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Strings and Supermoduli

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Polyakov's prescription for fermionic closed string amplitudes requires that we integrate over gauge-inequivalent geometries on a 2d supermanifold. These inequivalent geometries are parameterized by a finite-dimensional superspace of moduli. We describe this space and propose an integration measure on it which comes from gauge-fixing the heterotic string. The measure thus obtained is free of conformal and Lorentz anomalies and so can be used to compute invariant string amplitudes.

12/85

1. Introduction

In Polyakov's formulation of bosonic string theory [1] one integrates over string coordinates x^μ and metrics g_{mn} on the string world sheet M. The path integral can be given meaning by gauge fixing the symmetry group of the action, *i.e.*, factoring out the gauge group volume.

The problem with gauge-fixing is that it is not quite complete. Locally on M one can use the gauge symmetries of the theory to transform any metric to a standard one [1][2][3]. This is not quite true globally. In general, there is a finite-dimensional "moduli space," mod M, of gauge-inequivalent metrics.

Thus, factoring out the gauge group reduces the infinite-dimensional integral over g_{mn} to a finite-dimensional integral over moduli space [4][5]. The explicit measure on moduli space needed for the computation of closed bosonic string amplitudes has been derived in [6][7][8]. In the present letter we will consider the analogous problem of integrating over all 2-geometries for a fermionic string theory. We choose the heterotic string [9] mainly to simplify the equations; the superstring should be a straightforward extension.

2. Superspace

The dynamical variables and geometry relevant to the heterotic string are conveniently phrased in terms of a chiral two-dimensional superspace, which is a modification of the framework studied in [3][10]. Thus, the dynamical variables are functions on a superspace with two even coordinates $z, \bar{z}$ and one odd coordinate θ. The superspace frame ${E_M}^A$ is therefore an invertible 3×3 graded matrix. Heterotic geometry can be defined by imposing appropriate torsion constraints. If $\nabla_A = {E_A}^M \nabla_M$ is the covariant derivative and $\nabla_M V^A = \partial_M V^A + \Omega_M V^B L_B^A$ (see appendix) then define curvature and torsion via[1] $[\nabla_A, \nabla_B] = -{T_{AB}}^C \nabla_C + R_{AB} \cdot L$ and impose the minimal constraints

$$ {T_{++}}^a = 2i\gamma^a_{++}; \quad {T_{++}}^+ = {T_{z+}}^A = {T_{z+}}^+ = 0. \tag{2.1} $$

The above constraints together with the Bianchi identities imply that all the field strengths can be expressed in terms of R_{z+}.

The geometry defined by (2.1) admits three important group actions. First, the constraints are coordinate and Lorentz invariant. Next, one can define super-Weyl transformations [3] $\delta {E_M}^a = w {E_M}^a$; $\delta {E_M}^+ = \frac{1}{2} w {E_M}^+ - \frac{i}{\sqrt{2}} {E_M}^z \nabla_+ w$ for a scalar superfield parameter w. These also preserve (2.1).

Locally all frames are equivalent under these three group actions. There are $3 \times 3 + 3 = 12$ superfield degrees of freedom in the frame and connection. The seven constraints reduce this to five, corresponding to the $1 + 1 + 3$ gauge degrees of freedom in Weyl, Lorentz, and coordinate transformations.

We can now write an action in superspace which respects these three symmetries. Let X^μ be d scalar superfields neutral under super-Weyl, and let Λ_I^+ be 32 spinor superfields, with $\delta_W \Lambda^+ = -\frac{1}{2} w \Lambda^+$. The action for the matter fields is then

$$ S = \int_M \left[i(\nabla_z X)(\nabla_+ X)\gamma^z_{--} + \Lambda^+ \nabla_+ \Lambda^+ \right] (\operatorname{sdet} {E_M}^A)\, d^2\sigma\, d\theta \tag{2.2} $$

This is the supergravity version of the sigma-model actions discussed in [11][12][13]. It has all the required local symmetries to describe covariantly "supersymmetric surfaces" in spacetime.

3. Wess-Zumino Gauge

To quantize we need measures on the various field spaces and gauge groups. The most convenient way to define such measures while preserving as much symmetry as possible is to put metrics on the various function spaces. In the heterotic string the existence of only a single odd coordinate θ makes it impossible to find appropriate supersymmetric metrics on the space of frames and gauge groups. This problem is equally serious in the NSR superstring where θ and $\bar{\theta}$ must be given independent spin structures. Thus we will decompose the superfields in terms of components and put metrics on the component field spaces. Manifest supersymmetry is lost at this point and will have to be checked later.

The constrained frame contains ten component fields which can be described as gauge transformations away from a gauge slice. As a first step, consider a partial gauge-fixing to a Wess-Zumino gauge similar to the one in [3]: We gauge away the $\mathcal{O}(\theta^0)$ terms of ${E_\theta}^a$, Ω_θ, and $({E_\theta}^+ - 1)$. We also use super-Weyl transformations to set the spin-$\frac{1}{2}$ bit of the gravitino to zero. This intermediate step simplifies the algebra considerably. Since the passage to WZ gauge does not involve solving any differential equations (unlike *e.g.* the

[1] These conventions agree with [3] but not with [10].

passage to superconformal gauge), we encounter no topological obstructions at this step and hence no moduli. Since we do not quantize until after making the restriction, the issue of anomalies in the transformations taking us to WZ gauge will not arise.

These five constraints take the ten components of E down to five components, the graviton $e_m{}^a$ and gravitino $\chi_{\bar z}^+$. There remain five residual symmetries under ordinary Lorentz, Weyl, and coordinate transformations. Explicitly we have

$$E_M{}^A = \begin{pmatrix} e_m{}^z + i\sqrt{2}\theta\chi_m^+ & e_m{}^{\bar z} & \frac{1}{2}\chi_m^+ + \frac{i}{2}\theta\omega_m \\ -i\sqrt{2}\theta & 0 & 1 \end{pmatrix}; \qquad (3.1)$$

$$\Omega_A = \begin{pmatrix} \epsilon^{nm}\partial_n e_m{}^{\bar z} \\ \epsilon^{nm}\partial_n e_m{}^z + \sqrt{2}\theta D_z\chi_{\bar z}^+ \\ 0 \end{pmatrix}.$$

Let $X^\mu = y^\mu(\sigma) + \theta\lambda_+^\mu(\sigma)$ be d scalar superfields and $\Lambda_I^+ = \psi_I^+ + \theta F_I$ be 32 spinor superfields. Then the action is

$$S_{WZ} = \int \left[-\partial y\cdot\bar\partial y + 2\chi_{\bar z}^+\lambda_+\partial_z y - i\sqrt{2}\lambda_+ D_{\bar z}\lambda_+ + F^2 - i\sqrt{2}\psi^+ D_z\psi^+\right] e\,\mathrm{d}^2\sigma\ , \qquad (3.2)$$

where $e = \det e_m{}^a$. The residual symmetries in WZ gauge are:

$$\begin{aligned} &(a)\text{supergravity: } \delta y = \epsilon^+\lambda_+, \quad \delta\lambda_+ = i\sqrt{2}\epsilon^+\partial_z y, \quad \delta\chi_{\bar z}^+ = 2D_{\bar z}\epsilon^+, \\ &\quad \delta e_m{}^z = i\sqrt{2}\epsilon^+\chi_m^+, \quad \delta e_m{}^{\bar z} = 0, \quad \delta\psi^+ = \epsilon^+ F, \\ &\quad \delta F = i\sqrt{2}\epsilon^+ D_z\psi^+. \qquad (3.3) \\ &(b)\text{Weyl: } \delta y = 0, \quad \delta\lambda_+ = -\tfrac{1}{2}w\lambda_+, \quad \delta e_m{}^a = we_m{}^a, \quad \delta\chi_{\bar z}^+ = -\tfrac{1}{2}w\chi_{\bar z}^+, \\ &\quad \delta\psi^+ = -\tfrac{1}{2}w\psi^+, \quad \delta F = -wF. \\ &(c),(d)\text{coordinate, Lorentz: } \delta\lambda_+ = \xi^m D_m\lambda_+ - \frac{i}{2}\ell\lambda_+, \\ &\quad \delta e_a{}^m = -(D_a\xi^b)e_b{}^m + \ell e^{bm}\epsilon_{ba}, \quad etc. \end{aligned}$$

4. Supermoduli

We now ask to what extent further gauge fixing is possible. On M let $\mathcal{T}_n$ be the space of tensors $\mathbf{t}$ with n lower θ indices. We then have the inner product

$$\langle \mathbf{t}, \mathbf{s}\rangle = \int \mathrm{d}^2\sigma\ e\,(g^{uu})^{n/2}\mathbf{t}(\sigma)^*\mathbf{s}(\sigma) \quad , \qquad (4.1)$$

which we use to define adjoints of the covariant derivatives on M. For example $D_u^{(n)} : \mathcal{T}_n \to \mathcal{T}_{n+2}$, and $(D_u^{(n)})^\dagger = D^u_{(n+2)}$. By the index theorem

$$dim_{\mathbb{C}} ker D_u^{(n)} - dim_{\mathbb{C}} ker D^u_{(n+2)} = (|n|+1)(1-g) \quad , \qquad (4.2)$$

where g is the genus of M.

For $g > 1$, $D_u^{(2)}$ has no kernel (there are no conformal Killing vectors [5]) so (4.2) implies the existence of $3g-3$ complex deformations of $e_m{}^z$ which cannot be written as gauge variations [5], and similarly for $e_m{}^{\bar z}$. These are called *moduli* of the Riemann surface M. Similarly, for $g > 1$ $D_u^{(1)}$ has no kernel, since the product of two such conformal Killing spinors would be a conformal Killing vector. Thus there are $2g-2$ complex deformations of a given gravitino $\chi_{\bar z}^+$ which are orthogonal to the image of $D^u_{(-1)}$, and hence by (3.3) are not gauge. We will call these "supermoduli." Since the difference between spin structures amounts to the multiplication by a flat bundle, the number of supermoduli is independent of spin structure for $g > 1$. Hence a slice in *complex* frame space transverse to all gauge deformations will be a finite dimensional superspace with $6g-6$ commuting and $2g-2$ anticommuting dimensions.

For the sphere ($g = 0$) there are neither moduli nor supermoduli. For the torus moduli space is $2|0$ or $2|\,1$dimensional, depending on spin structure. We will defer discussion of these cases to [14] since they present the added wrinkle of conformal Killing spinors.

5. Change of Variables

We would now like to consider the string partition function as an integral over the quotient of fields modulo gauge symmetries, represent this quotient by a slice, and find the appropriate slice measure. This strategy will work only if the answer is independent of the slice chosen.

To begin, we need to define more carefully the configuration space. In Minkowski space the heterotic string contains a Majorana-Weyl gravitino and a real frame $e_m{}^a$ on M. In Euclidean space there are no MW spinors; moreover, the supergravity transformations (3.3) do not preserve reality of the frame, $e_{\bar z} = (e_z)^*$. We now address each of these problems in turn.

First, consider the quantization of a single MW fermion, with action $S = \int e\,\mathrm{d}^2\sigma\,\phi_+\partial_{\bar z}\phi_+$ in Minkowski space. Formally this system has the partition function $\int[\mathrm{d}\,\phi_+]\,e^{-S} = \text{Pfaff}\ \partial_{\bar z}$, the square root of $\det\partial_{\bar z}$. We can reproduce this result in Euclidean space if we interpret $[d\phi_+]$ not as the integral over a real (Grassmann) variable but as a contour integral in the complex ϕ_+ space. Which "contour" we choose is reflected in an overall phase.

Sometimes it is useful to define measures using metrics on field space. In our example we can take $\|\phi_+\|^2 = \int e\, d^2\sigma \phi_+^*(\sigma)\phi_+(\sigma)$, and using it define $[d\phi_+\, d\phi_+^*]$ by requiring $1 = \int [d\phi_+\, d\phi_+^*]\, e^{\|\phi_+\|^2}$. To apply this measure to our problem, we simply discard half the degrees of freedom to get $[d\phi_+]$ and integrate it as above. Similarly any jacobian computed with this complex measure will be the absolute square of the correct one for $[d\phi_+]$.

In the heterotic string we thus choose a "contour" for our integral over the slice. We then get the jacobian for the change to collective coordinates by working on the tangent space, *i.e.* small variations of frames, just as in the bosonic string [6][7]. As in the example above we will work with complex variations in order to have a metric, thus doubling the effective number of degrees of freedom. The desired jacobian will then be a square root of the expression thus obtained, as we discuss in sect. 6.

Specifically, our real slice will be of the form

$$\hat e_{\bar z}^{\ m}(t),\ \hat e_z^{\ m}(t) \equiv (\hat e_{\bar z}^{\ m}(t))^*,\ \hat\chi_{\bar z}^+ = \varsigma^\rho \nu_{\bar z-}^\rho, \tag{5.1}$$

where $\vec t$ are $6g-6$ real coordinates parameterizing a slice for ordinary real moduli space, ς^ρ are $2g-2$ real Grassmann coordinates, and $\nu_{\bar z-}^\rho$, $\rho = 1,\dots 2g-2$ are c-number spinors spanning a subspace of $\mathcal{T}_{-3}$ transverse to gauge directions (*i.e.* to the image of $D_{\bar u}^{(-1)}$). We can and do choose ν^ρ independent of ς^ρ.

Now consider the second problem. We simply define our configuration space to be the real gauge orbit of the slice. Since we never actually integrate away from the slice, it does not matter that the supergravity transformations are complex.

We can express complex deformations $\delta e_{\bar z}^{\ m}$, $\delta\chi_{\bar z}^+$ of the geometry in terms of a complex vector field ξ^a, a supersymmetry parameter ϵ^+, Weyl and Lorentz parameters w, ℓ, and moduli $\delta t_1, \delta t_2, \delta\varsigma$. Here δe_z, which depends on δt_1, is independent of $\delta e_{\bar z}$, which depends on another variation δt_2. Similarly ξ^z, $\xi^{\bar z}$ are independent. For example, letting $h_a^{\ b} = (\delta e_a^{\ m}) e_m^{\ b}$ and $(T^r)_a^{\ b} = (\frac{\partial}{\partial t^r} e_a^{\ m}) e_m^{\ b}$, we have

$$h_{zz} = -D_z\xi^z + w - i\ell + \delta t_1^r T_{zz}^r$$
$$h_{\bar z\bar z} = -D_{\bar z}\xi^{\bar z} + \delta t_2^r T_{\bar z\bar z}^r + i\sqrt{2}\epsilon^+\chi_{\bar z}^+, \quad etc.$$

Using the fact that the functional determinant for an *algebraic* change of variables equals unity [6], it is easy to show that the jacobian we seek is the product of two simpler ones:

$$h_{zz} = \begin{pmatrix} -D_z & T_{zz}^r \end{pmatrix} \begin{pmatrix} \xi^z \\ \delta t_1^r \end{pmatrix};$$

$$\begin{pmatrix} h_{\bar z\bar z} \\ \delta\chi_{\bar z}^+ \end{pmatrix} = \begin{pmatrix} -D_{\bar z} & -i\sqrt{2}\chi_{\bar z}^+ & T_{\bar z\bar z}^r & 0 \\ D_z\chi_{\bar z}^+ - \frac{1}{2}\chi_{\bar z}^+ D_z & +2D_{\bar z} & 0 & \nu_{\bar z-}^\rho \end{pmatrix} \begin{pmatrix} \xi^{\bar z} \\ \epsilon^+ \\ \delta t_2^r \\ \delta\varsigma^\rho \end{pmatrix}.$$

The first change of variables can be carried out much as in the bosonic string. Let S_{zz}^r, $r = 1,\dots 3g-3$ be solutions to $D_{\bar u} S_{zz}^r = 0$. Then [7]

$$[d h_{zz}\, d h_{zz}^*] = \det(D_u^{(2)\dagger} D_u^{(2)}) \det{}^{-1}\langle S^r, S^s\rangle\, |\det\langle S^r, T^s\rangle|^2\ [d\xi^z\, d\xi^{z*}\, d\delta t_1\, d\delta t_1^*] .$$

This expression is independent of the choice of basis S^r. The contribution from this sector to the heterotic string functional is now the chiral square root

$$J_1 = \det D_u^{(2)} \det\langle S^r, T^s\rangle \det{}^{-1/2}\langle S^r, S^s\rangle \quad . \tag{5.2}$$

The contribution from the other sector can be carried out by defining zero modes of the operator

$$Q_2 = \begin{pmatrix} -D_{\bar z} & -i\sqrt{2}\chi_{\bar z}^+ \\ D_z\chi_{\bar z}^+ - \frac{1}{2}\chi_{\bar z}^+ D_z & +2D_{\bar z} \end{pmatrix} = \begin{pmatrix} -D_{\bar z} & \\ & +2D_{\bar z} \end{pmatrix} + \mathcal{O}(\varsigma),$$

the counterpart of $Q_1 \equiv -D_z$. Let $\mathcal{S}^r$ denote $3g-3$ solutions to $Q_2^\dagger \mathcal{S}^r = 0$, such that $\mathcal{S}^r = \begin{pmatrix} (S_{zz}^r)^* \\ 0 \end{pmatrix} + \mathcal{O}(\varsigma)$. Let $\mathcal{P}^\rho$ be $2g-2$ solutions to the same equation such that $\mathcal{P}^\rho = \begin{pmatrix} 0 \\ P_{\bar z-}^\rho \end{pmatrix} + \mathcal{O}(\varsigma)$, where P^ρ are supermoduli, *i.e.* solutions to $D_u P_{\bar z-}^\rho = 0$. Let $H(Q_2^\dagger)$ be the graded matrix of inner products of these zero modes. Then just as before,

$$|J_2|^2 = \text{sdet}\, Q_2^\dagger Q_2\ \text{sdet}^{-1} H(Q_2^\dagger) \left|\text{sdet} \begin{pmatrix} \langle \mathcal{S}_{\bar z\bar z}, T_{\bar z\bar z}\rangle & \langle \mathcal{S}_{\bar z-}, \nu_{\bar z-}\rangle \\ \langle \mathcal{P}_{\bar z\bar z}, T_{\bar z\bar z}\rangle & \langle \mathcal{P}_{\bar z-}, \nu_{\bar z-}\rangle \end{pmatrix}\right|^2 \tag{5.3}$$

Unlike J_1, this expression's square root is a function of ς^ρ.

6. Anomalies

Thus for the higher-loop vacuum to vacuum amplitude W we have

$$W = \int \prod_{1}^{6g-6} \frac{dt^r}{\sqrt{2\pi}} \prod_{1}^{2g-2} d\varsigma^\rho \left(\frac{\det D_u^{(2)}}{\det^{1/2}\langle S^r, S^s\rangle} \det\langle S^r, T^s\rangle \right)$$
$$\times \sum_{\alpha,\beta} \eta_{\alpha\beta} \left[\frac{\text{sdet}_\alpha Q_2}{\text{sdet}_\alpha^{1/2} H(Q_2^\dagger)} \text{sdet}_\alpha \begin{pmatrix} \langle S^r, T^s\rangle & \langle S^r, \nu^\rho\rangle \\ \langle p^\rho, T^s\rangle & \langle p^\rho, \nu^\sigma\rangle \end{pmatrix} \right]$$
$$\times (\det_\alpha D_{\bar{u}}^{(1)})^5 \left(\frac{\det' - \nabla^2}{2\pi \int e} \right)^{-5} \left(\det_\beta D_u^{(-1)} \right)^{16} , \qquad (6.1)$$

where α labels the spin structures and the phases $\eta_{\alpha\beta}$ are needed for (symplectic) modular invariance[9]. Expression (6.1) involves chiral determinants which we have not yet defined rigorously. Quillen has shown in a different context that a canonical square root of the nonchiral determinant exists for any genus [15]. (Such a square root is easy to find for $g = 1$.) We will not solve here the important problem of defining $J_{1,2}$, but we expect that the approach of Quillen can be generalized to do the job. For the present purposes it suffices to know that (a) whatever definition we choose for sdet $Q_{1,2}$, their absolute squares will be given up to local counterterms by sdet $Q_{1,2}^\dagger Q_{1,2}$, which in turn can be defined via heat-kernel regularization, and that (b) any infinitesimal anomalies in the phase of sdet $Q_{1,2}$ will be local on the world sheet. We can then use consistency conditions[2] and Alvarez's family index calculation of the Lorentz and Weyl anomalies [17].

We begin by setting $\chi = 0$. Then $Q_{1,2}$ become simply covariant derivatives. Using

$$\begin{aligned} \delta_{W,L} \log\det D_z^{(n)} &= \frac{i}{48\pi}[2+3n(n+2)] \int e\, \tfrac{1}{2}\ell R \\ &\quad - \frac{1}{48\pi}[2+3n(n+2)] \int e\, wR \\ \delta_{W,L} \log\det D^{z}_{(n+2)} &= \frac{-i}{48\pi}[2+3n(n+2)] \int e\, \tfrac{1}{2}\ell R \\ &\quad - \frac{1}{48\pi}[2+3n(n+2)] \int e\, wR \quad , \end{aligned} \qquad (6.2)$$

[2] Somewhat different arguments appear in [16].

where R is the scalar curvature, one can check that the above combination of determinants has no anomalies under Weyl and Lorentz (hence gravitational) transformations. It is not entirely obvious that this cancellation will persist away from $\chi = 0$. Nevertheless we will argue that it is true as follows.

First we consider the supersymmetry anomaly. It must at least be globally Lorentz invariant, and also coordinate invariant if we choose to shift the anomalies to be strictly Lorentz.[3] Furthermore, it must be dimensionless. Since χ and ϵ^+ have mass dimensions $\pm\frac{1}{2}$ respectively, the only possible terms are

$$\begin{matrix} \int e\, D_z \chi_{\bar{z}}^+ D_z \epsilon^+ & \int e\, \chi_{\bar{z}}^+ \epsilon^+ (\omega_z)^2 \\ \int e\, D_z \chi_{\bar{z}}^+ \epsilon^+ \omega_z & \int e\, \chi_{\bar{z}}^+ D_z \epsilon^+ \omega_z \end{matrix} \qquad (6.3)$$

Of these, only the first satisfies the consistency condition

$$[\delta_S(\epsilon_1^+), \delta_S(\epsilon_2^+)] = \delta_{GC}(\xi) + \delta_L(\ell) + \delta_W(w) \ ,$$

where $\xi^z = -2i\sqrt{2}\epsilon_1^+\epsilon_2^+$, $\xi^{\bar{z}} = 0$, $\ell = \frac{i}{2} D_z \xi^z$, $w = -\frac{1}{2} D_z \xi^z$. No linear combination of the other three terms can satisfy these. Call the first term $\mathcal{A}_S$.

For the various functional determinants sdet $\mathcal{O}_i$ in (6.1) set $\delta_S \log\text{sdet}\mathcal{O}_i = c_i \mathcal{A}_S$. Then the consistency conditions say

$$\left(\sum c_i\right) \int e\, D_z(\epsilon_1^+\epsilon_2^+) R = [\delta_{GC} + \delta_L + \delta_W] \log \prod_i \text{sdet}\mathcal{O}_i \ .$$

Evaluating this at $\chi = 0$ shows that $\sum c_i = 0$. Since the c_i are just constants, the total supersymmetry anomaly cancels even away from $\chi = 0$. This in turn means that we can always gauge away χ locally, so that the coordinate, Lorentz, and Weyl anomalies all vanish too.

Prior to quantization we shifted the spin-$\frac{1}{2}$ part of the gravitino, χ_z^+, to zero. Had we not done so we would have retained the supersymmetric partner of conformal symmetry, gravitino shifts, as a symmetry of the theory. Since in the nonchiral string the anomaly in gravitino shifts is the supersymmetric partner of the conformal anomaly [10], we should find no gravitino shift anomaly here either.

[3] If we assume that the *local* Lorentz anomaly also cancels at $\chi \neq 0$, as suggested by the family index argument, then we can at once eliminate all but the first term in (6.3).

Finally we should verify the consistency of properties (a) and (b) above by checking that the nonlocal Weyl dependence in the zero-mode inner products cancels that in sdet $Q_2^\dagger Q_2$. In isothermal coordinates, where $g_{u\bar u} = e^{2\varphi}$,

$$Q_2 : \begin{pmatrix} v^u(\frac{\partial}{\partial u}) \\ p^\theta(\frac{\partial}{\partial u})^{1/2} \end{pmatrix} \longrightarrow \begin{pmatrix} h_{\bar u\bar u}(\mathrm{d}\,\bar u)^2 \\ m_{\bar u\bar\theta}(\mathrm{d}\,\bar u)^{3/2} \end{pmatrix} \tag{6.4}$$

and its adjoint in the component metrics are given by

$$\begin{aligned} Q_2 &= \begin{pmatrix} e^{2\varphi} & \\ & e^{\varphi} \end{pmatrix} \begin{pmatrix} -\partial_{\bar u} & -i\sqrt{2}\hat\chi_{\bar z}^{+} \\ -\frac{1}{2}\hat\chi_{\bar z}^{+}\partial_u + \partial_u\hat\chi_{\bar z}^{+} & 2\partial_{\bar u} \end{pmatrix} \\ Q_2^\dagger &= \begin{pmatrix} e^{-4\varphi} & \\ & e^{-3\varphi} \end{pmatrix} \begin{pmatrix} \partial_u & \frac{1}{2}((\chi_{\bar z}^{+})^*\partial_{\bar u} + 3\partial_{\bar u}(\hat\chi_{\bar z}^{+})^*) \\ i\sqrt{2}(\hat\chi_{\bar z}^{+})^* & -2\partial_u \end{pmatrix} \quad , \end{aligned} \tag{6.5}$$

where $\hat\chi_{\bar z}^{+} \equiv e^{\varphi/2}\chi_{\bar z}^{+}$ is φ-independent, so in the basis (6.4) the zero modes $\mathcal{S}$, $\mathcal{P}$ are φ-independent too. Since

$$\begin{aligned} \delta \log \mathrm{sdet}\,{}'Q^\dagger Q &= -\delta \int_\epsilon^\infty str'(e^{-tQ^\dagger Q})\frac{\mathrm{d}\,t}{t} \\ &= str'\begin{pmatrix} 2\delta\varphi & \\ & \delta\varphi \end{pmatrix} e^{-\epsilon Q\,Q^\dagger} - str'\begin{pmatrix} 4\delta\varphi & \\ & 3\delta\varphi \end{pmatrix} e^{-\epsilon Q^\dagger Q} \end{aligned} \tag{6.6}$$

and Q_2 has no zero modes for $g > 1$, the combination

$$\mathrm{sdet}\, Q_2^\dagger Q_2 / |\mathrm{sdet}\, H(Q_2^\dagger)|^2$$

depends only locally on φ. Under a Weyl transformation of the slice (5.1), $T_{\bar u\bar u}^r \mapsto e^{2\varphi}T_{\bar u\bar u}^r$, $\nu_{\bar u\bar\theta}^{\rho} \mapsto e^{\varphi}\nu_{\bar u\bar\theta}^{\rho}$. The third determinant in (5.2) and in (5.3) is Weyl-independent, so the nonlocal Weyl dependence in (6.1) cancels as expected.

7. Conclusion

We have generalized the analysis of [5][6][7] to give a discussion of supermoduli in string theory. We have worked out the appropriate integration measure on the superspace of moduli which remains after gauge fixing the heterotic string. The general answer is expressed in terms of some functional determinants and the solutions to a zero-mode problem. We have shown that this

answer is independent of the slice chosen to represent moduli space, *i.e.* that it is free from gauge anomalies, but we have not addressed global anomalies or modular invariance[18].

In [14] we will discuss the low-genus case, introduce vertex operators, and compute zero- and one-loop amplitudes. Much more remains to be done. We expect that our work will extend to the case with boundaries, and will therefore yield an off-shell propagator and Green functions as in [19].

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As this work was being completed we received [20], where supermoduli are discussed in the context of conformal field theory. It would be interesting to understand the connection between the two approaches.

Appendix

M is a real 2d riemannian manifold with coordinates σ^m and Euclidean metric g_{mn}. The metric defines a complex structure, and we choose complex coordinates $u, \bar u$ in which $g_{uu} = 0$. The tensors of type T^u form a bundle, and its square root consists of chiral spinors in a particular spin structure. These will have coordinate index θ.

We will integrate over frames $e^a \equiv \mathrm{d}\,x^m\, e_m{}^a \quad a = 1, 2$, chosen such that $e_m{}^a\, e_n{}^b\, \delta_{ab} = g_{mn}$. Thus letting e^z, $e^{\bar z} = \frac{1}{\sqrt{2}}(e^1 \pm i e^2)$ we have $e_u{}^{\bar z} = e_{\bar u}{}^{z} = 0$. We can then let $e_\theta{}^{+}$ be the square root of $e_u{}^{z}$ in the chosen spin structure, and similarly $e_{\bar\theta}{}^{-}$. Let α denote either $+$ or $-$. Thus $m = 1, 2; u, \bar u; \theta, \bar\theta$ are coordinate indices while $a = 1, 2; z, \bar z; \alpha = \pm$ are frame indices. The covariant derivative on M with the usual riemannian spin connection is called D_m.

We can also construct $\mathcal{M}$, a supermanifold with complex coordinates $\theta, \bar{\theta}$. For the heterotic string we make no use of $\bar{\theta}$. The generic index M can mean any of $m = 1, 2; \theta$. We will also use the dynamical frame $E_M{}^A$, where A can be $z, \bar{z}, +$. The covariant derivative on superspace is called ∇_M.

Finally our Dirac matrices are $\gamma^z_{++} = -\sqrt{2}$, $(\gamma_5)_\pm{}^\pm = \pm i$; the local Lorentz generator is $L_a{}^b = \epsilon_a{}^b$, $L_\alpha{}^\beta = \frac{1}{2}(\gamma_5)_\alpha{}^\beta$. Thus $\delta_L v^z = i\ell v^z$, *etc.*

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