# PADS: An approach to modeling resource demand and supply for the formal analysis of hierarchical scheduling

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**Abstract**

As real-time embedded systems become more complex, resource partitioning is increasingly used to guarantee real-time performance. Recently, several compositional frameworks of resource partitioning have been proposed using real-time scheduling theory with various notions of real-time tasks running under restricted resource supply environments. However, these real-time scheduling-based approaches are limited in their expressiveness in that, although capable of describing resource-demand tasks, they are unable to model resource supply. This paper describes a process algebraic framework PADS for reasoning about resource demand and resource supply inspired by the timed process algebra ACSR. In ACSR, real-time tasks are specified by enunciating their consumption needs for resources. To also accommodate resource-supply processes in PADS, given a resource $cpu$, we write $\overline{cpu}$ to denote the availability of $cpu$ for a requesting task process. Using PADS, we define a supply-demand relation where a pair $(T, S)$ belongs to the relation if the demand process $T$ can be scheduled under supply $S$. We develop a theory of compositional schedulability analysis as well as a technique for synthesizing an optimal supply process for a set of tasks. Furthermore, we define ordering relations between supplies which describe when a supply offers more resource capacity than another. With this notion it is possible to formally represent hierarchical scheduling approaches that assign more "generous" resource allocations to tasks in exchange for a simple representation. We illustrate our techniques via a number of examples.

## 1. Introduction

Component-based design has been widely accepted as a compositional approach to facilitate the design of complex systems. It provides means for decomposing a complex system into simpler components and for composing the components using interfaces that abstract component complexities. Such approaches are increasingly used in practice for real-time systems. For example, ARINC-653 standards by the Engineering Standards for Avionics and Cabin Systems committee specify partition-based design

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of avionics applications. Also, hypervisors for real-time virtual machines provide temporal partitions to guarantee real-time performance [15, 11].

To take advantage of the component-based design of real-time systems, schedulability analysis should support compositional analysis using component interfaces. These interfaces should abstract the timing requirements of a component with a minimum resource supply that is needed to meet the resource demand of the component. Component-based real-time systems often involve hierarchical scheduling frameworks that support resource sharing among components as well as associated scheduling algorithms [5, 22]. To facilitate the analysis of such systems, resource component interfaces and their compositional analysis have been proposed [16, 23, 24, 8, 25, 12]

Process algebras are abstract and compositional methodologies for system specification and analysis. They allow to systematically build complex systems from smaller ones via the use of a small number of operators, as well as to reason compositionally about system correctness. As such, they provide a promising framework in which to study compositional scheduling. This paper presents a formal treatment of the problem of compositional hierarchical scheduling by introducing a process algebraic framework, PADS, for modeling resource demand and supply inspired by the timed process algebra ACSR [13, 14]. The notions of resource demand and resource supply are fundamental in defining the meaning of compositional real-time scheduling analysis. Our proposed framework formally defines both of these notions. As in ACSR, a task in our process algebra is specified by describing its consumption needs for resources. To also accommodate resource-supply processes, we extend the notion of a *resource* and given a resource $cpu$ we use $\overline{cpu}$ to denote the availability of the resource for consumption by a requesting task. Our formalism then addresses the following issues:

1. *Schedulability*: We define a *supply simulation relation* $\models$ that captures when a task $T$ is schedulable by a supply $S$, $S \models T$.
2. *Compositionality*: We explore conditions under which we may safely compose schedulable systems. Specifically, we are interested to define functions on supplies, $\circ$, and appropriate conditions, $f$, such that if $T_1$ is schedulable by $S_1$ and $T_2$ by $S_2$ then the parallel composition of $T_1$ and $T_2$ is schedulable by $S_1 \circ S_2$, assuming that condition $f$ holds:
$$\frac{S_1 \models T_1, S_2 \models T_2}{S_1 \circ S_2 \models T_1 \| T_2}, \quad f(S_1, S_2)$$
3. *Supply Synthesis*: We propose a method by which we can generate a supply process to schedule a set of tasks, assuming that such a scheduler exists. Our method is based on the notion of a *demand* of a task which is a supply that can schedule the task and, at the same time, it is optimal in the sense that (1) it does not reserve more resources than those required and (2) it captures all possible ways in which a task can be scheduled. We then prove that two or more tasks are schedulable if and only if they can be scheduled by the composition of their demands.
4. *Task and Supply Orderings*: We propose an ordering between tasks which defines when a task is more "demanding" than another, meaning that it requires more resources in order to execute correctly. We also propose two orderings between supplies which define when a supply is more "generous" than another

meaning that it offers a greater resource allocation. The main result accompanying these notions is that any supply that schedules a more demanding task may also schedule a less demanding task and that any task schedulable by a less generous supply is also schedulable by a more generous supply. This result comes to complement our supply synthesis approach since it allows us to check whether a supply $S$ schedules a task set as follows: We begin by constructing the optimal supply/demand, $D$, for the task set and then check whether $S$ is more generous than $D$. In the affirmative case we may conclude that the task set is also schedulable by $S$.

*Related work.* As mentioned above, this work brings together two long-standing lines of research. On the one hand, there has been much work on compositional hierarchical scheduling based on real-time scheduling theory [16, 23, 24, 8, 6, 7]. Typically, such approaches to schedulability analysis rely on over-approximations of task demand using, for example, demand bound functions and under-approximations of resource supply using supply bound functions. Efficient algorithms are developed to ensure that demand never exceeds supply. On the other hand, several formal approaches to scheduling based on process algebras [3, 14, 13, 20, 18, 19], task automata [10, 9], preemptive Petri nets [4], etc., have been developed. To the best of our knowledge, none of these approaches consider the modeling of resource supply explicitly. Instead, sharing of a continuously available processing resource between a set of tasks has been considered.

Our approach to supply synthesis is conceptually similar to the work of Altisen *et al.* on applying controller synthesis to scheduling problems [1, 2]. The difference is that we are not aiming to generate schedulers, but rather an *interface* for a task set, an abstraction that can be used in a component-based approach to real-time system design.

The present paper extends our previous work of [21] as follows. It introduces priorities to the framework, thus allowing us to represent schedulability with respect to particular schedulers and it contains all the proofs missing from [21] adopted for the extended framework. Furthermore, it introduces ordering relations between tasks and supplies and associated results that enable us to formally represent techniques for over-approximating optimal resources as can be found in e.g. [23].

The rest of the paper is structured as follows. Section 2 presents our process algebra and its semantics. Section 3 contains our results on compositional schedulability analysis and interface construction, followed by examples illustrating the application of the theory in Section 3.3. Section 4 presents hierarchies between tasks and supplies and develops their properties and, finally, Section 5 concludes the paper.

## 2. The Language

In our calculus, PADS (Process Algebra for Demand and Supply), we consider a system to be a set of processes operating on a set of serially reusable resources denoted by $\mathsf{R}$. These processes are (1) the *tasks* of the system, which require the use of resources in order to complete their jobs, and (2) the *supplies*, that specify when each resource is available to the tasks. Based on this, each resource $r \in \mathsf{R}$ can be requested by a task, $r$, granted by a supply, $\overline{r}$, or consumed, $\overset{\leftrightarrow}{r}$, when a supply and a request for the resource are simultaneously available.

An action in PADS is a set relating to resource requests, grants and consumptions, where each resource may be represented at most once. Resource requests and consumptions are associated with a priority, where priorities are drawn from the nonnegative integers. These priorities are used to arbitrate between actions, the intention being that an action with a higher priority always wins. Supplies of resources are not associated with priorities since a resource can either be supplied or not supplied to a component and cannot be simultaneously offered to two or more tasks in a system. For example, the action $\{(r_1, 1), (r_2, 3)\}$ represents a request for the resources $r_1$ and $r_2$ at priorities 1 and 3, respectively, whereas the action $\{\overline{r_1}, (\overset{\leftrightarrow}{r}_2, 2), (r_3, 1)\}$ involves the granting of resource $r_1$, consumption of resource $r_2$ at priority level 2 and request for resource $r_3$ at priority level 1.

Our framework is intended to capture real-time, resource-aware systems. Such systems have a limited number of shared resources each of which is capable of participating in at most one action at a time. To capture this view and enable reasoning about scheduling such systems, our process algebra contains the notion of time. In particular, we take a discrete time approach: we assume that all actions require one unit of time to complete measured on a global clock, with action $\emptyset$ representing idling for one time unit since no resource is being employed.

We write *Act*, ranged over by $\alpha$ and $\beta$, for the set of all actions and distinguish $Act_R$, the set of actions involving only resource requests, ranged over by $\rho$, and $Act_G$, the set of actions involving only resource grants, ranged over by $\gamma$. Given $\alpha \in Act$ we write $\alpha^\flat$ to remove all priorities from resource-priority pairs in $\alpha$, e.g. $\{(r_1, 2), \overline{r_2}, (\overset{\leftrightarrow}{r}_3, 1)\}^\flat = \{r_1, \overline{r_2}, \overset{\leftrightarrow}{r}_3\}$ and $res(\alpha)$ for the set of resources occurring in $\alpha$, e.g. $res(\{(r_1, 2), \overline{r_2}, (\overset{\leftrightarrow}{r}_3, 1)\}) = \{r_1, r_2, r_3\}$. Finally, given an action $\alpha$ and a resource $r$, we write $\pi_\alpha(r)$ for the priority at which resource $r$ is employed within action $\alpha$, where we consider all supplied resources to be employed at priority level 0, e.g. for $\alpha = \{(r_1, 2), \overline{r_2}, (\overset{\leftrightarrow}{r}_3, 4)\}$, we have $\pi_\alpha(r_1) = 2$, $\pi_\alpha(r_2) = 0$ and $\pi_\alpha(r_3) = 4$. (Note that $\pi_\alpha$ is well defined since we have assumed that each resource may be represented in an action at most once.)

### *2.1. Syntax*

The following grammars define the set of tasks $\mathsf{T}$, the set of supplies $\mathsf{S}$ and the set of timed systems $\mathsf{P}$, where we recall that $\rho \in Act_R$ and $\gamma \in Act_G$. Furthermore, $C$ ranges over a set of *task constants*, each with an associated definition of the form $C \overset{\text{def}}{=} T$, where $T$ may contain occurrences of $C$ as well as other task constants and $D$ ranges over a similar set of *supply constants*.

$$\begin{array}{rcl} T & ::= & \text{FIN} \mid \rho : T \mid T + T \mid C \\ S & ::= & \text{FIN} \mid \gamma : S \mid S + S \mid D \\ P & ::= & \delta \mid T \mid S \mid P \| P \end{array}$$

We consider FIN to be the well-terminated process. Then a task process can be FIN, a task constant $C$, an action-prefixed process $\rho : T$ which executes $\rho$ during the first time unit and then behaves as $T$, or a nondeterministic choice $T_1 + T_2$ which offers the choice between executing $T_1$ or $T_2$. Similarly, a supply process can be FIN, a supply constant $S$, an action-prefixed process $\gamma : S$, or a nondeterministic choice $S_1 + S_2$. We make the assumption that all constants are guarded by an action, that is, it is not possible to define a process such as $C \overset{\text{def}}{=} D + ....$

Finally, a system can be a deadlocked system, $\delta$, or an arbitrary composition of tasks and supplies. In a parallel composition $P_1 \| P_2$, $P_1$ and $P_2$ run concurrently and synchronize while executing their actions. Furthermore, whenever one process requests a resource granted by the other, we obtain a consumption of the resource in question. Note that the difference between FIN and $\delta$ is that while FIN allows time to pass, $\delta$ does not. As a shorthand notation we will write $\Sigma_{i \in I} P_i$ for $P_{i_1} + \ldots + P_{i_n}$, where $I = \{i_1, \ldots, i_n\}$. Note that, given our assumption that all process constants occur guarded by an action, any task or supply different to FIN is in fact a guarded choice of the form $\Sigma_{i \in I} \alpha_i : P_i$.

### *2.2. Semantics*

The semantics of PADS is given in two steps. First, we develop a transition relation in which nondeterminism is resolved in all possible ways, the unprioritized transition relation $\twoheadrightarrow$. Then, we refine $\twoheadrightarrow$ into $\longrightarrow$, the prioritized transition relation, on the basis of a preemption relation which implements a type of "angelic" behavior in the way in which tasks resolve their nondeterminism, choosing the best possible outcome given the available supply and taking priorities into account.

We proceed to consider the unprioritized transition relation $\twoheadrightarrow$ defined in Table 1. FIN being a well-terminated (and not a deadlocked) process, it allows time to pass (axiom (IDLE)). Action-prefixed processes first execute their initial action and then proceed according to the continuation ((ActT) and (ActS)). Nondeterministic choice behaves as either of its constituent summands ((SumT) and (SumS)). A constant behaves as the process in its defining equation ((ConstT) and (ConstS)). Finally, rule (Par) specifies the way in which a parallel system evolves. To begin with, we recall that all actions take one time unit thus every step of a parallel composition should capture the actions of each of its components during the first time unit. To achieve this, the components of a parallel composition evolve synchronously and the composition advances only if both of the constituent processes are willing to take a step. Furthermore, the rule enunciates the outcome of the synchronization between two parallel processes, the most important aspect being that a request within one component is satisfied by an available grant in the other. The condition of rule (Par) imposes a restriction on when two actions may take place simultaneously within a system. Specifically, we say that actions $\alpha_1$ and $\alpha_2$ are *compatible* with each other if, whenever $r$ occurs in both actions then one occurrence must be a request and the other a supply of the resource. So, for example, it is not possible to simultaneously offer a resource in one component and consume or offer it in another, nor to request it by two different tasks. We capture this requirement as follows:

$$\mathsf{compatible}(\alpha_1, \alpha_2) \quad = \bigwedge_{r \in res(\alpha_1) \cap res(\alpha_2)} (r \in \alpha_1^\flat \wedge \overline{r} \in \alpha_2^\flat) \vee (r \in \alpha_2^\flat \wedge \overline{r} \in \alpha_1^\flat)$$

Table 1: **Transition rules for tasks, supplies and systems**

| | | | |
|---|---|---|---|
| (Idle) | $\mathrm{FIN} \overset{\emptyset}{\twoheadrightarrow} \mathrm{FIN}$ | | |
| (ActT) | $\rho : T \overset{\rho}{\twoheadrightarrow} T$ | (ActS) | $\gamma : S \overset{\gamma}{\twoheadrightarrow} S$ |
| (SumT) | $\dfrac{T_i \overset{\alpha}{\twoheadrightarrow} T, i \in \{1,2\}}{T_1 + T_2 \overset{\alpha}{\twoheadrightarrow} T}$ | (SumS) | $\dfrac{S_i \overset{\alpha}{\twoheadrightarrow} S, i \in \{1,2\}}{S_1 + S_2 \overset{\alpha}{\twoheadrightarrow} S}$ |
| (ConstT) | $\dfrac{T \overset{\alpha}{\twoheadrightarrow} T'}{C \overset{\alpha}{\twoheadrightarrow} T'} \quad C \overset{\mathrm{def}}{=} T$ | (ConstS) | $\dfrac{S \overset{\alpha}{\twoheadrightarrow} S'}{D \overset{\alpha}{\twoheadrightarrow} S'} \quad D \overset{\mathrm{def}}{=} S$ |
| (Par) | $\dfrac{P_1 \overset{\alpha_1}{\twoheadrightarrow} P_1' \quad P_2 \overset{\alpha_2}{\twoheadrightarrow} P_2'}{P_1 \| P_2 \overset{\alpha_1 \oplus \alpha_2}{\twoheadrightarrow} P_1' \| P_2'}$ | compatible$(\alpha_1, \alpha_2)$ | |

We may now combine compatible actions by transforming a simultaneous request and supply of the same resource into a consumption:

$$\begin{aligned} \alpha_1 \oplus \alpha_2 \quad = \quad & \{(r,p) \in \alpha_1 \cup \alpha_2 | \overline{r} \notin \alpha_1 \cup \alpha_2\} \cup \{\overline{r} \in \alpha_1 \cup \alpha_2 | (r,p) \notin \alpha_1 \cup \alpha_2\} \\ & \cup \{(\overset{\leftrightarrow}{r}, p) | (r,p) \in \alpha_i, \overline{r} \in \alpha_{3-i}, i \in \{1,2\} \text{ or } (\overset{\leftrightarrow}{r}, p) \in \alpha_1 \cup \alpha_2\} \end{aligned}$$

We may show the parallel composition operator to be associative with respect to $\twoheadrightarrow$ in the sense that, $P_1 \| (P_2 \| P_3) \overset{\alpha}{\twoheadrightarrow} P_1' \| (P_2' \| P_3')$ if and only if $(P_1 \| P_2) \| P_3 \overset{\alpha}{\twoheadrightarrow} (P_1' \| P_2') \| P_3'$. This can be shown by establishing that (1) compatible$(\alpha_1, \alpha_2)$ and compatible$(\alpha_1 \oplus \alpha_2, \alpha_3)$ if and only if compatible$(\alpha_2, \alpha_3)$ and compatible$(\alpha_1, \alpha_2 \oplus \alpha_3)$, and (2) the associativity of $\oplus$. Both of these properties are easy to prove by referring to the definitions.

**Example 2.1.** Consider the supply $S \overset{\mathrm{def}}{=} \{\overline{r_1}, \overline{r_2}\} : S$ which offers resources $r_1$ and $r_2$ simultaneously and the following task processes:

$$\begin{aligned} T_1 \quad & \overset{\mathrm{def}}{=} \{(r_1, 2)\} : \mathrm{FIN} + \emptyset : \{(r_1, 2)\} : \mathrm{FIN} \\ T_2 \quad & \overset{\mathrm{def}}{=} \{(r_1, 1), (r_2, 1)\} : \mathrm{FIN} + \{(r_2, 1), (r_3, 1)\} : T_2 \\ T_3 \quad & \overset{\mathrm{def}}{=} \{(r_1, 1), (r_2, 1)\} : \mathrm{FIN} + \{(r_2, 1)\} : \{(r_1, 1)\} : \mathrm{FIN} \end{aligned}$$

Task $T_1$ places a demand for resource $r_1$ at priority level 2 during either the first or the second time unit. Task $T_2$ requires the use of two resources simultaneously during the first time unit, either $r_1$ and $r_2$ or $r_2$ and $r_3$. Finally, task $T_3$ requires the use of resources $r_1$ and $r_2$ either simultaneously or in sequence. The transition systems of $T_1 \| S$, $T_2 \| S$ and $(T_1 \| S) \| T_3$ are depicted in Figure 1.

Note that $(T_2 \| S) \| T_3$ has no transitions altogether since both $T_2$ and $T_3$ require $r_2$ during the first time unit. □

Before we proceed to define the prioritized transition relation of PADS let us draw some motivation from the example above. We may note that these unprioritized transition systems include some unexpected and even undesirable behaviors. For example,

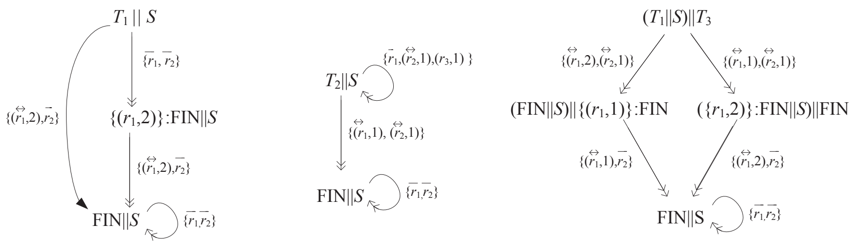

$T_1 \| S$
$\{\overline{r_1}, \overline{r_2}\}$
$\{(\overset{\leftrightarrow}{r_1},2),\overline{r_2}\}$
$\{(r_1,2)\}$:FIN$\|S$
$\{(\overset{\leftrightarrow}{r_1},2),\overline{r_2}\}$
FIN$\|S$
$\{\overline{r_1},\overline{r_2}\}$
$T_2\|S$
$\{\overline{r_1},(\overset{\leftrightarrow}{r_2},1),(r_3,1)\}$
$\{(\overset{\leftrightarrow}{r_1},1),(\overset{\leftrightarrow}{r_2},1)\}$
FIN$\|S$
$\{\overline{r_1},\overline{r_2}\}$
$(T_1\|S)\|T_3$
$\{(\overset{\leftrightarrow}{r_1},2),(\overset{\leftrightarrow}{r_2},1)\}$
$\{(\overset{\leftrightarrow}{r_1},1),(\overset{\leftrightarrow}{r_2},1)\}$
(FIN$\|S)\|\{(r_1,1)\}$:FIN
$(\{r_1,2)\}$:FIN$\|S)\|$FIN
$\{(\overset{\leftrightarrow}{r_1},1),\overline{r_2}\}$
$\{(\overset{\leftrightarrow}{r_1},2),\overline{r_2}\}$
FIN$\|$S
$\{\overline{r_1},\overline{r_2}\}$

Figure 1: The unprioritized transition systems of $T_1\|S$, $T_2\|S$ and $(T_1\|S)\|T_3$

consider task $T_1$. Our intention in writing this process is to express that $T_1$ requests resource $r_1$ during the first or the second time unit. More precisely, if $r_1$ is available during the first time unit, then $T_1$ should employ it and, if not, then it should idle and reiterate its request during the second time unit. However, when placing $T_1$ in parallel with a process like $S$ that offers $r_1$ (and $r_2$) immediately, we observe that $T_1\|S \overset{\{\overline{r_1},\overline{r_2}\}}{\twoheadrightarrow} \{(r_1,2)\} : \text{FIN}\|S$, i.e. the semantics allow for $T_1$ to choose its second alternative of idling instead of employing the available resource, contrary to our intention. Furthermore, consider $T_2\|S$. Again, here we observe that, contrary to what one might expect, the process may choose to execute its action $\{(r_2,1),(r_3,1)\}$, thus iterating its resource request for $r_3$, instead of consuming the available $r_1$ and $r_2$. Finally, in the transition system of $(T_1\|S)\|T_3$, we observe that the initial state enables two transitions whose actions contain the same resources but with one having higher priorities than the other: a treatment of priority is needed to ensure that higher-priority actions take precedence over lower-priority ones.

In order to capture the intended behavior of systems, as discussed above, we define a preemption relation on actions that prunes away undesirable behaviors. This preemption relation focuses on nondeterminism within tasks and it ensures that it is resolved based on the priorities of the resource requests and the following two assumptions:

1. Given a supply, a task should respond "angelically" and, given a nondeterministic set of enabled transitions, it should choose only between the ones that are satisfied by the available supply, assuming that such options exist. For example, $T_2\|S$ above should retain only transition $\{(\overset{\leftrightarrow}{r_1},1),(\overset{\leftrightarrow}{r_2},1)\}$ in its initial state.
2. In addition, we assume that a task behaves greedily and, at each step, it employs as many of the supplied resources as possible. For example, the composition $T_1\|S$ above should only retain transition $\{(\overset{\leftrightarrow}{r_1},2),\overline{r_2}\}$ in its initial state.

Given the above, we define the preemption relation as follows:

**Definition 2.2.** *We define the* preemption relation $\prec \in \text{Act} \times \text{Act}$ *so that* $\alpha \prec \beta$ *if one of the following holds:*

1. $\{r|\overline{r} \in \alpha^\flat or \overset{\leftrightarrow}{r} \in \alpha^\flat\} = \{r|\overline{r} \in \beta^\flat or \overset{\leftrightarrow}{r} \in \beta^\flat\}$, $\alpha^\flat \cap \mathsf{R} \neq \emptyset$ *and* $\beta^\flat \cap \mathsf{R} = \emptyset$*, that is,* $\alpha$ *and* $\beta$ *use the same consumed and offered resources and* $\alpha$ *contains some additional resource requests whereas* $\beta$ *does not.*

2. $\mathrm{res}(\alpha) = \mathrm{res}(\beta)$, *$\alpha^\flat \cap \mathsf{R} = \beta^\flat \cap \mathsf{R} = \emptyset$ and $\{r | \overset{\leftrightarrow}{r} \in \alpha^\flat\} \subset \{r | \overset{\leftrightarrow}{r} \in \beta^\flat\}$, that is, $\alpha$ and $\beta$ involve the same resources, neither of them makes any resources requests, but $\beta$ consumes more resources that $\alpha$.*
3. *$\alpha^\flat = \beta^\flat$, for all $r \in \mathrm{res}(\alpha)$ $\pi_\alpha(r) \leq \pi_\beta(r)$, and there exists $r \in \mathrm{res}(\alpha)$, $\pi_\alpha(r) < \pi_\beta(r)$, that is, $\alpha$ and $\beta$ contain the same resources with $\beta$ giving greater or equal priority to all resource usages, and there exists at least one resource which is associated with a strictly greater priority in $\beta$ than in $\alpha$.*

Intuitively, an action precludes another if it makes better usage of the same offered resources: According to clause (1), an action that involves no resource requests for an available resource supply preempts an action that makes further requests given the same supply which implies that tasks should behave in an "angelic" manner according to the first assumption above. According to clause (2), given a resource supply as much resource should be consumed as possible, thus tasks behave greedily according to the second assumption above. And, finally, the third clause implements our treatment of priority: if two resources contain exactly the same resources and in the same mode (request, grant or consume) then $\beta$ preempts $\alpha$ if each resource is employed by $\beta$ at a priority higher than or equal to $\alpha$, with at least one resource being implemented at a higher priority.

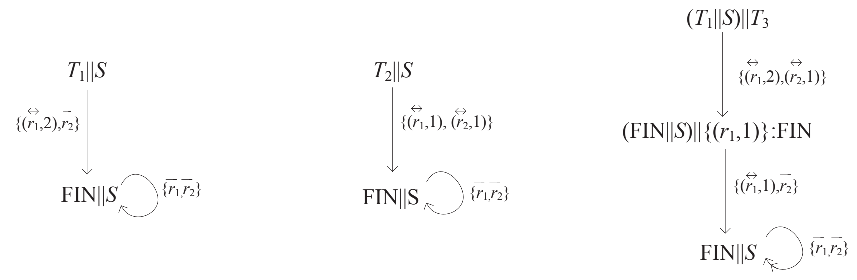
Note that preemption takes place between two actions only if they contain the same consumed and offered resources. For example, $\{(\overset{\leftrightarrow}{r_1}, 2), (\overline{r_2}, 1)\} \prec \{(\overset{\leftrightarrow}{r_1}, 1), (\overset{\leftrightarrow}{r_2}, 1)\}$ but $\{(\overset{\leftrightarrow}{r_1}, 1), (\overline{r_2}, 1)\} \not\prec \{(\overset{\leftrightarrow}{r_1}, 2))\}$ and $\{(\overset{\leftrightarrow}{r_1}, 1), (\overline{r_2}, 1), (r_3, 1)\} \not\prec \{(\overset{\leftrightarrow}{r_1}, 2))\}$. In other words, our semantics makes an asymmetric treatment between resource requests and resource supplies and, consequently, between task and supply processes. Intuitively, this asymmetry captures the understanding that while supplies control their nondeterminism and may choose to offer any one of their available actions, tasks respond to the supply available and resolve their nondeterminism based on the environment.

We may now define the prioritized transition relation $\xrightarrow{\alpha}$ by the following rule:

$$\boxed{\dfrac{P \overset{\alpha}{\twoheadrightarrow} Q}{P \xrightarrow{\alpha} Q}, \quad \text{there is no } P \overset{\beta}{\twoheadrightarrow}, \alpha \prec \beta}$$

Figure 2 presents the refined versions of the transition systems in Figure 1 after preemption is implemented.

We conclude this section by introducing some notations. We write $P \longrightarrow$ if there exists $\alpha$ such that $P \xrightarrow{\alpha}$. If $P \not\xrightarrow{\alpha}$ for all actions $\alpha$, we write $P = \delta$, where $\delta$ is the deadlocked process. We write $P \Longrightarrow P'$ if there exist $\alpha_1, \ldots, \alpha_n$ and $P_1, \ldots, P_n$, $n \geq 1$, such that $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \ldots P_{n-1} \xrightarrow{\alpha_n} P_n = P'$. The set of traces of $P$, $\mathsf{traces}(P)$, is defined to be the set of all infinite sequences $\alpha_1^\flat \alpha_2^\flat \ldots$ such that $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \ldots$. Furthermore, we write $\kappa$ for elements of $2^{\mathsf{R}}$ and $\overline{\kappa}$ to transform all resource requests in $\kappa$ into resource grants, so, $\overline{\{r_1, r_2\}} = \{\overline{r_1}, \overline{r_2}\}$. Extending this notation to traces of the form $w = \kappa_1 \kappa_2 \ldots$, we write $\overline{w}$ for $\overline{\kappa_1}\,\overline{\kappa_2} \ldots$. Finally, given $\alpha \in Act_G$, we write $\alpha^\natural$ to transform all resource grants into resource requests, so, $\{\overline{r_1}, \overline{r_2}\}^\natural = \{r_1, r_2\}$.



$T_1\|S$
$\{(\overleftrightarrow{r_1},2),\overline{r_2}\}$
$\mathrm{FIN}\|S$
$\{\overline{r_1},\overline{r_2}\}$
$T_2\|S$
$\{(\overleftrightarrow{r_1},1),(\overleftrightarrow{r_2},1)\}$
$\mathrm{FIN}\|\mathrm{S}$
$\{\overline{r_1},\overline{r_2}\}$
$(T_1\|S)\|T_3$
$\{(\overleftrightarrow{r_1},2),(\overleftrightarrow{r_2},1)\}$
$(\mathrm{FIN}\|S)\|\{(r_1,1)\}{:}\mathrm{FIN}$
$\{(\overleftrightarrow{r_1},1),\overline{r_2}\}$
$\mathrm{FIN}\|S$
$\{\overline{r_1},\overline{r_2}\}$

Figure 2: The prioritized transition systems of $T_1\|S$, $T_2\|S$ and $(T_1\|S)\|T_3$ from Example 2.1

## 3. Schedulability

In this section we present a theory of schedulability for our calculus. We begin by defining when a set of tasks is considered to be schedulable by a supply. Then we present an alternative characterization based on a type of simulation relation and we prove the two definitions to be equivalent. In what follows we write $\mathsf{T}^\star$ for the set containing all processes of the form $T_1\|\ldots\|T_n$, $n \geq 1$, and $\mathsf{S}^\star$ for the set containing all processes of the form $S_1\|\ldots\|S_n$, $n \geq 1$. For simplicity, we refer to elements of $\mathsf{T}^\star$ and $\mathsf{S}^\star$ simply as tasks and supplies, respectively.

**Definition 3.1.** *A task $T \in \mathsf{T}^\star$ is* schedulable by *supply $S \in \mathsf{S}^\star$ if whenever $T\|S \Longrightarrow P$ then (i) $P \longrightarrow$ and (ii) for all $P \xrightarrow{\alpha}$ we have $\alpha^\flat \cap \mathsf{R} = \emptyset$.*

According to this definition, a task $T$ is schedulable by supply $S$ if at no point during their interaction does the system deadlock (clause (i)) and, moreover, no request for a resource remains unsatisfied (clause (ii)).

**Example 3.2.** Let

$$
\begin{array}{lcllcl}
S_1 & \overset{\mathrm{def}}{=} & \{\overline{r_2}\} : \mathrm{FIN} & T_1 & \overset{\mathrm{def}}{=} & \{(r_1,1)\} : \mathrm{FIN} \\
S_2 & \overset{\mathrm{def}}{=} & \{\overline{r_1},\overline{r_2}\} : \mathrm{FIN} & T_2 & \overset{\mathrm{def}}{=} & \{(r_1,1)\} : \mathrm{FIN} + \{(r_2,1)\} : \mathrm{FIN} \\
S_3 & \overset{\mathrm{def}}{=} & \{\overline{r_1}\} : \mathrm{FIN} + \{\overline{r_2}\} : \mathrm{FIN} & T_3 & \overset{\mathrm{def}}{=} & \{(r_2,1)\} : \mathrm{FIN}
\end{array}
$$

Consider $T_1$. We observe that $T_1$ is not schedulable by $S_1$ since $T_1\|S_1 \xrightarrow{\{(r_1,1),\overline{r_2}\}}$. Clearly, this is so because $S_1$ does not offer $r_1$ as required by $T_1$, while $S_2$, by offering simultaneously $r_1$ and $r_2$, schedules $T_1$ as shown in the transition $T_1\|S_2 \xrightarrow{\{(\overleftrightarrow{r_1},1),\overline{r_2}\}}$ $\mathrm{FIN}\|\mathrm{FIN}$. However, this is not the case for supply $S_3$: although it offers both $r_1$ and $r_2$ during the first time unit, it does so in two distinct actions. If the nondeterminism is resolved according to the first summand, we obtain $T_1\|S_3 \xrightarrow{\{(r_1,1),\overline{r_2}\}}$ $\mathrm{FIN}\|\mathrm{FIN}$ (note that $\{(r_1,1),\overline{r_2}\}$ and $\{(\overleftrightarrow{r_1},1)\}$ are incomparable by $\prec$ thus both actions are enabled in $T_1\|S_3$).

Moving on to task $T_2$ we observe that this is schedulable by all three supplies. In particular, $T_2 \| S_1 \xrightarrow{\{(\overleftrightarrow{r_2},1)\}} \mathrm{FIN} \| \mathrm{FIN}$ and this is the only transition of $T_2 \| S_1$ since $\{\overline{r_2}, (r_1, 1)\} \prec \{(\overleftrightarrow{r_2}, 1)\}$. Finally, $T_3$ is schedulable by $S_1$ and $S_2$ but not $S_3$. $\Box$

Following this example we can make a number of observations regarding the defined notion of schedulability. Regarding supplies, we note that adding resources to the actions of a supply (as $S_2$ introduces $r_2$ in the action of supply $S_1$) appears to increase the supply's ability to schedule tasks since this implies that more resources are offered ($S_2$ schedules task $T_1$ whereas $S_1$ cannot). However, introducing nondeterministic alternatives in a supply reduces this ability; for example $S_1$ schedules $T_3$ but $S_3$ does not. The opposite holds for tasks: extending the actions of a task with resources decreases its ability to be scheduled by a supply since this implies that more resources are required, while extending a task with nondeterministic alternatives increases its ability to be schedulable since additional alternatives instill greater flexibility for the task to execute ($T_2$ is schedulable by $S_3$ unlike $T_1$ and $T_3$). These observations will be further studied and formalized in Section 4.

We now continue to provide an alternative characterization of schedulability via a type of simulation relations. This definition highlights the conditions under which a task is schedulable by a supply as well as the asymmetry between tasks and supplies discussed above. Before moving on to this definition we introduce some useful notations and results:

**Definition 3.3.** *Let $\alpha, \beta \in \mathrm{Act}$.*

- *We write $\mathsf{sat}(\beta, \alpha)$ if $\mathrm{res}(\beta) \subseteq \mathrm{res}(\alpha)$. In the case of $\beta \in \mathrm{Act}_R$ and $\alpha \in \mathrm{Act}_G$, we say that request action $\beta$ is satisfied by grant action $\alpha$.*
- *For a system $P$, we write $\beta \trianglelefteq_P \alpha$ if $\mathsf{sat}(\beta, \alpha)$ and there exists no $\gamma \in \mathrm{Act}$ such that $P \xrightarrow{\gamma} P'$, $\mathsf{sat}(\gamma, \alpha)$ and either $\beta^\flat \subset \gamma^\flat$ or $\beta^\flat = \gamma^\flat$ and $\beta \prec \gamma$. If $\beta \trianglelefteq_P \alpha$ we say that $\beta$ is a* maximal *response of $P$ with respect to $\alpha$.*

Note that, given a resource grant by some supply $S$, only maximal responses of a task $T$ are relevant responses to the supply. This is because, in the parallel composition of $T \| S$, any non-maximal responses will be pruned away by the preemption relation and thus they can be ignored. For example, if $T \overset{\mathrm{def}}{=} \{(r_1, 2), (r_2, 1)\} : T_1 + \{(r_1, 3)\} : T_2$ and $S \overset{\mathrm{def}}{=} \{\overline{r_1}, \overline{r_2}\} : S'$, we have $\{(r_1, 2), (r_2, 1)\} \trianglelefteq_T \{\overline{r_1}, \overline{r_2}\}$. We may in fact prove that:

**Lemma 3.4.** *For any $T \in \mathsf{T}^\star$, $S \in \mathsf{S}^\star$,*

1. *$T \| S \xrightarrow{\alpha} T' \| S'$ with $\alpha^\flat \cap \mathsf{R} = \emptyset$ if and only if $T \xrightarrow{\alpha_1} T'$, $S \xrightarrow{\alpha_2} S'$, $\alpha = \alpha_1 \oplus \alpha_2$ and $\alpha_1 \trianglelefteq_T \alpha_2$.*
2. *Suppose $T \| S \xrightarrow{\alpha} T' \| S'$, where $T \xrightarrow{\alpha_1} T'$, $S \xrightarrow{\alpha_2} S'$ and $\alpha = \alpha_1 \oplus \alpha_2$, and, furthermore, there exists $\beta$, $\mathsf{sat}(\beta, \alpha_2)$ with $T \xrightarrow{\beta} T''$. Then $\alpha_1 \trianglelefteq_T \alpha_2$.*

PROOF: For the first item of the lemma, consider $T \in \mathsf{T}^\star$, $S \in \mathsf{S}^\star$, such that $T \| S \xrightarrow{\alpha} T' \| S'$ with $\alpha^\flat \cap \mathsf{R} = \emptyset$. Then it must be that for some $\alpha_1 \in Act_R$ and $\alpha_2 \in Act_G$,

$T \xrightarrow{\alpha_1} T'$, $S \xrightarrow{\alpha_2} S'$ with $\alpha = \alpha_1 \oplus \alpha_2$. Since $\alpha^\flat \cap \mathsf{R} = \emptyset$ it must be that $\mathsf{sat}(\alpha_1, \alpha_2)$. Suppose that there exists $\gamma$ with $\mathsf{sat}(\gamma, \alpha_2)$ such that either $\alpha_2^\flat \subset \gamma^\flat$ or $\beta^\flat = \gamma^\flat$ and $\beta \prec \gamma$. In both cases we may see that $\alpha_1 \oplus \alpha_2 \prec \gamma \oplus \alpha_2$ which contradicts the existence of transition $T \| S \xrightarrow{\alpha}$. This implies that $\alpha_1 \trianglelefteq_T \alpha_2$ as required. The other direction of the property can be established using similar arguments.

For the second item of the lemma, suppose $T \| S \xrightarrow{\alpha} T' \| S'$, where $T \xrightarrow{\alpha_1} T'$, $S \xrightarrow{\alpha_2} S'$, and $\alpha = \alpha_1 \oplus \alpha_2$, and, furthermore, there exists $\beta$, $\mathsf{sat}(\beta, \alpha_2)$ with $T \xrightarrow{\beta} T''$. Suppose that $\alpha^\flat \cap \mathsf{R} \neq \emptyset$. Then $\alpha \prec \beta \oplus \alpha_2$ which contradicts the existence of the transition $S \| T \xrightarrow{\alpha} S' \| T'$. Thus, $\alpha^\flat \cap \mathsf{R} = \emptyset$, and by the first part of the lemma $\alpha_1 \trianglelefteq_T \alpha_2$. □

**Definition 3.5.** *A relation $\mathcal{S} \subseteq \mathsf{T}^\star \times \mathsf{S}^\star$ is a* supply simulation relation *if for all $(T, S) \in \mathcal{S}$, $S \longrightarrow$, and if $S \xrightarrow{\alpha} S'$ then*

1. *there exists $T \xrightarrow{\beta} T'$ with $\mathsf{sat}(\beta, \alpha)$ and $(T', S') \in \mathcal{S}$, and*
2. *whenever $T \xrightarrow{\beta} T'$ with $\beta \trianglelefteq_T \alpha$, then $(T', S') \in \mathcal{S}$.*

*If there exists a supply simulation relation between $T$ and $S$, then we write $S \models T$.*

That is, a task and a supply are related by a supply simulation relation if (i) the supply is able to offer resources to the task ($S \longrightarrow$), (ii) if a supply offers a set of resources then the task will be able to respond by an action that is satisfied by the available supply and to remain schedulable by the resulting state of the supply (clause 1), and (iii) given a set of resources offered by the supply, any maximal transition by which the task can accept the offered supply will result in a state that remains schedulable by the remaining supply (clause 2). Recall that, according to Lemma 3.4(2), only maximal transitions of $T$ with respect to some supply are relevant responses, all other transitions are pruned away by the preemption relation and can thus be ignored.

We may now prove that the two alternative schedulability notions coincide.

**Lemma 3.6.** *A task $T \in \mathsf{T}^\star$ is schedulable by supply $S \in \mathsf{S}^\star$ if and only if $S \models T$.*

PROOF: To begin with, suppose there exists a supply simulation relation $\mathcal{R}$ between $T$ and $S$. We will show that if $T \| S \xrightarrow{\alpha} T' \| S'$ then (i) $\alpha^\flat \cap \mathsf{R} = \emptyset$, (ii) $(T', S') \in \mathcal{R}$ and (iii) $T' \| S' \longrightarrow$. Then, by induction on the length of the transition of $T \| S \Longrightarrow P$, we may deduce that $T$ is schedulable by $S$, according to Definition 3.1.

So suppose that $T \| S \xrightarrow{\alpha} T' \| S'$ where $T \xrightarrow{\alpha_1} S'$ and $S \xrightarrow{\alpha_2} S'$, $\alpha = \alpha_1 \oplus \alpha_2$. We know that for some $\beta$, $\mathsf{sat}(\beta, \alpha_2)$, $T \xrightarrow{\beta} T''$ (Definition 3.5(1)). By Lemma 3.4(2) this implies that $\alpha_1 \trianglelefteq_T \alpha_2$, thus by Definition 3.5 $(T', S') \in \mathcal{R}$. Furthermore, by Lemma 3.4(1) we have that $\alpha^\flat \cap \mathsf{R} = \emptyset$. Finally, since $(T', S') \in \mathcal{R}$, by Definition 3.5 we have that $S' \longrightarrow$ and for each $S' \xrightarrow{\beta_1}$ there exists $T' \xrightarrow{\beta_2}$ with $\mathsf{sat}(\beta_2, \beta_1)$. This implies that $S' \| T' \longrightarrow$ which completes the first part of the proof.

Conversely, suppose that task $T$ is schedulable by supply $S$. We will show that

$$\mathcal{R} = \{(T, S) | T \text{ is schedulable by } S\}$$

is a supply simulation relation. Suppose $(T, S) \in \mathcal{R}$. Since $T \| S \longrightarrow$, $T \longrightarrow$ and $S \longrightarrow$. Furthermore, if $S \xrightarrow{\alpha} S'$ then, since $T$ is schedulable by $S$, there exists

$T \xrightarrow{\beta} T'$, $\mathsf{sat}(\beta, \alpha)$. If not, that is for all $T \xrightarrow{\beta} T''$, $\mathit{res}(\beta) - \mathit{res}(\alpha) \neq \emptyset$, then $T \| S \xrightarrow{\gamma}$, $\gamma^\flat \cap \mathsf{R} \neq \emptyset$ which contradicts our assumption of $T$ being schedulable by $S$. Next, suppose that $T \xrightarrow{\beta} T'$ and $\beta \trianglelefteq_T \alpha$. Since $S \xrightarrow{\alpha} S'$ and $T \xrightarrow{\beta} T'$ where $\beta \trianglelefteq_T \alpha$, by Lemma 3.4(1) $T \| S \xrightarrow{\alpha \oplus \beta} T' \| S'$, where $T'$ is schedulable by $S'$, which implies that $(T', S') \in \mathcal{R}$, as required. □

We define when a task is schedulable and this is done in the following obvious way.

**Definition 3.7.** *A task $T \in \mathsf{T}^\star$ is* schedulable *if there exists a supply $S$ with $S \models T$.*

We observe that the crux of the schedulability of a task by a supply lies in the ability of the task to operate acceptably for all possible actions of the supply and in doing so in all its enabled nondeterministic executions that can take place as a response to the supply available. The notion of a cylinder defined below is intended to capture all the possible ways in which a task can respond given an execution of a supply.

**Definition 3.8.** *Given a task $T \in \mathsf{T}^\star$ and an infinite trace $w = \kappa_1 \kappa_2 \ldots$, with $\kappa_i \subset \mathsf{R}$ for all $i$, we define the $w$-*cylinder *of $T$ to be the set $A = \cup_{i \geq 1} A_i$, where*

$$\begin{array}{rcl} A_1 & = & \{(T, \alpha_1, P_1) \mid T \xrightarrow{\alpha_1} P_1, \alpha_1 \trianglelefteq_T \overline{\kappa_1}\} \\ A_i & = & \{(P_i, \beta_i, P_i') \mid P_i \xrightarrow{\beta_i} P_i', \beta_i \trianglelefteq_{P_i} \overline{\kappa_i}, \exists (Q, \gamma, P_i) \in A_{i-1}\}, \quad i > 1 \end{array}$$

*Furthermore, we say that an $w$-cylinder $A = \cup_{i \geq 1} A_i$ is* live *if (i) for all $(Q, \alpha, Q') \in A$ then $Q \longrightarrow$, (ii) $A_i \neq \emptyset$ for all $i$ and (3) $\bigcup_{(P,\beta,Q) \in A_i} \beta^\flat = \kappa_i$.*

Thus, a $w$-cylinder, where $w = \kappa_1 \kappa_2 \ldots$, of a task $T$ contains all the possible/maximal responses of $T$ given the supply $\overline{\kappa_1}$ (set $A_1$), all possible responses of the resulting states given supply $\overline{\kappa_2}$ (set $A_2$), and so on. For example, consider task $T$ where

$$\begin{array}{rcl} T & \overset{\mathrm{def}}{=} & \{(r_1, 2)\} : T' \\ T' & \overset{\mathrm{def}}{=} & \{(r_1, 1)\} : \mathrm{FIN} + \{(r_2, 1)\} : \mathrm{FIN} + \{(r_3, 1)\} : \mathrm{FIN} \end{array}$$

For $w = \{r_1, r_2\}\{r_2, r_3\}\emptyset^*$, the $w$-cylinder of $T$ is $A = \cup_{i \geq 1} A_i$, where

$$\begin{array}{rcl} A_1 & = & \{(T, \{(r_1, 2)\}, T')\} \\ A_2 & = & \{(T', \{(r_2, 1)\}, \mathrm{FIN}), (T', \{(r_3, 1)\}, \mathrm{FIN})\} \\ A_i & = & \{(\mathrm{FIN}, \emptyset, \mathrm{FIN})\}, \quad i \geq 3 \end{array}$$

We observe that these are indeed the transitions that will be relevant when scheduling $T$ by a supply of the form $S \overset{\mathrm{def}}{=} \{\overline{r_1}, \overline{r_2}\}\{\overline{r_2}, \overline{r_3}\} : S'$. The following result relates live cylinders with task schedulability.

**Lemma 3.9.** *A task $T \in \mathsf{T}^\star$ is schedulable if and only if it possesses a live cylinder.*

PROOF: Suppose $T$ has a live $w$-cylinder where $w = \kappa_1\kappa_2\ldots$. Consider supply $S_0$ defined by the following set of equations $S_i \overset{\text{def}}{=} \overline{\kappa_{i+1}}{:}S_{i+1}$. Then, we may confirm that $S_0 \models T$. In particular we show that if $A = \cup_{i\geq 1}A_i$ is the $w$-cylinder of $T$, then

$$\mathcal{R} = \{(T, S_i) \mid (T, \beta, Q) \in A_i, i \geq 1\}$$

is a supply relation. So, consider $(T, S_i) \in \mathcal{R}$. To begin with, trivially $S_i \longrightarrow$. Further, if $S_i \xrightarrow{\alpha} S_{i+1}$, then since $A_i \neq \emptyset$, there exists $T \xrightarrow{\beta} Q$, $\beta \trianglelefteq_T \alpha$, and $(Q, S_{i+1}) \in \mathcal{R}$. In fact, this holds for all $T \xrightarrow{\beta} Q$, where $\beta \trianglelefteq_{T_i} \alpha$ and the result follows.

On the other hand, if $T$ is schedulable, then there exists a supply $S$ that schedules it. Let $w = \overline{\kappa_1}\,\overline{\kappa_2}\ldots \in \mathsf{traces}(S)$. We may construct a cylinder $A = \cup_{i\geq 1}A_i$ of $T$ as

$$\begin{array}{rcl} A_1 & = & \{(T, \alpha_1, P) \mid T \xrightarrow{\alpha_1} P, \alpha_1 \trianglelefteq_T \overline{\kappa_1}\} \\ A_i & = & \{(P, \beta_i, P') \mid P \xrightarrow{\beta_i} P', \beta_i \trianglelefteq_P \overline{\kappa_i}, (Q, \gamma, P) \in A_{i-1}\}, i > 1 \end{array}$$

Since $T$ is schedulable by $S$ it is straightforward to see that $A$ contains no triple of the form $(Q, \alpha, \delta)$ and also that $A_i \neq \emptyset$ for all $i$. Finally, if we take $\beta_i = \bigcup_{(P,\beta,Q)\in\mathcal{A}_i} \beta^\flat$, we may conclude that $A = \cup_{i\geq 1}A_i$ is a $w'$-cylinder of $T$, where $w' = \beta_1\beta_2\ldots$. □

*3.1. Matching Supplies to Tasks*

In this section we focus our attention to the problem of collecting the resource requirements of a task into a matching supply. Specifically, given a task, we would like to generate a supply process which schedules the task and at the same time is optimal in that (1) it does not reserve more resources than those required by the task and (2) it provides resource assignments to capture all possible ways in which the task can be scheduled. Both of these properties are important during the compositional scheduling of real-time tasks. The first property is clearly desirable since conservation of resources becomes critical when real-time components are composed. For the second property, we observe that capturing all possible ways of scheduling a task gives greater flexibility when one tries to compositionally schedule a set of tasks where the challenge is to share the resources between the tasks in ways that are acceptable to each one of them.

We begin by defining a function for combining supplies. This is helpful for a subsequent definition that considers matching supplies to tasks.

**Definition 3.10.** *Given supplies $S_1, S_2 \in \mathsf{S}$ we define $S_1 \otimes S_2$ as*

$$S_1 \otimes S_2 = \begin{cases} S_1 & \textit{if } S_2 = \mathrm{FIN} \\ S_2 & \textit{if } S_1 = \mathrm{FIN} \\ \Sigma_{i\in I}\Sigma_{j\in J} & \alpha_i \cup \beta_j{:}(\bigotimes_{k\in I, \alpha_k \trianglelefteq_{S_1} \alpha_i\cup\beta_j} P_k \otimes \bigotimes_{l\in J, \beta_l \trianglelefteq_{S_2} \alpha_i\cup\beta_j} Q_l) \\ & \textit{if } S_1 \overset{\text{def}}{=} \sum_{i\in I} \alpha_i{:}P_i \textit{ and } S_2 \overset{\text{def}}{=} \sum_{j\in J} \beta_j{:}Q_j \end{cases}$$

Essentially, the joined supply $S_1 \otimes S_2$ joins together the various summands of the individual supplies as follows: in its topmost summand it unites all available grants of $S_1$ with all available grants of $S_2$, while the continuation process consists of the join of those continuations of $S_1$ and $S_2$ which appear after "maximal" subsets of the initial action in question. For example we have:

$$\emptyset : \{\overline{r}\} : \emptyset : \mathrm{FIN} \otimes \emptyset : \emptyset : \{\overline{r}\} : \mathrm{FIN} \quad = \quad \emptyset : \{\overline{r}\} : \{\overline{r}\} : \mathrm{FIN}$$
$$\emptyset : \{\overline{r}\} : \emptyset : \mathrm{FIN} \otimes (\emptyset : \emptyset : \{\overline{r}\} : \mathrm{FIN} + \{\overline{r}\} : \emptyset : \emptyset : \mathrm{FIN})$$
$$= \quad \emptyset : \{\overline{r}\} : \{\overline{r}\} : \mathrm{FIN} + \{\overline{r}\} : \{\overline{r}\} : \emptyset : \mathrm{FIN}$$

Using this definition we now move to define the *demand* of a task. The demand of a task is intended to capture the optimal supply that can schedule a task in the sense we have already discussed. The main point to note in this definition is that we combine all same-prefixed nondeterministic choices of a task by a singly-prefixed supply.

**Definition 3.11.** *Given a task $T \in \mathsf{T}^\star$, we define its* demand *as the following element of $\mathsf{S}$:*

$$\mathsf{demand}(T) \overset{\mathrm{def}}{=} \sum_{\alpha : T \xrightarrow{\alpha}} \overline{\alpha^\flat}{:}[\bigotimes_{T' : T \xrightarrow{\alpha} T'} \mathsf{demand}(T')]$$

**Example 3.12.** Consider tasks

$$\begin{aligned}
T_1 &= \{(r,2)\} : \emptyset : \emptyset : T_1 + \emptyset : \{(r,1)\} : \emptyset : T_1 + \emptyset : \emptyset : \{(r,3)\} : T_1 \\
T_2 &= \{(r,1)\} : \emptyset : \emptyset : T_2 + \emptyset : (\{(r,2)\} : \emptyset : T_2 + \emptyset : \{(r,2)\} : T_2) \\
T_3 &= \{(r,1)\} : \{(r,1)\} : \mathrm{FIN} + \{(r,2)\} : \emptyset : T_3
\end{aligned}$$

Their demands are given by $X_1, X_2, X_3$ below, respectively.

$$\begin{aligned}
X_1 &= \{\overline{r}\} : \emptyset : \emptyset : X_1 + \emptyset : \{\overline{r}\} : \{\overline{r}\} : X_1 \\
X_2 &= \{\overline{r}\} : \emptyset : \emptyset : X_2 + \emptyset : (\{\overline{r}\} : \emptyset : X_2 + \emptyset : \{\overline{r}\} : X_2) \\
X_3 &= \{\overline{r}\} : \emptyset : X_3
\end{aligned}$$

$\square$

The next lemma considers the optimality of $\mathsf{demand}(T)$ following the requirements posed at the beginning of this section. We write $w \leq w'$ for the infinite traces $w = \alpha_1\alpha_2\ldots$ and $w' = \beta_1\beta_2\ldots$, if $\alpha_j \subseteq \beta_j$ for all $j \geq 1$.

**Lemma 3.13.** *If $w \in \mathsf{traces}(\mathsf{demand}(T))$ then $T$ possesses a live $w$-cylinder and if $w \in \mathsf{traces}(T)$ then there exists $w' \in \mathsf{traces}(\mathsf{demand}(T))$ such that $\overline{w} \leq w'$.*

PROOF: Suppose $\mathsf{demand}(T) \xrightarrow{\alpha_1} S_1 \xrightarrow{\alpha_2} S_2 \xrightarrow{\alpha_3} \ldots$. We will show that for the $w$-cylinder $A = \cup_{i\geq} A_i$ of $T$, where $w = \alpha_1^\natural \alpha_2^\natural \ldots$, we have $S_i = \bigotimes_{(P,\beta,Q)\in A_i} \mathsf{demand}(Q)$ and $A$ is live. Consider an arbitrary $S_i$ and suppose $S_i = \bigotimes_{(P,\beta,Q)\in A_i} \mathsf{demand}(Q)$ where $A_i \neq \emptyset$ and $A_i$ does not contain elements of the form $(P,\beta,\delta)$. Then, since $S_i \xrightarrow{\alpha_{i+1}}$, by the definition of $\otimes$, it must be that

$$\alpha_{i+1} = \bigcup\{\alpha \mid (P,\beta,Q) \in A_i, \mathsf{demand}(Q) \xrightarrow{\alpha}\}.$$

In addition, $S_i \xrightarrow{\alpha_{i+1}} \bigotimes_{T'\in B} \mathsf{demand}(T')$, $B = \{T' \mid (P,\beta,Q) \in A_i, Q \xrightarrow{\beta} T', \beta \trianglelefteq_Q \alpha_{i+1}\}$. But, $B = A_{i+1}$ and by the construction of $\alpha_{i+1}$, $A_{i+1} \neq \emptyset$ and $A$ is live, which completes the first part of the proof.

To establish the second part of the proof it is sufficient to note that if $T \xrightarrow{\alpha} T'$ then $\mathsf{demand}(T) \xrightarrow{\overline{\alpha}} \mathsf{demand}(T') \otimes S$ for some $S$ and, further, if $S_1 \xrightarrow{\alpha} S_1'$ then $S_1 \otimes S_2 \xrightarrow{\alpha'} S_1' \otimes S_2'$, where $\alpha \subseteq \alpha'$ for some $S_2'$. Then, by the definition of $\mathsf{demand}$, it is easy to see that if $T \xrightarrow{\alpha_1} T_1 \xrightarrow{\alpha_2} T_2 \xrightarrow{\alpha_3} \ldots$, then $\mathsf{demand}(T) \xrightarrow{\beta_1} \mathsf{demand}(T_1) \otimes S_1 \xrightarrow{\beta_2} \mathsf{demand}(T_2) \otimes S_2 \xrightarrow{\beta_3} \ldots$, where $\overline{\alpha_1^\flat \alpha_2^\flat \ldots} \leq \beta_1 \beta_2 \ldots$.

□

Thus, we may conclude that a task $T$ is schedulable by $\mathsf{demand}(T)$. Furthermore, $\mathsf{demand}(T)$ is an optimal supply for $T$ since each of its executions schedules exactly a cylinder of $T$, i.e. it offers exactly the resources necessary for scheduling the cylinder, and each possible schedule of $T$ is captured by an execution of $\mathsf{demand}(T)$.

## 3.2. Compositional Theory

We proceed to consider the schedulability problem of a set of task components. The first issue we tackle is the compositionality problem: If a component $T_1$ is schedulable by $S_1$ and an independent component $T_2$ by $S_2$ can we combine $S_1$ and $S_2$ into a collective supply that schedules $T_1 \| T_2$? We begin by noting a subtlety pertaining to this problem which we need to consider before answering it. Consider the two components below each consisting of one task:

$$\begin{array}{rcl} T_1 & = & \{(r,1)\}{:}\emptyset{:}\mathrm{FIN} + \emptyset{:}\{(r,1)\}{:}\mathrm{FIN} \\ T_2 & = & \{(r,1)\}{:}\emptyset{:}\mathrm{FIN} + \emptyset{:}\{(r,1)\}{:}\{(r,1)\}{:}\mathrm{FIN} \end{array}$$

These components are schedulable by supplies $S_1 = \emptyset{:}\{\overline{r}\}{:}\mathrm{FIN}$ and $S_2 = \{\overline{r}\}{:}\emptyset{:}\mathrm{FIN}$, respectively. That is, it is sufficient for component $T_1$ to obtain resource $r$ during the second time unit and for component $T_2$ during the first time unit. However, a supply $S = \{\overline{r}\}{:}\{\overline{r}\}{:}\mathrm{FIN}$, offering $r$ during both time units, fails to schedule $T_1 \| T_2$. This is due to the fact that the supply for resource $r$ during the first time unit is intended for component $T_2$ but may be consumed by component $T_1$ leading to a deadlock of the system during the third time unit. Moreover, if $T_1$ employed its resources at priority level 2, this would in fact be destined to happen.

To resolve this issue, we associate components with their matching supplies by annotating each resource reference by a number which distinguishes the component in which the resource is employed/supplied. Precisely, we assume that each component is associated with a component identifier and if resource $r$ is requested by a component with identifier $i$ we write $r[i]$ for the request and, similarly, if a supply of $r$ is intended for the component with identifier $i$ we write $\overline{r[i]}$ for the supply. So, we say that component $\{(r[1],1)\}{:}\mathrm{FIN}$ is schedulable by supply $\{\overline{r[1]}\}{:}\mathrm{FIN}$ and component $\{(r[2],3)\}{:}\mathrm{FIN}$ by supply $\{\overline{r[2]}\}{:}\mathrm{FIN}$. However, note that resources $r[1]$ and $r[2]$ do refer to the same resource and for all other purposes should be treated as the same. So, for example, $\{r[1]\} \cap \{r[2]\} \neq \emptyset$. To model this precisely we write:

- $P[i]$ for the process $P$ with all its resources $r$ renamed as $r[i]$ (and, thus, $\mathsf{demand}(P)[i]$ for the process $\mathsf{demand}(P)$ with all its resources $r$ renamed as $r[i]$).
- $\alpha \cap_{\mathsf{R}} \beta$ for $\{r \in \mathsf{R} \mid r[i] \in \mathit{res}(\alpha)$ and $r[j] \in \mathit{res}(\beta)\}$.

Furthermore, we use the notation $\alpha[i] = \{r[i] \mid r \in \alpha\}$ and, if $w = \alpha_1\alpha_2\ldots$, $w[i] = \alpha_1[i]\alpha_2[i]\ldots$. We have the following result:

**Lemma 3.14.** *If $T_1$ is schedulable by $S_1$, $T_2$ is schedulable by $S_2$ and $S_1\|S_2$ does not deadlock, then $T_1[1]\|T_2[2]$ is schedulable by $S_1[1]\|S_2[2]$.*

PROOF: We will show that $\mathcal{R}$, below, is a supply simulation relation.

$$\mathcal{R} = \{(T_1[1]\|T_2[2], S_1[1]\|S_2[2]) \mid S_1 \models T_1, S_2 \models T_2, S_1[1]\|S_2[2] \text{ does not deadlock}\}$$

Let $(T_1[1]\|T_2[2], S_1[1]\|S_2[2]) \in \mathcal{R}$. By the definition of $\mathcal{R}$, $S_1[1]\|S_2[2] \longrightarrow$. So consider $S_1[1]\|S_2[2] \xrightarrow{\alpha} S_1'[1]\|S_2'[2]$. It must be that $\alpha = \alpha_1[1] \oplus \alpha_2[2]$, where $S_1 \xrightarrow{\alpha_1} S_1'$, $S_2 \xrightarrow{\alpha_2} S_2'$ and $\alpha_1 \cap \alpha_2 = \emptyset$. Since $S_1 \models T_1$, $S_2 \models T_2$, we have $T_1 \xrightarrow{\beta_1} T_1'$, $S_1' \models T_1'$, and similarly $T_2 \xrightarrow{\beta_2} T_2'$, $S_2' \models T_2'$. In fact, for all $T_1 \xrightarrow{\beta_1} T_1'$, $\beta_1 \trianglelefteq_{T_1} \alpha_1$, it holds that $S_1' \models T_1'$, and for all $T_2 \xrightarrow{\beta_2} T_2'$, $\beta_2 \trianglelefteq_{T_2} \alpha_2$, it holds that $S_2' \models T_2'$. This implies that for all $T_1[1]\|T_2[2] \xrightarrow{\beta} T_1'[1]\|T_2'[2]$, $\beta \trianglelefteq_{T_1[1]\|T_2[2]} \alpha$, $(T_1'[1]\|T_2'[2], S_1'[1]\|S_2'[2]) \in \mathcal{R}$ and there exists at least one such $\beta$-transition. This completes the proof. □

However, note that even if $S_1\|S_2$ deadlocks, it is still possible that the supplies $S_1$ and $S_2$ can be combined to produce a supply for $T_1\|T_2$. In particular, we may suspect that every infinite trace of $S_1\|S_2$ is capable of scheduling $T_1\|T_2$, and in fact we can show that the part of the transition system that pertains to non-deadlocking behavior achieves exactly that. The following operator on supplies extracts this type of behavior.

**Definition 3.15.** *Given supplies $S_1$ and $S_2 \in \mathsf{S}$ we define their product $S_1 \times S_2$ by*

$$S_1 \times S_2 = \begin{cases} S_1 & \text{if } S_2 = \text{FIN} \\ S_2 & \text{if } S_1 = \text{FIN} \\ (\alpha \cup \beta){:}(S_1' \times S_2') & \text{if } S_1 = \alpha{:}S_1', S_2 = \beta{:}S_2', \alpha \cap_{\mathsf{R}} \beta = \emptyset, S_1' \times S_2' \neq \delta \\ \delta & \text{if } S_1 = \alpha{:}S_1', S_2 = \beta{:}S_2', \alpha \cap_{\mathsf{R}} \beta \neq \emptyset \text{ or } S_1' \times S_2' = \delta \\ \Sigma_{i\in I, j \in J}(S_1^i \times S_2^j) & \text{if } S_1 = \Sigma_{i\in I} S_1^i, S_2 = \Sigma_{j\in J} S_2^j \end{cases}$$

Note that the set of recursive equations used in the definition of $S_1 \times S_2$ may allow more than one solution. Consider, for example, $S_1 = \{\overline{r_1}\} : S_1$ and $S_2 = \{\overline{r_2}\} : S_2$. It is easy to see that $S_1 \times S_2 = \delta$ is a trivial solution. However, we are interested in the maximal solution to this set of equations, which in this case is $S_1 \times S_2 = \{\overline{r_1}, \overline{r_2}\} : S_1 \times S_2$. Intuitively, solutions can be ordered by the set of terms that are set to $\delta$: the fewer terms are deadlocked, the "larger" the solution. We use the following lemma to make this notion precise and show that, for finite-state processes, the maximal solution exists and can be computed iteratively:

**Lemma 3.16.** *Given supplies $S_1$ and $S_2$ the set of equations which arise through $S_1 \times S_2$ has a greatest fixed point.*

PROOF: Consider the term $S_1 \times S_2$. Let $S_{S_1,S_2} = \{S \mid S_1 \Longrightarrow S \text{ or } S_2 \Longrightarrow S\}$ and $S^{\times}_{S_1,S_2} = S_{S_1,S_2} \cup [S_{S_1,S_2} \times S_{S_1,S_2}]$. That is, $S^{\times}_{S_1,S_2}$ is a set containing all derivatives of $S_1$ or $S_2$ and all pairs of these derivatives. For finite-state processes, $S^{\times}_{S_1,S_2}$ is finite.

Consider the set of relations on $S^{\times}_{S_1,S_2}$, $\mathcal{W} = \{W \mid W \subseteq S^{\times}_{S_1,S_2} \times S^{\times}_{S_1,S_2}\}$, ordered by set inclusion. Because $\mathcal{W}$ is a powerset of a finite set, $\mathcal{W}$ is a complete lattice of finite height.

Consider relation $W$ where, if $(w_1, w_2) \in W$, then $w_1 \in [S_{S_1,S_2} \times S_{S_1,S_2}]$ and $w_2 \in S^{\times}_{S_1,S_2}$ appears on the right-hand side of the equation for $w_1$, disregarding the recursive part of the third clause and the fourth clause, so that

- if $S_i = \text{FIN}$ then $(S_1 \times S_2, S_{3-i}) \in W$,
- if $S_1 = \alpha{:}S'_1$, $S_2 = \beta{:}S'_2$ and $(S_1 \times S_2, S'_1 \times S'_2) \in W$, then $\alpha \cap_{\mathsf{R}} \beta \neq \emptyset$,
- if $S_1 = \Sigma_{i\in I} S^i_1$, $S_2 = \Sigma_{j\in J} S^j_2$ and $(S_1 \times S_2, S'_1 \times S'_2) \in W$ then $S'_1 = S^i_1$ for some $i \in I$ and $S'_2 = S^j_2$ for some $j \in J$.

Thus, such a relation $W$ relates a product $S_1 \times S_2$ with some of its possible derivatives according to the selected part of the definition. Further, suppose that whenever $w \in [S_{S_1,S_2} \times S_{S_1,S_2}]$ and there exists $w_1$ such that $(w_1, w) \in W$, then there also exists $w_2$ such that $(w, w_2) \in W$. Then, such $W$ is a fixed point of the set of equations defining $S_1 \times S_2$. This is because, according to the complete definition, $S_1 \times S_2$ has some derivative $w$, if and only if $w$ has a derivative (i.e. $w \neq \delta$).

Define a function $\mathcal{F} : \mathcal{W} \mapsto \mathcal{W}$ as $\mathcal{F}(W) = W - \{(w_1, w_2) \mid w_2 \in [S_{S_1,S_2} \times S_{S_1,S_2}] \wedge \forall w, (w_2, w) \notin W\}$. Since $\mathcal{F}$ can only remove elements from $W$, $\mathcal{F}(W) \leq W$. Furthermore, if $W_1 \leq W_2$, then $\mathcal{F}(W_1) \leq \mathcal{F}(W_2)$; that is, $\mathcal{F}$ is monotonic. Let us construct the set $W_0$ using the definition of $S_1 \times S_2$, again by omitting $S'_1 \times S'_2 \neq \delta$ from clause 3 and $S'_1 \times S'_2 = \delta$ from clause 4. Clearly, any fixed point of $S_1 \times S_2$, $W_{S_1 \times S_2}$ satisfies $W_{S_1\times S_2} \leq W_0$ since fewer terms are set to $\delta$ in $W_0$. Since $\mathcal{W}$ is a complete lattice, by the Tarski-Knaster theorem, the greatest fixed point exists and is unique. Since the lattice is of finite height, the fixed point can be computed starting from $W_0$ and iteratively applying $\mathcal{F}$ until the fixed point is reached. □

It is easy to see that, if $S_1 \| S_2$ does not deadlock then $S_1 \times S_2 \neq \delta$. However, the opposite is not true. By the construction of $\times$, $S_1 \times S_2$ selects the part of the transition system of $S_1 \| S_2$ that does not lead to deadlocked states. For example, consider $S_1 \overset{\text{def}}{=} \{\overline{r}\}{:}\{\overline{r}\}{:}\text{FIN} + \emptyset{:}\{\overline{r}\}{:}\{\overline{r}\}{:}\text{FIN}$ and $S_2 \overset{\text{def}}{=} \emptyset{:}\{\overline{r}\}{:}\text{FIN} + \{\overline{r}\}{:}\emptyset{:}\text{FIN}$. Then, although $S_1 \| S_2 \xrightarrow{\{\overline{r}\}} \{\overline{r}\}{:}\text{FIN} \| \{\overline{r}\}{:}\text{FIN} = \delta$, $S_1 \times S_2 = \{\overline{r}\}{:}(\{\overline{r}\}{:}\{\overline{r}\}{:}\text{FIN} \times \emptyset{:}\text{FIN})$, and $(\{\overline{r}\}{:}\{\overline{r}\}{:}\text{FIN} \times \emptyset{:}\text{FIN}) = \{\overline{r}\}{:}\{\overline{r}\}{:}\text{FIN}$.

**Lemma 3.17.** *If $T_1$ is schedulable by $S_1$, $T_2$ is schedulable by $S_2$ and $S_1 \times S_2 \neq \delta$, then $T_1[1] \| T_2[2]$ is schedulable by $S_1[1] \times S_2[2]$.*

PROOF: The proof is similar to that of Lemma 3.14. □

At this point we turn our attention to the problem of constructing an interface for a set of mutually schedulable tasks. To do this, we employ the notion of demands and we prove the following:

**Lemma 3.18.** *If $w \in \mathsf{traces}(T_1[1] \| T_2[2])$ then there exists a trace $w' \in \mathsf{traces}(\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2]))$ such that $\overline{w} \leq w'$.*

PROOF: Suppose that $w \in \mathsf{traces}(T_1[1] \| T_2[2])$. It is easy to see that $w[1]$ and $w[2]$ give rise to traces of $T_1[1]$ and $T_2[2]$. Then, by Lemma 3.13, there exist $w_1$ and $w_2$ such that $\overline{w[1]} \leq w_1$ and $\overline{w[2]} \leq w_2$ such that $w_1 \in \mathsf{traces}(\mathsf{demand}(T_1[1]))$ and $w_2 \in \mathsf{traces}(\mathsf{demand}(T_2[2]))$. Suppose that $w_1 = \alpha_1\alpha_2\ldots$ and $w_2 = \beta_1\beta_2\ldots$ and write $w' = \gamma_1\gamma_2\ldots$, where $\gamma_i = \alpha_i \cup \beta_i$. Then, from the definition of $\times$ we may conclude that $w'$ a trace of $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2])$, and, in addition, $\overline{w} \leq w'$, as required. □

This result implies that all alternatives of scheduling $T_1[1] \| T_2[2]$ will be explored by $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2])$. It can be extended to the composition of an arbitrary number of tasks. We are now ready to present our main theorem:

**Theorem 3.19.** *$T_1[1] \| T_2[2]$ is schedulable if and only if $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2]) \neq \delta$. Moreover, if it is schedulable, then it is schedulable by $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2])$.*

PROOF: Suppose $T_1[1] \| T_2[2]$ is schedulable. Then, by Lemma 3.9, it has a live $w$-cylinder. Let $w_1$ be the trace of an execution of $T_1[1] \| T_2[2]$ occurring within the cylinder. Then, by Lemma 3.18, there is a trace $w_2$, $\overline{w_1} \leq w_2$ such that $w_2$ is a trace of $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2])$. This implies that $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2]) \neq \delta$. On the other hand, if $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2]) \neq \delta$, then, since, additionally, $\mathsf{demand}(T_1[1])$ schedules $T_1[1]$ and $(T_2[2])$ schedules $T_2[2]$, then, by Lemma 3.17, $T_1[1] \| T_2[2]$ is schedulable by $\mathsf{demand}(T_1[1]) \times \mathsf{demand}(T_2[2])$. □

Based on this result we may determine the schedulability and a related scheduler for a set of tasks $T_1, \ldots, T_n$, as follows: For each task, extract its demand and compute the products $D_1 = \mathsf{demand}(T_1) \times \mathsf{demand}(T_2)$, $D_2 = D_1 \times \mathsf{demand}(T_3), \ldots$. If this process does not reduce to some $D_i = \delta$ then the tasks are schedulable by $D_{n-1}$. Furthermore, according to Theorem 3.19, if they are indeed schedulable then $D_{n-1} \neq \delta$. Thus, this method is guaranteed to produce a schedule if one exists.

## *3.3. Examples*

### *3.3.1. Scheduling periodic tasks*

We first consider a simple periodic task with period $p$ and execution time $w$ which requires usage of some resource $r$, $Task_{w,p} = T_{0,0,w,p}$. This is defined by the following equations where $e$ is the accumulated execution time of resource $r$ during the current period and $t$ the total elapsed time during the current period, and $\pi$ is the priority of the resource access.

$$T_{e,t,w,p} = \begin{cases} \emptyset : T_{e,t+1,w,p} & \text{if } e = w \text{ and } t < p \\ T_{0,0,w,p} & \text{if } e = w \text{ and } t = p \\ \emptyset : T_{e,t+1,w,p} + \{(r,\pi)\} : T_{e+1,t+1,w,p} & \text{if } e < w \text{ and } w - e < p - t \\ \{(r,\pi)\} : T_{e+1,t+1,w,p} & \text{if } e < w \text{ and } w - e = p - t \end{cases}$$

Note that in our definition, the task cannot idle if idling will make it miss the deadline. If the supply can avoid giving the resource to the task in this case, the system will have an unmet resource request transition that signals non-schedulability (by Definition 3.1). Let us consider an instance of a classical scheduling problem for a set of periodic tasks running on a single processor resource: $Task_{2,3} \| Task_{2,7} \| S$, where

$S = \{\overline{r}\} : S$. In the figure below, we show the initial part of the state space of the example where we assume that the priorities of all resource requests in both tasks are the same and equal to 1. Each state is represented as a tuple $ij|km$, where $i$ and $j$ are the first two parameters of the first task and $k$ and $m$ are the first two parameters of the second task. The other two parameters do not change and are omitted to avoid cluttering the figure. We also omit labels on the transitions: all transitions are labeled by $\{(\overset{\leftrightarrow}{r}, 1)\}$.

$$\begin{tikzcd}
00|00 \arrow[r] \arrow[dr] & 11|01 \arrow[r] \arrow[dr] & 22|02 \arrow[r] & 00|13 \arrow[r] \arrow[dr] & 11|14 \arrow[r] \arrow[dr] & 22|15 \arrow[r] & 00|26 \arrow[r] & 11|00 \arrow[r] \arrow[dr] & 22|01 \arrow[r] & 00|12 \arrow[r] \arrow[dr] & 11|13 \arrow[r] \arrow[dr] & \cdots \\
 & 01|11 \arrow[r] & 12|12 \arrow[ur] & & 01|24 \arrow[r] & 12|25 \arrow[ur] & & & 12|11 \arrow[ur] & & 01|23 \arrow[r] & \cdots
\end{tikzcd}$$

The tasks are schedulable according to the Definition 3.1 and the transition system of the composite process, shown above, can be seen as the specification of feasible schedulers for the task set. Nondeterminism in the transition system represents different decisions that a scheduler can make. For example, the trace along the top of the figure corresponds to the rate-monotonic scheduling policy, which gives higher priority to $Task_{2,3}$ as it has the smallest period. Indeed, to consider schedulability under a specific scheduling policy, we would simply need to specify the appropriate priorities and check for the schedulability of the system within the new transition system.

We now consider the demand of a periodic task defined above. It is easy to see that the task process is *deterministic*, that is, whenever $T_{e,t,w,p} \xrightarrow{\alpha} T_1$ and $T_{e,t,w,p} \xrightarrow{\alpha} T_2$ then $T_1 = T_2$. For a deterministic task, the demand is obtained by a straightforward replacement of requested resources by matching offered resources. Thus, $\mathsf{demand}(Task_{w,p}) = X_{0,0,w,p}$ is defined below:

$$X_{e,t,w,p} = \begin{cases} \emptyset : X_{e,t+1,w,p} & \text{if } e = w, t < p \\ X_{0,0,w,p} & \text{if } e = w, t = p \\ \emptyset : X_{e,t+1,w,p} + \{\overline{r}\} : X_{e+1,t+1,w,p} & \text{if } e < w, w - e < p - t \\ \{\overline{r}\} : X_{e+1,t+1,w,p} & \text{if } e < w, w - e = p - t \end{cases}$$

It is easy to check that $\mathsf{demand}(Task_{2,3}) \| \mathsf{demand}(Task_{2,7})$ does not deadlock and thus can schedule the two tasks according to Lemma 3.14.

Let us now consider a task with variable execution time which takes between $b$ and $w$ time units to complete: $Task^v_{b,w,p} = Task_{b,p} + Task_{b+1,p} + \ldots + Task_{w,p}$. One can see that $\mathsf{demand}(Task^v_{b,w,p}) = \mathsf{demand}(Task_{w,p})$. This observation matches the well-known fact from the real-time systems theory that for independent periodic tasks it is sufficient to consider the worst-case execution time of each task [17].

### *3.3.2. Scheduling with partial supplies*

To illustrate compositional analysis with partial supplies, we begin with a simple example of time-partitioned supplies that are widely used in practice. Consider a periodic time partition with period $P$, duration $D \le P$, and relative start time $t_0$, which essentially offers a resource $r$ for the interval $[t_0, t_0 + D)$ during each period: $Part_{t_0,D,P} = P_{0,t_0,D,P}$ is defined as follows where, again, addition is modulo $P$:

$$P_{t,t_0,D,P} = \begin{cases} \{\overline{r}\} : P_{t+1,t_0,D,P} & \text{if } t_0 \le t < t_0 + D \\ \emptyset : P_{t+1,t_0,D,P} & \text{otherwise} \end{cases}$$

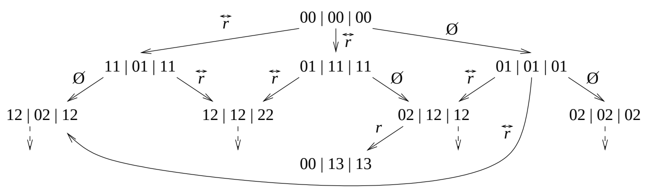


Figure 3: Scheduling with a periodic resource

It is clear that partitions with the same period and non-overlapping service intervals $[t, t + D)$ do not conflict. We can now analyze schedulability of tasks allocated to a partition separately from any other task in the system. It is, for example, trivial to see that partition $Part_{t_0,D,P}$ can schedule a task $Task_{D,P}$ for any $t_0$.

We can similarly define more complex partial supplies. Consider, for example, compositional scheduling based on periodic resource models [23, 24]. A periodic resource model is a supply that guarantees $w$ units of resource execution within a period $P$, however, the availability of the resource within the period is unknown *a priori*. We can straightforwardly model a periodic resource model as $PRM_{w,P} = \mathsf{demand}(Task_{w,P})$. We can then analyze whether a set of tasks is schedulable with respect to this supply. This analysis will not be limited to independent periodic or sporadic tasks, unlike existing approaches in the literature.

As an example, consider the system $T_1 = Task_{1,3} \| Task_{1,5} \| PRM_{3,5}$ where all priorities of resource requests are fixed to 1. Figure 3 shows the initial state space using the same notation as above, except now the state tuple also includes the parameters $e$ and $t$ of the supply. Note that, in this transition system we have actions pertaining to resource consumption, abbreviated by $\overset{\leftrightarrow}{r}$, actions pertaining to resource requests, abbreviated by $r$, and idling actions. Recall that idling and consumed resource actions are incomparable in the preemption relation, while idling preempts unsatisfied resource requests. We see that a poor scheduling decision can make $Task_{1,3}$ miss its deadline. The scenario is seen on the right side of the figure: in the first two time units, one unit of resource goes to $T_{1,5}$ and the other unit of resource is denied to both tasks (this can happen in any order). If on the third step the supply denies access to the resource again, the first task cannot idle, thus we reach a transition labeled by $\{r\}$, which implies that the task misses its deadline, leading to a violation of Definition 3.1.

If instead we wish to consider schedulability of the tasks under an EDF (earliest-deadline-first) policy, we would have to repeat our analysis for periodic tasks with priorities defined as below.

$$T_{e,t,w,p} = \begin{cases} \emptyset : T_{e,t+1,w,p} & \text{if } e = w, t < p \\ T_{0,0,w,p} & \text{if } e = w, t = p \\ \emptyset : T_{e,t+1,w,p} + \{(r, D_{max} - (p - t))\} : T_{e+1,t+1,w,p} & \\ & \text{if } e < w, w - e < p - t \\ \{(r, D_{max} - (p - t))\} : T_{e+1,t+1,w,p} & \text{if } e < w, w - e = p - t \end{cases}$$

where $D_{max}$ is a number exceeding the largest period in the task set. In this new setting, the composition $T_2 = Task_{1,3} \| Task_{1,5} \| PRM_{3,5}$, where $D_{max} = 6$, is schedulable

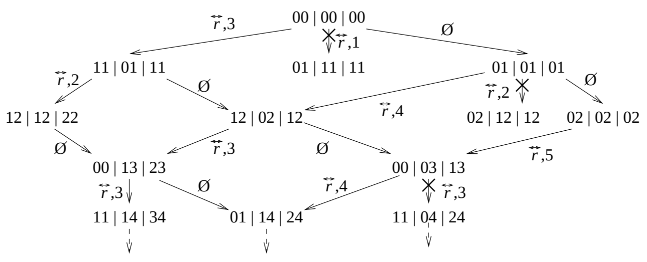

00 | 00 | 00
$\overleftrightarrow{r}$,3
$\overleftrightarrow{r}$,1
Ø
11 | 01 | 11
01 | 11 | 11
01 | 01 | 01
$\overleftrightarrow{r}$,2
Ø
$\overleftrightarrow{r}$,2
Ø
12 | 12 | 22
12 | 02 | 12
$\overleftrightarrow{r}$,4
02 | 12 | 12
02 | 02 | 02
Ø
$\overleftrightarrow{r}$,3
Ø
$\overleftrightarrow{r}$,5
00 | 13 | 23
00 | 03 | 13
$\overleftrightarrow{r}$,3
Ø
$\overleftrightarrow{r}$,4
$\overleftrightarrow{r}$,3
11 | 14 | 34
01 | 14 | 24
11 | 04 | 24

Figure 4: EDF scheduling with a periodic resource

as shown in Figure 4. In the figure, preempted transitions are crossed out. Note that the problematic action $r$ of the previous example is no longer present because, in the initial state, action $(\overleftrightarrow{r}, 1)$ is preempted by action $(\overleftrightarrow{r}, 3)$, and from state $01|01|01$ action $(\overleftrightarrow{r}, 2)$ is preempted by action $(\overleftrightarrow{r}, 4)$, and thus the trees pointed to by these preempted actions in the transition system are pruned away, including the request action $r$.

## 4. Hierarchies on tasks and supplies

In the previous section we defined an approach for scheduling a set of tasks via analysis of their demand processes which are supply processes capturing the precise resource allocation required by tasks to complete their execution. In this section we proceed to provide machinery that will allow us to reason about hierarchical approaches to scheduling that rely on approximating the necessary supply, making it more generous than necessary, in exchange to a simple representation. Specifically, we define an ordering relation between tasks and two ordering relations between supplies which describe when a task/supply requires/offers greater resource allocation than another.

### *4.1. Task demands*

We proceed to consider the notion of *task demand* and we define a relation on tasks which characterizes when a task is more "demanding" than another in the sense that it places more requirements on the available supply.

**Definition 4.1.** *A relation $\mathcal{D} \in \mathsf{T} \times \mathsf{T}$ is a* demand relation *if for all $(T_1, T_2) \in \mathcal{D}$, if $T_1 \xrightarrow{\alpha}$ then*

1. *there exist $T_2 \xrightarrow{\beta} T_2'$ with $\mathsf{sat}(\beta, \alpha)$, and $T_1 \xrightarrow{\alpha} T_1'$, such that $(T_1', T_2') \in \mathcal{D}$,*
2. *for all $T_2 \xrightarrow{\beta} T_2'$, if $\beta \trianglelefteq_{T_2} \alpha \cup \beta$, $\alpha \trianglelefteq_{T_1} \alpha \cup \beta$ and for no $\gamma$, $T_1 \xrightarrow{\gamma}$ and $\beta \trianglelefteq_{T_2} \gamma$ and $\gamma \trianglelefteq_{T_1} \alpha \cup \beta$, then there exists $T_1 \xrightarrow{\alpha} T_1'$ such that $(T_1', T_2') \in \mathcal{D}$.*

*We write $\preceq_D$ for the largest demand relation and we say that a task $T_1$ is* more demanding *than a task $T_2$, $T_2 \preceq_D T_1$, if there exists a demand relation $\mathcal{D}$ with $(T_1, T_2) \in \mathcal{D}$.*

According to this definition, if $T_1$ is more demanding than $T_2$ then for every action $\alpha$ enabled by $T_1$, (1) there is a move of $T_2$ which can be matched by some $\alpha$-move of $T_1$, (2) if $\beta$ and $\alpha$ are maximal moves of $T_2$ and $T_1$, respectively, with respect to $\alpha \cup \beta$, and, additionally, there is no $\gamma$-move of $T_1$ such that $\beta$ is a maximal move of $T_2$ with respect to $\gamma$ and $\gamma$ is a maximal move of $T_1$ with respect $\alpha \cup \beta$, then all $\beta$-derivatives of $T_2$ are related to some $\alpha$-derivative of $T_1$.

To better understand the definition, let us first consider the point relating to the *existence* of an $\alpha$ move of $T_1$ (instead of universality) as required by the first clause: let

$$\begin{aligned} T_1 &\stackrel{\text{def}}{=} \{(r_1,1)\} : \{(r_2,2)\} : \text{FIN} + \{(r_1,1)\} : \{(r_3,1)\} : \text{FIN} \\ T_2 &\stackrel{\text{def}}{=} \{(r_1,1)\} : \{(r_2,1)\} : \text{FIN} \end{aligned}$$

Although $T_2$ cannot match the second summand of $T_1$, it is intuitive that $T_1$ should be considered as more demanding than $T_2$. This is because for $T_1$ to be scheduled successfully it is imperative that after being offered $r_1$ it will be offered simultaneously both $r_2$ and $r_3$. Thus, it is sufficient for $T_2$ to match one of the $\{(r_1,1)\}$ actions of $T_1$.

The second clause of the definition is concerned with combinations of actions $\alpha \cup \beta$ where $T_1 \xrightarrow{\alpha}$ and $T_2 \xrightarrow{\beta}$, and it aims to ensure that, if a supply offers the resources in $\alpha \cup \beta$ then, if it is able to schedule the $\alpha$-continuation of $T_1$ it should also be able to schedule the $\beta$ continuation of $T_2$, that is, $T_1$ should continue to be more demanding than $T_2$. Clause (2) of the definition enunciates this requirement assuming that $\alpha$ and $\beta$ are maximal actions with respect to $\alpha \cup \beta$, since this is necessary for them to constitute relevant responses to a supply of $\alpha \cup \beta$ and furthermore, that no action of $T_1$, $\gamma$, lies between $\beta$ and $\alpha \cup \beta$, since, if such as $\gamma$ exists and $T_1 \xrightarrow{\gamma} T_1'$, it is sufficient that $T_1'$ is more demanding than $T_2'$. For example, for

$$\begin{aligned} T_1 &\stackrel{\text{def}}{=} \{(r_1,1),(r_3,1)\} : \{(r_2,1),(r_3,1)\} : \text{FIN} \\ T_2 &\stackrel{\text{def}}{=} \{(r_2,1)\} : \{(r_2,0)\} : \text{FIN} + \{(r_1,1),(r_3,1)\} : \{(r_3,0)\} : \text{FIN} \end{aligned}$$

we may check that, according to the definition, $T_1$ is more demanding than $T_2$. Note that supply $S \stackrel{\text{def}}{=} \{\overline{r_1},\overline{r_2},\overline{r_3}\} : \{\overline{r_2},\overline{r_3}\} : \text{FIN}$, schedules both tasks. Moreover, for

$$\begin{aligned} T_1 &\stackrel{\text{def}}{=} \{(r_1,1),(r_2,1),(r_3,1)\} : \{(r_2,1),(r_4,1)\} : \text{FIN} \\ &\quad + \{(r_1,1),(r_3,1),(r_4,1)\} : \{(r_2,1),(r_3,1)\} : \text{FIN} \\ T_2 &\stackrel{\text{def}}{=} \{(r_1,1)\} : \{(r_2,0)\} : \text{FIN} + \{(r_1,1),(r_2,1)\} : \{(r_2,1),(r_4,1)\} : \text{FIN} \end{aligned}$$

if we we apply the definition of a demand relation and take $\alpha = \{(r_1,1),(r_3,1),(r_4,1)\}$ and $\beta = \{(r_1,1),(r_2,1)\}$ it is not necessary for the $\alpha$-derivative of $T_1$ to be more generous than the $\beta$-derivative of $T_2$, which is not. This is because $T_1 \xrightarrow{\gamma}$, where $\gamma = \{(r_1,1),(r_2,1),(r_3,1)\}$ and $\beta \trianglelefteq_{T_2} \gamma$ and $\gamma \trianglelefteq_{T_1} \alpha \cup \beta$. Nonetheless, $T_1$ is *more* demanding than $T_2$ according to our definition and for a supply that offers the resources $\alpha \cup \beta$, if it may schedule $T_1$ then it must schedule both its $\alpha$ and $\gamma$ derivatives and, consequently, it must also schedule $T_2$.

Some further examples follow:

**Example 4.2.** Consider the following tasks.

$$\begin{array}{lcl}
T_1 & \overset{\text{def}}{=} & \{(r,1)\} : \emptyset : T_1 \\
T_2 & \overset{\text{def}}{=} & \{(r,1)\} : \emptyset : T_2 + \emptyset : \{(r,1)\} : T_2 \\
T_3 & \overset{\text{def}}{=} & \{(r,1)\} : \emptyset : \emptyset : T_3 + \emptyset : \{(r,1)\} : \emptyset : T_3 + \emptyset : \emptyset : \{(r,1)\} : T_3 \\
T_4 & \overset{\text{def}}{=} & \{(r,1)\} : \emptyset : \emptyset : T_4 + \emptyset : (\{(r,1)\} : \emptyset : T_4 + \emptyset : \{(r,1)\} : T_4) \\
T_5 & \overset{\text{def}}{=} & \emptyset : \emptyset : \{(r,1)\} : T_5 \\
T_6 & \overset{\text{def}}{=} & \emptyset : \emptyset : \{(r,1)\} : T_6 + \{(r,1)\} : \{(r,1)\} : \{(r,1)\} : T_6
\end{array}$$

$T_1$ and $T_2$ request resource $r$ once in every two time units with the distinction that $T_1$ requires the resource during the first time unit whereas $T_2$ is satisfied with an allocation during either time units. We may verify that $T_1$ is more demanding than $T_2$. Note that action $T_2 \xrightarrow{\emptyset}$ need not be matched by $T_1$ since, according to the definition, it is not a maximal move of $T_2$ with respect to $\emptyset \cup \{(r,1)\}$.

Moving on to tasks $T_3$ and $T_4$ we observe that they both require resource $r$ once in every three time units but they pose slightly different nondeterministic requirements: $T_3$ is defined as the nondeterministic choice between the options of using $r$ during one of the first three time units, whereas $T_4$ initially offers the choice between acquiring the resource and idling for two time units or idling and then acquiring the resource during one of the next two time units. We may check that $T_1$ is more demanding than both tasks $T_3$ and $T_4$ which demand $r$ once every three time units. In addition, $T_2$ is more demanding that $T_4$ but not of $T_3$ since $T_3$ may choose to respond to an initial $\emptyset$ action with the third summand which is *not* less demanding than $T_2$ given that it requests resource $r$ during the third time unit. A comparison between $T_3$ and $T_4$ shows that $T_3$ is more demanding than $T_4$. Finally, note that task $T_5$ is *not* more demanding than task $T_6$. Intuitively, we can see that task $T_5$ can be scheduled by supply $S \overset{\text{def}}{=} \{\overline{r}\} : \emptyset : \{\overline{r}\} : S$ but task $T_6$ cannot. Furthermore, according to the definition, action $T_6 \xrightarrow{\{(r,1)\}}$ needs to be examined as it is a maximal action of $T_6$ with respect to $\emptyset \cup \{(r,1)\}$ and clearly one that illustrates the absence of a demand relation between the two tasks. This example brings out the subtle treatment required for the actions of the less demanding task. □

We now proceed to justify our notion of *more demanding*. To begin with we may easily prove that $\preceq_D$ is reflexive and transitive. Furthermore, we may verify that more demanding tasks place more requirements on their supplies by proving that if task $T$ is more demanding than task $T'$ then a supply that can schedule $T$ can also schedule $T'$.

**Lemma 4.3.** *Suppose that task $T_1$ is schedulable by supply $S$ and that $T_1$ is more demanding than $T_2$. Then, task $T_2$ is also schedulable by supply $S$.*

PROOF: The proof consists of showing that the relation

$$\begin{array}{rcl}
\mathcal{S} & = & \{(T_2, S) \mid \exists \text{ demand relation } \mathcal{D}, \text{ supply simulation relation } \mathcal{R} \text{ and} \\
& & \qquad T_1 \in \mathsf{T}, (T_1, T_2) \in \mathcal{D}, (T_1, S) \in \mathcal{R}\}
\end{array}$$

is a supply simulation. Suppose $(T_2, S) \in \mathcal{S}$ and $T_1$ is a task such that $(T_1, T_2) \in \mathcal{D}$, where $\mathcal{D}$ is a demand relation, and $(T_1, S) \in \mathcal{R}$, where $\mathcal{R}$ is a supply simulation relation. Suppose $S \xrightarrow{\alpha} S'$. We confirm that the two clauses of Definition 3.5 are satisfied as follows:

- Since $(T_1, S) \in \mathcal{S}$, there exists $T_1'$ with $T_1 \xrightarrow{\beta} T_1'$, $\beta \trianglelefteq_{T_1} \alpha$ and $(T_1', S') \in \mathcal{R}$. Then, by clause (1) of Definition 4.1, there exists $T_2'$, such that $T_2 \xrightarrow{\beta'} T_2'$ with $\mathsf{sat}(\beta', \beta)$, and for some $T_1''$, $T_1 \xrightarrow{\beta} T_1''$, $(T_1'', T_2') \in \mathcal{D}$. By Definition 3.5 it is also the case that $(T_1'', S') \in \mathcal{R}$, while, clearly, $\mathsf{sat}(\beta', \alpha)$. This implies that $(T_2', S') \in \mathcal{S}$ as required.

- Next suppose $T_2 \xrightarrow{\beta_2} T_2'$. Two cases exist:

  If there exists $T_1 \xrightarrow{\gamma} T_1'$, $\gamma \trianglelefteq_{T_1} \alpha$ and $\beta_2 \trianglelefteq_{T_2} \gamma$. Then $(T_1', S') \in \mathcal{R}$ and $(T_1', T_2') \in \mathcal{D}$. Thus, $(T_2', S) \in \mathcal{S}$ as required.

  Now suppose there exists no $T_1 \xrightarrow{\gamma} T_1'$, $\gamma \trianglelefteq_{T_1} \alpha$ and $\beta_2 \trianglelefteq_{T_2} \gamma$. Nonetheless, since $S$ schedules $T_1$, there exists $T_1 \xrightarrow{\beta_1} T_1'$, $\beta_1 \trianglelefteq_{T_1} \alpha$ and $(T_1', S') \in \mathcal{R}$. Now consider $\beta_1 \cup \beta_2$. It must be the case that both $\beta_2 \trianglelefteq_{T_2} \beta_1 \cup \beta_2$ and $\beta_1 \trianglelefteq_{T_1} \beta_1 \cup \beta_2$, otherwise we would have contradictions to our assumptions that $\beta_2 \trianglelefteq_{T_2} \alpha$ and $\beta_1 \trianglelefteq_{T_1} \alpha$. Now, suppose there exists $\gamma$ such that $T_1 \xrightarrow{\gamma}$ with $\beta_2 \trianglelefteq_{T_2} \gamma$ and $\gamma \trianglelefteq_{T_1} \beta_1 \cup \beta_2$. Since $\gamma \trianglelefteq_{T_1} \beta_1 \cup \beta_2$, $\mathsf{sat}(\beta_1, \alpha)$ and $\mathsf{sat}(\beta_2, \alpha)$, we have that $\mathsf{sat}(\gamma, \alpha)$, which implies that either $\gamma \trianglelefteq_{T_1} \alpha$ or, if not, there exists $\gamma'$ with $\gamma \prec \gamma'$, $\mathsf{sat}(\gamma', \alpha)$ and $\gamma' \trianglelefteq_{T_1} \alpha$ while $\beta \trianglelefteq_{T_2} \gamma'$. This contradicts the assumption of the case and it implies that there exists no $\gamma$ as the one just described and, consequently, by Definition 4.1(2), there exists $T_1''$ such that $T_1 \xrightarrow{\alpha} T_1''$ and $(T_1'', T_2') \in \mathcal{D}$. By Definition 3.5 it is also the case that $(T_1'', S') \in \mathcal{R}$. Thus $(T_2', S') \in \mathcal{S}$ which completes the proof.

□

### *4.2. Supply generosity*

Similarly to demands, we now proceed to define a hierarchy on supplies. This hierarchy is built on the basis of simulation relations that capture when a supply is more "generous" than another, where the intended meaning of "generosity" is that the more generous a supply the more tasks it can schedule. Below we define two such notions.

#### *4.2.1. Strong generosity*

**Definition 4.4.** *A relation $\mathcal{R} \in \mathsf{S} \times \mathsf{S}$ is a* strong generosity relation *if for all $(S_1, S_2) \in \mathcal{R}$,*

1. *if $S_2 \longrightarrow$ then $S_1 \longrightarrow$.*
2. *if $S_2 \longrightarrow$ and $S_1 \xrightarrow{\alpha} S_1'$ then we have that $S_2 \xrightarrow{\alpha} S_2'$ and $(S_1', S_2') \in \mathcal{R}$.*

*We write $\preceq_S$ for the largest strong generosity relation and we say that supply $S_1$ is* strongly more generous than *supply $S_2$, $S_2 \preceq_S S_1$, if there exists $\mathcal{R}$ with $(S_1, S_2) \in \mathcal{R}$.*

According to the definition, $S_1$ is strongly more generous than $S_2$ if: (1) whenever $S_2$ is not deadlocked then $S_1$ is also not deadlocked, and (2) whenever $S_2$ is not deadlocked then any action enabled by $S_1$ is also enabled by $S_2$. Intuitively, this definition aims to establish that any task scheduled by the less generous supply, $S_2$, can also be scheduled by the more general supply, $S_1$. To implement this, $S_1$ is required to offer a subset of the behaviors of $S_2$, in this way it is guaranteed that each of $S_1$'s executions is also possible in $S_2$ and, thus, any task schedulable by $S_2$ will be schedulable by $S_1$. Thus, in Example 3.2, $S_1$ is a strongly more generous supply than $S_3$.

Note that the notion of strong generosity captures an earlier observation that the introduction of nondeterministic alternatives in supplies diminishes their potential of scheduling tasks. This is because, as viewed by a task, a supply with more choices constitutes an environment with more uncertainty, and the more ways in which a supply may offer resources implies a need for greater flexibility on behalf of a task. As an example consider

$$T \overset{\text{def}}{=} \emptyset : \{(r,1)\} : \text{FIN} + \{(r,1)\} : \{(r,2)\} : \{(r,1)\} : \text{FIN}$$

and

$$S_1 \overset{\text{def}}{=} \emptyset : \{\overline{r}\} : \text{FIN} + \{\overline{r}\} : \{\overline{r}\} : \text{FIN}, \quad S_2 \overset{\text{def}}{=} \emptyset : \{\overline{r}\} : \text{FIN}$$

Although $S_2$ can schedule $T$, this is not the case with $S_1$. The same is true in the case that we allow a supply to offer a wider range of resources. For example, $S_1' \overset{\text{def}}{=} \{\overline{r}\} : \{\overline{r}\} : \text{FIN}$ also fails to schedule task $T$.

It it easy to show that $\preceq_S$ is reflexive and transitive. Furthermore, the following result establishes that generosity preserves schedulability.

**Lemma 4.5.** *If task $T$ is schedulable by supply $S_2$ and $S_1$ is strongly more generous than supply $S_2$ then $T$ is also schedulable by supply $S_1$.*

PROOF: The proof consists of showing that the relation

$$\mathcal{S} = \{(T, S_1) | \exists\, S_2 \in \mathsf{S}, \text{supply simulation relation } \mathcal{R} \text{ and strong generosity relation } \mathcal{G}, (S_1, S_2) \in \mathcal{G} \text{ and } (T, S_2) \in \mathcal{R}\}$$

is a supply simulation relation. Suppose $(T, S_1) \in \mathcal{S}$ and $S_2$ is a supply such that $(T, S_2) \in \mathcal{R}$, where $\mathcal{R}$ is a supply simulation relation and $(S_1, S_2) \in \mathcal{G}$ where $\mathcal{G}$ is a strong generosity relation. Suppose $S_1 \xrightarrow{\alpha} S_1'$. By Definition 4.4(2), $S_2 \xrightarrow{\alpha} S_2'$ with $(S_1', S_2') \in \mathcal{G}$. Thus:

1. There exists $T \xrightarrow{\beta} T'$, $\beta \trianglelefteq_T \alpha$ with $(T', S_2') \in \mathcal{R}$. By definition, $(T', S_1') \in \mathcal{S}$ as required.
2. Suppose $T \xrightarrow{\beta} T'$, $\beta \trianglelefteq_T \alpha$. Again we have $(T', S_2') \in \mathcal{R}$ and $(T', S_1') \in \mathcal{S}$ which completes the proof. □

In fact, we can also show that:

**Lemma 4.6.** *$S_1$ is strongly more generous than $S_2$ if and only if each task schedulable by supply $S_2$ is also schedulable by supply $S_1$.*

PROOF: The '$\Rightarrow$' direction follows by the previous lemma. To demonstrate the '$\Leftarrow$' direction we will show that the following relation is a strong generosity relation.

$$\mathcal{R} = \{(S_1, S_2) | \forall T \cdot \ T \text{ schedulable by } S_2 \implies T \text{ schedulable by } S_1\}$$

Suppose $(S_1, S_2) \in \mathcal{R}$. We have the following:

1. Suppose $S_2 \xrightarrow{\alpha} S_2'$ and consider the set of tasks $\{\alpha : T | T \text{ schedulable by} S_2'\}$. Then, this set, being schedulable by $S_2$, is also schedulable by $S_1$, which implies that $S_1 \longrightarrow$ by Definition 3.5, as required.
2. Suppose $S_1 \xrightarrow{\alpha} S_1'$ and in order to reach a contradiction suppose further that $S_2 \not\xrightarrow{\alpha}$. Consider task $T \overset{\text{def}}{=} \sum_{S_2 \xrightarrow{\alpha_i}} \alpha_i \ : \ \text{FIN} + \alpha \ : \ T'$ where $T'$ is not schedulable by $S_1'$ nor by any of $S_2$'s derivatives. Then $T$ is schedulable by $S_2$ but not $S_1$, resulting in a contradiction. This implies that $S_2 \xrightarrow{\alpha} S_2'$ and $(S_1', S_2') \in \mathcal{R}$ as required. $\square$

As an example for strong generosity consider supplies $S_1$ and $S_2$ below

$$\begin{aligned} S_1 &\overset{\text{def}}{=} \{\overline{r}\} : \{\overline{r}\} : \emptyset : S_1 \\ S_2 &\overset{\text{def}}{=} \{\overline{r}\} : (\{\overline{r}\} : \emptyset : S_2 + \emptyset : \{\overline{r}\} : S_2) + \emptyset : \{\overline{r}\} : \{\overline{r}\} : S_2 \end{aligned}$$

where $S_1$ offers supply $r$ during the first two out of every three units of execution and $S_2$ offers $r$ for two out of every three time units where the precise timing of the offerings is nondeterministic. We may easily verify that $S_1$ is more generous than $S_2$ and, as such, it may schedule at least as many tasks as $S_2$. Thus, the deterministic nature of $S_1$ makes it more generous than $S_2$.

Generalizing this example, we may also see that a periodic time partition with period $P$, duration $D \leq P$, and relative start time $t_0$, $Part_{t_0,D,P}$, defined in Section 3.3.2, is strongly more generous than the periodic resource model $PRM_{D,P}$ that guarantees $D$ time units of resource usage within every period $P$. The former presents one of the possible behaviors of the latter, this making it more generous, and able to schedule at least as many tasks.

### *4.2.2. Weak generosity*

It turns out that the definition of strong generosity prevents us from comparing other supply models which one might be interested in comparing. For instance, supply $S_1$ above which offers a resource during the first two out of every three time units, would be intuitively considered as being more generous than supply $S_3 \overset{\text{def}}{=} \{\overline{r}\} : \emptyset : \emptyset : S_3$. However, $S_1$ is not strongly more generous than $S_3$, according to our definition and, for instance, although $S_1$ offers more resources than $S_3$ it fails to schedule task $T$ below which is in fact schedulable by the more stingy $S_3$:

$$T \overset{\text{def}}{=} \{(r, 0)\} : [\emptyset : \emptyset : \text{FIN} + \{(r, 0)\} : \{(r, 0)\} : \text{FIN}]$$

Nonetheless, we would like to relax the notion of supply generosity to encompass a wider set of supplies at the expense of Lemma 4.5. Specifically, below we define a weaker notion of generosity which is subsequently considered within a restricted class of tasks. This definition is as follows.

**Definition 4.7.** *A relation $\mathcal{R} \in \mathsf{S} \times \mathsf{S}$ is a* weak generosity relation *if for all $(S_1, S_2) \in \mathcal{R}$,*

1. *if $S_2 \longrightarrow$ then $S_1 \longrightarrow$.*
2. *if $S_2 \longrightarrow$ and $S_1 \xrightarrow{\alpha} S_1'$ then we have that $S_2 \xrightarrow{\beta} S_2'$, $\beta \subseteq \alpha$ and $(S_1', S_2')$.*

*We write $\preceq_W$ for the largest weak generosity relation and we say that supply $S_1$ is* weakly more generous than *supply $S_2$, $S_2 \preceq_W S_1$, if there exists a weak generosity relation $\mathcal{R}$ with $(S_1, S_2) \in \mathcal{R}$.*

This definition follows along the lines of that of strong generosity with the exception that we allow the less generous supply $S_2$ to match the supply of $S_1$ with a subset of its resources $\beta \subseteq \alpha$. Although we have shown that in this case $S_1$ is not guaranteed to schedule all tasks schedulable by $S_2$, this new notion allows to explore the intuition that offering more resources makes for more generous supplies. The following hold:

- Supplies $S_1$ and $S_3$ considered above are such that $S_1$ is weakly more generous than $S_3$.
- The partial supply $Part_{t_0,D,P}$ is weakly more generous than the partial supply $Part_{t_0,D',P}$, where $D' \leq D$.
- The periodic resource model $PRM_{w,P}$, defined in Section 3.3.2, is weakly more generous than the periodic resource model $PRM_{w',P}$, $w' \leq w$.
- The periodic resource model $PRM_{2,4}$ is *not* weakly more generous than the periodic resource model $PRM_{1,2}$. We may confirm this by considering the execution $PRM_{2,4} \xrightarrow{\{\overline{r}\}}\xrightarrow{\{\overline{r}\}}\xrightarrow{\emptyset}\xrightarrow{\emptyset} PRM_{2,4}$ and observing that it cannot be matched by $PRM_{1,2}$ as required by the definition of weak generosity. Note that task $Task_{1,2}$ is schedulable by supply $PRM_{1,2}$ but it is *not* schedulable by $PRM_{2,4}$.

Regarding the ability of weakly more generous supplies to schedule tasks we have the following result. Consider the class of periodic tasks $\mathcal{C}$ with period $p$ and execution time $w$, $Task_{w,p}$, defined in Section 3.3.1. We may prove that:

**Lemma 4.8.** *If task $T \in \mathcal{C}$ is schedulable by supply $S_2$ and $S_1$ is weakly more generous than supply $S_2$, then $T$ is also schedulable by supply $S_1$.*

PROOF: The proof consists of showing that the following relation is a supply simulation relation.

$$\begin{aligned}\mathcal{R} = \{(T_{e,t,w,p}, S_1) \ \mid \ & \exists S_2 \in \mathsf{S}, \text{ supply simulation relation } \mathcal{S} \text{ and weak generosity} \\ & \text{relation } \mathcal{W} \cdot (S_1, S_2) \in \mathcal{W} \text{ and } (T_{e',t,w,p}, S_2) \in \mathcal{S}, \text{ for some } e' \leq e\}\end{aligned}$$

So, consider $(T_{e,t,w,p}, S_1) \in \mathcal{R}$ and suppose there exist a supply $S_2$, a supply simulation relation $\mathcal{S}$ and a weak generosity relation $\mathcal{W}$, such that $(S_1, S_2) \in \mathcal{W}$ and $(T_{e',t,w,p}, S_2) \in \mathcal{S}$ for some $e' \leq e$. Suppose $S_1 \xrightarrow{\alpha} S_1'$. We will show that $T_{e,t,w,p} \xrightarrow{\beta} T$ where $\beta \trianglelefteq_{T_{e,t,w,p}} \alpha$ and $(T, S_1') \in \mathcal{R}$. First note that since $S_1 \xrightarrow{\alpha} S_1'$ and $(S_1, S_2) \in \mathcal{W}$, $S_2 \xrightarrow{\gamma} S_2'$, $\gamma \subseteq \alpha$ and $(S_1', S_2') \in \mathcal{W}$. The following cases exist:

- If $e = w$, $e = e'$ and $t < p$, then $T_{e',t,w,p} \xrightarrow{\emptyset} T_{e',t+1,w,p}$, $T_{e,t,w,p} \xrightarrow{\emptyset} T_{e,t+1,w,p}$, $(T_{e',t,w,p}, S'_2) \in \mathcal{S}$, and, thus, $(T_{e,t+1,w,p}, S'_1) \in \mathcal{R}$ as required.
- If $e = w$, $e < e'$ and $t < p$, then $T_{e',t,w,p} \xrightarrow{\beta'} T_{e'',t+1,w,p}$, where $e'' \in \{e', e' + 1\}$, depending on whether $\overline{r} \in \gamma$. In any case, $e'' \leq e$, $T_{e,t,w,p} \xrightarrow{\emptyset} T_{e,t+1,w,p}$, $(T_{e',t,w,p}, S'_2) \in \mathcal{S}$, and, thus, $(T_{e,t+1,w,p}, S'_1) \in \mathcal{R}$ as required.
- If $e = w$ and $t = p$, then since $T_{e',t,w,p}$ is schedulable by $S_2$ it must be that $e' = w$ and the proof follows as in the next case.
- If $e < w$, $w - e < p - t$, $w - e' < p - t$ the following cases exist. If $\overline{r} \in \gamma$, then $\overline{r} \in \alpha$ and $T_{e',t,w,p} \xrightarrow{(r,\pi)} T_{e'+1,t+1,w,p}$, $T_{e,t,w,p} \xrightarrow{(r,\pi)} T_{e+1,t+1,w,p}$, where $(T_{e'+1,t+1,w,p}, S'_2) \in \mathcal{S}$, and, thus, $(T_{e+1,t+1,w,p}, S'_1) \in \mathcal{R}$ as required. If $\overline{r} \notin \gamma$ and $\overline{r} \in \alpha$ then $T_{e',t,w,p} \xrightarrow{\emptyset} T_{e',t+1,w,p}$, $T_{e,t,w,p} \xrightarrow{(r,\pi)} T_{e+1,t+1,w,p}$, where $(T_{e',t+1,w,p}, S'_2) \in \mathcal{S}$, and, thus, $(T_{e+1,t+1,w,p}, S'_1) \in \mathcal{R}$ as required. Finally, if $\overline{r} \notin \gamma$ and $\overline{r} \notin \alpha$ then $T_{e',t,w,p} \xrightarrow{\emptyset} T_{e',t+1,w,p}$, $T_{e,t,w,p} \xrightarrow{\emptyset} T_{e,t+1,w,p}$, where $(T_{e',t+1,w,p}, S'_2) \in \mathcal{S}$, and, thus, $(T_{e,t+1,w,p}, S'_1) \in \mathcal{R}$ as required.
- If $e < w$, $w - e < p - t$ and $w - e' = p - t$, then the proof follows similarly to the first case of the previous clause.
- Finally, if $e < w$ and $w - e = p - t$, then, since $T_{e',t,w,p}$ is schedulable by $S_2$, $e = e'$, $\overline{r} \in \gamma$ and thus, $\overline{r} \in \alpha$ and $T_{e',t,w,p} \xrightarrow{(r,\pi)} T_{e'+1,t+1,w,p}$, $T_{e,t,w,p} \xrightarrow{(r,\pi)} T_{e+1,t+1,w,p}$, where $(T_{e'+1,t+1,w,p}, S'_2) \in \mathcal{S}$, and, thus, $(T_{e+1,t+1,w,p}, S'_1) \in \mathcal{R}$ which completes the proof. □

**Example 4.9.** Consider a system composed of two applications competing for the usage of a single resource, the first consisting of the task set $Task_{1,3} \| Task_{1,5}$ running under an EDF scheduler and the second consisting of the task set $Task_{1,6} \| Task_{1,5}$ running under a rate-monotonic (RM) scheduler (i.e. the smaller the period the higher the priority). We may verify that the assignment of supply $PRM_{3,5}$ to the first application and $PRM_{2,5}$ to the second application leads to the schedulability of the system. This can be achieved by constructing the demand-processes of the two applications and verifying that

1. $PRM_{3,5}$ is weakly more generous than demand$(Task_{1,3} \| Task_{1,5})$ and
2. $PRM_{2,5}$ is weakly more generous than demand$(Task_{1,6} \| Task_{1,5})$. □

As the above example illustrates, our study of generosity relations complement our compositionality results for schedulability analysis of real-time systems. Specifically, our framework represents a formal approach for hierarchical scheduling which allows us (1) to check compositionally whether a hierarchical system is schedulable and extract appropriate (optimal) supplies for its components via the demand function, and (2) to construct practical schedulers for the components in question by isolating simple supplies that are at least as generous as the component demands. Our framework may also be used to formally represent the hierarchical scheduling approaches based on resource models [23] that rely on approximating the necessary supply, making it more generous than necessary, in exchange for a simple representation.

## 5. Conclusions

In this paper, we have presented PADS, a process algebra for resource demand and supply. The algebra can be used to describe a task process and its demand on resources necessary for the execution of a real-time task as well as a supply process that describes the behavior of a resource allocator. We have defined precisely the notion of schedulability using demand and supply, that is, when a process can be scheduled under a supply process, and provided a compositional theory of demand-supply schedulability. We believe that PADS is the first process algebra that can describe the behavior of demand and supply processes and compositional schedulability between them.

There are several directions in which the current work can be extended. We are currently developing a tool which implements our techniques for schedulability analysis and compositional scheduling of real-time systems and we are developing the theory of the process algebra via the study of the precongruence properties and the axiomatizations of the preorders proposed in this paper. We plan to extend our work in order to handle dependencies between tasks. Furthermore, we would like to define the notion of a *residual* supply which captures the supply available after a system has its resource demands satisfied and which will enable to perform incremental scheduling of systems. It would also be interesting to explore how to extend the notion of schedulability to the notion of *resource satisfiability* between demand and supply of arbitrary resources that are not shared mutually exclusively. Another extension is to explore demand and supply processes in the presence of probabilistic behavior.

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