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Phase diagrams for the randomly diluted resistor network and XY model

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The randomly diluted resistor network and XY model at low temperature T are studied near the d-dimensional percolation threshold using the ε expansion, where $\varepsilon = 6-d$. The series expansion of the inverse susceptibility in powers of T for the XY model is identical to that of the appropriate resistive inverse susceptibility in powers of σ_0^{-1}, where σ_0 is the conductance of a bond. However, the temperature-dependent critical concentration $p_c(T)$ for the XY model has no analog in the resistor network, where p_c clearly does not depend on σ_0. This distinction arises from a rather subtle difference between the Fourier component representation of the Gaussian model for the resistor network and that of the bounded potential energy associated with the XY model. We introduce a family of models which provides a smooth interpolation between these two models and show that the phase boundary for the XY model satisfies certain simple self-consistency checks involving other susceptibilities. In particular we provide the first explicit calculation of the universal crossover function to finite temperature of the dilute XY model.

I. INTRODUCTION

It has long been realized that there exists a precise one-to-one correspondence between linearized spin-wave theory for continuous spins (such as the XY model) and Kirchhoff's equations for a resistor network on the same underlying lattice.[1] Therefore, one expects that the low-temperature (T) correlation functions for the former spin model are exactly the same as suitable resistance correlations at least to leading order in T. Indeed, for randomly diluted networks, these two problems have been shown to be described by the same crossover exponent, ϕ, near the percolation threshold.[2] An important problem which has not been addressed so far concerns the fact that in spite of this seeming equivalence between the two models, their phase diagrams differ in an important, but physically obvious way. This difference may be discussed in terms of the scaling variable, which for the randomly diluted XY (RXY) model is $T/[J(p-p_c)^\phi]$, where J is the exchange interaction associated with occupied bonds, and for the randomly diluted resistor network (RRN) is $\lambda^2/[\sigma_0(p-p_c)^\phi]$, where λ^2 is an adjustable parameter whose significance will be elucidated later and σ_0 is the conductance of an occupied bond. For the RXY model we will obtain the result for the concentration-dependent critical temperature, $T_c(p)$, which one expects on the basis of scaling, namely

$$T_c(p) = a[p-p_c]^\phi , \tag{1}$$

where a is a constant and within the ε expansion ϕ is given by[2-6]

$$\phi = 1 + \frac{\varepsilon}{42} , \tag{2}$$

where $\varepsilon = 6-d$, d being the spatial dimension. Alternatively, we may write Eq. (1) as

$$p_c(T) = p_c(0) + bT^{1/\phi} . \tag{3}$$

For the RRN we do not expect p_c to depend on σ_0, which is the parameter one would view as being the analog of J/T. The main result of this paper is to obtain phase boundaries for these two models consistent with these observations from scaling functions whose expansions in σ_0^{-1} and T/J, respectively, are identical. Furthermore we will introduce a model which interpolates smoothly between these two phase diagrams. We will also find universal relations between the coefficient b in Eq. (3) and other measurable amplitudes.

Explicitly the models we consider are the following. The RXY model is governed by the Hamiltonian

$$H = -\sum_{\langle \mathbf{x},\mathbf{x}'\rangle} J_{\mathbf{x},\mathbf{x}'}[\mathbf{S}(\mathbf{x})\cdot\mathbf{S}(\mathbf{x}')-1] , \tag{4}$$

where $\langle \mathbf{x},\mathbf{x}'\rangle$ indicates that the sum is over pairs of nearest neighboring sites, $\mathbf{x}$ and $\mathbf{x}'$, $\mathbf{S}(\mathbf{x}) \equiv (\cos\theta(\mathbf{x}), \sin\theta(\mathbf{x}))$ is an xy spin at the site $\mathbf{x}$, and $J_{\mathbf{x},\mathbf{x}'}$ is a quenched random variable which assumes the values J with probability p and 0 with probability $1-p$, corresponding, respectively to the bond $\mathbf{x}$-$\mathbf{x}'$ being occupied and vacant. The variable $\theta(\mathbf{x})$ is confined to the interval 0 to 2π. For this model we consider the kth-order spin susceptibility $\chi_{xy}^{(k)}$ defined as

$$\chi_{xy}^{(k)}(T,p) = \sum_{\mathbf{x}'} [\langle e^{ik\theta(\mathbf{x})} e^{-ik\theta(\mathbf{x}')}\rangle_T]_{\text{av}} , \tag{5}$$

where $\langle \cdots \rangle_T$ indicates a thermal average at temperature T and $[\cdots]_{\text{av}}$ an average over all configurations of occupied and vacant bonds. For the RRN the "Hamiltonian" is

$$H/T=\tfrac{1}{2}\sum_{\langle \mathbf{x},\mathbf{x}'\rangle}\sigma_{\mathbf{x},\mathbf{x}'}[V(\mathbf{x})-V(\mathbf{x}')]^2 , \tag{6}$$

where $V(\mathbf{x})$ is a Gaussian variable interpreted as the voltage at site $\mathbf{x}$ and $\sigma_{\mathbf{x},\mathbf{x}'}$ is a quenched random variable which assumes the values σ_0 with probability p and 0 with probability $1-p$, corresponding to the bond $\mathbf{x}$-$\mathbf{x}'$ respectively being occupied and vacant. For this model we consider the generalized resistive susceptibility $\chi_{\rm RN}$ defined as

$$\chi_{\rm RN}^{(\lambda)}(p)=\sum_{\mathbf{x}'}[\langle e^{i\lambda V(\mathbf{x})}e^{-i\lambda V(\mathbf{x}')}\rangle_T]_{\rm av} . \tag{7}$$

It can be shown[5,7] that

$$\begin{aligned}\chi_{\rm RN}^{(\lambda)}(p)&=\sum_{\mathbf{x}'}[e^{-(1/2)\lambda^2R(\mathbf{x},\mathbf{x}')}]_{\rm av}\\ &=\sum_{\mathbf{x}'}\sum_{k}(-\lambda^2/2)^k\frac{1}{k!}[\nu(\mathbf{x},\ \mathbf{x}')R(\mathbf{x},\mathbf{x}')^k]_{\rm av} ,\end{aligned} \tag{8}$$

where in a given configuration $R(\mathbf{x},\mathbf{x}')$ is the resistance between the nodes at $\mathbf{x}$ and $\mathbf{x}'$ and $\nu(\mathbf{x},\ \mathbf{x}')$ is an indicator function which is unity if the nodes $\mathbf{x}$ and $\mathbf{x}'$ are in the same cluster and is zero otherwise. We interpret νR^k to be zero when $\nu=0$ and $R=\infty$.

We will be interested in the behaviors of these susceptibilities for concentration, p, near the percolation threshold at $p=p_c$. For instance, for the RXY model we expect that

$$\chi_{xy}^{(k)}(T,p)^{-1}=A_1t^{\gamma}G_{xy}^{(k)}(A_2x) , \tag{9}$$

where $t=p_c-p$, γ is the susceptibility exponent for percolation, and x is the appropriate scaling variable, $x=T/(Jt^{\phi})$. If the nonuniversal constants A_1 and A_2 are chosen, e.g., by setting $G(0)=G'(0)=1$, then the scaling function $G_{xy}^{(k)}(x)$ is universal.[8] (We will use A_k to denote a nonuniversal quantity which is not necessarily the same in different occurrences.) For the RRN, it is known[5,9,10] that the average of the kth moment of the resistance, $R(\mathbf{x},\mathbf{x}')$ between two points $\mathbf{x}$ and $\mathbf{x}'$ in the same cluster, scales as $r^{k\phi/\nu}$ for $r=|\mathbf{x}-\mathbf{x}'|$ less than the percolation correlation length ξ, where $\xi\sim t^{-\nu}$. In view of Eq. (8), the fact that each power of R can be associated with a power of $r^{\phi/\nu}$ indicates that $\chi_{\rm RN}$ can be written in the scaling form

$$\chi_{\rm RN}^{(\lambda)}(p)^{-1}=A_1t^{\gamma}G_{\rm RN}(A_2x) , \tag{10}$$

where here the scaling variable is $x=\lambda^2/(\sigma_0t^{\phi})$ and $G_{\rm RN}$ is a universal function.

It is convenient to also consider a more general model which interpolates between the RXY model and the RRN. This generalized randomly diluted XY (GRXY) model has the Hamiltonian

$$H'_{xy}=-\sum_{\langle \mathbf{x},\mathbf{x}'\rangle}J_{\mathbf{x},\mathbf{x}'}\lambda_{\rm min}^{-2}(\cos\{[\theta(\mathbf{x})-\theta(\mathbf{x}')]\lambda_{\rm min}\}-1) , \tag{11}$$

where θ is restricted to the interval $0\le\theta\le2\pi/\lambda_{\rm min}$. To see the meaning of the Hamiltonian H'_{xy}, consider its expansion in powers of $[\theta(\mathbf{x})-\theta(\mathbf{x}')]$. To quadratic order in the θ's, H'_{xy} is independent of $\lambda_{\rm min}$. The role of $\lambda_{\rm min}$ is to set the scale over which the quadratic approximation to the cosine in Eq. (11) is valid, as shown in Fig. 1. Also, $\lambda_{\rm min}^{-2}$ sets the scale of the barrier in the potential when it is extended periodically to all θ. Intuitively it is clear that as $\lambda_{\rm min}$ approaches zero, the Hamiltonian of the GRXY in Eq. (11) approaches that of the RRN of Eq. (6). For this Hamiltonian we define the susceptibility

$$\hat{\chi}_{xy}^{(\lambda)}=\sum_{\mathbf{x}'}[\langle e^{i\lambda\theta(\mathbf{x})}e^{-i\lambda\theta(\mathbf{x}')}\rangle'_T]_{\rm av} , \tag{12}$$

where $\langle\cdots\rangle'_T$ indicates that the thermal average is taken with respect to H'_{xy}. Here we implicitly assume that λ is an integer multiple of $\lambda_{\rm min}$, to avoid any phase incompatibilities. As we shall show, $\hat{\chi}_{xy}^{(\lambda)}$ has the scaling representation

$$\hat{\chi}_{xy}^{(\lambda)}(T,p)^{-1}=A_1t^{\gamma}F(A_2T\lambda^2/(Jt^{\phi}),\lambda_{\rm min}/\lambda) , \tag{13}$$

where the function F is a universal function of both its arguments when A_1 and A_2 are fixed by, for example, requiring, $F(x,y)=\partial F(x,y)/\partial x=1$ for $x\to0$. It should be noted that the form in Eq. (13) is invariant with respect to the transformation applied to Eqs. (11) and (12), $\theta\to b\theta'$, $\lambda_{\rm min}\to b^{-1}\lambda_{\rm min}'$, $\lambda\to b^{-1}\lambda'$, and $J\to b^{-2}J'$. One sees that

$$\chi_{\rm RN}^{(\lambda)}(p)^{-1}=A_1t^{\gamma}F(A_2\lambda^2/(\sigma_0t^{\phi}),0) \tag{14a}$$

and

$$\chi_{xy}^{(1)}(T,p)=A_1t^{\gamma}F(A_2T/(Jt^{\phi}),1) . \tag{14b}$$

We will show that the power series expansion of $F(x,y)$ in powers of x is independent of y. Thus the coefficients of the expansion of $\chi_{\rm RN}$ in powers of σ_0^{-1} are the same as those of $\hat{\chi}_{xy}^{(\lambda)}$ in powers of T/J, independent of the value of $\lambda_{\rm min}$. This independence results from the fact that all these models have the same Hamiltonian to quadratic order in the θ's. However, we will show that the phase boundary in $F(x,y)$ occurs at $(xy^2)^{1/\phi}=-1/c$, where c is

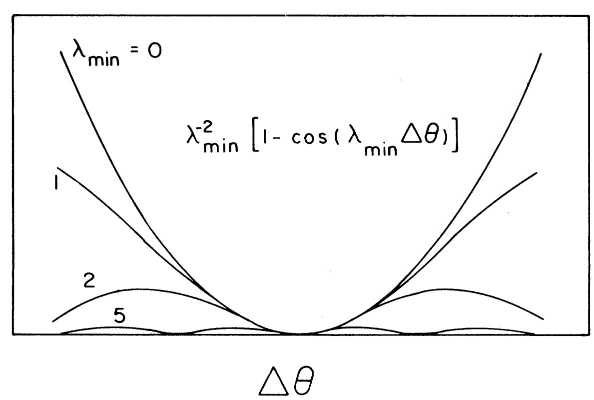

FIG. 1. Potential energy for the GRXY model as a function of θ for various values of $\lambda_{\rm min}$ as contrasted to that of the Gaussian (RRN) model.

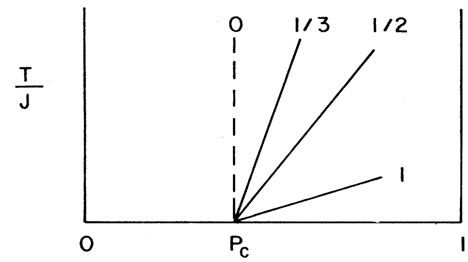

FIG. 2. Phase boundaries of the GRXY model as a function of p for indicated values of $\lambda_{\min}$. For the RXY model $\lambda_{\min}=1$ and for the RRN $\lambda_{\min}=0$.

a universal constant, so that

$$p_c(T)-p_c=c[A_2(T/J)\lambda_{\min}^2]^{1/\phi} , \tag{15}$$

as shown in Fig. 2. For the RRN, for which $\lambda_{\min}=0$, Eq. (15) gives correctly that p_c is independent of σ_0, the analog of J/T, whereas for the GRXY model it gives $\partial p_c/\partial T^{1/\phi}\sim\lambda_{\min}^{2/\phi}$, so that as $\lambda_{\min}$ varies from 1 to 0, we pass continuously from the RXY model to the RRN. It is tempting to associate thermal activation over the barriers of height $\lambda_{\min}^{-2}$ with this dependence of $p_c(T)$ on T.

Equation (15) provides a universal relation between the nonuniversal coefficients A_2 and b: the former can be obtained from measurements of $A_2=J\partial\ln\chi/\partial T\,|_{T=0}$, while the latter is found from Eq. (3). The ratio $c=b(A_2\lambda_{\min}^2/J)^{-1/\phi}$ is then universal.

Up to now there has been no calculation of the universal scaling function for the susceptibility of the RXY model. It has been known for some time[5,9] that the function $F(x,0)$ (i.e., the resistive susceptibility of the RRN) is indeed a function of the scaling variable $x=w\lambda^2/t^{\phi}$. Using the Rudnick-Nelson differential recursion relations[11] Harris and Lubensky[5] (HL) calculated the universal resistance ratios,

$$\rho_k\equiv\sum_{\mathbf{x}'}[\nu(\mathbf{x},\mathbf{x}')R(\mathbf{x},\mathbf{x}')^k]_{\rm av}\chi_0^{k-1}\times\left[\sum_{\mathbf{x}'}[\nu(\mathbf{x},\mathbf{x}')R(\mathbf{x},\mathbf{x}')]_{\rm av}\right]^{-k} , \tag{16}$$

where $\chi_0\equiv\sum_{\mathbf{x}'}[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}$ is the percolation susceptibility. From these ratios all the power-series coefficients in the expansion of $F(x,0)$ can be obtained and the resulting function explicitly constructed.[12] Meir *et al.*[12] also obtained $F(x,0)$ from a simpler direct diagrammatic calculation. We give here the first discussion of $F(x,y)$ for large x and its continuation to negative x near the phase boundary.

II. DIAGRAMMATIC EVALUATION OF $\chi_{xy}^{(\lambda)}$ FOR THE GENERALIZED *XY* MODEL

We now turn to the calculation of the inverse susceptibility for the GRXY model of Eq. (11). We use the field-theoretic formulation initiated by Stephen[7] as analyzed in detail by HL. Here we limit ourselves to zero momentum. The extension to nonzero momentum is immediate, but algebraically complicated, and will be discussed elsewhere.[13]

In the Stephen formalism one introduces the effective pair interaction $H_{\rm eff}(\mathbf{x},\mathbf{x}')$ as the average over the replicated Hamiltonian:

$$\exp[-H_{\rm eff}(\mathbf{x},\mathbf{x}')/T]=1-p+p\exp\left[[J/(T\lambda_{\min}^2)]\sum_{\alpha=1}^{n}(\cos\{[\theta_\alpha(\mathbf{x})-\theta_\alpha(\mathbf{x}')]\lambda_{\min}\}-1)\right] , \tag{17}$$

where the limit $n\to0$ is implied. This leads to the n-component Fourier representation of $H_{\rm eff}(\mathbf{x},\mathbf{x}')$:

$$H_{\rm eff}(\mathbf{x},\mathbf{x}')/T=\sum_{\lambda}B(\boldsymbol{\lambda})\exp\{i[\boldsymbol{\lambda}\cdot\boldsymbol{\theta}(\mathbf{x})-\boldsymbol{\lambda}\cdot\boldsymbol{\theta}(\mathbf{x}')]\} , \tag{18}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ are vectors with respective components $\lambda_1,\lambda_2,\ldots,\lambda_n$ and $\theta_1,\theta_2,\ldots,\theta_n$. Since θ ranges over the interval 0 to $2\pi/\lambda_{\min}$, we see that each component of $\boldsymbol{\lambda}$ in Eq. (18) is summed over the values $k\lambda_{\min}$, with k an integer. As discussed by HL we may omit the term with $\boldsymbol{\lambda}=0$ in Eq. (18), since it represents a constant. To obtain the RRN, one simply sets $\lambda_{\min}=0$ and $J/T=\sigma_0$, and the sum over $\boldsymbol{\lambda}$ becomes an integral. As Stephen[7] and HL have shown, $B(\boldsymbol{\lambda})$ is adequately represented as

$$B(\boldsymbol{\lambda})=\ln(1-p)+w\lambda^2 , \tag{19}$$

where $w=A_2T/J$. The only modification here from the formulation of HL is the introduction of the parameter $\lambda_{\min}$.

As discussed by HL, the major difference between the Gaussian model for the RRN and the GRXY model is that in the former the voltage variable ranges over the interval $-\infty$ to ∞ whereas in the latter the angle variable is confined to the interval 0 to $2\pi/\lambda_{\min}$. After a Stratonovich transformation to Gaussian fields $\Psi_\lambda(\mathbf{x})$ one finds the field theoretical Hamiltonian to be

$$H=\tfrac{1}{2}\sum_{\lambda}{}'\int\frac{d\mathbf{k}}{(2\pi)^d}\Psi_\lambda(\mathbf{k})\Psi_{-\lambda}(-\mathbf{k})[r_0+w\lambda^2+k^2]+\tfrac{1}{6}u_3\sum_{\lambda_1,\lambda_2}{}''\int\frac{d\mathbf{k}_1}{(2\pi)^d}\int\frac{d\mathbf{k}_2}{(2\pi)^d}\Psi_{\lambda_1}(\mathbf{k}_1)\Psi_{\lambda_2}(\mathbf{k}_2)\times\Psi_{-\lambda_1-\lambda_2}(-\mathbf{k}_1-\mathbf{k}_2) , \tag{20}$$

where $r_0\sim p_0-p$, with p_0 the mean-field value of p_c: $p_0\sim1-\exp(-1/z)$ where z is the coordination number of the lattice, and w for the RRN is of order $1/\sigma_0$, whereas w for the GRXY model is of order T/J. The primes on the sums in Eq. (20) indicate that terms which involve a $\Psi_0(\mathbf{k})$ are to be omitted and $\Psi_\lambda(\mathbf{k})$ is the spatial Fourier transform of $\Psi_\lambda(\mathbf{x})$. It is apparent at this stage that the only difference between the RRN and the GRXY model is the difference in mesh size, $\lambda_{\min}$. In writing Eq. (20) we have made several simplifications. We have omitted terms in H involving four or more Ψ's, since such terms are irrelevant, in the renormalization group (RG) sense, for d near 6. Also the coefficient in the quadratic term, $w\lambda^2$, is only the leading approximation to a series in powers of λ^2. However, the detailed analysis of HL indicates that terms of order $w_p\lambda^{2p}$ for $p>1$ give rise to corrections to scaling, which we will not consider here. The susceptibility of Eq. (12) is obtained from the correlation function with respect to the Hamiltonian (20) as follows. If $\langle \cdots \rangle_H$ denotes a canonical average with respect to the Hamiltonian H of (20), then

$$\hat{\chi}_{xy}^{(\lambda)}=\langle \Psi_\lambda(\mathbf{k}=0)\Psi_{-\lambda}(\mathbf{k}=0)\rangle_H \ . \tag{21}$$

Here we neglect amplitudes introduced in the various transformations required to obtain the Hamiltonian of Eq. (20). This susceptibility is somewhat more general than we need to discuss. We therefore restrict attention to the special case when $\boldsymbol{\lambda}=(\lambda,0,0,\ldots,0)$. Susceptibilities when two or more components of $\boldsymbol{\lambda}$ are nonzero are needed to generate higher order averages of the form $[\langle \cdots \rangle_T^m]_{\rm av}$ with $m\geq 2$. For instance, the case $m=2$ is the type of average used to describe spin-glass ordering. The susceptibility when $\boldsymbol{\lambda}=(\lambda,0,0,\ldots,0)$ is just the susceptibility $\hat{\chi}_{xy}^{(\lambda)}$ of Eq. (12). In this picture percolation at $T=0$ is described by the simultaneous criticality of correlation functions for all values of m. To get the RXY susceptibility one sets $\lambda=\lambda_{\min}$, whereas to get the resistive susceptibility one sets $\lambda_{\min}=0$.

For $w=0$, the Hamiltonian of Eq. (20) describes percolation. To see this for the RRN, note that $w=0$ corresponds to $\sigma_0=\infty$, so that $R(\mathbf{x},\mathbf{x}')$ assumes the values zero, if the sites $\mathbf{x}$ and $\mathbf{x}'$ are connected, and infinity, if they are not connected. Thus, in this limit the quantity $\exp[-\frac{1}{2}\lambda^2 R(\mathbf{x},\mathbf{x}')]$ is the indicator function for pair connectedness, $\nu(\mathbf{x},\mathbf{x}')$, which is used to define the percolation susceptibility. Similarly, for the GRXY model, $w=0$ is equivalent to the limit $T=0$, in which case all spins in a given cluster are parallel to one another, whereas spins in different clusters are totally uncorrelated. Thus $\langle \exp[ik\theta(\mathbf{x})]\exp[-ik\theta(\mathbf{x}')]\rangle_T$ assumes the values unity or zero depending on whether the sites $\mathbf{x}$ and $\mathbf{x}'$ are connected or not. Thus "turning on" w is a perturbation away from the percolation problem. We will present a calculation to first order in ε. To this order one has that ϕ is given in Eq. (2) and[14]

$$\gamma=1+\varepsilon/7 \ . \tag{22}$$

Before getting into the details of the calculation we remark on the expected analytic structure of $F(x,y)$, depending on whether or not $\lambda_{\min}=0$ (recall that $y=\lambda_{\min}/\lambda$). First of all, because of the analogy between Kirchhoff's equations and those for spin waves,[1] we expect that the coefficients in the expansion of $F(x,y)$ in powers of x should be independent of y. Now we consider the situation for large x. Here, for the RXY model ($\lambda_{\min}\neq 0$) the situation is similar to that for the dilute Ising model:[15,16] $F(x,y)$ must give rise to a susceptibility $\chi_{xy}(T,p)$ which for $T\neq 0$ is analytic for $p<p_c(T)$, where $p_c(T)>p_c(0)\equiv p_c$. For this to happen, $F(x,y)$ must have the large-x expansion,

$$F(x,y)=x^{\gamma/\phi}\sum_{n=0} a_n(y)x^{-n/\phi} \ . \tag{23}$$

Since $x\sim(p_c-p)^{-\phi}$, this form leads to a power series in integer powers of the variable p_c-p, which can therefore be analytically continued to negative values. In our calculation we are only going to obtain $F(x,y)$ to linear order in ε. It is therefore of interest to see what Eq. (23) is to that order. To order ε^0, we have that $F(x,y)=1+x$ independent of y. Thus in Eq. (23) all the a_n's for n not equal to 0 or 1 are of order ε. Accordingly, we set $a_n(y)=\delta_{n,0}+\delta_{n,1}+\varepsilon b_n(y)$, where δ is the Kronecker delta function, and the $b_n(y)$'s are independent of ε. Thus the analyticity requirement indicates that for the GRXY model one has the large x expansion

$$F(x,y)=x+1+\varepsilon\sum_{n=0}^{\infty} b_n(y)x^{1-n}+\varepsilon(\tfrac{5}{42}x+\tfrac{1}{7})\ln x \ , \tag{24}$$

where we have used the values of the exponents given by Eqs. (2) and (22). In contrast, consider the large-x behavior of the RRN, for which $\lambda_{\min}=0$. Here the situation is different than for the GRXY model, in that we expect the phase boundary to occur at $p=p_c$ for all λ. Thus for $\lambda_{\min}=0$ we expect to get a form inconsistent with Eq. (24). Therefore we expect that the function $F(x,y)$, although it has a power series for small x whose coefficients are independent of y, has a crucial dependence on y for large x.

We will calculate the inverse susceptibility of Eq. (12) to one loop order in the diagrammatic formulation[17] of the ε expansion, where $\varepsilon=6-d$ is small. We will obtain an explicit closed-form result for the RRN which agrees with the universal amplitude ratios previously found by HL. As we have just discussed this form is *not* consistent with Eq. (24) because the phase boundary for the RRN occurs at $p=p_c$. For the GRXY model we do not obtain a closed-form solution for $F(x,y)$, although it is reduced to quadrature. However, we do verify that (a) the small-x expansion for $F(x,y)$ in powers of x is independent of y, (b) for the GRXY model, unlike for the RRN, $F(x,y)$ is of the form of Eq. (24) for large x, and (c) for the GRXY model $F(x,y)$ is an analytic function of x for $(|x|y^2)^{1/\phi}<1/c+O(\varepsilon)$, where c is a constant from which a formal expression is given for $p_c(T)$ for the GRXY model, as in Eq. (15). Since $F(x,y)$ is universal, the constant c is also universal.

For the calculation to order ε we need evaluate only the contribution from the single diagram of Fig. 3, from which we get

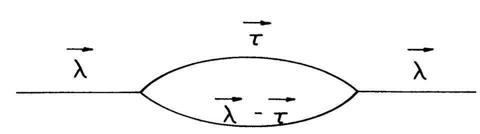

FIG. 3. Diagram which contributes to the susceptibility in one-loop order.

$$\hat{\chi}^{(\lambda)}(w,r_0)^{-1}=r_0+w\lambda^2-\tfrac{1}{2}u_3^2\int\frac{d\mathbf{k}}{(2\pi)^d}\sum_\tau{}'\frac{1}{(r_0+k^2+w\tau^2)[r_0+k^2+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]}\,, \tag{25}$$

where $\sum_\tau{}'$ indicates a sum over τ with the values $\tau=0$ and $\tau=-\lambda$ excluded. If r_{0c} is the value of r_0 for which $p=p_c$ and the susceptibility diverges, then we may write

$$\hat{\chi}_{xy}^{(\lambda)}(w,t)^{-1}=t+w\lambda^2-\tfrac{1}{2}u_3^2\int\frac{d\mathbf{k}}{(2\pi)^d}\sum_\tau{}'\left[\frac{1}{(r_0+k^2+w\tau^2)[r_0+k^2+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]}-\frac{1}{(r_{0c}+k^2)^2}\right] \tag{26a}$$

$$=t+w\lambda^2-\tfrac{1}{2}u_3^2\int\frac{d\mathbf{k}}{(2\pi)^d}\sum_\tau{}'\left[\frac{1}{(t+k^2+w\tau^2)[t+k^2+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]}-\frac{1}{k^4}\right]+O(u_3^4)\,, \tag{26b}$$

where $t=r_0-r_{0c}$. Since deviations in u_3 from its fixed point value u_3^* give rise to corrections to scaling,[11] which we do not consider, we set[14] $u_3=u_3^*$: $K_d u_3^2=2\varepsilon/7$, where K_d is the phase space factor in d dimensions. With $z=k^2$ we have that $d\mathbf{k}/(2\pi)^d=\frac{1}{2}K_d z^2 dz$ whence

$$\hat{\chi}_{xy}^{(\lambda)}(w,t)^{-1}=t+w\lambda^2+T_1+T_2+T_3\,, \tag{27}$$

with

$$T_1=-\frac{\varepsilon}{14}\int_0^\infty z^2dz\sum_\tau{}'\left[\frac{1}{(t+z+w\tau^2)[t+z+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]}-\frac{1}{(t+z+w\tau^2)^2}-\frac{1}{(z+w\tau^2)[z+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]}+\frac{1}{(z+w\tau^2)^2}\right], \tag{28a}$$

$$T_2=-\frac{\varepsilon}{14}\int_0^1 z^2dz\sum_\tau{}'\left[\frac{1}{(t+z+w\tau^2)^2}-\frac{1}{(z+w\tau^2)^2}-\frac{1}{z^2}\right], \tag{28b}$$

$$T_3=-\frac{\varepsilon}{14}\int_0^1 z^2dz\sum_\tau{}'\frac{1}{(z+w\tau^2)[z+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]}\,, \tag{28c}$$

where we replaced r_0 by t. We must keep all contributions which are at least as large as linear in t or in w. Higher order contributions do not contribute to the scaling function $F(x,y)$. Accordingly, since the integrand in Eq. (28a) is of order tw (it vanishes if either t or w is zero), its contributions which are linear in t or in w can only come from x near zero. We therefore extended the range of the z integration to $+\infty$ in Eq. (28a).

We now study the T_i's. For instance, we evaluate T_3 following the methods of HL:

$$T_3=-\frac{\varepsilon}{14}\int_0^1 z^2dz\sum_\tau{}'\int_0^\infty d\mu\int_0^\infty d\nu\exp\{-\mu(z+w\tau^2)-\nu[z+w(\boldsymbol{\lambda}+\boldsymbol{\tau})^2]\}\,. \tag{29}$$

We set $\mu=\xi(1-\rho)/2$ and $\nu=\xi(1+\rho)/2$, $\boldsymbol{\lambda}=(\lambda,0,0,\ldots,0)$. Then

$$T_3=-\frac{\varepsilon}{14}\int_0^1 z^2dz\int_0^\infty \xi\,d\xi\int_{-1}^1\frac{d\rho}{2}\exp[-\xi z-\tfrac{1}{2}\xi(1+\rho)w\lambda^2]\times\left[\sum_{m_1}\exp[-\xi w\lambda_{\min}^2 m_1^2-\xi w(1+\rho)\lambda_{\min}\lambda m_1]\right]\times\left[\left(\sum_{m_2}\exp(-\xi w\lambda_{\min}^2 m_2^2)\right)^{n-1}-1-\exp(w\xi\rho\lambda^2)\right], \tag{30}$$

in which form the limit $n\to0$ is easily taken. It is convenient to define

$$S(y)=\frac{\sum_m\exp[-\xi wy^2\lambda^2m^2-\xi w(1+\rho)y\lambda^2m]}{\sum_m\exp(-\xi wy^2\lambda^2m^2)}\,, \tag{31}$$

where $y=\lambda_{\min}/\lambda$, so that the RRN involves $S(0)\equiv\lim_{y\to0}S(y)$ and the RXY model involves $S(1)$. For the RRN we consider the limit $\lambda_{\min}\to0$ in which case the sums over m can be replaced by integrals and we have

$$S(0)=\exp[\xi(1+\rho)^2w\lambda^2/4]\,. \tag{32}$$

For the GRXY model, as we will discuss in a moment, one can not replace the sum $S(1)$ by the integral $S(0)$. With the replacement of Eq. (32) we can evaluate T_3 for the RRN, which we denote $T_{3;\mathrm{RRN}}$:

$$T_{3;\mathrm{RRN}}=\frac{\varepsilon}{7}\left[\frac{1}{2}-\frac{w\lambda^2}{12}+\frac{5}{6}w\lambda^2\ln(w\lambda^2)-Cw\lambda^2\right], \tag{33}$$

where

$$C=\tfrac{1}{8}\int_{-1}^{1}d\rho(1-\rho^2)\ln[(1-\rho^2)/4]=-\tfrac{5}{18}\ . \tag{34}$$

For the GRXY model we set $T_3=T_{3;\mathrm{RRN}}+\Delta T_3$ with

$$\Delta T_3=-\frac{\varepsilon}{14}\int_0^\infty z^2dz\int_0^\infty \xi d\xi\int_{-1}^{1}\frac{d\rho}{2}\exp[-\xi z-\xi(1+\rho)w\lambda^2/2][S(y)-S(0)]\ . \tag{35}$$

In writing Eq. (35) we extended the z integral to $+\infty$. This is justified because we only consider the case when w is small. In this case the quantity $[S(y)-S(0)]$ can be estimated by Poisson's summation formula to be of order $\exp[-a/(\xi w)]$. The value of ξ which then maximizes the product $\exp[-\xi z-a/(\xi w)]$ is of order $[a/(zw)]^{1/2}$, in which case the integrand of the z integral is of order $\exp[(-az/w)^{1/2}]$. Clearly contributions at least as large as linear in w come exclusively from small z and are not affected by extending the range of the z integral to $+\infty$. We then evaluate the z integral to obtain

$$\Delta T_3=-\frac{\varepsilon}{7}\int_0^\infty\frac{d\xi}{\xi^2}\int_{-1}^{1}\frac{d\rho}{2}\exp[-\xi(1+\rho)w\lambda^2/2][S(y)-S(0)]\ . \tag{36}$$

From this equation one can see that $\Delta T_3=-\varepsilon w\lambda^2C_3(y)/7$, where $C_3(y)$ vanishes as $y\to 0$. Thus, in all

$$T_3=\frac{\varepsilon}{7}\left[\tfrac{1}{2}+\tfrac{7}{36}w\lambda^2+\tfrac{5}{6}w\lambda^2\ln(w\lambda^2)-\int_0^\infty\frac{d\xi}{\xi^2}\int_{-1}^{1}\frac{d\rho}{2}\exp[-\xi(1+\rho)w\lambda^2/2][S(y)-S(0)]\right]. \tag{37}$$

Similarly we evaluate T_2 as

$$T_2=\frac{\varepsilon}{7}[-\tfrac{1}{2}+(t+w\lambda^2)\ln(t+w\lambda^2)-w\lambda^2\ln(w\lambda^2)+t/2]\ , \tag{38}$$

and for T_1 we obtain

$$T_1=\frac{\varepsilon}{7}\int_0^\infty\frac{d\xi}{\xi^2}(1-e^{-\xi t})\int_{-1}^{1}\frac{d\rho}{2}[e^{-\xi w\lambda^2}-e^{-\xi(1+\rho)w\lambda^2/2}-e^{-\xi(1-\rho)w\lambda^2/2}+S(y)e^{-\xi(1+\rho)w\lambda^2/2}]\ . \tag{39}$$

Collecting the above results we write, correct to order ε,

$$\begin{aligned}\hat{\chi}_{xy}^{(\lambda)}(w,t)^{-1}=t+w\lambda^2+\frac{\varepsilon}{7}\Big[&\tfrac{1}{2}t+\tfrac{7}{36}w\lambda^2-\tfrac{1}{6}w\lambda^2\ln(w\lambda^2)+(t+w\lambda^2)\ln(t+w\lambda^2)\\&-\int_0^\infty\frac{d\xi}{\xi^2}\int_{-1}^{1}\frac{d\rho}{2}[S(y)-S(0)]e^{-\xi w\lambda^2(1+\rho)/2}\\&+\int_0^\infty\frac{d\xi}{\xi^2}(1-e^{-\xi t})\int_{-1}^{1}\frac{d\rho}{2}[e^{-\xi w\lambda^2}-e^{-\xi(1+\rho)w\lambda^2/2}-e^{-\xi(1-\rho)w\lambda^2/2}+S(y)e^{-\xi w\lambda^2(1+\rho)/2}]\Big]\ .\end{aligned} \tag{40}$$

We set $t+(\varepsilon/7)t\ln t=t^\gamma$ and $w\lambda^2+(5\varepsilon/42)w\lambda^2\ln t=w\lambda^2t^{\gamma-\phi}$. Then, by proper choice of amplitudes we can express $\hat{\chi}_{xy}^{(\lambda)}$ in terms of a normalized scaling function, $F(x,y)$:

$$\hat{\chi}_{xy}^{(\lambda)}(w,t)^{-1}=t^\gamma(1+\varepsilon/14)F((1+3\varepsilon/28)w\lambda^2/t^\phi,y), \tag{41}$$

where

$$\begin{aligned}F(x,y)=1+x+\frac{\varepsilon}{7}\Big[&-\tfrac{19}{18}x-\tfrac{1}{6}x\ln x+(1+x)\ln(1+x)-x\int_0^\infty\frac{d\xi}{\xi^2}\int_{-1}^{1}\frac{d\rho}{2}[\hat{S}(y)-\hat{S}(0)]e^{-\xi(1+\rho)/2}\\&+x\int_0^\infty\frac{d\xi}{\xi^2}[1-e^{-\xi/x}]\int_{-1}^{1}\frac{d\rho}{2}[e^{-\xi}-e^{-\xi(1+\rho)/2}-e^{-\xi(1-\rho)/2}+\hat{S}(y)e^{-\xi(1+\rho)/2}]\Big]\ ,\end{aligned} \tag{42}$$

with

$$\hat{S}(y)=\frac{\sum_m\exp[-\xi y^2m^2-\xi(1+\rho)ym]}{\sum_m\exp(-\xi y^2m^2)}\ . \tag{43}$$

For the RRN we take the limit $y \to 0$, in which case the right-hand side of Eq. (42) can be evaluated in closed form. The ξ integrals can then be done using

$$\int_0^\infty \frac{d\xi}{\xi^2} \sum_i \alpha_i e^{-\beta_i \xi} = \sum_i \alpha_i \beta_i \ln\beta_i \ , \tag{44}$$

providing $\sum_i \alpha_i = \sum_i \alpha_i \beta_i = 0$, so that the integral converges. In this way we find that

$$F(x,0) = 1 + x + \frac{\varepsilon}{7}\left[\frac{(1+x)^2}{x}\ln(1+x) + \frac{1}{3} - \frac{19}{18}x - \frac{1}{6}\frac{(x+4)^{3/2}}{x^{1/2}}\ln\left[\frac{1+\sqrt{x/(x+4)}}{1-\sqrt{x/(x+4)}}\right]\right] , \tag{45}$$

in agreement with the result found by Meir *et al.*[12] and with the universal amplitude ratios of HL. As we shall show in a moment, $\hat{\chi}_{xy}^{(\lambda)}$ for small w has the same series expansion as $\chi_{\rm RN}$, so the normalization of $F(x,y)$, i.e., the choice of amplitudes in Eq. (41), does not depend on y. To elucidate its behavior for large x, we write $F(x,y)$ in the form

$$F(x,y) = 1 + x + \varepsilon(\tfrac{5}{42}x + \tfrac{1}{7})\ln x + \frac{\varepsilon}{7}\Bigg[-\tfrac{19}{18}x + (1+x)\ln\frac{(1+x)}{x} - C_3(y)x + x\int_0^\infty \frac{d\xi}{\xi^2}(1-e^{-\xi/x})\int_{-1}^{1}\frac{d\rho}{2}[e^{-\xi} - e^{-\xi(1+\rho)/2} - e^{-\xi(1-\rho)/2} + \hat{S}(y)e^{-\xi(1+\rho)/2}]\Bigg] , \tag{46a}$$

$$\equiv 1 + x + \varepsilon(\tfrac{5}{42}x + \tfrac{1}{7})\ln x + \frac{\varepsilon}{7}K(x,y), \tag{46b}$$

where $C_3(y)$ was introduced just after Eq. (36).

There are several points to be made about these results. First we can show that $F(x,y)$ has an expansion in powers of x for small x which is independent of y. That $F(x,y)$ depends on y is due to terms of order $\exp(1/x^{1/2})$. To see this we study $\Delta F(x,1) \equiv F(x,1) - F(x,0)$, which we obtain from Eq. (42) as

$$\Delta F(x,1) = -\frac{\varepsilon x}{7}\int_0^\infty \frac{d\xi}{\xi^2} e^{-\xi/x}\int_{-1}^{1}\frac{d\rho}{2}[\hat{S}(1) - \hat{S}(0)]e^{-\xi(1+\rho)/2} \ . \tag{47}$$

Due to the factor $\exp(-\xi/x)$ power series contributions in x can only come from small ξ. But for small ξ, the square bracket in Eq. (47) is at most of order $A\exp(-B/\xi)$, independent of ρ. Thus the maximum value of the integrand is of order $\exp[-(B/x)^{1/2}]$ and cannot contribute to a series expansion in powers of x. Thus the series expansions in powers of x for $F(x,y)$ for the RRN and the GRXY model are identical.

Next we show that these crossover functions differ in a physically important manner for large x. First of all, for large x, we see from Eq. (45) that

$$F(x,0) = \frac{\varepsilon}{7}\left[\frac{(1+x)^2}{x} - \frac{1}{6}\frac{(x+4)^{3/2}}{x^{1/2}}\right]\ln x + \sum_{n=-1}^{\infty} a_n x^{-n} + O(\varepsilon^2) \ , \tag{48}$$

which clearly is not of the form of Eq. (24). As we have discussed, this was expected, since the RRN has a phase transition at $p = p_c$ independent of $w\lambda^2$. In contrast, the situation for the GRXY model is different. Comparing Eqs. (24) and (46a), we see that they are of the same form providing the integral in Eq. (46a) has a series expansion in powers of $1/x$. Furthermore, the phase boundary is determined by the radius of convergence of this expansion. Thus, if this expansion converges for $|1/x| < R_c(y)$, then the phase boundary is located at $-xR_c(y) = 1$, or really, since we cannot distinguish terms of order ε, at $-[xR_c(y)]^{1/\phi} = 1$.

Therefore our intention is to put Eq. (46a) into the form of Eq. (24) and determine the radius of convergence of the resulting expansion in powers of $1/x$. The relevant terms in the first line of Eq. (46a) are analytic for $|1/x| = 1$. We next consider the Laurent expansion of

$$\delta F(x,y) = \int_0^\infty \frac{d\xi}{\xi^2}(1-e^{-\xi/x})\int_{-1}^{1}\frac{d\rho}{2}[e^{-\xi} - e^{-\xi(1+\rho)/2} - e^{-\xi(1-\rho)/2} + \hat{S}(y)e^{-\xi(1+\rho)/2}] \ . \tag{49}$$

If we break the integral over ξ into two integrals, the first from $\xi=0$ to $\xi=M$, and the second from $\xi=M$ to $\xi=\infty$, we observe that the expansion of the first integral in powers of $1/x$ has an infinite radius of convergence. Therefore we focus on the second term, and we consider large M such that $My^2 \gg 1$. The radius of convergence of the Laurent expansion of $\delta F(x,y)$ is thus the same as that of

$$\delta_2 F(x,y) = \int_M^\infty \frac{d\xi}{\xi^2}(1-e^{-\xi/x})\int_{-1}^{1}\frac{d\rho}{2}L(y,\xi,\rho) \ , \tag{50}$$

where $L(y,\xi,\rho)$ is the square bracket in Eq. (49). We will obtain the bound $L(y,\xi,\rho) < Ae^{-B\xi y^2}$, from which it follows that the Laurent expansion of $\delta_2 F(x,y)$ converges for $|1/x| < By^2$.

We have

$$L(y,\xi,\rho)=e^{-\xi}-e^{-\xi(1+\rho)/2}-e^{-\xi(1-\rho)/2}+\frac{\sum_m e^{-\xi\{m^2y^2+[my+(1/2)](1+\rho)\}}}{\sum_m e^{-\xi m^2y^2}} . \quad (51)$$

Since we have chosen M to be large, we set

$$\left[\sum_m e^{-\xi m^2y^2}\right]^{-1}=1+O(e^{-\xi y^2}) . \quad (52)$$

In the other sum over m we treat separately the dominant terms for $m=0$ and $m=-1/y$ (recall that $y^{-1}=\lambda/\lambda_{\rm min}$ is an integer):

$$L(y,\xi,\rho)=e^{-\xi}+\sum_{m\neq 0,-1/y} e^{-\xi\{m^2y^2+[my+(1/2)](1+\rho)\}}+O(e^{-y^2\xi}) . \quad (53)$$

For all ρ within the range $-1\le\rho\le+1$, each term in the sum over m in Eq. (53) is bounded by $e^{-\xi y^2}$. For large M we assume that such a term-by-term analysis suffices. Then we have established that $|L(y,\xi,\rho)| < Ae^{-\xi y^2}$ so that $R_c(y)=y^2$. Thus we conclude that the phase boundary in $F(x,y)$ occurs at

$$(xy^2)^{1/\phi}=-1-a_1\varepsilon=-c . \quad (54)$$

For the GRXY model this means, in view of Eq. (13), that $\hat{\chi}^{(\lambda)}_{xy}(w,t)$ becomes singular when Eq. (15) holds. For the RXY model (with $\lambda_{\rm min}=1$) this gives

$$p_c(T)=p_c+c(A_2T/J)^{1/\phi} \quad (55a)$$

and for the RRN ($\lambda_{\rm min}=0$)

$$p_c(1/\sigma_0)=p_c , \quad (55b)$$

as expected. This calculation is somewhat imprecise in that the value of a_1 in Eq. (54) is not calculated.

To obtain a_1 it is in principle possible to locate the singular point in $\hat{\chi}^{(\lambda)}_{xy}(w,t)$ for any λ, as we discuss below. However, it is simplest to consider $\hat{\chi}^{(\lambda)}_{xy}(w,t)$ for $\lambda=\lambda_{\rm min}$. In view of Eq. (13) we need to locate the zero of $F(x,1)$, i.e., we solve $F(x^*,1)=1$ for $x^*=-1-a_1\varepsilon$. For this purpose we use the large-x representation, as in Eq. (23) or Eq. (46). Thereby we obtain

$$a_1=\tfrac{1}{7}K(-1,1) \quad (56)$$

$$=\frac{1}{7}\left[\frac{19}{18}+C_3(1)+\int_0^\infty\frac{d\xi}{\xi^2}(1-e^{\xi})\int_{-1}^{1}\frac{d\rho}{2}\left[e^{-\xi}-e^{-\xi(1+\rho)/2}-e^{-\xi(1-\rho)/2}+\sum_m e^{-\xi m^2-\xi m(1+\rho)}\Big/\sum_m e^{-\xi m^2}\right]\right] . \quad (57)$$

This expression can be evaluated numerically: the convergence of the integrals is assured by the argument involving Eq. (53).

We now justify these results on less formal grounds. First, consider the GRXY susceptibility $\hat{\chi}^{(\lambda)}_{xy}(w,t)$. Within mean-field theory one has

$$\hat{\chi}^{(\lambda)}_{xy}(w,t)=[p_c-p+(T/J)\lambda^2]^{-1} . \quad (58)$$

This result seems to predict singular behavior in $\hat{\chi}^{(\lambda)}_{xy}(w,t)$ for

$$p_c(T)=p_c+(T/J)\lambda^2 , \quad (59)$$

where $\lambda=l\lambda_{\rm min}$, with $l=1,2,3,\ldots$. However, this result is analogous to the mean-field solution for the pure Ising model. There mean-field theory gives

$$\chi_{\rm MF}(q,T)\sim(T-T_c+q^2)^{-1} , \quad (60)$$

where q is the wave vector. In that case it is well known that the phase boundary is independent of q. For $T<T_c$ the nonzero order parameter causes singular behavior in $\chi(q,T)$ at $T=T_c$ for all q and removes the singularity in $\chi(q,T)$ at $T=T_c-q^2$ predicted in Eq. (60). Thus T_c is located as the singular point of $\chi_{\rm MF}(q,T)$ for $q=q_{\rm min}$, the minimum value of q, i.e., $q=0$. Here it is not possible to explicitly display the mean-field solution to show that all the $\hat{\chi}^{(\lambda)}_{xy}(w,t)$'s are singular when $\hat{\chi}^{(\lambda_{\rm min})}_{xy}(w,t)$ is singular. However, the result, Eq. (55a), which we obtained from a disordered-state calculation including fluctuations agrees with the result we would expect from the much more difficult ordered-phase calculation.

Finally, we may understand the distinction between the RXY and RRN models in the high dimensionality limit by considering these models on a Cayley tree. In this case the resistance always increases linearly in the chemical distance. In their large separation behavior, $\chi^{(\lambda)}_{\rm RN}$ and $\chi^{(\lambda)}_{xy}$ are thus identical on a Cayley tree, and in particular $\chi^{(\lambda)}_{\rm RN}$ is nonsingular for $p_c-p+A_1\lambda^2/\sigma_0>0$. It is only with the inclusion of loops that the two models differ. For a finite dimensional lattice we adopt the nodes links picture.[18] Here one sees that for $p>p_c$ the resistance to infinity is large but finite, since it is of order $R(\xi_p)$, the resistance between two points separated by a distance equal to the percolation correlation length ξ_p. The finiteness of $R(\mathbf{x},\mathbf{x}')$ for $|\mathbf{x}-\mathbf{x}'|\to\infty$ implies, by Eq. (8), that $\chi_{\rm RN}$ is divergent at $p=p_c$ for all values of λ^2/σ_0, whereas χ_{xy} is divergent at $p=p_c+A_1T/J$.

III. SUMMARY

In summary, our calculation gives the following intuitively satisfactory results.

(1) The resistive susceptibility,

$$\chi_{\mathrm{RN}}^{(\lambda)} \equiv \sum_{\mathbf{x}'} \left[e^{-(1/2)\lambda^2 R(\mathbf{x},\mathbf{x}')} \right]_{\mathrm{av}},$$

diverges at $p=p_c$ for all values of λ^2/σ_0. Mean-field theory for $p<p_c$, which gives $\chi_{\mathrm{RN}}^{(\lambda)} \sim [p_c - p + \lambda^2/\sigma_0]^{-1}$, fails to give a threshold value of p independent of λ^2/σ_0.

(2) We have introduced a family of models [in Eq. (11)] which interpolate smoothly between the random XY model (for $\lambda_{\min}=1$) and the random resistor network (for $\lambda_{\min}=0$). For $T/J \ll 1$ the phase boundary for these models obeys

$$p_c(T) = p_c + c(A_2 T \lambda_{\min}^2 / J)^{1/\phi}, \qquad (61)$$

where $\phi = 1 + \varepsilon/42 + O(\varepsilon^2)$ and c is a universal constant given in Eqs. (54) and (57). The nonuniversal constant A_2 must be determined experimentally.

(3) Our calculation is consistent with the spin-wave result that $\chi(T,p)$ for $T \ll |p_c - p|^{\phi}$ has a low temperature expansion whose coefficients are independent of $\lambda_{\min}$, even though $p_c(T)$ does depend on $\lambda_{\min}$.

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