $x\in[\tilde{p}_1, L_N]$ is upper bounded by the expected payoff at L_N. Now, from (4.42) (which also gives the expected payoff at L_N) we have already shown that the expected payoff at L_N is upper bounded by $\tilde{p}_3-c-s$. Hence, the maximum expected payoff attained by primary 1 in this case is $\tilde{p}_3-c-s$ and it is attained at price in the interval $[L_0, v]$.

Step iii: In Step (i), we have shown that under Y if primary 1 estimates that the channel state of primary 2 is 1, then the maximum expected payoff is $\tilde{p}_1-c-s$. In Step (ii), we have shown that if primary 1 estimates that the channel state of primary 2 is 0, then the maximum expected payoff is $\tilde{p}_3 - c - s$. Primary 1 estimates that the channel state of primary 2 is 1 w.p. $qq_s + (1-q)(1-q_s)$ and the channel state of primary 2 is 0 w.p. $(1-q)q_s + q(1-q_s)$. Thus, under Y, the maximum expected payoff that primary 1 can attain is

$$
\begin{aligned}
&(\tilde{p}_1 - c - s)(qq_s + (1-q)(1-q_s)) + (\tilde{p}_3 - c - s)((1-q)q_s + q(1-q_s)) \\
&= \tilde{p}_2 - c \quad \text{from (4.26).} \qquad (4.43)
\end{aligned}
$$

We have already shown that the payoff is achieved when primary 1 follows the strategy prescribed in the theorem.

Step iv: Now, we show that under N, the maximum expected payoff that primary 1 can attain is $\tilde{p}_2 - c$ and it is attained at every price in the interval $[L_N, L_0]$. Toward this end, we first show that if primary 1 selects a price from the interval $[L_N, L_0]$ it will fetch an expected payoff of $\tilde{p}_2 - c$ (Step iv.a). Subsequently, we show that if the primary selects any price from the interval $[L_0, v]$ or $[\tilde{p}_1, L_N]$ it can only get an expected payoff of at most $\tilde{p}_2 - c$ (Step iv.b. and Step iv.c. resp.).

Step iv.a: Suppose that primary 1 selects a price $x \in [L_N, L_0]$. When the channel of primary 2 is available, then primary 2 selects a price less than or equal to x if one of the following occurs–i) primary 2 selects Y and estimates the channel state of primary 1 is 1, ii) primary 2 selects N and selects a price less than or equal to x. (i) occurs with probability pq_s and (ii) occurs with probability $(1-p)\psi_N(x)$. Now, when primary 1 selects N it only knows that the channel of primary 2 is available w.p. q. Thus, the expected payoff of primary 1 at x is

$$(x-c)(1-pqq_s-(1-p)q\psi_N(x))=\tilde{p}_2-c \quad \text{from (4.20).} \tag{4.44}$$

Step iv.b: Note that when $q_s=1$, then $L_0=v$. Thus, we consider the case when $q_s<1$. Suppose primary 1 selects a price x in the interval $[L_0, v]$. If the channel of primary 2 is available, then, the primary 2 will select a price less than or equal to x if one of the following occurs–i) it selects Y and estimates that the channel state of primary 1 is 1, ii) primary 2 selects N, iii) primary 2 selects Y, estimates that the channel state of primary 1 is 0 and selects a price less than or equal to x. (i) occurs with probability pq_s since the channel of primary 1 is available. (ii) occurs with probability $1-p$. (iii) occurs with probability $p(1-q_s)\psi_{Y,0}x)$ (since the channel of primary 1 is available). When primary 1 selects N, it only knows that the channel of primary 2 is available w.p q. Thus, the expected payoff of primary 1 at x is

$$\begin{aligned}
&(x-c)(1-pqq_s-(1-p)q-pq(1-q_s)\psi_{Y,0}(x))\\
&=(x-c)(1-pqq_s-(1-p)q)\\
&-(x-c)\frac{(1-q)q_s+q(1-q_s)}{1-q_s}(1-\frac{\tilde{p}_3-c}{x-c}-\frac{pqq_s(1-q_s)+(1-p)q(1-q_s)}{(1-q)q_s+q(1-q_s)})\\
&=(x-c)(1-\frac{(1-q)q_s+q(1-q_s)}{1-q_s})+(\tilde{p}_3-c)\frac{(1-q)q_s+q(1-q_s)}{1-q_s}
\end{aligned}$$

Since $q_s>1/2$, the above is maximized at $x=L_0$. Thus, using (4.31), the above expression is upper bounded by

$$\begin{aligned}
&(L_0-c)(1-\frac{(1-q)q_s+q(1-q_s)}{1-q_s})+(L_0-c)\frac{pq(1-q_s)^2+q_s(1-q)}{1-q_s}\\
&=(L_0-c)(1-\frac{q(1-q_s)(1-p(1-q_s))}{1-q_s})=(L_0-c)(1-(1-p)q-pqq_s)=\tilde{p}_2-c.
\end{aligned} \tag{4.45}$$

where the last equality follows from (4.23).

Step iv.c: Now, suppose that primary 1 selects a price x in the interval $[\tilde{p}_1, L_N]$. Now, if the channel of primary 2 is available, then it selects a price less than or equal to x if it selects Y, estimates that the channel state of primary 1 is 1 and selects a price less than or equal to x. The above occurs with probability $pq_s\psi_{Y,1}(x)$. The channel of primary 2 is available w.p. q. Thus, at any price x in the interval $[\tilde{p}_1, L_N]$ the expected payoff of primary 1 is

$$\begin{aligned}
&(x-c)(1-pqq_s\psi_{Y,1}(x)) \\
&= (x-c)(1-\frac{qq_s+(1-q)(1-q_s)}{q_s}) + (\tilde{p}_1 - c)\frac{qq_s+(1-q)(1-q_s)}{q_s} \quad \text{cf (4.19).}
\end{aligned}$$

Since $q_s > 1/2$, the above is maximized at $x = L_N$. Thus, using (4.23), the maximum expected payoff is

$$\begin{aligned}
&(L_N-c)(1-\frac{qq_s+(1-q)(1-q_s)}{q_s}) + (L_N - c)\frac{qq_s(1-pq_s)+(1-q)(1-q_s)}{q_s} \\
&= (L_N-c)(1-\frac{qq_s pq_s}{q_s}) = \tilde{p}_2 - c. \qquad (4.46)
\end{aligned}$$

Again the last equality follows from (4.23).

Hence, the maximum expected payoff attained by primary 1 under N is $\tilde{p}_2 - c$. This is attained at any price in the interval $[L_N, L_0]$ which we have shown in Step iv.a.

Step v: Thus, either under Y or under N, the maximum expected payoff that primary 1 can attain is $\tilde{p}_2 - c$. Hence, any randomization between Y and N will also yield an expected payoff of $\tilde{p}_2 - c$. Primary 1 can attain the payoff of $\tilde{p}_2 - c$ following the strategy profile. Hence, primary 1 does not have any unilateral profitable deviation. Hence, the result follows. □

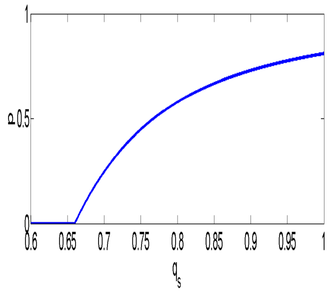


Figure 4.4: Variation of p with q_s for an example setting: $v = 50, c = 0, s = 4, q = 0.5$. p is 0 for $q_s \leq 0.67$, as s is above the threshold $q(v - c)(1 - q)(2q_s - 1)$ for this region.

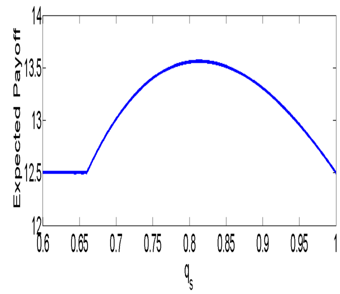


Figure 4.5: Variation of the expected payoff of a primary with q_s in the same example setting considered in Fig. 4.4.

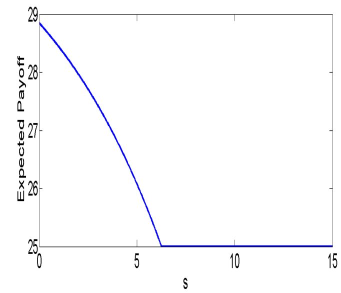


Figure 4.6: Variation of the expected payoff of a primary with s in an example setting: $v = 50, q = 0.5, c = 0, q_s = 3/4$.

4.3.5 Numerical Results

Fig. 4.4 shows that the probability p with which a primary selects Y increases as q_s increases. Intuitively, when q_s increases, the uncertainty of the channel state of the competitor decreases when a primary selects Y, thus, the primary selects Y with a higher probability. Additionally, Fig. 4.4 shows that the increment of p is sub-linear with q_s.

Fig. 4.5 shows the variation of the expected payoff of a primary with q_s. When $0.5 < q_s \leq 0.67$, a primary selects N w.p. 1, hence, the expected payoff is $(v - c)(1 - q)$ for $q_s \leq 0.67$. The expected payoff increases as q_s increases when $0.67 < q_s \leq 0.83$. After that the payoff decreases and ultimately the expected payoff again becomes equal to $(v - c)(1 - q)$ when $q_s = 1$. Thus, the payoff of a primary is higher when there is an error in estimation of the channel state compared to the setting where there is no error

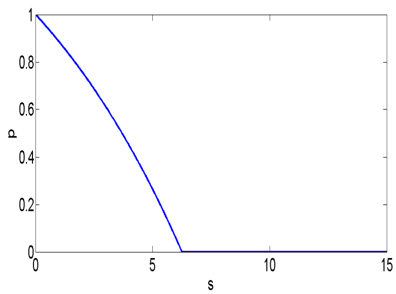


Figure 4.7: Variation of p with s in the same example setting as considered in Fig. 4.6.

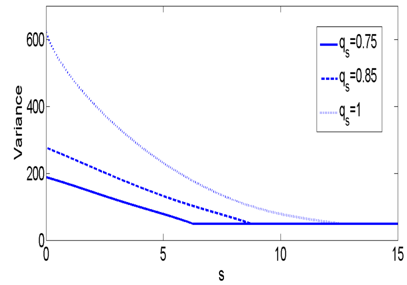


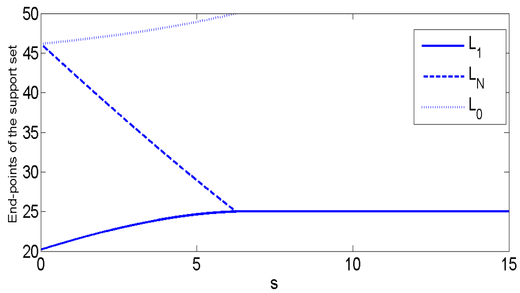
Figure 4.8: Variation of the variance of the price selected by a primary for an example setting:$v = 51, c = 1, q = 0.5$

in estimation which negates the conventional wisdom that the payoff should increases with the decreases in the error in the estimation. We have already provided the potential reasons behind this behavior in the section 4.3.2.

Fig. 4.6 shows the variation of the expected payoff of a primary with s. Note from Lemma 4.2 that the expected payoff of a primary increases as s decreases when a primary selects Y with a positive probability. Fig. 4.6 verifies the above result. Specifically, as s increases, the expected payoff decreases when $s < 6.5$. Additionally, the expected payoff decreases sub-linearly. When $s \geq 6.5$, a primary only selects N and attains an expected payoff of $(v - c)(1 - q)$, thus, the payoff becomes independent of s in this regime.

Fig. 4.7 shows that p, the probability with which a primary selects Y increases as s decreases. When $s \geq q(v - c)(1 - q)(2q_s - 1) = 6.25$, then the primary selects N w.p. 1 i.e. $p = 0$. Additionally, Fig. 4.7 shows that p decreases sub-linearly as s increases.

Fig. 4.8 shows the variation of the variance of the price selected by a primary with s and q_s. Note that the variance decreases as s decreases. Thus, when a primary selects N with a higher probability, the price volatility is lower. When $s \geq q(v - c)(1 - q)(2q_s - 1)$,



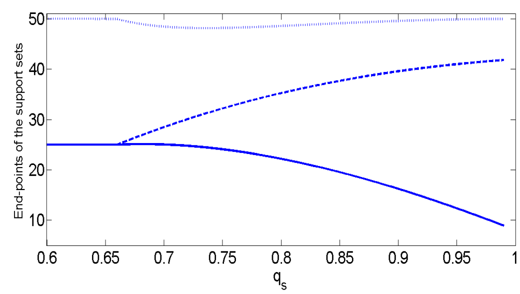


Figure 4.9: Variation of upper and lower end-points of the support sets of $\psi_{Y,1}$ (L_1 ($= \tilde{p}_1$) and L_N, resp.), ψ_N (L_N, and L_0 resp.) and $\psi_{Y,0}$ (L_0 and v) with s in the same example setting considered in Fig. 4.6.

Figure 4.10: Variation of upper and lower end-points of the support sets of $\psi_{Y,1}$ (L_1 and L_N, resp.), ψ_N (L_N, and L_0 resp.) and $\psi_{Y,0}$ (L_0 and v) with q_s in the same example setting considered in Fig. 4.4.

each primary selects N w.p. 1, thus, the variance becomes independent of s. This is because $\phi(\cdot)$, the price selection strategy from which a primary selects its price when $s \geq q(v-c)(1-q)(2q_s-1)$, is independent of s. Note that Fig. 4.8 also shows that the variance also decreases as q_s decreases. Intuitively, as q_s decreases, a primary selects N with a higher probability, thus, the variance decreases. Note that buyers in general do not like a market where the prices have higher variances. Thus, when s is low or q_s is high, a buyer may not like the setting.

Fig. 4.9 shows the variations of the end-points of the support sets of the price distributions. Note that when $s = 0$, $L_N = L_0$ as primaries select N with 0 probability. As s increases, L_N and L_0 increase as primaries select N with positive probability; primaries select prices from a larger interval when it selects N as s increases. Note that the lower end-point of $\psi_{Y,1}(\cdot)$, $\tilde{p}_1$ (L_1 in the figure) also increases as s increases. Thus, the price interval from which a primary selects its price Y decreases as s increases. Intuitively, as

s increases, a primary selects Y with a lower probability, thus the support also decreases. When $s \geq 6.25$, the primary only selects N, thus, $\tilde{p}_1 = L_N$ and $L_0 = v$.

Fig. 4.10 shows the variation of the end-points of the support sets of the price distributions. When $q_s \leq 0.67$, primaries only select N. Thus, $\tilde{p}_1 = L_N$ and $L_0 = v$. When $q_s > 0.67$, primaries select Y with positive probabilities. $\tilde{p}_1$ decreases as q_s increases. Thus, a primary selects a lower price when it selects Y and estimates that the channel state of the competitor is 1. Intuitively, as q_s increases, the uncertainty reduces, thus, the competitor's channel is more likely to be available when a primary estimates that the channel state of the primary is 1. Hence, the primary selects a lower price. Since primary selects Y when $q_s > 0.67$, L_0 decreases initially. However, L_0 increases when q_s becomes very high. Note that when q_s is very high, then a primary is aware that the channel of the competitor is more likely to be unavailable, hence it selects a high price. Thus, L_0 is close to v when q_s is very high. Note also that L_N increases with q_s. Intuitively, when q_s increases, a primary selects N with a lower probability, thus, a primary selects its price from a shorter interval when it selects N.

4.3.6 Price volatilities and payoffs in two settings

Note that in Chapter 2 we consider the setting where a primary can not acquire the CSI of the competitor. In this chapter, we consider the setting where a primary can acquire the CSI of the competitor by incurring a cost.

Note that when $q_s = 1$, the expected payoff of the primary is the same in both the settings. Thus, the primaries do not have any strict preference of one setting over another. However, Fig. 4.8 shows that the variance of the price increases when a primary acquires the CSI of the competitor. Thus, the regulator (e.g. FCC) and the secondary may prefer the setting where the primary *does not* acquire the CSI of the competitor.

Above remark also shows that the current auction framework in the TV white space trading (where the spectrum brokers gather the information of the unused spectrum) may have high price volatility. In such a trading, if the demand is low the price will be c, otherwise it will be v, (similar to the Bertrand model). Thus, the price will have higher volatility compared to the setting where the primary does not acquire the CSI of the competitor[7]. It shows one additional disadvantage of using Auction mechanism or the spectrum brokers in the real time secondary spectrum market.

When $q_s < 1$, the price volatility decreases but still the price volatility is higher compared to setting where the primary does not acquire the competitor's CSI. However, in this case the expected payoff of the primary is higher.

4.4 Unequal Costs

We, now, investigate the generalization of the basic model where each different primaries incur different costs to acquire the CSI of their respective competitors depicted in Section 4.1.2. Primary i incurs the cost s_i to acquire the CSI of its competitor. Without loss of generality we assume that $s_1 < s_2$.

[7]Though, in this chapter we have only considered the setting where $n = 1$, similar remark can also be made when $n > 1$. In Chapter 2 we have seen that the price volatility decreases as n increases as primaries select prices using pure strategies (Fig. 2.4).

4.4.1 Goals

The impact of different acquisition costs on the payoff of each primary and the frequency with which each primary selects Y is not apriori clear. For example, primary 1 which has a lower acquisition cost of CSI, can gain more compared to primary 2 by acquiring the CSI of primary 2 by paying a lower cost. However, primary 2 also acquires the CSI of primary 1 and selects a lower price when the channel of primary 1 is available, thus, primary 1 also selects a lower price in response, which in turn reduces the payoff of primary 1. The pricing decision of each primary also depends on the frequencies with which each primary selects Y. We resolve all these quandaries.

4.4.2 Results

We summarize our main findings here–

- The NE strategy is of the form $[T, p_i]$ for primary i with $T = q(v-c)(1-q)$. Note that T is the same as the basic model, however, since different primaries have different acquisition costs, p_is are different. For example, when $s_1 < T \leq s_2$, then primary 1 selects Y w.p. p_1, but primary 2 does not select Y. Even when $s_2 < T$, then primary 2 selects p_2 where $p_1 > p_2$ as $s_1 < s_2$.

- The difference in the acquisition costs lead to different payoffs for the primaries. In contrast to the basic model, primary 1 attains a higher payoff compared to the expected payoff of primary 2 when primary 1 selects Y with a positive probability (i.e. $s < T$) (Theorems 4.8, 4.9). The expected payoff of primary 1 becomes close to the payoff of the primary 2 as the difference between s_1 and s_2 decreases. The expected payoff of primary 2 is in fact independent of s_2. The expected payoff of the primaries are the same when $s_1 \geq T$, as both of them only select N.

- Primary i selects its price from the interval $[L, \tilde{p}_i]$ ($[\tilde{p}_i, v]$, resp.) , when the primary selects Y (N, resp.) and the channel of the competitor is available. However, there are also some differences in the pricing structure compared to the basic model because of different acquisition costs. Primary 2 selects v with a positive probability when it selects N when $s_1 < q(v-c)(1-q)$ and the probability decreases as the difference between s_1 and s_2 decreases (Theorems 4.9). Thus, the primary 2 has a discontinuity at v in contrast to the basic model where primaries select prices from continuous distribution. Additionally, we show that $\tilde{p}_1 > \tilde{p}_2$. Thus, primary 2 selects lower prices when it selects Y and the channel of primary 1 is available. On the other hand, when primary 2 selects N, its selects higher prices with higher probabilities.

4.4.3 High s_1, s_2

Our first result in this section shows that

Theorem 4.7. *When $s_1 \geq q(v-c)(1-q)$, then in the unique NE, both the primaries select N w.p. 1 and select their prices according to $\phi(\cdot)$ (given in (4.2)). Both the primaries attain an expected payoff of $q(v-c)(1-q)$.*

The proof is similar to the proof of Theorem 4.3, thus, we omit it here.

Note that since $s_2 > s_1$, $s_2 > q(v-c)(1-q)$. Thus, the above theorem shows that the expected payoff of primaries are identical when s_is are sufficiently high as both the primaries select N.

4.4.4 Low s_1, high s_2

Now, we consider the setting where $s_1 < q(v-c)(1-q)$, but $s_2 \geq q(v-c)(1-q)$. We show that there exists a NE where primary 1 randomizes between Y and N, and primary 2 selects N.

We first introduce some pricing distributions which we use throughout this section–

$$\begin{aligned} \psi_{1,Y}(x) =& 0, \quad \text{if } x < \tilde{p} \\ & \frac{1}{qp_1}(1 - \frac{\tilde{p}-c}{x-c}), \quad \text{if } \tilde{p} \leq x \leq \tilde{p}_1 \\ & 1, \quad \text{if } x > \tilde{p}_1 \end{aligned} \tag{4.47}$$

$$\begin{aligned} \psi_{1,N}(x) =& 0, \quad \text{if } x < \tilde{p}_1 \\ & \frac{1}{q(1-p_1)}(1 - \frac{\tilde{p}-c}{x-c} - qp_1) \text{if } \tilde{p}_1 \leq x \leq v \\ & 1, \quad \text{if } x > v \end{aligned} \tag{4.48}$$

$$\begin{aligned} \psi_2(x) =& 0, \quad \text{if } x < \tilde{p} \\ & (1 - \frac{\tilde{p}-c}{x-c}), \quad \text{if } \tilde{p} \leq x < \tilde{p}_1 \\ & \frac{1}{q}(1 - \frac{\tilde{p}_N - c}{x-c}), \text{if } \tilde{p}_1 \leq x < v \\ & 1, \quad \text{if } x \geq v. \end{aligned} \tag{4.49}$$

where

$$\begin{aligned} \tilde{p}_N &= (v-c)(1-q) + q(v-c)(1-q) - s_1 + c \\ \tilde{p} &= (v-c)(1-q) + c. \quad \tilde{p}_1 - c = \frac{(v-c)(1-q)}{1-qp_1}. \end{aligned} \tag{4.50}$$

and

$$p_1 = \frac{1}{q}(1 - \frac{(v-c)(1-q)^2}{(v-c)(1-q) - s_1}) \tag{4.51}$$

Note from (4.50) and (4.51) that

$$\tilde{p}_1 - c = \frac{(v-c)(1-q)[(v-c)(1-q) - s_1]}{(v-c)(1-q)^2} = \frac{(v-c)(1-q) - s_1}{1-q} \tag{4.52}$$

$\psi_2(\cdot)$ clearly has a jump at v as $s_1 < q(v-c)(1-q)$. From the expression of $\psi_2(\cdot)$ one may think that $\psi_2(\cdot)$ has a jump at $\tilde{p}_1$. We first rule out the above possibility.

Observation 4.2. $\psi_2(\cdot)$ does not have any jump except at v.

Proof. First, note that since $s_1 < q(v-c)(1-q)$, thus, $\psi_2(\cdot)$ has a jump at v.

Next, we show that $\psi_2(\cdot)$ does not have any jump at $\tilde{p}_1$. The continuity of $\psi_2(\cdot)$ at any other point can easily be observed.

Note from (4.52) the left hand limit is

$$1 - \frac{(v-c)(1-q)}{\tilde{p}_1 - c} = 1 - \frac{(v-c)(1-q)^2}{(v-c)(1-q) - s_1} \tag{4.53}$$

Again from (4.52), the right hand limit and the value of $\psi_2(\cdot)$ at $\tilde{p}_1$ is

$$\begin{aligned} &\frac{1}{q}(1 - \frac{(v-c)(1-q) + q(v-c)(1-q) - s_1}{\tilde{p}_1 - c} \\ &= \frac{1}{q}(1 - \frac{(1-q)[(v-c)(1-q) + q(v-c)(1-q) - s_1]}{(v-c)(1-q) - s_1} \\ &= 1 - \frac{(v-c)(1-q)^2}{(v-c)(1-q) - s_1} \end{aligned} \tag{4.54}$$

Hence, $\psi_2(\cdot)$ does not have any jump at $\tilde{p}_1$. Hence, the result follows. □

The continuity of $\psi_{1,Y}(\cdot)$ and $\psi_{1,N}(\cdot)$ can be easily concluded. Note that the variations of $\psi_{1,Y}(\cdot)$, $\psi_{1,N}(\cdot)$ and $\psi_2(\cdot)$ are similar they differ only in the support and the scaling parameters.

Now, we are ready to state the main result of this section.

Theorem 4.8. *Consider the following strategy profile: Primary* 1 *selects* Y *w.p.* p_1 *and* N *w.p.* $1-p_1$ *(*p_1 *is given in (4.51)) and primary* 2 *selects* N *w.p.* 1*. While selecting* Y*, if the channel of primary* 2 *is available, then primary* 1 *selects its price according to* $\psi_{1,Y}(\cdot)$*, otherwise it selects* v *w.p.* 1*. While selecting* N*, primary* 1 *selects its price according to* $\psi_{1,N}(\cdot)$*. Primary* 2 *selects its price according to* $\psi_2(\cdot)$*.*

The above strategy profile is an NE when $s_2 \geq q(v-c)(1-q)$ *and* $s_1 < q(v-c)(1-q)$*. The expected payoff that primary* 1 *attains is* $(v-c)(1-q)+q(v-c)(1-q)-s_1$ *and the expected payoff of primary* 2 *is* $(v-c)(1-q)$*.*

Discussion: Note that when $s_1 < q(v-c)(1-q)$ and $s_2 \geq q(v-c)(1-q)$, the payoff of primary 1 is higher compared to the primary 2. Apparently, when s_1 is low, then primary 1 takes advantage of the acquired CSI and gains more compared to primary 2. Primary 2 can not do the same as the cost s_2 is high. The expected payoff of primary 1 also increases with the decrease in s_1. Note that the threshold above which primary i only selects N is $q(v-c)(1-q)$; the threshold is the same for both the players.

The probability p_1 increases with decrease in s_1, hence, primary 1 is more likely to select Y with the decrease in s_1.

Since p_1 increases as s_1 decreases. $\tilde{p}_1$ also increases as s_1 decreases (from (4.52)) . Thus, $\psi_{1,Y}(\cdot)$ has larger support as s_1 decreases.

Under Y, primary 1 selects a price from the interval $[\tilde{p}, \tilde{p}_1]$ when the channel of primary 2 is available; under N, primary 1 selects a price from the interval $[\tilde{p}_1, v]$. Hence, primary 1 selects higher price under N as the uncertainty of the CSI of other primary increases.

Also note that $\psi_2(\cdot)$ overlaps both with $\psi_{1,Y}(\cdot)$ and $\psi_{1,N}(\cdot)$.

Also note that $\psi_2(\cdot)$ has a jump at v. Thus, primary 2 selects v with a positive probability. Intuitively, primary 1 selects Y with a higher probability. Thus, primary 1 knows the channel state of primary 2 with a higher probability and thus and selects a lower price. In response, primary 2 has two options– i) selects a high price with high probability (at least it can gain more when the channel of the primary 1 is not available), ii) selects a low price (it can increase the probability of winning). Our result shows that primary 2 selects the first option.

Proof of Theorem 4.8

First, we show that there is no profitable deviation for primary 1 when primary 2 follows the prescribed strategy stated in Theorem 4.8 (Case I), subsequently, we show that there is also no profitable deviation for primary 2 when primary 1 follows the prescribed strategy stated in Theorem 4.8 (Case II).

Case I: In the first step (i), we show that primary 1 can attain a maximum expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-s$ under Y. Next in step (ii), we show that primary 1 can attain a maximum expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-s_1$ under N. Finally in step (iii), we show that primary 1 attains the maximum expected payoff following the strategy which will show that primary 1 does not have any profitable unilateral deviation.

Step (i): Primary 1 selects Y. Suppose that the channel of primary 2 is available, then primary 1 will know that w.p. 1. At any x such that $\tilde{p}\leq x\leq \tilde{p}_1$ the primary 1 gets under Y is

$$(x-c)(1-\psi_2(x)) - s_1 = (v-c)(1-q) - s_1 \quad \text{from (4.49)\&(4.50).} \tag{4.55}$$

If primary 1 selects a price strictly less than $\tilde{p}$, then, its payoff is strictly less than $\tilde{p} - c - s_1 = (v-c)(1-q) - s_1$.

Now, at any $v > x \geq \tilde{p}_1$, the expected payoff of primary 1 in this setting is

$$\begin{aligned}
&(x-c)(1-\psi_2(x)) - s_1 \\
&= (x-c)(1-\frac{1}{q}(1-\frac{(v-c)(1-q)+q(v-c)(1-q)-s_1}{x-c})) - s_1 \text{from (4.49)} \\
&= (x-c)(1-\frac{1}{q}) + \frac{(v-c)(1-q)+q(v-c)(1-q)-s_1}{q} - s_1 \qquad (4.56)
\end{aligned}$$

Since $1/q > 1$, thus, the supremum is attained at $x = \tilde{p}_1$. Now from (4.52), the maximum value is

$$(v-c)(1-q) - s_1 \tag{4.57}$$

Since $\psi_2(\cdot)$ has a jump at v, thus, the expected payoff at v is strictly less than the value at a price close to v. Hence, when the channel of primary 2 is available, then the maximum expected payoff that primary 1 can attain at Y is $(v-c)(1-q) - s_1$ and it is attained at any price in the interval $[\tilde{p}, \tilde{p}_1]$.

Now, when the channel of primary 2 is unavailable, the expected payoff of primary 1 is $(v-c) - s_1$. Hence, the maximum expected payoff that primary 1 attains in Y is

$$\begin{aligned}
&(v-c-s_1)(1-q) + q[(v-c)(1-q) - s_1] \\
&= (v-c)(1-q) + q(v-c)(1-q) - s_1 \qquad (4.58)
\end{aligned}$$

Step (ii) Now suppose primary 1 selects N and a price x such that $\tilde{p}_1 \leq x < v$. Primary 2 selects a price less than x if the channel of primary 2 is available and selects a price less than or equal to x (it occurs w.p. $q\psi_2(x)$). By the continuity of $\psi_2(\cdot)$ in the interval $[\tilde{p}_1, v)$, the expected payoff of primary 1 at x is

$$(x-c)(1-q\psi_2(x)) = (v-c)(1-q) + q(v-c)(1-q) - s_1 \quad \text{from (4.49).} \tag{4.59}$$

Since $\psi_2(\cdot)$ has a jump at v, thus, the expected payoff of primary 1 is strictly less at a price close to v compared to v. Thus, the expected payoff of primary 1 at v is strictly less than $(v-c)(1-q) + q(v-c)(1-q) - s_1$.

Now, at any x such that $\tilde{p} \leq x \leq \tilde{p}_1$, the expected payoff of primary 1 under N is

$$\begin{aligned}(x-c)(1-q\psi_2(x)) &= (x-c)(1-q(1-\frac{(v-c)(1-q)}{x-c})) \\ &= (x-c)(1-q) + q(v-c)(1-q) \quad \text{from (4.49).}\end{aligned} \tag{4.60}$$

The supremum is attained at $x = \tilde{p}_1$. Putting the value of $\tilde{p}_1$ from (4.52) we obtain

$$(v-c)(1-q) - s_1 + q(v-c)(1-q) \tag{4.61}$$

The expected payoff at a price strictly less than $\tilde{p}$ will fetch a payoff which is strictly less than the payoff attained at $\tilde{p}$. Thus, primary 1 can attain at most an expected payoff of $(v-c)(1-q) + q(v-c)(1-q) - s_1$ under N. The maximum expected payoff is attained at any price in the interval $[\tilde{p}_1, v)$.

Step (iii): Hence, we show that the primary 1 can attain an expected payoff of $(v-c)(1-q) + q(v-c)(1-q) - s$ under either Y or N. Thus, any randomization between Y and N will also give an expected payoff of $(v-c)(1-q) + q(v-c)(1-q) - s_1$. Now, under the strategy profile the expected payoff is also $(v-c)(1-q)+q(v-c)(1-q)-s_1$, hence, primary 1 does not have any profitable deviation.

Case II: Now, we show that primary 2 does not have any profitable deviation when primary 1 selects the prescribed strategy stated in the theorem. Towards this end, we first show in Step (i) that any price in the interval $[\tilde{p}, \tilde{p}_1]$ will give an expected payoff of $(v-c)(1-q)$ to primary 2 when it selects N, subsequently, we show that any price in the interval $[\tilde{p}_1, v]$ will also provide an expected payoff of $(v-c)(1-q)$ to primary 2 when it selects N . In step (iii), we show that any price $x < \tilde{p}$ will give a strictly lower payoff compared to $(v-c)(1-q)$ when it selects N. Finally in step (iv), we show that if primary 2 selects Y, then it can only get a payoff of at most $(v-c)(1-q)$ when $s_2 \geq q(v-c)(1-q)$. This will show that primary 2 attains the maximum expected payoff of $(v-c)(1-q)$ and it is attained when it selects N and selects a price in the interval $[\tilde{p}, v]$.

Step (i): Note that when $x \in [\tilde{p}, \tilde{p}_1]$ primary 1 can select a price less than x only when the channel of primary 1 is available and primary 1 selects Y, thus, at any x such that $x \in [\tilde{p}, \tilde{p}_1]$, the expected payoff of primary 2 is

$$(x-c)(1-qp_1\psi_{1,Y}(x)) = (v-c)(1-q) \quad \text{from (4.47)}. \tag{4.62}$$

Step (ii) When $x \in [\tilde{p}_1, v]$, then primary 1 selects a price lower than x only if the channel of primary 1 is available and either it selects Y or while selecting N it selects a price less than x. Hence, the expected payoff of primary 2 is

$$(x-c)(1-qp_1-q(1-p_1)\psi_{1,N}(x)) = (v-c)(1-q) \quad \text{from (4.48)}. \tag{4.63}$$

(iii) At any price less than $\tilde{p}$ will fetch a payoff which is strictly less than $\tilde{p}-c$. However, $\tilde{p}-c = (v-c)(1-q)$. Thus, the expected payoff of primary 2 is strictly less than

$(v-c)(1-q)$ at any price less than $\tilde{p}$. Hence, primary 2 can only attain a maximum expected payoff of $(v-c)(1-q)$ and it is attained at the prices in the interval $[\tilde{p}, v]$.

(iv) Now, suppose primary 2 selects Y. If the channel of primary 1 is available, then at any $x \in [\tilde{p}, \tilde{p}_1]$, it will get an expected payoff of

$$\begin{aligned}(x-c)(1-p_1\psi_{1,Y}(x)) - s_2 &= (x-c)(1-\frac{1}{q}(1-\frac{(v-c)(1-q)}{x-c}) - s_2 \\ &= (x-c)(1-1/q) + (v-c)(1-q)/q - s_2 \end{aligned} \quad (4.64)$$

Since $1/q > 1$, thus the above is maximized at $x = \tilde{p}$, and the maximum expected payoff is $(v-c)(1-q) - s_2$ since $\tilde{p} - c = (v-c)(1-q)$.

Now, if primary 2 selects a price in the interval $[\tilde{p}_1, v]$ when the channel of primary 1 is available, then the expected payoff of primary 2 is

$$\begin{aligned}&(x-c)(1-(1-p_1)\psi_{1,N}(x) - p_1) - s_2 \\ &= (x-c)(1-p_1-\frac{1}{q}(1-\frac{(v-c)(1-q)}{x-c} - qp_1)) - s_2 \quad \text{from (4.48)} \\ &= (x-c)(1-1/q) + (v-c)(1-q)/q - s_2 \\ &< (\tilde{p}-c)(1-1/q) + (v-c)(1-q)/q - s_2 = (v-c)(1-q) - s_2 \\ &\text{since } \tilde{p} - c = (v-c)(1-q).\end{aligned}$$

Thus, primary 2 attains an expected payoff of at most $(v-c)(1-q) - s_2$ when it selects Y and the channel of primary 1 is available.

Now, when the channel of primary 1 is unavailable the payoff that primary 2 earns is $v - c - s_2$. Hence, the maximum expected payoff that primary 2 can earn by selecting Y

is

$$q[(v-c)(1-q)-s_2]+(1-q)(v-c-s_2)=q(v-c)(1-q)+(v-c)(1-q)-s_2 \tag{4.65}$$

when $s_2 \geq q(v-c)(1-q)$, thus, the primary attains at most a payoff of $(v-c)(1-q)$. Hence, primary 2 also does not have any profitable deviation. □

4.4.5 Low s_1, s_2

Lastly, we show that if $s_2 < q(v-c)(1-q)$ then, there exists an NE where primary 2 also randomizes between Y and N.

Again, we introduce some price distribution functions

$$\begin{aligned}\psi_{1,Y} =& 0, \quad x < L \\ & \frac{1}{p_1}(1-\frac{L-c}{x-c}), \quad L \leq x \leq \tilde{p}_2 \\ & \frac{1}{p_1 q}(1-\frac{(v-c)(1-q)}{x-c}), \quad \tilde{p}_2 < x \leq \tilde{p}_1 \\ & 1, \quad x > \tilde{p}_1\end{aligned} \tag{4.66}$$

$$\begin{aligned}\psi_{2,Y} =& 0, \quad x < L \\ & \frac{1}{p_2}(1-\frac{L-c}{x-c}), \quad L \leq x \leq \tilde{p}_2 \\ & 1, \quad x > \tilde{p}_2\end{aligned} \tag{4.67}$$

$$\begin{aligned}\psi_{1,N} =& 0, \quad x < \tilde{p}_1 \\ & \frac{1}{q(1-p_1)}(1-\frac{(v-c)(1-q)}{x-c}-p_1 q), \quad \tilde{p}_1 \leq x \leq v \\ & 1, \quad x > v\end{aligned} \tag{4.68}$$

and

$$\begin{aligned}
\psi_{2,N} =& 0, \quad x < \tilde{p}_2 \\
& \frac{1}{1-p_2}(1 - \frac{L-c}{x-c} - p_2), \quad \tilde{p}_2 \le x < \tilde{p}_1 \\
& \frac{1}{q(1-p_2)}(1 - \frac{\tilde{p}_{1,N}-c}{x-c} - p_2 q) \quad \tilde{p}_1 \le x < v \\
& 1, \quad x \ge v
\end{aligned} \tag{4.69}$$

where

$$\begin{aligned}
\tilde{p}_{1,N} - c = (v-c)(1-q) + s_2 - s_1, \quad L - c = \frac{s_2}{q}, \quad \tilde{p} = (v-c)(1-q) + c. \\
\tilde{p}_2 - c = (v-c)(1-q)/(1-p_2 q), \quad \tilde{p}_1 - c = \frac{(v-c)(1-q)}{1-p_1 q}
\end{aligned} \tag{4.70}$$

The values of p_1 and p_2 are

$$p_1 = \frac{q(v-c)(1-q) - s_1}{q(v-c)(1-q) - qs_1}, p_2 = \frac{q(v-c)(1-q) - s_2}{q(v-c)(1-q) - qs_2} \tag{4.71}$$

Since $s_1 < s_2$, $p_1 > p_2$. Note that $p_i, i = 1, 2$ only depends on s_i. Using the values of p_1 and p_2, we obtain from (4.70)

$$\tilde{p}_2 - c = \frac{(v-c)(1-q) - s_2}{1-q} = \frac{s_2}{q(1-p_2)} \tag{4.72}$$

and

$$\tilde{p}_1 - c = \frac{(v-c)(1-q) - s_1}{1-q} = \frac{s_1}{q(1-p_1)} \tag{4.73}$$

We also use the above equalities throughout this section.

It is easy to discern that $\psi_{1,N}(\cdot)$ and $\psi_{2,Y}(\cdot)$ are continuous. Now, we show that $\psi_{1,Y}(\cdot)$ is also continuous.

Observation 4.3. $\psi_{1,Y}(\cdot)$ is a continuous function.

Proof. We only show that $\psi_{1,Y}(\cdot)$ is continuous at $\tilde{p}_2$, it is easy to discern that $\psi_{1,Y}(\cdot)$ at other values. Note from (4.72) and (4.70), the left hand limit is

$$\frac{1}{p_1}(1 - \frac{L-c}{\tilde{p}_2 - c}) = \frac{1}{p_1}p_2$$

Now from (4.70), the right hand limit and the value of $\psi_{1,Y}(\cdot)$ at $\tilde{p}_2$ is

$$\frac{1}{qp_1}(1 - \frac{(v-c)(1-q)}{\tilde{p}_2 - c}) = \frac{1}{qp_1}p_2 q = p_2/p_1$$

Hence, $\psi_{1,Y}(\cdot)$ does not have a jump at $\tilde{p}_2$. □

Next, we show that $\psi_{2,N}(\cdot)$ is continuous everywhere but at v.

Observation 4.4. $\psi_{2,N}(\cdot)$ is continuous except at v.

Proof. Since $s_1 < s_2$, thus, it is easy to discern that $\psi_{2,N}(\cdot)$ has a jump at v.

Now, we show that $\psi_{2,N}(\cdot)$ does not have a jump at $\tilde{p}_1$. It is easy to discern that $\psi_{2,N}(\cdot)$ can not have a jump at any other point.

From (4.73), we have

$$\begin{aligned}
&\frac{(v-c)(1-q) + s_2 - s_1}{\tilde{p}_1 - c} \\
&= [(v-c)(1-q) + s_2 - s_1]\frac{1-q}{(v-c)(1-q) - s_1} \\
&= 1 - q + s_2\frac{1-q}{(v-c)(1-q) - s_1} \qquad (4.74)
\end{aligned}$$

Thus, the left hand limit at $\tilde{p}_1$ is

$$\begin{aligned}
&\frac{1}{q(1-p_2)}(1 - \frac{(v-c)(1-q) + s_2 - s_1}{\tilde{p}_1 - c} - p_2 q) \\
&= 1 - \frac{s_2(1-q)}{q(1-p_2)[(v-c)(1-q) - s_1]} \qquad (4.75)
\end{aligned}$$

Now, the right hand limit and the value of $\psi_{2,N}(\cdot)$ at $\tilde{p}_1$ is

$$\frac{1}{1-p_2}(1-\frac{L-c}{\tilde{p}_1-c}-p_2) = 1-\frac{s_2(1-q)}{q(1-p_2)[(v-c)(1-q)-s_1]} \quad \text{from (4.73).} \qquad (4.76)$$

Hence, $\psi_{2,N}(\cdot)$ does not have a jump at $\tilde{p}_1$. □

Note that though $\psi_{2,N}(\cdot)$ has a jump at v, the variation of $\psi_{2,N}$ with x is similar to the other distributions $\psi_{1,N}(\cdot), \psi_{1,Y}(\cdot)$ and $\psi_{2,Y}(\cdot)$.

Now, we are ready to state the main result in this section.

Theorem 4.9. *Consider the following strategy profile: Primary* 1 *selects* Y *w.p.* p_1 *and* N *w.p.* $1-p_1$ *and primary* 2 *selects* Y *w.p.* p_2 *and* N *w.p.* $1-p_2$ *where* p_1 *and* p_2 *are given in (4.71). While selecting* Y*, primary* $i=1,2$ *selects its price according to* $\psi_{i,Y}(\cdot)$ *when the channel of primary* $j, j \neq i$ *is available and will select the price* v *if the channel of primary* j *is unavailable; while selecting* N*, primary* i *selects its price according to* $\psi_{i,N}(\cdot)$*.*

The above strategy profile is an NE when $s_2 < q(v-c)(1-q)$*. The expected payoff that primary* 1 *attains is* $(v-c)(1-q)+s_2-s_1$ *and primary* 2 *attains is* $(v-c)(1-q)$*.*

Discussion: Since $s_2 > s_1$, thus, the expected payoff of primary 1 is higher compared to primary 2. Since $s_2 < q(v-c)(1-q)$, thus by Theorem 4.8 the expected payoff of primary 1 is lower compared to the setting where $s_2 \geq q(v-c)(1-q)$. Note also that the payoff of primary 1 decreases with s_2, but increases with s_1. Thus, if s_2 decreases it only impacts the payoff of primary 1, it does not affect the payoff of primary 2. The payoff of primary 1 also becomes closer to the payoff of primary 2 as s_2 becomes closer to s_1 and ultimately becomes equal when $s_2 = s_1$ which we have already seen in Section 4.2 where we analyze the scenario when both the primaries have identical cost to acquire the CSI of their respective competitors i.e. $s_2 = s_1 = s$.

p_1 (p_2,resp.) increases with the decrease in s_1 (s_2, resp.). Thus, primaries are more likely to select Y as the cost s_1, s_2 decrease. When $s \to 0$, then $p_1 \to 1$, and when $s_2 \to 0$, then $p_2 \to 1$. Since $s_1 < s_2$, thus, $p_1 > p_2$. Primary 1 is more likely to select Y compared to primary 2.

Note that under Y, primary 1 (primary 2, resp.) selects its price from the interval $[L, \tilde{p}_1]$ ($[L, \tilde{p}_2]$, resp.) when the channel of its competitor is available. Note that L is less than $\tilde{p}$ (given in (4.50)) as $s_2 < q(v-c)(1-q)$. Hence, each primary selects its price from a larger interval when both primaries randomize between Y and N. Also note that L decreases as s_2 decreases. However, L is independent of s_1. Since $\tilde{p}_1 > \tilde{p}_2$, hence, under Y primary 1 selects its price from a wider interval compared to primary 2. Also note from (4.72) and (4.73) $\tilde{p}_1$ and $\tilde{p}_2$ increase as s_1 and s_2 decrease respectively. Hence, $\psi_{i,Y}(\cdot)$ has larger supports as s_i decreases.

$\psi_{2,N}(\cdot)$ has a jump at v similar to the setting when $s_2 \geq q(v-c)(1-q)$ but $s_1 < q(v-c)(1-q)$.

Proof of Theorem 4.9

First, we show that there is no profitable unilateral deviation for primary 1 when primary 2 follows the strategy prescribed in Theorem 4.9 (Case I). Subsequently, we also show that there is no unilateral profitable deviation for primary 2 when primary 1 follows the strategy prescribed in Theorem 4.9 (Case II).

Case I: First, we show that under Y, when the channel of primary 2 is available, then the maximum expected payoff that primary 1 can attain is $L-c-s_1$ and it is attained at prices in the interval $[L, \tilde{p}_1]$ (Step i). Next, we show that when the channel of primary 2 is not available, then under Y the primary 1 attains a payoff of $(v-c)-s_1$ (by selecting price v). This shows that the maximum expected payoff that primary 1 attains under Y is $(v-c)(1-q)+s_2-s_1$ (Step ii). Subsequently, we show that under N, the maximum expected payoff that a primary can attain is $(v-c)(1-q)+s_2-s_1$ and it is attained only at the prices in the interval $[\tilde{p}_1, v)$ (Step iii). Finally, we show that the maximum expected payoff attained by primary 1 is $(v-c)(1-q)+s_2-s_1$ and it is attained when primary 1 follows the strategy (Step iv).

(i) Suppose that primary 1 selects Y and the channel of primary 2 is available. First, we show that at any price $x \in [L, \tilde{p}_2]$, the expected payoff of primary 1 in this case is $L-c-s_1$ (Step i.a.). Subsequently, we show that at any price $x \in [\tilde{p}_2, \tilde{p}_1]$ the expected payoff of primary 1 is $L-c-s_1$ (Step i.b.). Next, we show that at any price price $x \in [\tilde{p}_1, v]$ the expected payoff of primary 1 is at most $L-c-s_1$ (Step i.c.). Note that at a price less than L will fetch a payoff of strictly less than the payoff of $L-c-s_1$. Hence, this will show that when the channel of primary 2 is available, then under Y the maximum expected payoff attained by primary 1 is $L-c-s_1$ and it is attained only at prices $[L, \tilde{p}_1]$.

Step i.a.: Suppose that primary 1 selects a price $x \in [L, \tilde{p}_2]$. Since the primary 2 selects a price less than or equal to x if it selects Y and then selects a price less than or equal to x (it occurs w.p. $p_2\psi_{2,Y}(x)$). Thus, the expected payoff of primary 1 is

$$(x-c)(1-p_2\psi_{2,Y}(x))-s_1 = L-c-s_1 \quad \text{from (4.67)}.$$

Step i.b.: Now, suppose that primary 1 selects a price x in the interval $[\tilde{p}_2, \tilde{p}_1]$. Primary 2 selects a price less than or equal to x if it either selects Y or it selects N and then selects a price less than or equal to x. Thus, at any price x, the expected payoff of primary 1 is

$$(x-c)(1-(1-p_2)\psi_{2,N}(x)-p_2)-s_1 = L-c-s_1 \quad \text{from (4.69)}.$$

Thus, at any price in the interval $[\tilde{p}_2, \tilde{p}_1]$ fetches the primary an expected payoff of $L-c-s_1$.

Step i.c: Now, at any price in the interval $[\tilde{p}_1, v)$ the expected payoff of primary 1 is

$$\begin{aligned}
&(x-c)(1-(1-p_2)\psi_{2,N}(x)-p_2)-s_1 \\
&= (x-c)(1-\frac{1}{q}(1-\frac{(v-c)(1-q)+s_2-s_1}{x-c}-qp_2)-p_2)-s_1 \text{from (4.69)} \\
&= (x-c)(1-1/q)+((v-c)(1-q)+s_2-s_1)/q-s_1 \qquad (4.77)
\end{aligned}$$

Since the co-efficient of $(x-c)$ is negative, the above is maximized at $x=\tilde{p}_1$. Now, from (4.73)

$$(\tilde{p}_1-c)(1-1/q) = -((v-c)(1-q)-s_1)/q$$

Thus, (4.77) is upper bounded by $L-c-s_1$.

Since $\psi_{2,N}(\cdot)$ has a jump at v, thus, the expected payoff of primary 1 at v is strictly less than the expected payoff at a price close to v. Thus, the maximum expected payoff attained in the interval $[\tilde{p}_1, v]$ is $s_2/q - s_1 = L-c-s_1$ (by(4.70)).

Thus, under Y when the channel of primary 2 is available, the maximum expected payoff of primary 1 is $L-c-s_1$ and it is attained in every price in the interval $[L, \tilde{p}_1]$.

(ii) When the channel of primary 2 is unavailable, then the payoff of primary 1 is $(v-c)-s_1$ as primary 1 selects v and is still capable of selling its channel. Hence, the maximum expected payoff of primary 1 under Y is

$$qs_2/q + (v - c)(1 - q) - s_1 = (v - c)(1 - q) + s_2 - s_1 \tag{4.78}$$

(iii) Now, we show that under N, the maximum expected payoff of primary 1 is at most $(v-c)(1-q)+s_2-s_1$ and it is attained when it follows the strategy $\psi_{1,N}(\cdot)$. Toward this end, we first show that if primary 1 selects a price in the interval $[\tilde{p}_1, v]$ the expected payoff is $(v-c)(1-q)+s_2-s_1$ and it is attained at any price in the interval $[\tilde{p}_1, v)$ (Step iii.a). Subsequently, we show that if primary 1 selects a price in the interval $[\tilde{p}_2, \tilde{p}_1]$, then the expected payoff under N is at most $(v-c)(1-q)+s_2-s_1$ (Step iii.b.). Finally, we show that even if primary 1 selects a price in the interval $[L, \tilde{p}_2)$, then the expected payoff is also at most $(v-c)(1-q)+s_2-s_1$ under N (Step iii.c.). Note that at any price less than L will fetch a payoff which is strictly less than the payoff at L. Hence, this will complete the proof.

Step iii.a: Now, suppose primary 1 selects a price $x \in [\tilde{p}_1, v)$. Now, primary 2 selects a price less than or equal to x if the channel of primary 2 is available, and one of the two things occur–(i) primary 2 selects Y, and (ii) primary 2 selects N and selects a price less than or equal to x. (i) occurs w.p. p_2 and (ii) occurs w.p. $(1-p_2)\psi_{2,N}(x)$. The channel of primary 2 is available w.p. q. Thus, at x, the expected payoff of primary 1 is

$$(x - c)(1 - (1 - p_2)q\psi_{2,N}(x) - p_2 q) = (v - c)(1 - q) + s_2 - s_1 \quad \text{from (4.69)} \tag{4.79}$$

Since $\psi_{2,N}(\cdot)$ has a jump at v, hence, primary 1 attains strictly higher payoff at a price just below v compared to the payoff at v. Hence, the expected payoff at v is strictly less than $(v-c)(1-q)+s_2-s_1$.

Step iii.b: Now, if primary 1 selects any price in the interval $[\tilde{p}_2, \tilde{p}_1]$, then its expected payoff is

$$(x-c)(1-(1-p_2)q\psi_{2,N}(x)-p_2q) = (x-c)(1-q(1-\frac{L-c}{x-c})) \quad \text{from (4.69)}$$

$$= (x-c)(1-q)+[L-c]q \tag{4.80}$$

which is maximized at $x = \tilde{p}_1$. Now, from (4.73) $(\tilde{p}_1 - c)(1-q) = (v-c)(1-q) - s_1$. Since $(L-c)q = s_2$ (by (4.70), hence the maximum expected payoff is

$$(v-c)(1-q) - s_1 + s_2 \tag{4.81}$$

Step iii.c: Suppose that the primary 1 selects a price x in the interval $[L, \tilde{p}_2)$. Since primary 2 does not select any price in this interval when the channel of primary 2 is unavailable or primary 2 selects N. Thus, the expected payoff of primary 1 at x is

$$\begin{aligned}
&(x-c)(1-p_2q\psi_{2,Y}(x)) \\
&= (x-c)(1-q(1-\frac{L-c}{x-c})) \quad \text{(from (4.67))} = (x-c)(1-q)+(L-c)q \\
&< (\tilde{p}_1-c)(1-q)+(L-c)q = (v-c)(1-q)-s_1+s_2
\end{aligned} \tag{4.82}$$

Hence, under N, the maximum expected payoff that a primary can attain is $(v-c)(1-q)+s_2-s_1$ and this is attained at any price in the interval $[\tilde{p}_1, v)$.

(iv) Under Y or N, the maximum expected payoff that primary 1 can attain is $(v-c)(1-q)+s_2-s_1$. Thus, any randomization of Y and N also yields the same expected payoff. Under the strategy profile, the primary 1 attains the payoff of $(v-c)(1-q)+s_2-s_1$, hence, primary 1 does not have any profitable unilateral deviation.

Case II: We now show that primary 2 also does not have any profitable unilateral deviation. Toward this end we first show that when primary 2 selects Y and primary

1 is available, then the maximum expected payoff of primary 2 is $L - c - s_2$ and it is attained at any price in the interval $[L, \tilde{p}_2]$ (Step i). Subsequently, we show that under Y, the maximum expected attained by primary 2 is $(v - c)(1 - q)$ and it is attained when primary 2 follows the strategy (Step ii). Subsequently, we show that under N, the maximum expected payoff that primary 2 attains is $(v - c)(1 - q)$ and it is attained at a price in the interval $[\tilde{p}_2, v]$ (Step iii). Finally, we show that the maximum expected payoff that primary 2 attains is $(v - c)(1 - q)$ and it is attained when primary 2 follows the strategy (Step iv).

Step i: Suppose that primary 2 selects Y and primary 1 is available. We show that the maximum expected payoff that primary 2 attains is $L - c - s_2$ and it is attained at any price in the interval $[L, \tilde{p}_2]$. Toward this end, we first show that at any price $[L, \tilde{p}_2]$, the expected payoff is $L - c - s_2$ (Step i.a.). At any price in the interval $[\tilde{p}_2, \tilde{p}_1]$ and $[\tilde{p}_1, v]$ the expected payoff is at most $L - c - s_2$ (Steps i.b. and i.c. respectively). This will complete the proof.

Step i.a.: Suppose $x \in [L, \tilde{p}_2]$. Since primary 2 selects Y, primary 2 knows that primary 1 is available. Primary 1 selects a price in the interval $[L, \tilde{p}_2]$ if primary 1 selects Y (which occurs w.p. p_1). Thus, the expected payoff of primary 2 at x is

$$(x - c)(1 - p_1\psi_{1,Y}(x)) - s_2 = L - c - s_2 \quad \text{from (4.66).} \tag{4.83}$$

Thus, the expected payoff at any price $x \in [L, \tilde{p}_2]$ is $L - c - s_2$.

Step i.b.: Suppose $x \in [\tilde{p}_2, \tilde{p}_1]$. The expected payoff of primary 2 at x is

$$(x - c)(1 - p_1\psi_{1,Y}(x)) - s_2 = (x - c)(1 - \frac{1}{q}(1 - \frac{\tilde{p} - c}{x - c})) - s_2 \quad \text{from (4.66).}$$

$$= (x - c)(1 - 1/q) + (\tilde{p} - c)/q - s_2. \tag{4.84}$$

Since the coefficient of x is negative, the above is maximized at $\tilde{p}_2$. Thus, the expected payoff is upper bounded by

$$
\begin{aligned}
&(\tilde{p}_2 - c)(1 - 1/q) + (\tilde{p} - c)/q - s_2 \\
&= (\tilde{p}_2 - c)(1 - 1/q) + (v - c)(1 - q)/q - s_2 \quad \text{from (4.70)} \\
&= s_2/q - (v - c)(1 - q)/q + (v - c)(1 - q)/q - s_2 \quad \text{from (4.72)} \\
&= L - c - s_2 \quad \text{since } L - c = s_2/q \quad (cf.(4.70)).
\end{aligned} \tag{4.85}
$$

Thus, at any $x \in [\tilde{p}_2, \tilde{p}_1]$ the maximum expected payoff of primary 2 is $L - c - s_2$.

Step i.c.: Now, suppose $x \in [\tilde{p}_1, v]$. The expected payoff of primary 2 at x is

$$
\begin{aligned}
&(x - c)(1 - (1 - p_1)\psi_{1,N}(x) - p_1) - s_2 \\
&= (x - c)(1 - \frac{1}{q}(1 - \frac{\tilde{p} - c}{x - c} - p_1 q) - p_1) - s_2 \quad \text{from (4.68)} \\
&= (x - c)(1 - 1/q) + \frac{\tilde{p} - c}{q} - s_2 \\
&< (\tilde{p}_2 - c)(1 - 1/q) + \frac{\tilde{p} - c}{q} - s_2 \quad \text{since } \tilde{p}_2 < \tilde{p}_1, \quad = L - c - s_2 \quad \text{from (4.85).}
\end{aligned} \tag{4.86}
$$

Thus, from Steps i.a., i.b. and i.c. the maximum expected payoff attained by primary 2 in this case is $L - c - s_2$ and it is attained at the prices in the interval $[L, \tilde{p}_2]$.

Step ii: When primary 1 is unavailable, then primary 2 attains a payoff of $v - c - s_2$. Hence, under Y, the maximum expected payoff of primary 2 is

$$
\begin{aligned}
&(L - c - s_2)q + (v - c - s_2)(1 - q) = q(L - c) + (v - c)(1 - q) - s_2 \\
&= (v - c)(1 - q) \quad \text{from (4.70).}
\end{aligned} \tag{4.87}
$$

It is attained when primary 2 follows the strategy.

Step iii: Now, we show that when primary 2 selects N, then, its maximum expected payoff is $(v - c)(1 - q)$ and it is attained at any price in the interval $[\tilde{p}_2, v]$. Toward this end, we show that at any price in the intervals $[\tilde{p}_2, \tilde{p}_1]$ and $[\tilde{p}_1, v]$, the maximum expected payoff of primary 2 is $(v-c)(1-q)$ (Steps iii.a. and iii.b.). Subsequently, we show that the maximum expected payoff attained by the primary at any $x \in [L, \tilde{p}_2]$, the maximum expected payoff attained by primary 2 is $(v-c)(1-q)$.

Step iii.a: Suppose primary 2 selects a price $x \in [\tilde{p}_2, \tilde{p}_1]$. Since primary 2 selects N, thus, it only knows that primary 1 is available w.p. q. Thus, at x, the expected payoff of primary 2 at x is

$$(x-c)(1-p_1 q\psi_{1,Y}(x)) = (v-c)(1-q) \quad \text{from (4.66)} \tag{4.88}$$

Step iii.b.: Suppose primary 2 selects a price $x \in [\tilde{p}_1, v]$. Then the expected payoff of primary 2 at x is

$$(x-c)(1-p_1 q-(1-p_1)q\psi_{1,N}(x)) = (v-c)(1-q) \quad \text{from (4.68).} \tag{4.89}$$

From Steps iii.a. and iii.b., the expected payoff of primary 2 at $[\tilde{p}_2, v]$ is $(v-c)(1-q)$.

Step iii.c: Now, suppose primary 2 selects a price $x \in [L, \tilde{p}_2]$ is

$$(x-c)(1-p_1 q\psi_{1,Y}(x)) = (x-c)(1-q)+(L-c)q \quad \text{from (4.66)} \tag{4.90}$$

The above is maximized at $x = \tilde{p}_2$ as the coefficient of x is positive. Hence, the maximum expected payoff of primary 2 is upper bounded by

$$\begin{aligned}(\tilde{p}_2-c)(1-q)+(L-c)q &= (v-c)(1-q)-s_2+(L-c)q \quad \text{from (4.72).}\\ &= (v-c)(1-q) \quad \text{since } (L-c)q = s_2 \quad \text{from (4.70).}\end{aligned} \tag{4.91}$$

Thus, the maximum expected payoff of primary 2 is $(v-c)(1-q)$ and it is attained only at prices in the interval $x \in [\tilde{p}_2, v]$ (by Steps iii.a. and iii.b.).

Step iv: By Step (ii), the maximum expected payoff of primary 2 under Y is $(v-c)(1-q)$. From Step (iii), the maximum expected payoff of primary 2 under N is $(v-c)(1-q)$. Thus, any randomization between Y and N will yield a maximum expected payoff of $(v-c)(1-q)$. This maximum expected payoff is attained by primary 2 when it follows the strategy. □

4.5 Unequal Channel availability probabilities

We, now, consider the setting, where different primaries may have different availability probabilities depicted in Section 4.1.2. Without loss of generality, we assume that the channel of primary 1 is available w.p. q_1 and the channel of primary 2 is available w.p. q_2 where $q_1 > q_2$.

4.5.1 Goals

The impact of different availability probabilities on the frequency with which a primary selects Y can not be readily concluded. If primary 1 acquires the CSI of primary 2, it will more often find that that the channel of primary 2 is unavailable which may increase its payoff. However, primary 2 itself may also acquire the CSI of primary 1 and select a lower price, in response primary 1 selects a lower price which may reduce the payoff of primary 1. Even if the NE strategy is of the form $[T, p]$, the values of the thresholds may be different for different primaries. Additionally, it is not clear whether the threshold will be higher for primary 1. This is because if the availability probability of primary 2 is low, primary 1 may select Y for very small values of s, but primary 2 may still select Y for larger values of s as the channel availability probability of the primary 1 is higher.

The impact of different availability probabilities on the payoff of each primary is also not apriori clear. Conventional wisdom suggests that as s decreases the payoff of a primary should not decrease. However, the conventional wisdom is not definitive because of the following. Since the channel of primary 1 is available with a higher probability, when primary 2 acquires the CSI of primary 1, then, primary 1 selects a lower price more often which may reduce the payoff of primary 2. The pricing strategy also inherently depends on the frequency with which a primary selects Y. We resolve all these quandaries.

4.5.2 Results

We first discuss the main insights provided by our analysis.

- The NE strategy for primary i is of the form $[T_i, p_i]$ (Definition 4.4). However, T_i is different for different primaries due to different availability probabilities. Our result shows that $T_1 = q_2(v-c)(1-q_2)$ (Theorem 4.10) and $T_2 = q_2(v-c)(1-q_1)/(1-q_1+q_2)$ (Theorem 4.11) where $T_1 > T_2$. Note that in the basic model, we have shown that the threshold depends on the variance of the availability of the competitor's channel. Thus, $T_1 = q_2(v-c)(1-q_2)$ is expected. However, the expression of T_2 is surprising. Additionally, we show that $T_1 > T_2$ which is again not completely intuitive. Also note that when $T_2 \leq s < T_1$, primary 2 selects Y w.p. 0, but primary 1 selects Y w.p. p_1. Even when $s < T_2$, p_is are different with $p_1 > p_2$ (Theorem 4.12). Thus, primary 1 selects Y with a higher probability.

- Different availability probabilities also lead to different payoffs for the primaries. In contrast to the basic model, the expected payoff of primary 1 is higher than that of primary 2 when primary 1 selects Y with positive probability (Theorems 4.11 and 4.12). Additionally, the expected payoff of primary 2 decreases as s decreases. Thus, the expected payoff of a primary decreases with the ease of acquiring the CSI which negates the conventional wisdom. Intuitively, since primary 1 selects Y with a higher probability as s decreases, it selects a lower price when the channel of primary 2 is available. In response, primary 2 either must select a high price (so that, it can get a high payoff in the event when the channel of primary 1 is unavailable) or select a low price (so that, it can increase its probability of winning). Since the channel of primary 1 is available with a higher probability, the first option fetches a lower payoff compared to the latter one. Thus, primary 2 also selects a lower price. Thus, the expected payoff of primary 2 decreases as s decreases. The expected payoff of primary 2 becomes close to that of primary 1 as the difference between q_1 and q_2 decreases.

- Price strategies also exhibit some similarities with the basic model. Specifically, primary i selects its price from the interval $[L, \tilde{p}_i]$ ($[\tilde{p}_i, v]$, resp.) when it selects Y (N, resp.) and the channel of the competitor is available. However, since primaries have different availability probabilities, the price selection strategies also have some differences compared to the basic model. For example, $\tilde{p}_1 > \tilde{p}_2$. Thus, primary 2 selects a lower price when it selects Y and the channel of primary 1 is available. Additionally, primary 1 selects v with a positive probability when it selects N and the probability decreases as q_2 becomes close to q_1. Thus, primary 1 selects a price from a distribution function which has a discontinuity whereas in the basic model, each primary selects its price from a continuous distribution function. Intuitively, since primary 1 has higher channel availability probability, it selects a higher price when it selects N.

4.5.3 High s

Our first result shows that

Theorem 4.10. *If $s \geq q_2(v-c)(1-q_2)$ then in an NE, both the primaries select N w.p. 1. The expected payoff of both the primaries is $(v-c)(1-q_2)$*

Note that when $s \geq q_2(v-c)(1-q_2)$ both the primaries attain identical expected payoff though the availability probabilities are different.

Proof. When both players select N, then the setting becomes equivalent to the setting where primary 1 (primary 2, resp.) only knows that the channel of primary 2 (primary 1, resp.) is available w.p. q_1 (q_2 resp.). The above setting has already been considered in [45]. From [45],

Lemma 4.3. *In the unique NE pricing strategy under N, primary i should select its pricing strategy using $\psi_i(\cdot)$, where*

$$\psi_1(x) = \begin{cases} 0 & x < \bar{p} \\ \dfrac{1}{q_1}(1 - \dfrac{(v-c)(1-q_2)}{x-c}) & \bar{p} \leq x < v \\ 1 & x > v \end{cases}$$

$$\psi_2(x) = \begin{cases} 0 & x < \bar{p} \\ \frac{1}{q_2}(1 - \frac{(v-c)(1-q_2)}{x-c}) & \bar{p} \le x \le v \\ 1 & x > v \end{cases}$$

where $\bar{p} - c = (v-c)(1-q_2)$. $\psi_1(\cdot)$ *has a jump of* $\frac{q_1 - q2}{q_1}$ *at* v.

It is easy to show that each primary will attain an expected payoff of $(v-c)(1-q_2)$. Now, we show that each primary can not attain higher payoff by selecting Y. First, we show that primary 1 can not attain more by selecting Y (Step i). Subsequently, we show that primary 2 can not attain more by selecting Y (Step ii).

Step i: Suppose that primary 1 deviates and selects Y. When the channel of primary 2 is available, then the expected payoff of primary 1 at any $[\bar{p}, v]$ is

$$(x-c)(1-\psi_2(x)) - s = (x-c)(1-1/q_2) + (v-c)(1-q_2)/q_2 - s \tag{4.92}$$

The above is maximized at $\bar{p}$ as the co-efficient of x is negative. Since $\bar{p}-c = (v-c)(1-q_2)$ (from Lemma 4.3), hence, the above is upper bounded by

$$(v-c)(1-q_2)(1-1/q_2) + (v-c)(1-q_2)/q_2 - s = \bar{p} - c - s. \tag{4.93}$$

The price at any $x < \bar{p}$ will fetch an expected payoff of strictly less than $\bar{p} - c - s$. Thus, the maximum expected payoff that primary 1 attains in this setting is $\bar{p} - c - s$.

When the channel of primary 2 is not available, then the payoff of primary 1 is $(v - c) - s$. Hence, the maximum expected payoff that primary 1 can attain by deviating unilaterally is

$$q_2(v-c)(1-q_2) + (v-c)(1-q_2) - s \tag{4.94}$$

When $s \geq q_2(v-c)(1-q_2)$, then, primary 1 will attain an expected payoff of strictly less than $(v-c)(1-q_2)$. Hence, primary 1 does not have any profitable unilateral deviation.

Step ii: By applying the similar method we can show that the maximum expected payoff attained by primary 2 under Y is

$$q_1(v-c)(1-q_2)+(v-c)(1-q_1)-s \tag{4.95}$$

However, the above is strictly less than $q_2(v-c)(1-q_2)+(v-c)(1-q_2)-s$ since $q_1 > q_2$. If $s \geq q_2(v-c)(1-q_2)$, then the maximum expected payoff that primary 2 will attain under Y is strictly less than $(v-c)(1-q_2)$. However, primary 2 attains an expected payoff of $(v-c)(1-q_2)$ following the strategy profile at N, hence, primary 2 does not have any profitable unilateral deviation. □

4.5.4 s is neither too high nor too low

Now, we show that when $\frac{q_2(v-c)(1-q_1)}{1-q_1+q_2} \leq s < q_2(v-c)(1-q_2)$ then there is an NE where primary 1 randomizes between Y and N, however, primary 2 only selects N. First, we introduce some price distribution functions.

$$\begin{aligned}\psi_Y(x) =& 0, \quad x < L\\ & \frac{1}{p_1q_1}(1-\frac{L-c}{x-c}) \quad L \leq x \leq \tilde{p}\\ & 1, \quad x > \tilde{p}\end{aligned} \tag{4.96}$$

and

$$\psi_{1,N}(x) = 0, \quad x < \tilde{p}$$
$$\frac{1}{(1-p_1)q_1}(1 - \frac{L-c}{x-c} - p_1 q_1) \quad \tilde{p} \le x < v$$
$$1, \quad x \ge v \tag{4.97}$$

$$\psi_N(x) = 0, \quad x < L$$
$$(1 - \frac{L-c}{x-c}) \quad L \le x < \tilde{p}$$
$$\frac{1}{q_2}(1 - \frac{(v-c)(1-q_2)}{x-c}) \quad \tilde{p} \le x \le v$$
$$1, \quad x > v \tag{4.98}$$

where

$$L - c = \frac{s}{q_2} \tag{4.99}$$
$$\tilde{p} - c = \frac{L-c}{1-p_1 q_1} \tag{4.100}$$
$$p_1 = \frac{(v-c)(1-q_2) - s/q_2}{q_1(v-c)(1-q_2) - q_1 s} \tag{4.101}$$

Replacing the value of p_1 from (4.101) in $\tilde{p}$ we obtain

$$\tilde{p} - c = \frac{(s/q_2)(q_1(v-c)(1-q_2) - q_1 s)}{q_1(v-c)(1-q_2) - q_1 s - q_1(v-c)(1-q_2) + s q_1/q_2}$$
$$= \frac{(v-c)(1-q_2) - s}{(1-q_2)} \tag{4.102}$$

It is easy to discern that $\psi_Y(\cdot)$ is continuous. We, also, show that $\psi_N(\cdot)$ is a continuous function.

Observation 4.5. $\psi_N(\cdot)$ is a continuous function.

Proof. It is easy to discern that $\psi_N(\cdot)$ is continuous every except $x = \tilde{p}$. Now, we show that $\psi_N(\cdot)$ is also continuous at $\tilde{p}$. The left hand limit of $\psi_N(\cdot)$ is $1 - \dfrac{L-c}{\tilde{p}-c}$. Now, the right hand limit is

$$\begin{aligned}
&\frac{1}{q_2}(1 - \frac{(v-c)(1-q_2)}{\tilde{p}-c}) \\
&= \frac{1}{q_2}(1 - \frac{(\tilde{p}-c)(1-q_2) + (L-c)q_2}{\tilde{p}-c}) \quad \text{from (4.102) and (4.99)} \\
&= 1 - \frac{L-c}{\tilde{p}-c} \qquad (4.103)
\end{aligned}$$

Hence, $\psi_N(\cdot)$ does not have any jump at $\tilde{p}$. □

Now, we show that $\psi_{1,N}(\cdot)$ has a jump at v.

Observation 4.6. $\psi_{1,N}(\cdot)$ has a jump at v.

Proof. Note that $s \geq q_2(v-c)(1-q_1)/(1-q_1+q_2)$, hence, $\dfrac{L-c}{v-c} \geq \dfrac{1-q_1}{1-q_1+q_2} > 1-q_1$ as $q_1 > q_2$ and $L - c = s/q_2$. Thus,

$$1 - \frac{1}{(1-p_1)q_1}(1 - \frac{L-c}{v-c} - p_1q_1) > 1 - \frac{1}{(1-p_1)q_1}(1 - 1 + q_1 - p_1q_1) = 0 \qquad (4.104)$$

Hence, $\psi_{1,N}(\cdot)$ has a jump at v. □

Now, we are ready to state the main result.

Theorem 4.11. *Consider the following strategy profile: Primary* 1 *selects* Y *w.p.* p_1 *(given in (4.101)) and* N *w.p.* $1-p_1$ *and primary* 2 *selects* N *w.p.* 1*. While selecting* Y*, primary* 1 *selects its price according to* $\psi_Y(\cdot)$ *when the channel of primary* 2 *is available and selects* v *when the channel of primary* 2 *is unavailable. While selecting* N*, primary* 1 *selects its price according to* $\psi_{1,N}(\cdot)$*. Primary* 2 *selects its price according to* $\psi_N(\cdot)$*.*

The above strategy profile is an NE when $q_2(v-c)(1-q_2) > s \geq q_2(v-c)(1-q_1)/(1-q_1+q_2)$[8]*. The expected payoff of primary* 1 *is* $(v-c)(1-q_2)$ *and the expected payoff of primary* 2 *is* s/q_2*.*

Discussion: Since $s < q_2(v-c)(1-q_2)$, hence, $s/q_2 < (v-c)(1-q_2)$. Thus, the expected payoff of primary 2 is lower compared to the expected payoff of primary 1. The expected payoff of primary 2 decreases as s decreases. *This negates conventional wisdom which suggests that the expected payoff of a primary should increase when* s *decreases.*

Note that the support of ψ_Y is $[L, \tilde{p}]$ and the support of ψ_N is $[L, v]$. Thus, the support of ψ_Y and ψ_N overlap with each other. Also note that $\psi_{1,N}(\cdot)$ has a jump at v, where as $\psi_N(\cdot)$ does not have any jump. Intuitively, since primary 1 has higher availability probability, primary 1 selects higher prices with higher probabilities.

p_1 increases with decrease in s. L decreases when s decreases (by (4.99)). Thus, primaries select their prices from a larger interval as s decreases. Also note that L only depends on q_2, it is independent of q_1.

Also note from (4.102) that $\tilde{p}$ increases as s decreases. Thus, $\psi_Y(\cdot)$ has a larger support as s decreases and primary 1 selects its price from a larger interval when it selects Y, and the channel of primary 2 is available.

Now, we prove the above theorem.

Proof. First, we show that primary 1 does not have any profitable deviation when primary 2 follows the strategy prescribed in Theorem 4.11 (Case I). Subsequently, we show that primary 2 also does not have any profitable unilateral deviation when primary 2 follows

[8]Note that $(1-q_2)(1-q_1) + q_2(1-q_2) > (1-q_1)$ as $q_2 < q_1$, thus, $(1-q_2) > \dfrac{1-q_1}{1-q_1+q_2}$

the strategy prescribed in Theorem 4.11 (Case II).

Case I: First, we show that under Y, if the channel of primary 2 is available, then primary 1 can attain a maximum expected payoff of $L - c - s$ (Step i). When the channel of primary 2 is unavailable primary 1 will attain the payoff $v - c - s$. Thus, it shows that under Y, the expected payoff that primary 1 attains is $(v - c)(1 - q_2)$ (Step ii). Subsequently, we show that under N, the maximum expected payoff that primary 1 attains is $(v-c)(1-q_2)$ (Step iii). Finally, we show that primary 1 achieves the maximum expected payoff under the prescribed strategy (Step iv).

(i): Suppose primary 1 selects Y and the channel of primary 2 is available. The at any price $x \in [L, \tilde{p}]$, the expected payoff of primary 1 under Y is

$$(x - c)(1 - \psi_N(x)) - s = L - c - s \quad \text{from (4.98).} \tag{4.105}$$

At any price less than or equal to L will fetch a payoff which is strictly less than $L - c - s$.

At any price x in the interval $[\tilde{p}, v]$ the expected payoff of primary 1 is

$$\begin{aligned}(x - c)(1 - \psi_N(x)) - s &= (x - c)(1 - \frac{1}{q_2}(1 - \frac{(v - c)(1 - q_2)}{x - c})) - s \quad \text{from (4.98)}\\ &= (x - c)(1 - 1/q_2) + (v - c)(1 - q_2)/q_2 - s\end{aligned}$$

The above is maximized at $\tilde{p}$. Putting $x = \tilde{p}$, and from (4.102) we obtain

$$\begin{aligned}&\frac{(v - c)(1 - q_2) - s}{1 - q_2}(1 - 1/q_2) + (v - c)(1 - q_2)/q_2 - s = s/q_2 - s\\ &= L - c - s \quad \text{by (4.99)}\end{aligned} \tag{4.106}$$

Hence, the maximum expected payoff attained by primary 1 is $L - c - s$ and it is attained at any price in the interval $[L, \tilde{p}]$.

(ii): Now, when the channel of primary 2 is not available, then the payoff that primary 1 achieves under Y is $(v-c)-s$. Hence, the maximum expected payoff that primary 1 can achieve under Y is

$$q_2(L-c-s)+(v-c-s)(1-q_2)=(v-c)(1-q_2) \tag{4.107}$$

By following the strategy profile, primary 1 achieves the above payoff under Y.

(iii): When primary 1 selects N, then it only knows that the channel of the primary 2 is available w.p. q_2. Thus, under N, at any price x in the interval $[\tilde{p},v]$ the expected payoff of primary 1 is

$$(x-c)(1-q_2\psi_N(x))=(v-c)(1-q_2) \tag{4.108}$$

Similarly, at any price x in the interval $[L,\tilde{p}]$, the expected payoff of primary 1 is

$$\begin{aligned}(x-c)(1-q_2\psi_N(x)) &= (x-c)(1-q_2(1-\frac{L-c}{x-c})) \quad \text{from (4.98)}\\ &= (x-c)(1-q_2)+(L-c)q_2 \end{aligned} \tag{4.109}$$

The above is maximized at $x=\tilde{p}$. Putting the value of $\tilde{p}$ we obtain

$$\begin{aligned}(\tilde{p}-c)(1-q_2)+(L-c)q_2 &= (\tilde{p}-c)(1-q_2)+s \quad \text{by (4.99)}\\ &= (v-c)(1-q_2) \quad \text{by (4.102).}\end{aligned} \tag{4.110}$$

At any price less than L fetches a payoff of strictly less than L which is less than $(v-c)(1-q_2)$. Hence, the maximum expected payoff that primary 1 attains under N is $(v-c)(1-q_2)$. This is achieved at any price in the interval $[\tilde{p},v]$.

(iv): We have shown that under Y or under N, the maximum expected payoff that primary 1 can attain is $(v-c)(1-q_2)$. Thus, any randomization between Y and N also yields at most an expected payoff of $(v - c)(1 - q_2)$. Primary 1 attains the above payoff when it follows the prescribed strategy. Hence, primary 1 does not have any profitable deviation.

Case II: Now, we show that primary 2 does not have any profitable unilateral deviation. Toward this end we first show that under N, the maximum expected payoff that primary 2 attains is $L - c$ (Step i). Subsequently, we show that under N, the primary 2 attains the maximum expected payoff $L - c$ when it selects price in the interval $[L, v)$ (Step ii). Subsequently, we show that if primary 2 deviates and selects Y, then it can only attain a payoff of at most $L - c$ when $s \geq \dfrac{q_2(v-c)(1-q_1)}{1-q_1+q_2}$ (Step iii).

Step (i): Suppose that primary 2 selects N. Suppose that primary 2 selects a price x in the interval $[L, \tilde{p}]$. If the channel of primary 1 is available, then it selects a price less than or equal to x where $x \in [L, \tilde{p}]$ if primary 1 selects Y and then selects a price less than or equal to x. The above occurs w.p. $p_1\psi_Y(x)$. The channel of primary 1 is available w.p. q_1. Hence, by the continuity of $\psi_{1,Y}(\cdot)$ at x the expected payoff of primary 2 under N is

$$(x - c)(1 - p_1 q_1 \psi_Y(x)) = L - c \quad \text{from (4.96).} \tag{4.111}$$

Now, suppose that primary 2 selects a price x from the interval $[\tilde{p}, v)$. If the channel of primary 1 is available, then primary 1 selects a price less than or equal to x when $x \in [\tilde{p}, v)$ if–i) primary 1 selects Y or ii) primary 1 selects N and selects a price less than or equal to x. (i) occurs with probability p_1 and (ii) occurs with probability $(1 - p_1)\psi_{1,N}(x)$. The channel of primary 1 is available w.p. q_1. Since $\psi_{1,N}$ is continuous in $[\tilde{p}, v)$, the expected payoff of primary 2 at x is

$$(x-c)(1-(1-p_1)q_1\psi_{1,N}(x)-p_1q_1) = L-c \quad \text{from (4.97).} \tag{4.112}$$

Since $\psi_{1,N}(\cdot)$ has a jump at v, the expected payoff at v is strictly less than the expected payoff just below v. On the other hand a price less than L will fetch a payoff strictly less than $L-c$. Hence, the maximum expected payoff that primary 2 can attain under N is $L-c$.

Step ii: Primary 2 attains a payoff of $L-c$ under N and it is attained only at prices in the interval $[L, v)$.

Step (iii): Now, we show that if primary 2 selects Y, then it will not attain a payoff higher than s/q_2. Towards this end, we show when the channel of primary 1 is available, then the maximum expected payoff attained by primary 2 is $L-c-s$ (Step iii.a). When the channel of primary 1 is unavailable, then the payoff attained by primary 2 is $v-c-s$. Subsequently, we show that the maximum expected payoff attained under Y is at most $L-c$ when $s \geq \dfrac{q_2(v-c)(1-q_1)}{1-q_1+q_2}$ (Step iii.b.). This will complete the proof.

Step iii.a: When the channel of primary 1 is available then the expected payoff of primary 2 at any price x in the interval $[L, \tilde{p}]$ is

$$(x-c)(1-p_1\psi_Y(x))-s = (x-c)(1-1/q_1)+(L-c)/q_1-s \quad \text{cf (4.96).} \tag{4.113}$$

The above is maximized at $x = L$ since the co-efficient of x is negative, hence, the maximum value is $L-c-s$.

Similarly, the expected payoff of primary 2 at any price x in the interval $[\tilde{p}, v)$ is

$$
\begin{aligned}
&(x-c)(1-(1-p_1)\psi_{1,N}(x)-p_1)-s \\
&= (x-c)(1-1/q_1)+(L-c)/q_1-s \quad \text{from (4.97)} \\
&< (L-c)-s \quad \text{since } \tilde{p} > L. \qquad (4.114)
\end{aligned}
$$

The payoff at a price less than L fetches a payoff which is strictly less than $L-c-s$.

Hence, the maximum expected payoff attained by primary 2 when the channel of primary 1 is available is $L-c-s$.

Step iii.b: When the channel of primary 1 is unavailable, then the payoff that primary 2 attains is $(v-c)-s$. Hence, the expected payoff of primary 2 under Y is

$$
\begin{aligned}
&q_1(L-c-s)+(v-c-s)(1-q_1) = \frac{q_1 s}{q_2}+(v-c)(1-q_1)-s \\
&= (v-c)(1-q_1)+\frac{s(q_1-q_2)}{q_2} \\
&\leq s/q_2 \quad \text{as } q_2(v-c)(1-q_1)/(1-q_1+q_2) \leq s \\
&= L-c \qquad (4.115)
\end{aligned}
$$

But primary 2 attains $L-c$ under N following the strategy $\psi_N(\cdot)$. Thus, primary 2 does not have any profitable unilateral deviation. Hence, the result follows. □

4.5.5 Low s

Now, we show that when $s < \dfrac{q_2(v-c)(1-q_1)}{1-q_1+q_2}$ then there exists an NE where both the primaries randomize between Y and N. Again we first introduce some pricing distributions.

$$\begin{aligned}\psi_{1,Y}(x) =& 0, \quad x < L \\ & \frac{1}{p_1}(1 - \frac{L-c}{x-c}) \quad L \leq x < \tilde{p}_2 \\ & \frac{1}{p_1 q_1}(1 - \frac{\bar{p}-c}{x-c}) \quad \tilde{p}_2 \leq x \leq \tilde{p}_1 \\ & 1, \quad x > \tilde{p}_1 \end{aligned} \tag{4.116}$$

$$\begin{aligned}\psi_{2,Y}(x) =& 0, \quad x < L \\ & \frac{1}{p_2}(1 - \frac{L-c}{x-c}) \quad L \leq x \leq \tilde{p}_2 \\ & 1, \quad x > \tilde{p}_2 \end{aligned} \tag{4.117}$$

$$\begin{aligned}\psi_{1,N}(x) =& 0, \quad x < \tilde{p}_1 \\ & \frac{1}{(1-p_1)q_1}(1 - \frac{\bar{p}-c}{x-c} - p_1 q_1) \quad \tilde{p}_1 \leq x < v \\ & 1, \quad x \geq v \end{aligned} \tag{4.118}$$

$$\begin{aligned}\psi_{2,N}(x) =& 0, \quad x < \tilde{p}_2 \\ & \frac{1}{1-p_2}(1 - \frac{L-c}{x-c} - p_2), \quad \tilde{p}_2 \leq x < \tilde{p}_1 \\ & \frac{1}{(1-p_2)q_2}(1 - \frac{(v-c)(1-q_2)}{x-c} - p_2 q_2) \quad \tilde{p}_1 \leq x \leq v \\ & 1, \quad x > v \end{aligned} \tag{4.119}$$

where

$$\bar{p} - c = (v - c)(1 - q_1) + s(q_1 - q_2)/q_2 \tag{4.120}$$

$$p_1 = \frac{q_1(v - c)(1 - q_2) - s(q_1/q_2 - q_1 + q_2)}{q_1(v - c)(1 - q_2) - q_1 s} \tag{4.121}$$

$$p_2 = \frac{q_2(v - c)(1 - q_1) - s(1 - q_1 + q_2)}{q_2(v - c)(1 - q_1) - q_2 s} \tag{4.122}$$

$$L - c = s/q_2$$

$$\tilde{p}_2 - c = \frac{(v - c)(1 - q_1) + s(q_1 - q_2)/q_2}{1 - p_2 q_1}, \quad \tilde{p}_1 - c = \frac{\bar{p} - c}{1 - p_1 q_1} \tag{4.123}$$

First, we show some results which we use throughout this section. Replacing the value of p_2 in $\tilde{p}_2$ we have

$$\begin{aligned}
\tilde{p}_2 - c &= \frac{[q_2(v - c)(1 - q_1) - q_2 s][(v - c)(1 - q_1) + s(q_1 - q_2)/q_2]}{q_2(v - c)(1 - q_1)^2 + s(q_1 - q_1^2 + q_1 q_2 - q_2)} \\
&= \frac{[q_2(v - c)(1 - q_1) - q_2 s][(v - c)(1 - q_1) + s(q_1 - q_2)/q_2]}{q_2(v - c)(1 - q_1)^2 + s q_2(1 - q_1)(q_1 - q_2)/q_2} \\
&= \frac{(v - c)(1 - q_1) - s}{(1 - q_1)}
\end{aligned} \tag{4.124}$$

$$\begin{aligned}
\frac{L - c}{1 - p_2} = \frac{s}{q_2(1 - p_2)} &= \frac{((v - c)(1 - q_1) - s)s}{s(1 - q_1)} \\
&= \frac{(v - c)(1 - q_1) - s}{1 - q_1} = \tilde{p}_2 - c
\end{aligned} \tag{4.125}$$

Also note from (4.123) and (4.121) that

$$\begin{aligned}
\tilde{p}_1 - c &= \frac{[(v - c)(1 - q_1) + s(q_1 - q_2)/q_2][q_1(v - c)(1 - q_2) - q_1 s]}{q_1(v - c)(1 - q_2) - s q_1 - q_1^2(v - c)(1 - q_2) + s q_1^2/q_2 - s q_1^2 + s q_1 q_2} \\
&= \frac{[(v - c)(1 - q_1) + s(q_1 - q_2)/q_2][q_1(v - c)(1 - q_2) - q_1 s]}{q_1(1 - q_2)[(v - c)(1 - q_1) + s(q_1 - q_2)/q_2]} \\
&= \frac{(v - c)(1 - q_2) - s}{1 - q_2}
\end{aligned} \tag{4.126}$$

$\psi_{1,Y}(\cdot), \psi_{2,Y}(\cdot)$ and $\psi_{2,N}(\cdot)$ are continuous. However, $\psi_{1,N}(\cdot)$ is not continuous.

Observation 4.7. $\psi_{1,N}(\cdot)$ is continuous except at v.

Proof. It is easy to discern the continuity at every other point except v. Note from (4.118) $\psi_{1,N}(\cdot)$ has a jump of $\dfrac{s(q_1 - q_2)}{(v-c)(1-p_1)q_1q_2}$ at v. □

Observation 4.8. $\psi_{1,Y}(\cdot)$ is continuous.

Proof. It is easy to verify that $\psi_{1,Y}(\cdot)$ (cf. (4.116)) is continuous everywhere except at $\tilde{p}_2$. We now show that it is also continuous at $\tilde{p}_2$. The left hand limit at $\tilde{p}_2$ is

$$\begin{aligned} \frac{1}{p_1}(1 - \frac{L-c}{\tilde{p}_2 - c}) &= \frac{1}{p_1}(1-(1-p_2)) \quad \text{from (4.125)} \\ &= \frac{p_2}{p_1} \end{aligned} \tag{4.127}$$

The right hand limit (cf.(4.116)) is

$$\begin{aligned} \frac{1}{p_1q_1}(1 - \frac{\bar{p}-c}{\tilde{p}_2 - c}) &= \frac{1}{p_1q_1}(1-1+p_2q_1) \quad \text{from (4.123)} \\ &= \frac{p_2}{p_1} \end{aligned} \tag{4.128}$$

which is equal to the left hand limit. □

Observation 4.9. $\psi_{2,N}(\cdot)$ is continuous.

Proof. It is easy to verify that $\psi_{2,N}(\cdot)$ (cf. (4.119)) is continuous everywhere except at $\tilde{p}_1$. Now, we also show that $\psi_{2,N}(\cdot)$ is continuous at $\tilde{p}_1$. First note from (4.126) that

$$\frac{(v-c)(1-q_2)}{(1-q_2)(\tilde{p}_1 - c)} = 1 + \frac{s}{(1-q_2)(\tilde{p}_1 - c)} \tag{4.129}$$

The right hand limit at $\tilde{p}_1$ is

$$\begin{aligned}
&\frac{1}{(1-p_2)q_2}(1-\frac{(v-c)(1-q_2)}{\tilde{p}_1-c}-p_2q_2)\\
&=\frac{1}{(1-p_2)q_2}(1-(1-q_2)-\frac{s}{\tilde{p}_1-c}-p_2q_2) \quad \text{from (4.129)}\\
&=1-\frac{s}{(1-p_2)q_2(\tilde{p}_1-c)}=1-\frac{L-c}{(1-p_2)(\tilde{p}_1-c)} \quad \text{since } L-c=s/q_2\\
&=\frac{1}{(1-p_2)}(1-\frac{L-c}{\tilde{p}_1-c}-p_2) && (4.130)
\end{aligned}$$

which is the left hand limit (cf.(4.119)). Hence, the result follows. □

Note from (4.121) and (4.122) that p_i $i=1,2$ both depend on q_1 and q_2. Next, we show that $p_1 > p_2$.

Lemma 4.4. *$p_1 > p_2$ when $q_1 > q_2$.*

Proof. From (4.121) and (4.122), we need to show that

$$\frac{q_1(v-c)(1-q_2)-s(q_1/q_2-q_1+q_2)}{q_1(v-c)(1-q_2)-q_1s} > \frac{q_2(v-c)(1-q_1)-s(1-q_1+q_2)}{q_2(v-c)(1-q_1)-q_2s} \qquad (4.131)$$

By cross multiplication it is sufficient to show that

$$(v-c)sq_1q_2(q_2-q_1)-s(v-c)[(q_1-q_1q_2+q_2^2)(1-q_1)-(q_1-q_1^2+q_1q_2)(1-q_2)]$$
$$+s^2(q_2^2-q_1q_2-q_1q_2+q_1^2)>0$$

The last expression is $s^2(q_1-q_2)^2$ which is always positive when $q_1 > q_2$. Thus, it is sufficient to show that

$$(v-c)sq_1q_2(q_2-q_1)-$$

$$s(v-c)[q_1-q_1q_2+q_2^2-q_1^2+q_1^2q_2-q_1q_2^2-q_1+q_1^2-q_1q_2+q_1q_2-q_1^2q_2+q_1q_2^2]>0$$

$$\text{Or, } (v-c)sq_1q_2(q_2-q_1)-s(v-c)[q_2^2-q_1q_2]>0$$

$$(v-c)s(q_1-q_2)(q_2-q_1q_2)>0 \tag{4.132}$$

as $q_1 > q_2$, the above expression is indeed positive. Hence, the result follows. □

Now, we are ready to state the main result of this section.

Theorem 4.12. *Consider the following strategy profile: Primary i selects Y w.p. p_i (cf. (4.121)& (4.122))and N w.p. $1-p_i$. While selecting Y, primary $i=1,2$ selects its price according to $\psi_{i,Y}(\cdot)$ when the channel of primary $j \neq i$ is available and selects v when the channel of primary j is unavailable. While selecting N, primary i selects its price according to $\psi_{i,N}(\cdot)$.*

The above strategy profile is an NE when $s < q_2(v-c)(1-q_1)/(1-q_1+q_2)$. The expected payoff of primary 1 is $(v-c)(1-q_2)$ and the expected payoff of primary 2 is $(v-c)(1-q_1)+s(q_1-q_2)/q_2$.

Discussion: Since $p_1 > p_2$ (by Lemma 4.4), primary 2 selects Y with a lower probability compared to primary 1. Both p_1 and p_2 increase as s decreases. Both p_1 and p_2 go to 1 as $s \to 0$. Note that threshold T_i above which primary i selects only N is higher for primary 1 i.e. $T_1 > T_2$. Hence, primary 1 selects Y for a wider value of s.

Note that the expected payoff of primary 2 decreases with the cost of acquiring the CSI s. This negates *conventional wisdom which suggests that the expected payoff of a* *primary should increase as s decreases.* The expected payoff of primary 1 is independent of s. The expected payoff of primary 2 is lower than that of primary 1. The expected payoff of primary 2 becomes equal to that of the primary 1 when q_2 becomes equal to q_1.

Note that $\psi_{1,N}(\cdot)$ (see (4.118)) has a jump at v since $q_1 > q_2$. The jump decreases as the difference between q_1 and q_2 decreases. Since primary 1 has a higher availability probability, thus, it selects higher prices when it selects N. $\psi_{1,N}(\cdot)$ is continuous elsewhere. It is easy to show that $\psi_{1,Y}, \psi_{2,Y}$ and $\psi_{2,N}$ are continuous everywhere. Note that L decreases as s decreases. Thus, a primary selects its price from a larger interval as s decreases. $\tilde{p}_1$ and $\tilde{p}_2$ both decrease with s (from (4.124) and (4.126)). Hence, $\psi_{1,Y}(\cdot)$ and $\psi_{2,Y}(\cdot)$ have larger supports when s decreases.

Proof of Theorem 4.12

First, we show that primary 1 does not have any profitable unilateral deviation when primary 2 follows the strategy prescribed in the theorem (Case I). Subsequently, we show that primary 2 also does not have any profitable unilateral deviation when primary 1 follows the strategy prescribed in the theorem (Case II).

Case I: First, we show that under Y, the maximum expected payoff that primary 1 can attain is $(v-c)(1-q_2)$ (Step i). Toward this end, we first show that when primary 1 selects Y and the channel of the primary 2 is available, then the expected payoff that primary 1 will attain is at most $L-c-s$ (Step i.a.). When primary 1 selects Y and the channel of the primary 2 is unavailable, then the payoff of the primary 1 is $v-c-s$ which will in turn show that the maximum payoff attained by primary 1 under Y is $(v-c)(1-q_2)$ (Step i.b). Subsequently, we show that under N, the maximum expected payoff that primary 1 can attain is $(v-c)(1-q_2)$ (Step ii). Finally, we show that the maximum expected payoff is attained by primary 1 when it follows the strategy profile (Step iii).

Step i.a: Suppose that primary 1 selects Y and the channel of primary 2 is available. Then, at any price x in the interval $[L, \tilde{p}_2]$, the expected payoff of primary 1 is

$$(x-c)(1-p_2\psi_{2,Y}(x)) - s = L - c - s \quad \text{from (4.117)} \tag{4.133}$$

Now suppose that primary 1 selects a price x in the interval $[\tilde{p}_2, \tilde{p}_1]$. Primary 2 selects a price less than or equal to x if (i) primary 2 selects Y which occurs w.p. p_2 and (ii) primary 2 selects N and then selects a price less than or equal to x which occurs w.p. $(1-p_2)\psi_{2,N}(x)$. Thus, by the continuity of $\psi_{2,N}(\cdot)$ the expected payoff of primary 1 at x is

$$(x-c)(1-p_2-(1-p_2)\psi_{2,N}(x)) - s = L - c - s \quad \text{from (4.119).} \tag{4.134}$$

Similarly, when primary 1 selects a price x from the interval $[\tilde{p}_1, v]$, then its expected payoff is

$$\begin{aligned}&(x-c)(1-p_2-(1-p_2)\psi_{2,N}(x)) - s\\ &= (x-c)(1-\frac{1}{q_2}) + (v-c)(1-q_2)/q_2 - s \quad \text{from (4.119).}\end{aligned} \tag{4.135}$$

Thus, the above is maximized at $\tilde{p}_1$ since the coefficient of x is negative. Hence, the maximum value is

$$\begin{aligned}&(\tilde{p}_1-c)(1-1/q_2) + (v-c)(1-q_2)/q_2 - s\\ &= (\tilde{p}_1-c)(1-1/q_2) + (\tilde{p}_1-c)(1-q_2)/q_2 + s/q_2 - s \quad \text{(from (4.126))}\\ &= L - c - s \quad \text{from (4.123).}\end{aligned}$$

Any price which is strictly less than L will fetch a payoff of less than $L - c - s$. Hence, the maximum expected payoff that primary 1 can attain is $L - c - s$ when the channel of primary 2 is available and it is achieved at any price in the interval $[L, \tilde{p}_1]$.

Step i.b.: Note that the payoff that primary 1 attains when the channel of primary 2 is unavailable is $(v - c) - s$. Hence, the maximum expected payoff that primary 1 attains under Y is

$$
\begin{aligned}
(L - c - s)q_2 + (v - c - s)(1 - q_2) &= (v - c)(1 - q_2) + (L - c)q_2 - s \\
&= (v - c)(1 - q_2) \quad \text{from (4.123).} \qquad (4.136)
\end{aligned}
$$

Step ii: Now, we show that if primary 1 selects N, then, it will attain a maximum expected payoff of $(v-c)(1-q_2)$ and it is attained when it selects a price from the interval $[\tilde{p}_1, v]$. Towards this end, we first show that when primary 1 selects a price in the interval $[\tilde{p}_1, v]$, then its expected payoff is $(v - c)(1 - q_2)$ (Step ii.a.). Subsequently, we show that when primary 1 selects a price from the interval $[\tilde{p}_2, \tilde{p}_1]$, then its expected payoff is at most $(v - c)(1 - q_2)$ (Step ii.b.). Finally, we show that if primary 1 selects a price in the interval $[L, \tilde{p}_2]$, then its expected payoff is less than $(v - c)(1 - q_2)$ (Step ii.c.). Note that a price which is strictly less than L will fetch a payoff which is strictly less than the payoff at L, hence, this will show that under N the expected payoff of primary 1 is $(v-c)(1-q_2)$ and it is attained when at prices in the interval $[\tilde{p}_1, v]$.

Step ii.a: Suppose that primary 1 selects a price $x \in [\tilde{p}_1, v]$. Primary 2 selects a price less than or equal to x if the channel of primary 2 is available and either primary 2 selects Y or it selects N and then selects a price less than or equal to x. Thus, the probability that primary 2 selects a price less than or equal to x is $p_2q_2 + (1 - p_2)q_2\psi_{2,N}(x)$. Thus, by the continuity of $\psi_{2,N}(\cdot)$, the expected payoff at x is

$$(x-c)(1-(1-p_2)q_2\psi_{2,N}(x)-p_2q_2)=(v-c)(1-q_2) \quad \text{from (4.119)}. \tag{4.137}$$

Step ii. b.: Similarly, at price x in the interval $[\tilde{p}_2, \tilde{p}_1]$, the expected payoff of primary 1 is

$$(x-c)(1-p_2q_2-(1-p_2)q_2\psi_{2,N}(x))=(x-c)(1-q_2)+(L-c)q_2 \tag{4.138}$$

The above is maximized at $\tilde{p}_1$. From (4.123) $L-c=s/q_2$, thus, the maximum value is

$$(\tilde{p}_1-c)(1-q_2)+s=(v-c)(1-q_2) \quad \text{from (4.126)} \tag{4.139}$$

Step ii.c.: Now, suppose that primary 1 selects a price $x \in [L, \tilde{p}_2]$. Primary 2 does not select a price in this interval if it selects N. Hence, at x, the expected payoff of primary 1 is

$$\begin{aligned}
&(x-c)(1-p_2q_2\psi_{2,Y}(x))=(x-c)(1-q_2)+(L-c)q_2 \quad \text{from (4.117)}\\
&<(\tilde{p}_1-c)(1-q_2)+(L-c)q_2 \quad \text{as } \tilde{p}_1>\tilde{p}_2\\
&=(v-c)(1-q_2) \quad \text{from (4.126) and (4.123).}
\end{aligned} \tag{4.140}$$

Hence, the maximum expected payoff that primary 1 can attain under N is $(v-c)(1-q_2)$ and this is attained at every price in the interval $[\tilde{p}_1, v]$.

Step iii: The maximum expected payoff that primary 1 can attain is $(v-c)(1-q_2)$ either under Y or N. Hence, any randomization between Y and N will also yield the same expected payoff. The maximum expected payoff is attained by primary 1 when it follows the strategy profile. Hence, primary 1 does not have any profitable unilateral deviation.

Case II: We,now, show that primary 2 also does not have any profitable unilateral deviation. Toward this end, we first show that when primary 2 selects Y and the channel of primary 1 is available, then the maximum expected payoff that it can get is $L-c-s$ (Step i). When primary 2 selects Y and the channel of the primary 1 is unavailable, then the payoff of primary 2 is $(v-c-s)$. Subsequently, we show that under Y, the maximum expected payoff attained by primary 2 is $(v-c)(1-q_1)+s(q_1-q_2)/q_2$ (Step ii). Subsequently, we show that when primary 2 selects N then, the maximum expected payoff that primary 2 can get is also $(v-c)(1-q_1)+s(q_1-q_2)/q_2$ (Step iii). Finally, we show that primary 2 can attain the maximum expected payoff when it follows the strategy profile (Step iii).

Step i: Suppose primary 2 selects Y and the channel of primary 1 is available.

Primary 1 does not select a price from the interval $[L,\tilde{p}_2]$ when it selects N. Thus, at price x in the interval $[L,\tilde{p}_2]$, the expected payoff of primary 2 is

$$(x-c)(1-p_1\psi_{1,Y}(x))-s=L-c-s \quad \text{from (4.116)} \tag{4.141}$$

At price $x\in[\tilde{p}_2,\tilde{p}_1]$, the expected payoff of primary 2 is

$$\begin{aligned}&(x-c)(1-p_1\psi_{1,Y}(x))-s=\\&(x-c)(1-1/q_1)+\frac{(v-c)(1-q_1)+s(q_1-q_2)/q_2}{q_1}-s \quad \text{cf (4.116)\&(4.120).}\end{aligned} \tag{4.142}$$

Since the co-efficient of x is negative, the above is maximized at $x=\tilde{p}_2$. From (4.123) note that $(v-c)(1-q_1)+s(q_1-q_2)/q_2=(\tilde{p}_2-c)(1-p_2q_1)$. Thus, the expected payoff of primary 2 is upper bounded by

$$\begin{aligned}&(\tilde{p}_2-c)(1-1/q_1)+(\tilde{p}_2-c)(1-p_2q_1)/q_1-s=(\tilde{p}_2-c)(1-p_2)-s\\&=L-c-s \quad \text{from (4.125)}\end{aligned} \tag{4.143}$$

Now, suppose that primary 2 selects a price $x \in [\tilde{p}_1, v)$. At x, the expected payoff of primary 2 is

$$
\begin{aligned}
&(x-c)(1-p_1-(1-p_1)\psi_{1,N}(x))-s \\
&= (x-c)(1-1/q_1) + \frac{(v-c)(1-q_1)+s(q_1-q_2)/q_2}{q_1} - s \quad \text{cf. (4.118)\&(4.120)} \\
&< (\tilde{p}_2-c)(1-1/q_1) + \frac{(v-c)(1-q_1)+s(q_1-q_2)/q_2}{q_1} - s \quad \text{since } \tilde{p}_2 < \tilde{p}_1. \qquad (4.144)
\end{aligned}
$$

Note from (4.123) that $(v-c)(1-q_1)+s(q_1-q_2)/q_2 = (\tilde{p}_2-c)(1-p_2q_1)$, thus, the above can be written as

$$
(\tilde{p}_2-c)(1-1/q_1)+(\tilde{p}_2-c)(1-p_2q_1)/q_1 - s = L-c-s \quad \text{from (4.143).} \qquad (4.145)
$$

Since $\psi_{1,N}(\cdot)$ has a jump at v, thus, the expected payoff at v is strictly lower compared to a price close to v. Thus, the expected payoff of primary 2 at v is strictly less than $L-c-s$. Similarly, a price which is strictly less than L fetches a payoff of at most $L-c-s$ under Y.

Hence, when the channel of primary 1 is available, then, under Y the maximum expected payoff that primary 2 can attain is $L-c-s$. It is attained at any price in the interval $[L, \tilde{p}_2]$.

Step ii: When the channel of primary 1 is unavailable, then the payoff that primary 2 attains is $(v-c-s)$. Hence, the maximum expected payoff of primary 2 under Y is

$$
\begin{aligned}
q_1(L-c-s)+(1-q_1)(v-c-s) &= (v-c)(1-q_1)+(L-c)q_1 - s \\
= (v-c)(1-q_1)+q_1 s/q_2 - s &= (v-c)(1-q_1)+s(q_1-q_2)/q_2 \quad \text{since } L-c = s/q_2.
\end{aligned}
$$

Step iii: Now, we show that if primary 2 selects N, then the maximum expected payoff attained by primary 2 is $(v-c)(1-q_1)+s(q_1-q_2)/q_2$. Toward this end, we first show that the maximum expected payoff attained by primary 2 at any price in the interval $[\tilde{p}_2, \tilde{p}_1]$ and $[\tilde{p}_1, v]$ is $(v-c)(1-q_1)+s(q_1-q_2)/q_2$ and the maximum expected payoff is attained at any price in the interval $[\tilde{p}_2, v)$ (Step iii.a. and Step iii.b. resp.). Subsequently, we show that if primary 2 selects any price less than $\tilde{p}_2$, then the maximum expected payoff is at most $(v-c)(1-q_1)+s(q_1-q_2)/q_2$ (Step iii.c.).

Step iii.a: Suppose that primary 2 selects a price x in the interval $[\tilde{p}_2, \tilde{p}_1]$. Primary 1 selects a price less than or equal to $x \in [\tilde{p}_2, \tilde{p}_1]$ if the channel of primary 1 is available, it selects Y and a price which is less than or equal to x. Thus, primary 1 selects a price less than or equal to x w.p. $p_1 q_1 \psi_{1,Y}(x)$. Since $\psi_{1,Y}(\cdot)$ is continuous in $[\tilde{p}_2, \tilde{p}_1]$, the expected payoff of primary 2 at x is

$$(x-c)(1-p_1 q_1 \psi_{1,Y}(x)) = (v-c)(1-q_1)+s(q_1-q_2)/q_2 \quad \text{from (4.116)\&(4.120).} \tag{4.146}$$

Step iii.b: Now suppose primary 1 selects a price x from the interval $[\tilde{p}_1, v)$. Primary 1 selects a price less than or equal to x w.p. $p_1 q_1 + (1-p_1) q_1 \psi_{1,N}(x)$. Thus, the expected payoff of primary 2 at x is

$$(x-c)(1-p_1 q_1-(1-p_1) q_1 \psi_{1,N}(x)) = (v-c)(1-q_1)+s(q_1-q_2)/q_2 \quad \text{cf. (4.118)} \tag{4.147}$$

to primary 2. The expected payoff at v is strictly less than the expected payoff at a price just below v since $\psi_{1,N}(\cdot)$ has a jump at v. Thus, the expected payoff at v is strictly less than $(v-c)(1-q_1)+s(q_1-q_2)/q_2$.

Step iii.c: Now suppose that primary 2 selects a price x from the interval $[L, \tilde{p}_2]$. Primary 1 selects a price in the interval only when it selects Y. Thus, the expected payoff of primary 2 at x is

$$(x-c)(1-p_1q_1\psi_{1,Y}(x)) = (x-c)(1-q_1)+(L-c)q_1 \quad \text{from (4.116)} \tag{4.148}$$

Since the co-efficient of x is positive, the above is maximized at $x = \tilde{p}_2$. By (4.125) $L-c = (1-p_2)(\tilde{p}_2-c)$. Hence, the maximum value is

$$(\tilde{p}_2-c)(1-q_1)+(\tilde{p}_2-c)(1-p_2)q_1 = (\tilde{p}_2-c)(1-p_2q_1)$$

$$= (v-c)(1-q_1)+s(q_1-q_2)/q_2 \quad \text{from (4.123).} \tag{4.149}$$

On the other hand a price which is strictly less than L fetches a payoff which is strictly less than the payoff at L as L is the lowest end-point of the support of primary 1. Thus, under N, the maximum expected payoff attained by primary 2 is $(v-c)(1-q_1)+s(q_1-q_2)/q_2$ and it is attained at any price in the interval $[\tilde{p}_2, v)$.

Step iv: Hence, the maximum expected payoff attained by primary 2 is $(v-c)(1-q_1)+s(q_1-q_2)/q_2$ and it is attained if primary 2 follows the strategy profile. Thus, primary 2 also does not have any profitable unilateral deviation. Hence, the result follows. □

4.5.6 Numerical Results

Fig. 4.11 shows the variation of p_i, the probability with which primary i selects Y. As S increases, p_is decrease. Additionally, $p_1 > p_2$ when $0 < s < q_2(v-c)(1-q_2)$, when $s \geq q_2(v-c)(1-q_2)$ both the primaries select N w.p. 1 and thus, $p_i = 0$. When $s = 0$, then $p_i = 1$. When $s \geq q_2(v-c)(1-q_1)/(1-q_1+q_2)$ p_2 is also 0 but p_1 is positive. p_1 decreases at a slower rate compared to the p_2. The difference between p_1 and p_2 is maximum at $s = q_2(v-c)(1-q_1)/(1-q_1+q_2)$. When $s \geq q_2(v-c)(1-q_1)/(1-q_1+q_2)$, p_1 decreases at a faster rate.

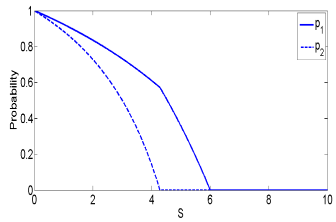


Figure 4.11: Variation of p_i, $i = 1, 2$ with S for an example setting: $v = 25$, $c = 0$, $q_1 = 0.7$, $q_2 = 0.4$.

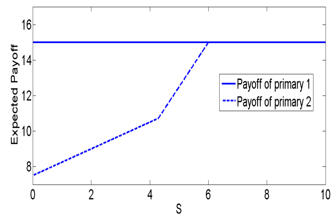


Figure 4.12: Variation of the expected payoffs of primaries with s in the same example setting considered in Fig. 4.11.

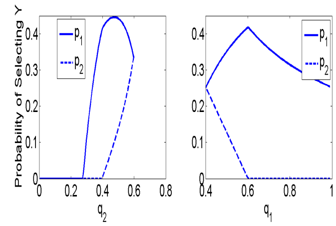


Figure 4.13: Variation of p_1 and p_2 with q_1 and q_2. In the left hand figure we use $q_1 = 0.6, c = 0, v = 50, s = 4$ and for the right hand figure we use $q_2 = 0.4, v = 50, c = 0, s = 4$.

Fig. 4.12 shows the variation of the expected payoffs of the primaries with s, the cost of acquiring the CSI. The expected payoff of primary 1 is independent of s. However, the expected payoff of primary 2 decreases as s when $s < q_2(v - c)(1 - q_2)$. Thus, as s decreases the payoff of primary 2 decreases which contradicts the conventional wisdom which suggests that the payoff of a primary *should increase* as s decreases.

Fig. 4.13 shows the variations of $p_i, i = 1, 2$ with $q_i, i = 1, 2$. Note from the left hand figure of Fig. 4.13 that when q_2 is low, both the primaries select N w.p. 1, thus, $p_i = 0$. When $q_2 > 0.25$, p_1 becomes positive, but p_2 is 0. Due to high q_1, primary 1 selects Y with a higher probability and gains more compared to primary 2. When $q_2 > 0.4$, p_2 becomes positive and the difference between p_1 and p_2 decreases. As q_2 becomes close to q_1, primary 2 also selects Y with a higher probability. Eventually, when $q_2 \to q_1$, $p_2 \to p_1$.

Note from the right hand figure of Fig. 4.13 that when $q_1 = q_2$, $p_1 = p_2$. As q_1

increases p_2 decreases and eventually it becomes 0. Note that p_1 initially increases with q_1. In this regime p_2 decreases, however, q_1 is not so high, thus, primary 1 can gain more by selecting Y, hence, p_1 increases. However, eventually p_2 becomes 0 and q_1 is high, thus, p_1 decreases.

4.6 Future Work

In this chapter, we consider the scenario where there are only two primaries. In future, we consider the scenario where there are more than two primaries. When there are more than two primaries, each primary can acquire the CSIs of any number of primaries it wants. A primary needs to select a price based on how many CSIs of the competitors it has acquired and how many channels are available among the acquired CSIs. The price of the primary will not only based on the information it has but also on the information that its competitors have. However, the primary itself is not aware of the decision of the competitors. The competitors may have acquired CSIs of different primaries and thus, may have different information compared to the primary. The primary, thus, itself is not aware of the information that its competitors have even they acquire the CSIs. This makes the characterization of an NE strategy a computationally challenging task.

We also consider that the available channels are statistically identical. The characterization of the NE when the available channel may belong to different states is also work for the future. In future, we also consider the setting where the market operates over multiple locations.

4.A Uniqueness of Results in the Basic Model

Here, we show that there can not be any other NE strategy profile apart from those described in Theorems 4.3 and 4.4 in the basic model. Note that when $s \geq q(v-c)(1-q)$, then the NE strategy profile is the one described in Theorem 4.3 and when $s < q(v-c)(1-q)$, the NE strategy profile is the one described in Theorem 4.4.

Structure of the Pricing strategies

We first investigate the key structure of the NE pricing strategies (if it exists).

Note that under Y, if a primary knows that its competitor's channel is not available then it will choose v w.p. 1. We thus, investigate the structure of $F_1(\cdot)$ and $F(\cdot)$ in an NE strategy. Recall that $F_1(\cdot)$ is the pricing distribution that a primary chooses when it selects Y and knows that the channel of its competitor is available, while $F(\cdot)$ is the pricing distribution that a primary chooses when it selects N.

Theorem 4.13. *In an NE strategy profile, neither $F(\cdot)$ nor $F_1(\cdot)$ can have a jump at any price which is less than v. Additionally, $F_1(\cdot)$ can not have a jump at v.*

Proof. First, we show that neither $F(\cdot)$ nor $F_1(\cdot)$ can have a jump at any price which is less than v. Subsequently, we show that $F_1(\cdot)$ can not have a jump at v.

Note that a primary can only have a jump at a price if it is a best response. First, note that $F(\cdot)$ can not have a jump at a price less than or equal to c. This is because at a price less than or equal to c will fetch a negative profit, however, if the primary selects v, then it will get an expected payoff of $(v-c)(1-q)$.

Similarly, if $F_1(\cdot)$ has a jump at a price less than or equal to c, then its payoff under

$F_1(\cdot)$ is at most $(c - c) - s = -s$. Note that when the channel of the competitor is unavailable, then the primary will attain the payoff of $(v - c) - s$. Hence, the expected payoff under Y is thus, $(v-c-s)(1-q)-sq = (v-c)(1-q)-s$. However, if the primary selects N and the price v which will fetch an expected profit of $(v - c)(1 - q)$.

Now if either $F_1(\cdot)$ or $F(\cdot)$ has a jump at $c < x < v$, then the other primary can select a price $x - \epsilon$ and still can gain higher payoff compared to x. Thus, the other primary will not select any price in the interval $(x - \epsilon, x + \epsilon)$ as it will get a strictly higher payoff at $x - \epsilon$ compared to any price in the interval. Hence, the primary itself can gain strictly higher payoff by selecting a price at $y \in (x, x + \epsilon)$ compared to x. It contradicts the fact that either $F_1(\cdot)$ or $F(\cdot)$ will have a jump at $x < v$.

Next, we show that $F_1(\cdot)$ can not have a jump at v. Suppose $F_1(\cdot)$ has a jump at v, then the other primary will never select v with positive probability when its channel is available as it can get strictly higher payoff by selecting a price slightly less than v. Thus, at v, the primary is never going to sell its channel when the channel of other primary is available. Thus, the expected payoff that the primary will get under $F_1(\cdot)$ is $-s$. Thus, under Y, the expected payoff that the primary will attain is $(v - c)(1 - q) - s$. Again, the primary will have an incentive to deviate to select N and select the price v which will fetch a payoff of at least $(v - c)(1 - q)$. □

The above theorem shows that if the channel of a primary is available then it can not have a jump at any price other than v.

Now, we show an important property of $F_1(\cdot)$ and $F(\cdot)$ when a primary randomizes between Y and N in an NE strategy.

Theorem 4.14. *Suppose that primary* 1 *selects* Y *w.p.* p *and* N *w.p.* $1-p$ *in an NE. Then, the upper end point of the support set of* $F_1(\cdot)$ *must be lower than or equal to to the lower end-point of the support set of* $F(\cdot)$.

Proof. Note from Theorem 4.13 that $F_1(\cdot)$ can not have a jump at v. Thus, the lower end point of $F_1(\cdot)$ can never be v. If the lower end-point of the support set of $F(\cdot)$ is v, then the statement is trivially true. So, we consider the setting where the lower end-point of the support set of $F(\cdot)$ is less than v. Suppose the statement is false. Thus, there must exist a $x < y < v$ such that x is in the support set of $F(\cdot)$ and y is in the support set of $F_1(\cdot)$. Now, suppose that the maximum expected payoff of primary 1 when it selects $F_1(\cdot)$ under Y is $\bar{p}_1$. Also let $\bar{p}_2$ be the maximum expected payoff primary 1 gets when it selects $F(\cdot)$ under N.

Since $x < v$, thus, if the channel of competitor is available, it can not have any jump at x. Hence, while choosing N, the probability of winning at x is $(1-q\phi_2(x))$ where $\phi_2(\cdot)$ is the probability that the primary 2 will select a price less than or equal to x when its channel is available. Since $x < v$ and primary 2 does not have a jump at x, thus, x is a best response to primary 1 under N. Thus,

$$(x-c)(1-q\phi_2(x)) = \bar{p}_2 \tag{4.150}$$

Since $\bar{p}_1$ is the maximum expected payoff that primary 1 gets under $F_1(\cdot)$, thus, if primary 1 selects x under $F_1(\cdot)$, then its payoff would be

$$(x-c)(1-\phi_2(x)) \leq \bar{p}_1$$

$$\frac{1-\phi_2(x)}{1-q\phi_2(x)} \leq \frac{\bar{p}_1}{\bar{p}_2} \quad \text{from (4.150)} \tag{4.151}$$

Similarly, since $y < v$, thus, primary 2 will not have a jump at y when its channel is available. Thus, primary 1's expected payoff under $F_1(\cdot)$ at the price y is

$$(y - c)(1 - \phi_2(y)) = \bar{p}_1 \tag{4.152}$$

If primary 1 selects N and the price y, then its expected payoff is

$$\begin{aligned}(y - c)(1 - q\phi_2(y)) &= \bar{p}_1 \frac{1 - q\phi_2(y)}{1 - \phi_2(y)} \quad \text{from (4.152)} \\ &\geq \frac{(1 - q\phi_2(y))(1 - \phi_2(x))}{(1 - \phi_2(y))(1 - q\phi_2(x))} \bar{p}_2 \text{from (4.151)}\end{aligned} \tag{4.153}$$

Now, note that $\phi_2(y) \geq \phi_2(x)$ as $y > x$. If $\phi_2(y) = \phi_2(x)$, then the expected payoff at y must be greater than the expected payoff at x, hence, x can not be a best response at N for primary 1. However, if $\phi_2(y) > \phi_2(x)$, then the expected payoff at y at N is strictly higher than $\bar{p}_2$ by (4.153). Thus, this leads to a contradiction since $\bar{p}_2$ is the maximum expected payoff at N. Hence, the result follows. □

Now, we show that both $F(\cdot)$ and $F_1(\cdot)$ are contiguous. Additionally, if a primary randomizes between Y and N, then there is no "gap" between $F(\cdot)$ and $F_1(\cdot)$.

Theorem 4.15. *(i) In a NE strategy if a primary selects Y w.p. 1, and it selects $F_1(\cdot)$ when it knows that the channel of other primary is available, then $F_1(\cdot)$ must be contiguous and the upper end-point of $F_1(\cdot)$ must be v.*

(ii) In a NE strategy if a primary selects N w.p. 1, and if it selects $F(\cdot)$ when it knows that channel of other primary is available, then $F(\cdot)$ must be contiguous and the upper end-point of $F(\cdot)$ must be v.

(iii) In a NE strategy if the primary randomizes between Y and N, both $F_1(\cdot)$ and $F(\cdot)$

must be contiguous, there must not be any gap between the support sets of $F_1(\cdot)$ *and* $F(\cdot)$. *Moreover, the upper-end point of* $F(\cdot)$ *must be* v.

Proof. We only show the proof of part (i). The proof of the other parts will be similar.

Part (i): Suppose that primary 1 selects $F_1(\cdot)$ such that $F_1(x) = F_1(y)$ for some $v \geq y > x$ such that both y, x are under the support set of $F_1(\cdot)$. Since $x < v$ thus, primary 2 does not have a jump at x when its channel is available. Hence, x is a best response for primary 1 under $F_1(\cdot)$. By Theorem 4.14 if a primary randomizes between Y and N, then the lower end-point of $F(\cdot)$ must be greater than or equal to the lower end-point of $F_1(\cdot)$. Thus, $F(x) = F(y) = 0$. Thus, primary 2 will attain a strictly higher payoff at any value $z \in (x, y)$ compared to at x. Thus, there is an $\epsilon > 0$ where primary 2 will never select any price in the interval $[x, x + \epsilon]$, hence, x itself is not a best response for primary 1. But the above contradicts the fact that x is in the support set of $F_1(\cdot)$. Hence, the result follows. □

Special Property where primaries randomize between Y and N

Next theorem shows that in an NE if both the primaries randomize between Y and N. Then both of them should put the same probability mass on Y (and N, resp.).

Theorem 4.16. *Suppose primary* 1 *selects* Y *w.p.* $1 > p_1 > 0$ *and* N *w.p.* $1 - p_1$. *Primary* 2 *selects* Y *w.p.* $1 > p_2 > 1$ *and* N *w.p.* $1 - p_2$. *Then,* $p_1 = p_2$ *in an NE strategy profile.*

Proof. Suppose that at Y, primary 1 (2, resp.) selects a price using the distribution $F_1(\cdot)$ ($\bar{F}_1(\cdot)$, resp.) when it knows that the channel of primary 2 (1, resp.) is available for sale.

At N, suppose that primary 1 (2, resp.) selects a price using the distribution $F(\cdot)$ ($\bar{F}(\cdot)$, resp.).

Let L_1 ($\bar{L}_1$, resp.) and U_1 ($\bar{U}_1$, resp.) be respectively the lower and upper end-points of the support of F_1 ($\bar{F}_1$, resp.). Let L ($\bar{L}$, resp.) and U ($\bar{U}$, resp.) be the lower and upper end-point of the support of $F(\cdot)$ ($\bar{F}$, resp.) respectively. By Theorem 4.14 $L_1 < L$ and $\bar{L}_1 < \bar{L}$. Note also from Theorem 4.15 that $U_1 = L$ and $\bar{U}_1 = \bar{L}$.

First, we show that $L_1 = \bar{L}_1$. Suppose not. Without loss of generality assume that $L_1 < \bar{L}_1$. Thus, primary 2 does not select any price in the interval $(L_1, \bar{L}_1)$. Thus, the primary 1 will get a strictly higher payoff at $\bar{L}_1 - \epsilon$ for some $\epsilon > 0$ compared to L_1. Hence, primary 1 must select prices close to L_1 with probability 0 which contradicts that L_1 is the lower end-point of F_1. Thus, $L_1 = \bar{L}_1$.

By Theorem 4.13 L_1 can not be equal to v. Thus, $L_1 = \bar{L}_1 < v$. Thus, both L_1 and $\bar{L}_1$ are best responses to primary 1 and primary 2 respectively at Y. Since $L_1 = \bar{L}_1$, thus, the expected payoff at Y must be the same for both players. Also note that since primaries randomize between Y and N, thus, the payoffs at Y and N must be the same. Hence, the expected payoff of the primaries at N also must be the same. Thus, no primary can have a jump at v under N. This is because if a primary has a jump at N, then the other primary would get a strictly higher payoff at a price just below v which contradicts that both the primaries must have the same payoff under N. Thus, $L, \bar{L} < v$.

Now, we show that $L = \bar{L}$, towards this end, we introduce few more notations. Let $\bar{p}_1 - c$ be the maximum expected payoff of primary 1 (2, resp.) under $F_1(\cdot)$ ($\bar{F}_1(\cdot)$, resp.) and $\bar{p}_2 - c$ be the expected payoff of primary 1 (2, resp.) under $F(\cdot)$ ($\bar{F}(\cdot)$, resp.).

Suppose $L \neq \bar{L}$. Without loss of generality assume that $L > \bar{L}$. Thus, $\bar{L} < v$. Since

$\bar{L}$ is the upper end-point of $\bar{F}_1(\cdot)$ and $\bar{L} < v$, thus, the expected payoff of primary 2 at $\bar{L}$ under $\bar{F}_1(\cdot)$ is $\bar{p}_1 - c$. Thus,

$$(\bar{L} - c)(1 - p_1 F_1(\bar{L})) = \bar{p}_1 - c \tag{4.154}$$

$\bar{L}$ is also a best response of primary 2 at N, thus,

$$(\bar{L} - c)(1 - qp_1 F_1(\bar{L})) = \bar{p}_2 - c \tag{4.155}$$

Since $v > L > \bar{L}$ and L is the upper end-point of $F_1(\cdot)$, thus, L is also a best response of primary 1 under Y.

$$(L - c)(1 - p_2 - (1 - p_2)\bar{F}(L)) = \bar{p}_1 - c \tag{4.156}$$

Since L is the lower end point of $F(\cdot)$, thus, under N, the expected payoff of primary 1 at L is

$$(L - c)(1 - qp_2 - q(1 - p_2)\bar{F}(L)) = \bar{p}_2 - c \tag{4.157}$$

Also note that since $L > \bar{L}$, thus, L is in the support of $\bar{F}(\cdot)$, thus, under N, the expected payoff to primary 2 at L is

$$(L - c)(1 - qp_1) = \bar{p}_2 - c \tag{4.158}$$

as $F_1(L) = 1$ and $F(L) = 0$.

Thus, from (4.158) and (4.157) $p_1 = p_2 + (1 - p_2)\bar{F}(L)$. Now, the expected payoff of primary 2 at L when it selects Y and the channel of primary 1 is available, is

$$\begin{aligned}(L - c)(1 - p_1) &= (L - c)(1 - p_2 - (1 - p_2)\bar{F}(L)) \\ &= \bar{p}_1 - c \quad from(4.156)\end{aligned} \tag{4.159}$$

Hence, from (4.154), (4.159), (4.155) and (4.158) that

$$\frac{1 - p_1 F_1(\bar{L})}{1 - p_1} = \frac{1 - q p_1 F_1(\bar{L})}{1 - q p_1} \tag{4.160}$$

which leads to a contradiction as neither q is not equal to 1 nor $F_1(\bar{L}) = 1$. Hence, we must have $L = \bar{L}$.

Now, at L, the expected payoff of primary 2 at Y is $(L - c)(1 - p_1) = \bar{p}_1 - c$. Similarly, at $\bar{L}$, the expected payoff of primary 1 at Y is $(\bar{L} - c)(1 - p_2) = \bar{p}_1 - c$. Since $L = \bar{L}$, thus, we must have $p_1 = p_2$. Hence, the result follows. □

Next, we determine the probability with which the primaries must randomize between Y and N in an NE strategy.

Observation 4.10. If both the primaries randomize between Y and N, they should do it w.p. p where $p = \dfrac{q(v - c)(1 - q) - s}{q(v - c)(1 - q) - sq}$.

Proof. Suppose that a primary selects its price from $F_1(\cdot)$ under Y and when it knows that the channel of other primary is available. Suppose that under $F_1(\cdot)$ the expected payoff is $\tilde{p}_1 - c$. Thus, the expected payoff of primary 1 under Y is

$$(v - c)(1 - q) + q(\tilde{p}_1 - c) - s \tag{4.161}$$

Suppose that the primary selects its price from $F(\cdot)$ under N. Since no primary has any jump at v when both the primaries randomize between Y and N and v is the upper end-point of $F(\cdot)$ by Theorem 4.15, thus, the expected payoff under N is $(v - c)(1 - q)$. Since the primary randomizes between Y and N, thus, the expected payoff under Y and under N must be the same. Hence, we must have $s = q(\tilde{p}_1 - c)$.

Suppose L be the upper end point of the support of $F_1(\cdot)$ (and thus, also the lower endpoint of $F(\cdot)$). Hence, the expected payoff at L is

$$(L-c)(1-qp) = (v-c)(1-q) \tag{4.162}$$

Thus, $L = (v-c)(1-q)/(1-qp) + c$. Also note that L is also a best response at $F_1(\cdot)$. Thus,

$$\begin{aligned}(L-c)(1-p) &= \frac{s}{q}\\ \frac{(v-c)(1-q)(1-p)}{1-qp} &= \frac{s}{q}\end{aligned} \tag{4.163}$$

Obtaining p from the above expression will give the desired result. □

Does there exists an NE where one player selects Y w.p. 1?

Theorem 4.17. *There is no NE where a primary selects Y w.p. 1 and the other primary selects Y w.p. p and N w.p. $1-p$.*

Proof. Without loss of generality, assume that primary 1 selects Y w.p. 1 and the primary 2 selects Y w.p. p and N w.p. $1-p$.

Now suppose that primary 1 selects a price using the distribution function $F_1(\cdot)$ when it knows that the channel of its competitor is available for sale. Let at Y, primary 2 selects a price using distribution function $F_2(\cdot)$ when it knows that the channel of its competitor is available for sale, and at N, it selects a price using distribution function $\bar{F}_2(\cdot)$.

Let L_1 be the lower end-point of the support of $F_1(\cdot)$ and L_2 ($\bar{L}_2$, resp.) be the lower end-point of $F_2(\cdot)$ ($\bar{F}_2$, resp.).

Note from Theorem 4.14 that $\bar{L}_2 > L_2$. Now, we show that $L_1 = L_2$. Suppose that $L_1 > L_2$, then, primary 2 can attain strictly higher payoff at any price close to L_1

compared to at L_2 which shows that L_2 can not be a lower end-point of F_2. By symmetry, it also follows that L_1 can not be less than L_2, hence $L_1 = L_2$. Thus, the expected payoff at Y must be equal for both the primaries.

Now, note that $F_1(\cdot)$ can not have a jump at v by Theorem 4.13. Note that the upper end-point of $\bar{F}_2(\cdot)$ is v by Theorem 4.15. Since $F_1(\cdot)$ does not have a jump at v, thus, v is a best response of primary 2 under N. Thus, the expected payoff of primary 2 under N is $(v-c)(1-q)$. Since primary 2 randomizes between N and Y, thus, the expected payoff of primary 2 is $(v-c)(1-q)$ under Y. Thus, the expected payoff of primary 1 is also $(v-c)(1-q)$.

At any $x \in [\bar{L}_2, v)$ primary 2 does not have any jump, thus, x is a best response for primary 1. Thus, at any $x \in [\bar{L}_2, v)$ the expected payoff of primary 1 is

$$(x-c)(1-p-(1-p)\bar{F}_2(x)) = \tilde{p}_1 - c$$

$$\bar{F}_2(x) = \frac{1}{1-p}(1-p-\frac{1}{1-p}\frac{\tilde{p}_1-c}{x-c}) \tag{4.164}$$

Note that under Y, the expected payoff of primary 1 is $(v-c)(1-q)+q(\tilde{p}_1-c)-s$.

Now, if primary 1 selects N and a price $x \in [\bar{L}_2, v)$, then its expected payoff is

$$\begin{aligned}&(x-c)(1-qp-q(1-p)\bar{F}_2(x))\\&=(x-c)(1-qp-q(1-p-\frac{\tilde{p}_1-c}{x-c}))\\&=(x-c)(1-q)+(\tilde{p}_1-c)q\end{aligned} \tag{4.165}$$

Thus, for any small enough $\epsilon > 0$, we have $(v-c-\epsilon)(1-q)+(\tilde{p}_1-c)q > (v-c)(1-q)(1-q)+q(\tilde{p}_1-c)-s$¿ Hence, primary 1 has profitable unilateral deviation. Hence, such a strategy profile can never be an NE. □

Note that we have already ruled out the possibility of the NE strategy profile where a primary selects Y w.p. 1 and the other selects either N or Y w.p. 1. Hence, there is no NE where a primary selects Y w.p. 1.

Does there exist a NE where one player selects N w.p. 1?

Theorem 4.18. *There is no NE where a primary selects N w.p. 1 and the other primary randomizes between Y and N.*

Proof. Without loss of generality assume that primary 1 selects N w.p. 1 and primary 2 randomizes between Y and N.

Suppose that primary 1 selects its price using $F(\cdot)$. Let L be the lower end-point of the support of $F(\cdot)$. Let $\tilde{p}_1 - c$ be the expected payoff of primary 1. Let primary 2 selects $F_2(\cdot)$ when it selects Y and it knows that the channel of primary 1 is available. Let L_2 be the lower end-point of $F_2(\cdot)$. First, note that L_1 must be equal to the L_2. Since $L_2 < v$ by Theorem 4.15 and $L_1 = L_2$, thus, L_2 is a best response for both primary 1 and 2. The expected payoff of primary 2 under Y when the channel of primary 1 is $L_2 - c - s$. Similarly, the expected payoff of primary 1 is $L_2 - c$. Thus, $\tilde{p}_1 - c = L_2 - c$. Expected payoff of primary 2 under Y is, $q(L_2 - c) + (v - c)(1 - q) - s$.

Also let L be the lower end point of $\bar{F}_2$ where $\bar{F}_2$ be the pricing strategy that primary 2 uses when it selects N. From Theorem 4.15 the upper end-point of the support of $F_2(\cdot)$ is also L. From Theorem4.15 also note that the upper end-point of $\bar{F}_2(\cdot)$ is v.

First, note that under N the expected payoff of primary 2 must be at least $(v-c)(1-q)$ as this is the payoff that primary 2 can at least get when it selects v. Now, we show that under N, the expected payoff of primary 2 must be equal to $(v - c)(1 - q)$. Suppose not, i.e. primary 2 attains an expected payoff of larger than $(v-c)(1-q)$. Since the upper end-point of $\bar{F}_2$ is v, thus, primary 1 must have a jump at v. Since primary 1 has a jump at v, thus, v is a best response for primary 1. Thus, primary 1 attains an expected payoff of $(v-c)(1-q)$ under N. Thus, $\tilde{p}_1 - c = (v-c)(1-q)$. Since primary 2 is randomizing between Y and N, thus, the primary 2's expected payoff is also greater than $(v-c)(1-q)$ when it selects Y. Thus, if the primary 1 select Y and price L_2 w.p. 1 when the channel of primary 2 is available and v w.p. 1 otherwise; then it will also get an expected payoff of $q(L_2-c)+(v-c)(1-q)-s$ which is higher compared to $(v-c)(1-q)$. Hence, this is not possible.

Thus, the expected payoff of primary 2 must be equal to $(v-c)(1-q)$. Since primary 1 gets an expected payoff of at least of $(v-c)(1-q)$, thus, $L_2 - c \geq (v-c)(1-q)$. Since L is the upper end-point of the support of $F_2(\cdot)$ and L is also the lower end-point of the support of $\bar{F}_2$, thus,

$$(L-c)(1-F_1(L)) - s \geq (v-c)(1-q) - s$$

$$(L-c)(1-qF_1(L)) = (v-c)(1-q) \tag{4.166}$$

both can not be true simultaneously since $q \neq 1$. Hence, the result follows. □

Chapter 5

Co-existence of Multiple Secondaries in the Primary's channel

We now consider the setting where the primary allows multiple secondaries to access the spectrum at a location. In a secondary network a secondary-base station (secondary-BS) transmits to a secondary-user terminal (secondary-UT) with certain power. Secondary-BSs are constrained to allocate transmitting powers such that the total interference at each primary-UT is below a given threshold. We formulate the power allocation problem as a concave non cooperative game with secondary-BSs as players and multiple primary-UTs enforcing the coupled constraints. The equilibrium selection is based on the concept of normalized Nash equilibrium (NNE). When the interference at a secondary-UT from adjacent secondary-BSs is negligible, the NNE is shown to be unique for any strictly concave utility. We propose a distributed algorithm which converges to the unique NNE. When the interference at a secondary-UT from adjacent secondary-BSs is not negligible, an NNE may not be unique and the computation of the NNE may be computationally challenging. To avoid these drawbacks, we introduce the concept of a *weakly normalized Nash equilibrium* (WNNE) which keeps most of the NNEs' interesting properties but, in contrast to the latter, the WNNE can be determined with low complexity.

The chapter is organized as follows: In Section 5.1 we describe the system model. In Section 5.2 we formulate the problem as a non cooperative game with secondary-BSs as the players; secondary-BSs need to select the power with which they will transmit subject to the constraint that the interference must be limited at each primary user terminal. We describe Normalized Nash equilibrium (NNE) as a solution concept. In Section 5.3 we characterize the utility functions which admit a unique NNE. In Section 5.4 we introduce the concept of Weakly normalized Nash equilibrium (WNNE). In Section 5.5 we show that the NNE is equivalent to an optimal solution of a convex optimization problem in a concave potential game. Leveraging on that we propose a distributed algorithm which converges to the NNE in the concave potential games. If the game is not concave potential, then we show that we can use the WNNE as an equilibrium concept which can be computed easily compared to the NNE in those games. We illustrate the significance of WNNE in a specific example setting in Section 5.6. In Section 5.7 we numerically analyze the properties of the NNE and WNNE for some widely used utility functions.

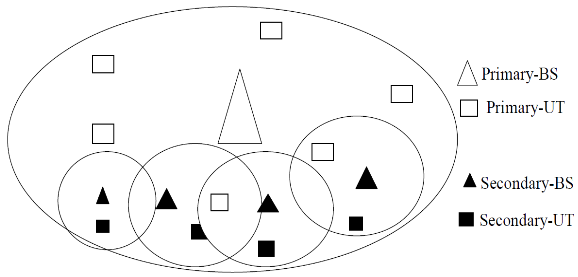


Figure 5.1: Primary-BS, primary-UTs, secondary-BSs and secondary-UTs in a region. Circles represent the range of base stations. Primary-BS has higher coverage compared to secondary-BS. Each secondary-BS serves only one secondary-UT. Secondary-UT is placed close to its serving secondary-BS. Secondary-BSs cause interference at a primary-UT as well as the secondary-UTs.

5.1 System Model

Notation. Vectors and matrices are denoted by bold lower case and bold capital letters, respectively; $\cdot^T$ denotes the transpose operator; the notation $\mathbf{x} \succeq \mathbf{0}$ stays for component-wise inequality; $\mathbf{I}_M$ denotes the $M \times M$ identity matrix and $\mathbf{1}_M$ and $\mathbf{0}_M$ are the M-dimensional column vectors of ones and zeros, respectively. Given the real x, $(x)^+ = \max(x, 0)$. Additionally, $\mathbf{H}_K$ is the sub-matrix of $\mathbf{H}$ containing only the rows and columns in an index set K. This notation is immediately extended to vectors. The vector $\mathbf{v}_{-i}$ is obtained from vector $\mathbf{v}$ by suppressing the ith component. The matrix operator $\odot$ denotes the Hadamard or component wise product; $\mathbf{D} = \text{diag}(\mathbf{v})$ maps vector $\mathbf{v}$ onto a diagonal matrix with diagonal component $d_{\ell,\ell} = v_\ell$. The set of nonnegative real numbers in denoted by $\mathbb{R}_+$.

We consider a cognitive radio network consisting of F secondary-BSs and M primary-UTs (Fig. 5.1) [3]. In each secondary network, a secondary-BS serves a single secondary-UT. Thus, there are F secondary-UTs. We do not make any assumptions regarding the

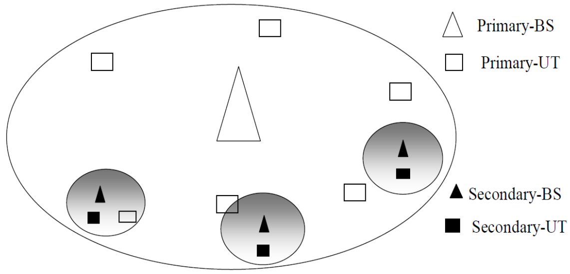


Figure 5.2: Primary-BS, primary-UTs, secondary-BSs and secondary-UTs in a region. Circles represent the range of base stations. The range of secondary-BS is smaller compared to Figure 5.1. The number of secondary networks is lower. In this setting, secondary-BSs do not cause interference at the adjacent secondary-UTs. However, they still cause interference at primary-UTs.

distribution of secondary-BSs, secondary-UTs and secondary-UTs except the fact that each secondary-UT is located close to its secondary-BS since secondary-BS has smaller coverage area. We consider the secondary spectrum access model where secondary networks and a primary network use the same channel for downlink communications. Let h_f be the channel gain between the secondary-BS f and its served secondary-UT f and $\widehat{h}_m^f$ is the channel gain between secondary-BS f and primary-UT m. Finally, $\widetilde{h}_k^f$ is the channel gain between secondary-BS k and secondary-UT f, with $f \neq k$. The secondary-BS f transmits with power $p_f \geq 0$. For future use, it is convenient to define the following vectors $\mathbf{p} = (p_1, p_2, \ldots, p_F)^T$, $\mathbf{h} = (h_1, h_2, \ldots, h_F)^T$, $\widehat{\mathbf{h}}_m = (\widehat{h}_m^1, \widehat{h}_m^2, \ldots, \widehat{h}_m^F)^T$, with $m = 1, \ldots, M$, $\widehat{\mathbf{h}}^f = (\widehat{h}_1^f, \widehat{h}_2^f, \ldots, \widehat{h}_M^f)^T$, and $\widetilde{\mathbf{h}}^f = (\widetilde{h}_1^f, \widetilde{h}_2^f, \ldots, \widetilde{h}_{f-1}^f, \widetilde{h}_{f+1}^f, \ldots \widetilde{h}_F^f)^T$, with $f = 1, \ldots, F$.

The primary network operates in the time division duplexing (TDD) mode i.e. the primary-UTs transmit and receive in the same frequency band. This feature implies that secondary-BS f can estimate $\widehat{h}_m^f$ by sensing a pilot signal sent by primary-UT m due to the *channel reciprocity* assumption [90] . Hence, the channel feedback by primary-UTs to secondary-BSs is not required to estimate $\widehat{h}_m^f$. Also note that secondary-BS f can acquire $\widehat{h}_m^f$ without communicating with other secondary-BSs or secondary-UTs. The interference from secondary-BSs at a primary-UT m is

$$I_m = \mathbf{p}^T \widehat{\mathbf{h}}_m \qquad\qquad m = 1, \ldots, M. \tag{5.1}$$

The signal to interference and noise ratio (SINR) at secondary-UT f is given by

$$\gamma_f = \frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f} \tag{5.2}$$

where σ^2 is the variance of the additive white Gaussian noise that also accounts for interference from primary-BSs. In general γ_f is a function of $\mathbf{p}$. When it is convenient, we explicitly point out this dependence by writing $\gamma_f(\mathbf{p})$, otherwise we omit it and use the short notation γ_f.

Until now, we have discussed a general model *where interference at a secondary-UT from other secondary-BSs is not negligible*. We also consider the setting depicted in Fig. 5.2 *where the interference at a secondary-UT from other secondary-BSs is negligible*. The above setting arises when the number of secondary networks is small and the secondary-BS has small coverage area. In this setting, it is reasonable to assume that $\widetilde{\mathbf{h}}^f \cong \mathbf{0}$, and thus, γ_f reduces to the signal to noise ratio (SNR) $\gamma'_f = \frac{p_f h_f}{\sigma^2}$. We show that in this case, the NNE has favorable properties which may not remain valid in a more general setting such as in Fig. 5.1.

Note that in both of these settings, primary-UTs are ubiquitous and in general are distant from the base station due to the wide coverage area of the primary-BS (consider, for example, a macro cell). Thus, primary-UTs are likely to be present relatively close to secondary-BSs. Additionally, since the distance from the primary-BS to primary-UT is large, the received signal power is very low at a primary-UT compared to the signal power received by the secondary-UTs. As a result, secondary-BSs can generate significant performance degradation to primary-UTs even when the interference at each secondary-UT caused by adjacent secondary-BSs is negligible. Thus, the performance degradation at primary-UT due to the transmission of secondary-BSs can be severe in both the studied settings. In order to keep the quality of the downlink communications in each primary-UT acceptable, total interference from all secondary-BSs must be below an acceptable limit in both of these settings. Specifically, we assume

$$\mathbf{p}^T\widehat{\mathbf{h}}_m \leq I_T \qquad\qquad m = 1, \ldots, M. \tag{5.3}$$

In this article we are interested only in keeping the interference caused by secondary-base stations to each primary-UT below a certain acceptable limit. In fact, in practice, the interference from primary-BSs to primary-UTs is efficiently controlled by proper beamforming design and coordinated beamforming. We consider identical thresholds at different primary-UTs in order to keep notations simple. The extension to the general case with different thresholds is straightforward.

Additionally, the transmission powers are constrained to a maximum value P_{MAX} such that

$$\mathbf{p} \preceq P_{\text{MAX}} \mathbf{1}_F. \tag{5.4}$$

5.2 Problem Formulation and Solution concept

5.2.1 Problem Formulation

Secondary-BS f selects its transmission power with the objective of maximizing the quality of its communication in downlink. Its communication quality is characterized by $U_f(\gamma_f)$, where $U_f(\cdot)$ is a concave nondecreasing function.

We formulate the power allocation at secondary-BSs as a non-cooperative game where each secondary-BS aims to maximize its utility $U_f(\gamma_f)$ under constraints[1] (5.3).

More specifically, we define this non-cooperative game in a strategic form as

$$\mathcal{G} = \{\mathcal{F}, \mathcal{P}, \{u_f(\mathbf{p})\}_{f \in \mathcal{F}}\} \tag{5.5}$$

where the elements of the game are

- Player set: Set of the secondary-BSs $\mathcal{F} = \{1, ..., F\}$;
- Strategy set: $\mathcal{P} = \{\mathbf{p} | \mathbf{p} \in \mathbb{R}_+^F \text{ and } \mathbf{p}^T \widehat{\mathbf{h}}_m \leq I_T,\ m = 1, \ldots M\}$, where $\mathbb{R}_+^F$ is the product space of F nonnegative real spaces $\mathbb{R}_+$.
- Utility set: the functions $u_f(\mathbf{p})$ are defined as $u_f(\mathbf{p}) \equiv U_f(\gamma_f(\mathbf{p})) = U_f\left(\frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f}\right)$, $U_f(\cdot)$ is a concave nondecreasing function in $\mathbb{R}_+$.

We adopt a NE of the non-cooperative game $\mathcal{G}$ as a power allocation policy for the secondary-BSs. More specifically, the power allocation vector $\mathbf{p}^*$ is a Nash Equilibrium

[1] Throughout the rest of this chapter, to keep notation and equations compact, we do not consider constraints (5.4). They are orthogonal and can be immediately embedded in the proposed game theoretical framework. The extension of all the results presented here to the case including these additional constraints is straightforward.

(NE), i.e., for every $f \in \mathcal{F}$ and p_f such that $(p_1^*, \dots p_{f-1}^*, p_f, p_{f+1}^*, p_F^*) \in \mathcal{P}$

$$U_f\left(\frac{p_f^* h_f}{\sigma^2 + \mathbf{p}_{-f}^{*T}\widetilde{\mathbf{h}}^f}\right) \geq U_f\left(\frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^{*T}\widetilde{\mathbf{h}}^f}\right). \tag{5.6}$$

By observing that each $U_f\left(\frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^{T}\widetilde{\mathbf{h}}^f}\right)$ is continuous in $\mathbb{R}_+^F$ and concave for $p_f \in \mathbb{R}_+$ and the set $\mathcal{P}$ is convex and closed, we conclude that $\mathcal{G}$ falls in the class of concave games with coupled constraints studied in [67] and a NE exists [67].

5.2.2 Normalized Nash Equilibrium

The strategy set $\mathcal{P}$ is closed, convex, and bounded[2]. Under the further assumption that the functions U_f, for all $f = \mathcal{F}$, possess continuous first derivatives, we can use the necessary and sufficient KKT conditions for constrained maxima [11] to obtain conditions satisfied by a NE $\mathbf{p}^*$. If $\mathbf{p}^*$ is a NE in $\mathcal{P}$, then, there exist F vectors $\boldsymbol{\lambda}^f = (\lambda_1^f, \lambda_2^f \dots, \lambda_M^f)$ with $\boldsymbol{\lambda}^f \geq \mathbf{0}$ such that $\mathbf{p}^*$ satisfies the following system of equations

$$\lambda_m^f(\mathbf{p}^T\widehat{\mathbf{h}}_m - I_T) = 0, \quad m = 1, \dots M$$

$$\text{and } f = 1, \dots F \tag{5.7}$$

$$\frac{\partial U_f(\gamma_f)}{\partial p_f} - \sum_{m=1}^{M} \lambda_m^f \frac{\partial}{\partial p_f}(\mathbf{p}^T\widehat{\mathbf{h}}_m - I_T) = 0 f = 1, \dots F \tag{5.8}$$

We can write (5.8) as

$$U_f'(\gamma_f)\frac{\partial \gamma_f}{\partial p_f} - \boldsymbol{\lambda}^{f\,T}\widehat{\mathbf{h}}^f = 0, \qquad f = 1, \dots F. \tag{5.9}$$

[2] Boundedness can be immediately verified if each secondary-BS's transmitted signal impinges on at least one of the primary-UT, i.e. there exists no secondary-BS f such that $\widehat{h}_m^f = 0$ for all $m = 1, \dots, M$. If this will not be the case, it is not of a practical interest to include a secondary-BS that does not cause interference on any primary-UT, as player in the game $\mathcal{G}$.

A NE in the coupled game $\mathcal{G}$ is not unique in general. The uniqueness of an NE in a constrained game is guaranteed only for orthogonal constraints, i.e., when the strategy of a player is independent of other players' strategies [67]. Rosen in [67] has introduced the concept of NNE that provides a useful equilibrium selection criterion for coupled constrained games when the strategy of a player poses restrictions on the strategy of other players as in our setting.

The strategy $\mathbf{p}^*$ is a *normalized Nash equilibrium* (NNE) if the KKT conditions in (5.7) and (5.9) are satisfied with[3] $\boldsymbol{\lambda}_m = (\lambda_m^1, \ldots, \lambda_m^F) = \lambda_m \mathbf{1}_F^T$, i.e., the Lagrangian multipliers are identical for all the players for each given constraint enforced by primary-UT m. The concept of NNE has several advantages described in the following.

The Lagrangian multiplier λ_m^f can be viewed as the price per unit of interference caused by player f at primary-UT m. Thus, as first advantage, a primary-UT does not have to select different prices for different players in an NNE. Additionally, as it will be clear from the decentralized implementation proposed in Section 5.5.2, the above property considerably reduces the cost and the complexity of the signaling among primary-UTs and secondary-BSs. A second benefit appears in obtaining a distributed algorithm where each primary-UT only needs to track the sum of the interferences in order to calculate the price and does not need to track the interference from each user reducing the communication and signaling costs further.

[3] In [67] an equilibrium is an NNE if the KKT conditions are satisfied for some vector $\mathbf{r} > 0$ and $\boldsymbol{\lambda} >= 0$, $\lambda_m^f = \lambda_m / r_f$. The NNE that we consider is in accordance with the above definition when $\mathbf{r} = \mathbf{1}$. In [67], $\mathbf{r}$ is used to find an NNE for a weighted utility function, i.e., U_f is scaled by $r_f U_F$. In our setting, $\mathbf{r} = \mathbf{1}$ since we do not consider any weighted utility function.

Since NNEs have favorable properties to be implemented in a decentralized fashion, we henceforth examine the computing and the uniqueness of the NNEs.

5.3 On the Uniqueness of an NNE

The uniqueness of an NNE for concave games with coupled constrained has been studied in [67]. In the following proposition we summarize the results useful for further developments.

Proposition 5.1. *[67] Let*

$$\mathbf{G}(\mathbf{p}) = \begin{pmatrix} \frac{\partial^2 u_1}{\partial p_1^2} & \frac{\partial^2 u_1}{\partial p_2 \partial p_1} & \cdots & \frac{\partial^2 u_1}{\partial p_F \partial p_1} \\ \frac{\partial^2 u_2}{\partial p_1 \partial p_2} & \frac{\partial^2 u_2}{\partial p_2^2} & \cdots & \frac{\partial^2 u_2}{\partial p_F \partial p_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 u_F}{\partial p_1 \partial p_F} & \frac{\partial^2 u_F}{\partial p_2 \partial p_F} & \cdots & \frac{\partial^2 u_F}{\partial p_F^2} \end{pmatrix}. \tag{5.10}$$

If the symmetric matrix $\mathbf{G}(\mathbf{p}) + \mathbf{G}^T(\mathbf{p})$ is negative definite for all $\mathbf{p} \in \mathcal{P}$, then there exists a unique vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$ and a unique NE $\mathbf{p}^*$ which satisfy all the KKT conditions in (5.7)-(5.9) with $\lambda_m^f = \lambda_m$ for all $f \in \mathcal{F}$ and $m = 1, \dots, M$, i.e., the NNE is unique[4].

In order to study the uniqueness of an NNE, throughout this article, we assume that the utility set $U_f(\cdot)$ are twice differentiable and strictly concave. We analyze under which

[4]The condition defined in Proposition 5.1 is sufficient, but not necessary. In [67] a weaker sufficient condition is provided for the uniqueness of NNE. We do not consider that conditona since it is very difficult to verify in practice.

conditions the game $\mathcal{G}$ defined for the cognitive network with secondary-BSs as players admits a unique NNE.

Initially, we focus on the case when the interference from adjacent secondary-BSs is negligible at all the secondary-UTs, i.e. $u_f(\mathbf{p}) = U_f(\gamma'_f) = U_f\left(\frac{p_f h_f}{\sigma^2}\right)$. Then, $\mathbf{G}(\mathbf{p})$ is *diagonal* and given by

$$\mathbf{G}(\mathbf{p}) \quad = \quad \mathrm{diag}\left(\frac{h_1^2}{\sigma^4}U_1''\left(\frac{p_1 h_1}{\sigma^2}\right), \frac{h_2^2}{\sigma^4}U_2''\left(\frac{p_2 h_2}{\sigma^2}\right), \ldots \frac{h_F^2}{\sigma^4}U_F''\left(\frac{p_F h_F}{\sigma^2}\right)\right). \quad (5.11)$$

Thanks to the assumption of strict concavity of $U_f(\cdot)$, for $f \in \mathcal{F}$ and for every $\mathbf{p} \in \mathcal{P}$, all the diagonal elements of the matrix $\mathbf{G}(\mathbf{p})$ are strictly negative and according to Proposition 5.1, *NNE is unique.*

Then, we consider the case where the interference from adjacent secondary-BSs is not negligible. In this case,

$$\frac{\partial^2 U_f}{\partial p_f^2} = \kappa_f U_f''(\gamma_f) h_f \quad (5.12)$$

$$\frac{\partial^2 U_f}{\partial p_\ell \partial p_f} = -\kappa_f \widetilde{h}_\ell^f \left(\gamma_f U_f''(\gamma_f) + U_f'(\gamma_f)\right) \quad (5.13)$$

with $\kappa_f = \dfrac{h_f}{(\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f)^2}$. Then, the general expression of matrix $\mathbf{G}(\mathbf{p})$ is presented in (5.14) and the properties of the matrix $\mathbf{G}(\mathbf{p})$ strictly depend on the selected functions $U_f(\cdot)$, with $f \in \mathcal{F}$, and on the realizations of the channels $\mathbf{h}$ and $\widetilde{\mathbf{h}}^f$, $f \in \mathcal{F}$. In general, it is not clear if the matrix $\mathbf{G}(\mathbf{p}) + \mathbf{G}^T(\mathbf{p})$ is negative definite for every $\mathbf{p} \in \mathcal{P}$ and thus, the *uniqueness of the NNE is not guaranteed.* Even for given functions $U_f(\cdot)$, $f \in \mathcal{F}$, and channel $\mathbf{h}$ and $\widetilde{\mathbf{h}}^f$, it is not clear if the condition of Proposition 5.1 is satisfied. Nevertheless, in the next section we provide a class of utility functions which are of practical interest and possess unique NNEs when the interference at a secondary-UT from adjacent secondary-BSs is not negligible.

5.4 Weakly normalized NE (WNNE)

First of all, we state a proposition whose proof is in Appendix.

Proposition 5.2. *Let* $\mathcal{G} \equiv \{\mathcal{F}, \mathcal{P}, \{U_f(\gamma_f)\}_{f\in\mathcal{F}}\}$ *and* $\mathcal{G}' \equiv \{\mathcal{F}, \mathcal{P}, \{V_f(\gamma_f)\}_{f\in\mathcal{F}}\}$ *be two games of the kind defined in (5.5) with identical player and strategy sets and different utility sets. Let the functions* V_f *and* U_f, $f \in \mathcal{F}$, *be strictly increasing functions. Then, if* $\mathbf{p}^*$ *is a NE of game* $\mathcal{G}'$, *then it is also a NE for game* $\mathcal{G}$.

By using the above proposition, we formally define the *weakly normalized NE (WNNE) of game* $\mathcal{G}$ *induced by the utility set* $\mathcal{V} = \{V_f(\gamma_f)\}_{f\in\mathcal{F}}$.

Definition 5.1. Let game $\mathcal{G}'$ with utility set $\mathcal{V} \equiv \{V_f(\gamma_f)|f \in \mathcal{F}\}$ and strictly increasing $V_f(\cdot)$ have an NNE $\overline{\mathbf{p}}$. Then, $\overline{\mathbf{p}}$ is also a NE of the game $\mathcal{G}$ with utility set $\mathcal{U} \equiv \{U_f(\gamma_f)|f \in \mathcal{F}\}$ and strictly increasing $U_f(\cdot)$. This NE $\overline{\mathbf{p}}$ is be denoted as the *Weakly Normalized Nash equilibrium of* $\mathcal{G}$ *induced by the utility set* $\mathcal{V}$.

Note that the WNNE depends on the set of utility functions $\mathcal{V}$. If the game $\mathcal{G}'$ with the specified set of utility functions $\mathcal{V}$ admits a unique NNE, then there is a unique WNNE of the game $\mathcal{G}$ induced by the utility set $\mathcal{V}$.

For some game $\mathcal{G}$ the NNE may not be unique and the computation of an NNE can be highly costly in terms of the computational complexity, while it can be possible to identify a unique WNNE induced on game $\mathcal{G}$ by a different set of utility functions with lower complexity. Thus, we can obtain a NE of the game that retains some of the appealing properties of an NNE. We enlighten the benefit in detail in Sections 5.5.4 and 5.6.

In cognitive radio networks with non negligible inter-secondary-network interference, the concept of WNNE can be illustrated by selecting the functions $V_f(x) = \log(x)$ and defining the utility set

$$\mathcal{V} = \left\{ v_f(\mathbf{p}) = \log(\gamma) = \log\left(\frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f} \right) \right\}. \tag{5.15}$$

Note that it is worthwhile to consider such a utility function thanks to the following features:

- $\sum_{f \in \mathcal{F}} \log(\gamma_f)$ is the utility function underlying a proportionally fair SINR allocation.
- When SINR $\gg 1$, then the maximum achievable rate of each secondary-UT, shortly referred as Shannon capacity, $\log(1 + \text{SINR})$ can be approximated by $\log(\text{SINR})$, i.e. $\log(1 + \text{SINR}) \approx \log(\text{SINR})$.
- For certain applications (e.g. voice transmission) the utility functions increase with SINR in a logarithmic manner.

By observing that

$$x \frac{d^2 \log(x)}{dx^2} + \frac{d \log(x)}{dx} = 0$$

the matrix $\mathbf{G}(\mathbf{p})$ in (5.14) reduces to a diagonal matrix with strictly negative diagonal elements for every $\mathbf{p} \in \mathcal{P}$ when the utility set $\mathcal{V}$ is adopted. Thus, by Proposition 5.1, the NNE $\overline{\mathbf{p}}$ *is unique* for the above game. We can adopt $\overline{\mathbf{p}}$ as a unique WNNE induced by the set $\mathcal{V}$ to any game $\mathcal{G}$ of the kind defined in (5.5) with strictly increasing utility functions $U_f(\cdot)$, $f \in \mathcal{F}$. For example, we can consider the set $\mathcal{V} = \{V_f(\gamma_f(\mathbf{p})) = \log(1 + \gamma_f)\}$ i.e. the set of Shannon Capacity functions.

5.5 Computing a Normalized Nash Equilibrium

In this section, we present the concept of coupled constrained potential games. Subsequently, we show that there are games of the type introduced in Section 5.2-where finding an NNE is equivalent to solving a concave potential game. We propose a distributed algorithm which converges to the unique NNE of a strictly concave potential game in Section 5.5.2. Subsequently, we identify a class of utility functions for which the distributed algorithm can be applied to attain the unique NNE in Sections 5.5.3 and 5.5.4. Finally, in Section 5.5.4, we discuss the significance of obtaining an NNE as it induces a WNNE in a broad class of games.

5.5.1 Constrained Concave Potential Games

Constrained potential games have been discussed in [92] and [71]. They found application to rate and power allocation in multiple access channels (MAC). To the best of our knowledge our work is the first one to provide a relationship between potential games and NNE in a cognitive network. In contrast to [92] and [71], we consider here an interference channel which presents more challenging issues than a MAC. For example, the Shannon capacity in a MAC setting admits a potential function and this property has been widely exploited in literature. However, the same utility does not admit a potential function when there is interference at a secondary-UT from other secondary-BSs in the interference channel. Additionally, in contrast to previous works, we also identify a broad class of utility functions which admit concave potential games. Finally, we propose a distributed algorithm by leveraging on the concave potential game. This distributed algorithm enables us to implement NNE or WNNE in a distributed fashion in cognitive radio networks which were not considered in [92] and [71].

We first introduce the following definitions which we use throughout.

Definition 5.2. [92] A non-cooperative game $\mathcal{G}$ with utility set $\{u_f(\mathbf{p})|f \in \mathcal{F}\}$ is an exact potential game[5] if there exists a function $\Phi(\mathbf{p})$ such that for all $f \in \mathcal{F}$ and $(p_f, \mathbf{p}_{-f})$, $(p'_f, \mathbf{p}_{-f}) \in \mathcal{P}$:

$$\Phi(p_f, \mathbf{p}_{-f}) - \Phi(p'_f, \mathbf{p}_{-f}) = u_f(p_f, \mathbf{p}_{-f}) - u_f(p'_f, \mathbf{p}_{-f}).$$

Definition 5.3. [92] A potential game is called a concave potential game if the potential function $\Phi(\mathbf{p})$ is concave in $\mathbf{p} \in \mathcal{P}$. If $\Phi(\mathbf{p})$ is strictly concave, it is called *strictly concave potential game.*

Remark 5.1. [61] For a differentiable utility function $u_f(\cdot)$, $\Phi(\cdot)$ is a potential function of the game if and only if (iff)

$$\frac{\partial u_f}{\partial p_f} = \frac{\partial \Phi}{\partial p_f} \qquad \forall f \in \mathcal{F}. \tag{5.16}$$

The utility of introducing the concave potential game is shown in the following proposition.

Proposition 5.3. *Suppose there exists a potential function $\Phi(\mathbf{p})$ of the game $\mathcal{G}$ defined in (5.5) as $\mathcal{G} = \{\mathcal{F}, \mathcal{P}, \{u_f(\mathbf{p})|f \in \mathcal{F}\}\}$. The solution of the following optimization problem, referred to as CCPG, is an NNE.*

$$\begin{array}{lll} \textit{CCPG} & \textit{maximize}_{\mathbf{p}} & \Phi(\mathbf{p}) \\ & \textit{subject to} & \mathbf{p} \in \mathcal{P} \end{array}$$

[5]Hitherto, we refer to an exact potential game as a potential game.

If the potential function $\Phi(\mathbf{p})$ is concave, then the optimal solution of the convex optimization problem CCPG is an NNE. Note that if an NNE is unique then the solution of CCPG is the unique NNE. If a coupled constraint concave game $\mathcal{G}$ admits a potential function, in general, not every NNE can be expressed as a solution of CCPG. However, if the potential function is concave, then, each NNE is a solution of the CCPG optimization problem since the KKT conditions are not only necessary but also sufficient for optimality for concave potential games.

5.5.2 A Distributed Algorithm to Determine an NNE

In Section 5.5.1, we showed that, when a coupled constrained concave game $\mathcal{G}$ has a unique NNE and admits a potential function, we can solve the potential game in order to achieve the unique NNE. When the potential function $\Phi(\cdot)$ is strictly concave and $\dfrac{\partial U_f}{\partial p_f}$ only depends on the local information measurable at secondary-BS f, i.e., $p_f, h_f, \sigma^2\, \forall f \in \mathcal{F}$, then, there exists a distributed algorithm which converges to the unique optimal solution $\mathbf{p}^*$ and the dual optimal solution $\boldsymbol{\lambda}^*$. The distributed algorithm is described in the following.

Algorithm DIST

Initially primary-UT m selects $\boldsymbol{\lambda}^0 \in \mathbb{R}_+^M \setminus \{\mathbf{0}\}$ randomly[6].

At iteration $k+1 = 1, 2, \ldots$, the following tasks are performed:

1. Each secondary-BS f sets

$$p_f^{k+1} = \text{argmax}_{p_f \geq 0} \Phi - p_f \boldsymbol{\lambda}^{k\,T} \hat{\mathbf{h}}^f \tag{5.17}$$

[6] $\boldsymbol{\lambda}^0$ is initialized arbitrarily.

Then, all the secondary-BSs transmit with updated power level

$\mathbf{p}^{k+1} = (p_1^{k+1}, p_2^{k+1}, \dots p_F^{k+1})$.

2. Primary-UT m sets

$$\lambda_m^{k+1} = (\lambda_m^k + \delta(\widehat{\mathbf{h}}_m^T \mathbf{p}_f^{k+1} - I_T))^+ \tag{5.18}$$

where $\delta > 0$ is a sufficiently small constant. Primary-UT m reports the updated cost λ_m^{k+1} to all the secondary-BSs.

Since computing an NNE is equivalent to solving the convex optimization problem CCPG by Proposition 5.3, thus, the convergence of Algorithm DIST follows immediately from known results in [9] and it is stated in the following:

Proposition 5.4. *Algorithm DIST converges to the unique optimal primal solution* $\mathbf{p}^*$ *and dual solution* $\boldsymbol{\lambda}^*$ *when* $\Phi(\cdot)$ *is strictly concave in* $\mathbf{p}$*.*

In algorithm DIST Secondary-BS f needs to find the optimal p_f. In order to find the optimal p_f, secondary-BS f needs to evaluate $\frac{\partial \Phi}{\partial p_f}$. Since in a potential game, $\frac{\partial \Phi}{\partial p_f} = \frac{\partial U_f}{\partial p_f}$, thus, a secondary-BS f does not need to know the utility functions of other secondary-BSs. Equivalently we can write step 1 as

$$p_f^{k+1} = \text{argmax}_{p_f \geq 0} U_f(\gamma_f(\mathbf{p})) - p_f \boldsymbol{\lambda}^{k\,T} \hat{\mathbf{h}}^f$$

In step 1, secondary-BS f needs to know $\widehat{\mathbf{h}}^f$. Note that $\widehat{\mathbf{h}}^f$ consists of the values of channel gains between secondary-BS f and all primary-UTs. As discussed in Section 5.1, Secondary-BS f can obtain those values locally through sensing of pilot signal sent from the primary-UTs assuming the channel reciprocity since the primary network operates in TDD mode. Thus, when $\frac{\partial U_f}{\partial p_f}$ only depends on p_f, h_f, σ^2, then secondary-BS f can update its power in step 1 without any costly feedback exchange with other secondary-BSs and secondary-UTs in algorithm DIST. Thus, even though the solution of potential game CCPG requires to know all the channel gains, a secondary-BS needs not know channel gains regarding other secondary-BSs and UTs in Algorithm DIST. Hence, a secondary-BS does not need to communicate with other secondary-BSs and UTs. In the following Sections 5.5.3 and 5.5.4, we show that for a wide class of utility functions concave potential games exist and $\frac{\partial U_f}{\partial p_f}$ only depends on p_f, h_f and σ^2, i.e., for a wide class of utility functions, algorithm DIST can be applied to obtain the unique NNE.

Note that primary-UTs need to broadcast the prices $\boldsymbol{\lambda}$. Thus, a cooperation from primary-UTs is required. However, we need very limited cooperation from primary-UTs. The costly channel feedback from primary-UTs to secondary-BSs is not required since secondary-BS f can acquire the vector $\widehat{\mathbf{h}}^f$ locally as discussed in Section 5.1. Each primary-UT only needs to track the total interference[7]. This tracking can be performed by using a known test signal sent by the primary-BS periodically. The primary-UT does not need to track interference from each secondary-BS. Hence, it does not need to communicate with each secondary-BS. Thus, a primary-UT is oblivious of the number of secondary-UTs, their utilities, the channel parameters and the transmitted power $\mathbf{p}$. Hence, the signaling and communication cost is greatly reduced. Primary-UT m is also compensated by the price λ_m for per unit of interference caused by the secondary-BSs. Thus, an incentive is also provided to primary-UTs for the minimal amount of cooperation

[7]It is reasonable to assume that the interference from secondary-BSs situated at far-off locations from a primary-BS is negligible in order to avoid communication overhead.

required in Algorithm DIST. Thus, the distributed algorithm DIST is readily scalable and implementable in practice.

5.5.3 Negligible Inter-Secondary Network Interference

In this section, we show that when the interference from other secondary-BSs at each secondary-UT is negligible , then the game is a strictly concave potential game.

In this setting, SINR reduces to the SNR $\gamma'_f(\mathbf{p}) = \dfrac{p_f h_f}{\sigma^2}, \forall f \in \mathcal{F}$. Next lemma shows that in *this scenario* every game $\mathcal{G}$ defined in (5.5) is a strictly concave potential game.

Lemma 5.1. *Let* $\Phi(\mathbf{p}) = \sum_f U_f(\gamma'_f(\mathbf{p}))$*. Then,* $\Phi(\mathbf{p})$ *is a potential function for game* $\mathcal{G}$ *defined in (5.5) with utility set* $\{U_f(\gamma'_f) | f \in \mathcal{F}\}$*. Moreover, if* $U_f(\cdot)$ *is strictly concave, then so* $\Phi(\cdot)$ *is.*

If we focus on strictly concave functions $U_f(\cdot)$, the NNE is unique as we have shown in Section 5.3. We can evaluate the unique NNE by solving the potential game CCPG by Proposition 5.3. Moreover, we can apply Algorithm DIST, which will converge to optimal $\mathbf{p}^*$ and dual variable $\boldsymbol{\lambda}^*$ since γ'_f is only a function of p_f, h_f and σ^2 and $U_f(\cdot)$ is a function of γ'_f in this setting. It is worth noting that *the NNE in this setting also maximizes the sum of the utility functions of players under the set of constraints* $\mathcal{P}$ *since* $\Phi = \sum_{f=1}^{F} U_f$.

5.5.4 Presence of Inter-Secondary Network Interference

When the interference from adjacent secondary-BSs is not negligible at all the secondary-UTs, then we have already seen that an NNE may be not unique. However, in Section 5.4 we have identified a utility set $\mathcal{V}$ defining a game $\mathcal{G}$, whose NNE is unique and can be used to define a unique WNNE for games with different utility sets. In this section, we show that the game with utility set $\mathcal{V}$ defined in (5.15) admits a potential function.

Let us consider again the utility set $\mathcal{V}$ defined in (5.15). Then, $V_f(\mathbf{p})$ is a strictly increasing concave function in p_f. The following lemma shows that the game $\mathcal{G}$ with the utility set $\mathcal{V}$ defined in (5.15) is a potential game.

Lemma 5.2. *Let $\Phi(\mathbf{p}) = \sum_f \log(p_f)$. $\Phi(\mathbf{p})$ is a potential function for the game $\mathcal{G}$ defined in (5.5) and the utility set (5.15). Moreover, the potential game is strictly concave.*

Hence, the solution of CCPG will provide the unique NNE to game $\mathcal{G}$ defined in (5.5) with the utility set (5.15). Note that though U_f depends on $\mathbf{p}_{-f}$ and $\widetilde{\mathbf{h}}^f$, $\frac{\partial \Phi}{\partial p_f}$ or $\frac{\partial U_f}{\partial p_f}$ only depends on p_f, h_f and σ^2, thus, we can use Algorithm DIST described in Section 5.5.2 to obtain the unique NNE for a game $\mathcal{G}$ with utility set (5.15). Note that the NNE may not maximize the sum of the utility functions of players unlike the NNEs of the games discussed in Section 5.5.3.

Now, we show that, in general, a potential game does not exist in this setting by using the following result.

Proposition 5.5. *[61] For twice continuously differentiable utility functions $u_f, u_\ell, f, \ell \in \mathcal{F}$, there exists a potential function iff $\frac{\partial^2 u_f}{\partial p_\ell \partial p_f}(\mathbf{p}) = \frac{\partial^2 u_\ell}{\partial p_f \partial p_\ell}(\mathbf{p})$, $\forall \mathbf{p} \in \mathcal{P}$.*

It is easy to verify from (5.13) that in general the utilities $U_f(\gamma_f)$ do not satisfy the conditions stated in Proposition 5.5. Thus, in general a potential function does not exist. There exist large classes of function $U_f(\gamma_f(\mathbf{p}))$ which are strictly concave functions of p_f but still they do not admit a potential function. One such examples is $U_f(\gamma_f) = \log(1+\gamma_f)$.

For all these cases when the functions $U_f(\cdot)$ are strictly increasing, it is convenient to invoke to Proposition 5.2 and resort to the concept of WNNE with respect to the utility set $\mathcal{V}$ in (5.15). We have already shown in Lemma 5.2 that the game $\mathcal{G}$ with utility set $\mathcal{V}$ in (5.15) is a strictly concave potential game and we can attain the unique NNE using Algorithm DIST proposed in Section 5.5.2. Hence, we can easily obtain the WNNE for any game $\mathcal{G}$ including the one where $U_f(\gamma_f) = \log(1+\gamma_f)$ even though this latter game is not a potential game.

5.6 A Relevant Case

In this section we focus on the Shannon capacity as utility, i.e. $U_f(\gamma_f) = B\log(1+\gamma_f)$, B being the channel frequency bandwidth. First, we determine conditions which are sufficient to conclude about uniqueness of an NNE. In Section 5.5.4 we saw that there is no potential function for this game. Thus, we cannot utilize the results regarding potential games in order to obtain the NNE. Nevertheless, we provide an algorithm which returns the unique NNE for a system with a single primary-UT when the uniqueness condition is satisfied and under some technical conditions detailed throughout this section.

Let us start introducing a first assumption that guarantees the uniqueness of the NNE.

Assumption 5.1. *The following matrix is row wise and column wise diagonally dominant*

$$\bar{H} = \begin{bmatrix} k_1 h_1 & k_1 \tilde{h}_2^1 & \dots & k_1 \tilde{h}_F^1 \\ \dots & \dots & \dots & \dots \\ k_F \tilde{h}_1^F & k_F \tilde{h}_2^F & \dots & k_F h_F \end{bmatrix}$$

for every $k_f, f \in \mathcal{F}$ *such that* $\dfrac{Bh_f}{(\bar{p}_f h_f + \bar{\mathbf{p}}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2)^2} \leq k_f \leq \dfrac{Bh_f}{\sigma^4}$ *where* $\bar{p}_f = \min_m \dfrac{I_T}{\hat{h}_m^f}, f \in \mathcal{F}$.

Remark 5.2. Note that the following conditions are sufficient to conclude that the matrix $\bar{H}$ is row wise and column wise diagonally dominant:

$$h_f > \mathbf{1}_{F-1}^T \widetilde{\mathbf{h}}^f \tag{5.19}$$

$$\frac{Bh_f^2}{(\bar{p}_f h_f + \bar{\mathbf{p}}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2)^2} > \sum_{\ell \neq f} \frac{Bh_\ell \widetilde{h}_f^\ell}{\sigma^4}. \tag{5.20}$$

Though the above conditions are technical but the above conditions are satisfied for sufficiently small $\widetilde{h}_f^\ell$ and $\widetilde{h}_\ell^f, \ell \neq f$ compared to h_f.

The following lemma shows that if Assumption 5.1 holds, then the NNE is unique.

Lemma 5.3. *If Assumption 5.1 holds, then there exists a unique NNE .*

Henceforth, we assume that Assumption 5.1 is satisfied and we examine how to compute the unique NNE. Since there is a unique NNE $\mathbf{p}^* = (p_1, \ldots, p_F) \in \mathcal{P}$, thus there is a unique $\boldsymbol{\lambda} \geq 0$ satisfying the system of equations (5.9) and the complementary slackness equations

$$\lambda_m(\mathbf{p}^{*T} \widehat{\mathbf{h}}_m - I_T) = 0 \quad \forall m = 1, \ldots, M. \tag{5.21}$$

By simple algebraic calculation we obtain the optimal p_f^* for a given $\boldsymbol{\lambda}$:

$$p_f^* = \frac{1}{h_f} \left(\frac{Bh_f}{\boldsymbol{\lambda}^T \widehat{\mathbf{h}}^f} - \sigma^2 - \mathbf{p}_{-f}^T \widetilde{\mathbf{h}^f} \right)^+. \tag{5.22}$$

The optimal strategy of a secondary-BS depends on the strategies adopted by the other secondary-BSs and $\boldsymbol{\lambda}$. It is not known a priori which p_f^* is zero and which constraints are active, i.e. which components of $\boldsymbol{\lambda}$ are strictly positive. The solution requires us to consider all the possible combinations. Thus, it is computationally demanding to compute the unique NNE. The complexity of the problem reduces to some extent when we focus on the scenario with one single primary-UT, i.e., $M = 1$. Thus, we consider such a scenario. Since there is only one constraint *with a slight abuse of notation* we shortly denote $\hat{h}_1^f$ as $\hat{h}^f$ $\forall f \in \mathcal{F}$ and $\widehat{\mathbf{h}}_1$ as $\widehat{\mathbf{h}}$.

The following observation shows that the single constraint must be active at the NNE in this setting.

Observation 5.1. At NNE $\mathbf{p}$, we must have $\mathbf{p}^T\widehat{\mathbf{h}} = I_T$.

Since there is a single constraint, thus (5.22) reduces to

$$p_f^* = \frac{1}{h_f}\left(\frac{Bh_f}{\lambda\hat{h}^f} - \mathbf{p}_{-f}^{*T}\widetilde{\mathbf{h}}^f - \sigma^2\right)^+ \tag{5.23}$$

From Observation 5.1 we also must have

$$\mathbf{p}^T\widehat{\mathbf{h}} = I_T. \tag{5.24}$$

Then, the NNE $\mathbf{p}^*$ and the corresponding λ have to satisfy (5.23) and (5.24).

In the following, we provide an algorithm yielding $\mathbf{p}^*$ for a given λ. Let

$$\mathbf{H} = \begin{bmatrix} h_1 & \tilde{h}_2^1 & \ldots & \tilde{h}_F^1 \\ . & . & \ldots & . \\ \tilde{h}_1^F & \tilde{h}_2^F & \ldots & h_F \end{bmatrix}$$

and

$$\overline{\mathbf{h}} = \begin{pmatrix} \frac{Bh_1}{\hat{h}^1} & \ldots & \frac{Bh_F}{\hat{h}^F} \end{pmatrix}^T$$

We assume that

Assumption 5.2. *The matrix* $\mathbf{H}$ *is row wise and column wise strictly diagonally dominant.*

Remark 5.3. Since $\mathbf{H}$ is row wise and column wise strictly diagonally dominant, thus the square matrix $\mathbf{H}_K$ is invertible for any index subset K of $\mathcal{F}$.

Under Assumptions 5.1 and 5.2, we can state the following algorithm.

Algorithm OPTI:

Given $\lambda > 0$, we execute the following steps:

1. Initialize K^0 to the empty set, i.e. set $K^0 = \emptyset$. For each index $i \in \mathcal{F}$, if $\frac{Bh_i}{\lambda \hat{h}^i} \leq \sigma^2$ set $p_i^* = 0$ otherwise assign $K^0 \leftarrow K^0 \bigcup \{i\}$.

2. If K^0 is empty exit; otherwise go to the next step.

3. Let $I = 0$.

4. Solve the system of equations[8]

$$\mathbf{H}_{K^I} \mathbf{p}_{K^I} = \frac{1}{\lambda} \bar{\mathbf{h}}_{K^I} - \sigma^2 \mathbf{1}_{|K^I|} \tag{5.25}$$

   and assign the solution to $\mathbf{p}^0_{K^I}$.

5. If $\mathbf{p}^0_{K^I} \succeq \mathbf{0}$, then set $\mathbf{p}^*_{K^I} = \mathbf{p}^0_{K^I}$ and exit. Otherwise go to the next step.

6. Assign $I \leftarrow I + 1$ and set $K^I = \emptyset$. For each index $i \in K^{I-1}$, if $p_i^0 < 0$ set $p_i^* = 0$ otherwise assign $K^I \leftarrow K^I \bigcup \{i\}$ and go to step 4.

First note that if K^0 is empty, then $\mathbf{p}_i^* = \mathbf{0}_F$ is the only possible solution. Additionally, the algorithm stops at most after F iterations. Thus, the algorithm scales *linearly* with the number of secondary-BSs. Finally, if $i \notin K^I$ at some iteration I, then $p_i^* = 0$.

Algorithm OPTI converges to the desired $\mathbf{p}^*$ under the following assumptions and this property is stated in Proposition 5.6.

[8]Since $\mathbf{H}_{K^0}$ is invertible, the solution is linear and the solution unique.

Assumption 5.3. *Fix any index k. For any $i = 1, \ldots, k-1, k+1, \ldots, F$, we have*

$$\sum_{j \neq k,i} \frac{\tilde{h}_j^k \tilde{h}_i^j}{h_j} \leq \tilde{h}_i^k$$

and

$$\sum_{j \neq k} \frac{\tilde{h}_j^k \tilde{h}_k^j}{h_k h_j} \leq 1$$

Proposition 5.6. *For a given $\lambda > 0$, Algorithm OPTI converges to $\mathbf{p}^*$, the solution to the system of equations (5.23) for every $f \in \mathcal{F}$.*

Now, we present Algorithm DIST-INT to update λ in a suitable way to attain the NNE $\overline{\mathbf{p}}$ and the corresponding λ. In the following, we denote by $\mathbf{p}^*(\lambda)$ the result of Algorithm OPTI for a given λ.

Algorithm DIST-INT:

1. Set the accuracy ϵ to a desired positive value. Initialize λ^0 to positive value[9] and set $J = 0$.

2. Apply Algorithm OPTI to determine $\mathbf{p}^*(\lambda^J)$.

3. Assign $J \to J + 1$ and set

$$\lambda^J = \left(\lambda^{(J-1)} + \delta_J(\widehat{\mathbf{h}}^T \mathbf{p}^*(\lambda^{J-1}) - I_T)\right)^+$$

   where $\delta_J = 1/(J+1)$ is the stepsize.

4. If $|\widehat{\mathbf{h}}^T \mathbf{p}^*(\lambda^{J-1}) - I_T| \leq \epsilon$, then exit, otherwise go to step 2.

[9] In the case it is necessary to avoid that the constraints on the communications are violated in during the transient of the algorithm, then λ^0 has to be initialized to a sufficiently high value.

The primary-UT only needs to track the total interference like in Algorithm DIST. The primary-UT does not need to track individual interference from a secondary-BS. Hence, computation and signaling costs between primary-UTs and secondary-BSs are greatly reduced. Also note that primary-UT is compensated by λ which gives the primary-UTs incentives to cooperate during the convergence of Algorithm DIST-INT.

Note that p_i^* is a continuous decreasing function of λ if the interference from other players is small. From (5.23) it is clear that for a sufficiently small λ, at least one p_i^* will be sufficiently large such that $p_i^* \hat{h}^i > I_T$. On the other hand, for sufficiently large λ, for all $i \in \mathcal{F}$ $p_i^* = 0$ and thus $\widehat{\mathbf{h}}^T \mathbf{p}^*(\lambda) < I_T$. Thus, by the intermediate theorem of continuity, there exists surely a $\lambda > 0$ such that $\widehat{\mathbf{h}}^T \mathbf{p}^*(\lambda) = I_T$. Hence, we obtain the following result.

Proposition 5.7. *If Assumption 5.1 holds, Algorithm DIST-INT converges to an optimal λ, which in turn yields an NNE $\overline{\mathbf{p}}$ by Algorithm OPTI.*

Discussion: Algorithm DIST-INT converges to the unique NNE with a single primary-UT and when Assumptions 5.1-5.3 are satisfied. Hence, if Assumptions 1-3 are not satisfied, we have to leverage on the concept of WNNE as explained in Section 5.4.

Additionally, Algorithm OPTI can be implemented if one of the following occurs –i) each secondary-BS knows the channel parameters $\mathbf{H}$, $\overline{\mathbf{h}}$, and $\widetilde{\mathbf{h}}^f$, ii) a central controller has the above information and coordinates among secondary-BSs, iii) using Gauss-Siedel method[10] Algorithm OPTI can be implemented in a distributed manner at each

[10]In the Gauss Siedel algorithm, secodnary-BS f basically solves the equation for each f in the linear equation (5.25) assuming that $\mathbf{p}_{-f}$ is fixed. Then, secondary-BS f transmits with p_f, and other secondary-BSs then update the power vector $\mathbf{p}_{-f}$. Secondary-BS f then again updates p_f until no more update is

secondary-BS. However, in all these approaches signalling and communication costs are greatly increased as secondary-BSs need to communicate with each other. On the other hand, WNNE can be implemented using Algorithm DIST where secondary-BSs do not need to exchange information among themselves. In contrast to the Gauss-Siedel method, secondary-BSs do not need to repeatedly update their powers to solve the linear equation (5.25), in Algorithm DIST. Hence, the convergence of the Algorithm DIST is also fast. Hence, we can leverage on WNNE if we want to reduce the signaling and communication cost or the time for finding an equilibrium solution and when Assumptions 5.1-5.3 are not satisfied.

5.7 Numerical Results

We numerically evaluate the characteristics of an NNE strategy profile for several utility functions. We consider two scenarios: i) The interference at each secondary-UT from adjacent secondary-BSs is negligible ($\gamma_i \approx \mathrm{SNR}_i$), ii) The interference is not negligible ($\gamma_i = \mathrm{SINR}_i$).

To generate $\hat{\mathbf{h}}^i_m, \tilde{\mathbf{h}}^i_f$, we first randomly place secondary-BSs and primary-UTs in a disc of radius r_1. Then, we randomly place a secondary-UT in a disc of radius r_2 around each secondary-BS as in Fig. 5.1. We take $r_1 > r_2$ because in practice, a secondary-UT is in a close vicinity of its secondary-BS compared to the range of primary-BS as a secondary-BS has smaller coverage area compared to a primary-BS. Since we consider that

possible. The above algorithm converges to a solution of the linear equation (5.25). However, it needs a longer time to converge and a secondary-BS needs to transmit its power to other secondary-BSs during the convergence process of the algorithm OPTI.

primary-UTs and secondary-BSs are placed randomly, our numerical analysis also includes the possible setting where secondary-BSs and primary-UTs may be close to each other. We compute the channel gain between two nodes according to the formula: $Kd^{-\beta}$ where K is a parameter which depends on the frequency, the antenna gains of the transmitter and the receiver d is the distance between two entities and β is the path loss exponent. Similar simulation setup has been considered in several papers including [86, 78, 52]. For all simulations we take $r_1 = 20$, $K = 1$, $\beta = 2$ and $\sigma^2 = 1$. We consider $r_2 = 2$ for all simulations except for simulation in Fig. 5.9. Throughout this section, we use the following notations:

$$U_{\text{OPT}} = \max_{\mathbf{p} \in \mathcal{P}} \sum_{f \in \mathcal{F}} U_f(\gamma_f(\mathbf{p})) \tag{5.26}$$

$$U_{\text{NNE}} = \sum_{f \in \mathcal{F}} U_f(\gamma_f(\mathbf{p}_{NNE})) \tag{5.27}$$

where $\mathbf{p}_{NNE}$ is the NNE strategy profile.

5.7.1 Negligible Interference at secondary-UT from adjacent secondary-BSs

In this setting $\gamma_i = \gamma_i' = \dfrac{p_i h_i}{\sigma^2}$.

We consider the following utility functions

1. *Shannon Capacity*: Here $U_f(\gamma_f') = B \log(1 + \gamma_f')$.

2. *Bit Error Rate*: From[17] we can approximate bit error rate (BER) for K-QAM modulation at secondary-BS f as follows:

$$\text{BER} = 0.2 \exp\left(-\frac{3\gamma_f'}{2(K-1)}\right). \tag{5.28}$$

Since each secondary-BS wants to minimize the BER, we define the utility function as

$$U_f(\gamma'_f) = -0.2 \exp\left(-\frac{3\gamma'_f}{2(K-1)}\right). \tag{5.29}$$

The above utility function is strictly concave in γ'_f and thus, the NNE is unique.

We set $I_T = 5dB$, $B = 1MHz$ and assume a 4-QAM modulation for all simulations in this subsection. Recall from Section 5.5.3 that $\mathbf{p}_{NNE}$ is also the optimal solution for U_{OPT} i.e. $U_{OPT} = U_{NNE}$.

First, we study the variation of maximum achievable total utility U_{OPT} with the number of primary-UTs when the Shanon capacity is the utility function. Fig. 5.3 shows that, as the number of primary-UTs increases, U_{OPT} and the individual utilities decrease. Intuitively, when the number of primary-UTs increases the strategy set $\mathcal{P}$ reduces. The decrement of U_{OPT} becomes small as the number of primary-UTs increases. Thus, an increase in the number of primary-UTs does not affect the utility significantly when it exceeds a certain threshold.

In Fig. 5.4 we adopt as utility function the BER in (5.28). Fig. 5.4 reveals that the mean BER and each secondary-BS's BER increase as the number of primary-UTs increases since each secondary-BS transmits with lower power as the number of primary-UTs increases. The rate of its increment slows down as the number of primary-UTs increases. Intuitively, as the number of primary-UTs increases, the strategy set $\mathcal{P}$ remains almost identical. Thus, the power remains almost the same even when the number of primary-UTs increases.

Fig.5.5 shows the convergence of Algorithm DIST for systems with different number of secondary-BSs, $F = 3, 5$, and 8. Numerical computations reveal that the convergence rate is higher for smaller number of secondary-BSs. Fig. 5.6 shows the convergence of Algorithm DIST for systems with different number of primary-UTs. Numerical analysis reveals that the convergence rate increases as the number of primary-UTs increase since it decreases the strategy space $\mathcal{P}$.

5.7.2 Non-Negligible Interference at Secondary-UTs from Adjacent Secondary-BSs

When we adopt the following utility $U_f(\gamma_f) = \log(\gamma_f), f \in \mathcal{F}$, then it is easy to verify that the maximization of $\sum_i U_i(\gamma_i)$ for $\mathbf{p} \in \mathcal{P}$ is a geometric programming [10]. Hence, we can employ standard optimization tools to compute U_{OPT}.

Fig.5.7 reveals that as the number of secondary-BSs increases the difference between U_{OPT} and U_{NNE} increases. Intuitively, when the number of secondary-BSs is small, then the interference at a secondary-UT is not significant and U_{NNE} closely matches U_{OPT}. However, as the number of secondary-BSs increases, U_{NNE} decreases and the difference between U_{NNE} and U_{OPT} increases. Note that U_{OPT} also decreases with the number of secondary-BSs. Intuitively, as $U_f(\gamma_f) = \log(\gamma_f)$, it must be $p_f > 0$ for any $f \in \mathcal{F}$[11]. Thus, as the number of secondary-BSs increases the interference at a secondary-UT becomes significant as each additional secondary-BS transmits with nonzero power. Thus, U_{OPT} decreases with the number of secondary-BSs as the interference at a secondary-UT from other secondary-BSs becomes significant.

Shannon capacity: We also numerically evaluate the characteristics of the NNE strat-

[11] Otherwise, U_{OPT} would be negative infinity.

egy profile when secondary-BS f's utility is $U_f(\gamma_f) = B\log(1+\gamma_f)$.

We obtain the NNE using the algorithms OPTI and DIST-INT. We consider $B = 1$, $\delta = 0.01$, $\lambda^0 = 5$ and $\epsilon = 5\times 10^{-4}$. Assumptions 5.1-5.3 are satisfied for most of the randomly generated instances of the system for $F \leq 8$. Let U_{WNNE} denote the total utility at a WNNE $\mathbf{p}_{WNNE}$, i.e., $\mathbf{p}_{WNNE}$ is the NNE for a game with utility function $U_f(\gamma_f) = \log(\gamma_f)$. Fig.5.8 reveals that U_{WNNE} closely matches U_{NNE} when the number of secondary-BSs is very small. But as the number of secondary-BSs increases the difference between U_{NNE} and U_{WNNE} increases since the interference at a secondary-UT from other secondary-BSs increases. Fig.5.8 also shows that the fluctuation of the utilities across players is higher for power allocations based on the NNE. Intuitively, $\sum_f \log(\gamma_f)$ induces a proportional fair SINR allocation, thus the utilities at the WNNE vary on a smaller range compared to the utilities corresponding to the power allocation based on an NNE.

Large r_2 and large F: Finally, Fig. 5.9 shows the U_{WNNE} as the number of secondary-BSs increase. We still consider the Shannon capacity function as the utility function. We consider $r_2 = 8$, hence, it models the setting where the coverage area of secondary-BS is large. We consider the number of primary-UTs (M) as 10. Note that, the NNE may not be unique in this setting since Assumption 5.1 may not be satisfied. Additionally, we cannot use Algorithm DIST-INT since $M > 1$. Instead, we can obtain WNNEs using Algorithm DIST. Fig. 5.9 reveals that the sum of the utilities (U_{WNNE}) increases initially with the number of secondary-BSs, but it decreases after it reaches a certain threshold. Intuitively, the characteristic of U_{WNNE} is similar to the characteristic observed in Fig. 5.7 since $\boldsymbol{p}_{\text{WNNE}}$ is an NNE when the utility function is $\log(\gamma_f)$.

5.8 Conclusion

We investigated the power allocation problem in cognitive radio networks using a game theoretic setting. Each secondary-BS selects a transmission power subject to the global constraint that the total interference should be below a given threshold at each primary-UT. This game falls into the category of the coupled constrained concave games. We adopted the NNE as an equilibrium concept since it caters to distributed settings. We showed that the NNE is unique when the interference at a secondary-UT from adjacent secondary-BSs is negligible. But an NNE may not be unique when the interference at a secondary-UT from adjacent secondary-BSs is not negligible. We identified a class of utility functions for which the NNE is unique in the latter setting. We also proposed a distributed algorithm which converges to the unique NNE for those utility functions. In the distributed algorithm, secondary-BSs do not exchange information among themselves and each primary-UT only needs to track the total interference. When it is computationally difficult to obtain an NNE or its uniqueness cannot be guaranteed, we leveraged on the concept of WNNE as an equilibrium concept which retains most of the properties of the NNE but it can be obtained with lower complexity compared to NNE. We showed the importance of WNNE when the utility function is Shannon Capacity function and there is inter secondary network interference.

WNNE is easy to compute, however, the performance of the WNNE can be poor compared to the NNE. The characterization of the difference between the WNNE and NNE, and the impact of the interference on the difference is a work for the future.

5.A Appendix

5.A.1 Proof of Proposition 5.2

Assume that $\mathbf{p}^*$ is a NE for $\mathcal{G}'$ but not for $\mathcal{G}$. Then, there exists a p_f such that $(p_f, \mathbf{p}^*_{-f}) \in \mathcal{P}$ and

$$U_f(\gamma_f(p_f, \mathbf{p}^*_{-f})) > U_f(\gamma_f(\mathbf{p}^*)).$$

Since U_f is increasing then $\gamma_f(p_f, \mathbf{p}^*_{-f}) > \gamma_f(\mathbf{p}^*)$. But also V_f is an increasing function in γ_f and

$$V_f(\gamma_f(p_f, \mathbf{p}^*_{-f})) > V_f(\gamma_f(\mathbf{p}^*)).$$

This contradicts the assumption that $\mathbf{p}^*$ is a NE for $\mathcal{G}'$. Thus, $\mathbf{p}^*$ is a NE for both $\mathcal{G}$ and $\mathcal{G}'$. ■

5.A.2 Proof of Proposition 5.3

Let $\mathbf{p}^*$ be an optimal solution to CCPG. First, note that $\mathbf{p}^*$ is an NE. If it was not, then there would exist a p'_f such that $(p'_f, \mathbf{p}_{-f}) \in \mathcal{P}, f \in \mathcal{F}$ such that

$$u_f(p^*_f, \mathbf{p}^*_{-f}) < u_f(p'_f, \mathbf{p}^*_{-f}). \tag{5.30}$$

Since $\Phi(\cdot)$ is a potential function, (5.30) implies that $\Phi(p^*_f, \mathbf{p}^*_{-f}) < \Phi(p'_f, \mathbf{p}^*_{-f})$. This contradicts the fact that $\mathbf{p}^*$ is an optimal solution.

Since $\mathbf{p}^*$ is an optimal solution, thus, according to the KKT conditions there exists a $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_M)^T \succeq \mathbf{0}_M$ such that

$$\frac{\partial \Phi}{\partial p_f} - \boldsymbol{\nu}\widehat{\mathbf{h}}^f = 0 \qquad \forall f \in \mathcal{F} \tag{5.31}$$

at $\mathbf{p} = \mathbf{p}^*$, with

$$\nu_m(\mathbf{p}^T\widehat{\mathbf{h}}_m - I_T) = 0 \quad \forall m. \tag{5.32}$$

Identifying $\boldsymbol{\lambda}$ in (5.7)–(5.9) with $\boldsymbol{\nu}$ we can easily discern that $\mathbf{p}^*$ is indeed an NNE. Hence, the result follows. □

5.A.3 Proof of Lemma 5.1

Since $U_f(\gamma'_f)$ does not depend on $\mathbf{p}_{-f}$, for any $f \in \mathcal{F}$

$$\Phi(p_f, \mathbf{p}_{-f}) - \Phi(p'_f, \mathbf{p}_{-f}) =$$
$$U_f(\gamma'_f(p_f, \mathbf{p}_{-f})) - U_f(\gamma'_f(p'_f, \mathbf{p}_{-f}))$$

for any $(p_f, \mathbf{p}_{-f}), (p'_f, \mathbf{p}_{-f}) \in \mathcal{P}$. This proves that $\Phi(\cdot)$ is a potential function.

By the definition of $\Phi(\cdot)$ and γ'_f, it is clear that if $U_f(\cdot)$ is strictly concave $\forall f \in \mathcal{F}$, then so $\Phi(\cdot)$ is. □

5.A.4 Proof of Lemma 5.2

Note that

$$\Phi(p_f, \mathbf{p}_{-f}) - \Phi(p'_f, p_{-f}) = \log(p_f) - \log(p'_f). \tag{5.33}$$

But,

$$\begin{aligned}&\log(\gamma_f(p_f, \mathbf{p}_{-f})) - \log(\gamma_f(p'_f, p_{-f})) = \\ &\log\left(\frac{p_f h_f}{\mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2}\right) - \log\left(\frac{p'_f h_f}{\mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2}\right) = \\ &\log(p_f) - \log(p'_f).\end{aligned} \tag{5.34}$$

Thus, comparing (5.33) and (5.34) we conclude that $\Phi(\mathbf{p})$ is a potential function. It is easy to verify that $\Phi(\mathbf{p})$ is a strictly concave function in $\mathbf{p}$. □

5.A.5 Proof of Lemma 5.3

Note that

$$\frac{\partial^2 U_f}{\partial p_f^2} = -\frac{B h_f^2}{(p_f h_f + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2)^2}$$
$$\frac{\partial^2 U_f}{\partial p_f \partial p_\ell} = -\frac{B h_f \widetilde{h}_f^\ell}{(p_f h_f + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2)^2}.$$

Additionally, the maximum of the denominator of $\frac{\partial^2 U_f}{\partial p_f^2}$ is obtained when $p_f, f \in \mathcal{F}$ attains its maximum value in $\mathcal{P}$. Note that for all feasible $p_f, f \in \mathcal{F}$ we must have

$$p_f \leq \min_m \frac{I_T}{\widehat{h}_m^f} = \bar{p}_f.$$

On the other hand, the minimum value[12] of the denominator of $\frac{\partial^2 U_f}{\partial p_f \partial p_\ell}$ and $\frac{\partial^2 U_f}{\partial^2 p_f}$ is σ^4. Since Assumption 5.1 is satisfied, thus, by identifying k_f with $\frac{B h_f}{(p_f h_f + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2)^2}$ we conclude that the matrix $-\mathbf{G}(\mathbf{p})$ defined in (5.10) is row-wise and column wise diagonally dominant for all $\mathbf{p} \in \mathcal{P}$. Hence, $-(\mathbf{G}(\mathbf{p}) + \mathbf{G}^T(\mathbf{p}))$ is positive definite $\forall \mathbf{p} \in \mathcal{P}$. Hence, the result follows from Proposition 5.1. □

5.A.6 Proof of Observation 5.1

Let us assume we have an NNE $\bar{\mathbf{p}} = (\bar{p}_1, \ldots, \bar{p}_F)$ such that $\bar{\mathbf{p}}^T \widehat{\mathbf{h}} < I_T$. Now, consider the following unilateral deviation of player f : It increases its power $\bar{p}_f$ to $\tilde{p}_f$ such that $\sum_{\ell \neq f} \bar{p}_\ell \hat{h}^\ell + \tilde{p}_f \hat{h}^f = I_T$, which is a feasible solution. Since $U_f(\gamma_f)$ is a strictly increasing

[12]The minimum value is obtained when $\mathbf{p} = \mathbf{0}_F$.

in p_f thus, secondary-BS f gets strictly higher utility at $\tilde{p}_f$ compared to $\bar{p}_f$. Thus, player f has an incentive to deviate unilaterally which, indeed, entails that $\bar{\mathbf{p}}$ cannot be an NE. Hence, the result follows. □

5.A.7 Proof of Proposition 5.6

We use the following result to prove proposition 5.6. Note that if $i \notin K^I$ for some I,then $p_i = 0$, thus from (5.25) we obtain for any index $k \in K^I$

$$p_k h_k + \mathbf{p}_{-k}^T \widetilde{\mathbf{h}}^k = \frac{Bh_k}{\lambda \hat{h}^k} - \sigma^2 \tag{5.35}$$

Now we are ready to prove the proposition.

If K^0 is empty, the optimal solution is $\mathbf{p}^* = 0$. Hence, the proposition is trivially true. Thus, we consider the case when K^0 is not empty.

Let $L + 1$ be the last iteration and thus $\mathbf{p}^{L+1} \geq 0$. Note that for p_k, $k \notin K^{L+1} = 0$. Thus, in order to conclude the optimality we have to check whether p_k, $k \notin K^{L+1}$ satisfies (5.22)i.e. $p_k, k \notin K^{L+1}$ must satisfy the following

$$\begin{aligned} \sum_{j \neq k} p_j^{L+1} \tilde{h}_j^k &\geq \frac{Bh_k}{\lambda \hat{h}^k} - \sigma^2 \\ \sum_{j \neq k} p_j^{L+1} \frac{\tilde{h}_j^k}{h_k} &\geq \frac{B}{\lambda \hat{h}^k} - \frac{\sigma^2}{h_k} \end{aligned} \tag{5.36}$$

First we show that for $k \in K^L, \notin K^{L+1}$, the above inequality holds. Later we show the above for ks which do not belong to K^L.

Note that since $k \in K^L$ and $p_i^L = 0, i \notin K^L$, thus we can write from (5.35)

$$p_k^L h_k + \mathbf{p}_{-k}^T \widetilde{\mathbf{h}}^k = \frac{Bh_k}{\lambda \hat{h}^k} - \sigma^2 \tag{5.37}$$

Since $k \in K^L$ and $k \notin K^{L+1}$ thus $p_k^L < 0, p_k^{L+1} = 0$.

Let $j \in K^{L+1}$. Thus, from (5.35)

$$p_j^{L+1} + \sum_{i \neq j} p_i^{L+1} \frac{\tilde{h}_i^j}{h_j} = \frac{B}{\lambda \hat{h}^j} - \frac{\sigma^2}{h_j} \tag{5.38}$$

Since $j \in K^{L+1}$, thus $j \in K^L$. Thus, from (5.35)

$$p_j^{L} + \sum_{i \neq j} p_i^{L} \frac{\tilde{h}_i^j}{h_j} = \frac{B}{\lambda \hat{h}^j} - \frac{\sigma^2}{h_j} \tag{5.39}$$

Thus, subtracting (5.39) from (5.38) we obtain

$$p_j^{L+1} + \sum_{i \neq j} p_i^{L+1} \frac{\tilde{h}_i^j}{h_j} - p_j^L - \sum_{i \neq j} p_i^{L} \frac{\tilde{h}_i^j}{h_j} = 0 \tag{5.40}$$

Note that the above expression is true for any $j \in K^{L+1}$. Let $P_1 = \{i : p_i^{L+1} \geq p_i^L\}$ and $P_2 = P_1^C$. Since $p_k^{L+1} > p_k^L$ (as $p_k^{L+1} = 0, p_k^L < 0$), thus P_1 is not empty. From (5.40) we obtain

$$p_j^{L+1} - p_j^L \geq \sum_{i \in P_1, \neq j} (p_i^L - p_i^{L+1}) \frac{\tilde{h}_i^j}{h_j} \tag{5.41}$$

If $i \notin K^{L+1}$, then $p_i^{L+1} = 0$ and $p_i^L \leq 0$, thus by the definition of P_2, all indices $j \in P_2$ must belong to K^{L+1}. Thus, (5.41) is valid for any $j \in P_2$. Thus,

$$\begin{aligned}
&\sum_{j \in P_2} (p_j^{L+1} - p_j^L) \frac{\tilde{h}_j^k}{h_k} + \sum_{i \in P_1/\{k\}} (p_i^{L+1} - p_i^L) \frac{\tilde{h}_i^k}{h_k} + p_k^{L+1} - p_k^L \\
&\geq \sum_{j \in P_2} (\sum_{i \in P_1} (p_i^L - p_i^{L+1}) \frac{\tilde{h}_i^j \tilde{h}_j^k}{h_k h_j}) \\
&+ \sum_{i \in P_1/\{k\}} (p_i^{L+1} - p_i^L) \frac{\tilde{h}_i^k}{h_k} + p_k^{L+1} - p_k^L \quad \text{(from (5.41))} \\
&= (p_k^{L+1} - p_k^L)(1 - \sum_{j \in P_2} \frac{\tilde{h}_k^j \tilde{h}_j^k}{h_k h_j}) \\
&+ \sum_{i \in P_1/\{k\}} (p_i^{L+1} - p_i^L)(- \sum_{j \in P_2} \frac{\tilde{h}_j^k \tilde{h}_i^j}{h_k h_j} + \frac{\tilde{h}_i^k}{h_k})
\end{aligned} \tag{5.42}$$

Since $p_i^{L+1} - p_i^L \geq 0 \ \forall i \in P_1$, thus, from assumption 5.3 and (5.42) we have

$$\begin{aligned}
&\sum_{j \in P_2} p_j^{L+1} \frac{\tilde{h}_j^k}{h_k} + \sum_{j \in P_1/\{k\}} p_j^{L+1} \frac{\tilde{h}_j^k}{h_k} + p_k^{L+1} \\
&\geq p_k^L + \sum_{j \neq k} p_j^L \frac{\tilde{h}_j^k}{h_k} \\
&= \frac{B}{\lambda \hat{h}^k} - \frac{\sigma^2}{h_k} \quad \text{(from (5.37))}
\end{aligned} \tag{5.43}$$

Since $p_k^{L+1} = 0$,hence from (5.43) we obtain (5.36) is valid for all $k \in K^L$ but not in K^{L+1}.

Now, suppose that $k \in K^{L-1}$ but $k \notin K^L$. Thus, $p_k^L = p_k^{L+1} = 0$. Since $k \in K^{L-1}$, thus from (5.35)

$$p_k^{L-1} h_k + \sum_{j \neq k} p_j^{L-1} \tilde{h}_j^k = \frac{B h_k}{\lambda \hat{h}^k} - \sigma^2 \tag{5.44}$$

Suppose $j \in K^{L+1}$. Then, from (5.35)

$$p_j^{L+1} h_j + \sum_{i \neq j} p_i^{L+1} \tilde{h}_i^j = \frac{B h_j}{\lambda \hat{h}^j} - \sigma^2 \tag{5.45}$$

Since $j \in K^{L+1}$, thus $j \in K^{L-1}$. Hence,

$$p_j^{L-1} h_j + \sum_{i \neq j} p_i^{L-1} \tilde{h}_i^j = \frac{B h_j}{\lambda \hat{h}^j} - \sigma^2 \tag{5.46}$$

Thus, from (5.45) and (5.46) we obtain

$$\begin{aligned}
&p_j^{L+1} - p_j^{L-1} + \sum_{i \neq j} (p_i^{L+1} - p_i^{L-1}) \frac{\tilde{h}_i^j}{h_j} = 0 \\
&p_j^{L+1} - p_j^{L-1} \geq \sum_{i \in K_1, \neq j} (p_i^{L-1} - p_i^{L+1}) \frac{\tilde{h}_i^j}{h_j}
\end{aligned} \tag{5.47}$$

where $K_1 = \{i : p_i^{L+1} \geq p_i^{L-1}\}$; let $K_2 = K_1^C$. Since $k \in K^{L-1}$ but $k \notin K^L$, thus $p_k^{L-1} < 0 = p_k^{L+1}$. Thus, K_1 is not empty. Now, suppose that $i \in K^L$, but $i \notin K^{L+1}$, we have already shown that (5.36) is valid for those indices, thus from (5.36) we have

$$p_i^{L+1} + \sum_{l \neq i} p_l^{L+1} \frac{\tilde{h}_l^i}{h_i} \geq \frac{B}{\lambda \hat{h}^i} - \frac{\sigma^2}{h_i} \tag{5.48}$$

where $p_i^{L+1} = 0$.

Again since $i \in K^L$, thus, $i \in K^{L-1}$. Thus from (5.35), we obtain

$$\begin{aligned}
& p_i^{L-1} h_i + \sum_{l \neq i} p_l^{L-1} \tilde{h}_l^i = \frac{B h_i}{\lambda \hat{h}^i} - \sigma^2 \\
& p_i^{L+1} - p_i^{L-1} + \sum_{l \neq i} (p_l^{L+1} - p_l^{L-1}) \frac{\tilde{h}_l^i}{h_i} \geq 0 \quad \text{(from (5.48))} \\
& p_i^{L+1} - p_i^{L-1} \geq \sum_{l \in K_1, \neq i} (p_l^{L-1} - p_l^{L+1}) \frac{\tilde{h}_l^i}{h_i} \qquad (5.49)
\end{aligned}$$

Note that if $a \notin K^L$, then $a \notin K^{L+1}$. Thus, $p_a^{L+1} = 0$ and $p_a^{L-1} \leq 0$. Thus, an index $i \in K_2$ only if $i \in K^L$ or $i \in K^{L+1}$. Thus, (5.49) is valid for any index $i \in K^2$. Hence

$$\begin{aligned}
& \sum_{j \in K_2} (p_j^{L+1} - p_j^{L-1}) \frac{\tilde{h}_j^k}{h_k} \\
& + \sum_{j \in K_1, \neq k} (p_j^{L+1} - p_j^{L-1}) \frac{\tilde{h}_j^k}{h_k} + p_k^{L+1} - p_k^{L-1} \\
& \geq \sum_{j \in K_2} \Big(\sum_{i \in K_1} (p_i^{L-1} - p_i^{L+1}) \frac{\tilde{h}_i^j \tilde{h}_j^k}{h_j h_k} \Big) \\
& + \sum_{i \in K_1, i \neq k} (p_i^{L+1} - p_i^{L-1}) \frac{\tilde{h}_i^k}{h_k} + p_k^{L+1} - p_k^{L-1} \quad \text{(by (5.47)\&(5.49))} \\
& = \sum_{i \in K_1 / \{k\}} (p_i^{L+1} - p_i^{L-1}) \Big(\frac{\tilde{h}_i^k}{h_k} - \sum_{j \in K_2} \frac{\tilde{h}_i^j \tilde{h}_j^k}{h_j h_k} \Big) \\
& + (p_k^{L+1} - p_k^{L-1}) \Big(1 - \sum_{j \in K_2} \frac{\tilde{h}_i^j \tilde{h}_j^k}{h_j h_k} \Big) \qquad (5.50)
\end{aligned}$$

Since $p_i^{L+1} - p_i^{L-1} \geq 0$ for $i \in K_1$, thus, from assumption 5.3 and (5.50) we have

$$\begin{aligned}
&\sum_{j \in K_2} p_j^{L+1} \frac{\tilde{h}_j^k}{h_k} + \sum_{j \in K_1/\{k\}} p_j^{L+1} \frac{\tilde{h}_j^k}{h_k} + p_k^{L+1} \\
&\geq p_k^{L-1} + \sum_{j \in K_2} p_j^{L-1} \frac{\tilde{h}_j^k}{h_k} + \sum_{j \in K_1/\{k\}} p_j^{L-1} \frac{\tilde{h}_j^k}{h_k} \\
&= \frac{B}{\lambda \hat{h}_k} - \sigma^2 \quad \text{(from (5.44))}
\end{aligned} \tag{5.51}$$

Since $p_k^{L+1} = 0$, thus, (5.51) implies that (5.36) is valid for $k \in K^{L-1}$, but $\notin K^{L+1}, K^L$.

Note that we have only used the fact that (5.36) is valid for any $i \in K^L$ but $i \notin K^{L+1}$ in order to show that (5.36) is valid for a $k \in K^{L-1}$ but $k \notin K^L, K^{L+1}$. Hence, using the same argument we can show that (5.36) is valid for $k \in K^I$ but $k \notin K^{I+1}$ for any $I < L - 1$. Hence, the result follows. □

$$\mathbf{G}(\mathbf{p}) = \begin{pmatrix} \kappa_1 h_1 & -\kappa_1 \widetilde{h}_2^1 & \dots & -\kappa_1 \widetilde{h}_F^1 \\ -\kappa_2 \widetilde{h}_1^2 & \kappa_2 h_2 & \dots & -\kappa_2 \widetilde{h}_F^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_F \widetilde{h}_1^F & -\kappa_F \widetilde{h}_2^F & \dots & -\kappa_F h_F \end{pmatrix} \odot$$

$$\begin{pmatrix} U_1''(\gamma_1) & \gamma_1 U_1''(\gamma_1) + U_1'(\gamma_1) & \dots & \gamma_1 U_1''(\gamma_1) + U_1'(\gamma_1) \\ \gamma_2 U_2''(\gamma_2) + U_2'(\gamma_2) & U_2''(\gamma_2) & \dots & \gamma_2 U_2''(\gamma_2) + U_2'(\gamma_2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_F U_F''(\gamma_F) + U_F'(\gamma_F) & \gamma_F U_F''(\gamma_F) + U_F'(\gamma_F) & \dots & U_F''(\gamma_F) \end{pmatrix}. \quad (5.14)$$

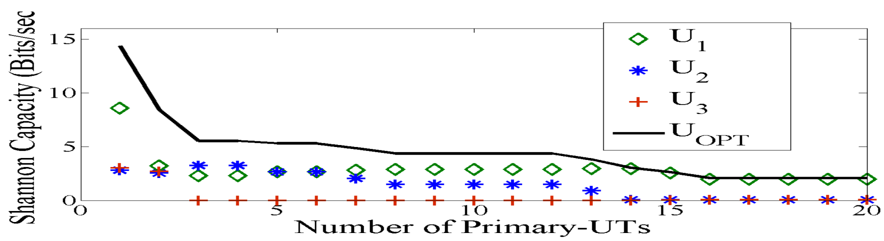


Figure 5.3: U_{OPT} and $U_f, f = 1, 2, 3$ versus number of primary-UTs for Shannon capacity and $F = 3$.

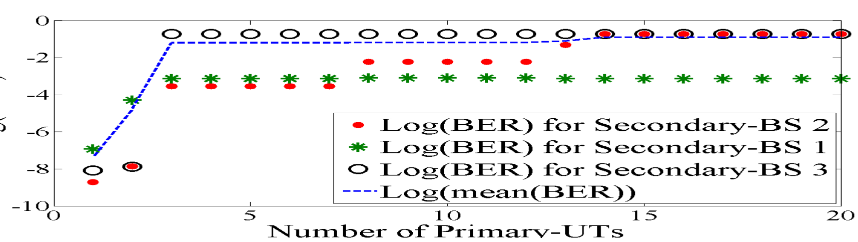


Figure 5.4: log(BER) versus number of primary-UTs for $F = 3$.

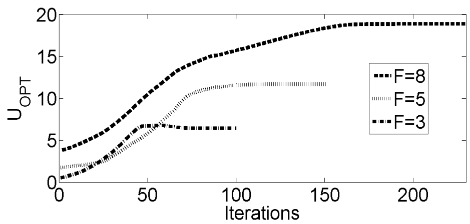


Figure 5.5: Convergence analysis of Algorithm DIST for $F = 3, 5$ and 8 secondary-BSs with $M = 5$.

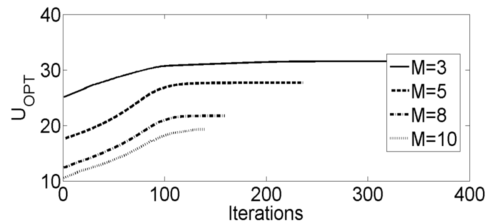


Figure 5.6: Convergence analysis of Algorithm DIST versus the number of primary-UTs for $F = 3$.

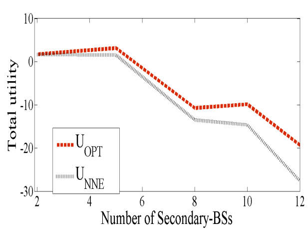


Figure 5.7: U_{OPT} and U_{NNE} versus number of secondary-BSs with $M = 5$.

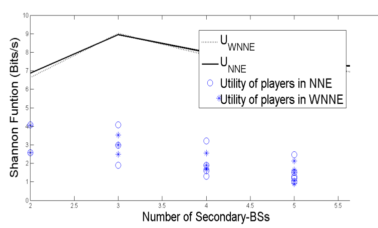


Figure 5.8: U_{WNNE}, U_{NNE} and individual utilities versus the number of secondary-BSs for $I_T = 10$dB.

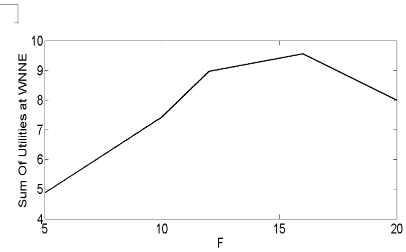


Figure 5.9: The sum of the utilities of Secondary-BSs at WNNE with $M = 10$ & $I_T = 10$dB.

Chapter 6

Conclusions

6.1 Summary

In this thesis, we have addressed several technical issues which have inhibited the deployment of the real time secondary spectrum market. Specifically, in the first part of the thesis (Chapters 2 to 4), we investigated a readily implementable secondary spectrum market where a primary announces its price for its available channel in a database. The price of the primary depends on several factors such as– the channel states, the information that the primaries have, the various stages of the deployment and the cost of acquiring additional information. We characterize the NE pricing strategies of the primaries which will enable them to select prices by considering each such factor. We also characterize the payoffs of the primaries and the possible social implications of the NE strategies.

In the last part of the thesis (Chapter 5), we consider the setting where multiple self-interested secondaries can use the spectrum of a primary. In such a scenario, the interference must be limited at each primary-UT in order to avoid the "tragedy of commons". However, since the secondaries are self-interested entities, the cooperation among the secondaries can not be expected which make the interference mitigation a challenging task. We provide a computationally efficient equilibrium which can redress the challenge. In the following, we briefly state the setting that we considered and key results of each of the chapter.

In Chapter 2 we consider the setting a primary is aware of its own channel state, but is unaware of the channel states of other primaries while selecting its price. The secondaries buy among the channels depending on the prices and the channel qualities. A primary allows at most one secondary to use the channel. We already discussed the complications arise in analyzing this setting in detail in Section 1.2. We show that the NE is unique in this setting and also compute the same. We show that the primaries select prices such that they will render the higher quality channels more preferable to the secondaries. Thus, though we consider that each primary only maximizes its own payoff, it gives rise a *socially efficient allocation* for a large class of penalty functions i.e. prices are selected in such a way that the secondaries always buy higher quality channels compared to the low quality ones (Please see Section 2.6 for complete discussion on results).

In Chapter 3 we analyze the setting where a primary owns a channel over multiple locations. A primary needs to select a set of non interfering locations apart from the price. We model the interference relationship as a conflict graph. Each primary selects a price and an independent set (which represents the set of non interfering locations in the conflict graph) where it will sell its channel. We already discussed the complications arise in analyzing the setting in Section 1.6.4. We consider two scenarios-i) the number of locations is small, ii) the number of locations is large. Since the spectrum leasing is in short term basis, thus, a computationally efficient strategy is also required. We show that when the number of locations is small, there exists a computationally efficient NE strategy where a primary selects an independent set whose cardinality is above a certain threshold. The threshold increases with the quality of the channel. The symmetric NE strategy is also unique in the linear conflict graph. When the number of locations is large, a primary only selects among the maximum independent sets. The symmetric NE strategy is also not unique in the linear conflict graph (Sections 1.6.4 and 3.7).

In Chapter 4 we consider a setting where a primary can acquire the CSI of the competitor by incurring a cost. We formulate the problem with two primaries as a non cooperative game. Each primary has to decide whether to acquire the CSI of the competitors or not, and then a price depending on the information it has. However, a primary is unaware of the decision of the other primaries. Thus, a primary is not aware whether its channel state is known to the competitors whereas in the setting considered in Chapter 2 a primary knows that its channel state is not known to the other primaries. Error also occurs while estimating the CSI, which also complicates the analysis. In Section 1.4 we discuss in detail all the complications arise in this setting. We characterize the NE strategies and the impact of estimation errors, different costs of acquiring the CSI, and different availability probabilities on the NE strategies. We have also obtained some counter intuitive results (Section 1.6.3 and the introduction of Chapter 4).

In Chapter 5 we consider the setting where multiple secondaries use the spectrum of a primary at a given location. We formulate the problem as a non cooperative game with the secondary-BSs as the players. Each secondary-BS selects a power to maximize its own utility subject to the constraint that the interference must be limited at each primary-UT. We use the normalized Nash equilibrium (NNE) as an equilibrium concept. We show that for a large class of utility functions, there exists a distributed algorithm which converges to the unique NNE. In the distributed algorithm a secondary-BS needs not to know the channel parameters of other secondary-BSs. We also introduce the concept of the weakly normalized Nash equilibrium (WNNE) which proved to be invaluable when the NNE is not unique or difficult to compute as WNNE retains most of the favorable properties of the NNE.

6.2 Future Work

We first state some generic directions in which our work can be extended. Subsequently, we state some specific directions in which the setting studied in each chapter can be extended.

Throughout this thesis, we mainly consider the scenario where the primaries interact only once. When the spectrum will be leased in a short term basis, the primaries may interact multiple times. Consideration of the above will give rise several future research directions. For example, the channel states of a primary may be correlated at different times, the characterization of the NE in such a setting is a work for the future. When the primaries interact multiple times, a primary may update the belief of the occurrence of the channel states of the other primaries based on their actions. A primary has to characterize the pricing strategy based on its own belief function. On the other hand, the primary will also know that its competitors will learn about its channel state based on its action. The primary may deliberately mislead its competitors for potential gain. The characterization of an NE in such a setting will require the tools from both the learning theory, and the game theory and it has been deferred to the future. We also consider that the secondaries are statistically identical in Chapters 2 to 4; the characterization of the NE strategies when the secondaries are heterogeneous is also a work for the future.

We now describe the specific directions in which the setting studied in each chapter can be extended. In Chapter 3 we consider that the available channels are statistically identical when the number of locations is large. The characterization of an NE where the available channel may belong to multiple states in the scenario where the number of locations is large remains open. We also assume that the demand at different locations are independent. The characterization of an NE when the demand at different locations are correlated is a work for the future.

In Chapter 4 we consider the scenario where there are only two primaries. When there are more than two primaries the complexity to characterize a NE strategy significantly increases (Section 4.6). The characterization of the NE strategy profile in the scenario where there are more than two primaries is deferred to the future. The characterization of an NE strategy profile in the setting where the primary owns a channel over multiple locations is a work for the future.

In Chapter 5, we consider the scenario where there is a constraint that the interference must be limited at each primary-UT. The characterization of the equilibrium strategies in the setting where primaries can relax the above constraint (by paying a penalty to the primary-UTs for the violation of the interference constraint) by quoting additional prices remains open.

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