

# Probability and Problems in Euclidean Combinatorial Optimization

J. Michael Steele

**Abstract.** This article summarizes the current status of several streams of research that deal with the probability theory of problems of combinatorial optimization. There is a particular emphasis on functionals of finite point sets. The most famous example of such functionals is the length associated with the *Euclidean traveling salesman problem (TSP)*, but closely related problems include the minimal spanning tree problem, minimal matching problems and others. Progress is also surveyed on (1) the approximation and determination of constants whose existence is known by subadditive methods, (2) the central limit problems for several functionals closely related to Euclidean functionals, and (3) analogies in the asymptotic behavior between worst-case and expected-case behavior of Euclidean problems.

No attempt has been made in this survey to cover the many important applications of probability to linear programming, arrangement searching or other problems that focus on lines or planes.

**Key words and phrases:** Matching, minimal spanning trees, Steiner trees, subadditive Euclidean functionals, traveling salesman problem, worst-case analyses.

## 1. INTRODUCTION

Almost all of the results surveyed here owe a debt of one sort or another to the classic theorem of Beardwood, Halton and Hammersley that lays out the basic behavior of the length of the shortest tour through a random sample from a general distribution in  $\mathbb{R}^d$ .

**THEOREM 1** (Beardwood, Halton and Hammersley 1959). *If  $X_i$ ,  $1 \leq i < \infty$  are independently and identically distributed random variables with bounded support in  $\mathbb{R}^d$ , then the length  $L_n$  under the usual Euclidean metric of the shortest path through the points  $\{X_1, X_2, \dots, X_n\}$  satisfies*

$$L_n/n^{(d-1)/d} \rightarrow \beta_{TSP,d} \int_{\mathbb{R}^d} f(x)^{(d-1)/d} dx \text{ almost surely.}$$

Here,  $f(x)$  is the density of the absolutely continuous part of the distribution of the  $X_i$ , and  $\beta_{TSP,d}$  is a positive constant that depends on  $d$  but not on the distribution of the  $X_i$ .

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The Beardwood, Halton and Hammersley (BHH) theorem has led to developments of several different types. The first developments that we review concern generalizations of the BHH theorem that aim to provide insight into larger classes of processes. It turns out that the essential geometric features of the traveling salesman problem (TSP) are present in many of the problems that have been investigated in the theory of combinatorial optimization, and there is also a close connection between these properties and the notions of subadditivity. These subadditivity-driven generalizations of the BHH are addressed primarily in Section 2.

The second type of problem considered looks at the BHH theorem from the perspective of precision rather than generality. It turns out that  $L_n$ , the length of the shortest tour through an  $n$ -sample, is highly concentrated about its mean. The main result of this development, the theorem of Rhee and Talagrand, is spelled out in Section 3. It tells us that the tails of the TSP functional  $L_n$  decay as rapidly as those of the normal distribution. This remarkable result has evolved through several intermediate stages, each of which offered new tools for the analysis of functionals of finite point sets in the plane.

Section 4 is the last to focus explicitly on the theory of the TSP, and, though we start to strain the notion

of a Euclidean problem, the development casts light on problems like those just described. The main purpose of Section 4 is to introduce the connection between *Euclidean* optimization problems and self-similar sets of fractional dimension; the principal result reviewed is a theorem by S. Lalley on the TSP. This is an intriguing result that suggests a rich connection between the rapidly developing theory of sets of fractals and many classic problems.

In Sections 5 and 6 the set of problems studied is widened. We first consider the theory of the minimal spanning tree (MST) and review how one can construct an infinite analog to the MST that leads to answers of a number of classic questions. One of the best reasons for introducing any new mathematical structure is that it simplifies and strengthens earlier results, and this is exactly what occurs in this development of an “infinite MST.” The new structure permits one to make many deductions that are apparently difficult (or perhaps impossible) to address without such a tool.

Section 6 gives a brief survey of some of the new developments in the theory of matching. This subject began with the theorem of Ajtai, Komlós and Tusnády on the two-sample matching problem, but it has developed rapidly because of its rich connections to such topics as bin packing, scheduling and the theory of empirical processes. There are many engaging special results in this active area, and it is not possible to offer much more than a taste. We particularly want to call attention to a new theory of majorizing measure initiated by M. Talagrand. This theory unifies and deepens many earlier results and also suggests many new concrete problems.

In the last three sections the pace is quickened, but the work that is surveyed represents many recent developments that can be expected to lead to much further research. In Section 7, recent progress is reviewed in the calculations of the constants that appear in results such as the BHH theorem. Section 8 then considers the very recent progress on the central limit problem for some geometric problems closely related to Euclidean functionals. Finally, in Section 9 we see some tight analogies between the probabilistic analysis of Euclidean functionals and their worst-case behavior. The concluding section mainly focuses on the omissions that have been occasioned by our focus on functionals on points rather than on the full range of Euclidean structures.

**The Connection to Algorithms.** Before going to the survey just outlined, there is one top-level observation about the BHH theorem that should be made. It is likely that this remarkable 1959 result would have remained relatively undeveloped if it had not been for the striking use of the BHH theorem in the polynomial-time probabilistic TSP algorithm of Karp (1976, 1977). The TSP was among the first of the traditional prob-

lems of geometric optimization to be proved to be NP-hard, and it is also a problem of genuine practical interest, so it is not surprising that considerable excitement was generated when Karp showed that the TSP is perfectly tractable under the plausible stochastic assumption that the sites to be visited by the tour can be modeled as a random uniform sample. In this context, Karp showed that, given any  $\varepsilon > 0$ , there is a (simple!) polynomial time algorithm that produces a tour of length no more than  $(1 + \varepsilon)$  times the optimal tour length with probability that converges to one as the size of the problem is increased. This discovery launched an important branch of the field of probabilistic algorithms. Moreover, it generated powerful interest in the BHH theorem, its generalizations and sharpenings that are at the center of this survey.

## 2. SUBADDITIVE EUCLIDEAN FUNCTIONALS

By abstracting from the traveling salesman tour just a few of its basic properties, it is possible to suggest a very general result that provides information comparable to that given by the BHH theorem for a large number of problems of combinatorial optimization in Euclidean space.

Let  $L$  be a function that associates a real number to each finite subset  $\{x_1, x_2, \dots, x_n\} \subset \mathbf{R}^d$ . To spell out the most innocent properties of  $L$  that mimic the behavior of the TSP, we first note that for the TSP,  $L$  exhibits *homogeneity* and *translation invariance*; that is,

$$(1) \quad L(\alpha x_1, \alpha x_2, \dots, \alpha x_n) = \alpha L(x_1, x_2, \dots, x_n) \quad \text{for all } \alpha > 0,$$

and

$$(2) \quad \begin{aligned} L(x_1 + y, x_2 + y, \dots, x_n + y) \\ = L(x_1, x_2, \dots, x_n) \text{ for all } y \in \mathbf{R}^d. \end{aligned}$$

The TSP's length also has some strong smoothness and regularity properties, but these turn out not to be too important for most purposes. All that is needed is that  $L$  be Borel-measurable when viewed as a function from  $\mathbf{R}^{nd}$  to  $\mathbf{R}$ . This condition is almost always trivial to obtain, but still one has to have it to be able to talk honestly about probabilities involving  $L$ .

Functions on the finite subsets of  $\mathbf{R}^d$  that are measurable in the sense just described and that are homogeneous of order one and translation-invariant are called *Euclidean functionals*. These three properties are commonplace but bland. One should not expect to be able to prove much in such a limited context, but, with the addition of just a couple of other structural features, a rich and useful theory emerges.

The first additional property of the TSP functional that we consider is that it is *monotone* in the sense that

$$(3) \quad L(x_1, x_2, \dots, x_n) \leq L(x_1, x_2, \dots, x_n, x_{n+1}) \\ \text{for } n \geq 1, \text{ and } L(\emptyset) = 0.$$

The final feature of the TSP functional that we abstract is the most substantial one. It expresses both the geometry of the underlying space and the fundamental suboptimality of one of the most natural TSP heuristics, the partitioning heuristic. If  $\{Q_i\}_{i=1}^{m^d}$  is a partition of  $[0, t]^d$  into smaller cubes of edge length  $t/m$ , the *subadditive property* says there exists a constant  $B$  independent of  $m$  and  $t$  such that

$$(4) \quad L(\{x_1, x_2, \dots, x_n\} \cap [0, t]^d) \\ \leq \sum_{i=1}^{m^d} L(\{x_1, x_2, \dots, x_n\} \cap Q_i) + Btm^{d-1}$$

for all integers  $m \geq 1$  and real  $t \geq 0$ .

Euclidean functionals that satisfy equations (3) and (4) will be called *monotone subadditive Euclidean functionals*. This class of processes seems to abstract the most essential features of the TSP that are needed for an effective asymptotic analysis of the functional applied to finite samples of independent random variables with values in  $\mathbb{R}^d$ .

To see how subadditive Euclidean functionals arise (and to see how some problems just barely elude the framework), it is useful to consider two additional examples.

The first is the *Steiner minimum tree*. For any finite set of  $n$  points,  $S = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^d$ , a Steiner minimum tree for  $S$  is a tree  $T$  whose vertex set contains  $S$  such that the sum of the lengths of the edges in  $T$  is minimal over all such trees. Note that the vertex set of  $T$  may contain points not in  $S$ ; these are called *Steiner points*. If  $L_{ST}(x_1, x_2, \dots, x_n)$  is the length of a Steiner tree of  $x_1, x_2, \dots, x_n$ , and if we let  $|e|$  be the length of an edge  $e$ , another way of defining  $L_{ST}$  is just

$$L_{ST}(S) = \min_T \left\{ \sum_{e \in T} |e| : T \text{ is a tree} \right. \\ \left. \text{containing } S \subset \mathbb{R}^d, S \text{ finite} \right\}.$$

The second example is closely related, yet it points out that the innocuous monotonicity property of the TSP can fail even in natural problems. The example we have in mind is the *minimum spanning tree*. For  $\{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^d$ , let  $L_{MST}(x_1, x_2, \dots, x_n) = \min \sum_{e \in T} |e|$ , where the minimum is over all connected graphs  $T$  with vertex set  $\{x_1, x_2, \dots, x_n\}$ . It is an easy matter to check that any minimizing graph must indeed be a spanning tree.

The functional  $L_{MST}$  is easily seen to be homogeneous, translation-invariant and measurable. One can also check without much trouble that it is subadditive in the sense required above. Still, many simple examples such as the sets  $S = \{(0, 0), (0, 2), (2, 0), (2, 2)\}$  and  $S \cup \{(1, 1)\}$  will show that  $L_{MST}$  fails to be monotone.

One should suspect that this failure is of an exceptional sort that would not have great influence on asymptotic behavior, and this suspicion is well justified. Still, the example puts us on warning that nonmonotone functionals can require delicate considerations that are not needed in cases that mimic the TSP more closely.

The main theorem of this section shows that the properties (1) through (4), together with a modest moment condition, are sufficient to determine the asymptotic behavior of  $L(X_1, X_2, \dots, X_n)$  when the  $X_i$  are independent and identically distributed.

**THEOREM 2** (Steele, 1981b). *Let  $L$  be a monotone subadditive Euclidean functional. If  $\{X_i\}$ ,  $i = 1, 2, \dots$ , are independent random variables with the uniform distribution on  $[0, 1]^d$  and  $\text{var} \{L(X_1, X_2, \dots, X_n)\} < \infty$  for each  $n \geq 1$ , then as  $n \rightarrow \infty$ ,*

$$L(X_1, X_2, \dots, X_n)/n^{(d-1)/d} \rightarrow \beta_{L,d}$$

*with probability 1, where  $\beta_{L,d} \geq 0$  is a constant depending only on  $L$  and  $d$ .*

The restrictions that this theorem imposes on a Euclidean functional are as few as one can reasonably expect to yield a generally useful limit theorem, and because of this generality the restriction to uniformly distributed random variables is palatable. Moreover, since many of the probabilistic models studied in operations research and computer science also focus on the uniformly distributed case, the theorem has immediate applications. Still, one cannot be long content with a theory confined to uniformly distributed random variables. Fortunately, with the addition of just a couple of additional constraints, the limit theory of subadditive Euclidean functionals can be extended to quite generally distributed variables.

To get to the essence of the extension, we first consider the case of random variables with a singular component, that is, variables such that there is a measurable set  $E \subset \mathbb{R}^d$  with Lebesgue measure zero such that  $P(X_i \in E) > 0$ . Random variables with a singular component provide a kind of geometrical worst case and hence provide a useful test case.

To deal with such random variables, we need to draw out three additional properties shared by many subadditive Euclidean functionals. The first of these, called *scale-boundedness*, holds provided there is a constant  $C$  such that

$$(5) \quad L(x_1, x_2, \dots, x_n) \leq Ctn^{(d-1)/d}$$

for all  $\{x_1, x_2, \dots, x_n\} \subset [0, t]^d$ . The second property, called *simple subadditivity*, holds provided there is a constant  $D$  such that

$$(6) \quad L(A_1 \cup A_2) \leq L(A_1) + L(A_2) + Dt$$

for any finite subsets  $A_1$  and  $A_2$  contained in  $[0, t]^d$ .

These two properties hold for many natural examples, and in particular they are easily checked for the TSP. The key benefit of scale-boundedness and simple subadditivity comes from their showing that the singular component of a distribution makes a contribution that is of lower order than that of the absolutely continuous component.

**LEMMA 1.** *Let  $L$  be a monotone subadditive Euclidean functional that is scale-bounded (5) and simply subadditive (6). If  $\{X_i\}$  are independent identically distributed (i.i.d.) random variables and  $E$  is any bounded set of Lebesgue measure zero, then as  $n \rightarrow \infty$*

$$L(\{X_1, X_2, \dots, X_n\} \cap E) / n^{(d-1)/d} \rightarrow 0$$

with probability 1.

The last property we will require for the extension of this theorem to general distributions calls on a type of restricted converse of simple subadditivity. A Euclidean functional  $L$  is called *upper-linear* provided we have the following: for every finite collection of cubes  $Q_j$ , where  $1 \leq j \leq M$  and the edges of the  $Q_j$  are parallel to the axes, and for every infinite sequence  $x_i$ , where  $1 \leq i < \infty$  and  $x_i \in \mathbb{R}^d$ , we have

$$(7) \quad \sum_{j=1}^M L(\{x_1, x_2, \dots, x_n\} \cap Q_j) \leq L(\{x_1, \dots, x_n\} \cap \bigcup_{j=1}^M Q_j) + o(n^{(d-1)/d}).$$

### 3. TAIL PROBABILITIES

The theory just outlined has a number of extensions and refinements. The first of these that we consider is the work of Rhee and Talagrand (1989) on the behavior of the tail probabilities of the TSP and related functionals under the model of independent uniformly distributed random variables in the unit  $d$ -cube. In Steele (1981a), it was observed that in dimension 2  $\text{Var}(L_n)$  is bounded independently of  $n$ . This somewhat surprising result motivated the search for a more detailed understanding of the tail probabilities  $P(L_n \geq t)$ , and, in particular, it opened the question of determining if these probabilities might decay at the Gaussian rate  $\exp(-ct^2/2)$ .

After introducing several methods from martingale theory and interpolation theory that led to interesting intermediate results, Rhee and Talagrand (1989) finally provided a remarkable proof that in  $d = 2$  the TSP (and many related functionals) do indeed have Gaussian tail bounds.

**THEOREM 3** (Rhee and Talagrand, 1989). *Suppose  $f$  is a Borel measurable function that assigns to each finite subset  $F \subset [0, 1]^2$  a real value  $f(F)$  such that*

$$f(F) \leq f(F \cup x) \leq f(F) + \min(d(x, y) : y \in F).$$

*There is then a constant  $K = K_f$  such that if  $X_i$  are independent and uniformly distributed in  $[0, 1]^2$ , then the random variables defined by  $U_n = f(\{X_1, X_2, \dots, X_n\})$  satisfy*

$$P(|U_n - E(U_n)| > t) \leq \exp(-t^2/K).$$

The proof of this result rests on the systematic exploitation of the martingale versions of the Hoeffding and Prokhorov large-deviation inequalities together with a powerful bare-hands construction that helps articulate technical features of a random sample being well spread out.

### 4. THE TSP IN FRACTAL SPACES

The BHH theorem may seem like a result that is wedded to  $\mathbb{R}^d$ ; certainly the growth rate  $n^{(d-1)/d}$  emerges as a natural characteristic of the dimension. Lalley (1990) has provided a set of results that cast new light on the role of dimension in problems like the TSP by developing limit results for TSP tours in fractal spaces. Lalley considered finite random samples chosen uniformly (in an appropriate sense) from self-similar subsets of  $\mathbb{R}^2$  that have Hausdorff dimension strictly less than two. Lalley obtained an almost-sure limit law for the length  $L_n$  of the shortest tour through  $\{X_1, X_2, \dots, X_n\}$ , but now the normalizing denominator is no longer  $\sqrt{n}$  but a power of  $n$  that depends on the structure of the self-similar set.

To give the flavor of the results obtained by Lalley, we will consider one specific example. The main condition for a compact set  $K$  to be called a strongly self-similar subset of  $\mathbb{R}^2$  is that there is a finite collection of transformations  $T_i : K \rightarrow K$  so that

$$K = \bigcup_{i=1}^m T_i(K),$$

where each  $T_i$  is a transformation that can be written as a *strict* affine contraction (with contraction factor  $r_i < 1$ ) followed by a rigid motion. The strong self-similarity of  $K$  requires moreover that the intersections  $T_i(K) \cap T_j(K)$ , where  $i \neq j$ , be negligibly small in a technical sense that we will not detail. Self-similar sets can be provided with a natural “uniform measure” that generalizes the measure one gets on the product of Cantor sets by “choosing the ternary expansion 0 or 2 according to fair coinflips.”

**THEOREM 4** (Lalley, 1990). *If  $X_i$ ,  $1 \leq i < \infty$ , are independent random variables with the “uniform distribution” on a strongly self-similar set  $K$  that has representation*

$$K = \bigcup_{i=1}^m T_i(K),$$

*where  $\|T_i(x) - T_j(y)\| = r_i \|x - y\|$ . If  $K$  has Hausdorff*

dimension strictly larger than 1, then the length  $L_n$  of the shortest tour through the point set  $\{X_i : 1 \leq i \leq n\}$  satisfies

$$\lim_{n \rightarrow \infty} L_n/n^\theta = C_K$$

with probability 1, where  $C_K > 0$  and  $0 < \theta \leq 1/2$  is a constant uniquely determined by the similarity ratios  $r_i$ .

## 5. MINIMAL SPANNING TREES

If  $\mathbf{x} = \{x_1, \dots, x_n\}$  is a finite set of points in  $\mathbb{R}^d$  for  $d \geq 2$ , a *minimal spanning tree* (MST)  $t$  of  $\mathbf{x}$  is a connected graph with vertex set  $\mathbf{x}$  such that the sum of the edge lengths of  $t$  is minimal; that is,

$$\sum_{e \in t} |e| = \min_G \sum_{e \in G} |e|,$$

where  $|e| = |x_i - x_j|$  is the Euclidean length of the edge  $e = (x_i, x_j)$  and the minimum is over all connected graphs  $G$ .

Minimal spanning trees are among the most studied objects in combinatorial optimization, and even the probability theory of MST is rather well developed. For example, in Steele (1989) one finds results that cover the basic almost-sure asymptotic behavior of the MST that parallels the BHH theorem and even deals with edge weights that are more complicated than those given by the basic edge lengths. In particular, it is shown that if  $X_i$  are i.i.d. with compactly supported density  $f$  and  $0 < \alpha < d$ , then

$$n^{-(d-\alpha)/d} \sum_{e \in t} |e|^\alpha \rightarrow c(\alpha, d) \int_{\mathbb{R}^d} f(x)^{d-\alpha/d} dx \text{ a.s.,}$$

where  $c(\alpha, d)$  depends only on  $\alpha$  and  $d$ .

One peculiar aspect of this result is that it fails to cover the extreme case  $\alpha = d$ . This failure was particularly intriguing since the case  $\alpha = d$  for  $d = 2$  had been the source of an interesting conjecture by R. Bland, who by direct computational experience had been led to believe that if one takes the sum of the squares of the edges of the MST of a sample of points chosen randomly from the unit square, then as  $n \rightarrow \infty$  the sum converges to a constant. This result is hinted at by the limit cited above, but the specifically desired result fell just outside of its domain.

In Aldous and Steele (1992) an approach to such limit problems was taken that differs completely from the subadditivity-based analyses such as that mentioned in Section 2 or the more ad hoc method used in Steele (1989). The new approach is based on the study of an analog of the MST for an *infinite* set of points, in particular the points of a Poisson process on all  $\mathbb{R}^d$ . The basic philosophy that guided the analysis is that because the empirical distributions of any independent uniform sequences look locally like a realization of the Poisson process, one should be able to relate the MST

of a uniform sample to the MST of the (unbounded) Poisson process. Though this sounds a little like the ‘‘Poissonization’’ trick that goes back even as far as Beardwood, Halton and Hammersley, it is really radically different in concept and detail. To give a forward hint of that difference, note that we have no idea how to apply the method to the TSP.

To sketch the basic idea, let  $\mathbf{x} = (x_i)$  be a finite or countably infinite subset of  $\mathbb{R}^d$ ,  $d \geq 2$ . We call  $\mathbf{x}$  *nice* if (1)  $\mathbf{x}$  is *locally finite*, that is, has only finitely many elements in bounded subsets of  $\mathbb{R}^d$ , and (2) the interpoint distances  $(|x_j - x_i|, i < j)$  are all distinct. Now, given a pair  $(x, \mathbf{x})$  with  $\mathbf{x}$  nice and  $x \in \mathbf{x}$ , we can define trees  $t_m(x, \mathbf{x})$  with vertices from  $\mathbf{x}$  as follows: Let  $\xi_1 = x$ , and let  $t_1$  be the single vertex  $\xi_1$ . Let  $t_2$  be the tree consisting of the vertex  $\xi_1$  and the vertex  $\xi_2 \in \mathbf{x} \setminus \{\xi_1\}$  that is closest to  $\xi_1$  in Euclidean distance, together with the edge (straight line segment) connecting  $\xi_1$  and  $\xi_2$ . We then proceed inductively in a greedy fashion and define  $t_m = t_m(x, \mathbf{x})$  to be  $t_{m-1}$  together with a new edge  $(\xi_{j_m}, \xi_m)$ , where  $j_m \leq m-1$  and  $\xi_m \in \mathbf{x} \setminus \{\xi_1, \dots, \xi_{m-1}\}$  are chosen so that the edge length  $|\xi_m - \xi_{j_m}|$  is minimal (over all possible edges connecting  $t_{m-1}$  to  $\mathbf{x} \setminus t_{m-1}$ ). In the finite case where  $n = |\mathbf{x}| < \infty$ , this procedure terminates with the tree  $t_n(x, \mathbf{x})$ , and in that circumstance the tree obtained does not depend on the choice of the starting vertex  $x$ .

When  $\mathbf{x}$  is infinite, the situation is more complex. We write  $t_\infty(x, \mathbf{x})$  for the set  $\bigcup_n t_n(x, \mathbf{x})$ , but some work is required to obtain useful structural information about this graph. In fact, it turns out to be technically useful to look at a different but related graph, described in the next lemma.

**LEMMA 2.** *Let  $g = g(\mathbf{x})$  be the graph on an infinite nice vertex set  $\mathbf{x}$  defined by taking any pair  $(x_1, x_2)$  as an edge in  $g$  if it is an edge in either  $t_\infty(x_1, \mathbf{x})$  or  $t_\infty(x_2, \mathbf{x})$ . Then the graph  $g$  is a forest and each component of  $g$  is an infinite tree.*

The main object of Aldous and Steele (1992) is the random tree  $\mathfrak{J}$  constructed by taking a Poisson point process  $\mathfrak{N} = \{\eta_i\}$  of rate 1 in  $\mathbb{R}^d$ , letting  $\mathfrak{N}^0 = \mathfrak{N} \cup \{0\}$ . Let  $\mathcal{G} = g(\mathfrak{N}^0)$  be the forest constructed as in the lemma, and finally take the connected graph  $\mathfrak{J}$  to be the largest tree containing 0.

It is natural to conjecture that  $\mathcal{G}$  is itself a *tree* with probability 1, but this possibility seems to be related to deep issues in continuum percolation, and the introduction of  $\mathfrak{J}$  permits one to finesse that subtle issue.

**LEMMA 3.** *Let  $D$  be the degree of vertex 0 in  $\mathfrak{J}$  and let  $L_1, \dots, L_D$  be the lengths of the edges of  $\mathfrak{J}$  incident to 0. We then have:*

1.  $D \leq b_d$ , a constant,
2.  $ED = 2$ , and
3.  $l_d \equiv \frac{1}{2} \sum_i EL_i^d < \infty$ .

Now let  $\mathfrak{N}_n$  denote the point process consisting of  $n$  points  $(\eta_i : 1 \leq i \leq n)$  that are independent and uniformly distributed on the unit cube  $[0, 1]^d$ . With probability 1,  $\mathfrak{N}_n$  is a nice subset of  $\mathbb{R}^d$ . Let  $S_n = t_n(\eta_1, \mathfrak{N}_n)$  be the minimal spanning tree of these  $n$  vertices. It is intuitive that  $S_n$ , after a suitable rescaling, should converge locally to  $\mathcal{G}$ .

**THEOREM 5.** (Aldous and Steele, 1992). (a) *If  $\{|e_i|\}$  denotes the set of lengths of the edges of  $S_n$ , then*

$$\sum_{i=1}^{n-1} |e_i|^d \xrightarrow{L^2} l_d \text{ as } n \rightarrow \infty.$$

(b) *If  $\Delta_{n,i}$  denotes the proportion of vertices of  $S_n$  with degree  $i$ , then for each  $i$ ,*

$$E\Delta_{n,i} \rightarrow P(D = i) \text{ as } n \rightarrow \infty.$$

The existence of limits in (b) was proved by different methods in Steele, Shepp and Eddy (1987), where one also finds a proof of almost sure convergence. The analogous question in the model in which the cost assigned to each edge is independent and identically distributed is solved in Aldous (1990) using “limit process” arguments such as those reviewed above.

## 6. MATCHING PROBLEMS

The problem of determining the least weight matching in a graph  $(G, E)$  for which there is a function that assigns a real value to the edges of  $G$  is a central problem of combinatorial optimization. The matching problem is intermediate in complexity between the MST problem and the TSP. The minimal matching problem cannot be solved by a naive greedy algorithm such as that used for the MST, but there is still a polynomial-time algorithm that does solve the problem—in contrast to the TSP.

There are two natural Euclidean versions of the matching problem. The simplest version considers  $2n$  points in  $\mathbb{R}^d$  and asks for the perfect matching of the set for which the total edge length is minimum. This matching problem has a theory that is almost perfectly parallel to the theory of the MST.

A more subtle set of issues arises when one instead considers two distinguished subsets  $\{x_1, x_2, \dots, x_n\}$  and  $\{y_1, y_2, \dots, y_n\}$ . Here the probability theory becomes much trickier, and new phenomena arise. A first taste of the new behavior is given by the results of Ajtai, Komlós and Tusnády, one of which we note below:

**THEOREM 6** (Ajtai, Komlós and Tusnády, 1984). *If  $X_i$  and  $Y_i$  are independent and uniformly distributed in  $\mathbb{R}^2$  for  $1 \leq i \leq n$ , then there are constants  $K_1$  and  $K_2$  such that*

$$K_1 \sqrt{n \log n} \leq M_n = \min_{\sigma} \sum_{i=1}^n |X_i - Y_{\sigma(i)}| \leq K_2 \sqrt{n \log n}$$

*with probability that approaches 1 as  $n \rightarrow \infty$ .*

The  $\sqrt{\log n}$  term in this result adds a new spin to the theory of Euclidean functionals for several reasons. First, it shows that the scaling arguments and self-similarity ideas that were useful for the TSP and the MST are no longer accurate enough to lead to the right orders. Still, the basic motivation remains substantially intact, and for  $d \geq 3$  one again finds the characteristic growth rates of  $n^{(d-1)/d}$ .

As an added impetus to the theory of Euclidean matching, there are some powerful connections with other parts of combinatorial optimization. It was through their work on bin packing and scheduling that Leighton and Shor were led to the investigation of the maximum length that must exist in a two-sample matching. The following example from their work again exhibits a curious logarithmic term.

**THEOREM 7** (Leighton and Shor, 1989). *If  $X_i$  and  $Y_i$  are independent and uniformly distributed in  $[0, 1]^2$  for  $1 \leq i \leq n$ , then there is a constant  $K$  such that*

$$\min_{\sigma} \max_{1 \leq i \leq n} |X_i - Y_{\sigma(i)}| \leq K n^{-1/2} (\log n)^{3/4}$$

*with probability that approaches 1 as  $n \rightarrow \infty$ .*

In one of the most recent and most sweeping studies reviewed for this survey, Talagrand (1991) has unified the two preceding results as well as many others. The details of that development are too substantial to review here, but it is possible to indicate some of the sources of the new power.

Begin by noting that there is a basic relation between the behavior of the two-sample matching problems and the theory of empirical discrepancy. We recall that if  $\mathcal{C}$  is any class of functions from  $[0, 1]$  to  $[0, 1]$ , the *empirical discrepancy* associated with  $\mathcal{C}$  is defined by

$$D_n(\mathcal{C}) = \sup_{f \in \mathcal{C}} \left| \sum_{i=1}^n (f(X_i) - \int_{\mathbb{R}^2} f(x) dx) \right|,$$

where the  $X_i$  are i.i.d. random variables with density  $f$ . Even before the work of Ajtai, Komlós and Tusnády, it was well understood that there was a close connection between  $M_n$  and  $D_n$  when  $\mathcal{C}$  is taken to be the class of Lipschitz functions. It was also known that the theory associated with  $D_n$  was closely linked to the epsilon entropy of the class  $\mathcal{C}$ , and finally people were beginning to understand that the issues addressed by epsilon entropy could in some cases be more effectively addressed by an emerging method of majorizing measures.

Unfortunately, even defining all these terms would take us rather far afield, but hopefully it is meaningful to say that epsilon entropy is a tool for measuring the complexity of a class of functions by counting the number of epsilon balls that are needed to cover certain compact subsets, whereas the method of majorizing



measures refines the theory of epsilon entropy to deal more effectively with aspects of inhomogeneity in  $\mathcal{C}$ . The substantial achievement of Talagrand (1991) was to develop the concrete tools of the theory of majorizing measures that make it directly useful in the theory of  $M_n$  and  $D_n$  and then to show how those tools could be used to deepen and unify the investigations initiated by Ajtai, Komlós and Tusnády and broadened by Leighton and Shor.

## 7. THE VALUES OF THE CONSTANTS

There is a long history of effort devoted to the determination of the limiting constraints in results like the BHH theorem. There is also a substantial literature that addresses the constraints associated with worst-case upper bounds.

Work by Supowit, Reingold and Plaisted (1983), Moran (1984) and Goldstein and Reingold (1987) provide recent progress on the worst-case bounds and provide surveys of the older literature. Two particularly noteworthy contributions for the TSP are those of Goddyn (1990) and Karloff (1987). The latter broke the  $\sqrt{2}$  barrier in dimension 2 by showing that  $\alpha_{TSP,2} \leq 0.984\sqrt{2}$ , whereas Goddyn (1990) obtained the best bounds in general dimension by means of a powerful technique that uses an infinite number of translations of quantizers other than cubical cylinders.

After the constraints on the TSP, the best studied value is  $\alpha_{RST,d}$ , the constant associated with the worst-case length of a rectilinear Steiner minimum tree in the unit  $d$ -cube. Chung and Graham (1981) proved that  $\alpha_{RST,2} = 1$ , which is significant in that it is the only nontrivial worst-case constant for which we have an exact expression. The problem of determining  $\alpha_{RST,d}$  in higher dimensions is still open, the bounds being  $\max\{1, d/(4e)\} \leq \alpha_{RST,d} \leq d4^{(1-d)/d}$  for  $d \geq 1$  (cf. Snyder, 1990, 1991; Salowe, 1991).

Other than these bounds, little progress concerning the  $\beta_{L,d}$  was made until Bertsimas and van Ryzin (1990) developed exact expressions for the probabilistic minimum spanning tree and matching constants as  $d$  gets large. Specifically, they showed that  $\beta_{MST,d} \sim \sqrt{d}/2\pi e$  and  $\beta_{M,d} \sim (1/2)\sqrt{d}/2\pi e$  as  $d \rightarrow \infty$ .

The crowning achievement concerning the probabilistic constants was the determination of an exact expression for  $\beta_{MST,d}$  for all  $d \geq 2$  by Avram and Bertsimas (1992). The expression for  $\beta_{MST,d}$  is a series expansion, each term of which requires a rather difficult integration. Still, the first few terms of the series in dimension 2 have been computed to yield a numerical lower bound of  $\beta_{MST,2} \geq 0.599$ , which agrees well with experimental data. We note that the proof employed by Avram and Bertsimas relies strongly on the fact that a greedy construction is guaranteed to yield an MST. Unfortunately, such a construction fails for many objects of interest, including the TSP. It is therefore

evident that entirely new methods probably will be required to develop exact expressions for constants such as  $\beta_{TSP,d}$  that arise from NP-hard problems.

As a final note, we return to the rectilinear Steiner minimum tree problem. Bern (1988) showed that, for a Poisson process with intensity  $N$  on the unit square, the value of the rectilinear MST constant is at least that of the rectilinear Steiner tree constant plus 0.0014. This separation of the constants was increased by Hwang and Yao (1990), who also extended the results to include points uniformly distributed in the unit square.

## 8. THE CENTRAL LIMIT PROBLEM

The recent work of Avram and Bertsimas (1992) provides enticing progress on the important and long-standing problem of providing a central limit theory (CLT) for Euclidean functionals. Their method falls short of providing a CLT for the MST problem or the TSP, but it provides very useful CLTs for somewhat easier problems, including the  $k$ th nearest neighbor problem, the Delauney triangulation and the length of the Voronoi diagram.

There are two keys to the approach used by Avram and Bertsimas. The first is the observation of the applicability of the relatively recent CLT for dependent random variables due to Baldi and Rinott (1989). This new CLT deals with the dependence relations within a collection of random variables through the structure of a dependency graph  $G_n$  and offers an explicit Berry-Essen type bound where the maximal degree  $D_n G_n$  plays a critical role. The second key observation is that one achieves a considerable simplification of the CLT problem by first conditioning on the event  $A_n$  that, in the subdivision of  $[0, 1]$  into subcubes of approximate size  $n/\log n$ , each subcube contains at least one and at most  $e \log n$  of the sample points. The specifics of the problems handled by the Avram-Bertsimas method are such that once one conditions on  $A_n$ , one finds a problem that is within range of the Baldi-Rinott CLT. Any given problem has a number of details that must be directly resolved.

## 9. WORST-CASE GROWTH RATES

One engaging aspect of the asymptotic theory of combinatorial optimization of point functionals is the persistent similarity between the probabilistic rates of growth that have been reviewed and the behavior of the corresponding functionals in worst-case settings.

As a primary example of a worst-case growth rate, consider the *worst-case length* of an optimal traveling salesman tour in the unit  $d$ -cube:

$$(8) \quad \rho_{TSP}(n) = \max_{\substack{S \subset [0,1]^d \\ |S|=n}} \min_{T \in \mathcal{T}} \left( \sum_{e \in T} l|e| : T \text{ is a tour of } S \right).$$

In words,  $\rho_{TSP}(n)$  is the maximum length over all  $n$  point sets in  $[0, 1]^d$  that an optimal traveling salesman tour can attain. There is absolutely no probability theory here, for the point sets and tours are deterministic, yet there is a theory for this function that parallels the BHH theorem as closely as one could hope.

THEOREM 8 (Steele and Snyder, 1989). *As  $n \rightarrow \infty$ ,*

$$\rho_{TSP}(n) \sim \alpha_{TSP,d} n^{(d-1)/d},$$

where  $\alpha_{TSP,d} > 0$  is a constant depending only on the dimension  $d$ .

The proof of this theorem and several related results all call on the subadditivity and smoothness properties of the underlying functional. The following elementary lemma often helps focus one's investigation of these problems by putting a finger on one clear set of sufficient conditions.

LEMMA 4. *If  $\rho(1) = 0$  and there exists a constant  $c \geq 0$  such that for all integers  $m \geq 1$  and  $k \geq 1$ , we have:*

1.  $\rho(n+1) \leq \rho(n) + cn^{-1/d}$  and
2.  $m^{d-1} \rho(k) - m^{d-1} k^{(d-1)/d} r(k) \leq \rho(m^d k)$ ,

where  $r(k) \rightarrow 0$  as  $k \rightarrow \infty$ , then as  $n \rightarrow \infty$ ,

$$\rho(n) \sim \alpha n^{(d-1)/d}$$

for a constant  $\alpha \geq 0$ .

For the analysis of any particular problem, there is almost always serious work to be done to make the problem amenable to this lemma or its variant. Still, the lemma has already proved its effectiveness in a number of problems, including minimum matchings (Snyder, 1987), minimum spanning trees (Steele and Snyder, 1989), greedy matchings (Snyder and Steele, 1990) and rectilinear Steiner minimum trees (Snyder, 1991).

## 10. CONCLUDING REMARKS

In the areas in which this survey has been forced to make the most substantial omissions, there are good sources to which one can turn. One important class of results that we have touched upon concerns the study of discrepancies of sequences that are more general than random samples. This subject goes under the name of *irregularity of distribution*, and it is closely allied in many technical ways with the work surveyed here. The volume by Beck and Chen (1987) gives one a view of the subject that it would be wrong to try to reprise, though it would have been interesting to make more explicit some of the relationships to questions such as the MST and Steiner trees.

Another huge area of related problems that have not been engaged here falls in the domain of the recent book *Intersection and Decomposition Algorithms for Planar Arrangements*, by F. K. Agarwal (1991). The

work of K. Clarkson, D. Dobkin, H. Edelsbrunner, L. Guibas, E. Welzl and many others is surveyed there, and the volume offers a compelling view of the power that emerges from a detailed understanding of the way a set of lines cut up the plane. Duality is one of the powerful tools of that subject, so the restriction that is made here to focus on problems involving only "sets of points" and not "sets of lines" would not show up as a valid distinction in the context of that theory. Still, a division had to be drawn, and there can be practical differences even where there are formal isomorphisms.

A third class of related results that have only been touched on here concern bin packing and its relation to scheduling. This topic is surveyed by Coffman et al. in this issue as well as in the beautiful recent book by Coffman and Lueker (1991) that is devoted to the topic.

Galileo is supposed to have said, "To study science, one must speak the language of science, and the language of science is written in circles, lines, and squares." With luck, this survey reveals that there is something a priori appropriate about the probability theory of combinatorial optimization. The subject has not strayed far from motivations that would have had meaning for Galileo. Still, the development has been extensive, and the methods have become powerful and diverse.

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