Nets As Tensor Theories

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Abstract

This report is intended to describe and motivate a relationship between a class of nets and the fragment of linear logic built from the tensor connective. In this fragment of linear logic a net may be represented as a *theory* and a computation on a net as a *proof*. A rigorous translation is described and a soundness and completeness theorem is stated. The translation suggests connections between concepts from concurrency such as causal dependency and concepts from proof theory such as cut elimination. The main result of this report is a "cut reduction" theorem which establishes that any proof of a sequent can be transformed into another proof of the same sequent with the property that all cuts are "essential". A net-theoretic reading of this result tells that unnecessary dependencies from a computation can be eliminated resulting in a maximally concurrent computation. We note that it is possible to interpret proofs as arrows in the strictly symmetric strict monoidal category freely generated by a net and establish soundness of our proof reduction rules under this interpretation. Finally, we discuss how other linear connectives may be related to the concepts of internal and external choice.

1   Introduction

In this paper we explore the idea of describing the operational semantics of a net (the so-called "token game") in proof-theoretic terms. Under our approach, a net will correspond to a logical theory, and the token games on the net will be represented as proof trees in the "logic" of the net. This correspondence reveals an interesting relationship between concepts of proof theory (such as cut elimination) and fundamental concepts in concurrency (such as causal dependency) as they are illustrated by net theory. Our proof-theoretic representation works for a certain class of nets in which events are uniquely determined by their pre and post conditions. Such nets are represented as sets of sequents in a fragment of linear logic based on the tensor connective.

Linear logic is a system introduced by J. Y. Girard based on the inspiration of his work on a class of mathematical domains called *coherence spaces* [7,8]. One way of understanding propositional linear logic is to see it as a modification of propositional logic which takes seriously the concept of a *resource*. As such it is related to such systems as *relevance logic* which incorporate this concept as well (see [3] for a full discussion). Resources are also a familiar aspect of the theory of Petri nets. In what follows, we will attempt to convince the reader that the senses in which linear logic and Petri nets deal with resources have many things in common. Indeed, we will demonstrate a translation which characterizes the relationship exactly.

However, the way linear logic and nets represent resources is only a part of what we feel is a much more important common characteristic of the two theories: the way in which they illustrate *true concurrency*. It is well-known that nets provide an intuitive and pictorial way of seeing many fundamental ideas of concurrent computation. In what follows, we will show how this intuition may also be seen in the theories and proof trees of (a fragment of) linear logic.

Other researchers have independently looked at the relationship between Petri nets and linear logic. The work of Asperti [1,2] follows much the same basic intuition that we discuss below for the tensor connective. Carolyn Brown at the University of Edinburgh has proven a result similar to our soundness and completeness theorem and studied a fragement of linear logic formulae with additional connectives [4]. Narciso Martí-Oliet and José Meseguer [11] have discussed the relationship between Petri nets and linear logic from the point of view of category theory. We would like to acknowledge the assistance of Jean Yves Girard, who provided much of the inspiration for this investigation. We also thank Dexter Kozen, Prakash Panangaden, and Andre Scedrov for ideas and encouragement and acknowledge helpful discussions with Eike Best, Ursula Goltz, Ugo Montanari, and Wolfgang Reisig.

Throughout the rest of the paper we will assume some familiarity with net theory and proof theory. Concepts and notations related to former can be found in [15] and [6]. For the latter, [18] and [17] are excellent references.

2 Relating Nets and Theories

In this section we outline the fragment of linear logic on which this paper will be concentrating. The theory will be given in the form of a Gentzen style sequent calculus.

A *tensor formula* is either a propositional atom or the tensor product $A \otimes B$ of tensor formulas A and B. A *tensor sequent* is a pair $\Gamma \vdash A$ where Γ is a list of tensor formulae. A *tensor theory* is a set of tensor sequents. Of course, any set of sequents T will generate a tensor theory $\text{Th}(T)$ which is the least set of sequents containing T and closed under the rules in Figure 1. We say that $\Gamma \vdash A$ is *provable in T* if $\Gamma \vdash A$ is in $\text{Th}(T)$. We say that $\Gamma \vdash A$ is provable if it is in $\text{Th}(\emptyset)$. Let us say that a pair $A \vdash\dashv B$ is provable if $A \vdash B$ and $B \vdash A$ are both provable. It is not hard to see from these axioms that the tensor connective is associative and commutative:

Proposition 1 *For any A, B, C, the sequents $A \otimes B \vdash\dashv B \otimes A$ and $(A \otimes B) \otimes C \vdash\dashv A \otimes (B \otimes C)$ are provable.* ∎

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$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \text{(Exchange)} \qquad \frac{}{A \vdash A} \text{(Identity)} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \text{(Cut)}$$

Logical Rules

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} (\otimes R) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} (\otimes L)$$

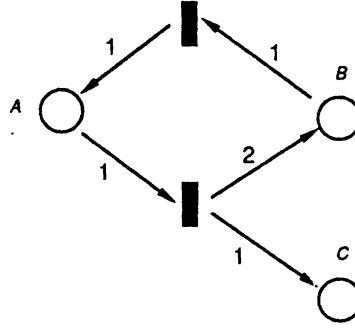Figure 1: Structural and logical rules for a fragment of linear logic.

Figure 2: Net N_0.

However, the tensor connective is *not* absorptive; for example, the sequent $A \otimes A \vdash A$ is not provable. It is therefore possible to think of a tensor formula as a *multi-set* (or "bag") of propositional atoms. Given a tensor formula A, let $M(A)$ be the multi-set of propositional atoms determined by A. It follows from the proposition that tensor formulae A and B such that $M(A) = M(B)$ are equivalent, *i.e.* $A \vdash\dashv B$. Moreover, sequents $\Gamma \vdash A$ and $\Delta \vdash A$ are equivalent in the sense that each can be derived from the other if the lists Γ and Δ determine the same multi-set of propositions. For this reason, we will treat sequents as pairs $\Gamma \vdash A$ where Γ is a multi-set.

For the purposes of this paper, a *net* N is a set S_N of *places* together with a set T_N of pairs of multi-sets over S_N. A pair $t = ({}^{\cdot}t, t^{\cdot}) \in N$ is called a *transition* of the net with *pre-condition* ${}^{\cdot}t$ and *post-condition* $t^{\cdot}$. Of course, this is only one of the many flavors of nets that have been studied in the rich literature on such structures. Nets, as defined here, are similar to place/transition-systems as defined, for example, in [15]. However, our notion of net has less structure since there are no capacities and a transition is uniquely determined by its pre and post conditions. Moreover, a net in our sense does not have a specified initial marking. One of the appealing characteristics of nets is the way they lend themselves to pictorial representation. For example, the net N_0 consisting of the pairs $(\{A\}, \{B, B, C\})$ and $(\{B\}, \{A\})$ is pictured as a labelled graph in Figure 2.

Before we offer a technical definition of just how a net determines a theory, we will attempt to motivate the basic idea by means of examples. Consider the net N_1 pictured in Figure 3. In this net, if we are given a token on the condition A, then it is possible to fire the event r Firing this event, exhausts
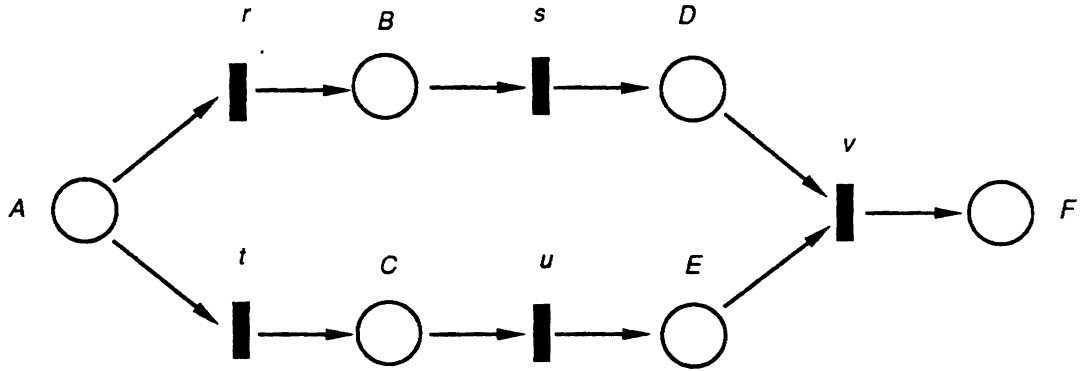
Figure 3: A net N_1 with concurrency and choice.

the token on A but provides a token on B. Logically, let us read the event r as an axiom $A \vdash B$ meaning "from A it is possible to obtain B." Similar ideas apply to the events s, t and u which we may read as $B \vdash D$ and $A \vdash C$ and $C \vdash E$ respectively. Now, event v requires a token on D and a token on E in order to fire and produce a token on F. We might therefore take $D, E \vdash F$ as the logical content of v. In summary, let T_1 be the set of axioms

$$A \vdash B \quad B \vdash D$$
$$D, E \vdash F$$
$$A \vdash C \quad C \vdash E$$

Do these axioms somehow characterize the net "logically"? If one interprets the comma between the D and E in the way that one ordinarily does in logic, this tempts one to think of $D, E \vdash F$ as $D \wedge E \vdash F$. But something is now wrong with the proposed "logical interpretation" of the net. In particular, it is easy to check that $A \vdash F$ is provable from the axioms T_1. However, if one's interpretation of $A \vdash F$ is "from the resource A one is able to obtain the resource F," then the deduction is evidently incorrect. The problem lies in the fact that ordinary propositional logic does not support properly a concept of "proof resource." The culprit (in this case) is the rule from first order logic which gives us:

$$\frac{A \vdash D \quad A \vdash E}{A \vdash D \wedge E}$$

This rule clearly does not reflect the desired intuition about resources. If I can use \$1 to buy a pepsi and \$1 to buy a coke, then I can't expect to use \$1 to buy both a pepsi *and* a coke. Of course, one can also write the conjunction rule as

$$\frac{A \vdash D \quad A \vdash E}{A, A \vdash D \wedge E}$$

but this only begs the issue, since some instance of the thinning rule:

$$\frac{\Gamma, X, X \vdash Y}{\Gamma, X \vdash Y}$$

would be used at a later step in the proof to remove the second copy of A and this rule is just as suspect as the earlier version of the conjunction rule. To deal with this problem, one needs a logic in which the
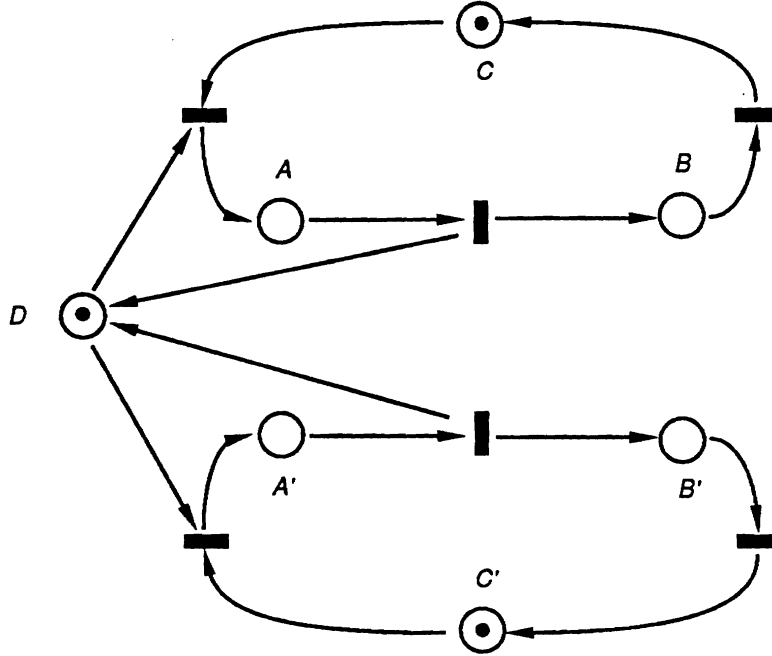
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Figure 4: A net N_2 with a critical region.

thinning rule is omitted and the second of the conjunction rules is used for the "and" connective that we have in mind.

The proper rules are those given in Figure 1 for the linear logic tensor connective $\otimes$. These rules keep track of the resources as needed. In linear logic, the sequent $A \vdash F$ is *not* provable in T_1. However, it is possible to check that $A, A \vdash F$ *is* provable in T_1, as we expect it should be. There are, in fact, several proofs of $A, A \vdash F$ in T_1; three of these are listed in Figure 5 (on page 7). We will come back to these proofs later to discuss how they relate to the net token games that move a token from the marking A, A to the marking F.

To give a slightly larger example, which we hope will suffice in giving the reader the general idea, consider the net N_2 in Figure 4. This net corresponds to the tensor theory T_2 with the following six axioms:

$$C \otimes D \vdash A \qquad\qquad B \vdash C$$
$$A \vdash B \otimes D$$
$$A' \vdash B' \otimes D$$
$$C' \otimes D \vdash A' \qquad\qquad B' \vdash C'$$

As one might expect, it will never be the case that from starting marking C, C', D, the resource $A \otimes A'$ is obtained. More precisely, one can show that $C \otimes C' \otimes D \not\vdash A \otimes A' \otimes A$ for any choice of linear proposition A.

A formal definition may be now be expressed as follows. Let N be a net and let S be the set of places of N. These will be the propositional atoms over which we form a set of tensor sequents as follows:

$$\mathcal{L}(N) = \{A \vdash B \mid \mathsf{M}(A) = {}^\cdot t \text{ and } \mathsf{M}(B) = t^\cdot \text{ for some } t \in T_N\}$$

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We will refer to $\mathcal{L}(N)$ as the *tensor theory determined by* N.

On the other hand, let T be a set of tensor sequents in a language with propositional atoms S. The theory T determines a net $\mathcal{N}(T)$ as the set $T_{\mathcal{N}(T)}$ of pairs $(\mathsf{M}(A), \mathsf{M}(B))$ such that $A \vdash B$ is in T. It is clear that $\mathcal{N}(\mathcal{L}(N)) = N$ for any net N. If $A' \vdash B'$ is an element of the set T whenever there is a sequent $A \vdash B$ in T such that $\mathsf{M}(A) = \mathsf{M}(A')$ and $\mathsf{M}(B) = \mathsf{M}(B')$, then it will also be the case that $\mathcal{L}(\mathcal{N}(T)) = T$. For example, the net N_1 in Figure 3 has $\mathcal{L}(N_1) = \{A \vdash B,\; B \vdash D,\; A \vdash C,\; C \vdash E,\; D {\otimes} E \vdash F,\; E {\otimes} D \vdash F\}$.

As the reader can guess from the examples, a marking M on a net N corresponds to a linear proposition A such that $\mathsf{M}(A) = M$. For example, the marking of the net N_1 in Figure 4 is represented by the proposition $C \otimes C' \otimes D$. In general, we have the following:

Theorem 2 (Soundness and Completeness) *Given a net N and markings M and M', the marking M' is in the forward marking set $[M\rangle$ of M if and only if the sequent $A \vdash A'$ is provable in the linear theory $\mathcal{L}(N)$ associated with N for any tensor formulae A and A' such that $\mathsf{M}(A) = M$ and $\mathsf{M}(A') = M'$.* ∎

We may apply the Soundness and Completeness Theorem to show how a non-trivial result from net theory leads to a result for a fragment of linear logic. Given a finite net N, it is decidable whether $M' \in [M\rangle$ for markings M and M'. This result is the culmination of a body of research which began with van Leeuwen [19] and has been worked on by a number of researchers [16 9,12,10,13]. Here is an immediate consequence:

Corollary 3 *Let N be a finite net and $\mathcal{L}(N)$ its associated linear theory. It is decidable whether $A \vdash B$ is provable in the theory $\mathcal{L}(N)$ for tensor formulae A and B.* ∎

Of course, the Corollary holds only for linear formulae in the small fragment of the system that we have discussed. Getting an assessment of how this result compares to known results about linear logic involves expanding our discussion to a larger fragment of the calculus. Since rules from $\mathcal{L}(N)$ may be used arbitrarily often, they must be represented as linear logic propositions using the "of course" operator, written $!A$. (Given a linear proposition A, the proposition $!A$ represents the "pure propositional content" of A. In the current context we may think of it as an unlimited resource of A's.) Linear propositional logic with the $!$ operator is not known to be decidable. The result above suggests that the decision procedure for this calculus, if it exists, will not be easy to find.

3 Proofs as Computations

Let us return now to our discussion of the net N_1 in Figure 3 (on page 4). This net displays some of the intuitive representations of concepts which have made nets an appealing model for both theoreticians and practitioners. The events r and t "compete" for the resource A and the events s and u are capable of running concurrently if they have the necessary resources B and C. There is a causal dependency between r and s: if r fires then s will be enabled. A similar dependency holds between t and u. If there is a line of computation which passes through r, s and another which passes through t, u, then these must "synchronize" before v is enabled. Most of these intuitions are represented in one form or another in the

Proof 1

$$
\cfrac{
\cfrac{
\cfrac{A \vdash A \qquad A \vdash B}{A,A \vdash A \otimes B}\ {\scriptstyle\otimes R}
\qquad
\cfrac{\cfrac{A \vdash C \qquad B \vdash B}{A,B \vdash C \otimes B}\ {\scriptstyle\otimes R}}{A \otimes B \vdash C \otimes B}\ {\scriptstyle\otimes L}
}{A,A \vdash C \otimes B}\ {\scriptstyle Cut}
\qquad
\cfrac{
\cfrac{\cfrac{B \vdash D \qquad C \vdash C}{B,C \vdash D \otimes C}\ {\scriptstyle\otimes R}}{C \otimes B \vdash D \otimes C}\ {\scriptstyle\otimes L}
}{\ }
}{\ }
$$

$$
\cfrac{
\cfrac{A,A \vdash C \otimes B \qquad C \otimes B \vdash D \otimes C}{A,A \vdash D \otimes C}\ {\scriptstyle Cut}
\qquad
\cfrac{\cfrac{\cfrac{D \vdash D \qquad C \vdash E}{D,C \vdash D \otimes E}\ {\scriptstyle\otimes R}}{D \otimes C \vdash D \otimes E}\ {\scriptstyle\otimes L}}{\ }
}{
\cfrac{
\cfrac{A,A \vdash D \otimes E \qquad D \otimes E \vdash F}{A,A \vdash F}\ {\scriptstyle Cut}
}{\ }
}
$$

Proof 2.

$$
\cfrac{
\cfrac{
\cfrac{A \vdash B \qquad A \vdash C}{A,A \vdash B \otimes C}\ {\scriptstyle\otimes R}
\qquad
\cfrac{\cfrac{B \vdash D \qquad C \vdash E}{B,C \vdash D \otimes E}\ {\scriptstyle\otimes R}}{B \otimes C \vdash D \otimes E}\ {\scriptstyle\otimes L}
}{A,A \vdash D \otimes E}\ {\scriptstyle Cut}
\qquad
D \otimes E \vdash F
}{A,A \vdash F}\ {\scriptstyle Cut}
$$

Proof 3.

$$
\cfrac{
\cfrac{
\cfrac{A \vdash 1 \qquad B \vdash D}{A \vdash D}\ {\scriptstyle Cut}
\qquad
\cfrac{A \vdash C \qquad C \vdash E}{A \vdash E}\ {\scriptstyle Cut}
}{A,A \vdash D \otimes E}\ {\scriptstyle\otimes R}
\qquad
D \otimes E \vdash F
}{A,A \vdash F}\ {\scriptstyle Cut}
$$

Figure 5: Three proofs that $A, A \vdash F$.

proof trees of the linear theory $\mathcal{L}(N_1)$. In particular, the *cut rule* corresponds to the concept of causal dependency or sequentialization. For example, to prove that $A \vdash D$, it is essential to use a cut. This relates to the fact that the event r must take place before the event s can be enabled. Basically, the only situation in which the cut rule is never needed for a proof is for a net whose theory is *trivial* since only in this case are there no causal dependencies! Hence, for the theory determined by a non-trivial net, we cannot expect that cut elimination is possible.

Given an initial marking $\{A, A\}$ on the net N_1, consider the following sequence of firings to produce F: first fire r, then fire t, then fire s, then fire u and then fire v. We can represent this by the following expression:

$$((((1_A \parallel r);(t \parallel 1_B));(s \parallel 1_C));(1_D \parallel u));v$$

where the semi-colon represents sequentialization, the parallel operator represents con urrency and an expression 1_X is the "idle event" on X. This computation is "maximally sequential" in the sense that it makes no real use of the possibility of doing two things "at the same time." This c responds to a linear logic proof in which there are many applications of the cut rule. This proof is give 1 as Proof 1 in Figure 5. But there are other ways the firing sequence from A, A to F could be carried out For example:

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first fire r and t, then fire s and u, and after thi., fire v. The following expression represents this firing sequence:

$$((r \parallel t);(s \parallel u));v.$$

This computation, which corresponds to Proof 2 in Figure 5, has still not made "maximal" use of concurrency, although it is better than the first firing sequence. Although r and t are not constrained to fire in any particular order, the event s, for example, is not permitted to fire until t has fired. This restriction is not really intrinsic to the causal dependencies of the net. On the other hand, it is clear that *no* firing sequence will allow v to be fired before both s and t have done so. The "best" or most concurrent firing sequence is therefore the following: fire r and then s while also firing t and then u, after this, fire v. This is represented by the expression:

$$((r;s) \parallel (t;u));v$$

which corresponds the Proof 3 in Figure 5.

Following these intuitions, it is desirable to provide a set of rewrite rules which will take proofs such as 1 and 2 and convert them to a "maximally concurrent" proof such as 3. This process resembles the cut elimination results from proof theory, but must differ in some ways since the cut elimination is being carried out in a theory in which cut *elimination* is impossible. A similar situation arises for cut elimination in a theory with equality where all but the cuts involving equational axioms can be eliminated. However, the "maximally concurrent" proof we desire cannot be obtained by a straight-forward translation of these ideas. Instead, it is necessary to rely on other intuitions about the correct forms.

4 Cut reduction.

In this section, we formalize the concepts intuitively discussed in the previous section. Our goal is to demonstrate a set of rewrite rules for transforming a given proof into a "maximally concurrent" proof of the same sequent. We begin by defining essential cuts and then state and prove the cut reduction theorem. The proof is based on giving a finite set of proof reduction rules which is shown to be strongly normalizing.

Definition 1 An instance of the cut rule in a proof is *trivial* if at least one of the premisses is an axiom of the form $A \vdash A$.

Definition 2 An instance of a cut rule in a proof is called *essential* if it is non-trivial and has the form

$$\frac{\Gamma \vdash A \qquad A \vdash B}{\Gamma \vdash B} \; Cut$$

where A is a netformula.

Theorem 4 (Cut-Reduction) *Given a net N and its associated deductive system $\mathcal{L}(N)$. If a sequent $\Gamma \vdash A$ is provable in $\mathcal{L}(N)$, then there is a proof of this sequent in $\mathcal{L}(N)$ such that all cuts are essential.*

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Intuitively, essential cuts seem to capture dependencies exactly as dictated by the underlying net. A proof is *cut-reduced* if all instances of cuts in it are essential.

We will give a collection of rewrite rules for proofs and show the existence of a normalizing sequence. We will then strengthen this result by establishing that the set of reduction rules is strongly normalizing . The theorem above will immediately follow from the proposition that every normal proof is cut-reduced.

Remark: Prawitz [14] distinguishes "normal form theorem", "normalization theorem", and "strong normalization theorem". In his terminology then, our cut-reduction theorem is a normal form theorem, the second theorem will be a normalization theorem, and the last one will be a strong normalization theorem.

We begin by enumerating transformations on proofs. Assume that a proof $\mathcal{P}$ ends with an inessential cut, *i.e.* it has the following form:

$$\mathcal{P}: \quad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} Cut{:}A$$

We will refer to the left and right sub-proofs as $\mathcal{P}'$ and $\mathcal{P}''$, respectively. The various transformations, based on the form of $\mathcal{P}'$ and $\mathcal{P}''$, are:

1. *Axioms*. This case is applicable when at least one of the sub-proofs is an axiom.

1.1 $\mathcal{P}'$ is an axiom. We have the following transformation:

$$\frac{A \vdash A \qquad A, \Delta \vdash B}{A, \Delta \vdash B} Cut{:}A \qquad \Rightarrow \qquad A, \Delta \vdash B$$

1.2 $\mathcal{P}''$ is an axiom. We transform $\mathcal{P}$ as follows:

$$\frac{\Gamma \vdash A \qquad A \vdash A}{\Gamma \vdash A} Cut{:}A \qquad \Rightarrow \qquad \Gamma \vdash A$$

2. *Permutation*. This rule is applied when at least one of the sub-proofs $\mathcal{P}'$ and $\mathcal{P}''$ terminates with a logical rule with the main formula being different from the cut formula A or with an essential cut. Following are the various possibilities.

2.1 Endsequent of $\mathcal{P}'$ is obtained by an essential cut or a logical rule whose main formula is different from the cut formula. We distinguish the following cases.

2.1.1 The last rule is a $\otimes L$. We obtain the new proof as follows:

$$\frac{\dfrac{\Gamma, B, C \vdash A}{\Gamma, B \otimes C \vdash A} \otimes L \qquad \Delta, A \vdash D}{\Gamma, B \otimes C, \Delta \vdash D} Cut{:}A \qquad \Rightarrow \qquad \dfrac{\dfrac{\Gamma, B, C \vdash A \qquad \Delta, A \vdash D}{\Gamma, B, C, \Delta \vdash D} Cut{:}A}{\Gamma, B \otimes C, \Delta \vdash D} \otimes L$$

2.1.2 The last rule in $\mathcal{P}'$ is an essential cut. Note that since we allow at most one formula in the succedent of a sequent, the last rule of $\mathcal{P}'$ cannot be a $\otimes R$ in 2.

$$\frac{\dfrac{\Gamma \vdash B \qquad B \vdash A}{\Gamma \vdash A} Cut{:}B \qquad A \vdash C}{\Gamma \vdash C} C_{t:A}$$

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$$\Rightarrow \qquad \cfrac{\Gamma \vdash B \qquad \cfrac{B \vdash A \qquad A \vdash C}{B \vdash C}\ Cut{:}A}{\Gamma \vdash C}\ Cut{:}B$$

Note in above that B is a netformula.

2.2 Similar to 2.1 above but for the sub-proof $\mathcal{P}''$. We distinguish the following cases.

2.2.1 The last rule is a $\otimes$L.

$$\cfrac{\Gamma \vdash A \qquad \cfrac{\Delta,C,D,A \vdash B}{\Delta,C \otimes D,A \vdash A}\ \otimes\text{L}}{\Gamma,\Delta,C \otimes D \vdash B}\ Cut{:}A \qquad \Rightarrow \qquad \cfrac{\cfrac{\Gamma \vdash A \qquad \Delta,C,D,A \vdash B}{\Gamma,\Delta,C,D \vdash B}\ Cut{:}A}{\Gamma,\Delta,C \otimes D \vdash B}\ \otimes\text{L}$$

2.2.2 The last rule is a $\otimes$R. We distinguish following two cases.

2.2.2.1 Cut formula A in upper left sequent of the last rule of $\mathcal{P}''$.

$$\cfrac{\Gamma \vdash A \qquad \cfrac{\Delta',A \vdash B \qquad \Delta'' \vdash C}{\Delta',\Delta'',A \vdash B \otimes C}\ \otimes\text{R}}{\Gamma,\Delta',\Delta'' \vdash B \otimes C}\ Cut{:}A \qquad \Rightarrow \qquad \cfrac{\cfrac{\Gamma \vdash A \qquad \Delta',A \vdash B}{\Gamma,\Delta' \vdash B}\ Cut{:}A \qquad \Delta'' \vdash C}{\Gamma,\Delta',\Delta'' \vdash B \otimes C}\ \otimes\text{R}$$

2.2.2.2 Cut formula A in upper right sequent of the last rule of $\mathcal{P}''$.

$$\cfrac{\Gamma \vdash A \qquad \cfrac{\Delta' \vdash B \qquad \Delta'',A \vdash C}{\Delta',\Delta'',A \vdash B \otimes C}\ \otimes\text{R}}{\Gamma,\Delta',\Delta'' \vdash B \otimes C}\ Cut{:}A \qquad \Rightarrow \qquad \cfrac{\Delta' \vdash B \qquad \cfrac{\Gamma \vdash A \qquad \Delta'',A \vdash C}{\Gamma,\Delta'' \vdash C}\ Cut{:}A}{\Gamma,\Delta',\Delta'' \vdash B \otimes C}\ \otimes\text{R}$$

2.2.3 The last rule of $\mathcal{P}''$ is an essential cut. In this case, the cut formula cannot come from the upper right sequent of the essential cut above. Thus we have only one case to consider.

$$\cfrac{\Gamma \vdash A \qquad \cfrac{\Delta',A \vdash B \qquad B \vdash C}{\Delta',A \vdash C}\ Cut{:}B}{\Gamma,\Delta' \vdash C}\ Cut{:}A$$

$$\Rightarrow \qquad \cfrac{\cfrac{\Gamma \vdash A \qquad \Delta',A \vdash B}{\Gamma,\Delta' \vdash B}\ Cut{:}A \qquad B \vdash C}{\Gamma,\Delta' \vdash C}\ Cut{:}B$$

Note once again that B belongs to some netaxiom in the two cases above.

3. *Logical*. This is the case where the cut formula is the main formula of a logical rule in both $\mathcal{P}'$ and $\mathcal{P}''$ and is introduced only by this instance of the rule. The transformation in this case depends on the outermost logical symbol of the cut formula and since we only have one logical connective, there is only one case to consider here.

$$\cfrac{\cfrac{\Gamma' \vdash A_1 \qquad \Gamma'' \vdash A_2}{\Gamma',\Gamma'' \vdash A_1 \otimes A_2}\ \otimes\text{R} \qquad \cfrac{A_1,A_2,\Delta \vdash B}{A_1 \otimes A_2,\Delta \vdash B}\ \otimes\text{L}}{\Gamma',\Gamma'',\Delta \vdash B}\ Cut{:}A_1 \otimes A_2$$

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$$\Rightarrow \quad \cfrac{\Gamma'' \vdash A_2 \qquad \cfrac{\Gamma' \vdash A_1 \qquad A_1, A_2, \Delta \vdash B}{\Gamma', A_2, \Delta \vdash B} \; Cut:A_1}{\Gamma'', \Gamma', \Delta \vdash B} \; Cut:A_2$$

Remark: It may seem that the rule 2.1.2 does not appear in its most general form and one may be tempted to consider the following as its most general form:

$$\cfrac{\cfrac{\Gamma \vdash B \qquad B \vdash A}{\Gamma \vdash A} \; Cut:B \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C} \; Cut:A$$

However, such a form is not only redundant but incorrect too. First, note that in the situation as above, the comma suggests that $\Delta, A \vdash C$ is obtained by a $\otimes R$ or $\otimes L$, and hence 2.2.1 or 2.2.2 would be applicable. An attempt to give a reduction rule based on the form above by permuting the the two cuts will make the cut on B inessential (unless Δ is empty, in which case 2.1.2 applies), thus destroying an important invariance property of these transformations. Also, note that in the case 3 above, the transformation splits a cut into two cuts but with cut formulas with less number of logical symbols. The transformation as presented first performs a cut on A_1 and then on A_2. However, we could have done a cut on A_2 before A_1 giving us another transformation. But including one or the other or both does not affect our results.

The following lemma singles out an important property of the above transformations.

Lemma 5 *Let $\mathcal{P}$ be a proof and let $\mathcal{P}'$ be a proof obtained from $\mathcal{P}$ by the applications of the transformations above, then the number of sequents (nodes) in $\mathcal{P}'$ (viewed as a tree) is less than or equal to the number of nodes in $\mathcal{P}$, i.e., the number of nodes in a proof is never increased by the application of the transformations above.*

Proof: Immediate. ∎

Definition 3 A proof $\mathcal{P}$ is in *normal* form if there does not exist a proof $\mathcal{P}'$ such that $\mathcal{P} \Rightarrow \mathcal{P}'$ (one step reduction) by the transformations above.

Lemma 6 *A proof P is in normal form iff it is cut-reduced.* ∎

Proof: (if part) Clearly $\mathcal{P}$ is trivially normal if it does not contain any inessential cut.

(only if part) Assume on the contrary that $\mathcal{P}$ is normal and contain inessential cuts. In $\mathcal{P}$ choose an inessential cut above which there is no other inessential cut. Clearly then one of the reduction rules given above is applicable to this (sub) proof $\mathcal{P}'$ in $\mathcal{P}$ depending on how the premisses of the (only) inessential cut in $\mathcal{P}'$ are obtained. This contradicts the assumption that $\mathcal{P}$ is normal. Hence a normal proof is cut-reduced. ∎

The following lemma is our main lemma which shows the existence of a normalizing sequence of reduction.

Lemma 7 *If $\mathcal{P}$ is a proof of $\Gamma \vdash A$ which contains only one (inessential) cut occurring as the last inference, then $\Gamma \vdash A$ is provable with no inessential cut.*

The proof of the theorem then immediately follows from the above lemma by an easy induction on the number of inessential cuts appearing in a proof. In any proof consider an inessential cut above whose lower sequent no inessential cuts appear; thus satisfying the condition of the lemma. According to the lemma this (sub) proof can be transformed into another (equivalent) proof which does not contain this cut. In doing so, rest of the proof remains unchanged. We get a cut-reduced (equivalent) proof by repeating this process until all the inessential cuts have been eliminated.

Proof: (of the main lemma) Easy induction on the number of nodes in a proof satisfying the condition of the lemma. ∎

The following is now immediate.

Theorem 8 *Let $\mathcal{P}$ be a proof. Then there exists a sequence of reductions such that $\mathcal{P} \Rightarrow^* \mathcal{P}'$, and $\mathcal{P}'$ is in normal form.* ∎

The following definition will be used in the proof of our next theorem.

Definition 4 The *grade* g of a formula A is the number of $\otimes$ contained in A. The grade of an inessential cut is the grade of its cut formula.

Thus, by the definition above, grade of an essential cut is 0.

Theorem 9 (Strong Normalization) *There is no infinite reduction sequence beginning with any proof $\mathcal{P}'$.*

Proof: We define a measure on proofs and show that each one step transformation reduces this measure. Let the *complexity* of a proof be a pair (a, b), where

- a = sum of the grade g of cut formulas of all inessential cuts in the proof.

- b = sum of the nodes above all inessential cuts (including the premisses and conclusion of the cut).

Clearly, a cut-reduced proof has complexity (0,0).

Now consider the three (main) classes of the transformations above. It is easy to see that application of these transformations in each case to a proof reduces its complexity.

Axiom: Both a and b are reduced.

Permutation: b is reduced keeping a he same.

Logical: a is reduced.

Thus, all reduction sequences terminate. ∎

In Appendix A we have written out how the rewriting works on Proof 1 and . in Figure 5.

5 Proofs as arrows.

A variety of publications have focused on the category-theoretic characteristics of Petri nets. In this section we hope to demonstrate that the proof transformations which we describe in the previous section are compatible with at least one elegant theory of nets as categories. To this end, we note how proofs can be interpreted as arrows in the category $T[\mathcal{N}]$ of Degano, Meseguer, and Montanari [5] and then show that the proof reduction rules as presented above are sound with respect to this interpretation, *i.e.* they transform arrows to equal arrows.

Let N be a net and $\mathcal{L}(\mathcal{N})$ be the tensor theory determined by it. Also, let $Proofs\,(N)$ denote the class of proofs in the theory $\mathcal{L}(\mathcal{N})$ and let $Mor\,(N)$ denote the class of morphisms of the strictly symmetric strict monoidal category $T[\mathcal{N}]$ freely generated by N. We define our interpretation $\mathbf{I} : Proofs\,(N) \longrightarrow Mor\,(N)$ as follows, where in writing

$$\frac{\Pi}{\Gamma \vdash A}$$

we mean a proof Π with conclusion $\Gamma \vdash A$.

1.
$$\mathbf{I}(A \vdash A) = i_A : A \to A$$

2.
$$\mathbf{I}(A_1 \otimes A_2 \cdots \otimes A_n \vdash B_1 \otimes B_2 \cdots \otimes B_m) = t : A_1 \otimes A_2 \cdots \otimes A_n \to B_1 \otimes B_2 \cdots \otimes B_m,$$

where t is in N.

3.
$$\mathbf{I}\left(\frac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ \Gamma \vdash A & \Delta \vdash B \end{array}}{\Gamma, \Delta \vdash A \otimes B} \right) = f \otimes g : \Gamma \otimes \Delta \to A \otimes B,$$

where $f : \Gamma \to A = \mathbf{I}\left(\begin{smallmatrix} \Pi_1 \\ \Gamma \vdash A \end{smallmatrix} \right)$ and $g : \Delta \to B = \mathbf{I}\left(\begin{smallmatrix} \Pi_2 \\ \Delta \vdash B \end{smallmatrix} \right)$.

4.
$$\mathbf{I}\left(\frac{\begin{array}{c} \Pi \\ \Gamma, A, B \vdash C \end{array}}{\Gamma, A \otimes B \vdash C} \right) = \mathbf{I}(\Gamma, A, B \vdash C)$$

5.
$$\mathbf{I}\left(\frac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ \Gamma \vdash A & A, \Delta \vdash B \end{array}}{\Gamma, \Delta \vdash B} \right) = (f \otimes i_\Delta) \circ g : \Gamma \otimes \Delta \to B,$$

where $f : \Gamma \to A = \mathbf{I}\left(\begin{smallmatrix} \Pi_1 \\ \Gamma \vdash A \end{smallmatrix} \right)$ and $g : A \otimes \Delta \to B = \mathbf{I}\left(\begin{smallmatrix} \Pi_2 \\ A, \Delta \vdash B \end{smallmatrix} \right)$.

13

Proposition 10 *The proof reduction rules are sound with respect to the interpretation above.*

Proof: We just consider an illustrative case here. Consider the reduction rule 2.2.2.1. The function I yields an arrow corresponding to the left hand side as follows. Let $f : \Gamma \to A$, $g : \Delta' \otimes A \to B$, and $h : \Delta'' \to C$. Then we have:

$$(f \otimes i_{\Delta' \otimes \Delta''}) \circ (g \otimes h) : \Gamma \otimes (\Delta' \otimes \Delta'') \to B \otimes C$$
$$= (f \otimes (i_{\Delta'} \otimes i_{\Delta''})) \circ (g \otimes h)$$
$$= ((f \otimes i_{\Delta'}) \otimes i_{\Delta''}) \circ (g \otimes h)$$
$$= ((f \otimes i_{\Delta'}) \circ g) \otimes (i_{\Delta''} \circ h)$$
$$= ((f \otimes i_{\Delta'}) \circ g) \otimes h$$
$$= \mathbf{I}\,(\text{right hand side})\ \blacksquare$$

In view of the above proposition and the strong normalization theorem, the following is immediate.

Corollary 11 *Every proof reduces to a unique normal form modulo the interpretation.* $\blacksquare$

It has long been argued by proof theorists that a notion of equivalence of proofs based on mere provability is too extensional and inadequate. But the question of the right notion of equivalence of proofs still remains open. Prawitz [14], for the system of Natural Deduction and his set of reduction rules, conjectured that two derivations represent the same proof if and only if they reduce to the same normal form. Now in view of corollary 11 above we may say something similar for the identification of the derivations in a tensor theory. However, it seems that such an identification does not quite capture the intuitive sense of equivalence (based on processes) that we have in mind for net computations and is still too extensional. For example, proof 2 and proof 3 of section 3 would be identified as the corresponding arrows are equal because $\_ \otimes \_$ is a bifunctor. However, the process interpretation that we have in mind should not result in such an identification. Thus the sense in which proof 3 is not equivalent to proof 2 (and in fact better) is lost in the denotational view that we have presented in this section. We are currently looking at how to attach such intensional interpretations to proofs in our setting. We have made some partial progress towards this, though mostly via some *ad hoc* means.

6 Choice Situations

We have so far restricted attention to a rather small fragment of linear logic because this fragment is already sufficient to illustrate several important concepts that suggest interesting relationships between concurrent computations and proofs. However, we believe that this is really only the beginning of the story. To give the reader a taste of how the theory can be further developed, we will give two simple examples that illustrate the potential role of the linear connectives known as "direct product" and "direct sum."

Given linear formulas, A and B, the expression $A \,\&\, B$ is a linear formula pronounced "A direct product B". The $\&$R rule is

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

Intuitively, $A \& B$ can be obtained from the resources Γ provided these resources can be used to obtain A and can also be used to obtain B. Note carefully how this differs from the $\otimes R$ rule where the resources must be divided in two parts—one part for proving A and the other for proving B. The $\&L$ rules are

$$\frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}$$

Intuitively, the resources C can be obtained from $A \& B$ provided they can be obtained *either* from A *or* from B.
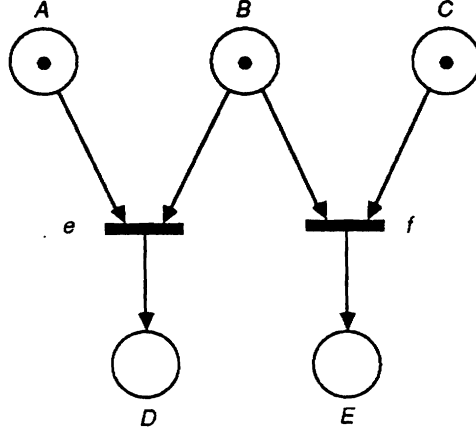
Figure 6: $A \otimes B \otimes C \vdash (D \otimes C) \& (A \otimes E)$

The intuitive explanations given above are meant to suggest to the reader the idea that the *direct product* operator represents a form of *choice*. To see a very simple example which we hope will be convincing enough to capture the reader's interest, consider the net pictured in Figure 6. Following the theory that we have developed in the preceding sections, this net is represented by the linear theory consisting of the sequents $A \otimes B \vdash D$ and $B \otimes C \vdash E$. Now, given a starting marking of one token on each of A, B and C, it is clear that a token can be moved to *at most one* of the conditions D and E. One might say that "$D \vee E$" is an obtainable marking, but "$D \wedge E$" is not. On the other hand, it seems that "$D \wedge E$" *is* obtainable in the sense that either of the conditions D and E *may* be fulfilled by some firing. In linear logic one may express this state of affairs with the formula $(D \otimes C) \& (A \otimes E)$. Here is a proof of the proper statment:

$$\frac{\dfrac{\dfrac{A \otimes B \vdash D \qquad C \vdash C}{A \otimes B, C \vdash D \otimes C} \otimes R}{A \otimes B \otimes C \vdash D \otimes C} \otimes L \qquad \dfrac{\dfrac{A \vdash A \qquad B \otimes C \vdash E}{A, B \otimes C \vdash A \otimes E} \otimes R}{A \otimes B \otimes C \vdash A \otimes E} \otimes L}{A \otimes B \otimes C \vdash (D \otimes C) \& (A \otimes E)} \& R$$

It is our feeling that the *direct product* operator captures a form of *external* choice. On the other hand, the *linear disjunction* captures a concept of *internal* choice. Given two linear propositions A and B, one proves the linear disjunction $A \oplus B$ of A and B from hypotheses Γ by using one of the following rules:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$
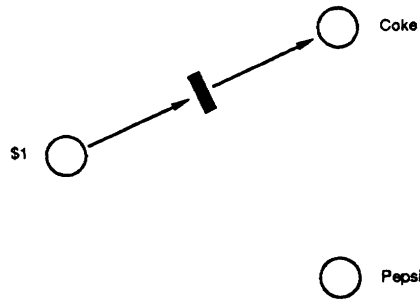
Figure 7: Coca Cola implements $1 \vdash$ coke $\oplus$ pepsi.

In other words, the resource $A \oplus B$ can be obtained from Γ just in case *either* A *or* B can be. On the other hand, if one wishes to obtain C from Γ and resource $A \oplus B$, then it must be shown that C can be obtained from *both* A *and* B. The rule is

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}$$
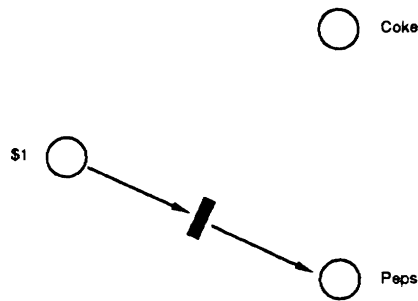
Figure 8: Pepsi Cola implements $1 \vdash$ coke $\oplus$ pepsi.
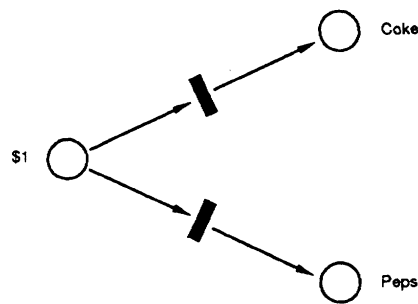
Figure 9: An implementation with choice: $1 \vdash$ coke & pepsi.

The internal/external distinction can be illustrated by a simple example which takes linear propositions as a specification language. Let us assume that we wish to contract a vendor to build us a machine for dispensing soft drinks for $1 and there are two possible flavors of drink that are available: coke and pepsi. If we do not actually care which of these is dispensed when a dollar is given to the machine, we may make the specification $1 \vdash$ coke $\oplus$ pepsi. The choice of which beverage we are given in exchange for $1 will

then be *internal* to the machine. For example, the machine which the Coca Cola company might design is pictured in Figure 7 and would dispense only coke. On the other hand, the Pepsi Cola company might design the machine in Figure 8 which dispenses only pepsi. In both cases, the tensor theories determined by the nets are strong enough that it is possible to prove the proposition $1 \vdash coke \oplus pepsi$. Thus the machines have met the specification. If, on the other hand, we wish to insure that there is an *external* choice so that a user can pick the flavor of his preference, then we might have used the specification $1 \vdash coke \& pepsi$. This specification is not met by either of the machines in Figures 7 and 8 since this sequent is not provable in either of the linear theories associated with these machines. On the other hand, the net in Figure 9 *does* meet the specification, since its associated theory is strong enough to prove the specifying sequent.

More research is needed to understand the possible significance of linear connectives in specification languages. While the example seems reasonable, it is worth keeping in mind that several other nets would meet the desired specification. For example, the net associated with the following theory

$$1 \vdash coke$$
$$1 \vdash pepsi$$
$$pepsi \vdash \$1$$

which gives a $1 in exchange for a pepsi, also meets the specification! On the other hand, the net associated with the sequent $1 \vdash coke \otimes pepsi$ which dispenses both a coke *and* a pepsi for each $1 does *not* implement the specification.

There are several more linear connectives which we will not discuss. No account with which we are familiar has addressed the linear logic negation, which seems to represent the dual of a resource—a "debt" perhaps. A treatment of negation would lead to an understanding of the dual of the tensor called the *par* which seems to represent a concept of "concurrent debts". We mentioned earlier the unary operator ! which represents an unlimited resource. This operator plays a subtle role in the theory we have exposited; work of Carolyn Brown [4] provides helpful insight. All of the linear logic connectives seem to have their own significance in terms of computation on nets. (We have included a list of some of the rules of linear logic in Appendix B.) Work on the exploitation of these ideas is likely to be a profitable for the study of both concurrency and proof theory.

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A Sample Proof Transformation

We give some examples of cut-reduction below. At each step of the reduction, the inessential cut to which a reduction rule is applied is denoted $\boxed{Cut}$. Other choices of inessential cuts, if any, at a step to which a rule could have been applied are denoted $\overline{Cut}$. Remaining inessential cuts are denoted $\underline{Cut}$.

Example 1

$$
\cfrac{
  \cfrac{A \vdash B \qquad A \vdash C}{A,A \vdash B \otimes C}\;{\otimes\mathrm{R}}
  \qquad
  \cfrac{
    \cfrac{\cfrac{B \vdash D \qquad C \vdash E}{B,C \vdash D \otimes E}\;{\otimes\mathrm{R}}}{B \otimes C \vdash D \otimes E}\;{\otimes\mathrm{L}}
  }{}
}{
  \cfrac{A,A \vdash D \otimes E \qquad\qquad\qquad\qquad D \otimes E \vdash F}{A,A \vdash F}\;{Cut}
}\;\boxed{Cut}
$$

$$
\overset{3}{\Longrightarrow}\quad
\cfrac{
  A \vdash C \qquad
  \cfrac{A \vdash B \qquad \cfrac{\cfrac{B \vdash D \qquad C \vdash E}{B,C \vdash D \otimes E}\;{\otimes\mathrm{R}}}{\;}\;\boxed{Cut}}{A,C \vdash D \otimes E}\;{\underline{Cut}}
}{
  \cfrac{A,A \vdash D \otimes E \qquad\qquad D \otimes E \vdash F}{A,A \vdash F}\;{Cut}
}
$$

$$
\overset{2.2.2.1}{\Longrightarrow}\quad
\cfrac{
  A \vdash C \qquad
  \cfrac{\cfrac{A \vdash B \qquad B \vdash D}{A \vdash D}\;{Cut} \qquad C \vdash E}{A,C \vdash D \otimes E}\;{\otimes\mathrm{R}}\;\boxed{Cut}
}{
  \cfrac{A,A \vdash D \otimes E \qquad\qquad D \otimes E \vdash F}{A,A \vdash F}\;{Cut}
}
$$

$$
\overset{2.2.2.2}{\Longrightarrow}\quad
\cfrac{
  \cfrac{\cfrac{A \vdash B \qquad B \vdash D}{A \vdash D}\;{Cut} \qquad \cfrac{A \vdash C \qquad C \vdash E}{A \vdash E}\;{Cut}}{A,A \vdash D \otimes E}\;{\otimes\mathrm{R}} \qquad D \otimes E \vdash F
}{
  A,A \vdash F
}\;{Cut}
$$

Example 2.

$$
\cfrac{
  \cfrac{A \vdash A \qquad \cfrac{\cfrac{A \vdash C \quad B \vdash B}{A,B \vdash C \otimes B}\;{\otimes\mathrm{R}}}{A \otimes B \vdash C \otimes B}\;{\otimes\mathrm{L}}}{A,A \vdash C \otimes B}\;{\otimes\mathrm{R}}\;\boxed{Cut}
  \;\;
  \cfrac{\cfrac{\cfrac{B \vdash D \quad C \vdash C}{B,C \vdash D \otimes C}\;{\otimes\mathrm{R}}}{C \otimes B \vdash D \otimes C}\;{\otimes\mathrm{L}}}{}\;{\underline{Cut}}
}{
  \cfrac{A,A \vdash D \otimes C \qquad \cfrac{\cfrac{D \vdash D \quad C \vdash E}{D,C \vdash D \otimes E}\;{\otimes\mathrm{R}}}{D \otimes C \vdash D \otimes E}\;{\otimes\mathrm{L}}\;{\underline{Cut}} \qquad D \otimes E \vdash F}{\cfrac{A,A \vdash D \otimes E}{A,A \vdash F}}\;{Cut}
}
$$

$$
\overset{3}{\Longrightarrow}\quad
\cfrac{
  A \vdash B \qquad
  \cfrac{A \vdash A \qquad \cfrac{\cfrac{A \vdash C \quad B \vdash B}{A,B \vdash C \otimes B}\;{\otimes\mathrm{R}}}{A,B \vdash C \otimes B}\;\boxed{Cut}}{A,A \vdash C \otimes B}\;{\underline{Cut}}
  \;\;
  \cfrac{\cfrac{\cfrac{B \vdash D \quad C \vdash C}{B,C \vdash D \otimes C}\;{\otimes\mathrm{R}}}{C \otimes B \vdash D \otimes C}\;{\otimes\mathrm{L}}}{}\;{\underline{Cut}}
}{
  \cfrac{A,A \vdash D \otimes C \qquad \cfrac{\cfrac{D \vdash D \quad C \vdash E}{D,C \vdash D \otimes E}\;{\otimes\mathrm{R}}}{D \otimes C \vdash D \otimes E}\;{\otimes\mathrm{L}}\;{\underline{Cut}} \qquad D \otimes E \vdash F}{\cfrac{A,A \vdash D \otimes E}{A,A \vdash F}}\;{Cut}
}
$$

19

$$\overset{1.1}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{A\vdash B \quad \cfrac{A\vdash C \quad B\vdash B}{A,B\vdash C\otimes B}\,\otimes\mathrm{R}}{A,A\vdash C\otimes B}\,\boxed{Cut}
      \quad
      \cfrac{\cfrac{B\vdash D \quad C\vdash C}{B,C\vdash D\otimes C}\,\otimes\mathrm{R}}{C\otimes B\vdash D\otimes C}\,\otimes\mathrm{L}
    }{A,A\vdash D\otimes C}\,\underline{Cut}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\underline{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$\overset{2.2.2.2}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{A\vdash C \quad \cfrac{A\vdash B \quad B\vdash B}{A\vdash B}\,\boxed{Cut}}{A,A\vdash C\otimes B}\,\otimes\mathrm{R}
      \quad
      \cfrac{\cfrac{B\vdash D \quad C\vdash C}{B,C\vdash D\otimes C}\,\otimes\mathrm{R}}{C\otimes B\vdash D\otimes C}\,\otimes\mathrm{L}
    }{A,A\vdash D\otimes C}\,\underline{Cut}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\underline{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$\overset{1.2}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{A\vdash C \quad A\vdash B}{A,A\vdash C\otimes B}\,\otimes\mathrm{R}
      \quad
      \cfrac{\cfrac{B\vdash D \quad C\vdash C}{B,C\vdash D\otimes C}\,\otimes\mathrm{R}}{C\otimes B\vdash D\otimes C}\,\otimes\mathrm{L}
    }{A,A\vdash D\otimes C}\,\boxed{Cut}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\underline{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$\overset{3}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      A\vdash B \quad
      \cfrac{A\vdash C \quad \cfrac{B\vdash D \quad C\vdash C}{B,C\vdash D\otimes C}\,\otimes\mathrm{R}}{A,B\vdash D\otimes C}\,\boxed{Cut}
    }{A,A\vdash D\otimes C}\,\underline{Cut}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\underline{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$\overset{2.2.2.1}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      A\vdash B \quad
      \cfrac{\cfrac{A\vdash C \quad C\vdash C}{A\vdash C}\,\boxed{Cut} \quad B\vdash D}{A,B\vdash D\otimes C}\,\otimes\mathrm{R}
    }{A,A\vdash D\otimes C}\,\overline{Cut}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\underline{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$\overset{1.2}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      A\vdash B \quad
      \cfrac{A\vdash C \quad B\vdash D}{A,B\vdash D\otimes C}\,\otimes\mathrm{R}
    }{A,A\vdash D\otimes C}\,\boxed{Cut}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\underline{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$\overset{2.2.2.2}{\Longrightarrow}\qquad
\cfrac{
  \cfrac{
    \cfrac{
      A\vdash C \quad
      \cfrac{A\vdash B \quad B\vdash D}{A\vdash D}\,Cut
    }{A,A\vdash D\otimes C}\,\otimes\mathrm{R}
    \quad
    \cfrac{\cfrac{D\vdash D \quad C\vdash E}{D,C\vdash D\otimes E}\,\otimes\mathrm{R}}{D\otimes C\vdash D\otimes E}\,\otimes\mathrm{L}
  }{A,A\vdash D\otimes E}\,\boxed{Cut}
  \quad
  D\otimes E\vdash F
}{A,A\vdash F}\,Cut$$

$$
\overset{3}{\Longrightarrow}\qquad
\cfrac{
\cfrac{\dfrac{A \vdash B \qquad B \vdash D}{A \vdash D}\ Cut \qquad
\cfrac{A \vdash C \qquad \dfrac{\dfrac{D \vdash D \qquad C \vdash E}{D, C \vdash D \otimes E}\,{\otimes}\text{R}}{}}{A, D \vdash D \otimes E}\ \boxed{Cut}
}{A, A \vdash D \otimes E}\qquad D \otimes E \vdash F
}{A, A \vdash F}\ Cut
$$

$$
\overset{2.2.2.2}{\Longrightarrow}\qquad
\cfrac{
\cfrac{\dfrac{A \vdash B \qquad B \vdash D}{A \vdash D}\ Cut \qquad
\cfrac{D \vdash D \qquad \dfrac{A \vdash C \qquad C \vdash E}{A \vdash E}\ Cut}{A, D \vdash D \otimes E}\ {\otimes}\text{R}\ \boxed{Cut}
}{A, A \vdash D \otimes E}\qquad D \otimes E \vdash F
}{A, A \vdash F}\ Cut
$$

$$
\overset{2.2.2.2}{\Longrightarrow}\qquad
\cfrac{
\cfrac{\dfrac{\dfrac{A \vdash B \qquad B \vdash D}{A \vdash D}\ Cut \qquad D \vdash D}{A \vdash D}\ \boxed{Cut} \qquad
\dfrac{A \vdash C \qquad C \vdash E}{A \vdash E}\ Cut
}{A, A \vdash D \otimes E}\ {\otimes}\text{R}\qquad D \otimes E \vdash F
}{A, A \vdash F}\ Cut
$$

$$
\overset{1.2}{\Longrightarrow}\qquad
\cfrac{
\cfrac{\dfrac{A \vdash C \qquad C \vdash E}{A \vdash E}\ Cut \qquad
\dfrac{A \vdash B \qquad B \vdash D}{A \vdash D}\ Cut
}{A, A \vdash D \otimes E}\ {\otimes}\text{R}\qquad D \otimes E \vdash F
}{A, A \vdash F}\ Cut
$$

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B Rules of Linear Logic

Structural Rules

$$\frac{\Gamma, A, B, \Delta \vdash \Theta}{\Gamma, B, A, \Delta \vdash \Theta} \text{ Exchange} \qquad \frac{\Gamma \vdash A, \Theta \qquad \Delta, A \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda} \, Cut$$

Logical Rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \perp R \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} \perp L$$

$$\frac{\Gamma \vdash A, \Theta \qquad \Delta \vdash B, \Lambda}{\Gamma, \Delta \vdash A \otimes B, \Theta, \Lambda} \otimes R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes L$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \,\wp\, B} \,\wp\, R \qquad \frac{\Gamma, A \vdash \Theta \qquad \Delta, B \vdash \Lambda}{\Gamma, \Delta, A \,\wp\, B \vdash \Theta, \Lambda} \,\wp\, L$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \multimap R \qquad \frac{\Gamma \vdash A, \Theta \qquad \Delta, B \vdash C, \Lambda}{\Gamma, \Delta, A \multimap B \vdash C, \Theta, \Lambda} \multimap L$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta} \,\&\, R \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \,\&\, L \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \,\&\, L$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus L$$

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