GLOBAL CLASSICAL SOLUTIONS TO THE RELATIVISTIC BOLTZMANN EQUATION WITHOUT ANGULAR CUT-OFF

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ABSTRACT

GLOBAL CLASSICAL SOLUTIONS TO THE RELATIVISTIC BOLTZMANN EQUATION WITHOUT ANGULAR CUT-OFF

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We prove the unique existence and exponential decay of global in time classical solutions to the special relativistic Boltzmann equation without any angular cut-off assumptions with initial perturbations in some weighted Sobolev spaces. We consider perturbations of the relativistic Maxwellian equilibrium states. We work in the case of a spatially periodic box. We consider the general conditions on the collision kernel from Dudyński and Ekiel-Jeżewska (Commun Math Phys **115**(4):607–629, 1985). Additionally, we prove sharp constructive upper and coercive lower bounds for the linearized relativistic Boltzmann collision operator in terms of a geometric fractional Sobolev norm; this shows that a spectral gap exists and that this behavior is similar to that of the non-relativistic case as shown by Gressman and Strain (Journal of AMS **24**(3), 771–847, 2011). We also derive the relativistic analogue of Carleman dual representation of Boltzmann collision operator. Lastly, we explicitly compute the Jacobian of a collision map $(p, q) \to (\theta p' + (1-\theta)p, q)$ for a fixed $\theta \in (0, 1)$, and it is shown that the Jacobian is bounded above in p and q. This is the first global existence and stability result for relativistic Boltzmann equation without

angular cutoff and this resolves the open question of perturbative global existence for the relativistic kinetic theory without the Grad's angular cut-off assumption.

Contents

Chapter 1

Introduction

1.1 The relativistic Boltzmann equation

In 1872, Boltzmann [12] derived an eqution which mathematically models the dynamics of a gas represented as a collection of molecules. This was a model for the collisions between non-relativistic particles. For the collisions between relativistic particles whose speed is comparable to the speed of light, Lichnerowicz and Marrot [38] have derived the relativistic Boltzmann equations in 1940. This is a fundamental model for fast moving particles. Understanding the behavior of relativistic particles is crucial in describing many astrophysical and cosmological processes [37]. Although the classical non-relativistic Boltzmann kinetic theory has been widely and heavily studied, the relativistic kinetic theory has received relatively less attention because of its complicated structure and computational difficulty on dealing with relativistic post-collisional momentums. The relativistic Boltzmann equation is written as

$$p^{\mu}\partial_{\mu}f = p^0\partial_t f + cp\cdot\nabla_x f = C(f,f), \tag{1.1.1}$$

where c is the speed of light and the collision operator $C(f,f)$ can be written as

$$C(f,h) = \int_{\mathbb{R}^3}\frac{dq}{q^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}\int_{\mathbb{R}^3}\frac{dp'}{p'^0}W(p,q|p',q')[f(p')h(q') - f(p)h(q)]. \tag{1.1.2}$$

Here, the transition rate $W(p,q|p',q')$ is

$$W(p,q|p',q') = \frac{c}{2}s\sigma(g,\theta)\delta^{(4)}(p^{\mu}+q^{\mu}-p'^{\mu}-q'^{\mu}),$$

where $\sigma(g,\theta)$ is the scattering kernel measuring the interactions between particles and the Dirac δ function expresses the conservation of energy and momentum.

1.2 Notation

The relativistic momentum of a particle is denoted by a 4-vector representation p^{μ} where $\mu = 0,1,2,3$. Without loss of generality we normalize the mass of each particle $m = 1$. We raise and lower the indices with the Minkowski metric $p_{\mu} = g_{\mu\nu}p^{\nu}$, where the metric is defined as $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. The signature of the metric throughout this paper is $(-+++)$. With $p\in\mathbb{R}^3$, we write $p^{\mu} = (p^0,p)$ where p^0 which is the energy of a relativistic particle with momentum p is defined as $p^0 = \sqrt{c^2+|p|^2}$. The product between the 4-vectors with raised and lowered indices is the Lorentz inner product which is given by

$$p^\mu q_\mu = -p^0 q^0 + \sum_{i=1}^{3} p_i q_i.$$

Note that the momentum for each particle satisfies the mass shell condition $p^\mu p_\mu = -c^2$ with $p^0 > 0$. Also, the product $p^\mu q_\mu$ is Lorentz invariant.

By expanding the relativistic Boltzmann equation and dividing both sides by p^0 we write the relativistic Boltzmann equation as

$$\partial_t F + \hat{p} \cdot \nabla_x F = Q(F, F)$$

where $Q(F, F) = C(F, F)/p^0$ and the normalized velocity of a particle $\hat{p}$ is given by

$$\hat{p} = c\frac{p}{p^0} = \frac{p}{\sqrt{1 + |p|^2/c^2}}.$$

We also define the quantities s and g which respectively stand for the square of the energy and the relative momentum in the *center-of-momentum* system, $p+q = 0$, as

$$s = s(p^\mu, q^\mu) = -(p^\mu + q^\mu)(p_\mu + q_\mu) = 2(-p^\mu q_\mu + 1) \geq 0, \tag{1.2.1}$$

and

$$g = g(p^\mu, q^\mu) = \sqrt{(p^\mu - q^\mu)(p_\mu - q_\mu)} = \sqrt{2(-p^\mu q_\mu - 1)}. \tag{1.2.2}$$

Note that $s = g^2 + 4c^2$.

Conservation of energy and momentum for elastic collisions is described as

$$p^\mu + q^\mu = p'^\mu + q'^\mu. \tag{1.2.3}$$

The scattering angle θ is defined by

$$\cos\theta = \frac{(p^\mu - q^\mu)(p'_\mu - q'_\mu)}{g^2}. \tag{1.2.4}$$

Together with the conservation of energy and momentum as above, it can be shown that the angle and $\cos\theta$ are well-defined [24].

Here we would like to introduce the relativistic Maxwellian which models the steady state solutions or equilibrium solutions also known as Jüttner solutions. These are characterized as a particle distribution which maximizes the entropy subject to constant mass, momentum, and energy. They are given by

$$J(p) = \frac{e^{-\frac{cp^0}{k_B T}}}{4\pi c k_B T K_2(\frac{c^2}{k_B T})},$$

where k_B is Boltzmann constant, T is the temperature, and K_2 stands for the Bessel function $K_2(z) = \frac{z^2}{2}\int_1^\infty dt e^{-zt}(t^2-1)^{\frac{3}{2}}$. Throughout this paper, we normalize all physical constants to 1, including the speed of light $c = 1$. Then we obtain that the relativistic Maxwellian is given by

$$J(p) = \frac{e^{-p^0}}{4\pi}.$$

We now consider the *center-of-momentum* expression for the relativistic collision operator as below. Note that this expression has appeared in the physics literature; see [14]. For other representations of the operator such as Glassey-Strauss coordinate expression, see [1], [27], and [25]. Also, see [49] for the relationship between those two representations of the collision operator. As in [46] and [14], one can reduce the collision operator (1.1.2) using Lorentz transformations and get

$$Q(f,h) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)[f(p')h(q') - f(p)h(q)], \tag{1.2.5}$$

where $v_\phi = v_\phi(p,q)$ is the Mϕller velocity given by

$$v_\phi(p,q) = \sqrt{\left|\frac{p}{p^0} - \frac{q}{q^0}\right|^2 - \left|\frac{p}{p^0} \times \frac{q}{q^0}\right|^2} = \frac{g\sqrt{s}}{p^0 q^0}.$$

Comparing with the reduced version of collision operator in [1], [27], and [25], we can notice that one of the advantages of this *center-of-momentum* expression of the collision operator is that the reduced integral (1.2.5) is written in relatively simple terms which only contains the Mϕller velocity, scattering kernel, and the cancellation between gain and loss terms.

The post-collisional momentums in the *center-of-momentum* expression are written as

$$p' = \frac{p+q}{2} + \frac{g}{2}\Big(w + (\gamma - 1)(p+q)\frac{(p+q)\cdot w}{|p+q|^2}\Big), \tag{1.2.6}$$

and

$$q' = \frac{p+q}{2} - \frac{g}{2}\Big(w + (\gamma - 1)(p+q)\frac{(p+q)\cdot w}{|p+q|^2}\Big). \tag{1.2.7}$$

The energy of the post-collisional momentums are then written as

$$p'^0 = \frac{p^0 + q^0}{2} + \frac{g}{2\sqrt{s}}(p+q)\cdot w,$$

and

$$q'^0 = \frac{p^0 + q^0}{2} - \frac{g}{2\sqrt{s}}(p+q)\cdot w.$$

These can be derived by using the conservation of energy and momentum (1.2.3); see [48].

For f, g smooth and small at infinity, it turns out [24] that the collision operator satisfies

$$\int Q(f,g)dp = \int pQ(f,g)dp = \int p^0 Q(f,g)dp = 0$$

and

$$\int Q(f,f)(1+\log f)dp \ \leq 0. \tag{1.2.8}$$

Using (1.2.8), we can prove the famous Boltzmann H-theorem that the entropy of the system $-\int f \log f dp\, dx$ is a non-decreasing function of t. The expression $-f \log f$ is called the *entropy density*.

1.3 A brief history of previous results in the relativistic Boltzmann theory

The full relativistic Boltzmann equation appeared first in the paper by Lichnerowicz and Marrot [38] in 1940. In 1967, Bichteler [11] showed the local existence of the solutions to the relativistic Boltzmann equation. In 1989, Dudynski and Ekiel-Jezewska [20] showed that there exist unique L^2 solutions to the linearized equation. Afterwards, Dudynski [17] studied the long time and small-mean-free-path limits of these solutions. Regarding large data global in time weak solutions, Dudynski and Ekiel-Jezewska [19] in 1992 extended DiPerma-Lions renormalized solutions

[16] to the relativistic Boltzmann equation using their causality results from 1985 [18]. Here we would like to mention the work by Alexandre and Villani [10] on renormalized weak solutions with non-negative defect measure to non-cutoff non-relativistic Boltzmann equation. In 1996, Andreasson [1] studied the regularity of the gain term and the strong L^1 convergence of the solutions to the Jüttner equilibrium which were generalizations of Lions' results [39, 40] in the non-relativistic case. He showed that the gain term is regularizing. In 1997, Wennberg [52] showed the regularity of the gain term in both non-relativistic and relativistic cases.

Regarding the Newtonian limit for the Boltzmann equation, we have a local result by Cercignani [13] and a global result by Strain [49]. Also, Andreasson, Calogero and Illner [2] proved that there is a blow-up if only with gain-term in 2004. Then, in 2009, Ha, Lee, Yang, and Yun [33] provided uniform L^2-stability estimates for the relativistic Boltzmann equation. In 2011, Speck and Strain [44] connected the relativistic Boltzmann equation to the relativistic Euler equation via the Hilbert expansions.

Regarding problems with the initial data nearby the relativistic Maxwellian, Glassey and Strauss [25] first proved there exist unique global smooth solutions to the equation on the torus $\mathbb{T}^3$ for the hard potentials in 1993. Also, in the same paper they have shown that the convergence rate to the relativistic Maxwellian is exponential. Their assumptions on the differential cross-section covered the case of hard potentials. In 1995 [26], they extended their results to the whole space and have shown that the convergence rate to the equilibrium solution is polynomial. Under reduced restrictions on the cross-sections, Hsiao and Yu [34] gave results on the asymptotic stability of Boltzmann equation using energy methods in 2006. Recently, in 2010, Strain [47] showed that unique global-in-time solutions to the relativistic Boltzmann equation exist for the soft potentials which contains more singular kernel and decay with any polynomial rate towards their steady state relativistic Maxwellian under the conditions that the initial data starts out sufficiently close in L^∞.

In addition, we would like to mention that Glassey and Strauss [27] in 1991 computed the Jacobian determinant of the relativistic collision map. Also, we notice that there are results by Guo and Strain [50, 51] on global existence of unique smooth solutions which are initially close to the relativistic Maxwellian for the relativistic Landau-Maxwell system in 2004 and for the relativistic Landau equation in 2006. In 2009, Yu [54] proved the smoothing effects for relativistic Landau-Maxwell system. In 2010, Yang and Yu [53] proved time decay rates in the whole space for the relativistic Boltzmann equation with hard potentials and for the relativistic Landau equation.

Chapter 2

Carleman dual representation of the relativistic collision operator

2.1 Dual representation

In this section, we develop the Carleman representation of the relativistic gain and loss terms which arise many times throughout this paper represented as an integral over $E^p_{q-p'}$ where the set is defined as:

$$E^p_{q-p'} \stackrel{\text{def}}{=} \{p \in \mathbb{R}^3 | (p'^{\mu} - p^{\mu})(q_{\mu} - p'_{\mu}) = 0\}. \tag{2.1.1}$$

We first derive the Carlelman dual representation of the relativistic gain term. Initially, suppose that $\int_{\mathbb{S}^2} dw\, |\sigma_0(\theta)| < \infty$ and that $\int_{\mathbb{S}^2} dw\, \sigma_0(\theta) = 0$. Then, the relativistic gain term part of the inner product $\langle\Gamma(f,h),\eta\rangle$ is written as

$$
\begin{aligned}
&\langle\Gamma^{+}(f,h),\eta\rangle\\
&=\frac{c}{2}\int_{\mathbb{R}^3}\frac{dp}{p^0}\int_{\mathbb{R}^3}\frac{dp'}{p'^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}s\sigma(g,w)\delta^{(4)}(p^\mu+q^\mu-p'^\mu-q'^\mu)\\
&\quad\times f(q)h(p)\sqrt{J(q')}\eta(p')
\end{aligned}
$$

where $\delta^{(4)}$ is the delta function in four variables. We will reduce the integral by evaluating the delta function. Note that we have

$$
\int_{\mathbb{R}^3}\frac{dp}{p^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}=\int_{\mathbb{R}^4}dp^\mu\int_{\mathbb{R}^4}dq'^\mu\delta(p^\mu p_\mu+1)\delta(q'^\mu q'_\mu+1)u(p^0)u(q'^0)
$$

where $u(x)=1$ if $x\geq 1$ and $=0$ otherwise. Then, we obtain that

$$
\begin{aligned}
&\langle\Gamma^{+}(f,h),\eta\rangle\\
=&\frac{c}{2}\int_{\mathbb{R}^3}\frac{dp'}{p'^0}\eta(p')\int_{\mathbb{R}^3}\frac{dq}{q^0}f(p)\int_{\mathbb{R}^4}dp^\mu h(p)\int_{\mathbb{R}^4}dq'^\mu e^{-\frac{q'^0}{2}}u(p^0)u(q'^0)\\
&\times\delta(p^\mu p_\mu+1)\delta(q'^\mu q'_\mu+1)s\sigma(g,w)\delta^{(4)}(p^\mu+q^\mu-p'^\mu-q'^\mu).
\end{aligned}
$$

We reduce the integral $\int_{\mathbb{R}^4}dq'^\mu$ by evaluating the last delta function and obtain

$$
\begin{aligned}
&\langle\Gamma^{+}(f,h),\eta\rangle\\
=&\frac{c}{2}\int_{\mathbb{R}^3}\frac{dp'}{p'^0}\eta(p')\int_{\mathbb{R}^3}\frac{dq}{q^0}f(q)\int_{\mathbb{R}^4}dp^\mu s\sigma(g,w)h(p)e^{-\frac{p^0+q^0-p'^0}{2}}u(p^0)\\
&\times u(q^0-p'^0+p^0)\delta(p^\mu p_\mu+1)\delta((q^\mu-p'^\mu+p^\mu)(q_\mu-p'_\mu+p_\mu)+1)
\end{aligned}
$$

The terms in the second delta function can be rewritten as

$$
\begin{aligned}
(q^\mu-p'^\mu+p^\mu)(q_\mu-p'_\mu+p_\mu)+1&=(q^\mu-p'^\mu)(q_\mu-p'_\mu)+2(q^\mu-p'^\mu)p_\mu\\
&=\tilde{g}^2+2p^\mu(q_\mu-p'_\mu).
\end{aligned}
$$

Therefore, by evaluating the first delta function, we finally obtain the dual representation of the gain term as

$$
\begin{aligned}
&\langle \Gamma^{+}(f,h), \eta \rangle \\
&= \frac{c}{2} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \eta(p') \int_{\mathbb{R}^3} \frac{dq}{q^0} f(q) \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \frac{s}{2\tilde{g}} \sigma(g,w) h(p) e^{-\frac{p^0+q^0-p'^0}{2}}
\end{aligned}
\tag{2.1.2}
$$

where the measure $d\pi_p$ is defined as

$$
d\pi_p = u(p^0 + q^0 - p'^0)\delta(\frac{\tilde{g}^2 + 2p^\mu(q_\mu - p'_\mu)}{2\tilde{g}}).
$$

We also want to compute the dual representation for the loss term. We start from the following.

$$
\begin{aligned}
&\langle \Gamma(f,h), \eta \rangle \\
&= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi f(q) h(p) \sigma(g,w) (\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p)) \\
&= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi f(q) h(p) \Phi(g) \sigma_0(\theta) (\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p)).
\end{aligned}
$$

Initially, suppose that $\int_{\mathbb{S}^2} dw \, |\sigma_0(\theta)| < \infty$ and that $\int_{\mathbb{S}^2} dw \, \sigma_0(\theta) = 0$. Then, the loss term vanishes and we obtain

$$
\langle \Gamma(f,h), \eta \rangle = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi f(q) h(p) \sigma(g,w) \sqrt{J(q')}\eta(p').
$$

This is the relativistic Boltzmann gain term and its dual representation is shown above to be the following:

$$
\frac{c}{2} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \frac{s\sigma(g,\theta)}{\tilde{g}} f(q) h(p) \sqrt{J(q')}\eta(p'). \tag{2.1.3}
$$

On the geometry $E^p_{q-p'}$, $(p'^\mu - p^\mu)(q_\mu - p'_\mu) = 0$ Thus we have $\bar{g}^2 + \tilde{g}^2 = g^2$. Note that

$$
\begin{aligned}
(p'^\mu - q'^\mu)(p_\mu - q_\mu) &= (2p'^\mu - p^\mu - q^\mu)(p_\mu - q_\mu) \\
&= (p'^\mu - p^\mu + p'^\mu - q^\mu)(p_\mu - p'_\mu + p'_\mu - q_\mu) \\
&= (p'^\mu - p^\mu)(p_\mu - p'_\mu) + (p'^\mu - q^\mu)(p'_\mu - q_\mu) \\
&= -\bar{g}^2 + \tilde{g}^2.
\end{aligned}
$$

Since $\cos\theta \stackrel{\text{def}}{=} \frac{(p'^\mu - q'^\mu)(p_\mu - q_\mu)}{g^2}$, we have that

$$\cos\theta \stackrel{\text{def}}{=} \frac{-\bar{g}^2 + \tilde{g}^2}{\bar{g}^2 + \tilde{g}^2}.$$

Define $t = \frac{-\bar{g}^2 + \tilde{g}^2}{\bar{g}^2 + \tilde{g}^2}$. Then, we obtain $dt = d\bar{g}\frac{-4\bar{g}\tilde{g}}{(\bar{g}^2 + \tilde{g}^2)^2}$. Since $\int_{-1}^1 dt\, \sigma_0(t) = 0$, we have

$$\int_0^\infty \frac{4\bar{g}\tilde{g}^2}{(\bar{g}^2 + \tilde{g}^2)^2}\sigma_0\Big(\frac{-\bar{g}^2 + \tilde{g}^2}{\bar{g}^2 + \tilde{g}^2}\Big)d\bar{g} = 0.$$

From the estimation part for the inequality on the set $E^p_{q-p'}$, we may find a proper variable $w' \in H^2$ such that $\mathbb{R}^+_0 \times H^2 = E^p_{q-p'}$. Then, the integral is now

$$\int_{H^2} dw\,' \int_0^\infty d\bar{g}\frac{4\bar{g}\tilde{g}^2}{(\bar{g}^2 + \tilde{g}^2)^2}\sigma_0\Big(\frac{-\bar{g}^2 + \tilde{g}^2}{\bar{g}^2 + \tilde{g}^2}\Big) = 0.$$

Then, we obtain

$$\int_{E^p_{q-p'}} d\pi_p \frac{\tilde{g}^2}{(\bar{g}^2 + \tilde{g}^2)^2}\sigma_0\Big(\frac{-\bar{g}^2 + \tilde{g}^2}{\bar{g}^2 + \tilde{g}^2}\Big) = 0.$$

Therefore by multiplying constant terms with respect to p, we have

$$\int_{E^p_{q-p'}} \frac{d\pi_p}{p^0}\frac{s\sigma(g,\theta)}{\tilde{g}}\frac{\tilde{s}\tilde{g}^4\Phi(\tilde{g})}{sg^4\Phi(g)}f(q)h(p')\eta(p')\sqrt{J(q)} = 0.$$

Now we subtract this expression from the Carleman representation just written for $\langle\Gamma(f,h),\eta\rangle$ must equal the usual representation. This will be called the relativistic dual representation. Thus,

$$
\begin{aligned}
&\langle\Gamma(f,h),\eta\rangle \\
=&\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi f(q)h(p)\sigma(g,w)(\sqrt{J(q')}\eta(p')-\sqrt{J(q)}\eta(p)) \\
=&\frac{c}{2}\int_{\mathbb{R}^3}\frac{dp'}{p'^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}\int_{E^p_{q-p'}}\frac{d\pi_p}{p^0}\frac{s\sigma(g,\theta)}{\tilde{g}}f(q)\eta(p') \\
&\qquad\qquad \times\{h(p)\sqrt{J(q')}-\frac{\tilde{s}\tilde{g}^4\Phi(\tilde{g})}{sg^4\Phi(g)}h(p')\sqrt{J(q)}\}.
\end{aligned}
\tag{2.1.4}
$$

We claim that this representation holds even when the mean value of σ_0 is not zero. Suppose that $\int_{\mathbb{S}^2} dw\,|\sigma_0(\theta)|<\infty$ and that $\int_{\mathbb{S}^2} dw\,\sigma_0(\theta)\neq 0$. Define

$$\sigma_0^\epsilon(t)=\sigma_0(t)-1_{[1-\epsilon,1]}(t)\int_{-1}^1 dt'\frac{\sigma_0(t')}{\epsilon}.$$

Then, we have $\int_{-1}^1\sigma_0^\epsilon(t)dt=0$ vanishing on $\mathbb{S}^2$. Now, define

$$
\begin{aligned}
&\langle\Gamma_\epsilon(f,h),\eta\rangle \\
&=\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi f(q)h(p)\sigma_0^\epsilon(\cos\theta)(\sqrt{J(q')}\eta(p')-\sqrt{J(q)}\eta(p)).
\end{aligned}
$$

Note that $t=\cos\theta$. Then,

$$
\begin{aligned}
&|\langle\Gamma(f,h),\eta\rangle-\langle\Gamma_\epsilon(f,h),\eta\rangle| \\
=&\Bigg|\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi f(q)h(p)\Phi(g) \\
&\cdot(\sqrt{J(q')}\eta(p')-\sqrt{J(q)}\eta(p))1_{[1-\epsilon,1]}(\cos\theta)\frac{1}{\epsilon}\int_{-1}^1\sigma_0(t')dt'\Bigg|.
\end{aligned}
\tag{2.1.5}
$$

Here, we briefly discuss some properties under the condition $\cos\theta=1$. By the definition, we have

$$\cos\theta=\frac{(p^\mu-q^\mu)(p'_\mu-q'_\mu)}{g^2}.$$

Thus, if $\cos\theta = 1$,

$$(p^\mu - q^\mu)(p'_\mu - q'_\mu) = g^2$$
$$= (p^\mu - q^\mu)(p_\mu - q_\mu).$$

Then we have

$$(p^\mu - q^\mu)(p'_\mu - p_\mu) = 0.$$

By the collision geometry $(p'^\mu - p^\mu)(p'_\mu - q_\mu) = 0$, we have

$$(p^\mu - p'^\mu)(p_\mu - p'_\mu) = \bar{g}^2 = 0.$$

Thus, we get $\bar{g} = 0$. Equivalently, this means that

$$(p'^0 - p^0)^2 = |p' - p|^2.$$

And this implies that $p^0 = p'^0$ and $p = p'$ because

$$|p'^0 - p^0| = \left| \frac{|p'|^2 - |p|^2}{p'^0 + p^0} \right| < |p' - p|.$$

Therefore, if $\cos\theta = 1$, we have $p'^\mu = p^\mu$ and $q'^\mu = q^\mu$. Thus, as $\epsilon \to 0$, the difference term in (2.1.5)$\to 0$ because the integrand vanishes on the set $\cos\theta = 1$. Therefore, we can call (2.1.4) as the dual representation because if we define

$$T_f\eta(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, \sigma(g,\theta) f(q)(\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p)),$$

$$T_f^* h(p') = \frac{1}{p'^0}\frac{c}{2}\int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \frac{s\sigma(g,\theta)}{\tilde{g}} f(q)$$
$$\times \{h(p)\sqrt{J(q')} - \frac{\tilde{s}\tilde{g}^4\Phi(\tilde{g})}{sg^4\Phi(g)} h(p')\sqrt{J(q)}\},$$

then

$$\langle \Gamma(f,h),\eta\rangle = \langle T_f\eta, h\rangle = \langle \eta, T_f^* h\rangle.$$

2.2 Alternative forms of the collision operator

The collision integral below can be represented in other variables:

$$\int_{\mathbb{R}^3}\frac{dp}{p^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}\int_{\mathbb{R}^3}\frac{dp'}{p'^0}s\sigma(g,\theta)\delta^{(4)}(p'^\mu+q'^\mu-p^\mu-q^\mu)A(p,q,p') \qquad (2.2.1)$$

where A has a sufficient vanishing condition so the integral is well-defined.

Here we can write the above integral as one on the set $\mathbb{R}^3\times\mathbb{R}^3\times E^{p'}_{p+q}$ where $E^{p'}_{p+q}$ is the hyperplane

$$E^{p'}_{p+q} = \{p'\in\mathbb{R}^3 : (p'^\mu - p^\mu)(p_\mu+q_\mu)=0\}.$$

We rewrite eq.(2.2.1) as

$$\int_{\mathbb{R}^3}\frac{dp}{p^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}B(p,q,p')$$

where $B = B(p,q,p')$ is defined as

$$\begin{aligned} B &= \int_{\mathbb{R}^3}\frac{dp'}{p'^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}s\sigma(g,\theta)\delta^{(4)}(p'^\mu+q'^\mu-p^\mu-q^\mu)A(p,q,p') \\ &= \int_{\mathbb{R}^4\times\mathbb{R}^4} d\Theta(p'^\mu,q'^\mu)s\sigma(g,\theta)\delta^{(4)}(p'^\mu+q'^\mu-p^\mu-q^\mu)A(p^\mu,q^\mu,p'^\mu)\end{aligned}$$

where $d\Theta(p'^\mu,q'^\mu)\overset{\text{def}}{=} dp'^{\,\mu}dq'^{\,\mu}u(q'^0)u(p'^0)\delta(s-g^2-4)\delta((p'^\mu-q'^\mu)(p'^\mu+q'^\mu))$ and $u(r)=0$ if $r<0$ and $u(r)=1$ if $r\geq 0$. Now we apply the change of variable

$$\bar{q'}^\mu = q'^\mu - p'^\mu.$$

Then with this change of variable the integral becomes

$$B = \int_{\mathbb{R}^4 \times \mathbb{R}^4} d\Theta(\bar{q}'^{\mu}, p'^{\mu}) s\sigma(g,\theta) \delta^{(4)}(2p'^{\mu} + \bar{q}'^{\mu} - p^{\mu} - q^{\mu}) A(p^{\mu}, q^{\mu}, p'^{\mu})$$

where $d\Theta(\bar{q}'^{\mu}, p'^{\mu}) \stackrel{\text{def}}{=} dp'^{\ \mu} d\bar{q}'^{\mu} u(\bar{q}'_0 + p'^0) u(p'^0) \delta(s - g^2 - 4) \delta(\bar{q}'^{\mu}(p'^{\mu} + q'^{\mu}))$. This change of variables gives us the Jacobian$= 1$. Finally we evaluate the delta function to obtain

$$B = \int_{\mathbb{R}^4} d\Theta(p'^{\mu}) s\sigma(g,\theta) A(p^{\mu}, q^{\mu}, p'^{\mu})$$

where we are now integrating over the four vector p'^{μ} and $d\Theta(p'^{\mu}) = dp'^{\ \mu} u(q^0 + p^0 - p'^0) u(p'^0) \delta(s - g^2 - 4) \delta((p^{\mu} + q^{\mu})(p_{\mu} + q_{\mu} - 2p'_{\mu}))$. We conclude that the integral is given by

$$B = \int_{E_{p+q}^{p'}} \frac{d\pi_{p'}}{2\sqrt{s}p'^0} s\sigma(g,\theta) A(p,q,p') \qquad (2.2.2)$$

where $d\pi_{p'} = dp' \, u(p^0 + q^0 - p'^0) \delta\left(-\frac{s}{2\sqrt{s}} - \frac{p'^{\mu}(p_{\mu}+q_{\mu})}{\sqrt{s}}\right)$. This is an 2 dimensional surface measure on the hypersurface $E_{p+q}^{p'}$ in $\mathbb{R}^3$.

Additionally, we can write the integral (2.2.1) as one on the set $\mathbb{R}^3 \times \mathbb{R}^3 \times E_{p'-p}^{q}$ where $E_{p'-p}^{q}$ is the hyperplane

$$E_{p'-p}^{q} = \{q \in \mathbb{R}^3 : (p'^{\mu} - p^{\mu})(p_{\mu} + q_{\mu}) = 0\}.$$

We rewrite (2.2.1) as

$$\int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} B(p,q,p')$$

where $B = B(p, q, p')$ is defined as

$$B = \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{\mathbb{R}^3} \frac{dq'}{q'^0} s\sigma(g,\theta)\delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu)A(p,q,p')$$
$$= \int_{\mathbb{R}^4\times\mathbb{R}^4} d\Theta(q^\mu, q'^\mu) s\sigma(g,\theta)\delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu)A(p^\mu, q^\mu, p'^\mu)$$

where $d\Theta(q^\mu, q'^\mu) \stackrel{\text{def}}{=} dq\,^\mu dq'\,^\mu u(q'^0)u(q^0)\delta(s - g^2 - 4)\delta((q^\mu - q'^\mu)(q^\mu + q'^\mu))$ and $u(r) = 0$ if $r < 0$ and $u(r) = 1$ if $r \geq 0$. Now we apply the change of variable

$$\bar{q}^\mu = q'^\mu - q^\mu.$$

Then with this change of variable the integral becomes

$$B = \int_{\mathbb{R}^4\times\mathbb{R}^4} d\Theta(\bar{q}^\mu, q^\mu) s\sigma(g,\theta)\delta^{(4)}(p'^\mu + \bar{q}^\mu - p^\mu)A(p^\mu, q^\mu, p'^\mu)$$

where $d\Theta(\bar{q}^\mu, q^\mu) \stackrel{\text{def}}{=} dq\,^\mu d\bar{q}^\mu u(\bar{q}^0 + q^0)u(q^0)\delta(s - g^2 - 4)\delta(\bar{q}^\mu(2q^\mu + \bar{q}^\mu))$. This change of variables gives us the Jacobian$= 1$. Finally we evaluate the delta function to obtain

$$B = \int_{\mathbb{R}^4} d\Theta(q^\mu) s\sigma(g,\theta)A(p^\mu, q^\mu, p'^\mu)$$

where we are now integrating over the four vector q^μ and

$d\Theta(q^\mu) = dq\,^\mu u(p^0 - p'^0 + q^0)u(q^0)\delta(s - g^2 - 4)\delta((p^\mu - p'^\mu)(2q_\mu + p_\mu - p'_\mu))$. We conclude that the integral is given by

$$B = \int_{E^q_{p'-p}} \frac{d\pi_q}{2\bar{g}q^0} s\sigma(g,\theta)A(p,q,p') \tag{2.2.3}$$

where $d\pi_q = dq\; u(p^0 + q^0 - p'^0)\delta\left(\frac{\bar{g}}{2} + \frac{q^\mu(p_\mu - p'_\mu)}{\bar{g}}\right)$. This is an 2-dimensional surface measure on the hypersurface $E^q_{p'-p}$ in $\mathbb{R}^3$.

We also want to introduce another way of writing the collision operator. The 12-fold integral (2.2.1) will be written in 9-fold integral in this section in $(p, p', \bar{q})$ where we define $\bar{q}$ as below. We write (2.2.1) using Fubini as follows

$$I \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{\mathbb{R}^3} \frac{dq'}{q'^0} s\sigma(g,\theta)\delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu)A(p,q,p').$$

By adding two delta functions and two step functions, we can express the integral above as follows

$$I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq^\mu \int_{\mathbb{R}^4} dq'^\mu \; u(q^0 + q'^0)u(\underline{s} - 4)\delta(\underline{s} - \underline{g}^2 - 4)$$
$$\times s\sigma(g,\theta)\delta((q^\mu + q'^\mu)(q_\mu - q'_\mu))\delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu)A(p,q,p')$$

where we are now integrating over the 14-vector (p, p', q^μ, q'^μ), u is defined by $u(r) = 0$ if $r < 0$ and $u(r) = 1$ if $r \geq 0$, and we let $\underline{g} \stackrel{\text{def}}{=} g(q^\mu, q'^\mu)$ and $\underline{s} \stackrel{\text{def}}{=} s(q^\mu, q'^\mu)$. We will convert the integral over (q^μ, q'^μ) into the integral over $q^\mu - q'^\mu$ and $q^\mu + q'^\mu$. Now we apply the change of variables

$$q_s^\mu \stackrel{\text{def}}{=} q^\mu + q'^\mu, \qquad q_g^\mu \stackrel{\text{def}}{=} q^\mu - q'^\mu.$$

This will do the change $(q^\mu, q'^\mu) \to (q_s^\mu, q_g^\mu)$ with Jacobian $= 16$. With this change, the integral I becomes

$$I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq_s^\mu \int_{\mathbb{R}^4} dq_g^\mu \; u(q_s^0)u(-q_s^\mu q_{s\mu} - 4)\delta(q_s^\mu q_{g\mu})$$
$$\times \delta(-q_s^\mu q_{s\mu} - q_g^\mu q_{g\mu} - 4)s\sigma(g,\theta)\delta^{(4)}(p'^\mu - p^\mu - q_g^\mu)A(p, \frac{q_s + q_g}{2}, p').$$

Then we evaluate the third delta function to obtain

$$I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq_s^\mu \; u(q_s^0) u(-q_s^\mu q_{s_\mu} - 4) \delta(-q_s^\mu q_{s_\mu} - \bar{g}^2 - 4)$$
$$\times \, \delta(q_s^\mu (p'_\mu - p_\mu)) s\sigma(g,\theta) A(p, \frac{q_s + q_g}{2}, p').$$

Note that $-q_s^\mu q_{s_\mu} - 4 = \bar{g}^2 \geq 0$ by the first delta function, and thus we always have $u(-q_s^\mu q_{s_\mu} - 4) = 1$. Also, since $\bar{s} = \bar{g}^2 + 4$, we have

$$\begin{aligned} u(q_s^0)\delta(-q_s^\mu q_{s_\mu} - \bar{g}^2 - 4) &= u(q_s^0)\delta(-q_s^\mu q_{s_\mu} - \bar{s}) \\ &= u(q_s^0)\delta(({q_s}^0)^2 - |q_s|^2 - \bar{s}) \\ &= \frac{\delta(q_s^0 - \sqrt{|q_s|^2 + \bar{s}})}{2\sqrt{|q_s|^2 + \bar{s}}}. \end{aligned}$$

Then we finally carry out an integration using the first delta function and obtain

$$I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq_s}{2\sqrt{|q_s|^2 + \bar{s}}} \, \delta(q_s^\mu (p'_\mu - p_\mu))$$
$$\times \, s\sigma(g,\theta) A(p, \frac{q_s + q_g}{2}, p'). \tag{2.2.4}$$

Chapter 3

On the global classical Solutions to the relativistic Boltzmann equation without angular cut-off

3.1 Statement of the main results and remarks

3.1.1 Linearization and reformulation of the Boltzmann equation

We will consider the linearization of the collision operator and perturbation around the relativistic Jüttner equilibrium state

$$F(t,x,p) = J(p) + \sqrt{J(p)}f(t,x,p). \tag{3.1.1}$$

Without loss of generality, we suppose that the mass, momentum, energy conservation laws for the perturbation $f(t,x,p)$ holds for all $t \geq 0$ as

$$\int_{\mathbb{R}^3} dp \int_{\mathbb{T}^3} dx \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} \sqrt{J(p)} f(t,x,p) = 0. \tag{3.1.2}$$

We linearize the relativistic Boltzmann equation around the relativistic Maxwellian equilibrium state (3.1.1). By expanding the equation, we obtain that

$$\partial_t f + \hat{p} \cdot \nabla_x f + L(f) = \Gamma(f,f), \qquad f(0,x,v) = f_0(x,v), \tag{3.1.3}$$

where the linearized relativistic Boltzmann operator L is given by

$$\begin{aligned} L(f) &\stackrel{\text{def}}{=} - J^{-1/2} Q(J, \sqrt{J} f) - J^{-1/2} Q(\sqrt{J} f, J) \\ &= \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,w) \Big(f(q)\sqrt{J(p)} \\ &\quad + f(p)\sqrt{J(q)} - f(q')\sqrt{J(p')} - f(p')\sqrt{J(q')} \Big) \sqrt{J(q)}, \end{aligned}$$

and the bilinear operator Γ is given by

$$\begin{aligned} \Gamma(f,h) &\stackrel{\text{def}}{=} J^{-1/2} Q(\sqrt{J} f, \sqrt{J} h) \\ &= \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,\theta) \sqrt{J(q)} (f(q')h(p') - f(q)h(p)). \end{aligned} \tag{3.1.4}$$

Then notice that we have

$$L(f) = -\Gamma(f, \sqrt{J}) - \Gamma(\sqrt{J}, f).$$

We further decompose $L = N + K$. We would call N as norm part and K as compact part. First, we define the weight function $\tilde{\zeta} = \zeta + \zeta_K$ such that

$$\Gamma(\sqrt{J}, f) = \left(\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(f(p') - f(p))\sqrt{J(q')}\sqrt{J(q)} \right) - \tilde{\zeta}(p) f(p), \tag{3.1.5}$$

where

$$\begin{aligned}
\tilde{\zeta}(p) &= \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})\sqrt{J(q)} \\
&= \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})^2 \\
&\qquad + \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})\sqrt{J(q')} \\
&\stackrel{\text{def}}{=} \zeta(p) + \zeta_K(p).
\end{aligned}$$

Then the first piece in Γ in (3.1.5) contains a crucial Hilbert space structure and this is a similar phenomenon to the non-relativistic case as mentioned in Gressman and Strain [30]. To see this, we take a pre-post collisional change of variables $(p,q) \to (p',q')$ as

$$\begin{aligned}
&- \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(f(p') - f(p))h(p)\sqrt{J(q')}\sqrt{J(q)} \\
&= -\frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(f(p') - f(p))h(p)\sqrt{J(q')}\sqrt{J(q)} \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(f(p) - f(p'))h(p')\sqrt{J(q)}\sqrt{J(q')} \\
&= \frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma(g,\theta)(f(p') - f(p))(h(p') - h(p))\sqrt{J(q')}\sqrt{J(q)}.
\end{aligned} \tag{3.1.6}$$

Then, we define the compact part K of the linearized Boltzmann operator L as

$$
\begin{aligned}
Kf &= \zeta_K(p)f - \Gamma(f, \sqrt{J}) \\
&= \zeta_K(p)f - \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,\theta) \sqrt{J(q)} (f(q')\sqrt{J(p')} - f(q)\sqrt{J(p)}),
\end{aligned}
$$

where

$$
\zeta_K(p) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,\theta) (\sqrt{J(q)} - \sqrt{J(q')}) \sqrt{J(q')}. \tag{3.1.7}
$$

Then, the rest of L which we call as the norm part N is defined as

$$
\begin{aligned}
Nf &= -\Gamma(\sqrt{J}, f) - \zeta_K(p)f \\
&= -\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,w) (f(p') - f(p)) \sqrt{J(q')}\sqrt{J(q)} + \zeta(p)f(p),
\end{aligned}
$$

where

$$
\zeta(p) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,\theta) (\sqrt{J(q)} - \sqrt{J(q')})^2. \tag{3.1.8}
$$

Then, as in (3.1.6), this norm piece satisfies that

$$
\begin{aligned}
\langle Nf, f \rangle = \frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \sigma(g,\theta) (f(p') - f(p))^2 \sqrt{J(q')}\sqrt{J(q)} \\
+ \int_{\mathbb{R}^3} dp \, \zeta(p) |f(p)|^2.
\end{aligned}
$$

Thus, we define a fractional semi-norm as

$$
|f|_B^2 \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, v_\phi \, \sigma(g,\theta) (f(p') - f(p))^2 \sqrt{J(q)J(q')}.
$$

This norm will appear in the process of linearization of the collision operator.

For the first part of the compact piece $\langle Kf, f\rangle$ and the second part of the norm piece $\langle Nf, f\rangle$, the following asymtotics will be shown in Proposition 4.1.1 and Corollary 4.2.2:

$$|\zeta_K(p)| \lesssim (p^0)^{\frac{a+\gamma}{2}} \quad \text{and} \quad \zeta(p) \approx (p^0)^{\frac{a+\gamma}{2}}, \tag{3.1.9}$$

under our hypothesis on the collision kernel (3.1.10) and (3.1.11) only. They can be established with the similar machinery as in Section 3.2, 3.3, and 3.5. Pao [42] proved related estimates in non-relativistic case using special function arguments. This completes our main splitting of the linearized relativistic Boltzmann collision operator.

We can also think of the spatial derivative of Γ which will be useful later. Recall that the linearization of the collision operator is given by (3.1.4) and that the post-collisional variables p' and q' satisfies (1.2.6) and (1.2.7). Then, we can define the spatial derivatives of the bilinear collision operator Γ as

$$\partial^\alpha \Gamma(f,h) = \sum_{\alpha_1 \le \alpha} C_{\alpha,\alpha_1} \Gamma(\partial^{\alpha-\alpha_1} f, \partial^{\alpha_1} h),$$

where C_{α,α_1} is a non-negative constant.

3.1.2 Main hypothesis on the collision kernel σ

The Boltzmann collision kernel $\sigma(g,\theta)$ is a non-negative function which only depends on the relative momentum g and the scattering angle θ. Without loss of generality, we may assume that the collision kernel σ is supported only when $\cos\theta \ge 0$

throught this paper; i.e., $0 \leq \theta \leq \frac{\pi}{2}$. Otherwise, the following *symmetrization* [24] will reduce the case:

$$\bar{\sigma}(g, \theta) = [\sigma(g, \theta) + \sigma(g, -\theta)]1_{\cos\theta \geq 0},$$

where 1_A is the indicator function of the set A.

Throughout this paper we assume the collision kernel satisfies the following growth/decay estimates:

$$\begin{aligned} \sigma(g, \theta) &\lesssim (g^a + g^{-b})\sigma_0(\theta) \\ \sigma(g, \theta) &\gtrsim (\frac{g}{\sqrt{s}})g^a\sigma_0(\theta) \end{aligned} \tag{3.1.10}$$

Additionally, the angular function $\theta \mapsto \sigma_0(\theta)$ is not locally integrable; for $c > 0$, it satisfies

$$\frac{c}{\theta^{1+\gamma}} \leq \sin\theta \cdot \sigma_0(\theta) \leq \frac{1}{c\theta^{1+\gamma}}, \quad \gamma \in (0, 2), \quad \forall \theta \in (0, \frac{\pi}{2}]. \tag{3.1.11}$$

Here we have that $a + \gamma \geq 0$ and $\gamma < b < \frac{3}{2} + \gamma$. Note that we do not assume any cut-off condition on the angular function.

The assumptions on our collision kernel have been motivated from many important physical interactions; the Boltzmann cross-sections which satisfy the assumptions above can describe many interactions such as short range interactions [22, 43] which describe the relativistic analogue of hard-sphere collisions, Mϕller scattering [14] which describes electron-electron scattering, Compton scattering [14] which is an approximation of photon-electron scattering, neutrino gas interactions [15], and the interactions of Israel particles [35] which are the relativistic analogue of the interactions of Maxwell molecules. For explicit representations of the collision kernels, see Appendix. Conditions on our collision kernel is generic in the sense of Dudyński and Ekiel-Jeżewska [20]. Some of the collision cross-sections of those important physical interactions have high angular singularities, so the non-cutoff assumptions on the angular kernel are needed.

3.1.3 Spaces

We will use $\langle \cdot, \cdot \rangle$ to denote the standard $L^2(\mathbb{R}^3_p)$ inner product. Also, we will use $(\cdot, \cdot)$ to denote the $L^2(\mathbb{T}^3_x \times \mathbb{R}^3_p)$ inner product. As will be seen, our solutions depend heavily on the following weighted geometric fractional Sobolev space:

$$I^{a,\gamma} \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}^3_p) : |f|_{I^{a,\gamma}} < \infty\},$$

where the norm is described as

$$|f|^2_{I^{a,\gamma}} \stackrel{\text{def}}{=} |f|^2_{L^2_{\frac{a+\gamma}{2}}} + \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \, \frac{(f(p') - f(p))^2}{\bar{g}^{3+\gamma}} (p'^0 p^0)^{\frac{a+\gamma}{4}} 1_{\bar{g} \le 1}$$

where $\bar{g}$ is the relative momentum between p'^μ and p^μ in the *center-of-momentum* system and is defined as

$$\begin{aligned} \bar{g} = g(p'^\mu, p^\mu) &= \sqrt{(p'^\mu - p^\mu)(p'_\mu - p_\mu)} \\ &= \sqrt{2(-p'^\mu p_\mu - 1)} = \sqrt{2(p'^0 p^0 - p' \cdot p - 1)}. \end{aligned} \tag{3.1.12}$$

Here, we also define another relative momentum between p'^{μ} and q^{μ} as

$$\begin{aligned}\tilde{g} = g(p'^{\mu}, q^{\mu}) &= \sqrt{(p'^{\mu} - q^{\mu})(p'_{\mu} - q_{\mu})} \\ &= \sqrt{2(-p'^{\mu} q_{\mu} - 1)} = \sqrt{2(p'^0 q^0 - p' \cdot q - 1)}.\end{aligned}$$

Note that this space $I^{a,\gamma}$ is included in the following weighted L^2 space given by

$$|f|^2_{L^2_{\frac{a+\gamma}{2}}} \overset{\text{def}}{=} \int_{\mathbb{R}^3} dp\, (p^0)^{\frac{a+\gamma}{2}} |f(p)|^2.$$

p^0 is a convenient notation, but we do not include the speed of light as a parameter in our norms as we normalize the speed of light $c = 1$ throughout this paper.

The notation on the norm $|\cdot|$ refers to function space norms acting on $\mathbb{R}^3_p$ only. The analogous norm acting on $\mathbb{T}^3_x \times \mathbb{R}^3_p$ is denoted by $||\cdot||$. So, we have

$$||f||^2_{I^{a,\gamma}} \overset{\text{def}}{=} ||\, |f|_{I^{a,\gamma}}\, ||^2_{L^2(\mathbb{T}^3)}.$$

The multi-indices $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ will be used to record spatial derivatives. For example, we write

$$\partial^{\alpha} = \partial^{\alpha^1}_{x_1} \partial^{\alpha^2}_{x_2} \partial^{\alpha^3}_{x_3}.$$

If each component of α is not greater than that of α_1, we write $\alpha \leq \alpha_1$. Also, $\alpha < \alpha_1$ means $\alpha \leq \alpha_1$ and $|\alpha| < |\alpha_1|$ where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

We define the space $H^N = H^N(\mathbb{T}^3 \times \mathbb{R}^3)$ with integer $N \geq 0$ spatial derivatives as

$$||f||^2_{H^N} = ||f||^2_{H^N(\mathbb{T}^3\times\mathbb{R}^3)} = \sum_{|\alpha| \leq N} ||\partial^{\alpha} f||^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}.$$

We sometimes denote the norm $||f||^2_{H^N}$ as $||f||^2_H$ for simplicity.

We also define the derivative space $I^{a,\gamma}_N(\mathbb{T}^3 \times \mathbb{R}^3)$ whose norm is given by

$$||f||^2_{I^{a,\gamma}_N} = ||f||^2_{I^{a,\gamma}_N(\mathbb{T}^3\times\mathbb{R}^3)} = \sum_{|\alpha|\le N} ||\partial^\alpha f||^2_{I^{a,\gamma}(\mathbb{T}^3\times\mathbb{R}^3)}.$$

Now, we state our main result as follows:

Theorem 3.1.1. *(Main Theorem) Fix $N \ge 2$, the total number of spatial derivatives. Choose $f_0 = f_0(x,p) \in H^N(\mathbb{T}^3 \times \mathbb{R}^3)$ in (3.1.1) which satisfies (3.1.2). There is an $\eta_0 > 0$ such that if $||f_0||_{H^N(\mathbb{T}^3\times\mathbb{R}^3)} \le \eta_0$, then there exists a unique global strong solution to the relativistic Boltzmann equation (1.1.1), in the form (3.1.1), which satisfies*

$$f(t,x,p) \in L^\infty_t([0,\infty); H^N(\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^2_t((0,\infty); I^{a,\gamma}_N(\mathbb{T}^3 \times \mathbb{R}^3)).$$

Furthermore, we have exponential decay to equilibrium. For some fixed $\lambda > 0$,

$$||f(t)||_{H^N(\mathbb{T}^3\times\mathbb{R}^3))} \lesssim e^{-\lambda t}||f_0||_{H^N(\mathbb{T}^3\times\mathbb{R}^3))}.$$

We also have positivity; $F = J + \sqrt{J} f \ge 0$ if $F_0 = J + \sqrt{J} f_0 \ge 0$.

3.1.4 Remarks and possibilities for the future

Our main theorem assumes that the initial function has at least N spatial derivatives. The minimum number of spatial derivatives $N \ge 2$ is needed to use the Sobolev embedding theorems that $L^\infty(\mathbb{T}^3_x) \supset H^2(\mathbb{T}^3_x)$. Note that if the number of spatial derivatives is $N > 4$, the strong solutions in the existence theorem are indeed classical solutions by the Sobolev lemma [23] that if $N > 1 + \frac{6}{2}$ then $H^N(\mathbb{T}^3 \times \mathbb{R}^3) \subset C^1(\mathbb{T}^3 \times \mathbb{R}^3)$. For the lowest number of spatial derivatives, $N \geq 2$, we obtain that the equation is satisfied in the weak sense; however, the weak solution is called a strong solution to the equation because we show that the solution is unique.

Cancellation estimates. Here we want to record one of the main computational and technical difficulties which arise in dealing with relativistic collisions. While one of the usual techniques to deal with the cancellation estimates which contains $|\eta(p) - \eta(p')|$ is to use the fundamental theorem of calculus and the change of variables in the non-relativistic settings, this method does not give a favorable estimate in the relativistic theory because the momentum derivative on the post-collisional variables (1.2.6) and (1.2.7) creates additional high singularities which are tough to control in the relativistic settings. Even with the other different representation of post-collisional variables as in [25], it is known in much earlier work [27] that the growth of momentum derivatives is large enough and this high growth prevents us from using known the non-relativistic method from [31]. It is also worth it to mention that the Jacobian which arises in taking the change of variables from p to $u = \theta p + (1-\theta)p'$ for some $\theta \in (0, 1)$ has a bad singularity at some $\theta = \theta(p, p')$. Even if we take a non-linear path from p to p', the author has computed that the Jacobian always blows up at a point on the path and has concluded that there exists a 2-dimensional hypersurface between the momentums p and p' on which the

Jacobian blows up. We deal with this difficulty by isolating the term $|\eta(p) - \eta(p')|$ in one integral in the sense of the Cauchy-Schwarz inequality and bounding the integral from above in terms of the norm $|\eta|_{I^{a,\gamma}}$ and the extra factor $2^{-\frac{3k}{2}}$, which is a favorable factor for the case $k \geq 0$. See Section 3.3.

Non-cutoff results. Regarding non-relativistic results with non-cutoff assumptions, we would like to mention the work by Alexandre and Villani [10] from 2002 on renormalized weak solutions with non-negative defect measure. Also, we would like to record the work by Gressman and Strain [29, 30] in 2010-2011. We also want to mention that Alexandre, Morimoto, Ukai, Xu, and Yang [4, 6, 7, 8, 9] obtained a proof, using different methods, of the global existence of solutions with non-cutoff assumptions in 2010-2012. Lastly, we would like to mention the recent work by the same group of Alexandre, Morimoto, Ukai, Xu, and Yang [5] from 2013 on the local existence with mild regularity for the non-cutoff Boltzmann equation where they work with an improved initial condition and do not assume that the initial data is close to a global equilibrium.

We also want to remark that Theorem 3.1.1 is the first global existence and stability proof in the relativistic kinetic theory without angular cutoff conditions and this solves an open problem.

Future possibilities: We believe that our method can be useful for making further progress on the non-cutoff relativistic kinetic theory. Note that our kernel assumes the hard potential interaction. We can use the similar methods to prove another open problem on the global stability of the relativistic Boltzmann equations for the soft potentials without angular cutoff. We will soon address in a future work the generalization to the soft potential interaction which assumes $-b+\gamma<0$ and $-\frac{3}{2}<-b+\gamma$ in a subsequent paper [36]. For more singular soft potentials $-b+\gamma\leq -\frac{3}{2}$, we need to take the momentum-derivatives on the bilinear collision operator $\partial_\beta \Gamma$ which is written in the language of the derivatives of the post-collision maps of (1.2.6) and (1.2.7) and the estimates on those terms need some clever choices of splittings of kernels so that we reduce the complexity of the derivatives. This difficulty on the derivatives is known and expected in the relativistic kinetic theory, for the representations of the post-collisional momentums in the *center-of-momentum* expression in (1.2.6) and (1.2.7) contain many non-linear terms.

Furthermore, we expect to generalize our result to the whole space case $\mathbb{R}^3_x$ by combining our estimates with the existing cut-off technology in the whole space.

It is also possible that our methods could help to prove the global existences and stabilities for other relativistic PDEs such as relativistic Vlasov-Maxwell-Boltzmann system for hard potentials without angular cut-off.

3.1.5 Outline of the article

In the following section, we first introduce the main lemmas and theorems that are needed to prove the local existence in Section 3.6.

In Section 3.2, some size estimates on single decomposed pieces will be introduced. We start by introducing our dyadic decomposition method of the angular singularity and start making an upper bound estimate on each decomposed piece. Some proofs will be based on the relativistic Carleman-type dual representation which is introduced in the Appendix. Note that some proofs on the dual representation require the use of some new Lorentz frames.

In Section 3.3, we estimate the upper bounds of the difference of the decomposed gain and loss pieces for the $k \geq 0$ case.

In Section 3.4, we first split the main inner product of the non-linear collision operator Γ which is written as a trilinear form. Then, we use the upper bound estimate on each decomposed piece and the upper bound estimates on the difference terms that were proven in the previous sections to prove the main upper bound estimates.

In Section 3.5, we use the Carleman dual representation on the trilinear form and find the coercive lower bound. We also show that the norm part $\langle Nf, f\rangle$ is comparable to the weighted geometric fractional Sobolev norm $|\cdot|_I$.

In Section 3.6, we finally use the standard iteration method and the uniform energy estimate for the iterated sequence of approximate solutions to prove the local existence. After this, we derive our own systems of macroscopic equations and the local conservation laws and use these to prove that the local solutions should be global by the standard continuity argument and the energy estimates.

In the Appendix, we derive the relativistic Carleman-type dual representation of the gain and loss terms and obtain the dual formulation of the trilinear form which is used in many places from the previous sections.

3.1.6 Main estimates

Here we would like to record our main upper and lower bound estimates of the inner products that involve the operators Γ, L, and N. The proofs for the estimates are introduced in Section 3.2 through 3.5.

Theorem 3.1.2. *We have the basic estimate*

$$|\langle \Gamma(f,h),\eta\rangle| \lesssim |f|_{L^2}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{I^{a,\gamma}}.$$

Lemma 3.1.3. *Suppose that $|\alpha|\leq N$ with $N\geq 2$. Then we have the estimate*

$$|\,(\partial^\alpha\Gamma(f,h),\partial^\alpha\eta)\,| \lesssim ||f||_{H^N}||h||_{I^{a,\gamma}_N}||\partial^\alpha\eta||_{I^{a,\gamma}}.$$

Lemma 3.1.4. *We have the uniform inequality for K that*

$$|\langle Kf,f\rangle| \leq \epsilon|f|^2_{I^{a,\gamma}} + C_\epsilon|f|^2_{L^2}$$

where ϵ is any positive small number and $C_\epsilon>0$.

Lemma 3.1.5. *We have the uniform inequality for N that*

$$|\langle Nf, f\rangle| \lesssim |f|^2_{I^{a,\gamma}}.$$

Lemma 3.1.6. *We have the uniform coercive lower bound estimate:*

$$\langle Nf, f\rangle \gtrsim |f|^2_{I^{a,\gamma}}.$$

Lemma 3.1.5 and Lemma 3.1.6 together implies that the norm piece is comparable to the fractional Sobolev norm $I^{a,\gamma}$ as

$$\langle Nf, f\rangle \approx |f|^2_{I^{a,\gamma}}.$$

Finally, we have the coercive inequality for the linearlized Boltzmann operator L:

Lemma 3.1.7. *For some $C > 0$, we have*

$$\langle Lf, f\rangle \gtrsim |f|^2_{I^{a,\gamma}} - C|f|^2_{L^2}.$$

Note that this lemma is a direct consequence of Lemma 3.1.4 and Lemma 3.1.6 because $L = K + N$.

3.2 Estimates on the single decomposed piece

In this chapter, we mainly discuss about the estimates on the decomposed pieces of the trilinear product $\langle \Gamma(f, h), \eta\rangle$. Each decomposed piece can be written in two different representations: one with the usual 8-fold reduced integral in $\int dp \int dq \int dw$ and the other in Carleman-type dual representation as introduced in the Appendix.

For the usual 8-fold representation, we recall (3.1.4) and obtain that

$$
\begin{aligned}
&\langle \Gamma(f,h),\eta\rangle \\
&= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)\eta(p)\sqrt{J(q)}\left(f(q')h(p') - f(q)h(p)\right) \\
&= T_+ - T_-
\end{aligned}
$$

where the gain term T_+ and the loss term T_- are defined as

$$
\begin{aligned}
T_+(f,h,\eta) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)\eta(p)\sqrt{J(q)} f(q')h(p') \\
T_-(f,h,\eta) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)\eta(p)\sqrt{J(q)} f(q)h(p)
\end{aligned}
$$

In this chapter, we would like to decompose T_+ and T_- dyadically around the angular singularity as the following. We let $\{\chi_k\}_{k=-\infty}^{\infty}$ be a partition of unity on $(0,\infty)$ such that $|\chi_k| \leq 1$ and $\text{supp}(\chi_k) \subset [2^{-k-1}, 2^{-k}]$. Then, we define $\sigma_k(g,\theta) \stackrel{\text{def}}{=} \sigma(g,\theta)\chi_k(\bar{g})$ where $\bar{g} \stackrel{\text{def}}{=} g(p^\mu, p'^\mu)$. The reason that we dyadically decompose around $\bar{g}$ is that we have $\theta \approx \frac{\bar{g}}{g}$ for small θ. Then we write the decomposed pieces T_+^k and T_-^k as

$$
\begin{aligned}
T_+^k(f,h,\eta) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma_k(g,\theta)\eta(p)\sqrt{J(q)} f(q')h(p') \\
T_-^k(f,h,\eta) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma_k(g,\theta)\eta(p)\sqrt{J(q)} f(q)h(p).
\end{aligned}
\tag{3.2.1}
$$

For some propositions, we utilize the Carleman-type dual representation and write the operator T_+ on the set $E^p_{q-p'}$ as

$$T_*(f,h,\eta) \stackrel{\text{def}}{=} \frac{c}{2}\int_{\mathbb{R}^3} \frac{dp'}{p'^0}\eta(p') \int_{\mathbb{R}^3} \frac{dq}{q^0} f(q) \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \frac{s\sigma(g,\theta)}{\tilde{g}} \sqrt{J(q')}h(p),$$

where the set $E^p_{q-p'}$ is defined as (2.1.1) in the Appendix. We also take the dyadic decomposition on those integral above. Then, we define the following integral

$$T^k_*(f,h,\eta) \stackrel{\text{def}}{=} \frac{c}{2}\int_{\mathbb{R}^3} \frac{dp'}{p'^0}\eta(p') \int_{\mathbb{R}^3} \frac{dq}{q^0} f(q) \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \tilde{\sigma_k} \sqrt{J(q')}h(p), \tag{3.2.2}$$

where

$$\tilde{\sigma_k} \stackrel{\text{def}}{=} \frac{s\sigma(g,\theta)}{\tilde{g}}\chi_k(\bar{g}), \quad \bar{g} \stackrel{\text{def}}{=} g(p^\mu, p'^\mu), \quad \tilde{g} \stackrel{\text{def}}{=} g(p'^\mu, q^\mu).$$

Thus, for $f,h,\eta \in S(\mathbb{R}^3)$, where $S(\mathbb{R}^3)$ denotes the standard Schwartz space on $\mathbb{R}^3$:

$$\langle \Gamma(f,h),\eta\rangle = \sum_{k=-\infty}^{\infty} \{T^k_+(f,h,\eta) - T^k_-(f,h,\eta)\}.$$

Now, we start making some size estimates for the decomposed pieces T^k_- and T^k_+.

Proposition 3.2.1. *For any integer k, l, and $m \geq 0$, we have the uniform estimate:*

$$|T^k_-(f,h,\eta)| \lesssim 2^{k\gamma} |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{L^2_{\frac{a+\gamma}{2}}}. \tag{3.2.3}$$

Proof. The term T^k_- is given as:

$$T^k_-(f,h,\eta) = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, \sigma_k(g,w) v_\phi f(q)h(p)\sqrt{J(q)}\eta(p), \tag{3.2.4}$$

where $\sigma_k(g,w) = \sigma(g,w)\chi_k(\bar{g})$. Since $\cos\theta = 1 - 2\frac{\bar{g}^2}{g^2}$, we have that $\bar{g} = g\sin\frac{\theta}{2}$. Therefore, the condition $\bar{g} \approx 2^{-k}$ is equivalent to say that the angle θ is comparable to $2^{-k}g^{-1}$. Given the size estimates for $\sigma(g,w)$ and the support of χ_k, we obtain

$$\begin{aligned}\int_{\mathbb{S}^2} dw\ \sigma_k(g,w) &\lesssim (g^a+g^{-b})\int_{\mathbb{S}^2} dw\ \sigma_0(\cos\theta)\chi_k(\bar{g})\\ &\lesssim (g^a+g^{-b})\int_{2^{-k-1}g^{-1}}^{2^{-k}g^{-1}} d\theta\sigma_0\sin\theta\\ &\lesssim (g^a+g^{-b})\int_{2^{-k-1}g^{-1}}^{2^{-k}g^{-1}} d\theta\frac{1}{\theta^{1+\gamma}}\\ &\lesssim (g^a+g^{-b})2^{k\gamma}g^{\gamma}.\end{aligned} \tag{3.2.5}$$

Thus,

$$\begin{aligned}|T^k_-(f,h,\eta)| &\lesssim 2^{k\gamma}\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq(g^{a+\gamma}+g^{-b+\gamma})v_\phi|f(q)||h(p)|\sqrt{J(q)}|\eta(p)|\\ &\stackrel{\text{def}}{=} I_1+I_2.\end{aligned}$$

Here, I_1 and I_2 corresponds to $g^{a+\gamma}$ and $g^{-b+\gamma}$ part respectively. Note that $a+\gamma\geq 0$ and $-b+\gamma<0$. We first estimate I_1. Since $g\lesssim\sqrt{p^0q^0}$ and $v_\phi\lesssim 1$, we obtain

$$I_1\lesssim 2^{k\gamma}\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq\ (p^0q^0)^{\frac{a+\gamma}{2}}|f(q)||h(p)|\sqrt{J(q)}|\eta(p)|.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}I_1 \lesssim & 2^{k\gamma}(\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq\ |f(q)|^2|h(p)|^2\sqrt{J(q)}(p^0)^{\frac{a+\gamma}{2}})^{\frac{1}{2}}\\ &\times(\int_{\mathbb{R}^3} dp\ |\eta(p)|^2(p^0)^{\frac{a+\gamma}{2}}\int_{\mathbb{R}^3} dq\sqrt{J(q)}(q^0)^{a+\gamma})^{\frac{1}{2}}.\end{aligned} \tag{3.2.6}$$

Since $\int_{\mathbb{R}^3} dq\sqrt{J(q)}(q^0)^{a+\gamma}\approx 1$, we have

$$\begin{aligned}I_1 \lesssim & 2^{k\gamma}(\int_{\mathbb{R}^3} dp\int_{\mathbb{R}^3} dq\ |f(q)|^2|h(p)|^2\sqrt{J(q)}(p^0)^{\frac{a+\gamma}{2}})^{\frac{1}{2}}\\ &\times(\int_{\mathbb{R}^3} dp\ |\eta(p)|^2(p^0)^{\frac{a+\gamma}{2}})^{\frac{1}{2}}\\ \lesssim & 2^{k\gamma}|f|_{L^2_{-m_1}}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{L^2_{\frac{a+\gamma}{2}}}\quad \text{for } m_1\geq 0.\end{aligned} \tag{3.2.7}$$

For I_2, we have

$$I_2 = 2^{k\gamma}\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq\, g^{-b+\gamma}|f(q)||h(p)|\sqrt{J(q)}|\eta(p)|.$$

Since $g \geq \frac{|p-q|}{\sqrt{p^0q^0}}$ and $-b+\gamma<0$, this is

$$I_2 \lesssim 2^{k\gamma}\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq\, |p-q|^{-b+\gamma}(p^0q^0)^{\frac{b-\gamma}{2}}|f(q)||h(p)|\sqrt{J(q)}|\eta(p)|.$$

With the Cauchy-Schwarz inequality,

$$\begin{aligned}I_2 \lesssim& 2^{k\gamma}(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq\, |f(q)|^2|h(p)|^2\sqrt{J(q)}(p^0)^{\frac{1}{2}(-b+\gamma)}(q^0)^{b-\gamma})^{\frac{1}{2}}\\ &\times(\int_{\mathbb{R}^3} dp\, |\eta(p)|^2(p^0)^{-\frac{1}{2}(-b+\gamma)}(p^0)^{b-\gamma}\int_{\mathbb{R}^3} dq\sqrt{J(q)}|p-q|^{2(-b+\gamma)})^{\frac{1}{2}}.\end{aligned}$$

Since $\int_{\mathbb{R}^3} dq\sqrt{J(q)}|p-q|^m \approx (p^0)^m$ if $m>-3$ and $2(-b+\gamma)>-3$, we have

$$\begin{aligned}I_2 \lesssim& 2^{k\gamma}(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq\, |f(q)|^2|h(p)|^2\sqrt{J(q)}(p^0)^{\frac{1}{2}(-b+\gamma)}(q^0)^{b-\gamma})^{\frac{1}{2}}\\ &\times(\int_{\mathbb{R}^3} dp\, |\eta(p)|^2(p^0)^{\frac{1}{2}(-b+\gamma)})^{\frac{1}{2}}\\ \lesssim& 2^{k\gamma}|f|_{L^2_{-m_2}}|h|_{L^2_{\frac{1}{2}(-b+\gamma)}}|\eta|_{L^2_{\frac{1}{2}(-b+\gamma)}} \qquad \text{for some } m_2 \geq 0.\end{aligned}$$

This completes the proof. □

Before we do the size estimates for T^k_+ terms, we first prove a useful inequality as in the following proposition.

Proposition 3.2.2. *Suppose that the set $E^p_{q-p'}$ is defined as in (2.1.1). On the set $E^p_{q-p'}$, we have that*

$$\int_{E^p_{q-p'}} \frac{d\pi_p}{p^0}\tilde{g}(\bar{g})^{-2-\gamma}\chi_k(\bar{g}) \lesssim 2^{k\gamma}\sqrt{\frac{q^0}{p'^0}}, \tag{3.2.8}$$

where $d\pi_p$ is the Lebesgue measure on the set $E^p_{q-p'}$ and is defined as

$$d\pi_p = dp\, u(p^0 + q^0 - p'^0)\delta\Big(\frac{\tilde{g}^2 + 2p^\mu(q_\mu - p'_\mu)}{2\tilde{g}}\Big),$$

where δ is the delta function in four variables, and $u(x) = 1$ if $x \geq 1$ and 0 otherwise.

Proof. We first introduce our 4-vectors $\bar{p}^\mu$ and $\tilde{p}^\mu$ defined as

$$\bar{p}^\mu = p^\mu - p'^\mu \text{ and } \tilde{p}^\mu = p'^\mu - q^\mu.$$

Then, notice that the Lorentzian inner product of the two 4-vectors are given by

$$\bar{p}^\mu \bar{p}_\mu = \bar{g}^2 \text{ and } \tilde{p}^\mu \tilde{p}_\mu = \tilde{g}^2.$$

Similarly, we define some other 4-vectors which will be useful:

$$\underline{\mathrm{p}}^\mu = p^\mu + p'^\mu \text{ and } \hat{p}^\mu = p'^\mu + q^\mu.$$

The product is then given by

$$-\underline{\mathrm{p}}^\mu \underline{\mathrm{p}}_\mu = \bar{s} \text{ and } -\hat{p}^\mu \hat{p}_\mu = \tilde{s}.$$

Note that the four-dimensional delta-function occuring in the measure is derived from the following orthogonality equation

$$(p^\mu - q'^\mu)(p_\mu + q'_\mu) = 0$$

which tells that the total momentum is a time-like 4-vector orthogonal to the space-like relative momentum 4-vector. This orthogonality can be obtained from the following conservation laws

$$p^\mu + q^\mu = p'^\mu + q'^\mu.$$

We start with expanding the measure as

$$
\begin{aligned}
I &\stackrel{\text{def}}{=} \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \tilde{g}(\bar{g})^{-2-\gamma}\chi_k(\bar{g}) \\
&= \int_{\mathbb{R}^3} \frac{dp}{p^0}\, u(p^0+q^0-p'^0)\delta\Big(\frac{\tilde{g}^2+2p^\mu(q_\mu-p'_\mu)}{2\tilde{g}^2}\Big)(\bar{g})^{-2-\gamma}\chi_k(\bar{g})
\end{aligned}
$$

where $u(x)=1$ if $x\geq 1$ and 0 otherwise.

Here, the numerator in the delta function can be rewritten as

$$
\begin{aligned}
&\tilde{g}^2+2p^\mu(q_\mu-p'_\mu) \\
&= (q^\mu-p'^\mu+2p^\mu)(q_\mu-p'_\mu) \\
&= q^\mu q_\mu+p'^\mu p'_\mu-2p'^\mu q_\mu+2p^\mu q_\mu-2p^\mu p'_\mu \\
&= 2(p'^\mu p'_\mu-p'^\mu q_\mu+p^\mu q_\mu-p^\mu p'_\mu) \\
&= 2(p'^\mu-p^\mu)(p'_\mu-q_\mu).
\end{aligned}
$$

Now, define $\bar{p}=p-p'\in\mathbb{R}^3$ and $\bar{p}^0=p^0-p'^0\in\mathbb{R}$. We denote the 4-vector $\bar{p}^\mu=(\bar{p}^0,\bar{p})=p^\mu-p'^\mu$. We now apply the change of variables $p\in\mathbb{R}^3\to\bar{p}\in\mathbb{R}^3$. Note that our kernel I will be estimated inside the integral of $\int\frac{dq}{q^0}\int\frac{dp'}{p'^0}$ in the next propositions and this change of variables is indeed $(p',p)\to(p',\bar{p})=(p',p-p')$.

With this change of variables the integral becomes

$$
I=\int_{\mathbb{R}^3}\frac{d\bar{p}}{\bar{p}^0+p'^0}\, u(\bar{p}^0+q^0)\delta\Big(\frac{\bar{p}^\mu(p'_\mu-q_\mu)}{\tilde{g}^2}\Big)(\bar{g})^{-2-\gamma}\chi_k(\bar{g}).
$$

The remaining part of this estimate will be performed in the *center-of-momentum* system where $p+p'=0$; i.e., we take a Lorentz transformation such that $\underline{\mathrm{p}}^\mu=(\sqrt{s},0,0,0)$ and $\bar{p}^\mu=(0,\bar{p})=(0,\bar{p}_x,\bar{p}_y,\bar{p}_z)$. (This technique is similar to the ones found in [14] and [49].) Note that this gives us that $|\bar{p}| = \bar{g}$. Also, we choose the z-axis parallel to $\tilde{p} \in \mathbb{R}^3$. Then, we have $\tilde{p}_x = \tilde{p}_y = 0$ and $\tilde{p}_z = \tilde{g}$. Additionally, we introduce a polar-coordinates for $\bar{p}$, taking the polar-axis along the z-direction:

$$\bar{p} = |\bar{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Note that $\bar{g}$ and the measure $\frac{d\bar{p}}{\bar{p}^0+p'^0}$ are Lorentz invariant because

$$\begin{aligned}\frac{d\bar{p}}{\bar{p}^0+p'^0} =& 2d\bar{p}^\mu u(\bar{p}^0+p'^0)\delta(\bar{p}^\mu\bar{p}_\mu + 2\bar{p}^\mu p'_\mu)\\ =& 2d\bar{p}^\mu u(\bar{p}^0+p'^0)\delta((\bar{p}^\mu + p'^\mu)((\bar{p}_\mu + p'_\mu)+1)\end{aligned}$$

and these are Lorentz invariant. Then the measure of the integral is now

$$d\bar{p} = |\bar{p}|^2 d|\bar{p}|d(\cos\theta)d\phi = \bar{g}^2 d\bar{g}d(\cos\theta)d\phi$$

We now write the terms in the delta function in these variables and perform the integration with respect to $\cos\theta$. The delta function is now written as

$$\delta\Big(\frac{2\bar{p}^\mu(p'_\mu - q_\mu)}{\tilde{g}^2}\Big) = \delta\Big(\frac{2|\bar{p}||\tilde{p}|\cos\theta}{\tilde{g}^2}\Big) = \frac{\tilde{g}^2}{2|\bar{p}||\tilde{p}|}\delta(\cos\theta) = \frac{\tilde{g}^2}{2\bar{g}|p'-q|}\delta(\cos\theta).$$

After we evaluate the integral by reducing this delta function, we obtain that our integral is now

$$I = \int_0^\infty d\bar{g}(\bar{g})^{-\gamma}\chi_k(\bar{g})\frac{\tilde{g}^2}{2p'^0\bar{g}|p'-q|} = \frac{\tilde{g}^2}{2p'^0|p'-q|}\int_0^\infty d\bar{g}(\bar{g})^{-1-\gamma}\chi_k(\bar{g}).$$

We recall the inequality that $\tilde{g} \leq |p'-q|$ and that $\tilde{g} \lesssim \sqrt{p'^0q^0}$. Using this inequality and the support condition of χ, we obtain that the integral is bounded above by

$$I \lesssim \sqrt{\frac{q^0}{p'^0}}\int_0^\infty d\bar{g}(\bar{g})^{-1-\gamma}\chi_k(\bar{g}) \lesssim 2^{k\gamma}\sqrt{\frac{q^0}{p'^0}}.$$

This completes the proof for the proposition. $\square$

We are now ready to estimate the operator T_+^k. This is more difficult and requires more refined techniques because it contains post-collisional momentums.

Proposition 3.2.3. *Fix an integer k. Then, we have the uniform estimate:*

$$|T_+^k(f,h,\eta)| \lesssim 2^{k\gamma}|f|_{L^2}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{L^2_{\frac{a+\gamma}{2}}}. \tag{3.2.9}$$

Proof. By taking a pre-post change of variables, we obtain from (3.2.1) that the term T_+^k is equal to

$$T_+^k(f,h,\eta) = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; \sigma_k(g,w) v_\phi f(q) h(p) \sqrt{J(q')}\eta(p'), \tag{3.2.10}$$

where $\sigma_k(g,w) = \sigma(g,w)\chi_k(\bar{g})$. Thus,

$$\begin{aligned}
&|T_+^k(f,h,\eta)| \\
&\lesssim \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; (g^a + g^{-b}) v_\phi \sigma_0 \chi_k(\bar{g}) |f(q)||h(p)|\sqrt{J(q')}|\eta(p')| \\
&\stackrel{\text{def}}{=} I_1 + I_2.
\end{aligned}$$

Here, I_1 and I_2 corresponds to g^a and g^{-b} part respectively. We estimate I_2 first. By the Cauchy-Schwarz inequality,

$$\begin{aligned}
I_2 &\lesssim \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \frac{g^{-b}\sigma_0\chi_k(\bar{g})}{g^{-b+\gamma}} |f(q)|^2 |h(p)|^2 \sqrt{J(q')} (p'^0)^{\frac{-b+\gamma}{2}} \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi g^{-b}\sigma_0\chi_k(\bar{g}) g^{-b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{\frac{b-\gamma}{2}} \right)^{\frac{1}{2}} \\
&= I_{21} \cdot I_{22}.
\end{aligned}$$

For I_{21}, we split the region of p' into two: $p'^0 \le \frac{1}{2}(p^0+q^0)$ and $p'^0 \ge \frac{1}{2}(p^0+q^0)$.

If $p'^0 \le \frac{1}{2}(p^0+q^0)$, $p^0+q^0-q'^0 \le \frac{1}{2}(p^0+q^0)$ by conservation laws. Thus, $-q'^0 \le -\frac{1}{2}(p^0+q^0)$ and $J(q') \le \sqrt{J(p)}\sqrt{J(q)}$. Since $(p'^0)^{\frac{1}{2}(-b+\gamma)} \lesssim 1$ and the exponential decay is faster than any polynomial decay, we have

$$(p'^0)^{\frac{1}{2}(-b+\gamma)}\sqrt{J(q')} \lesssim (p^0)^{-m}(q^0)^{-m}$$

for any fixed $m>0$.

On the other region, we have $p'^0 \ge \frac{1}{2}(p^0+q^0)$ and hence $p'^0 \approx (p^0+q^0)$ because $p'^0 \le (p^0+q^0)$.

Also, we have $(p'^0)^{\frac{1}{2}(-b+\gamma)} \lesssim (p^0)^{\frac{1}{2}(-b+\gamma)}$ because $-b+\gamma<0$. Thus, we obtain

$$(p'^0)^{\frac{1}{2}(-b+\gamma)}\sqrt{J(q')} \lesssim (p^0)^{\frac{1}{2}(-b+\gamma)}.$$

Note that $\sigma_0(\theta) \approx \theta^{-2-\gamma} \approx (\frac{\bar{g}}{g})^{-2-\gamma}$ because $\sigma_0 \sin\theta \approx \theta^{-1-\gamma}$ and $\cos\theta = 1-2\frac{\bar{g}^2}{g^2}$. Similarly, we have that $\sigma_0(\tilde{\theta}) \approx (\frac{\bar{g}}{\tilde{g}})^{-2-\gamma}$. After computing dw integral as in (3.2.5) in both cases above, we obtain

$$\begin{aligned}
I_{21} &\lesssim \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \frac{g^{-b}2^{k\gamma}g^{\gamma}}{g^{-b+\gamma}}|f(q)|^2|h(p)|^2\sqrt{J(q')}(p'^0)^{\frac{-b+\gamma}{2}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq\, 2^{k\gamma}|f(q)|^2|h(p)|^2(p^0)^{\frac{1}{2}(-b+\gamma)}\right)^{\frac{1}{2}} \\
&\lesssim 2^{\frac{k\gamma}{2}}|f|_{L^2}|h|_{L^2_{\frac{-b+\gamma}{2}}}
\end{aligned}$$

by the Cauchy-Schwarz inequality.

Now we estimate I_{22}. Note that $v_\phi = \frac{g\sqrt{s}}{p^0q^0}$. Then, by (3.2.5),

$$I_{22} = \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, v_\phi g^{-b}\sigma_0\chi_k(\bar{g})g^{-b+\gamma}|\eta(p')|^2\sqrt{J(q')}(p'^0)^{\frac{b-\gamma}{2}}\right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \frac{g\sqrt{s}}{p^0q^0}2^{k\gamma}g^{2(-b+\gamma)}|\eta(p')|^2\sqrt{J(q')}(p'^0)^{\frac{b-\gamma}{2}}\right)^{\frac{1}{2}}$$

By a pre-post change of variables, we have

$$I_{22} \lesssim \left(\int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq'\, \frac{g\sqrt{s}}{p'^0q'^0}2^{k\gamma}g^{2(-b+\gamma)}|\eta(p')|^2\sqrt{J(q')}(p'^0)^{\frac{b-\gamma}{2}}\right)^{\frac{1}{2}}.$$

Since $g(p'^\mu, q'^\mu) \le 2\sqrt{p'^0q'^0}$ and $s = g^2+4$, we have

$$v_\phi = \frac{g\sqrt{s}}{p'^0q'^0} = \frac{g(p'^\mu, q'^\mu)\sqrt{s(p'^\mu, q'^\mu)}}{p'^0q'^0} \lesssim 1.$$

Since $g \ge \frac{|p'-q'|}{\sqrt{p'^0q'^0}}$ and $-b+\gamma < 0$,

$$I_{22} \lesssim \left(\int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq'\, 2^{k\gamma}\frac{|p'-q'|^{2(-b+\gamma)}}{(p'^0q'^0)^{-b+\gamma}}|\eta(p')|^2\sqrt{J(q')}(p'^0)^{\frac{b-\gamma}{2}}\right)^{\frac{1}{2}}.$$

Note that $(q'^0)^{b-\gamma}\sqrt{J(q')} \lesssim \sqrt{J^\alpha(q')}$ for some $\alpha > 0$. Thus,

$$I_{22} \lesssim \left(\int_{\mathbb{R}^3} dp'\, 2^{k\gamma}|\eta(p')|^2(p'^0)^{\frac{3}{2}(b-\gamma)} \int_{\mathbb{R}^3} dq'\, \frac{\sqrt{J^\alpha(q')}}{|p'-q'|^{2(b-\gamma)}}\right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_{\mathbb{R}^3} dp'\, 2^{k\gamma}|\eta(p')|^2(p'^0)^{\frac{3}{2}(b-\gamma)}(p'^0)^{2(-b+\gamma)}\right)^{\frac{1}{2}}$$
$$= 2^{\frac{k\gamma}{2}}|\eta|_{L^2_{\frac{-b+\gamma}{2}}}.$$

Together, we obtain that

$$I_2 \lesssim 2^{k\gamma}|f|_{L^2}|h|_{L^2_{\frac{-b+\gamma}{2}}}|\eta|_{L^2_{\frac{-b+\gamma}{2}}}.$$

Now, we estimate I_1. By the Cauchy-Schwarz inequality,

$$\begin{aligned} I_1 \lesssim & \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\; v_\phi \frac{g^a \sigma_0 \chi_k(\bar{g})}{\tilde{g}^{a+\gamma}} |f(q)|^2 |\eta(p')|^2 \sqrt{J(q')} (p^0)^{\frac{a+\gamma}{2}} \right)^{\frac{1}{2}} \\ & \times \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\; v_\phi g^a \sigma_0 \chi_k(\bar{g}) \tilde{g}^{a+\gamma} |h(p)|^2 \sqrt{J(q')} (p^0)^{\frac{-a-\gamma}{2}} \right)^{\frac{1}{2}} \\ & = I_{11} \cdot I_{12}. \end{aligned}$$

For I_{12}, we first take a pre-post change of variables and use the Carleman dual representation as (2.1.2) in the Appendix. Note that $g \approx \tilde{g}$ and $s \approx \tilde{s}$ on the set $E^p_{q-p'}$ because the identity on the set $g^2 = \bar{g}^2 + \tilde{g}^2$ gives $g^2 \gtrsim \tilde{g}^2$ and the assumption that σ_0 vanishes for $\theta \in (\frac{\pi}{2}, \pi]$ gives $\cos\theta \geq 0$ which hence gives $g^2 \leq 2\tilde{g}^2$. Also, we recall that $\sigma_0(\theta) \approx \theta^{-2-\gamma} \approx (\frac{\bar{g}}{g})^{-2-\gamma}$. Then we use Proposition 3.2.2 to obtain

$$I_{12} \lesssim \left(\int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq}{q^0}\; 2^{k\gamma} \tilde{s}\, \tilde{g}^{2a+2\gamma} |h(p')|^2 \sqrt{J(q)} (p'^0)^{\frac{-a-\gamma-1}{2}} \sqrt{q^0} \right)^{\frac{1}{2}}.$$

We further use $\tilde{g} \lesssim \sqrt{p'^0 q^0}$ and $\tilde{s} \lesssim p'^0 q^0$ to conclude that

$$\begin{aligned} I_{12} \lesssim & \left(2^{k\gamma} \int_{\mathbb{R}^3} dp'\, (p'^0)^{\frac{a+\gamma-1}{2}} |h(p')|^2 \int_{\mathbb{R}^3} dq\, (q^0)^{a+\gamma+\frac{1}{2}} \sqrt{J(q)} \right)^{\frac{1}{2}} \\ \lesssim & 2^{\frac{k\gamma}{2}} |h|_{L^2_{\frac{a+\gamma}{2}}}. \end{aligned}$$

For I_{11}, we split the region of p' into two as before: $p'^0 \leq \frac{1}{2}(p^0 + q^0)$ and $p'^0 \geq \frac{1}{2}(p^0 + q^0)$. If $p'^0 \leq \frac{1}{2}(p^0 + q^0)$, we have that $-q'^0 \leq -\frac{1}{2}(p^0 + q^0)$ and $J(q') \leq \sqrt{J(p)}\sqrt{J(q)}$. Then we obtain

$$(p^0)^{\frac{a+\gamma}{2}} \sqrt{J(q')} \lesssim (p^0)^{-m} (q^0)^{-m}$$

for any fixed $m > 0$. On the other region, we have $p'^0 \geq \frac{1}{2}(p^0 + q^0)$ and hence $p'^0 \approx (p^0 + q^0)$ because $p'^0 \leq (p^0 + q^0)$. In this case, we have $\sqrt{q^0}(p^0)^{\frac{a+\gamma}{2}} \lesssim (p'^0)^{\frac{a+\gamma+1}{2}}$

because $a+\gamma \geq 0$. Thus, we obtain

$$(p^0)^{\frac{a+\gamma}{2}}\sqrt{J(q')} \lesssim (p'^0)^{\frac{a+\gamma+1}{2}}(q^0)^{-\frac{1}{2}}.$$

In both cases, we obtain that

$$I_{11} \lesssim \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, v_\phi \frac{g^a \sigma_0 \chi_k(\bar{g})}{\tilde{g}^{a+\gamma}} |f(q)|^2 |\eta(p')|^2 (p'^0)^{\frac{a+\gamma+1}{2}} (q^0)^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

By the Carleman dual representation as in Appendix, the last upper bound is

$$\Big(\frac{1}{2}\int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \frac{s}{\tilde{g}} \frac{g^a \sigma_0 \chi_k(\bar{g})}{\tilde{g}^{a+\gamma}} |f(q)|^2 |\eta(p')|^2 (p'^0)^{\frac{a+\gamma+1}{2}} (q^0)^{-\frac{1}{2}}\Big)^{\frac{1}{2}} \tag{3.2.11}$$

where $d\pi_p = dp \cdot u(p^0+q^0-p'^0)\cdot \delta\Big(\frac{\tilde{g}^2+2p^\mu(q_\mu - p'_\mu)}{2\tilde{g}}\Big)$.

Note that $\sigma_0(\theta) \approx \theta^{-2-\gamma} \approx (\frac{\bar{g}}{g})^{-2-\gamma}$ and $g \approx \tilde{g}$ on the set $E^p_{q-p'}$.

By the inequality (3.2.8) and $s \approx \tilde{s} \lesssim p'^0 q^0$, we have

$$\int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} \frac{s}{\tilde{g}} \frac{g^a \sigma_0 \chi_k(\bar{g})}{\tilde{g}^{a+\gamma}} \approx \int_{E^p_{q-p'}} \frac{d\pi_p}{p^0} (\bar{g})^{-2-\gamma} \chi_k(\bar{g}) \tilde{s}\tilde{g} \lesssim 2^{k\gamma} (p'^0)^{\frac{1}{2}} (q^0)^{\frac{3}{2}}.$$

Then, we obtain

$$\begin{aligned} I_{11} &\lesssim \Big(\int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} |f(q)|^2 |\eta(p')|^2 (p'^0)^{\frac{a+\gamma}{2}} 2^{k\gamma} p'^0 q^0 \Big)^{\frac{1}{2}} \\ &\lesssim \Big(2^{k\gamma} \int_{\mathbb{R}^3} dp'\, (p'^0)^{\frac{a+\gamma}{2}} |\eta(p')|^2 \int_{\mathbb{R}^3} dq |f(q)|^2\Big)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{k\gamma}{2}} |f|_{L^2} |\eta|_{L^2_{\frac{a+\gamma}{2}}} \end{aligned} \tag{3.2.12}$$

by the Cauchy-Schwarz inequality. Thus,

$$I_1 \lesssim 2^{k\gamma} |f|_{L^2} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{L^2_{\frac{a+\gamma}{2}}}.$$

This completes the proof. $\square$

3.3 Cancellation with hard potential kernels

Our goal in this section is to establish an upper bound estimate for the difference $T_+^k - T_-^k$ for the case that $k \geq 0$. We would like it to have a dependency on the negative power of 2^k so we have a good estimate after summation in k. Note that $k \geq 0$ also implies that $\bar{g} \leq 1$.

Firstly, we define paths from p' to p and from q' to q. Fix any two $p, p' \in \mathbb{R}^3$ and consider $\kappa : [0,1] \to \mathbb{R}^3$ given by

$$\kappa(\theta) \stackrel{\text{def}}{=} \theta p + (1-\theta)p'.$$

Similarly, we define the following for the path from q' to q;

$$\kappa_q(\theta) \stackrel{\text{def}}{=} \theta q + (1-\theta)q'.$$

Then we can easily notice that $\kappa(\theta) + \kappa_q(\theta) = p' + q' = p + q$.

We define the length of the gradient as:

$$|\nabla|^i H(p) \stackrel{\text{def}}{=} \max_{0 \leq j \leq i} \sup_{|\chi| \leq 1} \left| \left(\chi \cdot \nabla\right)^j H(p) \right|, \quad i = 0, 1, 2, \tag{3.3.1}$$

where $\chi \in \mathbb{R}^3$ and $|\chi|$ is the usual Euclidean length. Note that we have $|\nabla|^0 H = |H|$.

Now we start estimating the term $|T_+^k - T_-^k|$ under the condition $\bar{g} \leq 1$. We recall from (3.2.4) and (3.2.10) that $|(T_+^k - T_-^k)(f, h, \eta)|$ is defined as

$$\begin{aligned}
&|(T_+^k - T_-^k)(f, h, \eta)| \\
&= \left| \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \, \sigma_k(g, w) v_\phi f(q) h(p) (\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p)) \right|,
\end{aligned}$$

The key part is to estimate $|\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p)|$.

We have the following proposition for the cancellation estimate:

Proposition 3.3.1. *Suppose η is a Schwartz function on $\mathbb{R}^3$. Then, for any $k \geq 0$ and for $0 < \gamma < 2$ and $m \geq 0$, we have the uniform estimate:*

$$
\begin{aligned}
&|(T_+^k - T_-^k)(f,h,\eta)| \\
&\lesssim 2^{(\gamma-2)k}|f|_{L^2_{-m}}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{L^2_{\frac{a+\gamma}{2}}} + 2^{\frac{(\gamma-3)}{2}k}|f|_{L^2_{-m}}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{I^{a,\gamma}}.
\end{aligned}
$$

We observe that the weighted fractional Sobolev norm $|\eta|_{I^{a,\gamma}}$ is greater than or equal to $|\eta|_{L^2_{\frac{a+\gamma}{2}}}$. Therefore, the direct consequence of this proposition is that

$$
|(T_+^k - T_-^k)(f,h,\eta)| \lesssim \max\{2^{(\gamma-2)k}, 2^{\frac{(\gamma-3)}{2}k}\}|f|_{L^2_{-m}}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{I^{a,\gamma}}. \tag{3.3.2}
$$

Proof. Note that $0 < \gamma < 2$. We want our kernel has a good dependency on 2^{-k} so we end up with the negative power on 2 as $2^{(\gamma-2)k}$. Note that under $\bar{g} \leq 1$, we have $p'^0 \approx p^0$ and $q^0 \approx q'^0$. Thus, it suffices to estimate $\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p)$ only. We now split the term into three parts as

$$
\begin{aligned}
&\sqrt{J(q')}\eta(p') - \sqrt{J(q)}\eta(p) \\
&\quad =\sqrt{J(q')}(\eta(p') - \eta(p)) + \eta(p)\left(\sqrt{J(q')} - \sqrt{J(q)} - (\nabla\sqrt{J})(q)\cdot(q'-q)\right) \\
&\qquad + \eta(p)\left((\nabla\sqrt{J})(q)\cdot(q'-q)\right) \\
&\quad =\mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{3.3.3}
$$

We estimate the part II first. By the mean-value theorem on $\sqrt{J}$, we have

$$\sqrt{J(q')} - \sqrt{J(q)} = (q' - q) \cdot (\nabla \sqrt{J})(\kappa_q(\theta_1))$$

for some $\theta_1 \in (0,1)$. Now with the fundamental theorem of calculus, we obtain

$$(\nabla\sqrt{J})(\kappa_q(\theta_1)) - (\nabla\sqrt{J})(q) = \left(\int_0^{\theta_1} D(\nabla\sqrt{J})(\kappa_q(\theta'))d\theta'\right) \cdot (\kappa_q(\theta_1) - q),$$

where $D(\nabla\sqrt{J})$ is the 3×3 Jacobian matrix of $\nabla\sqrt{J}$. With the definition on $|\nabla|$ from (3.3.1), we can bound the modulus of part II by

$$\begin{aligned}
|\mathrm{II}| &\leq |\eta(p)||q' - q| \left| \left(\int_0^{\theta_1} D(\nabla\sqrt{J})(\kappa_q(\theta'))d\theta'\right) \cdot (\kappa_q(\theta_1) - q)\right| \\
&\leq |\eta(p)||q' - q||\kappa_q(\theta_1) - q| \int_0^{\theta_1} |\nabla|^2\sqrt{J}(\kappa_q(\theta'))d\theta' \\
&\leq |\eta(p)||q' - q|^2 \int_0^{\theta_1} |\nabla|^2\sqrt{J}(\kappa_q(\theta'))d\theta'.
\end{aligned} \tag{3.3.4}$$

Note that $|\nabla|^2\sqrt{J} \lesssim \sqrt{J}$ and that $|q' - q| \leq g(q^\mu, q'^\mu)\sqrt{q^0 q'^0} = \bar{g}\sqrt{q^0 q'^0} \approx 2^{-k}\sqrt{q^0 q'^0}$. Also, we have that $(q^0 q'^0)\sqrt{J}(\kappa_q(\theta')) \lesssim (J(q)J(q'))^\epsilon$ for sufficiently small ϵ. Thus, the estimate for the integral with this kernel II follows exactly the same as in the proposition for $|T^k_-|$ as in (3.2.5), (3.2.3), and (3.2.7), and we get the first term in the right-hand side of the proposition.

For part III, we consider the integral $(T^k_{+,\mathrm{III}} - T^k_{-,\mathrm{III}})$ in the *center-of-momentum* frame $p + q = 0$ with the Lorentz transformation Λ such that $\Lambda(p^\mu + q^\mu) = (\sqrt{s}, 0, 0, 0)^t$ where we recall that s is defined as in (1.2.1). We recall that the difference $(T^k_{+,\mathrm{III}} - T^k_{-,\mathrm{III}})$ is given as

$$
\begin{aligned}
&(T^k_{+,\mathrm{III}} - T^k_{-,\mathrm{III}}) \\
&\overset{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, v_\phi \sigma_k(g,w) f(q) h(p) \eta(p) (\nabla \sqrt{J(q)}) \cdot (q' - q) \qquad (3.3.5) \\
&= \int_{\mathbb{R}^3} \frac{dp}{p^0}\, h(p)\eta(p) \int_{\mathbb{R}^3} \frac{dq}{q^0}\, f(q) \int_{\mathbb{S}^2} dw\, g\sqrt{s}\sigma_k(g,w)(\nabla \sqrt{J(q)}) \cdot (p - p').
\end{aligned}
$$

By writing $(p-p')\cdot(\nabla\sqrt{J})(q)$ as

$$(p-p')\cdot(\nabla\sqrt{J})(q) = (p^\mu - p'^\mu)(\nabla\sqrt{J})_\mu(q)$$

with $(\nabla\sqrt{J})^\mu(q) \overset{\text{def}}{=} (0, (\nabla\sqrt{J})(q))$, we observe that the integrand and the measures are Lorentz invariant. Also, note that our $\cos\theta$ which was defined in (1.2.4) is now redefined as

$$\cos\theta = \frac{[\Lambda(p^\mu - q^\mu)](p'_\mu - q'_\mu)}{g^2} = \frac{\bar{p}}{|\bar{p}|}\cdot w$$

where $\bar{p}$ is defined by $\Lambda(p^\mu - q^\mu) = (0,\bar{p})^t$ and we have used $g = |\Lambda(p^\mu - q^\mu)| = |\bar{p}|$. Then the symmetry of σ_k with respect to w around the direction $\frac{\bar{p}}{|\bar{p}|}$ forces all components of $p-p'$ to vanish except the component in the symmetry direction. Therefore, we may replace $p-p'$ with

$$\frac{\bar{p}}{|\bar{p}|}\langle p-p', \frac{\bar{p}}{|\bar{p}|}\rangle = \frac{\bar{p}}{|\bar{p}|}\frac{(p^\mu - p'^\mu)[\Lambda(p_\mu - q_\mu)]}{g}$$

in the expression for $(p-p')\cdot(\nabla\sqrt{J})(q)$. Since we have $(p^\mu - p'^\mu)(p'_\mu - q_\mu) = 0$, the vector further reduces to

$$\frac{\bar{p}}{|\bar{p}|}\frac{(p^\mu - p'^\mu)(p_\mu - p'_\mu)}{g} = \frac{\bar{p}}{|\bar{p}|}\frac{\bar{g}^2}{g}.$$

Hence we obtain

$$\left|\frac{\bar{p}}{|\bar{p}|}\frac{\bar{g}^2}{g}\right| \leq 2^{-2k}g^{-1}.$$

Thus we must control the following integral

$$2^{-2k}\int_{\mathbb{R}^3}\frac{dp}{p^0}\,h(p)\eta(p)\int_{\mathbb{R}^3}\frac{dq}{q^0}\,f(q)\int_{\mathbb{S}^2}dw\,\sqrt{s}\sigma_k(g,w)(|\nabla|\sqrt{J(q)}). \tag{3.3.6}$$

Note that $\sqrt{s}\lesssim p^0q^0$. Thus, we obtain that

$$|T^k_{+,\mathrm{III}}-T^k_{-,\mathrm{III}}|\lesssim 2^{(\gamma-2)k}|f|_{L^2_{-m}}|h|_{L^2_{\frac{a+\gamma}{2}}}|\eta|_{L^2_{\frac{a+\gamma}{2}}}$$

by following exactly the same argument as in (3.2.5), (3.2.3), and (3.2.7).

For the part I, we define $\tilde{\eta}(p,p')=\eta(p')-\eta(p)$. Since $\bar{g}\leq 1$, we have $q^0\approx q'^0$ and this gives that there is some uniform constant $c>0$ such that $\sqrt{J(q')}\leq (J(q))^c$. Thus, we have

$$\begin{aligned}|T^k_{+,\mathrm{I}}-T^k_{-,\mathrm{I}}| &\leq \left|\int_{\mathbb{R}^3}dp\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\,v_\phi\sigma_k(g,w)f(q)h(p)\tilde{\eta}(p,p')\sqrt{J(q')}\right|\\ &\leq \left|\int_{\mathbb{R}^3}dp\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\,v_\phi\sigma_k(g,w)f(q)h(p)\tilde{\eta}(p,p')(J(q))^c\right|.\end{aligned}$$

Now, we use the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}|T^k_{+,\mathrm{I}}-T^k_{-,\mathrm{I}}| \leq &\left(\int_{\mathbb{R}^3}dp\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\,v_\phi\sigma_k(g,w)|f(q)|^2|h(p)|^2(J(q))^c\right)^{\frac{1}{2}}\\ &\times\left(\int_{\mathbb{R}^3}dp\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\,v_\phi\sigma_k(g,w)|\tilde{\eta}(p,p')|^2(J(q))^c\right)^{\frac{1}{2}}.\end{aligned} \tag{3.3.7}$$

The first part on the right-hand side is bounded by $2^{\frac{k\gamma}{2}}|f|_{L^2_{-m}}|h|_{L^2_{\frac{a+\gamma}{2}}}$ for some $m\geq 0$ as in (3.2.6) and (3.2.7). For the second part, we rewrite this 8-fold integral as the following 12-fold integral:

$$\begin{aligned}
&\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma_k(g,w) |\tilde{\eta}(p,p')|^2 (J(q))^c \\
&= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \; s\sigma(g,w)\chi_k(\bar{g}) |\tilde{\eta}(p,p')|^2 \\
&\quad \times (J(q))^c \delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu).
\end{aligned}$$

As in (2.2.3), we reduce this integral to the integral on the set $E^q_{p'-p}$ as the following:

$$\begin{aligned}
&\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \; s\sigma(g,w)\chi_k(\bar{g}) |\tilde{\eta}(p,p')|^2 \\
&\quad \times (J(q))^c \delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu) \\
&= \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{E^q_{p'-p}} \frac{d\pi_q}{2\bar{g}q^0} s\sigma(g,\theta)\chi_k(\bar{g}) |\tilde{\eta}(p,p')|^2 (J(q))^c \\
&= \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{E^q_{p'-p}} \frac{d\pi_q}{2\bar{g}q^0} s\sigma(g,\theta)\chi_k(\bar{g}) \frac{|\tilde{\eta}(p,p')|^2}{\bar{g}^{3+\gamma}} \bar{g}^{3+\gamma} (J(q))^c \\
&\lesssim 2^{-k(3+\gamma)} \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{E^q_{p'-p}} \frac{d\pi_q}{2\bar{g}q^0} s\sigma(g,\theta)\chi_k(\bar{g}) \frac{|\tilde{\eta}(p,p')|^2}{\bar{g}^{3+\gamma}} (J(q))^c 1_{\bar{g}\leq 1} \\
&= 2^{-k(3+\gamma)} \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{E^q_{p'-p}} \frac{d\pi_q}{2\bar{g}q^0} s\sigma(g,\theta)\chi_k(\bar{g}) \frac{|\eta(p') - \eta(p)|^2}{\bar{g}^{3+\gamma}} (J(q))^c 1_{\bar{g}\leq 1}.
\end{aligned}$$

By following the proof of Proposition 3.2.2 in the different Lorentz frame $q+p'=0$ and recalling that $\sigma(g,w) \lesssim (g^a + g^{-b})\sigma_0(w) \approx (g^a + g^{-b}) \left(\frac{\bar{g}}{g}\right)^{-2-\gamma}$, we obtain

$$\frac{1}{p^0 p'^0} \int_{E^q_{p'-p}} \frac{d\pi_q}{2\bar{g}q^0} s\sigma(g,\theta)\chi_k(\bar{g})(J(q))^c \lesssim 2^{k\gamma} (p^0 p'^0)^{\frac{a+\gamma}{4}}.$$

Therefore, the second part of the right-hand side of (3.3.7) is bounded above by

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \; v_\phi \sigma_k(g,w) |\tilde{\eta}(p,p')|^2 (J(q))^c \right)^{\frac{1}{2}} \\
&\lesssim 2^{-\frac{3k}{2}} \left(\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' (p^0 p'^0)^{\frac{a+\gamma}{4}} \frac{|\eta(p') - \eta(p)|^2}{\bar{g}^{3+\gamma}} 1_{\bar{g}\leq 1} \right)^{\frac{1}{2}} \leq 2^{-\frac{3k}{2}} |\eta|_{I^{a,\gamma}}.
\end{aligned}$$

Therefore, we finally obtain that

$$|T^k_{+,\mathrm{I}} - T^k_{-,\mathrm{I}}| \leq 2^{\frac{(\gamma-3)}{2}k} |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{I^{a,\gamma}}.$$

Together with the previous estimates on part II and III, we obtain the proposition.

□

3.4 Main upper bound estimates

In this section, we finally establish the main upper bound estimates with the hard potential collision kernel.

We first make an upper bound estimate for the trilinear product $\langle \Gamma(f,h), \eta\rangle$. We consider the dyadic decomposition of gain and loss terms as the following.

$$\begin{aligned} \langle \Gamma(f,h), \eta\rangle &= \sum_{k=-\infty}^{\infty} \{T^k_+(f,h,\eta) - T^k_-(f,h,\eta)\} \\ &= \sum_{k=-\infty}^{0} \{T^k_+(f,h,\eta) - T^k_-(f,h,\eta)\} \\ &\quad + \sum_{k=1}^{\infty} \{T^k_+(f,h,\eta) - T^k_-(f,h,\eta)\} \\ &\stackrel{\text{def}}{=} S_1 + S_2. \end{aligned} \tag{3.4.1}$$

We first compute the upper bound for the sum S_2. In this sum, we note that $k \geq 0$ and $0 < \gamma < 2$. Then, by (3.3.2), we obtain

$$\begin{aligned} |S_2| &\lesssim \sum_{k=1}^{\infty} \max\{2^{(\gamma-2)k}, 2^{\frac{(\gamma-3)}{2}k}\} |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{I^{a,\gamma}} \\ &\lesssim |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{I^{a,\gamma}}. \end{aligned}$$

For the sum S_1, we note that $\sum_{k=-\infty}^{0} 2^{k\gamma} \lesssim 1$. Then, by (3.2.9) and (3.2.3), we obtain that

$$
\begin{aligned}
|S_1| \lesssim & \sum_{k=-\infty}^{0} 2^{k\gamma} |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{L^2_{\frac{a+\gamma}{2}}} \\
\lesssim & |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{L^2_{\frac{a+\gamma}{2}}}
\end{aligned}
$$

Thus, we can collect the estimates on S_1 and S_2 and conclude that

$$
|\langle \Gamma(f,h), \eta\rangle| \lesssim |f|_{L^2_{-m}} |h|_{L^2_{\frac{a+\gamma}{2}}} |\eta|_{I^{a,\gamma}}. \tag{3.4.2}
$$

This proves Theorem 3.1.2. Note that this immediately implies Lemma 3.1.3 by taking the spatial derivatives on the functions.

Here we also would like to mention a proposition that is used to prove other further compact estimates. Let $\{e_k\}_{k=1}^{14}$ consist of the following elements:

$$
\sqrt{J}, \quad \left(\frac{p_i}{p^0}\sqrt{J}\right)_{1\le i\le 3}, \quad p^0\sqrt{J}, \quad \left(p_i\sqrt{J}\right)_{1\le i\le 3}, \quad \left(\frac{p_i p_j}{p^0}\sqrt{J}\right)_{1\le i\le j\le 3}. \tag{3.4.3}
$$

We will see in (3.6.8) that this is the basis for the hydrodynamic part Pf.

Proposition 3.4.1. *Let $\{e_l\}_{l=1}^{14}$ be the basis for the hydrodynamic part Pf defined as in (3.4.3). Then we have that*

$$
|\langle \Gamma(e_l, f), h\rangle| \lesssim |f|_{L^2_{\frac{a+\gamma}{2}}} |h|_{I^{a,\gamma}}, \tag{3.4.4}
$$

and that

$$
|\langle \Gamma(f, e_l), h\rangle| \lesssim |f|_{L^2} |h|_{I^{a,\gamma}}. \tag{3.4.5}
$$

Additionally, for any $m \ge 0$, we have

$$
|\langle \Gamma(f,h), e_l\rangle| \lesssim |f|_{L^2_{-m}} |h|_{L^2_{-m}}. \tag{3.4.6}
$$

Proof. For (3.4.4), we expand the trilinear form as in (3.4.1) and the proof follows the same lines as the proof of (3.4.2). Here, we use Sobolev embeddings on the L^2-norm of e_l to bound it by L^∞-norm with some derivatives which are also bounded uniformly.

For (3.4.5), we write the trilinear form as the sum

$$\begin{aligned}\langle \Gamma(f,e_l),\eta\rangle &= \sum_{k=0}^{\infty}(T_+^k - T_-^k)(f,e_l,\eta) + \sum_{k=-\infty}^{-1}(T_+^k - T_-^k)(f,e_l,\eta)\\ &\stackrel{\text{def}}{=} S_1 + S_2.\end{aligned}$$

For the sum S_2, we obtain from (3.2.3) and (3.2.9) that

$$|T_+^k(f,e_l,\eta)| + |T_-^k(f,e_l,\eta)| \lesssim 2^{k\gamma}|f|_{L^2}|\eta|_{L^2_{\frac{a+\gamma}{2}}}.$$

On the other hand, if $k \geq 0$, we observe (3.3.2) and obtain that

$$|(T_+^k - T_-^k)(f,e_l,\eta)| \lesssim \max\{2^{(\gamma-2)k}, 2^{\frac{(\gamma-3)}{2}k}\}|f|_{L^2_{-m}}|\eta|_{I^{a,\gamma}}.$$

This gives the upper bound for the sum S_1 since $0<\gamma<2$.

Lastly, we prove (3.4.6). We write the trilinear form as the sum

$$\begin{aligned}\langle \Gamma(f,h),e_l\rangle &= \sum_{k=0}^{\infty}(T_+^k - T_-^k)(f,h,e_l) + \sum_{k=-\infty}^{-1}(T_+^k - T_-^k)(f,h,e_l)\\ &\stackrel{\text{def}}{=} S_1 + S_2.\end{aligned}$$

If e_l is defined as in (3.4.3), then both T_+^k and T_-^k have rapid decay in both p and q variables in (3.2.10) and (3.2.4). By applying Cauhy-Schwarz inequality, we obtain

$$|T_+^k(f,h,e_l)| + |T_-^k(f,h,e_l)| \lesssim 2^{k\gamma}|f|_{L^2_{-m}}|h|_{L^2_{-m}}$$

and this gives our upper bound for S_2. For S_1, we note that e_l has rapid decay because $|e_l(p)| \leq p^0\sqrt{J(p)} \lesssim J(p)^{(\frac{1}{2}-\epsilon)}$ for any choice of e_l in the basis (3.6.8) and for sufficiently small $\epsilon > 0$. Then, instead of decomposing the cancellation term as in (3.3.3), we do the following decomposition:

$$
\begin{aligned}
&\sqrt{J(q')}e_l(p') - \sqrt{J(q')}e_l(p') \\
=&\sqrt{J(q')}\left(e_l(p') - e_l(p) - (\nabla e_l)(p)\cdot(p'-p)\right) \\
&+ e_l(p)\left(\sqrt{J(q')} - \sqrt{J(q)} - (\nabla\sqrt{J})(q)\cdot(q'-q)\right) \\
&+ \sqrt{J(q')}(\nabla e_l)(p)\cdot(p'-p) + e_l(p)(\nabla\sqrt{J})(q)\cdot(q'-q) \\
\stackrel{\text{def}}{=}&D_1 + D_2 + D_3 + D_4.
\end{aligned}
$$

We follow the part II estimate as (3.3.4) for D_1 and D_2, and we follow the part III estimate as (3.3.5) for D_3 and D_4. Then we obtain that for any $m \geq 0$

$$|(T^k_+ - T^k_-)(f,h,e_l)| \lesssim 2^{(\gamma-2)k}|f|_{L^2_{-m}}|h|_{L^2_{-m}}$$

for $k \geq 0$ case. We use this for getting the upper bound for S_1 because $\gamma - 2 < 0$. This complete the proof.

$\square$

Note that (3.4.4) implies Lemma 3.1.5. Also, this proposition further implies the following lemma:

Lemma 3.4.2. *We have the uniform estimate*

$$|\langle Kf,h\rangle| \lesssim |f|_{L^2}|h|_{I^{a,\gamma}}. \tag{3.4.7}$$

We obtain this lemma from (3.4.5) and the estimate on $\zeta_K(p)$ as in (3.1.9). Note that this lemma implies Lemma 3.1.4 by letting $h = f$. More precisely, we use that the upper bound of the inequality in the lemma is bounded above by

$$|f|_{L^2}|f|_{I^{a,\gamma}} \leq \epsilon|f|^2_{I^{a,\gamma}} + C_\epsilon|f|^2_{L^2}.$$

Then we obtain Lemma 3.1.4.

3.5 Main coercive estimates

In this section, for any schwartz function f, we consider the quadratic difference arising in the inner product of the norm part Nf with f. The main part is to estimate the norm $|f|^2_B$ which arises in the inner product and will be defined as follows.

$$\begin{aligned}
|f|^2_B &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)(f(p') - f(p))^2 \sqrt{J(q)J(q')} \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)(f(p') - f(p))^2 \sqrt{J(q)J(q')} 1_{\bar{g}\leq 1}.
\end{aligned}$$

Note that if $\bar{g} \leq 1$, we have $q^0 \approx q'^0$ as well as $p^0 \approx p'^0$. Thus, we can bound $\sqrt{J(q)J(q')}$ below as $\sqrt{J(q)J(q')} \gtrsim e^{-Cq'^0}$ for some uniform constant $C > 0$.

By the alternative Carleman-type dual representation of the integral operator as in (2.2.4), we may write the lower bound of the norm as an integral of some kernel

$K(p,p')$ as

$$
\begin{aligned}
|f|_B^2 &\gtrsim \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \;\; v_\phi \sigma(g,\theta)(f(p')-f(p))^2 e^{-Cq'^0} 1_{\bar{g}\le 1} \\
&\approx \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} (f(p')-f(p))^2 1_{\bar{g}\le 1} \\
&\quad \times \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2+\bar{s}}}\, \delta(q_s^\mu(p'_\mu - p_\mu)) s\sigma(g,\theta) e^{-Cq'^0} \\
&= \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} (f(p')-f(p))^2 1_{\bar{g}\le 1} K(p,p'),
\end{aligned}
$$

where the kernel $K(p,p')$ is defined as

$$
K(p,p') \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2+\bar{s}}}\, \delta(q_s^\mu(p'_\mu - p_\mu)) s\sigma(g,\theta) e^{-Cq'^0}. \tag{3.5.1}
$$

Our goal in this section is to make a coercive lower bound of this kernel and hence the norm $|f|_B$. First of all, the delta function in (3.5.1) implies that $(p'^\mu - p^\mu)(p'_\mu - p_\mu + 2q'_\mu) = 0$. Then this implies that

$$
\begin{aligned}
2(p'^\mu - p^\mu)(q'_\mu - p_\mu) &= 2p'^\mu q'_\mu - 2p'^\mu p_\mu - 2p^\mu q'_\mu + 2p^\mu p_\mu \\
&= 2p'^\mu q'_\mu - 2p^\mu q'_\mu - p'^\mu p_\mu - p^\mu p'_\mu + p^\mu p_\mu + p'^\mu p'_\mu \\
&= (p'^\mu - p^\mu)(p'_\mu - p_\mu + 2q'_\mu) = 0.
\end{aligned}
$$

Then, we obtain that

$$
\begin{aligned}
\bar{g}^2 + \tilde{g}^2 &= (p'^\mu - p^\mu)(p'_\mu - p_\mu) - 2(p'^\mu - p^\mu)(q'_\mu - p_\mu) + (q'^\mu - p^\mu)(q'_\mu - p_\mu) \\
&= (p'^\mu - q'^\mu)(p'_\mu - q'_\mu) \stackrel{\text{def}}{=} g'^2,
\end{aligned}
$$

and we have $\bar{g}^2+\tilde{g}^2 = g'^2$ on this hyperplane as expected where $g' \stackrel{\text{def}}{=} g(p'^\mu, q'^\mu)$. Note that, from the assumptions on the collision kernel, we have $\sigma(g',\theta) = \Phi(g')\sigma_0(\theta)$ and

$$\sigma_0(\theta) \approx \frac{1}{\sin\theta \cdot \theta^{1+\gamma}} \approx \frac{1}{\theta^{2+\gamma}} \approx \Big(\frac{g'}{\bar{g}}\Big)^{2+\gamma}.$$

Thus,

$$\sigma(g',\theta) \approx \Phi(g')\Big(\frac{g'}{\bar{g}}\Big)^{2+\gamma}.$$

Together with this, we have

$$\begin{aligned}
K(p,p') &\approx \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2+\bar{s}}}\,\delta(q_s^\mu(p'_\mu - p_\mu)) s\Phi(g')\Big(\frac{g'}{\bar{g}}\Big)^{2+\gamma} e^{-Cq'^0}\\
&\gtrsim \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2+\bar{s}}}\,\delta(q_s^\mu(p'_\mu - p_\mu)) s\Big(\frac{g'}{\bar{g}}\Big)^{2+\gamma} e^{-Cq'^0}\frac{g'}{\sqrt{s}}g'^a\\
&\gtrsim \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2+\bar{s}}}\,\delta(q_s^\mu(p'_\mu - p_\mu)) e^{-Cq'^0}\frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}}\\
&\gtrsim \int_{\mathbb{R}^3} \frac{dq_s}{q'^0}\,\delta(q_s^\mu(p'_\mu - p_\mu)) e^{-Cq'^0}\frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}},
\end{aligned}$$

where the first inequality is from the assumption on the collision kernel (3.1.10) that $\Phi(g') \gtrsim \frac{g}{\sqrt{s}}g^a$ and that $s = g^2+4 > g^2$, and the last inequality is by that $\sqrt{|q_s|^2+\bar{s}} \lesssim q'^0$ if $\bar{g} \le 1$ by the geometry.

Here, we have the following lower bound for the kernel $K(p,p')$.

Proposition 3.5.1. *If $\bar{g} \le 1$, the kernel $K(p,p')$ is bounded uniformly from below as*

$$K(p,p') \gtrsim \frac{(p'^0)^{2+\frac{a+\gamma}{2}}}{\bar{g}^{3+\gamma}}.$$

With this proposition, we can obtain the uniform lower bound for the norm $|f|_B$ as below.

$$
\begin{aligned}
|f|_B^2 &\gtrsim \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \frac{(f(p')-f(p))^2}{\bar{g}^{3+\gamma}} (p'^0)^{2+\frac{a+\gamma}{2}} 1_{\bar{g}\leq 1} \\
&\gtrsim \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \, \frac{(f(p')-f(p))^2}{\bar{g}^{3+\gamma}} (p'^0)^{\frac{a+\gamma}{2}} 1_{\bar{g}\leq 1} \\
&\gtrsim \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \, \frac{(f(p')-f(p))^2}{\bar{g}^{3+\gamma}} (p'^0 p^0)^{\frac{a+\gamma}{4}} 1_{\bar{g}\leq 1}
\end{aligned}
$$

Thus, the proof for our main coercive inequality is complete because we have that

$$
|f|^2_{L^2_{\frac{a+\gamma}{2}}} + |f|_B^2 \gtrsim |f|^2_{I^{a,\gamma}}.
$$

Proof. Here we prove Proposition 3.5.1. We begin with

$$
K(p,p') \gtrsim \int_{\mathbb{R}^3} \frac{dq_s}{q'^0} \, \delta(q_s^\mu(p'_\mu - p_\mu)) e^{-Cq'^0} \frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}}.
$$

First, we take a change of variables from $q_s = p' - p + 2q'$ to q'. Then we obtain that

$$
\begin{aligned}
K(p,p') &\gtrsim \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \, \delta((p'^\mu - p^\mu + 2q'^\mu)(p'_\mu - p_\mu)) e^{-Cq'^0} \frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}} \\
&= \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \, \delta(\bar{g}^2 + 2q'^\mu(p'_\mu - p_\mu)) e^{-Cq'^0} \frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}}.
\end{aligned}
$$

Now we take a change of variables on q' into polar coordinates as $q' \in \mathbb{R}^3 \to (r, \theta, \phi)$ and choose the z-axis parallel to $p' - p$ such that the angle between q' and $p' - p$ is equal to ϕ. Then we obtain that

$$K(p,p') \gtrsim \int_1^\infty dq'^0 \int_0^\infty dr \int_0^{2\pi} d\theta \int_0^\pi d\phi\, r^2 \sin\phi \\ \times \frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}} \delta(\bar{g}^2 + 2q'^\mu(p'_\mu - p_\mu))\delta(r^2+1-(q'^0)^2)e^{-Cq'^0}. \tag{3.5.2}$$

The terms in the first delta function in (3.5.2) can be written as

$$\begin{aligned}\bar{g}^2 + 2q'^\mu(p'_\mu - p_\mu) &= \bar{g}^2 - 2q'^0(p'^0 - p^0) + 2q'\cdot(p'-p) \\ &= \bar{g}^2 - 2q'^0(p'^0 - p^0) + 2r|p'-p|\cos\phi.\end{aligned}$$

Also, note that the second delta function is

$$\begin{aligned}\delta(r^2+1-(q'^0)^2) &= \delta((r-\sqrt{(q'^0)^2-1})(r+\sqrt{(q'^0)^2-1})) \\ &= \frac{\delta(r-\sqrt{(q'^0)^2-1})}{2\sqrt{(q'^0)^2-1}},\end{aligned}$$

because $r > 0$. Now we reduce the integration against r using this delta function and get

$$K(p,p') \gtrsim \int_1^\infty dq'^0 \int_0^{2\pi} d\theta \int_0^\pi d\phi\, \frac{(q'^0)^2-1}{2\sqrt{(q'^0)^2-1}} \sin\phi\, \frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}} e^{-Cq'^0} \\ \times \delta(\bar{g}^2 - 2q'^0(p'^0-p^0) + 2\sqrt{(q'^0)^2-1}|p'-p|\cos\phi).$$

Now, let $v = \cos\phi$. Then, $dv = -\sin\phi\, d\phi$ and the integration is now rewritten as

$$K(p,p') \gtrsim \int_1^\infty dq'^0 \int_0^{2\pi} d\theta \int_{-1}^1 dv\, \frac{(q'^0)^2-1}{2\sqrt{(q'^0)^2-1}} \frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}} e^{-Cq'^0} \\ \times \delta(\bar{g}^2 - 2q'^0(p'^0-p^0) + 2\sqrt{(q'^0)^2-1}|p'-p|v).$$

Note that

$$
\begin{aligned}
&\delta(\bar{g}^2 - 2q'^0(p'^0 - p^0) + 2\sqrt{(q'^0)^2 - 1}|p' - p|v) \\
&= \frac{1}{2\sqrt{(q'^0)^2 - 1}|p' - p|}\delta\Big(v + \frac{\bar{g}^2 - 2q'^0(p'^0 - p^0)}{2\sqrt{(q'^0)^2 - 1}|p' - p|}\Big).
\end{aligned}
$$

We remark that $\left|\frac{\bar{g}^2 - 2q'^0(p'^0 - p^0)}{2\sqrt{(q'^0)^2 - 1}|p' - p|}\right| \le 1$. Then we further reduce the integration on v by removing this delta function and get

$$
\begin{aligned}
K(p,p') &\gtrsim \int_1^\infty dq'^0 \int_0^{2\pi} d\theta\, \frac{1}{|p'-p|}\frac{g'^{4+a+\gamma}}{\bar{g}^{2+\gamma}}e^{-Cq'^0} \\
&\gtrsim \int_1^\infty dq'^0 e^{-Cq'^0}\, \frac{g'^{4+a+\gamma}}{\bar{g}^{3+\gamma}q'^0} \\
&\gtrsim \int_1^\infty dq'^0 e^{-Cq'^0}\, \frac{|p'^0 - q'^0|^{4+a+\gamma}}{\bar{g}^{3+\gamma}(\sqrt{q'^0p'^0})^{4+a+\gamma}q'^0} \\
&\gtrsim \frac{1}{\bar{g}^{3+\gamma}(p'^0)^{2+\frac{a+\gamma}{2}}}\int_1^\infty dq'^0 e^{-Cq'^0}\, \frac{|p'^0 - q'^0|^{4+a+\gamma}}{(q'^0)^{3+\frac{a+\gamma}{2}}} \\
&\approx \frac{(p'^0)^{4+a+\gamma}}{\bar{g}^{3+\gamma}}\frac{1}{(p'^0)^{2+\frac{a+\gamma}{2}}} = \frac{(p'^0)^{2+\frac{a+\gamma}{2}}}{\bar{g}^{3+\gamma}}
\end{aligned}
\tag{3.5.3}
$$

where $q \stackrel{\text{def}}{=} p' + q' - p$ and the second inequality is by $\frac{|p'-p|}{q'^0} \approx \frac{|q-q'|}{\sqrt{q'^0q^0}} \lesssim \bar{g}(q^\mu, q'^\mu) = \bar{g}$, the third inequality is by $\frac{|p'^0 - q'^0|}{\sqrt{p'^0q'^0}} \le g'$, and the last equivalence is by

$$
\int_1^\infty d(q'^0)e^{-Cq'^0}\frac{|p'^0 - q'^0|^{4+a+\gamma}}{(q'^0)^k} \approx (p'^0)^{4+a+\gamma}
$$

for any $k \in \mathbb{R}$. This proves the proposition.

□

Note that Lemma 3.1.6 has been proven in this proof above.

3.6 Global existence

3.6.1 Local existence

In this section, we use the estimates that we made in the previous sections to show the local existence results for small data. We use the standard iteration method and the uniform energy estimate for the iterated sequence of approximate solutions. The iteration starts at $f^0(t,x,p)=0$. We solve for $f^{m+1}(t,x,p)$ such that

$$(\partial_t+\hat{p}\cdot\nabla_x+N)f^{m+1}+Kf^m=\Gamma(f^m,f^{m+1}),\qquad f^{m+1}(0,x,p)=f_0(x,p). \quad (3.6.1)$$

Using our estimates, it follows that the linear equation (3.6.1) admits smooth solutions with the same regularity in H^N as a given smooth small initial data and that the solution also has a gain of $L^2((0,T);I^{a,\gamma}_N)$. We will set up some estimates which is necessary to find a local classical solution as $m\to\infty$.

We first define some notations. We will use the norm $||\cdot||_H$ for $||\cdot||_{H^N}$ for convenience and also use the norm $||\cdot||_I$ for the norm $||\cdot||_{I^{a,\gamma}_N}$. Define the total norm as

$$M(f(t))=||f(t)||^2_H+\int_0^t d\tau||f(\tau)||^2_I.$$

We will also use $|f|_{I^{a,\gamma}}$ for $\langle Nf,f\rangle$.

Here we state a crucial energy estimate:

Lemma 3.6.1. *The sequence of iterated approximate solutions $\{f^m\}$ is well defined. There exists a short time $T^*=T^*(||f_0||^2_H)>0$ such that for $||f_0||^2_H$ sufficiently* *small, there is a uniform constant $C_0 > 0$ such that*

$$\sup_{m\geq 0}\sup_{0\leq\tau\leq T^*} M(f^m(\tau)) \leq 2C_0||f_0||_H^2.$$

Proof. We prove this lemma by induction over k. If $k = 0$, the lemma is trivially true. Suppose that the lemma holds for $k = m$. Let f^{m+1} be the solution to the linear equation (3.6.1) with given f^m. We take the spatial derivative ∂^α on the linear equation (3.6.1) and obtain

$$(\partial_t + \hat{p}\cdot\nabla_x)\partial^\alpha f^{m+1} + N(\partial^\alpha f^{m+1}) + K(\partial^\alpha f^m) = \partial^\alpha\Gamma(f^m, f^{m+1}).$$

Then, we take a inner product with $\partial^\alpha f^{m+1}$. The trilinear estimate of Lemma 3.1.3 implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\partial^\alpha f^{m+1}||^2_{L^2_pL^2_x} + ||\partial^\alpha f^{m+1}||^2_{I^{a,\gamma}} + (K(\partial^\alpha f^m), \partial^\alpha f^{m+1})\\
&= (\partial^\alpha\Gamma(f^m, f^{m+1}), \partial^\alpha f^{m+1}) \lesssim ||f^m||_H||f^{m+1}||^2_I.
\end{aligned}$$

We integrate over t we obtain

$$\begin{aligned}
\frac{1}{2}||\partial^\alpha f^{m+1}(t)||^2_{L^2_pL^2_x} &+ \int_0^t d\tau||\partial^\alpha f^{m+1}(\tau)||^2_{I^{a,\gamma}}\\
&+ \int_0^t d\tau(K(\partial^\alpha f^m), \partial^\alpha f^{m+1})\\
\leq \frac{1}{2}||\partial^\alpha f_0||^2_{L^2_pL^2_x} &+ C\int_0^t d\tau||f^m||_H||f^{m+1}||^2_I.
\end{aligned}\tag{3.6.2}$$

From the compact estimate (3.4.7), for any small $\epsilon > 0$ we have

$$\begin{aligned}
&\left|\int_0^t d\tau(K(\partial^\alpha f^m), \partial^\alpha f^{m+1})\right|\\
&\leq C_{\frac{1}{2}+\epsilon}\int_0^t d\tau||\partial^\alpha f^m(\tau)||^2_{L^2} + (\frac{1}{2}+\epsilon)\int_0^t d\tau||\partial^\alpha f^{m+1}(\tau)||^2_{I^{a,\gamma}}.
\end{aligned}$$

We use this estimate for (3.6.2) and take a sum over all the derivatives such that $|\alpha| \leq N$ to obtain

$$
\begin{aligned}
M(f^{m+1}(t)) \leq & C_0||f_0||_H^2 + 2C \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)) \sup_{0\leq\tau\leq t} M^{1/2}(f^m(\tau)) \\
& + 2C_{\frac{1}{2}+\epsilon} \int_0^t d\tau ||f^m(\tau)||_H^2 + 2\epsilon \int_0^t d\tau ||f^{m+1}(\tau)||_I^2 \\
\leq & C_0||f_0||_H^2 + 2C \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)) \sup_{0\leq\tau\leq t} M^{1/2}(f^m(\tau)) \\
& + 2C_{\frac{1}{2}+\epsilon} t \sup_{0\leq\tau\leq t} M(f^m(\tau)) + 2\epsilon \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)).
\end{aligned}
\tag{3.6.3}
$$

Then by the induction hypothesis on $M(f^m(\tau))$, we obtain that

$$
\begin{aligned}
M(f^{m+1}(t)) \leq & C_0||f_0||_H^2 + 2C\sqrt{2C_0}||f_0||_H \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)) \\
& + 4C_0 C_{\frac{1}{2}+\epsilon} t ||f_0||_H^2 + 2\epsilon \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)) \\
\leq & C_0||f_0||_H^2 + 2C\sqrt{2C_0}||f_0||_H \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)) \\
& + 4C_0 C_{\frac{1}{2}+\epsilon} T^* ||f_0||_H^2 + 2\epsilon \sup_{0\leq\tau\leq t} M(f^{m+1}(\tau)).
\end{aligned}
$$

Then we obtain that

$$
(1 - 2\epsilon - 2C\sqrt{2C_0}||f_0||_H) \sup_{0\leq\tau\leq t} M(f^{m+1}(t)) \leq (C_0 + 4C_0 C_{\frac{1}{2}+\epsilon} T^*)||f_0||_H^2.
$$

Then, for sufficiently small ϵ, T^* and $||f_0||_H$, we obtain that

$$
\sup_{0\leq\tau\leq t} M(f^{m+1}(t)) \leq 2C_0||f_0||_H^2.
$$

This proves the lemma by the induction argument. $\square$

Now, we prove the local existence theorem with the uniform control on each iteration.

Theorem 3.6.2. *For any sufficiently small $M_0 > 0$, there exists a time $T^* = T^*(M_0) > 0$ and $M_1 > 0$ such that if $||f_0||_H^2 \leq M_1$, then there exists a unique solution $f(t,x,p)$ to the linearized relativistic Boltzmann equation (3.1.3) on $[0,T^*) \times \mathbb{T}^3 \times \mathbb{R}^3$ such that*

$$\sup_{0\leq t\leq T^*} M(f(t)) \leq M_0.$$

Also, $M(f(t))$ is continuous on $[0,T^*)$. Furthermore, we have the positivity of the solutions; i.e., if $F_0(x,p) = J + \sqrt{J}f_0 \geq 0$, then $F(t,x,p) = J + \sqrt{J}f(t,x,p) \geq 0$.

Proof. Existence and Uniqueness. By letting $m \to \infty$ in the previous lemma, we obtain sufficient compactness for the local existence of a strong solution $f(t,x,p)$ to (3.1.3). For the uniqueness, suppose there exists another solution h to the (3.1.3) with the same initial data satisfying $\sup_{0\leq t\leq T^*} M(h(t)) \leq \epsilon$. Then, by the equation, we have

$$\{\partial_t + \hat{p}\cdot\nabla_x\}(f-h) + L(f-h) = \Gamma(f-h,f) + \Gamma(h,f-h). \tag{3.6.4}$$

Then, by Sobolev embedding $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$ and Theorem 3.1.2, we have

$$\begin{aligned}
&|(\{\Gamma(f-h,f)+\Gamma(h,f-h)\}, f-h)| \\
&\lesssim ||h||_{L^2_pH^2_x}||f-h||^2_{I^{a,\gamma}} \\
&\quad + ||f-h||^2_{L^2_{p,x}}||f||_{H^2_xI^{a,\gamma}}||f-h||_{I^{a,\gamma}} \\
&= T_1 + T_2.
\end{aligned}$$

For T_1, we have

$$\int_0^t d\tau\, T_1(\tau) \leq \sqrt{\epsilon} \int_0^t d\tau ||f(\tau) - h(\tau)||^2_{I^{a,\gamma}}$$

because we have $\sup_{0\leq t\leq T^*} M(h(t)) \leq \epsilon$. For T_2, we use the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}\int_0^t d\tau\, T_2(\tau) \leq& \sqrt{\epsilon}\left(\sup_{0\leq\tau\leq t} ||f(\tau) - h(\tau)||^2_{L^2_{p,x}} \int_0^t d\tau ||f(\tau) - h(\tau)||^2_{I^{a,\gamma}}\right)^{1/2}\\ \lesssim& \sqrt{\epsilon}\left(\sup_{0\leq\tau\leq t} ||f(\tau) - h(\tau)||^2_{L^2_{p,x}} + \int_0^t d\tau ||f(\tau) - h(\tau)||^2_{I^{a,\gamma}}\right)\end{aligned}$$

because f also satisfies $\sup_{0\leq t\leq T^*} M(f(t)) \leq \epsilon$. For the linearized Boltzmann operator L on the left-hand side of (3.6.4), we use Lemma 3.1.7 to obtain

$$(L(f-h), f-h) \leq c||f-h||^2_{I^{a,\gamma}} - C||f-h||^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}$$

for some small $c > 0$. We finally take the inner product of (3.6.4) and $(f-h)$ and integrate over $[0,t] \times \mathbb{T}^3 \times \mathbb{R}^3$ and use the estimates above to obtain

$$\begin{aligned}&\frac{1}{2}||f(t) - h(t)||^2_{L^2_{p,x}} + c\int_0^t d\tau\, ||f(\tau) - h(\tau)||^2_{I^{a,\gamma}}\\ &\lesssim \sqrt{\epsilon}\left(\sup_{0\leq\tau\leq t} ||f(\tau) - h(\tau)||^2_{L^2_{p,x}} + \int_0^t d\tau ||f(\tau) - h(\tau)||^2_{I^{a,\gamma}}\right)\\ &\quad + \int_0^t d\tau ||f(\tau) - h(\tau)||^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}.\end{aligned}$$

By the Gronwall's inequality, we obtain that $f = h$ because f and h satisfies the same initial conditions. This proves the uniqueness of the solution.

Continuity. Let $[a,b]$ be a time interval. We follow the simliar argument as in (3.6.2) and (3.6.3) with the time interval $[a,b]$ instead of $[0,t]$ and let $f^m = f^{m+1} = f$

and obtain that

$$
\begin{aligned}
|M(f(b)) - M(f(a))| &= |\frac{1}{2}||f(b)||_H^2 - \frac{1}{2}||f(a)||_H^2 + \int_a^b d\tau \ ||f(\tau)||_I^2 \\
&\lesssim \left(\int_a^b d\tau \ ||f(\tau)||_I^2\right)\left(1 + \sup_{a\le\tau\le b} M^{1/2}(f(\tau))\right).
\end{aligned}
$$

As $a \to b$, we obtain that $|M(f(b)) - M(f(a))| \to 0$ because $||f||_I^2$ is integrable in time. This proves the continuity of M.

Positivity. For the proof of positivity of the solution, we recall the paper [3] where we see the positivity of strong solutions to the non-relativistic Boltzmann equations without angular cut-off with the initial data $f_0 \in H^M$ for $M \geq 5$ and with moderate singularity $0 \leq \gamma \leq 1$. Similar to this proof, we consider the cut-off approximation F^ϵ to the relativistic Boltzmann equation except that the kernel σ has been replaced by σ_ϵ where the angular singularity has been removed and $\sigma_\epsilon \to \sigma$ as $\epsilon \to 0$. We obtain that F^ϵ is positive. If our initial data is nice enough to be in H^M for $M > 5$, we conclude that $F = J + \sqrt{J}f \geq 0$ using the compactness argument from the uniqueness of the solution. If our initial solution is not regular enough, then we use the density argument that H^M is dense in $H(\mathbb{T}^3 \times \mathbb{R}^3)$ and the approximation arguments and the uniqueness to show the positivity. If the angular cutoff is more singular as $1 \leq \gamma < 2$, then the positivity can be obtained by using higher derivative estimates and following the same compactness argument as in the case with lower singularity. $\square$

3.6.2 Global existence

In this section, we would like to derive the systems of macroscopic equations and balance laws with respect to the coefficients appearing in the expression for the hydrodynamic part Pf and prove an coercive inequality of the microscopic part $\{I-P\}f$. With this coercivity estimates for the non-linear local solutions to the relativistic Boltzmann system, we will show that these solutions must be global with the standard continuity argument and by proving energy inequalities. We will also show rapid time decay of the solutions.

For the relativistic Maxwellian solution J, we have normalized so that $\int_{\mathbb{R}^3} J(p)dp = 1$. Here we introduce the following notations for the integrals:

$$
\begin{gathered}
\lambda_0 = \int_{\mathbb{R}^3} p^0 J dp, \quad \lambda_{00} = \int_{\mathbb{R}^3} (p^0)^2 J dp, \quad \lambda_1 = \int_{\mathbb{R}^3} (p_1)^2 J dp, \\
\lambda_{10} = \int_{\mathbb{R}^3} \frac{p_1^2}{p^0} J dp, \quad \lambda_{12} = \int_{\mathbb{R}^3} \frac{p_1^2 p_2^2}{(p^0)^2} J dp, \quad \lambda_{11} = \int_{\mathbb{R}^3} \frac{p_1^4}{(p^0)^2} J dp, \\
\lambda_{100} = \int_{\mathbb{R}^3} \frac{p_1^2}{(p^0)^2} J dp.
\end{gathered}
$$

We also mention that the null space of the linearized Boltzmann operator L is given by the 5-dimensional space

$$
N(L) = \operatorname{span}\{\sqrt{J}, p_1\sqrt{J}, p_2\sqrt{J}, p_3\sqrt{J}, p^0\sqrt{J}\}.
$$

Then we define the orthogonal projection from $L^2(\mathbb{R}^3)$ onto $N(L)$ by P. Then we can write Pf as a linear combination of the basis as

$$
Pf = \left(\mathcal{A}^f(t,x) + \sum_{i=1}^{3} \mathcal{B}_i^f(t,x) p_i + \mathcal{C}^f(t,x) p^0 \right) \sqrt{J} \tag{3.6.5}
$$

where the coefficients are given by

$$\mathcal{A}^f = \int_{\mathbb{R}^3} f\sqrt{J}dp - \lambda_0 c^f, \quad \mathcal{B}_i^f = \frac{\int_{\mathbb{R}^3} fp\sqrt{J}dp}{\lambda_i}, \quad \mathcal{C}^f = \frac{\int_{\mathbb{R}^3} f(p^0\sqrt{J} - \lambda_0\sqrt{J})}{\lambda_{00} - \lambda_0^2}.$$

Then we can decompose $f(t,x,p)$ as

$$f = Pf + \{I - P\}f. \tag{3.6.6}$$

We start from plugging the expression (3.6.6) into (3.1.3). Then we obtain

$$\{\partial_t + \hat{p}\cdot\nabla_x\}Pf = -\partial_t\{I-P\}f - (\hat{p}\cdot\nabla_x + L)\{I-P\}f + \Gamma(f,f). \tag{3.6.7}$$

Note that we have expressed the hydrodynamic part Pf in terms of the microscopic part $\{I-P\}f$ and the higher-order term Γ. We define an operator $l = -(\hat{p}\cdot\nabla_x + L)$ here. Using the expression (3.6.5) of Pf with respect to the basis elements, we obtain that the left-hand side of the (3.6.7) can be written as

$$\partial_t\mathcal{A}\sqrt{J} + \sum_{i=1}^3 \partial_i(\mathcal{A} + \mathcal{C}p^0)\frac{p_i}{p^0}\sqrt{J} + \partial_t\mathcal{C}p^0\sqrt{J} + \sum_{i=1}^3 \partial_t\mathcal{B}_i p_i\sqrt{J}$$
$$+ \sum_{i=1}^3 \partial_i\mathcal{B}_i\frac{p_i^2}{p^0}\sqrt{J} + \sum_{i=1}^3\sum_{i\neq j}\partial_j\mathcal{B}_i\frac{p_ip_j}{p^0}\sqrt{J}$$

where $\partial_i = \partial_{x_i}$. For fixed (t,x) we can write the left-hand side with respect to the following basis, $\{e_k\}_{k=1}^{14}$, which consists of

$$\sqrt{J}, \quad \left(\frac{p_i}{p^0}\sqrt{J}\right)_{1\le i\le 3}, \quad p^0\sqrt{J}, \quad \left(p_i\sqrt{J}\right)_{1\le i\le 3}, \quad \left(\frac{p_ip_j}{p^0}\sqrt{J}\right)_{1\le i\le j\le 3}. \tag{3.6.8}$$

Then, we can rewrite the left-hand side as

$$\partial_t\mathcal{A}\sqrt{J} + \sum_{i=1}^3 \partial_i\mathcal{A}\frac{p_i}{p^0}\sqrt{J} + \partial_t\mathcal{C}p^0\sqrt{J} + \sum_{i=1}^3(\partial_i\mathcal{C} + \partial_t\mathcal{B}_i)p_i\sqrt{J}$$
$$+ \sum_{i=1}^3\sum_{j=1}^3((1-\delta_{ij})\partial_i\mathcal{B}_j + \partial_j\mathcal{B}_i)\frac{p_ip_j}{p^0}\sqrt{J}.$$

By a comparison of coefficients, we obtain a system of macroscopic equations

$$
\begin{aligned}
\partial_t \mathcal{A} &= -\partial_t m_a + l_a + G_a, \\
\partial_i \mathcal{A} &= -\partial_t m_{ia} + l_{ia} + G_{ia}, \\
\partial_t \mathcal{C} &= -\partial_t m_c + l_c + G_c, \\
\partial_i \mathcal{C} + \partial_t \mathcal{B}_i &= -\partial_t m_{ic} + l_{ic} + G_{ic}, \\
(1-\delta_{ij})\partial_i \mathcal{B}_j + \partial_j \mathcal{B}_i &= -\partial_t m_{ij} + l_{ij} + G_{ij},
\end{aligned} \tag{3.6.9}
$$

where the indices are from the index set defined as $D = \{a, ia, c, ic, ij | 1 \le i \le j \le 3\}$ and m_μ, l_μ, and G_μ for $\mu \in D$ are the coefficients of $\{I-P\}f$, $l\{I-P\}f$, and $\Gamma(f,f)$ with respect to the basis $\{e_k\}_{k=1}^{14}$ respectively.

We also derive a set of equations from the conservation laws. For the perturbation solution f, we multiply the linearized Boltzmann equation by $\sqrt{J}, p_i\sqrt{J}, p^0\sqrt{J}$ and integrate over $\mathbb{R}^3$ to obtain that

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}^3} f\sqrt{J} dp + \int_{\mathbb{R}^3} \hat{p}\cdot\nabla_x f\sqrt{J} dp = 0 \\
\partial_t \int_{\mathbb{R}^3} f\sqrt{J} p_i dp + \int_{\mathbb{R}^3} \hat{p}\cdot\nabla_x f\sqrt{J} p_i dp = 0 \\
\partial_t \int_{\mathbb{R}^3} f\sqrt{J} p^0 dp + \int_{\mathbb{R}^3} \hat{p}\cdot\nabla_x f\sqrt{J} p^0 dp = 0.
\end{aligned} \tag{3.6.10}
$$

These hold because $1, p_i, p^0$ are collisional invariants and hence

$$
\int_{\mathbb{R}^3} Q(f,f)dp = \int_{\mathbb{R}^3} Q(f,f)p_i dp = \int_{\mathbb{R}^3} Q(f,f)p^0 dp = 0.
$$

We will plug the decomposition $f = Pf + \{I-P\}f$ into (3.6.10). We first consider the microscopic part. Note that

$$\int_{\mathbb{R}^3} \hat{p}\cdot\nabla_x\{I-P\}f\sqrt{J}\begin{pmatrix}1\\ p_i\\ p^0\end{pmatrix}dp = \sum_{j=1}^{3}\int_{\mathbb{R}^3}\frac{p_j}{p^0}\partial_j\{I-P\}f\sqrt{J}\begin{pmatrix}1\\ p_i\\ p^0\end{pmatrix}dp$$

$$= \sum_{j=1}^{3}\partial_j\int_{\mathbb{R}^3}\{I-P\}f\sqrt{J}\begin{pmatrix}\frac{p_j}{p^0}\\ \frac{p_ip_j}{p^0}\\ p_j\end{pmatrix}dp = \sum_{j=1}^{3}\partial_j\langle\{I-P\}f,\sqrt{J}\begin{pmatrix}\frac{p_j}{p^0}\\ \frac{p_ip_j}{p^0}\\ 0\end{pmatrix}\rangle. \tag{3.6.11}$$

Also, we have that

$$\partial_t\int_{\mathbb{R}^3}\{I-P\}f\sqrt{J}\begin{pmatrix}1\\ p_i\\ p^0\end{pmatrix} = \partial_t\langle\{I-P\}f,\sqrt{J}\begin{pmatrix}1\\ p_i\\ p^0\end{pmatrix}\rangle = 0. \tag{3.6.12}$$

On the other hand, the hydrodynamic part $Pf = (\mathcal{A}+\mathcal{B}\cdot p+\mathcal{C}p^0)\sqrt{J}$ satisfies

$$\partial_t\int_{\mathbb{R}^3}\begin{pmatrix}1\\ p_i\\ p^0\end{pmatrix}Pf\sqrt{J}dp + \int_{\mathbb{R}^3}\hat{p}\cdot\nabla_x Pf\sqrt{J}\begin{pmatrix}1\\ p_i\\ p^0\end{pmatrix}dp$$

$$
= \partial_t \int_{\mathbb{R}^3} \begin{pmatrix} \mathcal{A} + \mathcal{B} \cdot p + \mathcal{C} p^0 \\ \mathcal{A} p_i + \mathcal{B} \cdot p p_i + \mathcal{C} p^0 p_i \\ \mathcal{A} p^0 + \mathcal{B} \cdot p p^0 + \mathcal{C} (p^0)^2 \end{pmatrix} \sqrt{J} dp
$$

$$
+ \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_j \begin{pmatrix} \frac{p_j}{p^0} (\mathcal{A} + \mathcal{B} \cdot p + \mathcal{C} p^0) \\ \frac{p_i p_j}{p^0} (\mathcal{A} + \mathcal{B} \cdot p + \mathcal{C} p^0) \\ p_j \mathcal{A} + \mathcal{B} \cdot p p_j + \mathcal{C} p^0 p_j \end{pmatrix} \sqrt{J} dp \tag{3.6.13}
$$

$$
= \begin{pmatrix} \partial_t \mathcal{A} + \lambda_0 \partial_t \mathcal{C} \\ \lambda_1 \partial_t \mathcal{B}_i \\ \lambda_0 \partial_t \mathcal{A} + \lambda_{00} \partial_t \mathcal{C} \end{pmatrix} + \begin{pmatrix} \lambda_{10} \nabla_x \cdot \mathcal{B} \\ \lambda_{10} \partial_i \mathcal{A} + \lambda_1 \partial_i \mathcal{C} \\ \lambda_1 \nabla_x \cdot \mathcal{B} \end{pmatrix}.
$$

Also, we have that $L(f) = L\{I - P\}f$. Together with (3.6.10), (3.6.11), (3.6.12), and (3.6.13), we finally obtain the local conservation laws satisfied by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$:

$$
\begin{aligned}
&\partial_t \mathcal{A} + \lambda_0 \partial_t \mathcal{C} + \lambda_{10} \nabla_x \cdot \mathcal{B} = -\nabla_x \cdot \langle \{I - P\} f, \sqrt{J} \frac{p}{p^0} \rangle, \\
&\lambda_1 \partial_t \mathcal{B} + \lambda_{10} \nabla_x \mathcal{A} + \lambda_1 \nabla_x \mathcal{C} = -\nabla_x \cdot \langle \{I - P\} f, \sqrt{J} \frac{p \otimes p}{p^0} \rangle, \\
&\lambda_0 \partial_t \mathcal{A} + \lambda_{00} \partial_t \mathcal{C} + \lambda_1 \nabla_x \cdot \mathcal{B} = 0.
\end{aligned}
$$

Comparing the first and the third conservation laws, we obtain

$$
\begin{aligned}
\partial_t \mathcal{A} \left(1 - \frac{\lambda_0^2}{\lambda_{00}} \right) + \nabla_x \cdot \mathcal{B} \left(\lambda_{10} - \frac{\lambda_0 \lambda_1}{\lambda_{00}} \right) &= -\nabla_x \cdot \langle \{I - P\} f, \sqrt{J} \frac{p}{p^0} \rangle, \\
\lambda_1 \partial_t \mathcal{B} + \lambda_{10} \nabla_x \mathcal{A} + \lambda_1 \nabla_x \mathcal{C} &= -\nabla_x \cdot \langle \{I - P\} f, \sqrt{J} \frac{p \otimes p}{p^0} \rangle, \qquad (3.6.14) \\
\left(\lambda_0 - \frac{\lambda_{00}}{\lambda_0} \right) \partial_t \mathcal{C} + \left(\lambda_{10} - \frac{\lambda_1}{\lambda_0} \right) \nabla_x \cdot \mathcal{B} &= -\nabla_x \cdot \langle \{I - P\} f, \sqrt{J} \frac{p}{p^0} \rangle.
\end{aligned}
$$

We also mention that we have the following lemma on the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$ by the conservation of mass, momentum, and energy:

Lemma 3.6.3. *Let $f(t,x,p)$ be the local solution to the linearized relativistic Boltzmann equation (3.1.3) which is shown to exist in Theorem 3.6.2 which satisfies the mass, momentum, and energy conservation laws (3.1.2). Then we have*

$$\int_{\mathbb{T}^3} \mathcal{A}(t,x)dx = \int_{\mathbb{T}^3} \mathcal{B}_i(t,x)dx = \int_{\mathbb{T}^3} \mathcal{C}(t,x)dx = 0,$$

where $i \in 1,2,3$.

We also list two lemmas that helps us to control the coefficients in the linear microscopic term l and the non-linear higher-order term Γ.

Lemma 3.6.4. *For any coefficient l_μ for the microscopic term l, we have*

$$\sum_{\mu \in D} ||l_\mu||_{H_x^{N-1}} \lesssim \sum_{|\alpha| \leq N} ||\{I-P\}\partial^\alpha f||_{L^2_{\frac{a+\gamma}{2}}(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

Proof. In order to estimate the size for H^{N-1} norm, we take

$$\langle \partial^\alpha l(\{I-P\}f), e_k\rangle = -\langle \hat{p}\cdot\nabla_x(\{I-P\}\partial^\alpha f), e_k\rangle - \langle L(\{I-P\}\partial^\alpha f), e_k\rangle.$$

For any $|\alpha| \leq N-1$, the L^2-norm of the first part of the right-hand side is

$$\begin{aligned} ||\langle \hat{p}\cdot\nabla_x(\{I-P\}\partial^\alpha f), e_k\rangle||^2_{L^2_x} &\lesssim \int_{\mathbb{T}^3\times\mathbb{R}^3} dxdp\,|e_k||\{I-P\}\nabla_x\partial^\alpha f|^2 \\ &\lesssim ||\{I-P\}\nabla_x\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}(\mathbb{T}^3\times\mathbb{R}^3)} \end{aligned}$$

Similarly, we have

$$\begin{aligned} ||\langle L(\{I-P\}\partial^\alpha f), e_k\rangle||^2_{L^2_x} &\lesssim \left|\left| |\{I-P\}\partial^\alpha f|_{L^2_{\frac{a+\gamma}{2}}} |\sqrt{J}|_{L^2_{\frac{a+\gamma}{2}}} \right|\right|^2_{L^2_x} \\ &\lesssim ||\{I-P\}\nabla_x\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}(\mathbb{T}^3\times\mathbb{R}^3)}. \end{aligned}$$

This completes the proof. □

Lemma 3.6.5. *Let $||f||_H^2 \leq M$ for some $M > 0$. Then, we have*

$$\sum_{\mu \in D} ||G_\mu||_{H_x^{N-1}} \lesssim \sqrt{M} \sum_{|\alpha| \leq N} ||\partial^\alpha f||_{L^2_{\frac{a+\gamma}{2}}(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

Proof. In order to estimate the size for H^{N-1} norm, we consider $\langle \Gamma(f,f), e_k \rangle$. By (3.4.6), for any $m \geq 0$,

$$\begin{aligned} ||\langle \Gamma(f,f), e_k \rangle||_{H_x^{N-1}} &\lesssim \sum_{|\alpha| \leq N-1} \sum_{\alpha_1 \leq \alpha} \left|\left| |\partial^{\alpha - \alpha_1} f|_{L^2_{-m}} |\partial^{\alpha_1} f|_{L^2_{-m}} \right|\right|_{L^2_x} \\ &\lesssim ||f||_{L^2_{-m} H_x^N} \sum_{|\alpha| \leq N} ||\partial^\alpha f||_{L^2_{\frac{a+\gamma}{2}}} \\ &\lesssim \sqrt{M} \sum_{|\alpha| \leq N} ||\partial^\alpha f||_{L^2_{\frac{a+\gamma}{2}}(\mathbb{T}^3 \times \mathbb{R}^3)}. \end{aligned}$$

This completes the proof. □

These two lemmas above, the macroscopic equations, and the local conservation laws will together prove the following theorem on the coercivity estimate for the microscopic term $\{I - P\}f$ which is crucial for the energy inequality which will imply the global existence of the solution with the continuity argument.

Theorem 3.6.6. *Given the initial condition $f_0 \in H$ which satisfies the mass, momentum, and energy conservation laws (3.1.2) and the assumptions in Theorem 3.6.2, we can consider the local solution $f(t,x,p)$ to the linearized relativistic Boltzmann equation (3.1.3). Then, there is a constant $M > 0$ such that if*

$$||f(t)||_H^2 \leq M_0,$$

then there are universal constants $\delta > 0$ and $C > 0$ such that

$$\sum_{|\alpha|\leq N} ||\{I-P\}\partial^\alpha f||^2_{I^{a,\gamma}}(t) \geq \delta \sum_{|\alpha|\leq N} ||P\partial^\alpha f||^2_{I^{a,\gamma}}(t) - C\frac{dI(t)}{dt},$$

where $I(t)$ is the interaction potential defined as

$$I(t) = \sum_{|\alpha|\leq N-1} \{I^\alpha_a(t) + I^\alpha_b(t) + I^\alpha_c(t)\}$$

and each of the sub-potentials $I^\alpha_a(t)$, $I^\alpha_b(t)$, and $I^\alpha_c(t)$ is defined as

$$\begin{aligned}
I^\alpha_a(t) &= \sum_{i=1}^3 \int_{\mathbb{T}^3} \partial_i\partial^\alpha m_{ia}\partial^\alpha \mathcal{A}(t,x)dx,\\
I^\alpha_b(t) &= -\sum_{i=1}^3 \sum_{j\neq i} \int_{\mathbb{T}^3} \partial_j\partial^\alpha m_{ij}\partial^\alpha \mathcal{B}_i dx,\\
I^\alpha_c(t) &= \int_{\mathbb{T}^3} (\nabla\cdot\partial^\alpha\mathcal{B})\partial^\alpha\mathcal{C}(t,x)dx + \sum_{i=1}^3 \int_{\mathbb{T}^3} \partial_i\partial^\alpha m_{ic}\partial^\alpha \mathcal{C}(t,x)dx.
\end{aligned}$$

Proof. Since $Pf = \mathcal{A} + \mathcal{B}\cdot p + \mathcal{C}p^0$, we have that

$$||P\partial^\alpha f(t)||^2_{I^{a,\gamma}} \lesssim ||\partial^\alpha \mathcal{A}(t)||^2_{L^2_x} + ||\partial^\alpha \mathcal{B}(t)||^2_{L^2_x} + ||\partial^\alpha \mathcal{C}(t)||^2_{L^2_x}.$$

Thus, it suffices to prove the following estimate:

$$\begin{aligned}
&||\partial^\alpha \mathcal{A}(t)||^2_{H^N_x} + ||\partial^\alpha \mathcal{B}(t)||^2_{H^N_x} + ||\partial^\alpha \mathcal{C}(t)||^2_{H^N_x}\\
&\qquad\lesssim \sum_{|\alpha|\leq N} ||\{I-P\}\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}} + M\sum_{|\alpha|\leq N} ||\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}} + \frac{dI(t)}{dt}.
\end{aligned} \tag{3.6.15}$$

Note that the term $M\sum_{|\alpha|\leq N}||\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}}$ can be ignored because we have

$$\sum_{|\alpha|\leq N}||\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}} \lesssim \sum_{|\alpha|\leq N}||P\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}} + \sum_{|\alpha|\leq N}||\{I-P\}\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}}$$

$$\lesssim ||\partial^\alpha \mathcal{A}(t)||^2_{H^N_x} + ||\partial^\alpha \mathcal{B}(t)||^2_{H^N_x} + ||\partial^\alpha \mathcal{C}(t)||^2_{H^N_x} + \sum_{|\alpha|\leq N}||\{I-P\}\partial^\alpha f(t)||^2_{L^2_{\frac{a+\gamma}{2}}}.$$

Therefore, with sufficiently small $M>0$, (3.6.15) will imply Theorem 3.6.6.

In order to prove (3.6.15), we will estimate each of the ∂^α derivatives of $\mathcal{A},\mathcal{B},\mathcal{C}$ for $0<|\alpha|\leq N$ separately. Later, we will use Poincaré inequality to estimate the L^2-norm of $\mathcal{A},\mathcal{B},\mathcal{C}$ to finish the proof.

For the estimate for $\mathcal{A}$, we use the second equation in the system of macroscopic equations (3.6.9) which tells $\partial_i\mathcal{A} = -\partial_t m_{ia} + l_{ia} + G_{ia}$. We take $\partial_i\partial^\alpha$ onto this equation for $|\alpha|\leq N-1$ and sum over i and obtain that

$$-\Delta\partial^\alpha\mathcal{A} = \sum_{i=1}^{3}(\partial_t\partial_i\partial^\alpha m_{ia} - \partial_i\partial^\alpha(l_{ia}+G_{ia})).$$

We now multiply $\partial^\alpha\mathcal{A}$ and integrate over $\mathbb{T}^3$ to obtain

$$||\nabla\partial^\alpha\mathcal{A}||^2_{L^2_x} \leq ||\partial^\alpha(l_{ia}+G_{ia})||_{L^2_x}||\nabla\partial^\alpha\mathcal{A}||_{L^2_x} + \frac{d}{dt}\sum_{i=1}^{3}\int_{\mathbb{T}^3}\partial_i\partial^\alpha m_{ia}\partial^\alpha\mathcal{A}(t,x)dx$$
$$-\sum_{i=1}^{3}\int_{\mathbb{T}^3}\partial_i\partial^\alpha m_{ia}\partial_t\partial^\alpha\mathcal{A}(t,x)dx.$$

We define the interaction functional

$$I^\alpha_a(t) = \sum_{i=1}^{3}\int_{\mathbb{T}^3}\partial_i\partial^\alpha m_{ia}\partial^\alpha\mathcal{A}(t,x)dx.$$

For the last term, we use the first equation of the local conservation laws (3.6.14) to obtain that

$$\int_{\mathbb{T}^3}\sum_{i=1}^3|\partial_i\partial^\alpha m_{ia}\partial_t\partial^\alpha\mathcal{A}(t,x)|dx\leq\zeta||\nabla\cdot\partial^\alpha\mathcal{B}||^2_{L^2_x}+C_\zeta||\{I-P\}\nabla\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}},$$

for any $\zeta>0$. Together with Lemma 3.6.4 and Lemma 3.6.5, we obtain that

$$\begin{aligned}&||\nabla\partial^\alpha\mathcal{A}||^2_{L^2_x}-\zeta||\nabla\cdot\partial^\alpha\mathcal{B}||^2_{L^2_x}\\&\lesssim C_\zeta\sum_{|\alpha|\leq N}||\{I-P\}\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}+\frac{dI^\alpha_a}{dt}+M\sum_{|\alpha|\leq N}||\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}.\end{aligned}\tag{3.6.16}$$

For the estimate for $\mathcal{C}$, we use the fourth equation in the system of macroscopic equations (3.6.9) which tells $\partial_i\mathcal{C}+\partial_t\mathcal{B}_i=-\partial_t m_{ic}+l_{ic}+G_{ic}$. We take $\partial_i\partial^\alpha$ onto this equation for $|\alpha|\leq N-1$ and sum over i and obtain that

$$-\Delta\partial^\alpha\mathcal{C}=\frac{d}{dt}(\nabla\cdot\partial^\alpha\mathcal{B})+\sum_{i=1}^3(\partial_t\partial_i\partial^\alpha m_{ic}-\partial_i\partial^\alpha(l_{ic}+G_{ic})).$$

We now multiply $\partial^\alpha\mathcal{C}$ and integrate over $\mathbb{T}^3$ to obtain

$$\begin{aligned}||\nabla\partial^\alpha\mathcal{C}||^2_{L^2_x}\leq&\frac{d}{dt}\int_{\mathbb{T}^3}(\nabla\cdot\partial^\alpha\mathcal{B})\partial^\alpha\mathcal{C}(t,x)dx-\int_{\mathbb{T}^3}(\nabla\cdot\partial^\alpha\mathcal{B})\partial_t\partial^\alpha\mathcal{C}(t,x)dx\\&+||\partial^\alpha(l_{ic}+G_{ic})||_{L^2_x}||\nabla\partial^\alpha\mathcal{C}||_{L^2_x}+\frac{d}{dt}\sum_{i=1}^3\int_{\mathbb{T}^3}\partial_i\partial^\alpha m_{ic}\partial^\alpha\mathcal{C}(t,x)dx\\&-\sum_{i=1}^3\int_{\mathbb{T}^3}\partial_i\partial^\alpha m_{ic}\partial_t\partial^\alpha\mathcal{C}(t,x)dx.\end{aligned}$$

We define the interaction functional

$$I^\alpha_c(t)=\int_{\mathbb{T}^3}(\nabla\cdot\partial^\alpha\mathcal{B})\partial^\alpha\mathcal{C}(t,x)dx+\sum_{i=1}^3\int_{\mathbb{T}^3}\partial_i\partial^\alpha m_{ic}\partial^\alpha\mathcal{C}(t,x)dx.$$

We also use the third equation of the local conservation laws (3.6.14) to obtain that

$$\int_{\mathbb{T}^3}\sum_{i=1}^3|\partial_i\partial^\alpha m_{ic}\partial_t\partial^\alpha\mathcal{C}(t,x)|dx\leq\zeta||\nabla\cdot\partial^\alpha\mathcal{B}||^2_{L^2_x}+C_\zeta||\{I-P\}\nabla\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}},$$

for any $\zeta > 0$. Together with Lemma 3.6.4 and Lemma 3.6.5, we obtain that

$$\begin{aligned}&||\nabla\partial^\alpha \mathcal{C}||^2_{L^2_x} - \zeta||\nabla\cdot\partial^\alpha\mathcal{B}||^2_{L^2_x}\\ &\lesssim C_\zeta \sum_{|\alpha|\leq N} ||\{I-P\}\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}} + \frac{dI^\alpha_c}{dt} + M\sum_{|\alpha|\leq N}||\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}.\end{aligned} \tag{3.6.17}$$

For the estimate for $\mathcal{B}$, we use the last equation in the system of macroscopic equations (3.6.9) which tells $(1-\delta_{ij})\partial_i\mathcal{B}_j + \partial_j\mathcal{B}_i = -\partial_t m_{ij} + l_{ij} + G_{ij}$. Note that when $i = j$, we have

$$\partial_i\mathcal{B}_i = -\partial_t m_{ii} + l_{ii} + G_{ii}.$$

Also, if $i \neq j$, we have

$$\partial_i\mathcal{B}_j + \partial_j\mathcal{B}_i = -\partial_t m_{ij} + l_{ij} + G_{ij}.$$

We take $\partial_j\partial^\alpha$ on both equations above for $|\alpha| \leq N-1$ and sum on j to obtain

$$\begin{aligned}\Delta\partial^\alpha\mathcal{B}_i &= -\partial_i\partial_i\partial^\alpha\mathcal{B}_i + 2\partial_i\partial^\alpha l_{ii} + 2\partial_i\partial^\alpha G_{ii}\\ &\quad + \sum_{j\neq i}(-\partial_i\partial^\alpha l_{jj} - \partial_i\partial^\alpha G_{jj} + \partial_j\partial^\alpha l_{ij} + \partial_j\partial^\alpha G_{ij} - \partial_t\partial_j\partial^\alpha m_{ij}).\end{aligned}$$

We now multiply $\partial^\alpha\mathcal{B}_i$ and integrate over $\mathbb{T}^3$ to obtain

$$\begin{aligned}||\nabla\partial^\alpha\mathcal{B}_i||^2_{L^2_x} &\leq -\frac{d}{dt}\sum_{j\neq i}\int_{\mathbb{T}^3}\partial_j\partial^\alpha m_{ij}\partial^\alpha\mathcal{B}_i dx + \sum_{j\neq i}\int_{\mathbb{T}^3}\partial_j\partial^\alpha m_{ij}\partial_t\partial^\alpha\mathcal{B}_i dx\\ &\quad + \sum_{\mu\in D}||\partial^\alpha(l_\mu + G_\mu)||_{L^2_x}.\end{aligned}$$

We define the interaction functional

$$I^\alpha_b(t) = -\sum_{i=1}^{3}\sum_{j\neq i}\int_{\mathbb{T}^3}\partial_j\partial^\alpha m_{ij}\partial^\alpha\mathcal{B}_i dx.$$

We also use the second equation of the local conservation laws (3.6.14) to obtain that

$$\sum_{i=1}^{3}\sum_{j\neq i}\int_{\mathbb{T}^3}|\partial_j\partial^\alpha m_{ij}\partial_t\partial^\alpha\mathcal{B}_i(t,x)|dx$$
$$\leq \zeta(||\nabla\cdot\partial^\alpha\mathcal{A}||^2_{L^2_x}+||\nabla\cdot\partial^\alpha\mathcal{C}||^2_{L^2_x})+C_\zeta||\{I-P\}\nabla\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}},$$

for any $\zeta>0$. Together with Lemma 3.6.4 and Lemma 3.6.5, we obtain that

$$\begin{aligned}&||\nabla\partial^\alpha\mathcal{B}||^2_{L^2_x}-\zeta(||\nabla\cdot\partial^\alpha\mathcal{A}||^2_{L^2_x}+||\nabla\cdot\partial^\alpha\mathcal{C}||^2_{L^2_x})\\ &\lesssim C_\zeta\sum_{|\alpha|\leq N}||\{I-P\}\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}+\frac{dI^\alpha_b}{dt}+M\sum_{|\alpha|\leq N}||\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}.\end{aligned}\tag{3.6.18}$$

Choose sufficiently small $\zeta>0$. Then, (3.6.16), (3.6.17), and (3.6.18) implies that

$$\begin{aligned}||\nabla\mathcal{A}||^2_{H^{N-1}_x}+&||\nabla\mathcal{B}||^2_{H^{N-1}_x}+||\nabla\mathcal{C}||^2_{H^{N-1}_x}\\ &\lesssim\sum_{|\alpha|\leq N}||\{I-P\}\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}+\frac{dI}{dt}+M\sum_{|\alpha|\leq N}||\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}.\end{aligned}\tag{3.6.19}$$

On the other hand, with the Poincaré inequality and Lemma 3.6.3, we obtain that

$$||\mathcal{A}||^2\lesssim\left(||\nabla\mathcal{A}||+\left|\int_{\mathbb{T}^3}\mathcal{A}(t,x)dx\right|\right)^2=||\nabla\mathcal{A}||^2\lesssim\sum_{|\alpha|\leq N}||\partial^\alpha f||^2_{L^2_{\frac{a+\gamma}{2}}}.$$

This same estimate holds for b and c. Therefore, the inequality (3.6.15) holds and this finishes the proof for the theorem. □

We now use this coercive estimate to prove that the local solutions from the Theorem 3.6.2 should be global-in-time solutions by standard continuity argument. We will also prove that the solutions have rapid exponential time decay.

Before we go into the proof for the global existence, we would like to mention a coercive lower bound for the linearized Boltzmann collision operator L which also gives the positivity of the operator:

Theorem 3.6.7. *There is a constant $\delta > 0$ such that*

$$\langle Lf, f\rangle \geq \delta |\{I-P\}f|^2_{I^{a,\gamma}}.$$

Proof. By following [41] with our assumptions on the relativistic long-range collision kernel and using that $g \geq \frac{|p-q|}{\sqrt{p^0 q^0}}$, we can obtain that

$$\langle Lf, f\rangle \geq \delta_1 |\{I-P\}f|^2_{L^2_{\frac{a}{2}}}.$$

The positive constant δ_1 is explicitly computable. Also, by Lemma 3.1.7, we have

$$\langle Lf, f\rangle \geq |f|^2_{I^{a,\gamma}} - C|f|^2_{L^2}$$

for some $C > 0$. If we suppose that $f = \{I-P\}f$, then we can conclude that

$$\begin{aligned}\langle Lf, f\rangle &= \delta_2\langle Lf, f\rangle + (1-\delta_2)\langle Lf, f\rangle \\ &\geq \delta_2 |f|^2_{I^{a,\gamma}} - C\delta_2|f|^2_{L^2} + (1-\delta_2)\delta_1|f|^2_{L^2_{\frac{a}{2}}}\end{aligned}$$

for any $\delta_2 \in (0,1)$. Note that we have $|f|^2_{L^2_{\frac{a}{2}}} \geq |f|^2_{L^2}$. By choosing $\delta_2 > 0$ sufficiently small, we obtain the theorem. $\square$

Now, we define the dissipation rate $\mathcal{D}$ as

$$\mathcal{D} = \sum_{|\alpha|\leq N} ||\partial^\alpha f(t)||^2_{I^{a,\gamma}}.$$

We will use the energy functional $\mathcal{E}(t)$ to be a high-order norm which satisfies

$$\mathcal{E}(t) \approx \sum_{|\alpha| \leq N} ||\partial^\alpha f(t)||^2_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}. \tag{3.6.20}$$

This functional will be precisely defined during the proof. Then, we would like to set up the following energy inequality:

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{D}(t) \leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

We will prove this energy inequality and use this to show the global existence.

Proof. (*Proof for Theorem 3.1.1*) We denote $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}_0$ and $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{E}_0$. By the definitions on interaction functionals, there is a sufficiently large constant $C'' > 0$ for any $C' > 0$ such that

$$||f(t)||^2_{L^2_p H^N_x} \leq (C''+1)||f(t)||^2_{L^2_p H^N_x} - C'I(t) \lesssim ||f(t)||^2_{L^2_p H^N_x}.$$

Note that C'' doesn't depend on $f(t, x, p)$ but only on C' and I. Here we define the energy functional $\mathcal{E}(t)$ as

$$\mathcal{E}(t) = (C''+1)||f(t)||^2_{L^2_p H^N_x} - C'I(t).$$

Then, the above inequalities show that the definition of $\mathcal{E}$ satisfies (3.6.20).

Recall the local existence Theorem 3.6.2 and Theorem 3.6.6 and choose $M_0 \leq 1$ so that both theorems hold. We choose $M_1 \leq \frac{M_0}{2}$ and consider initial data $\mathcal{E}(0)$ so that

$$\mathcal{E}(0) \leq M_1 < M_0.$$

From the local existence theorem, we define $T > 0$ so that

$$T = \sup\{t \geq 0 | \mathcal{E}(t) \leq 2M_1\}.$$

By taking the spatial derivative ∂^α onto the linearized relativistic Boltzmann equation (3.1.3), integrating over (x,p), and summing over α, we obtain

$$\frac{1}{2}\frac{d}{dt}||f(t)||^2_{L^2_p H^N_x} + \sum_{|\alpha|\leq N} (L\partial^\alpha f, \partial^\alpha f) = \sum_{|\alpha|\leq N} (\partial^\alpha \Gamma(f,f), \partial^\alpha f). \tag{3.6.21}$$

By the estimates from Lemma 3.1.3, we have

$$\sum_{|\alpha|\leq N} (\partial^\alpha \Gamma(f,f), \partial^\alpha f) \lesssim \sqrt{\mathcal{E}}\mathcal{D}.$$

Since our choice of M_1 satisfies $\mathcal{E}(t) \leq 2M_1 \leq M_0$, we see that the assumption for Theorem 3.6.6 is satisfied. Then, Theorem 3.6.6 and Theorem 3.6.7 tells us that

$$\begin{aligned}\sum_{|\alpha|\leq N} (L\partial^\alpha f, \partial^\alpha f) &\geq \delta ||\{I-P\}f||^2_{I^{a,\gamma}} \\ &\geq \frac{\delta}{2}||\{I-P\}f||^2_{I^{a,\gamma}} + \frac{\delta\delta'}{2}\sum_{|\alpha|\leq N} ||P\partial^\alpha f||^2_{I^{a,\gamma}}(t) - \frac{\delta C}{2}\frac{dI(t)}{dt}.\end{aligned}$$

Let $\delta'' = \min\{\frac{\delta}{2}, \frac{\delta\delta'}{2}\}$ and let $C' = \delta C$. Then, we have

$$\frac{1}{2}\frac{d}{dt}\Big(||f(t)||^2_{L^2_p H^N_x} - C' I(t)\Big) + \delta''\mathcal{D} \lesssim \sqrt{\mathcal{E}}\mathcal{D}.$$

We multiply (3.6.21) by $\frac{C''}{2}$ and add this onto the last inequality above using the positivity of L to conclude that

$$\frac{d\mathcal{E}(t)}{dt} + \delta''\mathcal{D}(t) \leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

for some $C > 0$. Suppose $M_1 = \min\{\frac{\delta''^2}{8C^2}, \frac{M_0}{2}\}$. Then, we have

$$\frac{d\mathcal{E}(t)}{dt} + \delta''\mathcal{D}(t) \le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t) \le C\sqrt{2M_1}\mathcal{D}(t) \le \frac{\delta''}{2}\mathcal{D}(t). \tag{3.6.22}$$

Now, we integrate over t for $0 \le t \le \tau < T$ and obtain

$$\mathcal{E}(\tau) + \frac{\delta''}{2}\int_0^\tau \mathcal{D}(t)dt \le \mathcal{E}(0) \le M_1 < 2M_1.$$

Since $\mathcal{E}(\tau)$ is continuous in τ, $\mathcal{E}(\tau) \le M_1$ if $T < \infty$. This contradicts the definition of T and hence $T = \infty$. This proves the global existence.

Also, notice that $\mathcal{E}(t) \lesssim \mathcal{D}(t)$. This and the equation (3.6.22) show the exponential time decay. □

Chapter 4

Further upper bound and coercivity estimates

In this chapter, we compute the upper bound estimates of $\zeta_K(p)$ and $\zeta(p)$ and the lower bound coercivity estimate of $\zeta(p)$. Recall from (3.1.7) and (3.1.8) that $\zeta_K(p)$ and $\zeta(p)$ are defined as

$$\zeta_K(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, v_\phi \sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})\sqrt{J(q')}$$

and

$$\zeta(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\, v_\phi \sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})^2.$$

Obtaining the asymptotics (3.1.9) is crucial for the estimates for the inner product with the norm part $\langle Nf, f\rangle$ and the one with the compact part $\langle Kf, f\rangle$ of the linearized Boltzmann collision operator.

4.1 Upper bound estimates

We first show the upper bound estimates for $\zeta_K(p)$ and $\zeta(p)$. More precisely, we have the following proposition for ζ_K:

Proposition 4.1.1.

$$|\zeta_K(p)| \lesssim (p^0)^{\frac{a+\gamma}{2}}.$$

Also, we have the following upper bound estimate for ζ:

Proposition 4.1.2.

$$\zeta(p) \lesssim (p^0)^{\frac{a+\gamma}{2}}.$$

Proof. (Proof for Proposition 4.1.1) We let $\{\chi_l\}_{l=-\infty}^{\infty}$ be a partition of unity on $(0,\infty)$ such that $|\chi_l| \le 1$ and $\mathrm{supp}(\chi_l) \subset [2^{-l-1}, 2^{-l}]$. Then, we define $\sigma_l(g,\theta) \overset{\text{def}}{=} \sigma(g,\theta)\chi_l(\bar{g})$ where $\bar{g} \overset{\text{def}}{=} g(p^\mu, p'^\mu)$. Define

$$\zeta_K^l(p) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})\sqrt{J(q')}.$$

If $l<0$, we can write

$$|\zeta_K^l(p)| \lesssim \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi (g^a + g^{-b})\sigma_0(\theta)\chi_l(\bar{g}) \left|\sqrt{J(q)} + \sqrt{J(q')}\right| \sqrt{J(q')},$$

since $\sigma(g,\theta) \lesssim (g^a+g^{-b})\sigma_0(\theta)$ and $|\sqrt{J(q)} - \sqrt{J(q')}| \le |\sqrt{J(q)} + \sqrt{J(q')}|$. Then,

$$|\zeta_K^l(p)| \lesssim \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi (g^a + g^{-b})\sigma_0(\theta)\chi_l(\bar{g}) \left(\sqrt{J(q)J(q')} + J(q')\right).$$

For the part with $\sqrt{J(q)J(q')}$, we use $\sqrt{J(q')} \le 1$ and $v_\phi \lesssim 1$ to obtain that

$$\begin{aligned} &\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi (g^a + g^{-b})\sigma_0(\theta)\chi_l(\bar{g})\sqrt{J(q)J(q')} \\ &\lesssim \int_{\mathbb{R}^3} dq \left((p^0q^0)^{\frac{a}{2}} + \frac{|p-q|^{-b}}{(p^0q^0)^{-\frac{b}{2}}}\right)\sqrt{J(q)} \int_{\mathbb{S}^2} dw\ \sigma_0(\theta)\chi_l(\bar{g}). \end{aligned} \tag{4.1.1}$$

For the last inequality, we used $\frac{|p-q|}{\sqrt{p^0q^0}} \le g \lesssim \sqrt{p^0q^0}$. Then we use

$$\int_{\mathbb{S}^2} dw\, \sigma_0(\theta)\chi_l(\bar{g}) \approx \int_{2^{-l-1}}^{2^{-l}} \frac{1}{\theta^{1+\gamma}} d\theta = 2^{l\gamma} g^\gamma \lesssim 2^{l\gamma}(p^0q^0)^{\frac{\gamma}{2}}$$

to obtain that the first part of $|\zeta_K^l(p)|$ is bounded above by

$$2^{l\gamma}\left(\int_{\mathbb{R}^3} dq (p^0q^0)^{\frac{a+\gamma}{2}}\sqrt{J(q)} + \int_{\mathbb{R}^3} dq (p^0q^0)^{\frac{b+\gamma}{2}}|p-q|^{-b}\sqrt{J(q)}\right).$$

Use the inequality that $\int_{\mathbb{R}^3} dq|p-q|^{-b}J(q)^c \approx (p^0)^{-b}$ for some $c > 0$ and $b < 3$ to conclude that the first part of $|\zeta_K^l(p)|$ is bounded above by $2^{l\gamma}(p^0)^{\frac{a+\gamma}{2}}$.

For the rest part of $|\zeta_K^l(p)|$, we use a dual representation to write the rest of the integral as

$$\frac{c}{2p^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}\int_{E^{p'}_{q'-p}}\frac{d\pi_{p'}}{p'^0}(g^a+g^{-b})\sigma_0(\theta)\chi_l(\bar{g})J(q'), \tag{4.1.2}$$

where

$$E^{p'}_{q'-p} \stackrel{\text{def}}{=} \{p' \in \mathbb{R}^3 | (p^\mu - p'^\mu)(q'_\mu - p_\mu) = 0\}$$

and $d\pi_{p'}$ is the Lebesgue measure on the set $E^{p'}_{q'-p}$ and is defined as

$$d\pi_{p'} = dp' u(p'^0 + q'^0 - p^0)\delta\left(\frac{\tilde{g}^2 + 2p'^\mu(q'_\mu - p_\mu)}{2\tilde{g}}\right)$$

Here $\tilde{g} \stackrel{\text{def}}{=} g(p^\mu, q'^\mu)$ and $u(x) = 1$ if $x \ge 1$ and 0 otherwise. By following the proof for Proposition 3.2.2 with the roles of p and p' and the roles of q and q' are reversed respectively, we obtain that

$$\int_{E^{p'}_{q'-p}} \frac{d\pi_{p'}}{p'^0}\tilde{g}(\bar{g})^{-2-\gamma}\chi_l(\bar{g}) \lesssim 2^{l\gamma}\sqrt{\frac{q'^0}{p^0}}.$$

Since we further have $g \approx \tilde{g}$ and $\sigma_0(\theta) \approx \left(\frac{\bar{g}}{g}\right)^{-2-\gamma}$, (4.1.2) is bounded above as

$$
\begin{aligned}
&\frac{c}{2p^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}\int_{E^{p'}_{q'-p}}\frac{d\pi_{p'}}{p'^0}(g^a+g^{-b})\sigma_0(\theta)\chi_l(\bar{g})J(q') \\
&\lesssim \frac{1}{p^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}J(q')\sqrt{\frac{q'^0}{p^0}}(\tilde{g}^{a+\gamma+1}+\tilde{g}^{-b+\gamma+1}) \\
&\lesssim \frac{1}{p^0}\int_{\mathbb{R}^3}dq'J(q')\left((p^0q^0)^{\frac{a+\gamma}{2}}+\frac{|p-q'|^{-b+\gamma}}{(p^0q^0)^{\frac{-b+\gamma}{2}}}\right) \\
&\lesssim (p^0)^{\frac{a+\gamma}{2}},
\end{aligned}
$$

where the last inequality is by $\int_{\mathbb{R}^3}dq'|p-q'|^{-b+\gamma}J(q')^c \approx (p^0)^{-b+\gamma}$ for some $c>0$. Therefore, we finally obtain that if $l<0$,

$$
|\zeta^l_K(p)| \lesssim 2^{l\gamma}(p^0)^{\frac{a+\gamma}{2}}.
$$

On the other hand, if $l \geq 0$, then we have $\bar{g} \leq 1$ and we obtain $q^0 \approx q'^0$. Then we further split the integral into three parts as the following:

$$
\begin{aligned}
\zeta^l_K(p) &= \int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\; v_\phi\sigma_l(g,\theta)(\sqrt{J(q)}-\sqrt{J(q')})\sqrt{J(q')} \\
&= -\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\; v_\phi\sigma_l(g,\theta)(\sqrt{J(q)}-\sqrt{J(q')})^2 \\
&\quad + \int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\; v_\phi\sigma_l(g,\theta)\Big(\sqrt{J(q)}-\sqrt{J(q')}-(\nabla\sqrt{J})(q)\cdot(q-q')\Big)\sqrt{J(q)} \\
&\quad + \int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}dw\; v_\phi\sigma_l(g,\theta)(\nabla\sqrt{J})(q)\cdot(q-q')\sqrt{J(q)} \\
&\stackrel{\mathrm{def}}{=} I_1+I_2+I_3.
\end{aligned}
$$

For the part I_1, we use mean-value theorem to write

$$
\sqrt{J(q)}-\sqrt{J(q')} = (q-q')\cdot(\nabla\sqrt{J})(\theta q+(1-\theta)q')
$$

for some $\theta \in (0,1)$. Note that

$$|(\nabla\sqrt{J})(\theta q + (1-\theta)q')| \lesssim \Big(J(q)J(q')\Big)^{\frac{\epsilon}{2}} \leq J(q)^{\frac{\epsilon}{2}}$$

for some small $\epsilon > 0$. Thus,

$$\begin{aligned}
|I_1| \lesssim & \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta)|q-q'|^2(J(q))^\epsilon \\
\leq & \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta)\bar{g}^2(q^0 q'^0)(J(q))^\epsilon \\
\lesssim & 2^{-2l} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta)(J(q))^{\epsilon'} \\
\lesssim & 2^{l(\gamma-2)}(p^0)^{\frac{a+\gamma}{2}}
\end{aligned}$$

where the last inequality is by the same argument as (4.1.1).

For the part I_2, we use the same argument as the one for part II in (3.3.3) and (3.3.4) to obtain the same result.

For the part I_3, we follow the same argument as the one for part III in (3.3.5) and (3.3.6) without having the functions f, η, and h. Therefore, if $l \geq 0$, we finally have

$$|\zeta_K^l(p)| \lesssim 2^{l(\gamma-2)}(p^0)^{\frac{a+\gamma}{2}}.$$

Consequently, we sum up the decomposed pieces over l and obtain

$$|\zeta_K(p)| \lesssim (p^0)^{\frac{a+\gamma}{2}}.$$

$\square$

We similarly show the upper bound estimate for $\zeta(p)$ as the following.

Proof. (Proof for Proposition 4.1.2) As in the previous proof for ζ_K, we decompose the function dyadically around the singularity. We define

$$\zeta^l(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})^2.$$

If $l \geq 0$, this is bounded above by $2^{l(\gamma-2)}(p^0)^{\frac{a+\gamma}{2}}$ by the same argument as the one for part I_1 of $\zeta_K^l(p)$ estimate above.

If $l < 0$, we observe that $(\sqrt{J(q)} - \sqrt{J(q')})^2 \lesssim \max\{J(q), J(q')\}$. If $J(q) \geq J(q')$, it suffices to estimate the following:

$$\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta) J(q).$$

This estimate is already done in (4.1.1) and we obtain that $\zeta^l(p) \lesssim 2^{l\gamma}(p^0)^{\frac{a+\gamma}{2}}$. If $J(q) \leq J(q')$, it suffices to estimate the following:

$$\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw\ v_\phi \sigma_l(g,\theta) J(q').$$

This is equal to (4.1.2) and we obtain that $\zeta^l(p) \lesssim 2^{l\gamma}(p^0)^{\frac{a+\gamma}{2}}$.

Finally, we sum up the decomposed pieces over l and obtain

$$\zeta(p) \lesssim (p^0)^{\frac{a+\gamma}{2}}.$$

$\square$

This completes our proofs for the upper bound estimates.

4.2 Lower bound coercivity estimate

In this section, we would like to obtain the following coercivity estimate on ζ:

Proposition 4.2.1.

$$\zeta(p) \gtrsim (p^0)^{\frac{a+\gamma}{2}}.$$

Together with Proposition 4.1.2, we obtain the following equivalence:

Corollary 4.2.2.

$$\zeta(p) \approx (p^0)^{\frac{a+\gamma}{2}}.$$

Proof. (Proof for Proposition 4.2.1) $\zeta(p)$ is defined as

$$\zeta(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} dw \ v_\phi \sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})^2.$$

We write this as an integral on the set $\mathbb{R}^3 \times E^{q'}_{p+q}$ where the set $E^{q'}_{p+q}$ is defined as

$$E^{q'}_{p+q} \stackrel{\text{def}}{=} \{q' \in \mathbb{R}^3 | (q'^\mu - p^\mu)(p_\mu + q_\mu) = 0\}.$$

Then we use a dual representation to write the integral as

$$\zeta(p) = \frac{c}{2p^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{E^{q'}_{p+q}} \frac{d\pi_{q'}}{q'^0} \sqrt{s}\sigma(g,\theta)(\sqrt{J(q)} - \sqrt{J(q')})^2$$

where $d\pi_{q'}$ is the Lebesgue measure on the set $E^{q'}_{p+q}$ and is defined as

$$d\pi_{q'} \stackrel{\text{def}}{=} dq' \delta\left(\frac{s + 2q'^\mu(p_\mu + q_\mu)}{2\sqrt{s}}\right).$$

We first observe that $\sigma(g,\theta) \gtrsim \frac{g}{\sqrt{s}} g^a \sigma_0(\theta) \approx \frac{g^{a+1}}{\sqrt{s}} \left(\frac{g}{\bar{g}}\right)^{2+\gamma}$. Also, there is some $\theta \in (0,1)$ such that

$$\begin{aligned}(\sqrt{J(q)} - \sqrt{J(q')})^2 &= \left(\frac{q^0}{2} - \frac{q'^0}{2}\right)^2 \exp(-(\theta q^0 + (1-\theta)q'^0)) \\ &\geq \frac{1}{4}(q^0 - q'^0)^2 e^{-q^0} e^{-q'^0}\end{aligned}$$

by the mean-value theorem. Therefore, we have

$$\zeta(p) \gtrsim \frac{1}{p^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}e^{-q^0}\int_{E^{q'}_{p+q}}\frac{d\pi_{q'}}{q'^0}\,\frac{g^{a+\gamma+3}}{\bar{g}^{2+\gamma}}(q^0-q'^0)^2e^{-q'^0}.$$

By writing the integral with respect to dq' as the integral with respect to dq'^μ by having extra delta function, the last lower bound is equal to

$$\begin{aligned}&\frac{1}{p^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}e^{-q^0}\int_{\mathbb{R}^4}dq'^\mu u(q'^0)\delta(q'^\mu q'_\mu+1)\delta\left(\frac{s+2q'^\mu(p_\mu+q_\mu)}{2\sqrt{s}}\right)\\ &\qquad\times\frac{g^{a+\gamma+3}}{\bar{g}^{2+\gamma}}(q^0-q'^0)^2e^{-q'^0}.\end{aligned}$$

We use $\sqrt{s}\geq g$ and $\bar{g}^{\gamma+2}\lesssim (q^0q'^0)^{\frac{\gamma}{2}+1}$ to decrease the last lower bound further as

$$\begin{aligned}&\frac{1}{p^0}\int_{\mathbb{R}^3}\frac{dq}{(q^0)^{\frac{\gamma}{2}+2}}e^{-q^0}g^{a+\gamma+4}\int_{\mathbb{R}^4}dq'^\mu u(q'^0)\delta(q'^\mu q'_\mu+1)\delta\left(s+2q'^\mu(p_\mu+q_\mu)\right)\\ &\qquad\times\frac{(q^0-q'^0)^2}{(q'^0)^{\frac{\gamma}{2}+1}}e^{-q'^0}.\end{aligned}$$

Here we write dq' integral in $\int_{\mathbb{R}^4}dq'^\mu u(q'^0)$ using polar coordinates ($q'\in\mathbb{R}^3\to(r,\theta,\phi)$) as

$$\int_1^\infty d(q'^0)\int_0^\infty dr\int_0^{2\pi}d\theta\int_0^\pi d\phi\, r^2\sin\phi$$

and choose the z-axis parallel to $p+q$ such that the angle between q' and $p+q$ is equal to ϕ. Then we follow a similar estimate as in (3.5.2) and (3.5.3), and use $\int_1^\infty d(q'^0)e^{-Cq'^0}(q^0-q'^0)^2\approx (q^0)^2$ for some $C>0$ to obtain that

$$\zeta(p)\gtrsim\frac{1}{p^0}\int_{\mathbb{R}^3}\frac{dq}{(q^0)^{\frac{\gamma}{2}+2}}e^{-q^0}g^{a+\gamma+4}\frac{(q^0)^2}{|p+q|}.$$

We observe that $a+\gamma+4>0$ and use $g\geq\frac{|p-q|}{\sqrt{p^0q^0}}$ to obtain that

$$\zeta(p)\gtrsim\frac{1}{p^0}\frac{1}{(p^0)^{\frac{a+\gamma}{2}+2}}\int_{\mathbb{R}^3}dq\,e^{-q^0}|p-q|^{a+\gamma+4}\frac{1}{|p+q|(g^0)^{\frac{a}{2}+\gamma+1}}.$$

Here we note that $|p+q| \leq 2\max\{p^0, q^0\} \lesssim p^0 q^0$. Then we write $\frac{e^{-q^0}}{(g^0)^{\frac{a}{2}+\gamma+1}} \gtrsim e^{-(1+\epsilon)q^0}$ for some small $\epsilon > 0$ to obtain

$$\begin{aligned}\zeta(p) &\gtrsim \frac{1}{(p^0)^{\frac{a+\gamma}{2}+4}} \int_{\mathbb{R}^3} dq\, e^{-(1+\epsilon)q^0} |p-q|^{a+\gamma+4} \\ &\approx \frac{1}{(p^0)^{\frac{a+\gamma}{2}+4}} (p^0)^{a+\gamma+4} = (p^0)^{\frac{a+\gamma}{2}}.\end{aligned}$$

This completes the proof for the lower bound coercive estimate for $\zeta(p)$. $\square$

Chapter 5

On the Derivative of the Relativistic Collision Map

5.1 Introduction

We consider a pair of relativistic particles with momenta p and q that collide and diverge with post-collisional momenta p' and q'. Using the Center-of-Momentum expression, we can represent the post-collisional variables p' and q' as (1.2.6) and (1.2.7). In this chapter, we are interested in the Jacobian of the collision map $(p, q) \to (u, q)$ where u is defined as $u \stackrel{\text{def}}{=} \theta p' + (1-\theta)p$ for some $\theta \in (0, 1)$. The Jacobian will be computed explicitly and it will be shown that the Jacobian is bounded above in the variable p and q.

In this chapter, our goal is to compute

$$\det\left(\frac{\partial u}{\partial p}\right)$$

where $u \stackrel{\text{def}}{=} \theta p' + (1-\theta)p$ for some $\theta \in (0,1)$. Recall that the post-collisional momentum in the center-of-momentum expression is defined as

$$p' \stackrel{\text{def}}{=} \frac{p+q}{2} + \frac{g}{2}\left(w + (\gamma-1)(p+q)\frac{(p+q)\cdot w}{|p+q|^2}\right),$$

where

$$(\gamma-1) \stackrel{\text{def}}{=} \frac{p_0+q^0-\sqrt{s}}{\sqrt{s}} = \frac{|p+q|^2}{\sqrt{s}(p_0+q^0+\sqrt{s})}.$$

Notice that $(\gamma-1) \geq 0$ from the construction.

5.2 The Jacobian of the collision map

We first state our main results:

Proposition 5.2.1. *The Jacobian determinant* $\det\left(\frac{\partial u}{\partial p}\right)$ *is equal to*

$$\det\left(\frac{\partial u}{\partial p}\right) = A^3 + P_2A^2 + P_3 \tag{5.2.1}$$

where $A \in (1-\theta, 1)$ is defined as

$$A \stackrel{\text{def}}{=} (1-\frac{\theta}{2}) + \frac{\theta}{2}\left(g\frac{(\gamma-1)(p+q)\cdot w}{|p+q|^2}\right),$$

and P_2 and P_3 are defined as in (5.2.2) and (5.2.3) and satisfy that

$$|P_2| \lesssim q^0\left(1+\frac{1}{g}\right) \quad \textit{and} \quad |P_3| \lesssim 1.$$

Since $A \in (1-\theta, 1)$, we obtain the following corollary:

Corollary 5.2.2. *The Jacobian determinant* $\det\left(\frac{\partial u}{\partial p}\right)$ *is bounded above as*

$$\left|\det\left(\frac{\partial u}{\partial p}\right)\right| \lesssim q^0\left(1+\frac{1}{g}\right).$$

A similar work on the relativistic Jacobian has been done by Glassey and Strauss [27] in 1993 with Glassey-Strauss coordinates without using the center-of-momentum system. More precisely, they showed that

$$\sum_{i,j}\int_{\mathbb{S}^2}\left\{\left|\frac{\partial p_i'}{\partial q_j}\right| + \left|\frac{\partial q_i'}{\partial q_j}\right|\right\} dw \lesssim (p^0)^5.$$

Proof. (Proof for Proposition 5.2.1) Now we compute the derivative

$$\frac{\partial u_i}{\partial p_j} = (1-\theta)\delta_{ij} + \theta\frac{\partial p_i'}{\partial p_j}$$

for any choices of $i, j \in \{1,2,3\}$. We observe that

$$\begin{aligned}\frac{\partial p_i'}{\partial p_j} = \frac{1}{2}\Bigg(&\delta_{ij} + \frac{\partial g}{\partial p_j}w_i + \frac{\partial g}{\partial p_j}(\gamma-1)(p_i+q_i)\frac{(p+q)\cdot w}{|p+q|^2} \\ &+ g\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})}\delta_{ij} + g(p_i+q_i)\frac{\partial}{\partial p_j}\left(\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})}\right)\Bigg)\end{aligned}$$

Here,

$$\begin{aligned}\frac{\partial g}{\partial p_j} &= \frac{\partial}{\partial p_j}(\sqrt{-(p^0-q^0)^2+|p-q|^2}) \\ &= \frac{1}{2g}\frac{\partial}{\partial p_j}(-(p^0-q^0)^2+|p-q|^2) \\ &= \frac{1}{2g}(-2(p^0-q^0)\frac{\partial p^0}{\partial p_j} + 2|p-q|\frac{\partial|p-q|}{\partial p_j}) \\ &= \frac{1}{g}(-(p^0-q^0)\frac{p_j}{p^0} + (p_j-q_j)) \\ &= \frac{1}{g}(\frac{q^0}{p^0}p_j - q_j).\end{aligned}$$

Also, we have

$$
\begin{aligned}
&\frac{\partial}{\partial p_j}\left(\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})}\right)\\
&=\frac{w_j(\sqrt{s}(p^0+q^0+\sqrt{s}))-(p+q)\cdot w\frac{\partial}{\partial p_j}(\sqrt{s}(p^0+q^0+\sqrt{s}))}{s(p^0+q^0+\sqrt{s})^2}
\end{aligned}
$$

Note that we have

$$
\begin{aligned}
\frac{\partial\sqrt{s}}{\partial p_j}&=\frac{\partial}{\partial p_j}(\sqrt{(p^0+q^0)^2-|p+q|^2})\\
&=\frac{1}{2\sqrt{s}}\frac{\partial}{\partial p_j}((p^0+q^0)^2-|p+q|^2)\\
&=\frac{1}{2\sqrt{s}}(2(p^0+q^0)\frac{\partial p^0}{\partial p_j}-2|p+q|\frac{\partial|p+q|}{\partial p_j})\\
&=\frac{1}{\sqrt{s}}((p^0+q^0)\frac{p_j}{p^0}-(p_j+q_j))\\
&=\frac{1}{\sqrt{s}}(\frac{q^0}{p^0}p_j-q_j),.
\end{aligned}
$$

Then we obtain that

$$
\begin{aligned}
\frac{\partial}{\partial p_j}(\sqrt{s}(p^0+q^0+\sqrt{s}))&=\frac{\partial\sqrt{s}}{\partial p_j}(p^0+q^0+\sqrt{s})+\sqrt{s}(\frac{\partial p^0}{\partial p_j}+\frac{\partial\sqrt{s}}{\partial p_j})\\
&=(\frac{q^0}{p^0}p_j-q_j)\frac{p^0+q^0+2\sqrt{s}}{\sqrt{s}}+\frac{\sqrt{s}}{p^0}p_j.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial p_i'}{\partial p_j}&=\left(\frac{1}{2}+\frac{1}{2}g\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})}\right)\delta_{ij}\\
&\quad+\frac{1}{2g}(\frac{q^0}{p^0}p_j-q_j)\left(w_i+(\gamma-1)(p_i+q_i)\frac{(p+q)\cdot w}{|p+q|^2}\right)\\
&\quad+\frac{1}{2}g(p_i+q_i)(\frac{w_i}{\sqrt{s}(p^0+q^0+\sqrt{s})})\\
&\quad-\frac{(p+q)\cdot w}{2s(p^0+q^0+\sqrt{s})^2}g(p_i+q_i)\left(\frac{p^0+q^0+2\sqrt{s}}{\sqrt{s}}(\frac{q^0}{p^0}p_j-q_j)+\frac{\sqrt{s}}{p^0}p_j\right).
\end{aligned}
$$

Putting all these together, we can write:

$$\frac{\partial u_i}{\partial p_j} = A\delta_{ij} + Bp_ip_j + Cq_iq_j + Dp_iq_j + Eq_ip_j + Fp_jw_j + Gq_iw_j + Hw_ip_j + Iw_iq_j,$$

where the scalars are

$$A = (1-\theta) + \frac{\theta}{2}\left(1 + g\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})}\right) = (1-\frac{\theta}{2}) + \frac{\theta}{2}\left(g\frac{(\gamma-1)(p+q)\cdot w}{|p+q|^2}\right),$$

$$B = \frac{(p+q)\cdot w}{2gp_o(p^0+q^0+\sqrt{s})^2 s^{\frac{3}{2}}}(q^0s(p^0+q^0+\sqrt{s}) - g^2q^0(p^0+q^0+2\sqrt{s}) - g^2s)$$

$$C = \frac{(p+q)\cdot w}{2g(p^0+q^0+\sqrt{s})^2 s^{\frac{3}{2}}}(-s(p^0+q^0+\sqrt{s}) + g^2(p^0+q^0+2\sqrt{s}))$$

$$D = C, \quad E = B$$

$$F = \frac{g}{2\sqrt{s}(p^0+q^0+\sqrt{s})}, \quad G = F$$

$$H = \frac{q^0}{2gp^0}, \quad I = -\frac{1}{2g}.$$

Notice that $(1-\theta) < A < 1$ because

$$g\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})} \in (-1,1).$$

We will use this to compute the determinant of the matrix $\mathbf{\Phi} = (\Phi_{ij})$ where $\Phi_{ij} = \frac{\partial u_i}{\partial p_j}$.

We first decompose the pre-collisional vector p as below:

$$p = (p\cdot w)w + w\times(p\times w).$$

Define $\bar{w} \stackrel{\text{def}}{=} \frac{w\times(p\times w)}{|p\times w|}$. Then, $\bar{w}\in\mathbb{S}^2$ and $\bar{w}\perp w$.

Also, define $\tilde{w} \stackrel{\text{def}}{=} \frac{p\times w}{|p\times w|}$. Then, $\tilde{w}\in\mathbb{S}^2$ and $\tilde{w}\perp w$ and $\tilde{w}\perp\bar{w}$. Thus, $\{w,\bar{w},\tilde{w}\}$ is an orthonormal basis for $\mathbb{R}^3$. Then, we can decompose q as below:

$$\begin{aligned}
q &= (q \cdot w)w + w \times (q \times w) \\
&= (q \cdot w)w + ((w \times (q \times w)) \cdot \bar{w})\bar{w} + \bar{w} \times ((w \times (q \times w)) \times \bar{w}) \\
&= (q \cdot w)w + (q \cdot \bar{w})\bar{w} + (q \cdot \tilde{w})\tilde{w} \\
&\stackrel{\text{def}}{=} aw + b\bar{w} + c\tilde{w}.
\end{aligned}$$

Similarly, write:

$$p = (p \cdot w)w + w \times (p \times w) = (p \cdot w)w + |p \times w|\bar{w} \stackrel{\text{def}}{=} dw + e\bar{w}.$$

Notice that $a^2 + b^2 + c^2 = |q|^2$ and $d^2 + e^2 = |p|^2$.

Then, we can rewrite the matrix element Φ_{ij}:

$$\begin{aligned}
\Phi_{ij} = A\delta_{ij} + B'w_iw_j + C'\bar{w}_i\bar{w}_j + D'\tilde{w}_i\tilde{w}_j + E'w_i\bar{w}_j + F'w_i\tilde{w}_j \\
+ G'\bar{w}_iw_j + H'\bar{w}_i\tilde{w}_j + I'\tilde{w}_iw_j + J'\tilde{w}_i\bar{w}_j,
\end{aligned}$$

where

$$\begin{aligned}
B' &= Bd^2 + Ca^2 + Dad + Ead + Fd + Ga + Hd + Ia \\
C' &= Be^2 + Cb^2 + Dce + Ece \\
D' &= Cc^2 \\
E' &= Bde + Cab + Dbd + Eae + He + Ib \\
F' &= Cac + Dcd + Ic
\end{aligned}$$

$$G' = Bde + Cba + Dae + Ebd + Fe + Gb$$

$$H' = Cbc + Dce$$

$$I' = Cac + Ecd + Gc$$

$$J' = Cbc + Ece.$$

Since $\{w, \bar{w}, \tilde{w}\}$ forms a basis for $\mathbb{R}^3$, the determinant of $\mathbf{\Phi}$ is equal to:

$$\det(\mathbf{\Phi}) = \begin{vmatrix} A + B' & E' & F' \\ G' & A + C' & H' \\ I' & J' & A + D' \end{vmatrix}$$

Subtracting (Column 3)$\times\frac{a}{c}$ from (Column 1) and subtracting (Column 3)$\times\frac{b}{c}$ from (Column 2) gives

$$\Phi_{11} = A + Bd^2 + Ead + Fd + Ga + Hd$$

$$\Phi_{21} = Bde + Ebd + Fe + Gb$$

$$\Phi_{31} = Ecd + Gc - \frac{a}{c}A$$

$$\Phi_{12} = Bde + Eae + He$$

$$\Phi_{22} = A + Be^2 + Ebe$$

$$\Phi_{32} = Ece - \frac{b}{c}A$$

There is no change on Column 3 by this column reduction. Notice that this reduction does not change the determinant.

Now, subtracting (Column 2)$\times\frac{d}{e}$ from (Column 1) gives

$$\Phi_{11} = A + Fd + Ga, \;\; \Phi_{21} = -\frac{d}{e}A + Fe + Gb, \;\; \Phi_{31} = (\frac{bd}{ce} - \frac{a}{c})A + Gc.$$

Now, we subtract (Row 3)$\times\frac{a}{c}$ from (Row 1) and (Row 3)$\times\frac{b}{c}$ from (Row 2) respectively. Then, we have the matrix elements to be:

$$\begin{aligned}
\Phi_{11} &= (1 - \frac{abd}{c^2e} + \frac{a^2}{c^2})A + Fd \\
\Phi_{21} &= (\frac{ab}{c^2} - \frac{d}{e} - \frac{b^2d}{c^2e})A + Fe \\
\Phi_{31} &= (\frac{bd}{ce} - \frac{a}{c})A + Gc \\
\Phi_{12} &= \frac{ab}{c^2}A + Bde + He \\
\Phi_{22} &= (1 + \frac{b^2}{c^2})A + Be^2 \\
\Phi_{32} &= -\frac{b}{c}A + Ece \\
\Phi_{13} &= -\frac{a}{c}A + Dcd + Ic \\
\Phi_{23} &= -\frac{b}{c}A + Dce \\
\Phi_{33} &= A + Cc^2.
\end{aligned}$$

We do one more row reduction: (Row 1)-(Row 2)$\times\frac{d}{e}$. This gives

$$\det(\mathbf{\Phi}) = \begin{vmatrix} a_{11}A & a_{12}A + He & a_{13}A + Ic \\ a_{21}A + Fe & a_{22}A + Be^2 & a_{23}A + Dce \\ a_{31}A + Gc & a_{32}A + Ece & a_{33}A + Cc^2 \end{vmatrix}$$

where

$$
\begin{aligned}
a_{11} &= 1 - \frac{abd}{c^2e} + \frac{a^2}{c^2} - \frac{abd}{c^2e} + \frac{d^2}{e^2} + \frac{b^2d^2}{c^2e^2} \\
a_{21} &= \frac{ab}{c^2} - \frac{d}{e} - \frac{b^2d}{c^2e} \\
a_{31} &= \frac{bd}{ce} - \frac{a}{c} \\
a_{12} &= \frac{ab}{c^2} - \frac{d}{e} - \frac{b^2d}{c^2e} \\
a_{22} &= 1 + \frac{b^2}{c^2} \\
a_{32} &= -\frac{b}{c} \\
a_{13} &= \frac{bd}{ce} - \frac{a}{c} \\
a_{23} &= -\frac{b}{c} \\
a_{33} &= 1.
\end{aligned}
$$

Since $B = E, C = D$,and $G = F$, we can do one more row reduction: (Row 2)-(Row 3)$\times\frac{e}{c}$. This gives

$$
\begin{aligned}
\Phi_{21} &= (\frac{ab}{c^2} - \frac{d}{e} - \frac{b^2d}{c^2e} - \frac{bd}{c^2} + \frac{ae}{c^2})A \stackrel{\text{def}}{=} a'_{21}A \\
\Phi_{22} &= (1 + \frac{b^2}{c^2} + \frac{be}{c^2})A \stackrel{\text{def}}{=} a'_{22}A \\
\Phi_{23} &= (-\frac{b}{c} - \frac{e}{c})A \stackrel{\text{def}}{=} a'_{23}A.
\end{aligned}
$$

Finally, we have

$$
\det(\mathbf{\Phi}) = \begin{vmatrix} a_{11}A & a_{12}A + He & a_{13}A + Ic \\ a'_{21}A & a'_{22}A & a'_{23}A \\ a_{31}A + Gc & a_{32}A + Ece & a_{33}A + Cc^2 \end{vmatrix}
$$

where, with $L \stackrel{\text{def}}{=} ae - bd$,

$$a_{11} = \frac{c^2|p|^2 + L^2}{c^2e^2}, \quad a'_{21} = \frac{L(b+e) - c^2d}{c^2e}, \quad a_{31} = -\frac{L}{ce},$$
$$a_{12} = \frac{bL - c^2d}{c^2e}, \quad a'_{22} = \frac{b^2 + c^2 + be}{c^2}, \quad a_{32} = -\frac{b}{c},$$
$$a_{13} = -\frac{L}{ce}, \quad a'_{23} = -\frac{b+e}{c}, \quad a_{33} = 1.$$

Then the determinant is

$$\begin{aligned}\det(\mathbf{\Phi}) = A\big(&a_{11}A(a'_{22}A + Cc^2a'_{22} - a'_{23}a_{32}A - a'_{23}Ece) \\ &- (a_{12}A + He)(a'_{21}A + a'_{21}Cc^2 - a'_{23}a_{31}A - a'_{23}Gc) \\ &+ (a_{13}A + Ic)(a'_{21}a_{32}A + a'_{21}Ece - a'_{22}a_{31}A - a'_{22}Gc)\big).\end{aligned}$$

Here we further reduce the determinant. First, notice that

$$a'_{22} = \frac{b^2 + c^2 + be}{c^2} = 1 + (-\frac{b}{c})(-\frac{b+e}{c}) = 1 + a_{32}a'_{23}.$$

Thus,

$$a'_{22}A - a'_{23}a_{32}A = A.$$

Also, we have

$$a'_{21} - a'_{23}a_{31} = \frac{L(b+e) - c^2d}{c^2e} - (\frac{-b-e}{c})(\frac{-L}{ce}) = -\frac{d}{e}$$

and

$$a'_{21}a_{32} - a'_{22}a_{31} = \frac{L(b+e) - c^2d}{c^2e} \cdot \frac{-b}{c} - (\frac{b^2 + be}{c^2} + 1) \cdot \frac{-L}{ce} = \frac{a}{c}.$$

Then the determinant is now

$$
\begin{aligned}
\det(\mathbf{\Phi}) = A\big(&a_{11}A(A + Cc^2 a'_{22} - a'_{23}Ece) \\
&- (a_{12}A + He)(-\frac{d}{e}A + a'_{21}Cc^2 - a'_{23}Gc) \\
&+ (a_{13}A + Ic)(\frac{a}{c}A + a'_{21}Ece - a'_{22}Gc)\big) \\
= &(a_{11} + a_{12}\frac{d}{e} + a_{13}\frac{a}{c})A^3 \\
&+ (a_{11}a'_{22}Cc^2 - a_{11}a'_{23}Ece + Hd - a_{12}a'_{21}Cc^2 \\
&\quad + a_{12}a'_{23}Gc + Ia + a_{13}a'_{21}Ece - a_{13}a'_{22}Gc)A^2 \\
&+ (-a'_{21}CHc^2e + a'_{23}GHce + a'_{21}IEc^2e - a'_{22}IGc^2)A \\
\stackrel{\text{def}}{=} &P_1A^3 + P_2A^2 + P_3A.
\end{aligned}
$$

We compute P_1 first.

$$
\begin{aligned}
P_1 &= a_{11} + a_{12}\frac{d}{e} + a_{13}\frac{a}{c} \\
&= \frac{c^2|p|^2 + L^2}{c^2e^2} + \frac{bdL - c^2d^2}{c^2e^2} - \frac{Lae}{c^2e^2} \\
&= \frac{1}{c^2e^2}(c^2e^2 + L^2 + L(bd - ae)) \\
&= \frac{1}{c^2e^2}(c^2e^2 + L^2 - L^2) \\
&= 1.
\end{aligned}
$$

Now let's simplify P_2;

$$
\begin{aligned}
P_2 = &Cc^2(a_{11}a'_{22} - a_{12}a'_{21}) + Ece(a_{13}a'_{21} - a_{11}a'_{23}) \\
&+ Gc(a_{12}a'_{23} - a_{13}a'_{22}) + \frac{1}{2g}(\frac{q^0}{p^0}d - a).
\end{aligned}
$$

Firstly,

$$
\begin{aligned}
&a_{11}a'_{22} - a_{12}a'_{21} \\
&= \frac{1}{c^4e^2}((c^2|p|^2 + L^2)(b^2 + c^2 + be) - (bL - c^2d)(L(e + b) - c^2d)) \\
&= \frac{1}{c^4e^2}(c^4(|p|^2 - d^2) + b^2c^2|p|^2 + bc^2(|p|^2e + 2Ld) + c^2(L^2 + Lde)) \\
&= 1 + \frac{1}{c^2e^2}(b^2|p|^2 + b(|p|^2e + 2Ld) + L^2 + Lde) \\
&= 1 + \frac{1}{c^2e^2}(b^2e^2 + |p|^2be + e^2a^2 + e^2ad - bd^2e) \\
&= \frac{1}{c^2e^2}(|q|^2e^2 + be^3 + e^2ad) \\
&= \frac{1}{c^2}(|q|^2 + be + ad).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&a_{13}a'_{21} - a_{11}a'_{23} \\
&= \frac{1}{b^3e^2}(-L^2(e + b) + Lc^2d + (c^2|p|^2 + L^2)(b + e)) \\
&= \frac{1}{b^3e^2}(Lc^2d + bc^2|p|^2 + c^2|p|^2e) \\
&= \frac{1}{b^3e^2}(c^2dea - bc^2d^2 + bc^2|p|^2 + c^2|p|^2e) \\
&= \frac{1}{be}(ad + be + |p|^2).
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
& a_{12}a'_{23} - a_{13}a'_{22} \\
&= \frac{1}{c^3 e}((bL - c^2 d)(-b-e) + L(b^2 + c^2 + be)) \\
&= \frac{1}{c^3 e}(c^2 d(b+e) + Lc^2) \\
&= \frac{1}{c^3 e}(c^2 db + c^2 de + c^2 ea - c^2 bd) \\
&= \frac{a+d}{c}
\end{aligned}
$$

Thus, we have

$$
P_2 = C(ad + be + |q|^2) + E(ad + be + |p|^2) + G(a+d) + \frac{1}{2g}(\frac{q^0}{p^0}d - a).
$$

We have that

$$
\begin{aligned}
B = E &= \frac{1}{p^0}\frac{(p+q)\cdot w}{2g(p^0 + q^0 + \sqrt{s})^2 s^{\frac{3}{2}}}(4q^0(p^0 + q^0 + \sqrt{s}) - g^2\sqrt{s}(q^0 + \sqrt{s})) \\
&= \frac{(a+d)}{2gp^0}\frac{(\gamma - 1)^2}{|p+q|^4}(4q^0(\gamma + 1) - g^2(q^0 + \sqrt{s}))
\end{aligned}
$$

and

$$
\begin{aligned}
C = D &= \frac{(p+q)\cdot w}{2g(p^0 + q^0 + \sqrt{s})^2 s^{\frac{3}{2}}}(-4(p^0 + q^0 + \sqrt{s}) + g^2\sqrt{s}) \\
&= \frac{(a+d)}{2g}\frac{(\gamma - 1)^2}{|p+q|^4}(-4(\gamma + 1) + g^2)
\end{aligned}
$$

and

$$
G = \frac{g}{2\sqrt{s}(p^0 + q^0 + \sqrt{s})} = \frac{(\gamma - 1)g}{2|p+q|^2}.
$$

Let us reduce P_2.

$$
\begin{aligned}
P_2 =& C(ad+be+|q|^2)+E(ad+be+|p|^2)+G(a+d)+\frac{1}{2g}(\frac{q^0}{p^0}d-a)\\
=& \frac{(\gamma-1)^2(p+q)\cdot w}{2gp^0|p+q|^4}\Big((|q|^2+ad+be)(p^0g^2-4p^0(\gamma+1))\\
&\qquad + (|p|^2+ad+be)(4q^0(\gamma+1)-g^2(q^0+\sqrt{s}))\\
&\qquad + \frac{|p+q|^2}{(\gamma-1)}p^0g^2+\frac{|p+q|^4}{(\gamma-1)^2}\frac{(q^0d-p^0a)}{a+d}\Big)\\
=& \frac{(\gamma-1)^2(p+q)\cdot w}{2gp^0|p+q|^4}\Big((ad+be)(4(q^0-p^0)(\gamma+1)+g^2(p^0-q^0)-g^2\sqrt{s})\\
&\qquad + (4\gamma+4-g^2)(|p|^2q^0-|q|^2p^0)-g^2|p|^2\sqrt{s}\\
&\qquad + \frac{|p+q|^2}{(\gamma-1)}p^0g^2+\frac{|p+q|^4}{(\gamma-1)^2}\frac{(q^0d-p^0a)}{a+d}\Big)\\
=& \frac{(\gamma-1)^2(p+q)\cdot w}{2gp^0|p+q|^4}\Big((ad+be)((q^0-p^0)(4\gamma+4-g^2)-g^2\sqrt{s})\\
&\qquad + (4\gamma+4-g^2)(p^0-q^0)(1+p^0q^0)-g^2|p|^2\sqrt{s}\\
&\qquad + \frac{|p+q|^2}{(\gamma-1)}p^0g^2+\frac{|p+q|^4}{(\gamma-1)^2}\frac{(q^0d-p^0a)}{a+d}\Big)\\
=& \frac{(\gamma-1)^2(p+q)\cdot w}{2gp^0|p+q|^4}\Big((1+p^0q^0-ad-be)(p^0-q^0)(4\gamma+4-g^2)\\
&\qquad - (ad+be+|p|^2)g^2\sqrt{s}\\
&\qquad + \frac{|p+q|^2}{(\gamma-1)}p^0g^2+\frac{|p+q|^4}{(\gamma-1)^2}\frac{(q^0d-p^0a)}{a+d}\Big).
\end{aligned}
$$

Thus, we obtain that

$$
\begin{aligned}
P_2 =& \frac{(\gamma-1)^2(p+q)\cdot w}{2gp^0|p+q|^4}\Big((1+p^0q^0-ad-be)(p^0-q^0)(4\gamma+4-g^2) \\
& \qquad\qquad - (ad+be+|p|^2)g^2\sqrt{s} \\
& \qquad\qquad + \frac{|p+q|^2}{(\gamma-1)}p^0g^2 + \frac{|p+q|^4}{(\gamma-1)^2}\frac{(q^0d-p^0a)}{a+d}\Big).
\end{aligned}
\tag{5.2.2}
$$

Here, we note that $|a|,|b|,|c| \lesssim q^0$ and $|d|,|e| \lesssim p^0$. Since we also have $g \leq \sqrt{s} \lesssim \sqrt{p^0q^0}$, we can conclude that $|P_2| \lesssim q^0\left(1+\frac{1}{g}\right)$.

We now simplify P_3.

$$
\begin{aligned}
P_3 &= -a'_{21}CHc^2e + a'_{23}GHce + a'_{21}IEc^2e - a'_{22}IGc^2 \\
&= a'_{21}c^2e(IE-CH) + G(a'_{23}Hce - a'_{22}Ic^2) \\
&= (abe - bde + ae^2 - c^2d - b^2d)(IE-CH) - G((b+e)He + I(b^2+c^2+be))
\end{aligned}
$$

Notice that

$$
\begin{aligned}
IE - CH &= -\frac{1}{2g}E - \frac{q^0}{2gp^0}C \\
&= -\frac{1}{2g}(-C\frac{q^0}{p^0} - \frac{(a+d)g\sqrt{s}}{2p^0}\frac{(\gamma-1)^2}{|p+q|^4} + C\frac{q^0}{p^0}) \\
&= \frac{(a+d)(\gamma-1)^2\sqrt{s}}{4p^0|p+q|^4}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
P_3 =& \frac{(a+d)(\gamma-1)^2\sqrt{s}}{4p^0|p+q|^4}(abe - bde + ae^2 - c^2d - b^2d) \\
&+ \frac{(\gamma-1)}{4p^0|p+q|^2}(q^0(-be-e^2) + p^0(b^2+c^2+be)) \\
=& \frac{(\gamma-1)}{4p^0|p+q|^2}\Big(\frac{(a+d)(abe - bde + ae^2 - c^2d - b^2d)}{p^0+q^0+\sqrt{s}} \\
&+ be(p_q - q^0) + p^0(|q|^2 - a^2) - q^0(|p|^2 - d^2)\Big) \\
=& \frac{(\gamma-1)}{4p^0|p+q|^2}\frac{1}{p^0+q^0+\sqrt{s}}((e^2a - bde + abe - b^2d - c^2d)(a+d) \\
&+ (p^0+q^0+\sqrt{s})(p^0(be + |q|^2 - a^2) + q^0(-be - |p|^2 + d^2))) \\
=& \frac{(\gamma-1)}{4p^0|p+q|^2}\frac{1}{p^0+q^0+\sqrt{s}}(I_1 + I_2)
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
I_1 &= (e^2a - bde + abe - b^2d - c^2d)(a+d) \\
&= (a^2-d^2)(ad+be) + (a^2-d^2)((p^0)^2 - 1) + (ad+d^2)((p^0)^2 - (q^0)^2) \\
&= (a^2-d^2)(ad+be-1) + (p^0)^2(a^2+ad) - q^2(ad+d^2)
\end{aligned}
$$

and

$$
I_2 = (p^0+q^0+\sqrt{s})(p^0(be+|q|^2-a^2) + q^0(-be-|p|^2+d^2)) \stackrel{\text{def}}{=} (p^0+q^0+\sqrt{s})I_3.
$$

Then,

$$\begin{aligned}
I_1 + I_2 =& \sqrt{s} I_3 + (ad+be)(a^2 - d^2 + (p^0)^2 - (q^0)^2) - (a^2 - d^2) \\
& + (p^0)^2|q|^2 - (q^0)^2|p|^2 + p^0 q^0(-|p|^2 + d^2 + |q|^2 - a^2) \\
=& \sqrt{s} I_3 + (ad + be - p^0 q^0)(a^2 - d^2 + (p^0)^2 - (q^0)^2) \\
& - (a^2 - d^2) + (p^0)^2|q|^2 - (q^0)^2|p|^2 \\
=& \sqrt{s} I_3 + (1 + p^0 q^0 - ad - be)(d^2 - a^2 - (p^0)^2 + (q^0)^2) \\
=& \sqrt{s} I_3 + (1 + p^0 q^0 - ad - be)(-e^2 + b^2 + c^2).
\end{aligned}$$

Therefore, we get

$$P_3 = \frac{(\gamma-1)^2}{4p^0\sqrt{s}|p+q|^4}(\sqrt{s}((be+e^2)(p^0-q^0)) + (1+p^0q^0-ad-be)(\sqrt{s}p^0 - e^2 + b^2 + c^2)). \tag{5.2.3}$$

By Cauchy-Schwarz inequality, it can be shown that

$$(1 + p^0 q^0 - ad - be) > 1.$$

$\square$

Remark that P_2 and P_3 are not independent of A because both still contain the term $\frac{g(a+d)(\gamma-1)}{|p+q|^2}$ which is equal to $\frac{2}{\theta}(A - 1 + \frac{\theta}{2})$. Let $K \stackrel{\text{def}}{=} \frac{g(a+d)(\gamma-1)}{|p+q|^2}$. Since

$$K = \frac{g(a+d)(\gamma-1)}{|p+q|^2} = \frac{((p+q)\cdot w)g}{(p^0+q^0+\sqrt{s})\sqrt{s}},$$

$|K|$ is bounded by 1. Then we can write the Jacobian as a cubic polynomial in K as in the following proposition:

Proposition 5.2.3.

$$\det\left(\frac{\partial u}{\partial p}\right) = D_1K^3 + D_2K^2 + D_3K + D_4$$

where $|K| \leq 1$, $|D_1| \lesssim q^0$ *and* $|D_2|, |D_3|, |D_4| \lesssim \frac{q^0}{g}$.

Thus, we obtain the following corollary:

Corollary 5.2.4.

$$\left|\det\left(\frac{\partial u}{\partial p}\right)\right| \lesssim q^0\left(1 + \frac{1}{g}\right).$$

Proof. (Proof for Proposition 5.2.3) Let's rewrite the constant coefficients A through I and the coefficients P_2 and P_3 in terms of K.

$$\begin{aligned}
A &= (1 - \frac{\theta}{2}) + \frac{\theta}{2}K \\
B = E &= K \cdot \frac{1}{2p^0g^2s}(4q^0 - \frac{g^2s(\sqrt{s}+q^0)(\gamma-1)}{|p+q|^2}) \\
C = D &= K \cdot \frac{1}{2g^2s}(-4 + \frac{g^2s(\gamma-1)}{|p+q|^2}) \\
F = G &= \frac{g(\gamma-1)}{2|p+q|^2} \\
H &= \frac{q^0}{2p^0g} \\
I &= \frac{-1}{2g}
\end{aligned}$$

Now, express P_2 in terms of K.

$$
\begin{aligned}
P_2 =& C(ad+be+|q|^2)+E(ad+be+|p|^2)+G(a+d)+\frac{1}{2g}(\frac{q^0}{p^0}d-a)\\
=& K\Big(\frac{(\gamma-1)}{2|p+q|^2p^0g^2}\{(1+p^0q^0-ad-be)(p^0-q^0)(4\gamma+4-g^2)\\
&-(ad+be+|p|^2)g^2\sqrt{s}\}+\frac{1}{2}\Big)+\frac{1}{2g}(\frac{q^0}{p^0}d-a)\\
=&: P_{21}K+P_{22}
\end{aligned}
$$

For P_3, we have

$$
\begin{aligned}
P_3 =& \frac{(a+d)(\gamma-1)^2\sqrt{s}}{4p^0|p+q|^4}(abe-bde+ae^2-c^2d-b^2d)\\
&+\frac{(\gamma-1)}{4p^0|p+q|^2}(q^0(-be-e^2)+p^0(b^2+c^2+be))\\
=& \frac{(\gamma-1)\sqrt{s}}{4p^0|p+q|^2g}\Big(K(abe-bde-ae^2-c^2d-b^2d)\\
&+\frac{g}{\sqrt{s}}\{(q^0(-be-e^2)+p^0(b^2+c^2+be)\}\Big)\\
=& \frac{(abe-bde-ae^2-c^2d-b^2d)}{4p^0(p^0+q^0+\sqrt{s})g}K\\
&+\frac{((q^0(-be-e^2)+p^0(b^2+c^2+be))}{4p^0(p^0+q^0+\sqrt{s})\sqrt{s}}\\
\overset{\text{def}}{=}& P_{31}K+P_{32}
\end{aligned}
$$

Now, the determinant is

$$\begin{aligned}
\det(\mathbf{\Phi}) =& A^3 + A^2 P_2 + A P_3 \\
=& \Big((1-\frac{\theta}{2}) + \frac{\theta}{2} K \Big)^3 + \Big((1-\frac{\theta}{2}) + \frac{\theta}{2} K \Big)^2 (P_{21} K + P_{22}) \\
& \qquad + \Big((1-\frac{\theta}{2}) + \frac{\theta}{2} K \Big)(P_{31} K + P_{32}) \\
=& \Big((\frac{\theta}{2})^3 + P_{21}(\frac{\theta}{2}) \Big) K^3 + \Big(\frac{\theta}{2}(1-\frac{\theta}{2})(\frac{3\theta}{2} + 2P_{21}) + \frac{\theta}{2}(P_{22}(\frac{\theta}{2}) + P_{31}) \Big) K^2 \\
& + \Big(3(1-\frac{\theta}{2})^2 \frac{\theta}{2} + (1-\frac{\theta}{2})^2 P_{21} + \theta(1-\frac{\theta}{2}) P_{22} + (1-\frac{\theta}{2}) P_{31} + \frac{\theta}{2} P_{32} \Big) K \\
& + \Big((1-\frac{\theta}{2})^3 + P_{22}(1-\frac{\theta}{2})^2 + P_{32}(1-\frac{\theta}{2}) \Big) \\
=& : D_1 K^3 + D_2 K^2 + D_3 K + D_4.
\end{aligned}$$

Since we have $|P_{21}| \lesssim q^0$ and $|P_{22}|, |P_{31}|, |P_{32}| \lesssim \frac{q^0}{g}$, we obtain that $|D_1| \lesssim q^0$ and $|D_2|, |D_3|, |D_4| \lesssim \frac{q^0}{g}$ and this completes the proof. $\square$

Chapter 6

Appendix

6.1 On the relativistic collsional scattering angle

Consider the *center-of-momentum* expression for the collision operator. Under the expression, note that

$$
\begin{aligned}
p' - q' &= gw + g(\gamma - 1)(p+q)\frac{(p+q)\cdot w}{|p+q|^2} \\
&= gw + \sqrt{s}(p'^0 - q'^0)(\gamma - 1)(p+q)\frac{1}{|p+q|^2}
\end{aligned}
$$

Thus, w can be represented as

$$
\begin{aligned}
w &= \frac{1}{g}(p' - q' - \sqrt{s}(p'^0 - q'^0)(\gamma - 1)(p+q)\frac{1}{|p+q|^2}) \\
&= \frac{1}{g}(p' - q' - (p'^0 - q'^0)\frac{p'+q'}{p^0+q^0+\sqrt{s}}) \\
&= \frac{(p'-q')(p'^0+q'^0+\sqrt{s}) - (p'^0-q'^0)(p'+q')}{g(p'^0+q'^0+\sqrt{s})} \\
&= \frac{(\sqrt{s}+2q'^0)p' - (\sqrt{s}+2p'^0)q'}{g(p^0+q^0+\sqrt{s})}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\cos\theta &= \frac{(p^\mu - q^\mu)(p'_\mu - q'_\mu)}{g^2} \\
&= \frac{-(p^0-q^0)(p'^0-q'^0) + (p-q)\cdot(p'-q')}{g^2} \\
&= \frac{1}{g^2}\Big(-(p^0-q^0)(\frac{g}{\sqrt{s}}w\cdot(p+q)) + (p-q)\cdot(gw + g(\gamma-1)(p+q)\frac{(p+q)\cdot w}{|p+q|^2})\Big) \\
&= \frac{1}{g^2}\Big(-(p^0-q^0)(\frac{g}{\sqrt{s}}w\cdot(p+q)) + g(p-q)\cdot w + g({p^0}^2 - {q^0}^2)\frac{(p+q)\cdot w}{\sqrt{s}(p^0+q^0+\sqrt{s})}\Big) \\
&= \frac{1}{g\sqrt{s}(p^0+q^0+\sqrt{s})}\Big(-(p^0-q^0)(p^0+q^0+\sqrt{s})w\cdot(p+q) \\
&\quad + \sqrt{s}(p-q)\cdot w(p^0+q^0+\sqrt{s}) + w\cdot(p+q)({p^0}^2-{q^0}^2)\Big) \\
&= \frac{-(p^0-q^0)w\cdot(p+q) + (p-q)\cdot w(p^0+q^0+\sqrt{s})}{g(p^0+q^0+\sqrt{s})} \\
&= \frac{(\sqrt{s}+2q^0)p - (\sqrt{s}+2p^0)q}{g(p^0+q^0+\sqrt{s})}\cdot w \\
&= k\cdot w.
\end{aligned}
$$

Note that $|k| = 1$. This expression on $\cos\theta$ gives us the intuition on the relationship between $\cos\theta$ expressed as the Lorentzian inner product of 4-vectors and that expressed as the usual Euclidean inner product of 3-vectors. Thus, we can see that even in the relativistic collisional kinetics, the geometry can be expressed by using the usual 3-vectors and the usual Euclidean inner product with the above translation.

6.2 Grad angular cut-off assumptions on the relativistic scattering kernel

In many papers, we suppose that the collision scattering kernel $\sigma(g,w)$ takes the form of a product in the relative momentum and the scattering angle as:

$$\sigma(g,w) = \Phi(g)\sigma_0(cos\theta). \tag{6.2.1}$$

Grad [28] announced a cut-off condition for the angular function $\sigma_0(cos\theta)$ which requires that the function $\sigma_0(cos\theta)$ is bounded. Afterwards, people started considering more lenient conditions and the $L^1(\mathbb{S}^2)$ bound has become popular which states

$$\int_{\mathbb{S}^2} \sigma_0(cos\theta)dw < \infty. \tag{6.2.2}$$

We remark that our angular kernel defined as in (3.1.11) is not integrable and does not assume any cut-off.

6.3 Assumptions on the relativistic scattering kernels from physics literature

In this section, we would like to introduce some different assumptions on the relativistic scattering kernels $\sigma(g,\theta)$. As we have seen in the previous section, Grad's angular cut-off conditions play important roles for the classical theory. Similarly, they are also important in the relativistic kinetic theory and we will see some scattering kernels from the physics literature.

6.3.1 Short Range Interactions [22, 43]

This is an analogue of the hard-sphere case in the Newtonian case. In this case, for short range interactions,

$$\sigma := \text{constant}$$

or

$$s\sigma := \text{constant}.$$

Then, we can check that this satisfies Grad's angular cut-off condition because it is bounded.

6.3.2 Mϕller Scattering [14]

This is an approximation of electron-electron scattering.

$$\sigma(g,\theta) = r_0^2 \frac{1}{u^2(u^2-1)^2} \frac{(2u^2-1)^2}{sin^4\theta} - \frac{2u^4-u^2-1/4}{sin^2\theta} + \frac{1}{4}(u^2-1)^2,$$

where $u = \frac{\sqrt{s}}{2mc}$ and $r_0 = \frac{e^2}{4\pi mc^2}$.

6.3.3 Compton Scattering [14]

This is an approximation of photon-electron scattering.

$$\sigma(g,\theta) = \frac{1}{2} r_0^2 (1-\xi) 1 + \frac{1}{4} \frac{\xi^2 (1-cos\theta)^2}{1-\frac{1}{2}\xi(1-cos\theta)} + (\frac{1-(1-\xi/2)(1-cos\theta)}{1-\frac{1}{2}\xi(1-cos\theta)})^2,$$

where $\xi = 1 - \frac{m^2c^2}{s}$.

6.3.4 Neutrino Gas [15]

In this case, the differential cross section does not depend on θ:

$$\sigma(g,\theta) = \frac{G^2}{\pi \hbar^2 c^2} g^2,$$

where G is the weak coupling constant and $\hbar$ is Planck's constant. Similarly, the angular function is bounded so it satisfies the Grad cut-off condition.

6.3.5 Israel Particles [35]

This is the analogue of the Maxwell molecules cross section in the Newtonian theory:

$$\sigma = \frac{m}{2g} \frac{b(\theta)}{1+(g/mc)^2}.$$

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