

Visual Servoing of Quadrotors for Perching by Hanging from Cylindrical Objects

Supplementary Material, 12/4/2015

Justin Thomas, Giuseppe Loianno, Kostas Daniilidis, and Vijay Kumar

I. INTRODUCTION

This work provides additional derivations and technical proofs related to [1] and is considered supplementary material.

II. PRELIMINARIES

The translational dynamics of a quadrotor can be expressed as

$$m\ddot{\mathbf{x}}_q = -mg\mathbf{e}_3 + fR_B^W \mathbf{e}_3 \quad (1)$$

where m is the mass of the vehicle, g is gravity, and f is the net thrust. Further, the angular dynamics are given by

$$\dot{R}_B^W = R_B^W \hat{\Omega} \quad (2)$$

$$\mathcal{I}\dot{\Omega} + \Omega \times \mathcal{I}\Omega = \mathbf{M} \quad (3)$$

where $\Omega \in \mathbb{R}^3$ is the angular velocity of the robot in the body frame, \mathcal{I} is the inertial tensor, $M \in \mathbb{R}^3$ is the control moments, and $\hat{\cdot} : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ is defined such that $\hat{\mathbf{a}}\mathbf{b} = \mathbf{a} \times \mathbf{b}$.

Let

$$\mathbf{f}_{des} = m \left(g\mathbf{e}_3 + J^{-1} (k_x \mathbf{e}_s + k_v \dot{\mathbf{e}}_s + \ddot{\mathbf{s}}_{des}) + J^{-1} \dot{\mathbf{s}} \right) \quad (4)$$

where

$$\mathbf{e}_s = \mathbf{s}_{des} - \mathbf{s}, \quad \dot{\mathbf{e}}_s = \dot{\mathbf{s}}_{des} - \dot{\mathbf{s}}$$

are the position and velocity errors in the image coordinates, and k_x and k_v are positive gains. Then, let the thrust be

$$f = \mathbf{f}_{des} \cdot R_B^W \mathbf{e}_3 \quad (5)$$

and the attitude controller be as defined in [2], the system defined by eq. (1) and eq. (3) is exponentially stable.

III. STABILITY OF THE ANGULAR DYNAMICS

Following the treatment in [3], the Lyapunov candidate is

$$\mathcal{V}_R = \frac{1}{2} \mathbf{e}_\Omega \cdot \mathcal{I} \mathbf{e}_\Omega + k_R \Psi(R, R_d) + c_2 \mathbf{e}_R \cdot \mathbf{e}_\Omega, \quad (6)$$

with c_2 being a positive scalar, such that,

$$\mathbf{z}_\theta^T M_\theta \mathbf{z}_\theta \leq \mathcal{V}_R \leq \mathbf{z}_\theta^T M_\Theta \mathbf{z}_\theta, \quad (7)$$

$$\dot{\mathcal{V}}_R \leq -\mathbf{z}_\theta^T W_\theta \mathbf{z}_\theta, \quad (8)$$

where $\mathbf{z}_\theta = [\|\mathbf{e}_R\|, \|\mathbf{e}_\Omega\|]^T$, and M_θ , M_Θ , and W_θ are positive definite. This in turn guarantees the asymptotic stability of the attitude dynamics.

IV. STABILITY OF TRANSLATIONAL DYNAMICS

The velocity of the image features is given by

$$\dot{\mathbf{s}} = \frac{\partial \Gamma(\mathbf{P}_1^\nu)}{\partial \mathbf{P}_1^\nu} \dot{\mathbf{P}}_1^\nu.$$

Since

$$\dot{\mathbf{P}}_1^\nu = -R_c^\nu R_{W\nu}^c \dot{\mathbf{x}}_q,$$

we can express the image feature velocities in terms of the robot velocity in the inertial frame and the point \mathbf{P}_1^ν

$$\dot{\mathbf{s}} = -\frac{\partial \Gamma(\mathbf{P}_1^\nu)}{\partial \mathbf{P}_1^\nu} R_c^\nu R_{W\nu}^c \dot{\mathbf{x}}_q \equiv J \dot{\mathbf{x}}_q. \quad (9)$$

Using (9), the acceleration of the robot can be expressed in terms of J , \mathbf{s} , and their derivatives

$$m \left(J^{-1} \ddot{\mathbf{s}} + J^{-1} \dot{\mathbf{s}} \right) = f R_B^W \mathbf{e}_3 - mg\mathbf{e}_3. \quad (10)$$

Rearranging, the dynamics of the image features are

$$\ddot{\mathbf{s}} = \frac{1}{m} J \left(f R_B^W \mathbf{e}_3 - mg\mathbf{e}_3 - m J^{-1} \dot{\mathbf{s}} \right). \quad (11)$$

Using (9), we can determine the image errors

$$\ddot{\mathbf{e}}_s = \ddot{\mathbf{s}}_{des} - \frac{1}{m} J \left(f R_B^W \mathbf{e}_3 - mg\mathbf{e}_3 - m J^{-1} \dot{\mathbf{s}} \right), \quad (12)$$

so that

$$m \ddot{\mathbf{e}}_s = m \ddot{\mathbf{s}}_{des} - f J R_B^W \mathbf{e}_3 + J m g \mathbf{e}_3 + m J J^{-1} \dot{\mathbf{s}}. \quad (13)$$

Defining

$$\mathbf{X} = J \frac{f}{\mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3} \left(R_c \mathbf{e}_3 - (\mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3) R_B^W \mathbf{e}_3 \right), \quad (14)$$

the error dynamics become

$$m \ddot{\mathbf{e}}_s = m \ddot{\mathbf{s}}_{des} - J \left(\frac{f}{\mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3} R_c \mathbf{e}_3 \right) + \mathbf{X} + J m g \mathbf{e}_3 + m J J^{-1} \dot{\mathbf{s}}. \quad (15)$$

Next, let

$$f = \mathbf{f}_{des} \cdot R_B^W \mathbf{e}_3, \quad (16)$$

and the commanded attitude be defined by

$$R_c \mathbf{e}_3 = \frac{\mathbf{f}_{des}}{\|\mathbf{f}_{des}\|}. \quad (17)$$

The authors are with the GRASP Lab, University of Pennsylvania, Philadelphia, PA, 19104 USA. e-mail: {jut, loiannog, kostas, kumar}@seas.upenn.edu.

Then, from the previous two equations, we have

$$f = \|\mathbf{f}_{des}\| \mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3. \quad (18)$$

Substituting this into (15) and using \mathbf{f}_{des} , we have

$$\begin{aligned} m\dot{\mathbf{e}}_s &= m\ddot{\mathbf{s}}_{des} - J \left(\frac{\|\mathbf{f}_{des}\| \mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3}{\mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3} R_c \mathbf{e}_3 \right) + \mathbf{X} \\ &+ Jm g \mathbf{e}_3 + mJ J^{-1} \dot{\mathbf{s}} \\ &= m\ddot{\mathbf{s}}_{des} - J (\|\mathbf{f}_{des}\| R_c \mathbf{e}_3) + \mathbf{X} \\ &+ Jm g \mathbf{e}_3 + mJ J^{-1} \dot{\mathbf{s}} \\ &= m\ddot{\mathbf{s}}_{des} - J \mathbf{f}_{des} + \mathbf{X} + Jm g \mathbf{e}_3 + mJ J^{-1} \dot{\mathbf{s}}, \end{aligned} \quad (19)$$

From eq. (4) the error equation finally becomes

$$m\ddot{\mathbf{e}}_s = -k_x \mathbf{e}_s - k_v \dot{\mathbf{e}}_s + \mathbf{X}. \quad (20)$$

Proof: We use the Lyapunov candidate from [3]

$$\mathcal{V}_s = \frac{1}{2} k_x \|\mathbf{e}_s\|^2 + \frac{1}{2} m \|\dot{\mathbf{e}}_s\|^2 + c_1 \mathbf{e}_s \cdot \dot{\mathbf{e}}_s. \quad (21)$$

Now, let $\mathbf{z}_s = [\|\mathbf{e}_s\|, \|\dot{\mathbf{e}}_s\|]^T$, then it follows that the Lyapunov function \mathcal{V}_v is bounded as

$$\mathbf{z}_s^T M_s \mathbf{z}_s \leq \mathcal{V}_v \leq \mathbf{z}_s^T M_S \mathbf{z}_s, \quad (22)$$

where $M_s, M_S \in \mathbb{R}^{2 \times 2}$ are defined as,

$$M_s = \frac{1}{2} \begin{bmatrix} K_p & -c_1 \\ -c_1 & m \end{bmatrix}, \quad M_S = \frac{1}{2} \begin{bmatrix} K_p & c_1 \\ c_1 & m \end{bmatrix}. \quad (23)$$

Then,

$$\dot{\mathcal{V}}_s = k_x (\dot{\mathbf{e}}_s \cdot \mathbf{e}_s) + m (\ddot{\mathbf{e}}_s \cdot \dot{\mathbf{e}}_s) + c_1 (\mathbf{e}_s \cdot \ddot{\mathbf{e}}_s + \dot{\mathbf{e}}_s \cdot \dot{\mathbf{e}}_s), \quad (24)$$

and incorporating (20),

$$\begin{aligned} \dot{\mathcal{V}}_s &= \frac{c_1 k_x}{m} \|\mathbf{e}_s\|^2 + (k_v - c_1) \|\dot{\mathbf{e}}_s\|^2 \\ &+ c_1 \frac{K_v}{m} (\mathbf{e}_s \cdot \dot{\mathbf{e}}_s) + \mathbf{X} \cdot \left(\frac{c_1}{m} \mathbf{e}_s + \dot{\mathbf{e}}_s \right). \end{aligned} \quad (25)$$

Now, we establish a bound on \mathbf{X} . From (14),

$$\begin{aligned} \mathbf{X} &= J \frac{f}{\mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3} \left((\mathbf{e}_3^T R_c^T R_B^W \mathbf{e}_3) R_c \mathbf{e}_3 - R_c \mathbf{e}_3 \right) \\ \|\mathbf{X}\| &\leq \|J\| \left\| \frac{\|\mathbf{f}_{des}\| R_c \mathbf{e}_3 \cdot R_B^W \mathbf{e}_3}{R_c \mathbf{e}_3 \cdot R_B^W \mathbf{e}_3} \right\| \|e_R\| \\ &\leq \|J\| \|\mathbf{f}_{des}\| \|e_R\| \\ &\leq \|J\| m \left\| g \mathbf{e}_3 + J (k_x \mathbf{e}_s + k_v \dot{\mathbf{e}}_s + \ddot{\mathbf{s}}_{des}) + J^{-1} \dot{\mathbf{s}} \right\| \|e_R\| \\ &\leq (k'_x \|\mathbf{e}_s\| + k'_v \|\dot{\mathbf{e}}_s\| + B) \|e_R\|, \end{aligned} \quad (26)$$

where k'_x, k'_v, B are as defined as

$$k'_x = m \|J\| \|J\| k_x, \quad (27)$$

$$k'_v = m \left(\|J\| \|J\| k_v + \|J^{-1}\| \right), \quad (28)$$

$$B = m \|J\| \left(g + \|J\| \|\ddot{\mathbf{s}}_{des}\| + \|J^{-1}\| \|\dot{\mathbf{s}}_{des}\| \right), \quad (29)$$

and from [3], $0 \leq \|e_R\| \leq 1$.

Next we will show that there exists positive constants $\gamma_1, \gamma_2, \gamma_3$ s.t., $\|J\| \leq \gamma_1$, $\|J^{-1}\| \leq \gamma_2$, and $\|J^{-1}\| \leq \gamma_3$. Since Γ is smooth (we only require C^2 here), J is smooth on the closed set S . This implies J is bounded on S , i.e., $\exists \gamma_1 > 0$, s.t. $\|J\| < \gamma_1$. Next, since J is smooth and nonsingular on S , the inverse is well defined and is smooth on S , which implies J^{-1} is bounded on S , i.e., $\exists \gamma_2 > 0$, s.t. $\|J^{-1}\| < \gamma_2$. Next, observe that $\frac{d}{dt} J^{-1}(\mathbf{x}_q) = \frac{\partial}{\partial \mathbf{x}_q} J^{-1}(\mathbf{x}_q) \dot{\mathbf{x}}_q$ is a composition of smooth functions on S , implying that it is bounded on S , i.e., $\exists \gamma_3 > 0$, s.t. $\|J^{-1}\| < \gamma_3$.

Then, similar to [4], we can express $\dot{\mathcal{V}}_v$ as

$$\begin{aligned} \dot{\mathcal{V}}_s &= \frac{c_1 k_x}{m} \|\mathbf{e}_s\|^2 + (k_v - c_1) \|\dot{\mathbf{e}}_s\|^2 \\ &+ c_1 \frac{K_v}{m} (\mathbf{e}_s \cdot \dot{\mathbf{e}}_s) + \mathbf{X} \cdot \left(\frac{c_1}{m} \mathbf{e}_s + \dot{\mathbf{e}}_s \right) \\ &\leq - [\|\mathbf{e}_s\| \quad \|\dot{\mathbf{e}}_s\|] W_{s1} \begin{bmatrix} \|\mathbf{e}_s\| \\ \|\dot{\mathbf{e}}_s\| \end{bmatrix} \\ &+ k'_p \|\mathbf{e}_s\| \|e_R\| \left(\frac{c_1}{m} \|\mathbf{e}_s\| + \|\dot{\mathbf{e}}_s\| \right) \\ &+ k'_v \|\dot{\mathbf{e}}_s\| \|e_R\| \left(\frac{c_1}{m} \|\mathbf{e}_s\| + \|\dot{\mathbf{e}}_s\| \right) \\ &+ B \|e_R\| \left(\frac{c_1}{m} \|\mathbf{e}_s\| + \|\dot{\mathbf{e}}_s\| \right). \end{aligned}$$

This expression can be written as,

$$\dot{\mathcal{V}}_s \leq -\mathbf{z}_s^T W_s \mathbf{z}_s + \mathbf{z}_s^T W_{s\theta} \mathbf{z}_\theta, \quad (30)$$

where W_s is defined in eq. (33).

$$W_{s1} = \begin{bmatrix} \frac{c_1 k_x}{m} & \frac{c_1 k_v}{2m} \\ \frac{c_1 k_v}{2m} & k_v - c_1 \end{bmatrix}, \quad W_{s\theta} = \begin{bmatrix} \frac{c_1}{m} B & 0 \\ B & 0 \end{bmatrix}, \quad (31)$$

$$W_{s2} = \begin{bmatrix} \frac{c_1 \alpha k'_x}{m} & \frac{\alpha}{2} \left(\frac{c_1}{m} k'_v + k'_x \right) \\ \frac{\alpha}{2} \left(\frac{c_1}{m} k'_v + k'_x \right) & \alpha k'_v \end{bmatrix}, \quad (32)$$

$$W_s = W_{s1} - W_{s2}. \quad (33)$$

Since $W_s = (W_s)^T$ and $W_s \in \mathbb{R}^{2 \times 2}$, it is sufficient to show that $\det(W_s) > 0$ and $W_s(1,1) > 0$ in order to claim that $W_s > 0$. Then, assuming $(1 - \alpha \gamma_1 \gamma_2 m) > 0$, we have $w_{11} > 0$. This is reasonable since α is a functional on the attitude error such that $\alpha \in [0, 1]$. Thus, this assumption is simply a bound on the attitude error. The determinant can be expressed as a quadratic function of k_v such that

$$\det(W_s) = \beta_0 + \beta_1 k_v + \beta_2 k_v^2, \quad (34)$$

and β_i is a function of $c_1, k_x, \gamma_1, \gamma_2, \gamma_3$, and m . The critical point of the quadratic occurs when

$$k_v = \frac{k_x m}{c_1} + \frac{k_x m + \alpha c_1 \gamma_1 \gamma_3}{c_1 (1 - \alpha \gamma_1 \gamma_2 m)}, \quad (35)$$

and has a value of

$$\det(W_s) = \frac{k_x (1 - \alpha \gamma_1 \gamma_2 m) (k_x m - c_1^2)}{m}. \quad (36)$$

In both equations, since $(1 - \alpha \gamma_1 \gamma_2 m) > 0$, the eq. (35) is positive, eq. (36) is positive and $W_s > 0$.

V. STABILITY OF THE UNDERACTUATED SYSTEM

Now, we consider the combined Lyapunov candidate for the translational and rotational error dynamics, $\mathcal{V} = \mathcal{V}_s + \mathcal{V}_R$. From (7) and (22), we have,

$$\mathbf{z}_s^T M_s \mathbf{z}_s + \mathbf{z}_\theta^T M_\theta \mathbf{z}_\theta \leq \mathcal{V} \leq \mathbf{z}_\theta^T M_\Theta \mathbf{z}_\theta + \mathbf{z}_s^T M_S \mathbf{z}_s. \quad (37)$$

Further, we see that

$$\dot{\mathcal{V}} \leq -\mathbf{z}_s^T W_s \mathbf{z}_s + \mathbf{z}_s^T W_{s\theta} \mathbf{z}_\theta - \mathbf{z}_\theta^T W_\theta \mathbf{z}_\theta, \quad (38)$$

$$\begin{aligned} &\leq -\lambda_{\min}(W_s) \|\mathbf{z}_s\|^2 + \|W_{v\theta}\| \|\mathbf{z}_s\| \|\mathbf{z}_\theta\| \\ &\leq -\lambda_{\min}(W_\theta) \|\mathbf{z}_\theta\|^2. \end{aligned} \quad (39)$$

Suppose that we choose positive constants c_1, k_x, k_v, k_R such that

$$k_x > \frac{c_1^2}{m}, \quad (40)$$

$$\lambda_{\min}(W_\theta) > \frac{4\|W_{s\theta}\|^2}{\lambda_{\min}(W_s)} \quad (41)$$

we have $\dot{\mathcal{V}}$ to be negative definite ensuring the equilibrium of the closed-loop system to be asymptotically stable. ■

REFERENCES

- [1] J. Thomas, G. Loianno, K. Daniilidis, and V. Kumar, "Visual Servoing of Quadrotors for Perching by Hanging from Cylindrical Objects," *IEEE Robotics and Automation Letters*, pp. 1–1, 2015. 1
- [2] T. Lee, M. Leoky, and N. H. McClamroch, "Geometric tracking control of a quadrotor UAV on SE(3)," in *IEEE Conference on Decision and Control*. IEEE, dec 2010, pp. 5420–5425. 1
- [3] T. Lee, M. Leok, and N. H. McClamroch, "Nonlinear Robust Tracking Control of a Quadrotor UAV on SE(3)," *Asian Journal of Control*, vol. 15, no. 2, pp. 391–408, mar 2013. [Online]. Available: <http://doi.wiley.com/10.1002/asjc.567> 1, 2
- [4] K. Sreenath, T. Lee, and V. Kumar, "Geometric control and differential flatness of a quadrotor UAV with a cable-suspended load," in *52nd IEEE Conference on Decision and Control*. IEEE, dec 2013, pp. 2269–2274. 2