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Series expansions for the Ising spin glass in general dimension

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We have developed 15th-order high-temperature series expansions for the study of the critical behavior of the Ising spin glass with nearest-neighbor exchange interactions each of which assumes the values $\pm J$ randomly. Series for the Edwards-Anderson spin-glass susceptibility (χ^{EA}) and two of its derivatives with respect to the ordering field have been evaluated for hypercubic lattices in general dimension, d. These extend previous general-dimension series by five terms. Certain measurable universal amplitude ratios have been estimated from the new series. Accurate critical data for $d=5$ and the first reliable estimates of the exponent β for $d=4$ and 5, are given. We quote $\gamma=1.73\pm0.03$, 2.00 ± 0.25, and $2.7^{+1.0}_{-0.6}$ and $\beta=0.95\pm0.04$, 0.9 ± 0.1, and 0.7 ± 0.2 in 5, 4, and three dimensions, respectively. Our results provide a smooth extrapolation between the mean-field results above six dimensions and experiments and simulations in physical dimensions. We relate our calculated derivatives of χ^{EA} to measurements of derivatives of the magnetization with respect to a uniform magnetic field.

I. INTRODUCTION

In this paper we report our results for Ising spin glasses[1–3] in general d-dimensional hypercubic lattices. Our results include series expansion estimates of critical exponents, critical temperatures, and certain universal amplitude ratios. Our results are compared to those from other series work, numerical simulations, and the ϵ expansion. Our results enable reliable smooth extrapolations to be made from mean-field results above six dimensions to the physically relevant case of three dimensions.

Spin-glass (SG) systems have been subjected to intensive study via experiments,[2–4] simulations,[5–7] series expansions,[8–13] the renormalization-group ϵ expansion,[14–18] and various approximate theories during the last decade. Magnetic SG's exhibit interesting phenomena that also occur in other materials such as orientational glasses[19] and superconductors.[20–23] A SG can arise when different magnetic interactions compete randomly with each other and thus cause individual magnetic spins to be frustrated. Many distinct ground states are possible, as well as many metastable states, and the system takes a long time to relax after any perturbation. Irreversibility is observed in experimental measurements of SG materials.[2–4] In some other glassy systems there is a multiplicity of choices leading to apathy,[24] rather than frustration, but the end results of multiple ground states, metastability, and long relaxation times are ubiquitous.

In a SG, the usual magnetic order parameter, i.e., the average magnetization, is zero, and the usual magnetic susceptibility does not diverge as temperature is reduced. However, new order parameters[1] that relate to time averages attain finite values below the spin-glass transition, where the related spin-glass susceptibilities diverge. The spin-glass transition can, in principle, be characterized by critical exponents just like those used to describe the transitions in the simple Ising spin model or the percolation process. The exponent γ is associated with the divergence of the SG order-parameter susceptibility (related to the usual nonlinear susceptibility) and β characterizes the singularity in the order parameter. In practice, the description of SG transitions is far more complicated than is the case for the usual Ising model or percolation process.

Our understanding of critical behavior for Ising models and percolation is now rather complete. Exact results exist[25,26] for two dimensions (2D) and at the upper critical dimensions, d_c, of four and six, respectively. For intermediate dimensions, exact series expansions have given high-quality numerical data for critical exponents,[27,28] amplitude ratios,[29] and critical temperatures and thresholds, that are in excellent agreement with the exact results in low dimensions, with the field theoretic results[29,30] in $d=d_c-\epsilon$ dimensions, and with simulation calculations,[31,32] of critical exponents and amplitude ratios, critical temperatures, and percolation thresholds. Different exponents can be measured independently to confirm the validity of scaling and hyperscaling. Interpolation between results in different dimensions is quite smooth, and the agreement with the exact results at both ends of the dimension range lends a great deal of certainty to the numerical values.

For SG's, the current situation is far less satisfactory. Extensive simulations[5-7] and series expansions[8-13] have given some numerical results for Ising SG's in 3D and 4D, and our new results are in broad agreement with these existing calculations. However, there appears to be no long-range SG order in two dimensions, and at present no exact results are available to guide numerical calculations from the lower end. Although a renormalization-group ϵ expansion for critical exponents exists near $d_c=6$ (Refs. 14–18) the asymptotic series for the critical exponents are ill converged even in the vicinity of five dimensions and therefore are of no real use for extrapolation towards 3D. We have tried to use Padé analysis on the existing[17] ϵ expansion to order ϵ^3, and found a very large scatter of the results, even in $d=5$.[33] There have also been problems within field theory.[34] To the best of our knowledge there exist no published SG critical amplitude ratios in any dimension. Experimental measurements of critical exponents of SG's have been made but are not very precise.

In order to provide interpolation between dimensions and to obtain reliable equilibrium estimates of critical exponents, amplitude ratios, and critical temperatures in general dimension we have undertaken a comprehensive study of the Ising SG defined by the Hamiltonian

$$\mathcal{H}=-\sum_{\langle ij\rangle} J_{ij}S_iS_j - H\sum_i S_i \ , \qquad (1.1)$$

where $\langle ij\rangle$ denotes a sum over pairs of nearest neighbors (i,j), and $S_i=\pm 1$, while the nearest-neighbor exchange variables, $J_{ij}=J_{ji}$, independently assume the values $\pm J$ randomly with equal probability. In Eq. (1.1) we include the effect of a uniform nonrandom external field, H. We treat the case of quenched randomness so that the quenched averaged free energy per spin, F, is given by

$$F=-(kT/N)[\ln \mathrm{Tr} e^{-\mathcal{H}/k_BT}]_{\mathrm{av}} \ , \qquad (1.2)$$

where $[\ ]_{\mathrm{av}}$ denotes an average over all configurations of the J's and N is the total number of spins. We have studied the phase transition at the critical temperature T_c, via 15th-order power-series expansions in the high-temperature variable $w=\tanh^2(J/k_BT)$.

A convenient way to calculate the configurational average in Eq. (1.2) is to introduce the replica Hamiltonian,[1]

$$\mathcal{H}^{(n)}=-\sum_{\langle ij\rangle}\sum_{\alpha=1}^{n} J_{ij}S_i^\alpha S_j^\alpha - h\sum_i \sum_{1\le\alpha<\beta\le n} S_i^\alpha S_i^\beta \ , \qquad (1.3)$$

where h is the field conjugate to the Edwards-Anderson SG order parameter $Q=[\langle S_i\rangle^2]_{\mathrm{av}}$, and there are n replicas. Then we define

$$F_{\mathrm{rep}}=\lim_{n\to 0}\left[-\frac{2k_BT}{n(n-1)N}\ln \mathrm{Tr}[e^{-\mathcal{H}^{(n)}/k_BT}]_{\mathrm{av}}\right] . \qquad (1.4)$$

We obtained expansions for Γ_k, for $k=2,3,4$, where we define Γ_k as the kth derivative of F_{rep} with respect to the SG ordering field:

$$\Gamma_k=-\left[\frac{\partial^k \tilde{F}_{\mathrm{rep}}}{\partial \tilde{h}^k}\right]\Bigg|_{h=0} , \quad k=2,3,4 \ , \qquad (1.5)$$

where $\tilde{F}_{\mathrm{rep}}=F_{\mathrm{rep}}/k_BT$ and $\tilde{h}=h/k_BT$. The second derivative, Γ_2, is the Edwards-Anderson (EA) susceptibility[1]

$$\Gamma_2\equiv\chi^{\mathrm{EA}}=N^{-1}\sum_{i,j}[\langle S_iS_j\rangle^2]_{\mathrm{av}} \ , \qquad (1.6)$$

where $\langle\ \rangle$ denotes a thermal average at a fixed configuration of the J's. Explicit expressions for Γ_3 and Γ_4 are given in Eqs. (3.1) and (3.2) below. As will be discussed in Sec. IV, the expansions

$$\Gamma_k(w)=a_k(0,0)+\sum_{m=1}^{15}\sum_{n=1}^{m} a_k(m,n)w^m d^n \ , \qquad (1.7)$$

are fitted to critical behavior with corrections to scaling of the form

$$\Gamma_k(w)=A_k(w_c-w)^{-\gamma_k}[1+a_k(w_c-w)^{\Delta_1} + b_k(w_c-w)+\cdots] \ , \qquad (1.8)$$

with $\gamma_k=\gamma+(k-2)\Delta$, where the gap exponent Δ is equal to $(\gamma+\beta)$. In Eq. (1.8) we have allowed for both nonanalytic and analytic corrections to scaling as is discussed in more detail in Sec. IV below. For $d>6$, $\gamma=\beta=1$. Our results for χ^{EA}, Γ_3, and Γ_4 give three independent estimates of the two exponents β and γ in general dimension and the possibility of studying universal amplitude ratios[29,35] such as

$$R\equiv\frac{\Gamma_2\Gamma_4}{(\Gamma_3)^2}\sim\frac{A_2A_4}{(A_3)^2} \ . \qquad (1.9)$$

Here and below the symbol $\sim$ means "asymptotically equal" for $w\to w_c$. We have also obtained the first ϵ-expansion results for amplitude ratios in the Ising SG.

The earliest steps in the generation of series for the SG were made by Fisch and Harris,[8] (hereafter denoted as FH) who generated 10th-order general dimension series for the EA susceptibility alone. The generation of these series represented a major breakthrough, but an unfortunate choice of analysis method led to problems of interpretation below five dimensions. While writing up our calculations we received preprints of the Singh-Fisher[36,37] calculations to 10th order for the gap exponents in general dimension; our results are in broad agreement with these values but are substantially more precise above three dimensions. Longer series for two, three, and four dimensions were obtained by Singh and Chakravarty[9,10] (SC) using the star graph approach, but the present study gives the first long series for the higher derivatives Γ_3 and Γ_4 in $d \neq 3$ and five extra terms in the series for χ^{EA} for general dimension. In 3D, SC also calculated a different type of susceptibility series,

$$\Gamma' = \frac{1}{N} \sum_{i,j} [\langle S_i S_j \rangle^2]^2_{\text{av}} , \tag{1.10}$$

which scaling indicates has a dominant critical exponent

$$\gamma' = (4 - d - 2\eta)\nu = \gamma - 2\beta$$

with $\gamma = (2-\eta)\nu$. The exponent ν describes the divergence of the correlation length and η the behavior of the pair correlation function at criticality, $\mathcal{G} \sim 1/r^{d-2+\eta}$. Together with our calculation of the Γ_k's, we therefore have four independent measurements of combinations of critical exponents. These determinations enable additional useful tests of self-consistency to be made. The new enumerations are part of a project to calculate extended series for many systems in general dimension, which has recently been reviewed in Ref. 38.

The earlier series expansion calculations and those that we will describe below, as well as the ϵ-expansion studies, are all carried out for the equilibrium state. This is in contrast with many experimental measurements that are made dynamically and with problems with equilibration that may arise in simulations. In general, quite good static experimental measurements of γ can be made for SG's, but many recent β estimates have been deduced from dynamic scaling analyses.[39] The long relaxation times appear to greatly complicate analysis of the experimental data when dynamic scaling is used, and it would be desirable to obtain accurate β estimates via experimental measurements of the different susceptibilities as we do in this study. We note that while writing up this paper we received several preprints that relate to improved analyses of experimental data.[40] These show both that there were problems in the past and that the situation is still not entirely clarified, especially with regard to correction to scaling terms.

The series for the SG are far more difficult to analyze than those for percolation or Ising models. Large corrections to scaling have been observed in simulations in the lower dimensions[41] and are probably also present in the series. In addition, the series may have substantial analytic corrections. We have undertaken test series studies on series that mimic the SG ones in these aspects, and our analysis is based, in part, on conclusions drawn from these. Our results above three dimensions are well converged, and we quote $\gamma = 1.73 \pm 0.03$ and $\beta = 0.95 \pm 0.04$ in five dimensions, $\gamma = 2.00 \pm 0.25$ and $\beta = 0.9 \pm 0.1$ in four dimensions, and $\gamma = 2.7^{+1.0}_{-0.6}$ and $\beta = 0.7 \pm 0.2$ in three dimensions. These give a smooth interpolation between the mean-field results in six dimensions and other calculations in three dimensions. For various dimensions we give a comprehensive summary of both extant results and our new estimates for critical exponents in Table I and for critical temperatures in Table II (Ref. 42). We have also determined the amplitude ratio R in all dimensions and find $R = 2.77 \pm 0.08$ in five dimensions, estimates of 1.9 and 3.8 leading to $R = 2.8 \pm 1.5$ in four dimensions, and $R = 1.7 \pm 0.4$ from a direct evaluation in 3D. We also obtain an indirect estimate of $R = 1.85 \pm 0.4$ in 3D, and develop a connection between R and an experimentally measurable quantity, related to the dependence of the magnetization on a uniform external field.

This paper is arranged as follows. Section II contains a discussion of experimental results, scaling for SG's, and the relation (derived in Appendix A) between experimental derivatives of the free energy with respect to the uniform field H and those with respect to the SG ordering field h. Here we also discuss the ϵ-expansion results, including new results for certain universal amplitude ratios. The generation of the new series is described in Sec. III, and the series coefficients are presented in Table III. Details of the series generation are given in Appendixes B and C. A discussion of analysis methods for the SG series is given in Sec. IV, and new results of the test series analysis are placed in Appendix D. We present our results for the values of the critical exponents in Sec. V and for the universal amplitude ratios in Sec. VI. A general discussion of our results and their comparison with other calculations is given in Sec. VII.

II. EXPERIMENTAL RESULTS, SCALING, AND THE ϵ EXPANSION

There is an excellent discussion of older SG measurements concerning critical exponents in Ref. 2. Although there are few natural SG's with Ising symmetry, it has been shown, for example,[43–45] that Ruderman-Kittel-Kasuya-Yosida SG's, which contain some randomly anisotropic Dzyaloshinsky-Moriya interactions crossover from Heisenberg to Ising critical behavior. Typical older estimates for such glasses, found experimentally in 3D, are[46] $\gamma = 2.2 \pm 0.1$, $\delta = 3.1 \pm 0.2$, and $\beta = 1.0 \pm 0.1$ for AgMn and $\gamma = 2.3 \pm 0.2$ and $\delta = 5.2 \pm 0.5$ (Ref. 47) for $Fe_{10}Ni_{70}Pd_{20}$. The exponent δ describes the dependence of the order parameter on the ordering field at T_c and is equal to $1 + \gamma/\beta$ by scaling. Recent reanalyses of some data for compounds including $Cd_{0.6}Mn_{0.4}Te$ by the AT&T group[40] have shown that earlier estimates of $\gamma = 3.3 \pm 0.3$ for this system were far too low. Values between 4.28 and 4.4 are now proposed from improved static scaling analyses based on linear rather than logarithmic plots. It is not clear whether $Cd_{0.6}Mn_{0.4}Te$ is a short-range Ising system that can be directly compared to our series results. The only estimates that we are aware of for systems that are explicitly claimed to be short-range Ising are $\beta = 0.7 \pm 0.1$ and $\beta = 0.4 \pm 0.1$ from two different types of dynamic scaling analyses of the same experiments on the Ising SG $Fe_{0.5}Mn_{0.5}TiO_3$ by Norblad *et al.*[39] The AT&T reanalysis of these data suggests $\beta=0.56$. We shall discuss these analyses further in Sec. VII, in the light of the suggestions that we make below for the measurement of β and of critical amplitude ratios.

The different order-parameter susceptibilities that we have calculated are defined above in Eq. (1.5). Since the experimentalists usually measure magnetization as a function of applied uniform magnetic field H, we give the connection between the measured quantities and our results. The magnetization (per spin) M is obtained by taking the first derivative of the configurationally averaged free energy F with respect to H so that

$$M=-\left[\frac{\partial F}{\partial H}\right]_{H=0}=\frac{1}{N}\left[\sum_i \langle S_i\rangle\right]_{\text{av}}=0\ . \qquad (2.1)$$

The second derivative yields the usual susceptibility, χ_1,

$$\chi_1\equiv-\left[\frac{\partial^2 F}{\partial H^2}\right]_{H=0}=\frac{1}{Nk_BT}\sum_{ij}[\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle]_{\text{av}}\ . \qquad (2.2)$$

As is well known,[1,2] χ_1 does not diverge for SG's but ex-

TABLE I. A selection of estimates of dominant critical exponents for $d<6$. When it has proved possible to deduce additional results (from the estimates of that calculation alone) via scaling and/or hyperscaling we have quoted the central values in parentheses.

Reference	γ	β	Δ	γ'	$\nu\ \left[=\frac{2\beta+\gamma}{d}\right]$	$\eta=2-\frac{\gamma}{\nu}$
			Five dimensions			
Epsilon expansion[a]						
First order	2					
Series						
Singh-Fisher[b]			2.0			
Fisch-Harris[c]	1.95					
Fisch-Harris[d]	2.23					
This work	1.73±0.03	0.95±0.04	2.68±0.05		(0.73)	(−0.38)
			Four dimensions			
Epsilon expansion[a]						
First order	3					
Series						
Singh-Chakravarty						
(All approximants)[e]	2.0±0.4					
(Highest approximants)[e]	1.855±0.041					
Singh-Fisher[b]			2.4			
This work	2.00±0.25	0.9±0.1	2.9±0.3		(0.95)	(−0.11)
Simulation[f]	1.8±0.4				0.8	−0.3±0.15
			Three dimensions			
Epsilon expansion[a]						
First order	4					
Series						
Singh-Chakravarty						
(All approximants)[e]	2.9±0.3	(0.47)	(3.37)	1.96±0.19	1.3±0.2	−0.25±0.17
(Highest approximants)[e]	2.94±0.13					
Singh-Fisher[b]			3.4			
This work	$2.7^{+1.0}_{-0.6}$	0.7±0.2	3.4±0.5	1.5±0.3	(1.37)	(0.03)
Simulation						
Bhatt-Young[g]					1.3±0.3	−0.3±0.2
Bhatt-Young[h]	3.2	0.5	3.7		1.4	−0.28
Ogielski-Morgenstern[i]					1.2±0.1	≈0
Ogielski[j]	2.9±0.3	(0.5)	(3.4)		1.3±0.1	−0.22±0.05
Experiment[k]	2.3±0.2					
Experiment[l]		0.4–0.7				

[a]Reference 17.
[b]Reference 36.
[c]Reference 8.
[d]Reference 8 using a fit to the Rudnick-Nelson form.
[e]Reference 9.
[f]Reference 7.
[g]Reference 5 ($T\geq 1.2$).
[h]Reference 5 ($T_c=1.2$).
[i]Reference 6.
[j]Reference 7.
[k]An average of values quoted in Ref. 2.
[l]Reference 39.

TABLE II. Critical values of $w_c = \tanh^2(J/kT_c)$.

Reference	$d=9$	$d \geq 6$ $d=8$	$d=7$	$d=6$
		$1/\sigma$ expansion[a]		
Five terms	0.060 73	0.0696	0.0818	0.1002
		Series expansion		
Singh-Fisher[a]		0.070±0.001	0.083±0.001	0.102±0.002
Fisch-Harris[b]				0.1023
Fisch-Harris[c]				0.1019
This work	0.059 66±0.000 15	0.067 98±0.000 15	0.079 14±0.000 15	0.101 69±0.000 3

Reference	$d=5$	$d \leq 5$ $d=4$	$d=3$
		$1/\sigma$ expansion[a]	
Five terms	0.1322	0.2133	1.2110
Four terms		0.1932	0.5036
Three terms		0.1772	0.3321
		Series expansion	
Singh-Chaakravarty[d]		0.21±0.01	0.48±0.04
Singh-Fisher[a]	0.139±0.002	0.21±0.01	0.48±0.04
Fisch-Harris[b]	0.1400		
Fisch-Harris[c]	0.1400		
Guttmann[e]		0.2±0.1	0.5±0.1
This work	0.1372±0.0008	0.207±0.008	$0.40^{+0.06}_{-0.04}$
		Simulation	
Bhatt-Young[f]			$0.46^{+0.12}_{-0.04}$
Ogielski-Morgenstern[g]			0.478±0.013

[a]Reference 36.
[b]Reference 8.
[c]Reference 8, using a fit to the Rudnick-Nelson form.
[d]Reference 9.
[e]Reference 42.
[f]Reference 5.
[g]Reference 6.

hibits a cusp at T_c. This cusp has its origin in the last term in Eq. (2.2), which behaves as the Edwards-Anderson order parameter $[\langle S_i \rangle^2]_{\rm av} \sim (T_c - T)^\beta$ for $T < T_c$ and is zero for $T > T_c$, of course.

One may likewise define higher derivatives of the free energy with respect to H. For $H=0$, only even derivatives are nonzero because odd derivatives are expressed in terms of products of averages of odd numbers of spin operators. Such averages vanish by symmetry in the disordered or in the SG phase, since these phases do not support any local order in $\langle S_i \rangle$. Thus, for instance, $\partial^3 F/\partial H^3$ contains contributions of the form $[\langle S_i S_j S_k \rangle]_{\rm av}$, $[\langle S_i S_j \rangle \langle S_k \rangle]_{\rm av}$ and $[\langle S_i \rangle \langle S_j \rangle \langle S_k \rangle]_{\rm av}$, all of which vanish.

The fourth derivative, or the nonlinear susceptibility, is the first to diverge as T_c is approached from above:

$$\chi_2 \equiv -\left[\frac{\partial^4 F}{\partial H^4}\right]_{H=0} = N^{-1}(k_B T)^{-3} \sum_{i,j,k,l} ([\langle S_i S_j S_k S_l \rangle]_{\rm av} - 3[\langle S_i S_j \rangle \langle S_k S_l \rangle]_{\rm av}$$
$$-4[\langle S_i \rangle \langle S_j S_k S_l \rangle]_{\rm av} + 12[\langle S_i \rangle \langle S_j \rangle \langle S_k S_l \rangle]_{\rm av}$$
$$-6[\langle S_i \rangle \langle S_j \rangle \langle S_k \rangle \langle S_l \rangle]_{\rm av}) \tag{2.3a}$$

$$= N^{-1}(k_B T)^{-3} \sum_{i,j,k,l} ([\langle S_i S_j S_k S_l \rangle]_{\rm av} - [\langle S_i S_j \rangle \langle S_k S_l \rangle]_{\rm av}$$
$$- [\langle S_i S_k \rangle \langle S_j S_l \rangle]_{\rm av} - [\langle S_i S_l \rangle \langle S_j S_k \rangle]_{\rm av}) \ . \tag{2.3b}$$

TABLE III. Coefficients, $a_k(m,n)$, of the expansions of Γ_k for $k=2,3$ and 4 as defined in Eq. (1.7).

m	n	$a(m,n)\times(15/2^n)$	m	n	$a(m,n)\times(15/2^n)$
Γ_2					
1	1	15	2	1	-15
2	2	15	3	1	15
3	2	-30	3	3	15
4	1	195	4	2	-60
4	3	-45	4	4	15
5	1	-885	5	2	810
5	3	-120	5	4	-60
5	5	15	6	1	-2445
6	2	390	6	3	930
6	4	-165	6	5	-75
6	6	15	7	1	30255
7	2	-28410	7	3	5505
7	4	1200	7	5	-195
7	6	-90	7	7	15
8	1	122115	8	2	-61380
8	3	-13185	8	4	4905
8	5	1605	8	6	-210
8	7	-105	8	8	15
9	1	-1570565	9	2	1760990
9	3	-599650	9	4	41650
9	5	4425	9	6	2130
9	7	-210	9	8	-120
9	9	15	10	1	-11416035
10	2	9358505	10	3	-1492160
10	4	-249290	10	5	25895
10	6	3840	10	7	2760
10	8	-195	10	9	-135
10	10	15	11	1	112166115
11	2	-149692660	11	3	70473785
11	4	-13176020	11	5	619285
11	6	9180	11	7	2940
11	8	3480	11	9	-165
11	10	-150	11	11	15
12	1	1356823945	12	2	-1411141535
12	3	449856665	12	4	-31880070
12	5	-6036925	12	6	536375
12	7	-9230	12	8	1530
12	9	4275	12	10	-120
12	11	-165	12	12	15
13	1	-10136173845	13	2	15758058690
13	3	-9255013080	13	4	2517305150
13	5	-299763115	13	6	8405330
13	7	478210	13	8	-29770
13	9	-570	13	10	5130
13	11	-60	13	12	-180
13	13	15	14	1	-193131348405
14	2	233990140580	14	3	-100234812405
14	4	17313953240	14	5	-504638515
14	6	-152530370	14	7	7600450
14	8	446785	14	9	-52590
14	10	-3525	14	11	6030
14	12	15	14	13	-195
14	14	15	15	1	1169084029295
15	2	-2047560422262	15	3	1409447709675
15	4	-483873855650	15	5	85884651315
15	6	-6964848988	15	7	109073405
15	8	7007120	15	9	445820
15	10	-77570	15	11	-7485
15	12	6960	15	13	105
15	14	-210	15	15	15
Γ_3					
1	1	-180	2	1	360
2	2	-360	3	1	-660
3	2	1260	3	3	-600
4	1	-3240	4	2	-900
4	3	2880	4	4	-900
5	1	26460	5	2	-23400
5	3	-900	5	4	5400

TABLE III. (*Continued*).

m	n	$a(m,n)\times(15/2^n)$	m	n	$a(m,n)\times(15/2^n)$
5	5	-1260	6	1	16200
6	2	67680	6	3	-58320
6	4	-1200	6	5	9000
6	6	-1680	7	1	-782100
7	2	630180	7	3	47160
7	4	-116280	7	5	-2700
7	6	13860	7	7	-2160
8	1	-2233080	8	2	-256500
8	3	1135440	8	4	97920
8	5	-207000	8	6	-6660
8	7	20160	8	8	-2700
9	1	45376860	9	2	-49862880
9	3	13869060	9	4	464520
9	5	266220	9	6	-341040
9	7	-14700	9	8	28080
9	9	-3300	10	1	303184440
10	2	-185950080	10	3	-21435360
10	4	20009160	10	5	278160
10	6	619920	10	7	-529200
10	8	-28800	10	9	37800
10	10	-3960	11	1	-3868113060
11	2	5289043980	11	3	-2423335740
11	4	372966120	11	5	-790680
11	6	772080	11	7	1253880
11	8	-781920	11	9	-51300
11	10	49500	11	11	-4680
12	1	-41567727720	12	2	37812686220
12	3	-6866812680	12	4	-1973610000
12	5	616364760	12	6	-25626960
12	7	2134560	12	8	2294460
12	9	-1108680	12	10	-84900
12	11	63360	12	12	-5460
13	1	420544286460	13	2	-666222762840
13	3	387263226660	13	4	-96890206800
13	5	7987077720	13	6	406796280
13	7	-56914080	13	8	4517040
13	9	3903120	13	10	-1517400
13	11	-132660	13	12	79560
13	13	-6300	14	1	6725410425000
14	2	-7518508002720	14	3	2568718933680
14	4	-104752561920	14	5	-99489656160
14	6	14478893760	14	7	198505920
14	8	-98288520	14	9	7992120
14	10	6279120	14	11	-2013840
14	12	-198000	14	13	98280
14	14	-7200	15	1	-58749828442740
15	2	103130213675364	15	3	-69984222095040
15	4	22832363075580	15	5	-3490184663700
15	6	152472320436	15	7	11881141020
15	8	-1438080	15	9	-154808340
15	10	12500220	15	11	9661320
15	12	-2601000	15	13	-284700
15	14	119700	15	15	-8160
Γ_4					
1	1	2760	2	1	-7890
2	2	8700	3	1	21240
3	2	-42240	3	3	21000
4	1	1050	4	2	117330
4	3	-135900	4	4	43050
5	1	-294600	5	2	145680
5	3	378360	5	4	-339600
5	5	78960	6	1	1090170
6	2	-2780490	6	3	1049400
6	4	961200	6	5	-725550
6	6	133560	7	1	-5012040
7	2	12001800	7	3	-12923880
7	4	3705600	7	5	2130600
7	6	-1391040	7	7	212400
8	1	-20893470	8	2	19333950
8	3	24002520	8	4	-36161940

TABLE III. (*Continued*).

m	n	$a(m,n)\times(15/2^n)$	m	n	$a(m,n)\times(15/2^n)$
8	5	9630450	8	6	4311090
8	7	-2462040	8	8	321750
9	1	747752920	9	2	-883558600
9	3	389990120	9	4	2933440
9	5	-84516480	9	6	20976000
9	7	8149680	9	8	-4096800
9	9	468600	10	1	4898862510
10	2	-3661546900	10	3	-112552340
10	4	695488390	10	5	-5266780
10	6	-179590350	10	7	40586280
10	8	14589360	10	9	-6489450
10	10	660660	11	1	-101839012680
11	2	133308759920	11	3	-64162173160
11	4	12443570320	11	5	133565440
11	6	47843880	11	7	-356552640
11	8	71971200	11	9	24953400
11	10	-9873600	11	11	906360
12	1	-1161392058170	12	2	1120230485700
12	3	-273440216220	12	4	-32583147760
12	5	19885870380	12	6	-334825220
12	7	225270200	12	8	-669170430
12	9	119173050	12	10	41040450
12	11	-14525940	12	12	1214850
13	1	14957584143000	13	2	-23266942206000
13	3	13819538965480	13	4	-3859450427120
13	5	462726617760	13	6	-4925598920
13	7	434008880	13	8	623665120
13	9	-1195894440	13	10	186500400
13	11	65230440	13	12	-20769840
13	13	1596000	14	1	252036632455770
14	2	-288401955236350	14	3	105433350441430
14	4	-7267585664430	14	5	-4130954100750
14	6	957354421610	14	7	-55751977060
14	8	3927101870	14	9	1402319190
14	10	-2046955530	14	11	278104200
14	12	100601280	14	13	-28978950
14	14	2060400	15	1	-2614889817754120
15	2	4562983300326792	15	3	-3139324953436640
15	4	1076653601309680	15	5	-186016795230280
15	6	11642649023368	15	7	799152280600
15	8	-130515881360	15	9	12164098440
15	10	2825306760	15	11	-3372378600
15	12	397370880	15	13	151056360
15	14	-39580800	15	15	2619360

To interpret the above expression, we simplify it for the special case of the short-range $\pm J$ model of Eq. (1.1). In this model the average interaction J_{ij} is zero, and there are no correlations between different J_{ij}'s. We will later indicate how our results should be modified for a more general SG model. For the short-range case, the first term, $[\langle S_iS_jS_kS_l\rangle]_{\rm av}$, is nonzero only for $i=j$ and $k=l$, for $i=k$ and $j=l$, or for $i=l$ and $j=k$. Thus, its contribution is $3N(N-1)+N$. The other terms are nonzero when $i=j$ and $k=l$, when $i=k$ and $j=l$, or when $i=l$ and $j=k$. Thus, their contribution is $-3N^2-6\sum_{i,j}[\langle S_iS_j\rangle^2]_{\rm av}(1-\delta_{i,j})$, where $\delta_{i,j}$ is the Kronecker δ. Therefore,

$$\chi_2=4-6N^{-1}(k_BT)^{-3}\sum_{i,j}[\langle S_iS_j\rangle^2]_{\rm av}\ . \tag{2.4a}$$

Asymptotically, the second term dominates so

$$\chi_2\sim -6N^{-1}(k_BT)^{-3}\sum_{i,j}[\langle S_iS_j\rangle^2]_{\rm av}\sim -6\chi^{\rm EA}/(k_BT)^3\ . \tag{2.4b}$$

Thus the divergence in the nonlinear susceptibility is proportional[48–50] to the Edwards-Anderson susceptibility, which diverges with the exponent γ. Similarly we derived the result

$$\chi_3\equiv-\left[\frac{\partial^6F}{\partial H^6}\right]_{H=0}=N^{-1}(k_BT)^{-5}\sum_{i,j,k,l,m,n}([\langle S_iS_jS_kS_lS_mS_n\rangle]_{\rm av}-15[\langle S_iS_j\rangle\langle S_kS_lS_mS_n\rangle]_{\rm av}+30[\langle S_iS_j\rangle\langle S_kS_l\rangle\langle S_mS_n\rangle]_{\rm av})\ . \tag{2.5a}$$

For the model of Eq. (1.1) this is

$$\chi_3=N^{-1}(k_BT)^{-5}\left[\sum_{i,j,k}240[\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_kS_i\rangle]_{\rm av}-480\sum_{i,j}[\langle S_iS_j\rangle^2]_{\rm av}+256N\right], \tag{2.5b}$$

which asymptotically is

$$\chi_3\sim 240N^{-1}(k_BT)^{-5}\sum_{i,j,k}[\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_kS_i\rangle]_{\rm av}. \tag{2.5c}$$

Referring to the explicit expression for Γ_3 in Eq. (3.1), below, we see that

$$\chi_3\sim -60\Gamma_3/(k_BT)^5 . \tag{2.6}$$

In Appendix A we show that all the even derivatives of the free energy with respect to H (which can be measured experimentally), are asymptotically related to the order-parameter susceptibilities Γ_k of Eq. (1.5). The result of Appendix A is

$$\frac{\chi_k}{k_BT}\equiv-\frac{1}{k_BT}\left[\frac{\partial^{2k}F}{\partial H^{2k}}\right]_{H=0}\sim-(\lambda/k_BT)^{2k}(2k-1)![(k-1)!]^{-1}\Gamma_k . \tag{2.7}$$

For the model of Eq. (1.1) the scale factor λ is unity. For more general models this nonuniversal constant need not be unity but, as noted in Appendix A, will reflect the range of the short-range correlations of the J_{ij}'s. Equation (2.7) indicates the equivalence (up to a scale factor) between H^2, the square of the uniform field, and h, which may be interpreted as the variance of the random field. This equivalence implies that our series expansions for the Γ_k provide information on the experimentally accessible derivatives of the free energy with respect to the uniform field.

Notice that the magnetization M is related to the χ_k simply through the Taylor expansion of M in terms of H,

$$M=\chi_0H+\frac{1}{3!}\chi_2H^3+\frac{1}{5!}\chi_3H^5+\frac{1}{7!}\chi_4H^7+\cdots . \tag{2.8}$$

A particular application of Eq. (2.7) is to elucidate the connection between universal amplitude ratios involving the Γ_k, such as R of Eq. (1.9), and experimentally observable universal quantities such as $\chi_2\chi_4/(\chi_3)^2$. We obtain

$$\chi_2\chi_4/(\chi_3)^2=7R/5 , \tag{2.9}$$

and stress that this relation does not depend on the nonuniversal parameter λ as long as there is a spin-glass phase transition.

We conclude this section with a brief review of the results obtained using the renormalization group in $6-\epsilon$ dimensions.[14–18] From the field theoretic formulation with replicas, values for critical exponents, amplitude ratios, and universal scaling functions can be found. First we quote the values of the critical exponents that were directly obtained[17]:

$$\eta=-0.3333\epsilon+1.2593\epsilon^2+2.5367\epsilon^3 \tag{2.10a}$$

and

$$\nu^{-1}=2-1.6666\epsilon+8.0185\epsilon^2+1.6969\epsilon^3 . \tag{2.10b}$$

Other exponents can be calculated by using scaling relations, in particular,

$$\gamma=1+\epsilon-3.8056\epsilon^2-9.2971\epsilon^3 , \tag{2.11a}$$

$$\beta=1+0.5\epsilon-3.2778\epsilon^2-4.9503\epsilon^3 . \tag{2.11b}$$

In principle, these expansions can be used to calculate numerical estimates in particular dimensions, but in practice different approximants give erratic results even for $\epsilon=1$. These difficulties were mentioned in Ref. 2 and will be discussed in detail in Ref. 33.

A particularly useful field theoretic calculation was made by Pytte and Rudnick,[18] who performed a renormalization-group analysis in the ordered phase and derived the equation of state. Elsewhere[51] we use their formulation to obtain results for a hierarchy of universal amplitude ratios of the type[29,35]

$$\Gamma_k\Gamma_l/(\Gamma_m\Gamma_n) \text{ with } k+l=m+n . \tag{2.12}$$

For instance, to order ϵ we find that the quantity defined in Eq. (1.9) is given as

$$R=3[1+\tfrac{1}{4}\epsilon+O(\epsilon^2)] . \tag{2.13}$$

This detailed renormalization-group calculation,[51] based on the work of Ref. 18, yields the result

$$\Gamma_k=A_{k,0}t^{-\gamma_{k,0}}[1+a_0(t^{-\epsilon/2}-1)/\epsilon]^{2(\gamma_k-\gamma_{k,0})/\epsilon}=A_kt^{-\gamma_k}(1+gt^{\epsilon/2})^{2(\gamma_k-\gamma_{k,0})/\epsilon} , \tag{2.14}$$

where $t=T-T_c$, and $\gamma_{k,0}=2k-3$ is the mean-field value of γ_k. A similar form was first derived for the susceptibility of an ordinary n-vector model by Rudnick and Nelson,[52] and gives logarithmic corrections at $d=6$. The constants appearing in Eq. (2.14) are nonuniversal, but it is clear that if a_k denotes the amplitude of the correction to scaling term [Eq. (1.8) with $\Delta_1=\epsilon/2$], then a_k is proportional to $(\gamma_k-\gamma_{k,0})$, and we have

$$a_2:a_3:a_4=2:5:8, \quad \text{etc.} \tag{2.15}$$

In addition to the equation of state, we also derived[51] the result that

$$\Gamma'\Gamma_3^2/\Gamma_2^5=-5.818/\epsilon+O(1) . \tag{2.16}$$

The field theoretical formulation can also be used[49,50] to write a scaling form for all the even derivatives, $\Gamma_k(T,h)$. Replacing h by H^2, we have

$$\chi_k(T,H)\sim(T-T_c)^{-\gamma_k}f_k^{\pm}(H^2/(T-T_c)^{\Delta}, g(T-T_c)^{\Delta_1}) , \tag{2.17}$$

where $\gamma_k=\gamma+(k-2)\Delta$ (with $\Delta=\beta+\gamma$) and the $\pm$ notation denotes the different functions for $T<T_c$ and $T>T_c$. Note that $\gamma_1=-\beta$ as expected for $T<T_c$. In Eq. (2.17) we incorporated the leading irrelevant parame-

ter, g, yielding the leading nonanalytic correction to scaling. The fact that all the derivatives $(\partial^{2k}F/\partial H^{2k})_{H=0}$ arise as derivatives with respect to H^2 of a single function

$$f_2(H^2/(T-T_c)^{\Delta}, g(T-T_c)^{\Delta_1})$$

indicates that all of them have corrections with the same exponent $\Delta_1>0$.

One can use the scaling form [Eq. (2.17)], as well as Eq. (2.8) to obtain the following relation:

$$M-\chi_0 H=H(T-T_c)^{\beta}f(H^2/(T-T_c)^{\Delta}) \ . \tag{2.18}$$

Therefore, experimental data, when plotted as $(M-\chi_0 H)/H(T-T_c)^{\beta}$ versus $H^2/|T-T_c|^{\Delta}$, will collapse onto a single universal scaling function (having one adjustable nonuniversal scale factor associated with each axis), and can be used to determine β and Δ.

III. GENERATION OF THE SERIES

We have obtained high-temperature 15th-order power-series expansions of $\Gamma_2=\chi^{\mathrm{EA}}$, Γ_3, and Γ_4 for the $\pm J$ Ising model on d-dimensional hypercubic lattices. These quantities are defined generally in Eq. (1.5). An explicit expression for Γ_2 is given in Eq. (1.6), and for Γ_3 and Γ_4 we have

$$N\Gamma_3=-4\sum_{i,j,k}[\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_kS_i\rangle]_{\mathrm{av}} \ , \tag{3.1}$$

$$N\Gamma_4=\sum_{i,j,k,l}[36\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_kS_l\rangle\langle S_lS_i\rangle-12(\langle S_iS_jS_kS_l\rangle-\langle S_iS_j\rangle\langle S_kS_l\rangle)\langle S_iS_j\rangle\langle S_kS_l\rangle$$
$$+\langle S_iS_jS_kS_l\rangle^2-\langle S_iS_j\rangle^2\langle S_kS_l\rangle^2-\langle S_iS_k\rangle^2\langle S_jS_l\rangle^2-\langle S_iS_l\rangle^2\langle S_jS_k\rangle^2]_{\mathrm{av}} \ . \tag{3.2}$$

The complexity of these expressions suggests that a direct evaluation of them is inconvenient. The series have been generated via the Harris[53] scheme that uses only no-free-end (NFE) diagrams. In this scheme it is necessary to obtain the dependence of various thermodynamic functions on suitably renormalized potentials but calculated only for NFE diagrams. This calculation can be algebraically quite complicated but yields enormous savings in computer time, because the number of NFE diagrams is much smaller than the total number of diagrams. For example, we have 13 NFE diagrams with at most 11 bonds on a hypercubic lattice, whereas the total number of diagrams with at most 11 bonds on this lattice is over 1500. The enumeration of all NFE diagrams up to 15th order for general dimension hypercubic lattices is given in Ref. 54.

The NFE scheme[53] applies when the free energy can be expressed in the form

$$e^{-F/k_BT}\equiv Z=\mathrm{Tr}\prod_i \rho_i \prod_{\langle ij\rangle} f_{ij} \ , \tag{3.3}$$

where ρ_i is a function of variables associated with site i, and f_{ij} involves interactions between sites i and j. We may rewrite Z as

$$Z=\mathrm{Tr}\prod_i \rho_i[g(\mathbf{S}_i,h)]^z\prod_{\langle ij\rangle}\frac{f_{ij}}{g(\mathbf{S}_i,h)g(\mathbf{S}_j,h)}$$
$$=\mathrm{Tr}\prod_i\tilde{\rho}_i\prod_{\langle ij\rangle}(1+V_{ij}) \ , \tag{3.4}$$

where $\tilde{\rho}_i=\rho_i g(\mathbf{S}_i,h)^z$,

$$V_{ij}=-1+f_{ij}[g(\mathbf{S}_i,h)g(\mathbf{S}_j,h)]^{-1} \ ,$$

$z=2d$, where d is the dimension of the hypercubic lattice, and $g(\mathbf{S}_i,h)$ is an arbitrary function, to be chosen below. As usual Z can be interpreted as traces over all possible diagrams, and if we calculate Z for a diagram Γ, then $Z(\Gamma)$ is the trace over all subdiagrams including Γ. The cumulant expansion $Z_c(\Gamma)$ is obtained by subtracting from $Z(\Gamma)$ traces of all subdiagrams not including Γ. Therefore

$$Z_c(\Gamma)=Z(\Gamma)-\sum_{\gamma\in\Gamma}Z_c(\gamma)=\mathrm{Tr}\prod_{i\in\Gamma}\tilde{\rho}_i\prod_{\langle ij\rangle\in\Gamma}V_{ij} \ . \tag{3.5}$$

As usual, the cumulant vanishes if the coupling constant J of any single bond is set equal to zero. As a result a diagram with b bonds gives contributions of order w^m, which are nonzero only for $m\geq b$.

In a free end diagram there is always a site j that is singly connected to a site i. In this case if

$$\mathrm{Tr}_j(\tilde{\rho}_jV_{ij})=0 \ , \tag{3.6}$$

where Tr_j indicates a trace over all operators at site j, then Z_c vanishes, and this diagram does not contribute to the cumulant expansion. Thus if we require the function $g(\mathbf{S}_i,h)$ to satisfy Eq. (3.6), we may use only NFE diagrams. In order to satisfy Eq. (3.6), $g(\mathbf{S}_i,h)$ should obey[53]

$$g(\mathbf{S}_i,h)=\frac{\mathrm{Tr}_j\{\rho_j[g(\mathbf{S}_j,h)]^{\sigma}f_{ij}\}}{\mathrm{Tr}_j\{\rho_j[g(\mathbf{S}_j,h)]^z\}} \ , \tag{3.7}$$

where $\sigma=z-1$.

In order to apply this formalism to the free energy of Eq. (1.2) (for $H=0$), we use the replica Hamiltonian of Eq. (1.3), and the definition of Eq. (1.4)

$$F_{\mathrm{rep}}=\lim_{n\to 0}\left[\frac{-2k_BT}{n(n-1)N}\ln Z^{(n)}\right] \ , \tag{3.8}$$

where

$$Z^{(n)}=[\mathrm{Tr}e^{-\mathcal{H}^{(n)}/k_BT}]_{\mathrm{av}}=\mathrm{Tr}\left[\prod_i \exp\left[(h/k_BT)\sum_{\alpha<\beta}S_i^\alpha S_i^\beta\right]\prod_{\langle ij\rangle}\cosh\left[(J/k_BT)\sum_\alpha S_i^\alpha S_j^\alpha\right]\right], \quad (3.9)$$

is of the form required in Eq. (3.3) with

$$\rho_i=\exp\left[h/k_BT)\sum_{\alpha<\beta}S_i^\alpha S_i^\beta\right]$$

and

$$f_{ij}=\cosh\left[(J/k_BT)\sum_\alpha S_i^\alpha S_j^\alpha\right].$$

The solution of Eq. (3.7) is carried out as an expansion in powers of h up to order h^4 in Appendix B. We thus determine the Γ_k's by using only NFE diagrams, and write

$$\Gamma_k=\Gamma_k^{\mathrm{CT}}+\sum_\Gamma W_d(\Gamma)[\delta\Gamma_k(\Gamma)]_c, \quad (3.10)$$

where Γ_k^{CT} is the calculated susceptibility on a Cayley tree that has the same coordination number ($2d$), as the d-dimensional hypercubic lattice, $W_d(\Gamma)$ is the weight of the NFE diagram Γ, and $[\delta\Gamma_k(\Gamma)]_c$ is the cumulant contribution of this diagram to Γ_k. Explicit expressions for these quantities are given in Appendix C. Since $W_d(\Gamma)$ is a polynomial in d whose order is the number of bonds in Γ, we obtain results of the form written in Eq. (1.7) by taking all diagrams having up to 15 bonds.

It is worth noting that the introduction of replicas is purely a mathematical convenience. We express the final results for $\delta\Gamma_k(\Gamma)$ in terms of configurational averages of thermally averaged quantities with respect to the physical Hamiltonian of Eq. (1.1), so that no replica indices appear in these expressions. The series that we have derived are presented in Table III and the constant terms are $a_2(0,0)=1$, $a_3(0,0)=-4$, and $a_4(0,0)=34$. The $d=2$, 3, and 4 dimensional χ^{EA} series agree with the previous calculations ones of SC.[9] For the other susceptibilities checks have been made on elements up to the eighth order by calculations from the complete graph lists as well as from NFE ones. Furthermore, important checks on the correctness of the expressions for $[\delta\Gamma_k(\Gamma)]_c$ are that (a) this quantity vanishes when evaluated for a diagram with free ends and (b) for a NFE diagram consisting of n_b bonds there are no contributions of order w^k with $k<n_b$.

IV. ANALYSIS

We have analyzed the series presented above as well as some of the series from SC. A general review of analysis of multidimensional low concentration series has recently been given by the authors of this paper,[38] and since the present high-temperature series are similar to the low-concentration ones, we refer the interested reader to our review for details. In our approach to the analysis of multidimensional series, each series at $d\neq d_c$ is analyzed with two different methods,[55–57] based on the assumption that for $d\neq d_c$ there are power-law corrections to scaling that become logarithmic for $d=d_c$.

The analysis for $d\neq d_c=6$ assumes that the series being studied, denoted by $H(w)$, in general, has the form

$$H(w)=At^{-h}(1+at^{\Delta_1}+bt+\cdots), \quad (4.1)$$

where $t=(w_c-w)$, and h is the critical exponent that we wish to determine. A more complete analysis would also include higher-order correction terms such as t^{Δ_m} and $t^{m+n\Delta_1}$, but in the interest of simplicity we will mostly consider Eq. (4.1). Some of our methods require an input value of the critical temperature. For those cases where this is not known, a wide range of trial values is used, with the actual value chosen as that where best convergence is obtained.

In the first method of analysis, denoted below as $M1$,[55] we study the logarithmic derivative of

$$B(w)=hH(w)-(w_c-w)\frac{dH(w)}{dw}, \quad (4.2a)$$

which is

$$\frac{1}{B(w)}\frac{\partial B}{\partial w}=-\frac{a\Delta_1(\Delta_1-h)t^{\Delta_1-1}+b(1-h)}{t(a\Delta_1 t^{\Delta_1-1}+b)}. \quad (4.2b)$$

Assuming that the amplitudes a and b are comparable, we see that for $\Delta_1<1$, the dominant singularity in the logarithmic derivative is a pole at $w=w_c$ with residue $(h-\Delta_1)$. For $\Delta_1>1$, the dominant singularity is a pole at $w=w_c$ but now with a residue $(h-1)$. We may summarize the conclusion as follows. If the coefficients in Eq. (4.1b) can all be considered to be comparable in magnitude, then the use of method $M1$ is consistent with the approximation

$$H(w)=At^{-h}(1+bt^{\delta}), \quad (4.3)$$

where

$$\delta=\min\{1,\Delta_1\}. \quad (4.4)$$

Of course, if one of the coefficients in Eq. (4.1b) is anomalously large, $M1$ will yield the estimate of Eq. (4.3) for $H(w)$, with δ being close to the exponent whose associated amplitude is large. In intermediate cases, δ should be interpreted as an effective correction to scaling exponent. We implement method $M1$ as follows: For a given value of w_c we obtain Δ_1 versus input h for all Padé approximants, and we choose the triplet w_c,h,Δ_1, where all Padé approximants yield as nearly as possible identical values of h.

In the second method, denoted below as $M2$,[56] we first transform the series in w into a series in the variable y, where

$$y=1-(1-w/w_c)^{\Delta_1}, \quad (4.5)$$

and then take Padé approximants to

$$G(y)=\Delta_1(y-1)\frac{d}{dy}\ln[H(w)], \quad (4.6)$$

which should converge to $-h$. Here we plot graphs of h

versus the input Δ_1 for different values of w_c and again choose the triplet w_c, h, Δ_1, where all Padé approximants converge to the same point. Both those methods have proven very useful for many problems but do require the simultaneous determination of three critical quantities. In addition to these sophisticated approaches, we have also carried out simple unbiased *D*log Padé analyses. These give results that are equivalent to setting $\Delta_1=1.0$ in the $M2$ method.

In addition to the analyses of the individual series, we have studied various combinations of the series for the different susceptibilities. These include series obtained via division of the series for successive derivatives and series obtained from term-by-term divisions. The former involve dividing the entire series, and have critical points at the same location as the individual series. The latter eliminate the need for prior knowledge of the critical-point location and are based on an old method (see, for example, Ref. 58) recently revived by Y. Meir.[59] If we begin with two series expansions

$$Y=\sum_{j=0} y_j w^j \sim (w_c - w)^{-\gamma_y}$$

and

$$Z=\sum_{j=0} z_j w^j \sim (w_c - w)^{-\gamma_z} ,$$

we denote the term-by-term divided series,

$$\sum_{j=0}^{n} (y_j/z_j) w^j ,$$

by $Y \div Z$. This divided series has critical behavior with a threshold at $w=1$, i.e.,

$$Y \div Z \sim (1-w)^{\gamma_z-\gamma_y-1} .$$

One way to obtain exponents from a single series $H(w)$ without knowing the exact critical temperature is by utilizing the above approach with series Y being $[H(w)]^2$, and series Z being $H(w)$ itself. We call the resultant series, which has a critical exponent of $h+1$, a "self-divided" series, and denote it by H^{SD}.

The SG series have corrections to scaling that are apparently larger than those of the Ising model and percolation. Some preliminary studies of these series with term-by-term divided methods gave some unexpected results, and therefore we decided to undertake test series studies to examine the reliability of the term-by-term divided series for systems with large corrections to scaling. In Appendix D we describe test series work for the $M1$ method. We illustrate the importance of using $M1$ and $M2$ in tandem and demonstrate the relative strengths of temperature biased and divided series. For the divided series, regardless of whether $\Delta_1<1$ or $\Delta_1>1$, we find that convergence occurs at the correct dominant exponent estimate, and for δ given by Eq. (4.4), as previously found for the method $M2$.[60] A corollary from the test series work is the result that when $\Delta_1>1$ simple *D*log Padé approximants (which assume $\Delta_1=1$ and are therefore unreliable in general) can give the correct dominant exponent because the introduced analytic term swamps the original nonanalytic correction. This result has limited practical application because of the poor convergence that often occurs in practice in such cases.

At the upper critical dimension, $d_c=6$ the logarithmic corrections are expected to have the behavior

$$H(w)=(w_c-w)^{-h}|\ln(w_c-w)|^{\theta} , \qquad (4.7)$$

which is a special case of the Rudnick-Nelson[52] form; see also Eq. (2.14) for $\epsilon\to 0$. We fitted this form with the method of Adler and Privman.[57] The analysis of the logarithmic form involves taking Padé approximants to the series

$$g(w)=-(w_c-w)\ln(w_c-w)\{[H(w)'/H(w)] -[h/(w_c-w)]\} . \qquad (4.8)$$

We can show that the limit of $g(w)$ as $w\to w_c$ is θ. We take Padé approximants to g at the best available estimate of w_c to obtain graphs of θ as a function of h.

V. EXPONENTS AND CRITICAL TEMPERATURES

A summary of our results from the series analysis is given in Tables I and II, and in this section we present some details. We have made a very serious attempt to undertake an analysis independently of the results of other SG studies, with some interesting conclusions. Comparisons with results from other calculations will be made in Sec. VII.

A. Above six dimensions

Our exponent estimates above the upper critical dimension ($d=6$) are in excellent agreement with the exact values from mean-field theory, $\gamma=\beta=1$. Both analysis methods $M1$ and $M2$ give optimal convergence at these estimates for temperatures of $w_c=0.059\,66$, $0.067\,98$, and $0.079\,14$, for dimensions 9, 8, and 7, respectively. The error is about $\pm 0.000\,15$ in all cases. These values are based on all series, with emphasis on χ^{EA}. Analyses of self-divided series and term-by-term division of the different series confirm that these high-dimensional series do give the mean-field exponents. We have also obtained estimates for the correction terms for these dimensions and quote $\Delta_1\approx 0.5$ at $d=7$, $\Delta_1\approx 1.0$ at $d=8$ and 9, again in agreement with the theoretical expectations.

B. Six dimensions

The series at $d_c=6$ is expected[8] to have behavior of the form of Eq. (3.4), with $\theta=2$. This is a special case of the behavior of the general Rudnick-Nelson[52] form, given for this problem in Eq. (11) of FH, see also Eq. (2.14). FH imposed the Rudnick-Nelson form below six dimensions, where it gave problematic results near $d=4$, but we expect that it should be reliable near $d_c=6$. The method described in Eq. (4.8) has given excellent results for other problems such as percolation,[38] but for the present problem we did not find optimal convergence in $\Gamma_2=\chi^{\mathrm{EA}}$ for the expected $\theta=2$. We find the best convergence for $w_c=0.101\,69$, with a θ of about 1.4. Near $w_c=0.102\,00$, some of the approximants are consistent with $\theta=2.0$, but the convergence is extremely poor. Γ_3 and Γ_4 did not converge well in this analysis.

C. Five dimensions

For this case we have substantially longer series than those of previous calculations for all quantities, and happily our results in five dimensions are extremely well converged. We find optimal convergence in χ^{EA} at $w_c=0.1368$, with $\gamma=1.70\pm0.01$ at this temperature, and an approximately analytic correction to scaling. The two other susceptibilities give optimal convergence for a similar temperature with exponents of $2\gamma+\beta=4.35\pm0.01$ and $3\gamma+2\beta=7.25\pm0.1$. As can be seen from the quoted errors (which are obtained by averaging over estimates from the different analysis estimates at the optimal threshold), convergence for Γ_4 is much weaker. We illustrate the results with graphs of the $M2$ analysis at $w_c=0.1368$; the analysis for χ^{EA} is given in Fig. 1(a) and that for Γ_3 in Fig. 1(b). We observe that the correction in the biased series is approximately analytic (or that $\Delta_1\approx1.0$). The same correction behavior is seen in both susceptibilities as discussed above below Eq. (2.14). The analytic correction leads us to expect that there will be no problems with the divided series analyses. Optimal convergence is indeed seen for the self-divided series near $\Delta_1=1.0$, and we find $\gamma=1.75\pm0.10$ from $(\chi^{\mathrm{EA}})^{\mathrm{SD}}$, and $2\gamma+\beta=4.2\pm0.2$ and $3\gamma+2\beta=7.0\pm0.2$ from the Γ_3^{SD} and Γ_4^{SD}. A graph of Padé approximants to $\gamma+1$ as a function of Δ_1 (obtained via $M2$) is given in Fig. 1(c).

The divided series estimate of $\gamma=1.75$ corresponds to a temperature value of just above 0.1375. At this temperature $2\gamma+\beta=4.45$ and $3\gamma+2\beta=7.3$. However, the estimate $2\gamma+\beta=4.2$, corresponds to a temperature a little below 0.1365, where $\gamma=1.65$. The range of $0.1365<w_c<0.1375$ encompasses much of the range where any convergence is seen and we therefore deduce that the temperature range for the $d=5$ Ising SG is 0.1372 ± 0.0008. We select as our best γ estimate an average of the best converged temperature biased and the self-divided values, and quote $\gamma=1.73\pm0.03$. From the different estimates for combinations of β and γ we have deduced that $\beta=0.95\pm0.04$, giving a gap exponent estimate of $\beta+\gamma=2.68\pm0.07$.

D. Four dimensions

In four dimensions our χ^{EA} series is not longer than that of SC, but we have new long series for the two other susceptibilities. Our preliminary temperature biased analyses showed that the correction to scaling exponent is larger than unity, and therefore the dominant exponent in $(\chi^{\mathrm{EA}})^{\mathrm{SD}}$ should be the value that corresponds to the introduced analytic correction. The $M2$ analysis for $(\chi^{\mathrm{EA}})^{\mathrm{SD}}$ is presented in Fig. 2(a), and we observe that there is no real convergence in the region of the analytic correction. There is, however, convergence near $\Delta_1=3.5$ at a value of $\gamma=1.9$, and some of the approximants are fairly flat throughout the region in the figure, therefore we may tentatively cite an unbiased estimate of $\gamma=1.9\pm0.2$. The Γ_3^{SD} and Γ_4^{SD} series were too poorly converged to make any estimates. The divided series $\Gamma_3\div\chi^{\mathrm{EA}}$ and $\Gamma_4\div\Gamma_3$ both have no convergence in the neighborhood of the analytic correction. The former gives an estimate of $\gamma+\beta=3.0\pm0.1$ near a large correc-

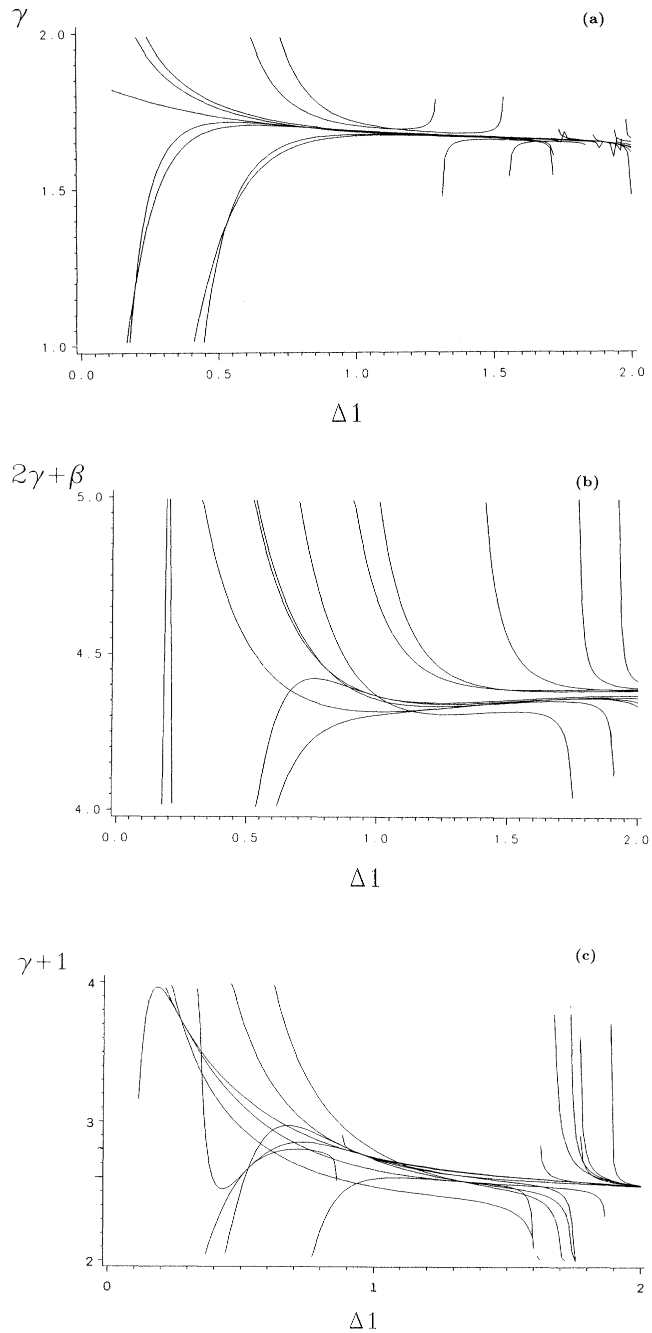

FIG. 1. Graph of Padé approximants for the dominant critical exponent in five dimensions as a function of trial Δ_1 estimate from the $M2$ analysis for the (a) χ^{EA} series and (b) Γ_3 series at $w_c=0.1368$. The $M2$ analysis for $(\chi^{\mathrm{EA}})^{\mathrm{SD}}$ is given in (c).

tion to scaling with a few approximants being flat into the region near the analytic correction and the latter gives a value of near 2.8 with no flatness. The results from the test series studies imply that we cannot place too much confidence on these values, as the correct values should be seen near the introduced analytic correction. If we do rely on both these values then they imply an average β estimate of 1.0 ± 0.2.

Optimal convergence is seen in the temperature biased $M1$ analyses for χ^{EA} and Γ_3 series near $w_c=0.205$, where $\gamma=2.0\pm0.2$ and $2\gamma+\beta=4.9\pm0.2$, respectively. The correction exponent Δ_1 is close to 3 for both cases. This gives a central β estimate of 0.9. The optimal convergence for $M2$ is closer to $w_c=0.210$, where $\gamma=2.2\pm0.1$, and $2\gamma+\beta=5.3\pm0.3$, again leading to a central estimate of $\beta=0.9$. At the lower temperature choices the individual $M1$ estimates were about 0.1 higher than the $M2$ ones, but for $w_c=0.210$ the values were similar. We have also studied derivatives of the χ^{EA} series in order to take account of the possibility of a large analytic correction. The central values of the $M2$ analysis of the first and second derivatives are not any different from those of the undifferentiated series. We illustrate the $M2$ analysis of the first derivative of χ^{EA} at $w_c=0.205$ in Fig. 2(b). The correction to scaling exponent appears to be about 3.0. A very interesting effect is seen in the $M1$ analysis of the first derivative. In the region $0.203<w_c<0.205$, in addition to the $M1$ estimate at about 0.1 above the $M2$ estimate, there are some indications that some $M1$ approximants are tending towards about the same value as the $M2$ estimate. This effect may be the precursor of a crossover of the $M1$ estimates from values above the $M2$ ones to estimates identical with those of $M2$. The splitting effect becomes smaller for shorter series and is absent completely for 12 terms.

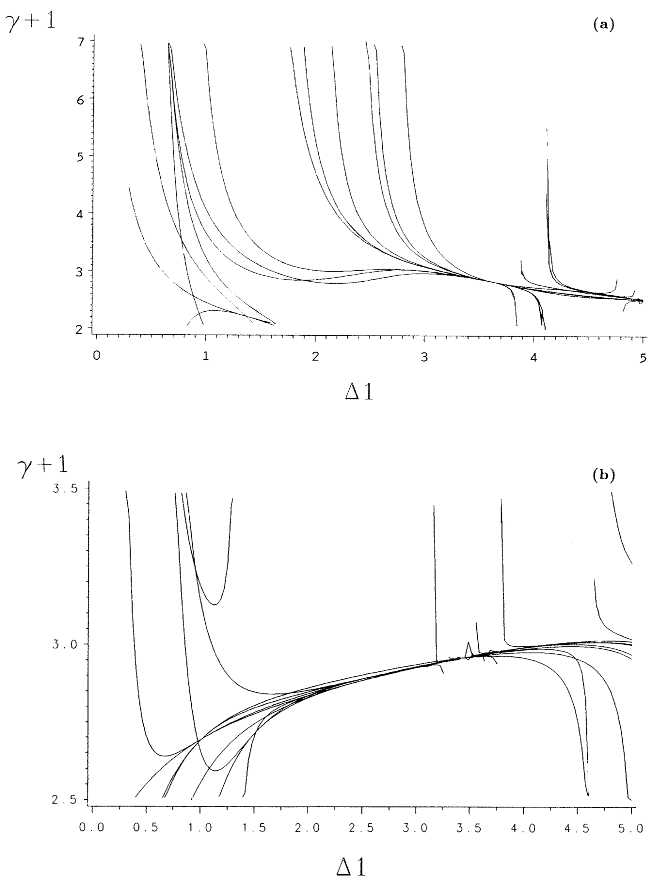

FIG. 2. Graph of Padé approximants for the dominant critical exponent in four dimensions as a function of trial Δ_1 estimate from the $M2$ analysis for the (a) $(\chi^{EA})^{SD}$ series and (b) first derivative of the χ^{EA} series at $w_c=0.205$.

With overall emphasis on the temperature biased analyses, we quote the overall estimates $w_c=0.207\pm0.008$, $\gamma=2.00\pm0.25$, $\beta=0.9\pm0.1$, and $\Delta=2.9\pm0.3$ from the 15-term series and note that it is possible that the γ value may be towards the bottom of the range if the splitting in the $M1$ derivative analysis is not a coincidence.

E. Three dimensions

Three dimensions is very close to the lower critical dimension for Ising SG's.[36] The quality of convergence is far poorer than in the case of higher-dimensional SG's or other systems in three dimensions. Fortunately, SC were able to obtain two 17th-order series for this dimension, one for χ^{EA} and one for Γ' [Eq. (1.10)], which has a critical exponent γ' that can be related to β via hyperscaling

$$\gamma'=(4-d-2\eta)\nu=\gamma-2\beta\ .$$

There is no reason to doubt hyperscaling for the SG, and therefore we have no reason to expect that there is any objection, in principle, to obtaining the β estimate via hyperscaling from the γ' series if the series are long enough. However, there is no reason to expect *a priori* that the Γ' and χ^{EA} series will have corrections to scaling of similar relative amplitudes, and, therefore, in finite series, differences in effective corrections could degrade the convergence and result in different estimates of β from the two series. As argued in some of our recent analyses,[61] and as required by the scaling form Eq. (2.16), we expect (and observed above for the SG in four and five dimensions) that series that are successive field derivatives, whose exponents differ by a constant gap, should have similar corrections to scaling. This means that the β estimate from a pair of such series could be reliable even if there is a slight systematic error in each of the dominant exponent estimates. Therefore, although we have only been able to obtain a 15-term series for Γ_3 and Γ_4, it appeared to be worthwhile to analyze these to obtain direct estimates of the gap exponent, and thence of $\beta=\Delta-\gamma$.

Our preliminary analyses showed that $\Delta_1>1$, and therefore term-by-term divided series analyses, should give the correct dominant exponent near an analytic correction. Since we failed to find convergence in the analytic region we shall not report in detail on these analyses. We have carried out extensive temperature-biased analyses of all four series using trial temperatures in the range $0.32<w_c<0.55$, with emphasis on the range $0.36<w_c<0.48$. In the tighter range indications that good convergence were observed in at least one analysis for at least one of the series. For χ^{EA} we found that in the best converged region in the $M2$ analysis γ decreased from 4.25 at $w_c=0.48$ to about 2.5 at $w_c=0.36$. There were some indications that the same splitting of estimates that occurred in the derivative series in four dimensions, occurs for $M1$ in the region of $w_c=0.40$. One estimate was in agreement with the converged $M2$ value of $\gamma=3.0$ at $\Delta_1\approx4.0$ and the second was consistent with a correction exponent estimate of $\Delta_1\approx2.0$ and a γ estimate of about 2.4. Derivatives of χ^{EA} give central estimates of γ that range from 4.25 at $w_c=0.48$ to 2.3 at $w_c=0.38$. These estimates appear to confirm the lower of the two γ estimates in the split case. An extensive study has also been made of the Γ_3 series, where for the same threshold range we find $2\gamma+\beta=7.0\pm2.0$ from the original series and $2\gamma+\beta=6.2\pm2.0$ from the second derivative series. Convergence is much better for the second derivative, and we may conclude that using this series we have a β estimate of about 0.8 at $w_c=0.40$. The Γ_3 series appear to have a large correction to scaling exponent estimates, again near 4.0. With a strong bias from the results of the derivative analysis we decided to exclude $w_c>0.46$ from our temperature range. We conclude that the χ^{EA} series give an exponent value that is strongly dependent on the threshold choice, and for $w_c=0.40^{+0.06}_{-0.04}$ (the range being chosen both from analyses of the series and of its derivative) we find $\gamma=2.7^{+1.0}_{-0.6}$.

For Γ' we found that γ' decreased from 3.0 (2.5) at $w_c=0.48$ to about 1.0 at $w_c=0.36$ for $M2$ ($M1$). The $M1$ and $M2$ analyses of this series at $w_c=0.42$ are given in Figs. 3(a) and 3(b), respectively. We have also studied second derivatives of the Γ' series. The $M2$ results failed to converge but the $M1$ results agreed quite nicely with the other estimates. The nature of the correction behavior in the Γ' series appears to differ from that of the other series; there is a correction exponent of about 2.0 at optimal convergence. From the scaling relation $\beta=(\gamma-\gamma')/2$ we may deduce $\beta=0.6$ at $w_c=0.40$.

At $w_c=0.40$, the two β estimates average at 0.7 ± 0.2, giving a gap exponent of 3.4. Attempts to obtain β estimates towards the extremes of the temperature regions quoted above lead to estimates of about 0.5 near $w_c=0.36$ values and 1.5 near $w_c=0.48$. We believe that, since it is most likely that β in three dimensions will be below the values in higher dimensions, this indicates that the true w_c is likely to be below 0.44. We quote $\gamma=2.7^{+1.0}_{-0.6}$, $\beta=0.7\pm0.2$ and $w_c=0.40^{+0.06}_{-0.04}$.

VI. AMPLITUDE RATIOS

We have studied the amplitude ratio R of Eq. (1.9) in all dimensions and the ratios $R_1=\Gamma'\Gamma_3^2/\Gamma_2^5$ and $R_2=\Gamma'\Gamma_4/\Gamma_2^4$ in three dimensions. The measurements of the amplitudes in the ratios R, R_1, and R_2 are all made on the same side of the transition, and this is of considerable importance because in many systems of interest such as the three-dimensional SG the error in the critical point is large. These ratios are less sensitive to the exact choice of transition temperature than ratios taken between amplitudes of, for example, the same susceptibility above and below the transition. For problems such as the SG where series have been obtained to date only on one side of the transition, these are, at present, the only amplitude ratios that can be calculated from the series approach.

We have evaluated these ratios using a method developed in Ref. 61, whereby the series are multiplied and divided in the appropriate combinations to give an expansion for the ratio in question. Padé approximants are then obtained for the expansion and evaluated at the critical point. Another method of determining amplitude ratios[59] was attempted here, but the results were unclear.

A graph of central and nearest diagonal highest Padé approximants for the ratio R is given in Fig. 4. Above six dimensions the numerical evaluation of R is in excellent agreement with the exact mean-field result of 3.0. We measure 3.00 ± 0.01 and 3.02 ± 0.02 at the critical temperatures in eight and seven dimensions, respectively. At six dimensions an average of central and near-diagonal highest approximants gives $R=3.08\pm0.08$. As we decrease towards five dimensions, this ratio increases a little, but the increase is very small relative to the scatter. We quote $R=3.14\pm0.20$ at the best threshold estimate in 5.5 dimensions. By the time five dimensions is reached

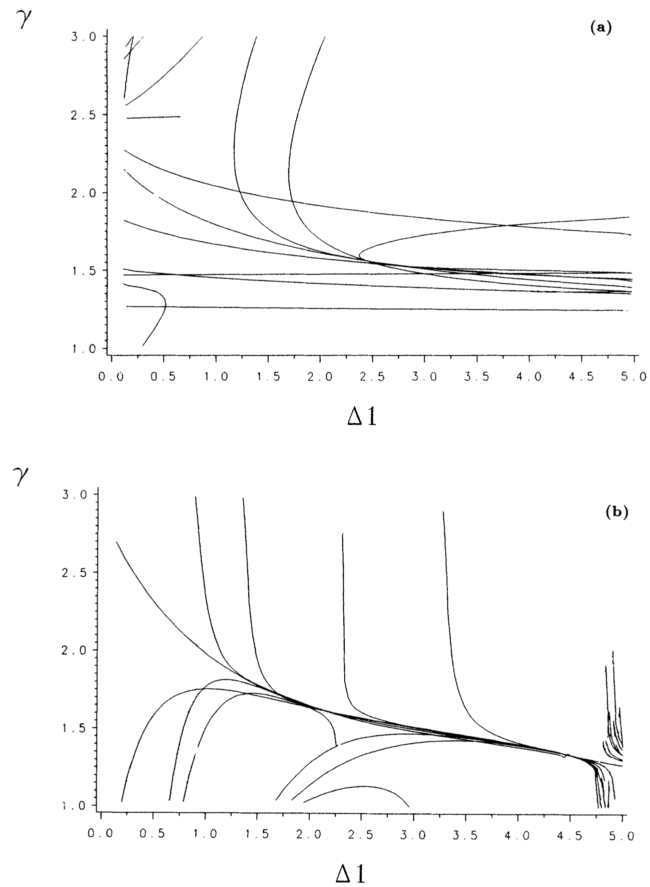

FIG. 3. Graph of Padé approximants for the dominant critical exponent in three dimensions as a function of trial Δ_1 estimate from the (a) $M1$ and (b) $M2$ analysis for the Γ' series at $w_c=0.42$.

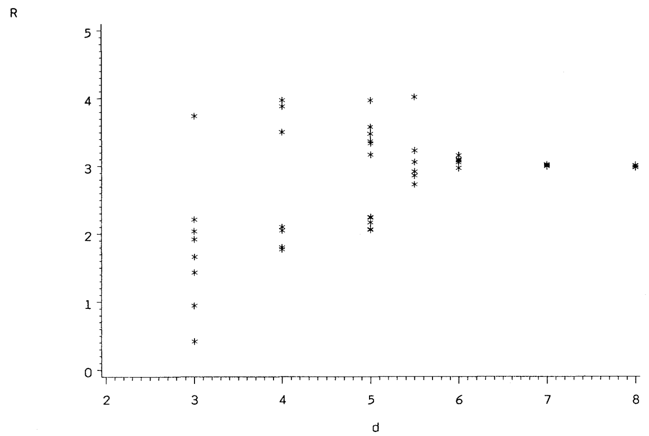

FIG. 4. Graph of Padé approximants for the amplitude ratio R as a function of dimension.

the ratio has clearly decreased, and we measure an average $R=2.77\pm0.08$, for central and nearest-diagonal approximants with two outlying approximants being discarded. The estimates are not uniformly scattered but rather peak around two values. About half the approximants give an average estimate of 3.4, and the others average at 2.1. The two peaks become even more pronounced as the dimension is further reduced, and at four dimensions there is an averaged estimate of $R=2.8\pm1.5$, with peaks at about 1.9 and 3.8. In this dimension four outlying approximants were discarded. This two-branched behavior of R estimates is very interesting, especially as the upper branch may be following the ϵ-expansion results of Eq. (2.11). In $4\leq d\leq 6$ we quote results averaged over our entire range for the critical temperature.

In three dimensions, the upper branch is represented only by the [6,6] Padé approximant, for any temperature choice within or near our range or that of SF. Other $[L,M]$ approximants with $L+M<13$ give negative or close to zero values of R. There are six high-diagonal and near-diagonal approximants, which give estimates between 0.9 and 2.3 within our critical temperature range, and we quote an average value of 1.7 ± 0.4 from these. This value decreases by about 0.1 at the top of the w range and increases by 0.1 at the bottom of the range. If we include the [6,6] approximant we have a central estimate of 1.9. At $w_c=0.40$, we estimate $R_1=39.4\pm0.4$ from approximants of degree 13 and higher. For $w_c=0.48$ this value is higher by about 3.0 and for $w_c=0.36$ lower by about 1.0. This ratio is much better behaved than either R or R_2. We estimate R_2 as 73 ± 20 at $w_c=0.40$ from 9 central and near-diagonal approximants. For $w_c=0.48$ this value is higher by about 8.0, and for $w_c=0.36$ lower by about 5.0. The ratio of R_2/R_1 is equal to R by elementary considerations. Using our direct estimates of R_1 and R_2 this leads to an indirect central estimates of $R=1.85(1.91,1.77)$, at $w_c=0.40(0.48,0.36)$, in excellent agreement with the direct estimates. In fact, given the wide error ranges, this agreement is almost "too good." We have tried to verify the estimates by changing the criterion of selection of approximants in both R_1 and R_2. Such changes do change the value of both central estimates (discarding more approximants in both cases lowers the average as in each case there are several approximants near the top of the error range), but the ratio remains unchanged. We note that the trend of the change as threshold changes is different for the direct and indirect estimates; therefore best agreement is obtained for the center of our range. This supports our choice of a central w_c estimate.

VII. TRENDS, COMPARISONS, AND CONCLUSIONS

The main trend that can be observed from our calculations is the smooth monotonic extrapolation that our new estimates give between the mean-field six-dimensional exponents and existing exponent estimates in three dimensions, (Table I). In the light of the convergence difficulties that the ϵ-expansion experiences, we believe that this is the first time that a smooth interdimensional extrapolation has been obtained for several exponents of the Ising SG. We have also obtained amplitude ratios that extrapolate fairly smoothly between dimensions.

Several comparisons can be made between our new results and existing calculations. In $d\geq 6$, the main interest is in the comparison of our critical temperatures with the other estimates of Table II. We see that there is broad general agreement with the shorter series and $1/\sigma$ expansion calculations of SF;[36,37] our results fall below the latter in $d>6$, whereas the series results of SF (Refs. 36 and 37) fall above. In $d=6$ our critical temperature is close to the FH value[8] that is found from fitting to the full Nelson-Rudnick form. This is a pleasing confirmation of the reliability of both algorithms at the upper critical dimension.

We have also been able to confirm the mean-field dominant exponent estimates in $d\geq 6$. Our correction exponents for $d=7$ and 8 are in agreement with the expected [see Eq. (2.14), which is exact above six dimensions] $\Delta_1=(d-6)/2$ for a ϕ^3 field theory. The detailed derivation of this result for percolation is given in Ref. 62, but it is equally valid in this case. For $d=9$ we measured the dominant analytic term rather than the expected $\Delta_1=1.5$. A similar phenomenon was seen in $d=9$ percolation.[28] In six dimensions the convergence to the expected logarithmic correction $\theta=2$ is less clear: This poor convergence may be caused by defective approximants or other causes such as the series being too short to capture all the details of the system behavior or higher-order corrections.

In $d=5$, we observe that our critical temperature falls between the SF and $1/\sigma$ estimates, which are all substantially below the FH estimates. Our γ value in this case is below the FH value but above the SF one. Another difference from the SF calculation is that our gap exponent of 2.68 is already substantially above the mean-field result; the value quoted in the table of SF is exactly the mean-field value of 2.0, but in a footnote they quote 2.4. From our results for γ and β, we use scaling to evaluate the central estimates $\nu\simeq0.73$ and $\eta\simeq-0.38$, the latter being in qualitative agreement with the first-order ϵ-expansion values of 0.91 and -0.33, respectively. Our results for five dimensions are well converged and indicate the potential for smooth extrapolation down to four dimensions.

Before comparing our lower dimensional results with previous studies it is necessary to pause and consider briefly the extrapolation of the $1/\sigma$ expansion in lower dimensions. If we follow the procedure suggested by SF for extrapolating this asymptotic expansion, then we should truncate the expansion after the smallest term. This means retaining only three terms in three and four dimensions. The estimates obtained from these truncations are quoted in Table I; they are considerably smaller than the values obtained from truncations after five terms, as quoted by SF.

Our four-dimensional results are not very different from those of SC for w_c and γ, but our gap exponent estimate is larger than that of SF. The larger gap exponent is based on longer series for the higher moment than those of SF. Since this value falls smoothly on the extrapolation from six dimensions to the SC value in three dimensions, we expect that our gap exponent of 2.9 ± 0.3 is reliable. Using scaling we deduce the estimates $\nu\simeq0.95$ and $\eta\simeq-0.11$, respectively. The former can be compared with a value of about 0.7 from the combined SC and SF estimates. Our estimate for η is very much smaller in magnitude than the corresponding estimates of -0.5 to -0.8 deduced from the SC-SF values.

In three dimensions, our central w_c estimate falls below all other central estimates given in Table I, except for that of the third order $1/\sigma$ expansion. Since three terms are apparently the right number to take in this dimension, this result is interesting. Our estimate overlaps with all calculations cited in the table except that of Ogielski and Morgenstern,[6] whose lower bound of 0.465 just misses our upper bound of 0.46. Our gap exponent is in excellent agreement with both the SF and SC values, but our γ value is lower by 0.2 and our β value higher by 0.2 than those of SC and of the simulations. As described in detail above, it has been observed that within our calculations, raising the central critical temperature estimate has the effect of raising our γ estimate. Such an increase is also seen in our γ' estimates, and if β is estimated from the difference of γ and γ' the change of critical temperature has relatively little effect on its value. Thus despite the marginally better convergence seen for the lower temperatures a slight decrease in w_c would lead to values in better agreement with those of other authors, and therefore would perhaps be justified. However, our long Γ_3 series enable us to take an alternative determination of β. This alternative determination increases very quickly as the trial critical temperature is increased; for example, at $w_c=0.48$ we see a β estimate of 1.5. Reasonable consistency of β estimates is only seen below $w_c=0.46$, and therefore we propose a central estimate of w_c about 10% below that of earlier calculations. Some support for a possible deviation of 10% can be gleaned from an extension of the discussion on p. 3995 of SF who present a mapping from the threshold of the $\pm J$ model to that of the Gaussian model. The Gaussian simulation estimate[5] is $T_c^{(G)}=0.9$, about 10% above the expected extrapolation of $T_c^{(G)}=0.81$ mapped from $\omega_c=0.48$. (Note that above in the w variable corresponds to below in the T variable.) In $d=4$, where our w_c result is in excellent agreement with SC the percentage difference is only 2% between the extrapolated series value and the Gaussian simulation result.

After completing our analysis, we heard about some recent calculations,[63] on damage spreading in spin glasses, which give a three-dimensional critical temperature estimate of w_c near 0.34. This is just below the bottom of our range and very close to the $1/\sigma$ value.

Our three-dimensional ν estimate derived via scaling is not very different from that of the other calculations, but our η estimate of ≈0 agrees only with the original Ogielski-Morgenstern[6] analyses and not with the later Ogielski[7] analysis. There is no way that our η values for the SG can be reconciled to be a monotonic function of dimension, as both our five-dimensional η estimate and the slope of the first-order ϵ expansion indicate a fairly rapid initial decrease as a function of decreasing dimension. Our results are fairly smooth, and the decrease and subsequent increase are reminiscent of the case of isotropic percolation, where η initially decreases and then increases as a function of dimension. An increase in η between four and three dimensions is also seen in the SC-SF results.

We finally consider the relationship between our new results and the experimental work. For the single case of clear overlap, namely, the β for the three-dimensional Ising glass, the new[40] value is a little below ours. We suggest that it would be of great interest if alternative experimental measurements could be made for β via a study of higher derivatives. It would also be very useful if experimental determination of the ratio R could be made, as this is likely to be less sensitive to the exact choice of critical temperature. It would also be of interest to study both the derivatives and the ratio in future simulations. We note that, despite the suggestions that corrections to scaling may be important in both simulations[5] and in experimental[40] data analyses, to the best of our knowledge our calculations are the only ones where correction effects have been systematically incorporated into the analysis.

The final question for discussion is the nature of the three-dimensional transition and the location of the lower critical dimension. The poor quality of the numerical convergence in 3D immediately leads to the question of whether it is justifiable to fit the three-dimensional SG susceptibilities by power-law singularities as in Eq. (1.5). In support of such doubts we quote two different groups that have speculated on this. Bhatt and Young[5] have suggested that the Monte Carlo data in 3D may indicate that the transition is of a different type than the power-law transitions that are clearly observed in simulations of the four-dimensional $\pm J$ Ising and Gaussian SG's. They found clear evidence of long-range order below T_c in four dimensions, but they suggest that the three-dimensional system may have no long-range order but rather have an infinite $\chi^{\rm EA}$ at all temperatures below T_c. This criticality was proposed on the basis of results for samples of up to

16^3 sites, and could be a finite-size effect. It is suggested that, since 3D is close to the lower critical dimension (SF have claimed a lower critical dimension of about 2.6 for the SG based on their 10-term general dimension series), corrections to finite-size effects (which are not taken into account) mask the ordering in 3D. One reason advanced by Bhatt and Young to support their data analysis is that the numerical estimates are close to the series estimates of SC; in both these cases correction effects were neglected. In an independent calculation, Guttmann[42] has noted that similar numerical problems are observed in series analysis when fitting susceptibilities of two-dimensional planar-rotator models (which do not have long-range order in zero field) and three-dimensional SG series to power-law divergences with first-order differential approximants. Guttmann[42] has shown that the critical behavior of the plane rotator model susceptibility, which is known to have an exponential singularity, actually gives a better fit to the power-law form. We have attempted to explore this matter by undertaking analyses of the 3D SG susceptibilities based on exponential singularities, but the results were quite inconclusive, as were attempts to fit the series to assorted other types of singularities.

Our own explanation for the poor convergence in 3D is rather similar to that of Bhatt and Young.[5] We suspect that the proximity to the lower critical dimension is such that corrections play a significant role. We have included correction to scaling effects, and it would be of great interest to see a reanalysis of the simulation data that allows for such correction effects.

In summary, we have been able to obtain a comprehensive set of critical exponents and temperatures for the Ising spin glass. New, long series for higher moments and comprehensive analyses that allow for corrections to scaling for all available data have led to many results that are in agreement with other estimates and to several new exponent values. For the first time a clear numerical picture has been obtained for $6>d>3$, and this indicates that despite the severe convergence problems of the ϵ expansion, smooth extrapolation from mean-field values down to those obtained from the extensive numerical and experimental studies in the lower dimensions is possible. Although some of our quantitative results in the three-dimensional case are a little different from those of some other calculations (see Tables I and II), and further work on the fine details of the quantitative behavior in three dimensions is desirable, the smoothness of our extrapolations clearly support the existence of a finite-temperature SG transition in three dimensions.

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APPENDIX A: DERIVATIVES OF THE FREE ENERGY WITH RESPECT TO A UNIFORM APPLIED MAGNETIC FIELD

In this appendix we demonstrate the connection between derivatives of the replica free energy with respect to a uniform applied magnetic field H and the derivatives of the free energy with respect to the SG ordering field h.

We start by considering a special model, namely, an Ising model with nearest-neighbor exchange interactions each of which is subject to a Gaussian distribution whose variance is J_0. Here we follow the approach of Bray and Moore[64] to convert the partition function in the presence of a uniform field into a field theoretical model. For a fixed configuration of the J_{ij}'s the partition function in the presence of a magnetic field H is

$$Z=\operatorname*{Tr}_{\{S_i=\pm 1\}}\exp\left[(k_BT)^{-1}\left[\sum_{\langle ij\rangle}J_{ij}S_iS_j+H\sum_i S_i\right]\right]. \tag{A1}$$

The configurationally averaged free energy F is obtained via the *replica procedure:*

$$F=\lim_{n\to 0}\left[-\frac{k_BT}{nN}\ln[Z^n]_{\rm av}\right]=\lim_{n\to 0}\left\{-\frac{k_BT}{nN}\ln\left[\operatorname*{Tr}_{\{S_i^\alpha=\pm 1\}}\exp\left[\frac{1}{k_BT}\left[\sum_{\langle ij\rangle;\alpha}J_{ij}S_i^\alpha S_j^\alpha+H\sum_{i,\alpha}S_i^\alpha\right]\right]\right]_{\rm av}\right\}, \tag{A2}$$

where α is summed over the range 1 to n. After configurational averaging we have

$$[Z^n]_{\rm av}=\operatorname*{Tr}_{\{S_i^\alpha=\pm 1\}}\exp\left[\frac{1}{4}\sum_{i,j;\alpha,\beta}(J_0/k_BT)^2\gamma_{i,j}S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta+(H/k_BT)\sum_{\alpha,i}S_i^\alpha\right], \tag{A3}$$

where $\gamma_{i,j}$ is 1 when i and j are nearest neighbors and 0 otherwise. We use the Hubbard-Stratonovich transformation,

$$\exp\left[\frac{1}{2}\sum_{i,j}x_iA_{ij}x_j\right]=\text{const}\times\int_{-\infty}^{\infty}\prod_i dy_i\exp\left[-\tfrac{1}{2}\sum_{i,j}y_i(A^{-1})_{ij}y_j+\sum_i x_iy_i\right] \tag{A4}$$

on Eq. (A3) and obtain, apart from some unimportant constant terms,

$$[Z^n]_{\rm av}=\int_{-\infty}^{\infty}\prod_{i,\,\alpha<\beta}\left\{dQ_i^{\alpha\beta}\exp\left[-\tfrac{1}{2}\sum_j\left[\frac{k_BT}{J_0}\right]^2Q_i^{\alpha\beta}(\gamma^{-1})_{ij}Q_j^{\alpha\beta}\right]\right\}\prod_i\operatorname*{Tr}_{\{S_i^\alpha=\pm 1\}}K_i\,, \tag{A5}$$

where K_i is

$$K_i = \exp\left[(H/k_BT)\sum_\alpha S_i^\alpha + \sum_{\alpha<\beta} Q_i^{\alpha\beta} S_i^\alpha S_i^\beta\right] . \tag{A6}$$

Thus,

$$\begin{aligned} \mathrm{Tr} K_i &= \mathrm{Tr}\exp\left[(H/k_BT)\sum_\alpha S_i^\alpha\right] \\ &\quad\times\left[1+\sum_{\alpha<\beta} Q_i^{\alpha\beta}S_i^\alpha S_i^\beta + \cdots + \frac{1}{k!}\sum_{\alpha_1<\beta_1}\sum_{\alpha_2<\beta_2}\cdots\sum_{\alpha_k<\beta_k} Q_i^{\alpha_1\beta_1}\cdots Q_i^{\alpha_k\beta_k} S_i^{\alpha_1} S_i^{\beta_1}\cdots S_i^{\alpha_k}S_i^{\beta_k}\right] \\ &= [2\cosh(H/k_BT)]^n\left[1+(h/k_BT)\sum_{\alpha<\beta} Q_i^{\alpha\beta} + O(Q^2)\right] , \end{aligned} \tag{A7}$$

with $h = k_BT\tanh^2(H/k_BT)$. Thus $h \sim H^2/k_BT$. An effective Hamiltonian, $\mathcal{H}_{\rm eff}$, can be defined through the relation

$$\int dQ_i^{\alpha\beta}\exp(-\mathcal{H}_{\rm eff}/k_BT) = [Z^n]_{\rm av} \tag{A8}$$

and we may observe that

$$\mathcal{H}_{\rm eff} = -h\sum_i\sum_{\alpha<\beta} Q_i^{\alpha\beta} + \sum_l \Lambda_l O_l , \tag{A9}$$

where O_l is an operator of order Q^l, $l \geq 2$, and the coefficients Λ_l may be H dependent. The scaling form of the free energy is now

$$F = b^{-d} f(hb^{\lambda_h}, tb^{\lambda_t}, \Lambda_l b^{\lambda_l}) . \tag{A10}$$

The exponent λ_l decreases with increasing l. Therefore $\lambda_l < \lambda_h$ and we also notice that near $d=6, \lambda_h = 4+O(\epsilon)$, whereas $\lambda_t = 2+O(\epsilon)$. Thus λ_h is the largest exponent. Therefore when we take derivatives with respect to H, the leading terms come from derivatives of the first argument of F, h, and we can neglect derivatives of all other arguments. The leading terms of the derivatives at $H=h=0$ of the free energy with respect to H are then

$$\begin{aligned} \frac{\partial^{2k}F}{\partial H^{2k}} &= \frac{\partial^{2k}}{\partial H^{2k}}\left[\frac{1}{k!}\frac{\partial^k F}{\partial h^k}h^k\right] \\ &= \frac{2k!}{k!}\left[\frac{1}{k_BT}\right]^k \frac{\partial^k F}{\partial h^k} = \frac{2k!}{k!}\left[\frac{1}{k_BT}\right]^{2k-1}\frac{\partial^k \tilde{F}_{\rm rep}}{\partial \tilde{h}^k} \\ &= -\frac{(2k-1)!}{(k-1)!}\left[\frac{1}{k_BT}\right]^{2k-1}\frac{\partial^k F_{\rm rep}}{\partial \tilde{h}^k} \\ &= -\frac{(2k-1)!}{(k-1)!}\left[\frac{1}{k_BT}\right]^{2k-1}\Gamma_k . \end{aligned} \tag{A11}$$

In obtaining this result we used the fact that in the $n \to 0$ limit, $F_{\rm rep} = -2F$.

We now discuss the fact that within the SG universality class we must replace β by $\lambda\beta$ in Eq. (A11), where λ is a nonuniversal constant. Consider the sum

$$S = N^{-1}\sum_{i,j,k,l}[\langle S_iS_j\rangle\langle S_kS_l\rangle]_{\rm av}$$

of Eq. (2.3b). We have said that this average is nonzero only if the indices are equal in pairs. However, consider a model in which $[J_{ij}]_{\rm av}$ is nonzero, but is small enough that the ordered state is still a SG state. Then, within mean-field theory the relation between the ferromagnetic correlation length ξ_F and the SG correlation length ξ is

$$\left[\frac{\xi_F}{\xi}\right]^2 = \frac{(k_BT)^2 - z[J^2]_{\rm av}}{(k_BT - z[J]_{\rm av})^2} \sim \frac{T-T_{\rm SG}}{T-T_F} , \tag{A12}$$

where $T_{\rm SG} \sim (z[J^2]_{\rm av}/k_B^2)^{1/2}$ and $T_F \sim [J]_{\rm av}/k_B$ are the mean-field spin-glass and ferromagnetic transition temperatures. S is now

$$S = N^{-1}\sum_{i,j,\delta_1,\delta_2}[\langle S_iS_j\rangle\langle S_{i+\delta_1}S_{j+\delta_2}\rangle]_{\rm av} \tag{A13}$$

$$= N^{-1}\lambda^2\sum_{i,j}[\langle S_iS_j\rangle^2]_{\rm av} , \tag{A14}$$

where δ_1 and δ_2 must be summed over a volume whose linear dimension is of order ξ_F. Thus λ is of order $(\xi_F)^d$. More generally this reasoning leads to the replacement of $(k_BT)^{-1}$ by $\lambda(k_BT)^{-1}$ as written in Eq. (2.7). In the renormalization-group formulation the appearance of λ is regulated by the irrelevant operators Q^l of Eq. (A9). To reproduce Eq. (A14) using the renormalization group is not easy.

APPENDIX B: THE FORM OF $g(\mathbf{S}_i, h)$

In this appendix we discuss the solution to Eq. (3.7). We write

$$g(\mathbf{S}_i, h) = c(h)\tilde{g}(\mathbf{S}_i, h) , \tag{B1}$$

where $c(h)$ is independent of $\mathbf{S}_i$ and $\tilde{g}(\mathbf{S}_i,h)$ is normalized so that it is unity for $S^\alpha = 0$. Then Eq. (3.7) becomes

$$c^2(h)\tilde{g}(\mathbf{S}_i,h)=\frac{\mathrm{Tr}_j\left\{\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^\sigma\cosh\left[(J/k_BT)\sum_\alpha S_i^\alpha S_j^\alpha\right]\right\}}{\mathrm{Tr}_j\left\{\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_i,h)]^z\right\}}\ . \tag{B2}$$

The part of this equation, which is independent of S_i^α (which can be found by setting $S_i^\alpha=0$), yields

$$c^2(h)=\frac{\mathrm{Tr}_j\left\{\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^\sigma\right\}}{\mathrm{Tr}_j\left\{\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^z\right\}}\ . \tag{B3a}$$

To evaluate thermodynamic properties we need to evaluate $c(h)$ for $n\to 0$ to order n. In this connection observe that for $n\to 0$ both the numerator and denominator in Eq. (B3a) approach unity. To see this, note that any term involving an S_j^α involves at least one sum over replica indices and is therefore proportional to n. We therefore rewrite Eq. (B3a) as

$$c^2(h)=\frac{1+\left[\mathrm{Tr}_j\left\{\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^\sigma-1\right\}\right]}{1+\left[\mathrm{Tr}_j\left\{\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^z-1\right\}\right]}\ . \tag{B3b}$$

Since the quantities in the large square brackets are of order n, we have the result correct to order n:

$$c(h)=1+\gamma(h)n+O(n^2)\ , \tag{B4}$$

where

$$\gamma(h)=\lim_{n\to 0}\left\{\frac{1}{2n}\mathrm{Tr}_j\left[\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^\sigma[1-\tilde{g}(\mathbf{S}_j,h)]\right]\right\}\ . \tag{B5}$$

We only need $\tilde{g}(S_i,h)$ to leading order in n. It is determined by setting

$$\tilde{g}(\mathbf{S}_i,h)=1+hav_1(\mathbf{S}_i)+h^2[v_2(\mathbf{S}_i)b_2+v_1(\mathbf{S}_i)b_1]+h^3[v_3(\mathbf{S}_i)c_3+v_2(\mathbf{S}_i)c_2+v_1(\mathbf{S}_i)c_1]$$
$$+h^4[v_4(\mathbf{S}_i)d_4+v_3(\mathbf{S}_i)d_3+v_2(\mathbf{S}_i)d_2+v_1(\mathbf{S}_i)d_1]+\cdots\ , \tag{B6}$$

where for $1\le 2k\le n$ we define

$$v_k(S_i)=\sum_{1\le\alpha_1<\alpha_2<\cdots<\alpha_{2k}\le n}S_i^{\alpha_1}S_i^{\alpha_2}\cdots S_i^{\alpha_{2k}}\ . \tag{B7}$$

The coefficients in Eq. (B6) are determined by substitution into Eq. (B2), which becomes in the limit $n\to 0$

$$\tilde{g}(\mathbf{S}_i,h)=\left[\mathrm{Tr}_j\exp\left[(h/k_BT)\sum_{\alpha<\beta}S_j^\alpha S_j^\beta\right][\tilde{g}(\mathbf{S}_j,h)]^\sigma\cosh\left[(J/k_BT)\sum_\alpha S_i^\alpha S_j^\alpha\right]\right]\Big/\left[\mathrm{Tr}_j 1\right]\ . \tag{B8}$$

To calculate the susceptibilities Γ_k up to $k=4$, we need $\gamma(h)$ up to order h^4, but we do not need to evaluate $\tilde{g}$ up to this order. To see this, consider the evaluation of Eq. (3.5) up to order h^4. Using the terms of order h^4 in Eq. (B6) will give contributions to $Z_c(\Gamma)$ involving the thermal averages of $v_k(\mathbf{S}_i)$. But such single-site averages vanish, since there is no broken symmetry. In general, one sees that nonzero contributions can come only from terms in which $\tilde{g}$ is expanded beyond the constant term of unity at two or more sites. Then even terms of order c_3 and c_2 do not enter the calculation up to order h^4. We could imagine having v_3 (or v_2) at one site correlated with v_1 at another site. But such an average vanishes in the absence of broken symmetry. Thus, for the calculation to order h^4 we only need to evaluate the following coefficients in Eq. (B6): a, b_1, b_2, and c_1. In addition we need to evaluate $\gamma(h)$ up to order h^4. We quote the evaluations

$$a=wD\ , \tag{B9a}$$

$$b_1=-2wD^3/E\ , \tag{B9b}$$

$$b_2=3w^2D^2\ , \tag{B9c}$$

$$c_1=wD^4[\tfrac{17}{3}(1-3\sigma w^2+2\sigma w^3)+\sigma(1-\sigma w^2)(1-w)(9w^2E+8wD)]\ , \tag{B9d}$$

where $D=(1-\sigma w)^{-1}$, $E=(1-\sigma w^2)^{-1}$, and $\sigma=2d-1$. We write $\gamma(h)=\sum_n \gamma_n h^n$ and have

$$\gamma_0=\gamma_1=0\ , \tag{B10a}$$

$$\gamma_2=\tfrac{1}{4}wD^2\ , \tag{B10b}$$

$$\gamma_3=-wD^4/E\ , \tag{B10c}$$

$$\begin{aligned}\gamma_4=\tfrac{1}{2}[&\sigma ac_1+\tfrac{1}{2}\sigma b_1^2+\tfrac{1}{4}\sigma b_2^2+\tfrac{9}{4}\sigma(\sigma-1)a^2b_2-3\sigma(\sigma-1)a^2b_1+\tfrac{17}{6}\sigma(\sigma-1)(\sigma-2)a^4+\tfrac{1}{2}c_1\\&+3\sigma ab_2-4\sigma ab_1+\tfrac{17}{2}\sigma(\sigma-1)a^3+\tfrac{3}{4}b_2+\tfrac{9}{2}\sigma a^2-b_1+4\sigma a^2+\tfrac{17}{6}a]\ .\end{aligned} \tag{B10d}$$

APPENDIX C: NO-FREE-END CONTRIBUTIONS TO THE HIGHER-ORDER SUSCEPTIBILITIES SERIES

We use the notation n_b for the number of bonds in the diagram Γ, n_s for the number of sites in the diagram Γ, z_i for the number of neighbors of the site in the diagram Γ, and $\sigma_i=z_i-1$, as well as that of Appendix B. In specifying the $\Gamma_k^{\rm CT}$ and $\delta\Gamma_k(\Gamma)$ that appear on the right-hand side of Eq. (3.10), for $k=2$, 3, and 4 it is convenient for presentation to break up their contributions. For $k=2$,

$$\Gamma_2^{\rm CT}=D(1+w) \tag{C1}$$

and

$$\delta\Gamma_2(\Gamma)=D^2\left[\sum_{i=1}^{n_s} z_i^2w^2-2n_bw(1+w)+\sum_{k,l=1}^{n_s}(1-\sigma_kw)(1-\sigma_lw)[\langle S_kS_l\rangle^2]_{\rm av}\right]\ . \tag{C2}$$

Here and below we use the notation that

$$[\langle A_1\rangle\langle A_2\rangle\cdots\langle A_p\rangle]_{\rm av} \tag{C3}$$

is calculated for the Hamiltonian restricted to the bonds of the cluster Γ, which we write as

$$H(\Gamma)=\sum_{\langle ij\rangle\in\Gamma} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j\ . \tag{C4}$$

Then

$$\langle A\rangle\equiv\frac{{\rm Tr}(e^{-H(\Gamma)/k_BT}A)}{{\rm Tr}e^{-H(\Gamma)/k_BT}}\ . \tag{C5}$$

Thus, in Eq. (C2) (and similarly below) n_s, n_b, z_i, σ_k, and $[\langle S_kS_l\rangle^2]_{\rm av}$ all explicitly depend on Γ.

For $k=3$,

$$\Gamma_3^{\rm CT}=-4D^3(1+3w-3\sigma w^2-\sigma w^3)\ , \tag{C6a}$$

$$\delta\Gamma_3(\Gamma)=D^3\Lambda_3(\Gamma)-12(1-w)\delta\Gamma_2(\Gamma)\ , \tag{C6b}$$

where

$$\begin{aligned}\Lambda_3(\Gamma)=&\left[48n_b-12\sum_{i=1}^{n_s}z_i^2\right]w^2+\left[4\sum_{i=1}^{n_s}z_i^3-12\sum_{i=1}^{n_s}z_i^2+16n_b\right]w^3\\&+\sum_{k<l}\left[24(\sigma_k+\sigma_l-1)w+12[\sigma_k+\sigma_l-(\sigma_k+\sigma_l)^2-2\sigma_k\sigma_l]w^2+12\sigma_k\sigma_l(\sigma_k+\sigma_l)w^3\right][\langle S_kS_l\rangle^2]_{\rm av}\\&-24\sum_{i<j<k}(1-\sigma_iw)(1-\sigma_jw)(1-\sigma_kw)[\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_kS_i\rangle]_{\rm av}\ .\end{aligned} \tag{C7}$$

For $k=4$,

$$\begin{aligned}\Gamma_4^{\rm CT}=&\,34+68Dzw+D^3zw\{48+[48(z-1)(z^2-3z+1)-68(z-1)(z-2)]w^2\}\\&+34D^4zw[2-6(z-1)w^2-(z-1)(z^2-5z+2)w^3]\\&+48D^5z(z-1)w^2[2-zw-2(z-1)w^2+z(z-1)w^3]+54D^2Ezw^2[1-(z-1)w^2]\\&+54D^4Ez(z-1)w^3[2-zw-2(z-1)w^2+z(z-1)w^3]\ ,\end{aligned} \tag{C8a}$$

$$\delta\Gamma_4(\Gamma)=240D^4(1-w)^2\delta\Gamma_2(\Gamma)+D^5\Lambda_4^1(\Gamma)+D^4\Lambda_4^2(\Gamma)\ , \tag{C8b}$$

$$\Lambda_4^1(\Gamma)=(1-w)\Big[56w\sum_i z_i+w^2(232\sum_i z_i^2-656\sum_i z_i)+w^3(-96\sum_i z_i^3+424\sum_i z_i^2-328\sum_i z_i)\Big]$$

$$+576\sum_{i<j<k}[\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_iS_k\rangle]_{\rm av}(1-w\sigma_i)(1-w\sigma_j)(1-w\sigma_k)$$

$$+\sum_{i<j}[\langle S_iS_j\rangle^2]_{\rm av}[-112+w(848-464\sigma_i-464\sigma_j)+w^2(-560\sigma_i-560\sigma_j+288\sigma_i^2+288\sigma_j^2+1040\sigma_i\sigma_j)$$

$$+w^3(272\sigma_i\sigma_j-288\sigma_i^2\sigma_j-288\sigma_i\sigma_j^2)] \qquad \text{(C9a)}$$

and

$$\Lambda_4^2=34w^4\sum_i z_i^4-136w^3(1+w)\sum_i z_i^3+12w^2(17w^2+42w+9)\sum_i z_i^2-6w^2(17w^2+92w+27)\sum_i z_i$$

$$+24\sum_{i<j}[\langle S_iS_j\rangle^2]_{\rm av}\Big[\tfrac{17}{3}(1-w\sigma_i)^3(1-w\sigma_j)+\tfrac{17}{3}(1-w\sigma_j)^3(1-w\sigma_i)-16(1-w)(1-w\sigma_i)(1-w\sigma_j)$$

$$+\frac{1-w}{3}(7-17w)[2-w(\sigma_i+\sigma_j)]$$

$$+4(1-w)^2-4(1-w)[(1-w\sigma_i)^2+(1-w\sigma_j)^2]+4(1-w\sigma_i)^2(1-w\sigma_j)^2\Big]$$

$$+216\sum_{j<k,\,i\neq j,k}[\langle S_iS_j\rangle^2\langle S_iS_k\rangle^2]_{\rm av}(1-w\sigma_i)^2(1-w\sigma_j)(1-w\sigma_k)$$

$$-192\sum_{j<k,\,i\neq j,k}[\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_iS_k\rangle]_{\rm av}(1-w\sigma_j)(1-w\sigma_k)[1-w-(1-w\sigma_i)^2]$$

$$+108\sum_{i<j}[\langle S_iS_j\rangle^4]_{\rm av}(1-w\sigma_i)^2(1-w\sigma_j)^2$$

$$+\sum_{i<j<k<l}(1-w\sigma_i)(1-w\sigma_j)(1-w\sigma_k)(1-w\sigma_l)$$

$$\times[288([\langle S_iS_j\rangle\langle S_jS_k\rangle\langle S_kS_l\rangle\langle S_lS_i\rangle]_{\rm av}+[\langle S_iS_k\rangle\langle S_kS_j\rangle\langle S_jS_l\rangle\langle S_lS_i\rangle]_{\rm av}$$

$$+[\langle S_iS_j\rangle\langle S_jS_l\rangle\langle S_lS_k\rangle\langle S_kS_i\rangle]_{\rm av})$$

$$-96([\langle S_iS_jS_kS_l\rangle\langle S_iS_j\rangle\langle S_kS_l\rangle]_{\rm av}+[\langle S_iS_jS_kS_l\rangle\langle S_iS_k\rangle\langle S_jS_l\rangle]_{\rm av}+[\langle S_iS_jS_kS_l\rangle\langle S_iS_l\rangle\langle S_jS_k\rangle]_{\rm av})$$

$$+72([\langle S_iS_j\rangle^2\langle S_kS_l\rangle^2]_{\rm av}+[\langle S_iS_k\rangle^2\langle S_jS_l\rangle^2]_{\rm av}+[\langle S_iS_l\rangle^2\langle S_jS_k\rangle^2]_{\rm av})+24[\langle S_iS_jS_kS_l\rangle^2]_{\rm av}]\ . \qquad \text{(C9b)}$$

APPENDIX D: TEST-SERIES STUDY

In this appendix we describe test-series work for the $M1$ method and illustrate the importance of using $M1$ and $M2$ in tandem. Our main motivation for this work was to test the usefulness of the term-by-term divided series for cases of large correction to scaling exponents. One hopes to avoid the uncertainty associated with the choice of critical point by using the term-by-term divided series. In addition, for those cases where the nonanalytic correction is small, one attempts to obtain the dominant exponent accurately from a strong confluence of Padés near the analytic correction introduced by term-by-term division. This requires that the amplitude of the introduced analytic correction be sufficient to swamp the nonanalytic correction of the individual series which, of course, is still present. If the two corrections have similar amplitudes then convergence will be poorer than in the original series, owing to competing effects of the two corrections.

It is important to use the two distinct $M1$ and $M2$ methods, because in $M2$ there are[60] resonant convergences at $\Delta_1 = \Delta_1^*/n$, for $n = 1,2,3,\ldots$, where we use Δ_1^* to denote the correction term of the series itself. These resonances were first identified (see the note added in proof in Ref. 60) in Ref. 65 for real series. The resonances have never been seen with $M1$, and there is no analytical reason to suspect their presence in the $M1$ algorithm. Thus use of both $M1$ and $M2$ allows one to distinguish the effect of this resonance. We do not use $M1$ alone, as $M1$ gives[66] a slight systematic error in the correction experiment when it is far from analytic. Although we have published[56] extensive test-series work for $M2$, we publish test-series results for $M1$ in this appendix. A comparison between $M1$ and $M2$, is also given.

We have carried out a comprehensive series of tests of our different methods of analysis (both term by term divided, and temperature biased) on test series with large correction to scaling exponents. The series that we chose to study are of the form of Eq. (1.8) for $k=2$ with different values of the γ_2, Δ_1, and a_2 parameters. The γ, β, and Δ_1 values were chosen to mimic the measured values of the low-dimensional series. a_2, the amplitude of the correction term, was varied $(1 \le a_2 \le 5)$ in order to see how this affected the convergence. All the test-series results reported here used $w_c = \frac{1}{2}$, which is near the three-dimensional SG value. We begin with a report on the cases where $\Delta_1 < 1.0$. Unless otherwise stated, we present graphs of 20-term series but have also looked at shorter series. For these the convergence is a little looser, but there is no real difference in central estimates for series above 12 or so terms. We have considered $\Delta_1 = \frac{1}{3}$ and $\frac{1}{2}$, with $\gamma = 2.7$ and $a_2 = 3.0$. The results for both $M1$ and $M2$ are shown in Figs. 5(a) and 5(b), respectively, for the case of $\Delta_1 = 0.5$, and for the critical temperature biased to the correct threshold of $\frac{1}{2}$. We observe that for $M1$, we simply have a straight line, that passes through (0.5, 2.700). On a finer grid we see that there is a convergence of the different Padé's that is centered at the point (0.5, 2.700). For $M2$ we observe that as we scan along decreasing trial values of Δ_1, we first observe that there is convergence to the point (0.5, 2.6999), which then degrades as Δ_1 is further decreased. Convergence returns at the resonance $\Delta_1^*/2 = 0.25$ with $\gamma = 2.7000$ for Δ_1 trial values below $\frac{1}{6}$ for some Padé's, and for all Padé's in the region of $\Delta_1^*/3 = \frac{1}{6}$. These results are completely in accord with previously published results for both test and

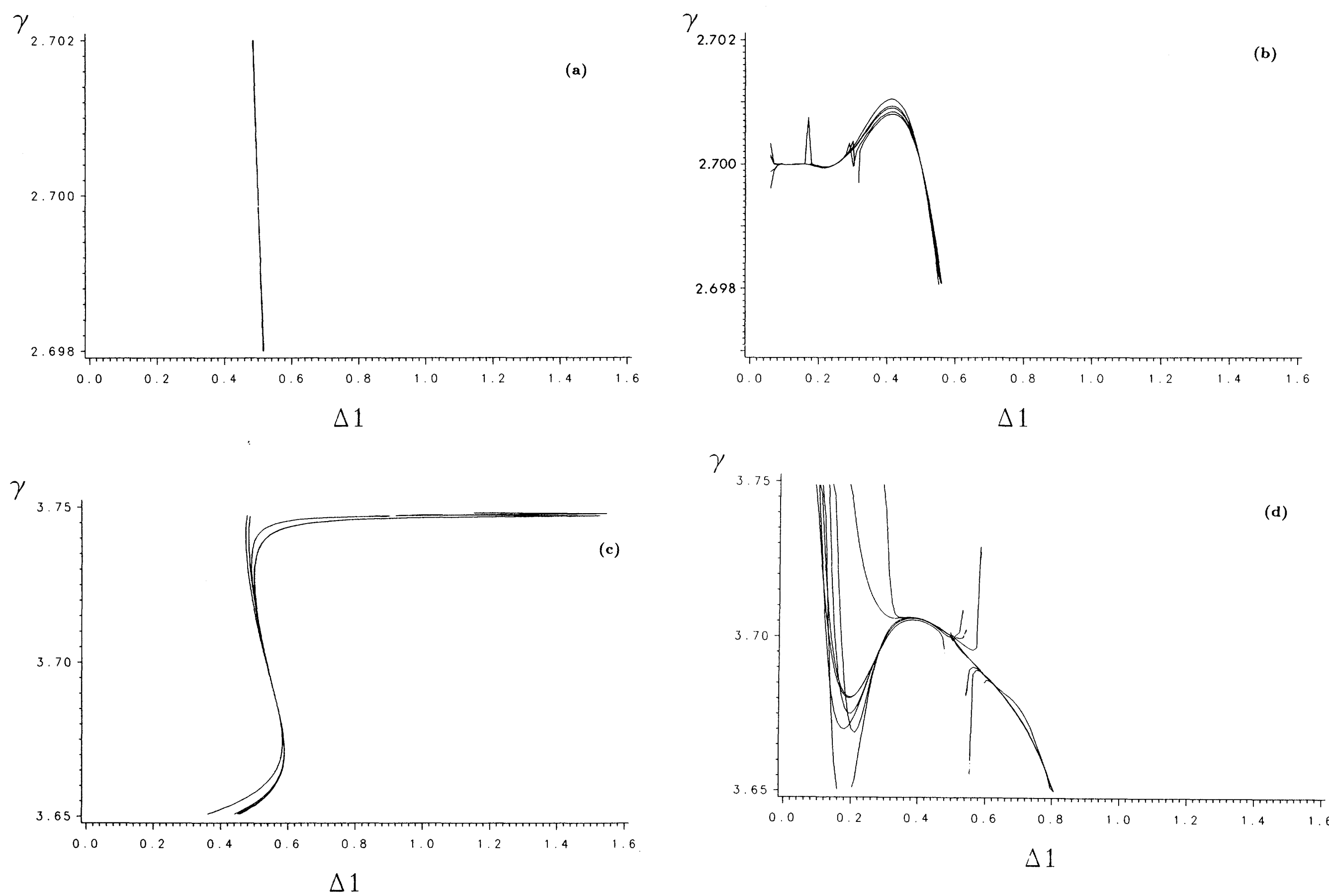

FIG. 5. Graphs of Padé approximants for the dominant critical exponent as a function of trial Δ_1 estimate for the test series with $\gamma = 2.7$, $\Delta_1 = 0.5$, and $a_2 = 3.0$ at the exact threshold of $w_c = 0.5$. We give the (a) $M1$ and (b) $M2$ analyses of the series itself and the (c) $M1$ and (d) $M2$ analyses for the self-divided series.

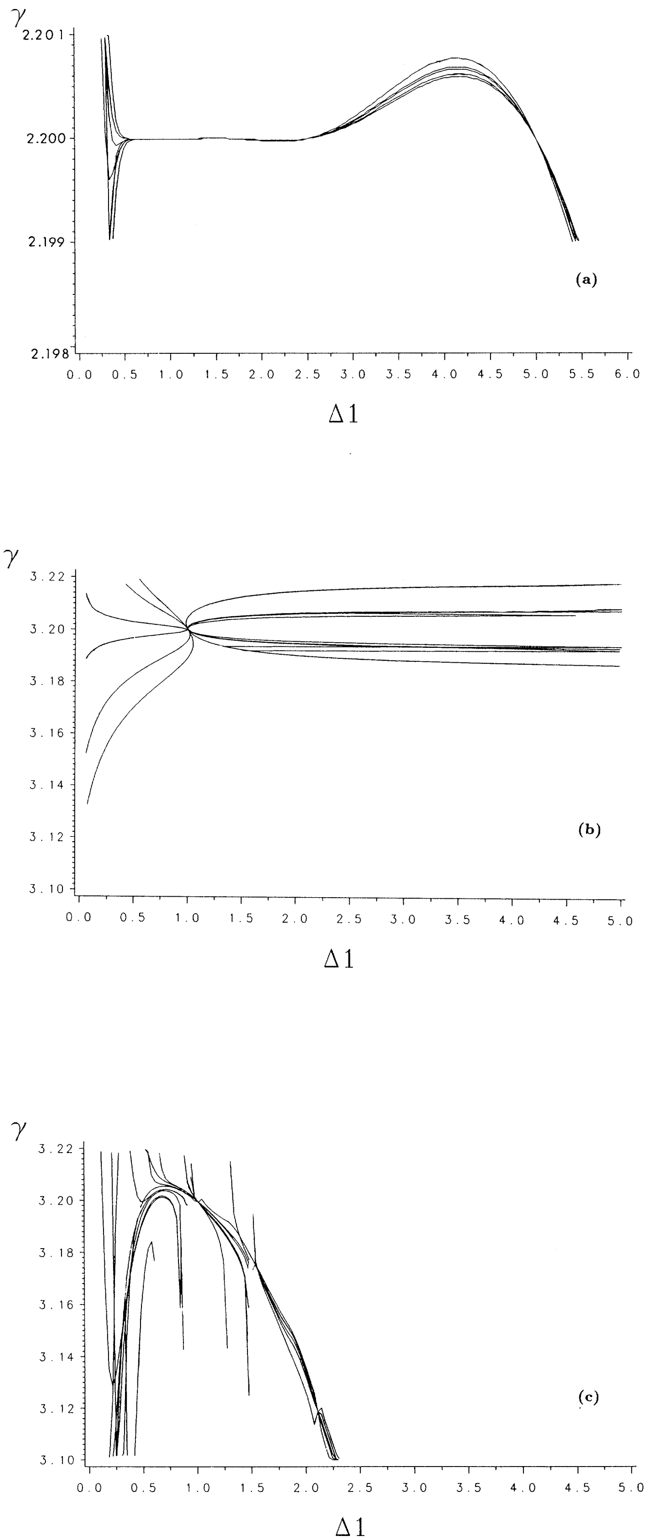

FIG. 6. Graphs of Padé approximants for the dominant critical exponent as a function of trial Δ_1 estimate for the test series with $\gamma=2.2$, $\Delta_1=5.0$, and $a_2=5.0$ at the exact threshold of $w_c=0.5$. We give the (a) $M2$ analyses of the series itself and the (b) $M1$, and (c) $M2$ analyses for the self-divided series.

real series. In Figs. 5(c) and 5(d), we present the results of the analysis for the term-by-term self-divided series. For the series studied above the expected values in the divided series should be $\gamma+1=3.7$ and $w_c=1.0$. In Fig. 5(c), we see that for $M1$ there is convergence only in the region of the original nonanalytic correction, with tightest convergence at (0.55, 3.69). Note the slight deviation from the exact values at optimal convergence. For $M2$, we observe resonances for $M2$ in Fig. 5(d); the main convergence occurs near the correct (0.5, 3.7) value close to that observed in the $M1$ graph in Fig. 5(c), and in addition there is a clear resonance in the region of (0.28, 3.7), where the 0.28 is an approximation to $\Delta_1^*/2$. The approximants pass near the value $\Delta_1=\Delta_1^*/3$ but do not converge. By comparing the results of both methods $M1$ and $M2$ we deduce that the four graphs concur that this series has $\gamma=2.7$ and $\Delta_1=0.5$ as expected. It is also clear that for this $\Delta_1<1.0$ case, the convergence in the divided series is at the Δ_1 value rather than at unity.

Similar studies have been made for the case of $\Delta_1>1$, with $\gamma=2.2$, 4.9, and 7.6, $\Delta=3.0$ and 5.0, and $a_2=1.0$ and 5.0. We select $\gamma=2.2$, $a_2=5.0$, and $\Delta_1=5.0$ as typical results for presentation in Fig. 6. Here w_c was again set at 0.5. The $M1$ graph is not presented as it is an even straighter line than that shown in Fig. 5(a). The $M2$ graph is given in Fig. 6(a), and we see convergence at (5.00, 2.20) with resonance at (2.5, 5.0) and below. The divided series graph of $M1$ and $M2$ are given in Figs. 6(b) and 6(c), respectively. Here we see convergence at (1.00, 3.20) in both cases. In the $M1$ case there is a clear single intersection at this point, but the $M2$ case is a little more complex. In Fig. 6(c) we observe additional intersections at other values of Δ_1, notably ≈ 1.5 and ≈ 2. There is a weak resonance at the dominant exponent value of 3.20, near $\Delta_1=\Delta_1^*/2=0.5$, but the approximants do not actually intersect. The existence of this resonance is a signal that 3.2 is indeed the correct dominant exponent, but true confirmation that 3.2 is the dominant exponent comes from comparison with the single intersection seen in the $M1$ results. We can deduce from these and similar graphs that for $\Delta_1>1$ the correct dominant exponent is seen at the introduced analytic correction, rather than at the original nonanalytic correction to scaling of the undivided series. At the original correction there is no convergence for $M1$. For $M2$ there is a rather interesting effect at Δ_1 values above unity. We see a series of spurious convergences at (1.5, 3.18) and at (2.0, 3.12). An in depth examination of this phenomenon is planned for a later paper. By the time we reach $\Delta_1=5.0$ (off the graph) the dominant exponent is far below its correct value.

We conclude from this test series analysis that use of both $M1$ and $M2$ is necessary, for both temperature biased and divided series. The results of the test on the divided series can be summarized as follows. For $\Delta_1<1$ the divided series converge at the correction of the original series. For $\Delta_1>1$ the divided series converge at the introduced analytic correction. In both cases convergence corresponds to the correct dominant exponent estimate. We note that this is in accord with the early test series work on $M2$ (Ref. 55) that we will always measure the correction to scaling with the lowest Δ_1 value.

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