

Percolation in negative field and lattice animals

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(Received 8 June 1988)

We study in detail percolation in a negative “ghost” field, and show that the percolation model crosses over, in the presence of a negative field h , to the lattice-animal model, as predicted by the field theory. This was done by exact solutions in one dimension and on a Cayley tree, and series expansions in general dimension. We confirm the scaling picture near the percolation threshold, and study the extended scaling ansatz for all values of h in terms of the nonlinear scaling field g_h . Estimates for g_h are obtained as a function of h in all dimensions. We also show how information on percolation clusters in all concentrations up to the percolation threshold may be extracted by studying the critical behavior of the generalized susceptibilities $\chi_k(p, h)$ near their critical point $p_c(h)$ as a function of h , and obtain data on the cluster distribution function and on the ratio of perimeter bonds to cluster bonds, for large clusters for all $0 \leq p \leq p_c$. The crossover function is studied in one dimension, mean-field theory and the ϵ expansion.

I. INTRODUCTION

The statistics of clusters on diluted lattices have attracted much attention in the last decade. In the site-lattice-animal model^{1,2} each site is assigned a fugacity K , and the number A_n of clusters (animals) with n sites, scales, for large n , as

$$A_n \sim n^{-\theta_a} K_c^n, \tag{1.1}$$

where K_c is a nonuniversal, lattice-dependent constant and θ_a is the animal critical exponent. In the site-percolation model^{1,3} each site is present with probability p and absent with probability $q = 1 - p$. Although the two models were defined above as site models, equivalent definitions apply for the bond models. From universality⁴ one expects the same critical behavior for both models, and a difference between the two kinds of models is expected only when dealing with nonuniversal quantities, such as K_c .

Near the percolation threshold p_c , the number per site, $B(n, p)$, of clusters with n sites, scales as

$$B(n, p) = n^{-\tau_p} f(n | p_c - p |^{\Delta_p}), \tag{1.2}$$

where τ_p and Δ_p are related to the order-parameter exponent β_p and the susceptibility exponent γ_p , of percolation, via

$$\Delta_p = \gamma_p + \beta_p, \tag{1.3a}$$

$$\tau_p = 2 + \beta_p / (\gamma_p + \beta_p). \tag{1.3b}$$

At the percolation threshold Eq. (1.2) reduces to

$$B(n, p_c) \sim n^{-\tau_p}, \tag{1.4}$$

while below p_c it was shown rigorously^{5,6} that

$$\ln B(n, p) \sim -n \tag{1.5a}$$

which leads to¹

$$B(n, p) \sim n^{-\theta(p)} e^{-A(p)n}. \tag{1.5b}$$

For $p \rightarrow 0$ the cluster numbers $B(n, p)$ reduce to $A_n p^n$, so one expects $\theta(0) = \theta_a$ and $A(p \rightarrow 0) = -\ln(p) + O(1)$.

As one might expect from universality arguments, $\theta(p)$ is not a continuous function of p , but rather $\theta(p) = \theta_a$ for all $p < p_c$ and $\theta(p_c) = \tau_p$. Indeed, this is the outcome of the field theory constructed by Harris and Lubensky⁷ (HL) and real-space renormalization-group studies.⁸ It has not been easy to numerically verify this discontinuous behavior. Monte Carlo methods can hardly help, since for $p < p_c$ the number of large clusters is exponentially small. For instance, Bauchspiess and Stauffer⁹ used the available perimeter polynomials¹⁰ to study the distribution of clusters in the whole range $0 < p < p_c$. They found that $\theta(p)$ changes continuously from $\theta(0) = \theta_a$ to $\theta(p_c) = \tau_p$, but they did not rule out the possibility of the discontinuity we now believe to occur. In fact, as we shall see, the series analysis we present here is perhaps the best numerical evidence currently available for the discontinuous scenario.

For such an analysis it is convenient to define the susceptibilities $\chi_k(p, h)$ as

$$\chi_k(p, h) = \sum_{n=0}^{\infty} n^k B(n, p) e^{-nh}. \tag{1.6}$$

h here is the usual ghost field introduced in the percolation problem.¹ Using (1.5b) we find

$$\begin{aligned} \chi_k(p, h) &\sim [A(p) + h]^{-[k+1-\theta(p)]} \\ &\sim [p_c(h) - p]^{-\gamma_k(h)}. \end{aligned} \quad (1.7)$$

At $h=0$ Eq. (1.2) implies that $\gamma_k(0) = \gamma_p + (k-2)\Delta_p$. For $h < 0$ these quantities all diverge at $p_c(h)$, where $p_c(h)$ is the solution of $A[p_c(h)] = -h$. In the limit $p \rightarrow 0$, one again uses $B(n, p) \sim A_n p^n$ to find the dominant singular behavior:

$$\chi_k(p, h) = \frac{\Gamma(k - \theta_a + 1)}{(1 - pK_c e^{-h})^{k+1-\theta_a}}, \quad (1.8)$$

where Γ is the usual gamma function. Thus, $\chi_k(p, h)$ diverge in this limit at $p_c(h) = \exp(h)/K_c$, with exponents $\gamma_k = k + 1 - \theta_a$, and the gap exponent, defined as the difference between γ_k and γ_{k+1} , is unity for the animal problem.¹¹ For $h=0$ the gap exponent is given by Eq. (1.3a).

The scaling form (1.2) for $B(n, p)$ implies, with $t \equiv (p_c - p)/p_c$,

$$\chi_k(p, h) = t^{-\gamma_p - (k-2)\Delta_p} F_k(-ht^{-\Delta_p}) \quad (1.9)$$

which should hold in the vicinity of the percolation critical point. Equation (1.9) is only a first-order approximation in terms of t and h of the exact scaling form^{12,13}

$$\chi_k(p, h) = g_t^{-\gamma_p - (k-2)\Delta_p} F_k(g_h/g_t^{\Delta_p}), \quad (1.10)$$

in terms of g_t and g_h , the nonlinear scaling fields for which the recursion relations of the renormalization group become exactly linear.¹³ g_t and g_h reduce in linear order to t and $-h$, respectively, and Eq. (1.10) is correct up to corrections due to irrelevant variables. The extended scaling ansatz¹⁴ implies that the scaling relation (1.10) holds for any h , near the appropriate threshold $p_c(h)$:

$$\chi_k(p, h) = g_t^{-\gamma_p - (k-2)\Delta_p} G_k(g_h/g_t^{\Delta_p}), \quad (1.11)$$

where now g_t reduces in linear order to $t(h) \equiv [p_c(h) - p]/p_c(h)$. For $h < 0$, i.e., $g_h/g_t^{\Delta_p} \rightarrow \infty$, one expects animal critical behavior [Eq. (1.8)] and thus, in this limit

$$G_k(x) \sim x^{-[\gamma_p - \gamma_a + (k-2)(\Delta_p - 1)]/\Delta_p}, \quad (1.12)$$

with $\gamma_a = 3 - \theta_a$, and we used the fact that the gap exponent for animals is unity. Combining Eqs. (1.11) and (1.12) we have

$$\chi_k(p, h) \sim g_t^{-\gamma_a - (k-2)} g_h^{-[\gamma_p - \gamma_a + (k-2)(\Delta_p - 1)]/\Delta_p}. \quad (1.13)$$

Taking the limit $h \rightarrow -\infty$ of (1.13) and comparing with Eq. (1.8) we find $g_h(h \rightarrow -\infty) \rightarrow \text{const.}$ Alternatively, one can use (1.12) to obtain

$$\begin{aligned} \chi_k(p, h) &= \frac{d^{k-2} \chi_2(p, h)}{dh^{k-2}} \\ &\sim g_t^{-\gamma_a - (k-2)} g_h^{-[\gamma_p - \gamma_a]/\Delta_p} \left[\frac{dg_t}{dh} \right]^{k-2}. \end{aligned} \quad (1.14)$$

Comparing with (1.13) we find $dg_t/dh \sim g_h^{(1-\Delta_p)/\Delta_p}$.

The crossover from percolation to animals is described as follows within the field theory. The model Hamiltonian contains three independent parameters, which can be expressed in terms of p , h , and q , the bond percolation variables, where q , the probability for a bond to be vacant, is taken to be independent of p . HL showed that in this parameter space there is a critical surface on which there are three nontrivial fixed points describing, respectively, percolation, lattice animals, and the theta point of branching polymers. If we vary p at $h=0$ we intersect the percolation fixed point which is unstable in both the thermal (i.e., p) and field directions, as usual for a continuous phase transition. If we vary p for $h < 0$ we hit a critical line $p = p_c(h)$ from which we flow to the animal fixed point. The theta point requires q to not be equal to $1-p$, and hence we need not consider this point here. The $q = 1-p$ plane of this three-dimensional phase space is displayed in Fig. 1. In this two-dimensional phase space the critical surface appears as a critical line $p_c(h)$, where the susceptibilities $\chi_k(p, h)$, defined in Eq. (1.6), diverge. We see that for $p < p_c$ the critical behavior is determined by the animal fixed point, with exponents independent of the percolation exponents. This also implies that (1.11) is obeyed for all $h \neq 0$ near the appropriate threshold. Some aspects of this phase space were independently confirmed by real-space renormalization-group studies.⁸

As was mentioned above, information on larger clusters below the percolation threshold is rather hard to extract by computer simulations. We will demonstrate below how such information can be obtained by studying percolation in the presence of a negative field. In particular we discuss the perimeter ratio,¹⁵ the ratio of the number of perimeter sites, n_p , to the number of cluster sites. This ratio becomes a nonuniversal, concentration depen-

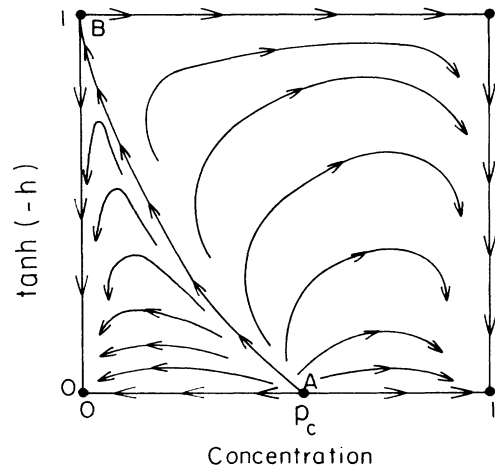


FIG. 1. The schematic (p, h) phase diagram of percolation in a negative field. Point A is the percolation fixed point, while point B corresponds to the animals fixed point.

dent constant, when averaged over all clusters having n sites for n large. At the percolation threshold it was proven⁵ that $n_p/n = (1-p_c)/p_c$. Using¹

$$n_p = \frac{1-p}{p} n - (1-p) \frac{d}{dp} \ln B(n,p), \quad (1.15)$$

leads, via Eq. (1.5), to

$$\frac{n_p}{n} = \frac{1-p}{p} + (1-p) \frac{dA(p)}{dp}, \quad (1.16)$$

with corrections of the order of $1/n$. Equations (1.15) and (1.16) hold also for the bond percolation case, where n_p and n should stand for the number of perimeter and cluster bonds, respectively. Thus, this ratio, for a given concentration, gives information on the clusters statistics in the same concentration and vice versa. In particular for small p we have

$$A(p) = -\ln(p) + C + (n_p/n)_{p=0} p + O(p^2), \quad (1.17)$$

where the ratio evaluated at $p=0$ is the animal perimeter ratio,¹⁶ and $C = -\ln(K_c)$ for (1.16) to be consistent with (1.1). An example of the dependence of this ratio on concentration can be seen in Ref. 1, p. 37, for site percolation in two dimensions.

In the next section we discuss in detail the above ideas in the case of the exactly soluble models in one dimension and on a Cayley tree. Section III contains the construction and analysis of series in general dimension and in Sec. IV we discuss in more detail the crossover between percolation and animals. Section V contains our conclusions.

II. EXACT SOLUTIONS IN ONE DIMENSION AND ON A CAYLEY TREE

Here we analyze exact solutions of the site-percolation problem. The results for the bond-percolation case are very similar.

In one dimension the number per site, $B(n,p)$, of clusters with n sites is $p^n(1-p)^2$, while $A_n \equiv 1$. Writing $B(n,p) = n^{-2} e^{n \ln(p)} [n(1-p)]^2$, we have, according to Eqs. (1.2) and (1.5b) $p_c = 1$, $\tau_p = 2$, $\Delta_p = 1$, and $A(p) = -\ln(p)$, while the scaling function in Eq. (1.2) is given by $f(x) = x^2$. For all $p \neq p_c$ we have $\theta(p) = \theta_a = 0$. Using definition (1.6) we find

$$\chi_k(p,h) = \frac{(1-p)^2 \Gamma(k+1)}{(1-pe^{-h})^{k+1}}, \quad (2.1)$$

leading, for $h \leq 0$, to $p_c(h) = e^h$ (see Fig. 2), consistent with $A(p_c(h)) = -h$, and $\gamma_p = 1$, as implied by the scaling relation (1.3a). Near the percolation threshold $p_c = 1$, χ_k obey the scaling form (1.9) with $F_k(x) = \Gamma(k+1)(1+x)^{-(k+1)}$. For $h \neq 0$ one can write χ_k to leading order in $t(h)$ in the extended scaling form $\chi_k(p,h) = t^{-\gamma_k}(h) G_k(g_h/t(h))$, with $G_k(x) = \Gamma(k+1)(1+x)^2$, consistent with the asymptotic animal behavior (1.12), and the scaling field g_h is given, to zeroth order in $t(h)$, by $g_h = 1 - e^h$ (see Fig. 3). Note that corrections to this scaling form arise from two sources. There are corrections to the exact scaling behavior [Eqs.

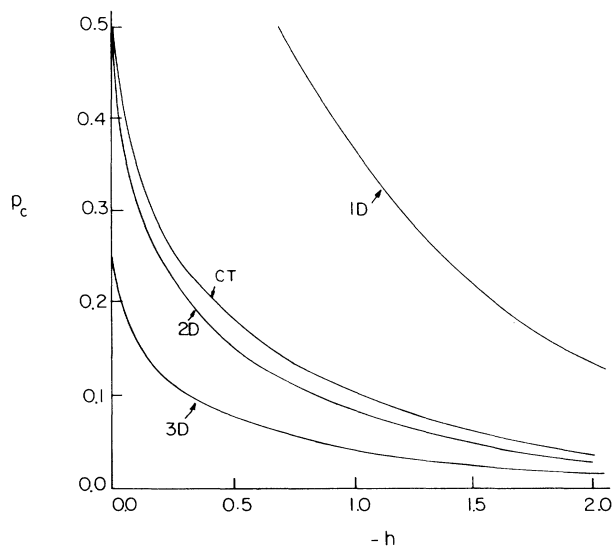


FIG. 2. The critical concentration $p_c(h)$ in 1D (one dimension) and Cayley tree ($\sigma=2$) (exact) and in 2D (two dimensions) and 3D (three dimensions) (series).

(1.10) and (1.11)] due to irrelevant variables and there are also corrections due to the fact that the scaling form was expressed in terms of t and not the exact scaling field g_t . Since all corrections in one dimension (1D) are analytic, we cannot differentiate between these two contributions and there is no unique definition of the scaling field g_h . We note, however, that χ_k can be written in the scaling form (1.11), without any corrections to scaling, for $g_h = 1 - e^h + t(h)e^h$ and $g_t = t(h)$.

Since the number of perimeter sites is always two in one dimension, the perimeter ratio goes to zero for large clusters for all concentrations, consistent with the above $A(p)$ and Eq. (1.16). The same applies for the bond-perimeter ratio, the ratio of the number of perimeter bonds to cluster bonds.

One can also calculate the universal amplitude ratios

$$R_{ij/kl}(h) = \frac{\chi_i \chi_j}{\chi_k \chi_l}, \quad (2.2)$$

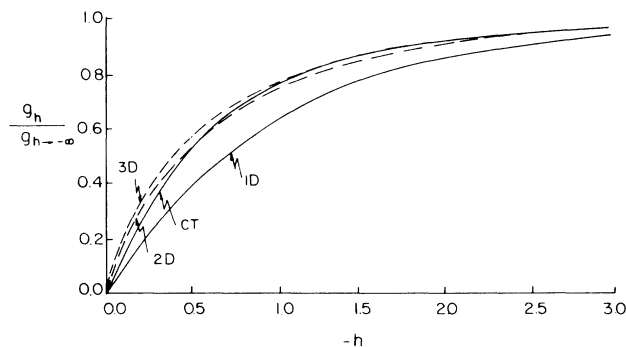


FIG. 3. The nonlinear scaling field g_h vs h for 1D and Cayley tree (exact, full line) and 2D and 3D (series, dashed line).

where the right-hand side is evaluated at the limit $p \rightarrow p_c(h)$. Much as the critical exponents, the amplitude ratios should also assume their percolation values for $h=0$, while for $h \neq 0$ these ratios assume the animal values. In one dimension one uses Eq. (2.1) to find $R_{ij/kl} = \Gamma(i+1)\Gamma(j+1)/\Gamma(k+1)\Gamma(l+1)$, independent of h .

It will be useful to define the following universal quantities:

$$S_{ij/kl} = R_{ij/kl} \frac{\Gamma(\gamma_k) \Gamma(\gamma_l)}{\Gamma(\gamma_i) \Gamma(\gamma_j)}. \quad (2.3)$$

From Eq. (1.8) it follows that for the animal model $S_{ij/kl} = 1$ for all i, j, k , and l and all dimensions. For the percolation model in one dimension we have

$$S_{ij/kl} = i(i-1)j(j-1)/[k(k-1)l(l-1)].$$

Exact results can also be obtained on a Cayley tree, which simulates an infinite-dimensional lattice. The cluster numbers are given by¹⁷

$$B(n, p) = (\sigma+1)(1-p)^2 \frac{(n\sigma)!}{n!(n\sigma-n+2)!} \times [p(1-p)^{\sigma-1}]^n, \quad (2.4)$$

where $\sigma+1$ is the coordination number. Using the Stirling formula for large n , Eq. (2.4) simplifies to

$$B(n, p) = C_1(1-p)^2 n^{-5/2} e^{-nA(p)}, \quad (2.5)$$

with

$$C_1 = (\sigma+1)[\sigma/2\pi(\sigma-1)]^{1/2} [e/(\sigma-1)]^2$$

and

$$A(p) = -\ln[\sigma^\sigma p(1-p)^{\sigma-1}/(\sigma-1)^{\sigma-1}],$$

in accordance with Eq. (1.5b), with $\theta=5/2$ for all p . Again, $A(p) \sim -\ln(p)$ for $p \rightarrow 0$. Near the percolation threshold, $p_c = 1/\sigma$, $A(p) = C_2 t^2$, with $C_2 = \sigma/2(\sigma-1)$, and we reconfirm the scaling form (1.2) with $\Delta_p = 2$ and $f(x) = C_1[(\sigma-1)/\sigma]^2 \exp(-C_2 x)$. The dominant singular behavior of χ_k is obtained from Eq. (2.4)

$$\chi_k(p, h) = C_1(1-p)^2 \frac{\Gamma(k - \frac{3}{2})}{[A(p) + h]^{k-5/2}}. \quad (2.6)$$

The critical threshold $p_c(h)$ is the solution of

$$e^h = \frac{p_c(h)[1-p_c(h)]^{\sigma-1}}{\frac{1}{\sigma} \left[1 - \frac{1}{\sigma}\right]^{\sigma-1}} = \frac{p_c(h)[1-p_c(h)]^{\sigma-1}}{p_c(0)[1-p_c(0)]^{\sigma-1}}. \quad (2.7)$$

For $\sigma=2$, for example, this equation yields

$$p_c(h, \sigma=2) = \frac{1}{2} [1 - (1-e^h)^{1/2}], \quad (2.8)$$

which is displayed in Fig. 2. At the percolation threshold $h=0$, and one can use the value obtained above for $A(p)$ to find

$$\chi_k(p) \sim \frac{\Gamma(k-3/2)}{(p_c-p)^{2k-3}}, \quad (2.9)$$

in accordance with $\gamma_p=1$ and $\Delta_p=2$ on a Cayley tree. Near the percolation threshold χ_k can be written in the scaling form $\chi_k(p, h) \sim t^{-(2k-3)} F_k(ht^{-\Delta_p})$, with $F_k(x) \sim (C_2+x)^{-(k-3/2)}$. For general h the algebra is quite cumbersome, and we quote the results only for the case $\sigma=2$. There one can write χ_k in the extended scaling form

$$\chi_k(p, h) \sim \frac{\Gamma(k-3/2)}{t^{2k-3}} G_k(g_h/t^2), \quad (2.10)$$

with

$$G_k(x) = (1+2\sqrt{x})^{-(k-3/2)},$$

and

$$g_h = (1-e^h)[1+(1-e^h)^{1/2}]^2/(2-e^h)^2$$

(see Fig. 3). The large- x behavior of G_k is again consistent with the prediction (1.12). Note also that the remark after Eq. (2.1) for one dimension applies also for the Cayley tree and we are not able to extract the scaling field g_t . The amplitude ratios are readily read from Eq. (2.9) leading to $R_{ij/kl} = \Gamma(i-3/2)\Gamma(j-3/2)/\Gamma(k-3/2)\Gamma(l-3/2)$ for all h .

The perimeter ratio is easily read from Eq. (2.4), leading to $n_p/n = \sigma-1$ for all concentrations below p_c , consistent with (1.15). Since the number of bonds and the number of sites differ on a Cayley tree by one, the bonds perimeter ratio is also equal to $\sigma-1$.

We expect that all universal quantities evaluated on the Cayley tree would hold for all dimensions larger than eight, the upper critical dimension for the animal problem.¹⁸ On the other hand, the universal quantities of the percolation model should hold for dimensions larger than six, the upper critical dimension in this model.^{19,20}

III. SERIES EXPANSIONS IN GENERAL DIMENSION

In dimensions larger than one the cluster numbers are not known exactly to all orders of n . Thus the susceptibilities χ_k cannot be evaluated analytically. We have calculated the sum (1.6) up to 11th order in p , for the bond-percolation problem. To this aim it is more convenient to rewrite (1.6) as a sum over all clusters Γ :

$$\chi_k(p, h) = \sum_{\Gamma} p^{n_b(\Gamma)} (1-p)^{n_p(\Gamma)} \chi_k(\Gamma, h), \quad (3.1)$$

where $n(\Gamma)$, $n_b(\Gamma)$, and $n_p(\Gamma)$ are the numbers of sites, bonds, and perimeter bonds of the cluster Γ , respectively, and $\chi_k(\Gamma, h) = n(\Gamma)^k e^{-n(\Gamma)h}$. This sum can be simplified to²¹

$$\chi_k(p, h) = \sum_{\Gamma} \chi_k^{(c)}(\Gamma, h) W(\Gamma; d) p^{n_b(\Gamma)}, \quad (3.2)$$

where the sum is now only on topologically inequivalent clusters and $W(\Gamma; d)$ is the number of ways per site a dia-

gram topologically equivalent to Γ can be realized on a d -dimensional lattice.²² $\chi_k^{(c)}$, the cumulant susceptibility, is defined by

$$\chi_k^{(c)}(\Gamma, h) = \chi_k(\Gamma, h) - \sum_{\gamma \subset \Gamma} \chi_k^{(c)}(\gamma, h), \quad (3.3)$$

where the sum is over all subdiagrams γ of Γ .

The factor $p^{n_b(\Gamma)}$ in Eq. (3.2) implies that evaluation of χ_k up to order p^N involves only clusters with up to N bonds. We have constructed the series (3.2) up to 11th order in p , for $0 \leq -h \leq 5$, in a discrete mesh. We show in Table I the coefficients of the series $\chi_2(p, h = -\ln 2)$, which are easy to check, since the contribution per cluster is given by $n(\Gamma)^2 2^{n(\Gamma)}$.

We first applied the nonhomogeneous differential Padé method²³ to the series (3.2). This method yielded ~ 40 approximants for p_c and γ_k for a given h . Our estimate for $p_c(h)$ was taken as the average of all approximants, and the dispersion yielded our estimate for the error bar, which was of the order of a few percent. The estimates for γ_k are obtained by interpolation of the (p_c, γ_k) line to the above value of p_c . In Fig. 2 we show the dependence of $p_c(h)$ on h for two and three dimensions.

Near $p_c(0)$ the scaling relation (1.9) should hold, which leads to $p_c(h) - p_c(0) \sim (-h)^{-1/\Delta_p}$. Indeed, by plotting h versus $p_c(h) - p_c(0)$ in a log-log plot, we obtained for small h straight lines, with slopes 2.4 ± 0.1 in two dimensions and 2.2 ± 0.1 in three dimensions, consistent with the exact result²⁴ $\Delta_p = \frac{43}{18}$ in two dimensions and the recent estimate²⁵ $\Delta_p = 2.23 \pm 0.05$ in three dimensions.

The estimates for $\gamma_k(h)$ are sensitive to the estimates of $p_c(h)$, and we preferred to analyze the crossover by using another method. We divided the series for χ_k by χ_{k-1} , term by term. This leads to a new series, which diverge at $p=1$, with exponents equal to the difference between $\gamma_k(h)$ and $\gamma_{k-1}(h)$ plus one.^{26,27} Thus the estimates for the gap exponents are unbiased by the value of the critical point. In Fig. 4 we show the dependence of the effective gap exponent Δ^{eff} on h , for two and three dimensions. We find a continuous but sharp change from Δ_p , for $h=0$, to $\Delta_a=1$ in the range $0 \leq -h \leq 0.05$, which is equivalent to a small neighborhood of p_c .

We also studied the crossover in the universal quantities $S_{ij/kl}$, defined by (2.3), using a method developed in Ref. 27. The estimates for S are not biased by the values of the critical point nor the critical exponents. Figure 5

TABLE I. The series coefficients for $\chi_2(p, h = -\ln 2, d) = \sum_{k,l} a_{kl} p^k d^l$. The notation $[x]$ denotes $\times 10^x$.

k	l	a_{kl}	k	l	a_{kl}
1	1	0.140 000 000 000 000 000 00[2]	2	1	-0.410 000 000 000 000 000 00[2]
2	2	0.820 000 000 000 000 000 00[2]	3	1	0.186 000 000 000 000 000 00[3]
3	2	-0.686 000 000 000 000 000 00[3]	3	3	0.628 000 000 000 000 000 00[3]
4	1	0.635 000 000 000 000 000 00[2]	4	2	0.378 583 333 333 333 333 33[4]
4	3	-0.832 200 000 000 000 000 00[4]	4	4	0.483 266 666 666 666 666 67[4]
5	1	-0.127 268 000 000 000 000 00[5]	5	2	0.524 966 666 666 666 666 67[4]
5	3	0.606 856 666 666 666 666 67[5]	5	4	-0.903 026 666 666 666 666 67[5]
5	5	0.380 541 333 333 333 333 33[5]	6	1	0.781 090 000 000 000 000 00[5]
6	2	-0.302 920 877 777 777 777 78[6]	6	3	0.601 666 666 666 666 666 67[2]
6	4	0.845 679 222 222 222 222 22[6]	6	5	-0.922 320 666 666 666 666 67[6]
6	6	0.303 857 155 555 555 555 56[6]	7	1	0.103 116 085 714 285 714 29[7]
7	2	0.465 339 000 000 000 000 00[6]	7	3	-0.402 511 422 222 222 222 22[7]
7	4	-0.157 333 366 666 666 666 67[7]	7	5	0.107 391 757 777 777 777 78[8]
7	6	-0.908 242 533 333 333 333 33[7]	7	7	0.245 134 158 730 158 730 16[7]
8	1	-0.737 971 225 000 000 000 00[7]	8	2	0.288 636 572 202 380 952 38[8]
8	3	0.371 467 303 888 888 888 89[7]	8	4	-0.446 657 222 027 777 777 78[8]
8	5	-0.408 743 517 777 777 777 78[8]	8	6	0.127 719 169 222 222 222 22[9]
8	7	-0.872 943 375 111 111 111 11[8]	8	8	0.199 316 002 603 174 603 17[8]
9	1	-0.357 233 655 333 333 333 33[9]	9	2	0.463 171 792 369 841 269 84[9]
9	3	0.869 989 674 473 544 973 54[8]	9	4	0.169 251 789 277 777 777 78[9]
9	5	-0.413 563 473 683 333 333 33[9]	9	6	-0.734 438 387 822 222 222 22[9]
9	7	0.144 753 178 102 222 222 22[10]	9	8	-0.824 749 212 825 396 825 40[9]
9	9	0.163 066 239 547 089 947 09[9]	10	1	0.232 993 558 220 000 000 00[10]
10	2	-0.132 247 042 697 065 079 37[11]	10	3	0.155 451 645 946 876 984 13[11]
10	4	-0.417 758 981 038 800 705 47[10]	10	5	0.397 066 005 978 055 555 56[10]
10	6	-0.279 148 542 186 185 185 19[10]	10	7	-0.111 160 454 351 777 777 78[11]
10	8	0.158 171 960 960 105 820 11[11]	10	9	-0.769 382 391 899 047 619 05[10]
10	10	0.134 077 700 344 578 483 25[10]	11	1	0.140 691 084 984 545 454 55[12]
11	2	-0.225 428 261 672 077 777 78[12]	11	3	-0.605 908 400 930 804 761 90[11]
11	4	0.230 012 260 330 640 652 56[12]	11	5	-0.104 002 727 784 415 432 10[12]
11	6	0.657 008 943 284 240 740 74[11]	11	7	-0.258 757 160 016 814 814 81[10]
11	8	-0.151 764 855 434 296 296 30[12]	11	9	0.167 977 606 262 738 624 34[12]
11	10	-0.710 772 321 146 906 525 57[11]	11	11	0.110 698 394 003 799 775 53[11]

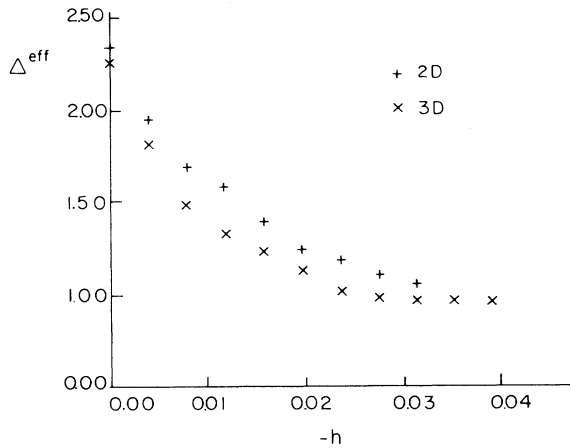


FIG. 4. The effective gap exponent $\Delta^{\text{eff}}(h)$ for 2D and 3D (series).

displays the dependence of $S_{24/33}$ on h for two and three dimensions. Again we see the crossover from $S_{24/33}(0)$ (see Ref. 25) to $S_{24/33} = 1$ [see remark after Eq. (2.3)] for the animal model in the somewhat larger range of $0 \leq -h \leq 0.15$. We do not see a sharp jump in the values of the exponents or the amplitude ratios due to the fact that we have only a finite number of terms in our series. This is equivalent to not being exactly at the critical point,^{28,29} and we come back to this point in the next section.

The mere fact, that for negative h we find divergence at an h -dependent critical point, implies that the cluster numbers $B(n, p)$ are exponential in n , and this confirms the prediction (1.5). Moreover from the values of the threshold as a function of h , one can extract the function $A(p)$, which is exactly the inverse function of $p_c(h)$, and is readily read from Fig. 2. The dependence of p_c on h also gives us information on the perimeter ratio, since the

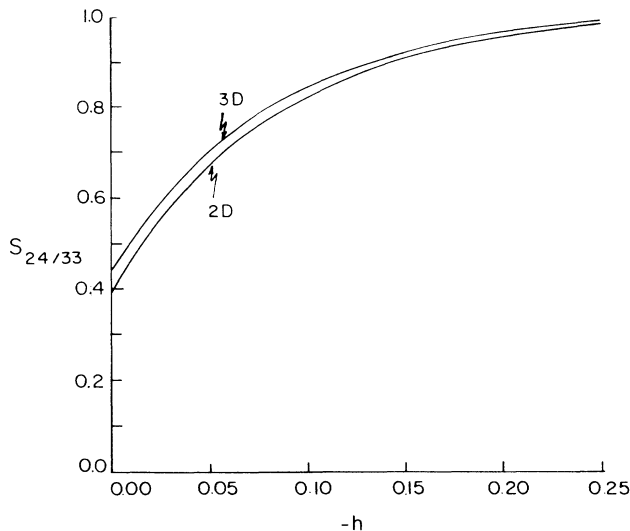


FIG. 5. The amplitude ratios $S_{24/33}(h)$ for 2D and 3D (series).

bond-percolation version of Eq. (1.16) can be recast in the form

$$\frac{n_p}{n_b} = \frac{1-p}{p} + \frac{1-p}{dp_c/dh}. \quad (3.4)$$

Instead of reading dp_c/dh from the graph, which may lead to large numerical uncertainties, we employed the following procedure, which uses the derivatives of χ_k with respect to p and to h :

$$\frac{d\chi_k}{dp} \sim -\gamma_k [p_c(h) - p]^{-\gamma_k - 1}, \quad (3.5a)$$

$$\frac{d\chi_k}{dh} = -\chi_{k+1} \sim -\gamma_k \frac{dp_c(h)}{dh} [p_c(h) - p]^{-\gamma_k - 1}. \quad (3.5b)$$

Then, the series obtained by dividing these series term by term behave as $(dp_c/dh)/(1-p)$. Analyzing for the residue of the series at $p=1$ in two and three dimensions gives rise to Fig. 6, where the errors in the estimates are of the order of a few percent. The fact that we find a finite value for $h \rightarrow -\infty$ ($p \rightarrow 0$) implies $A(p) \sim -\ln(p)$ [or $p_c(h) \sim e^h$], in agreement with the remark after Eq. (1.5b), while the value of the ratio at $p=0$ gives us the correction to this leading behavior, as depicted in Eq. (1.17). Note that since we are dealing with bond percolation, we have calculated the bond ratio and we cannot compare this nonuniversal graph to the site-percolation data.^{15,16}

Finally we investigate the extended scaling behavior (1.11). Using (1.13) we have for $|h| > 0$

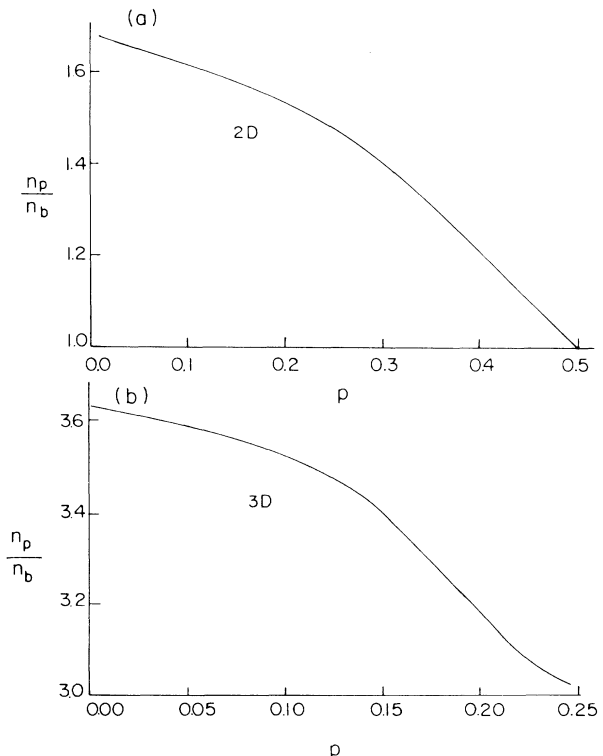


FIG. 6. The perimeter ratio for (a) 2D and (b) 3D.

$$\frac{\chi_k(p, h)}{\chi_{k-1}(p, h)} \sim \frac{g_h^{(1-\Delta_p)/\Delta_p}}{t} \quad (3.6)$$

We checked Eq. (3.6) by the procedure of dividing the two series term by term. Near p_c one expects $g_h \sim -h$, and indeed, by plotting the residue of Eq. (3.6) versus $-h$ for small $|h|$ on a log-log graph, we found linear curves with slopes -0.5 ± 0.1 in two dimensions and -0.5 ± 0.1 in three dimensions, to be compared with $(1-\Delta_p)/\Delta_p = -53/89 \sim -0.60$ in two dimensions²⁴ and -0.55 ± 0.01 in three dimensions.²⁵ We also used Eq. (3.6) to estimate the dependence of the nonlinear scaling field g_h on h for general h by plotting the residue to the power $\Delta_p/(1-\Delta_p)$. In view of the undetermined constant in Eq. (3.6) we could only determine g_h to within a constant. Accordingly, in Fig. 3 we display the resulting behavior of $g_h/g_{h \rightarrow -\infty}$ in two dimensions and three dimensions.

IV. CROSSOVER FROM PERCOLATION TO ANIMALS

It has been evident for a long time²⁸ that many experiments and computer simulations, even those with high precision,²⁹ do not attain the truly asymptotic regime. When one tries to measure the critical behavior of any quantity, such as χ_k in Eq. (1.7), one observes an effective exponent^{28,29}

$$\gamma_k^{\text{eff}} = \frac{d \ln \chi_k}{d \ln t} \Big|_{t=\bar{t}}, \quad (4.1)$$

where \bar{t} is a suitable average of t over the range of measurement. The corrections to the leading behavior also play a role in analyses aiming to extract γ_k by series expansions, as was demonstrated for model series.³⁰

Let us demonstrate the above ideas for the crossover from percolation to animals in one dimension. Applying (4.1) to Eq. (2.1) we have

$$\gamma_k^{\text{eff}} = -(k+1) + \frac{2\bar{t}}{\bar{t}+1-e^h} \sim -(k+1) + 2\bar{t}/(\bar{t}-h), \quad (4.2)$$

where we used that the crossover happens for $|h| \ll 1$. The same crossover is obtained for all universal quantities in one dimension. Note that we could arrive at (4.2) without any analysis, using some simple assumptions. Any quantity Q , should cross over as

$$Q^{\text{eff}} = Q_a + (Q_p - Q_a)F(\bar{t}, h), \quad (4.3)$$

$$\gamma_k^{\text{eff}} = \left[1 + \frac{\bar{t}}{t_1} \right] \left[\frac{2k-3}{2} - \frac{\epsilon}{12} \frac{B\sqrt{-h}}{[(\epsilon - B\sqrt{-h})\bar{t}_1^{\epsilon/4} + B\sqrt{-h}]} \right], \quad (4.6)$$

where \bar{t}_1 is the appropriate average of t_1 . A few details are noteworthy. For $\epsilon=0$ the above formula reduces to the mean-field formula (4.4). For $h=0$ we retrieve $\gamma_k = 2k-1$, as expected for the mean-field behavior of

where Q_a and Q_p are the values of Q for the animal model and the percolation model, respectively, and $F(\bar{t}, h)$ should obey $F(0, h)=0$ for all $h \neq 0$, and $F(\bar{t} \rightarrow 0, 0)=1$. If one assumes that the h dependence of the corrections to the scaling behavior (1.9) can be neglected, which is correct in one dimension, then F is only a function of $x = \bar{t}|h|^{-1/\Delta_p}$, and the simplest such function is $F(x) = x/(1+x)$, which is what we found in Eq. (4.2). Note that redefinition of \bar{t} eliminates any constant in front of x in the denominator. These arguments are applicable as long as the assumption concerning the corrections is justified and we will see below deviations from the scaling behavior of F in the ϵ expansion.

We checked (4.2) by analyzing series for $S_{24/33}$ in one dimension. Since the results are known analytically we can construct the series to all orders. Analysis of the series with 11, 15, and 20 terms yields graphs similar to Fig. 5, which are very well approximated by (4.2) with $\bar{t}=0.06, 0.035$, and 0.02 , respectively. As expected, \bar{t} decreases as the number of terms, N , increases. A detailed analysis of the dependence of \bar{t} on N might prove useful for future series analyses.

A similar analysis can be carried out in mean-field theory.⁷ HL obtained $\chi_k \sim t_1^{-(2k-3)/2}$ for $|h| \ll 1$, with $t_1 = t^2 + 2wh$, w being some nonuniversal constant, which leads to

$$\gamma_k^{\text{eff}} = \frac{2k-3}{2} \left[1 + \frac{\bar{t}}{\bar{t} + \sqrt{-h}} \right], \quad (4.4)$$

where all constants were incorporated in a redefinition of \bar{t} . As expected, Eq. (4.4) is of the type (4.3) with $F(x) = x/(1+x)$ again. From the above expression for χ_k it is clear that, for $h \neq 0$, χ_k has two critical points, $p_c(0) \pm (-2wh)^{1/2}$. Our series analysis was not fit to deal with such close singularities, and we were unable to check (4.4) numerically, for dimensions larger than 8. Calculations for the Cayley tree yield the same results as mean-field theory, as expected.

The above analysis can be extended to include corrections in the ϵ expansion scheme. For dimensions below 8, the upper critical dimension for the animal model, HL obtained, for $|h| \ll 1$, the following scaling function, to first order in $\epsilon = 8-d$:

$$\chi_k = A t_1^{-(2k+3)/2} \left[1 + \frac{B\sqrt{-h}}{\epsilon} (t_1^{-\epsilon/4} - 1) \right]^{1/3}. \quad (4.5)$$

where A and B are nonuniversal constants. Equation (4.5) leads, after tedious but straightforward algebra, to

the percolation model. For $h \neq 0$ we find for $\bar{t} \rightarrow 0$, $\gamma_k = (2k-1)/2 - \epsilon/12$, in agreement with the first-order ϵ expansion¹⁸ for γ_a and with $\Delta_a = 1$ in all dimensions. Note also that Eq. (4.6) contains two kinds of corrections.

We have the same corrections as in the mean field [Eq. (4.4)], since we are still above the upper critical dimension of the percolation model, but we also find corrections of the order $t_1^{\epsilon/4}$, which are in agreement with the ϵ expansion of the correction to scaling exponent for the animal model¹¹ (see also Ref. 2 and references therein). This is a confluent correction to the scaling behavior (1.11), and thus the crossover function cannot be written in a scaling form of $t/g_h^{1/\Delta_p}$. This result is consistent with HL who claimed that between six and eight dimensions one should have two scaling fields as a function of h . Below six dimensions one should also take into account the confluent corrections of the percolation model. Again, due to the same numerical problems, described after Eq. (4.4), we were unable to check Eq. (4.6) numerically.

V. CONCLUSIONS

We have investigated in detail percolation in a negative ghost field h and showed that in the presence of such a field, the percolation problem crosses over to the animal model, in accordance with renormalization-group ideas. This was done by exact solutions in one dimension and on a Cayley tree, where we showed that the critical exponents in the presence of any finite-negative field are an-

imal exponents, and by series expansions in all dimensions, where the estimates for the critical exponents and the amplitude ratios cross over, on a very small scale of h , from their percolation values to their animal ones (see Figs. 4 and 5). We have obtained the crossover function in one dimension and for infinite d by exact solution and also for $d = 8 - \epsilon$ using the ϵ expansion, and extracted the nonlinear scaling field in all dimensions, using the exact solutions and the series expansions. We showed in Fig. 3 that in dimensions larger than one g_h can be described very well by the Cayley tree result. We have shown how analysis of the singular points of the generalized susceptibilities χ_k as a function of h can yield information about the percolation model well below the percolation threshold. In particular we obtained information on the cluster distribution and the perimeter ratio in all dimensions, via series expansions.

ACKNOWLEDGMENTS

This work was supported by the Israel Academy of Sciences and Humanities and the U.S.—Israel Binational Scientific Foundation (BSF). One of us (A.B.H.) acknowledges partial support from the National Science Foundation under Grant No. DMR 85-20272.

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