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Critical Behavior of Random Resistor Networks*

A. B. Harris and R. Fisch
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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We present numerical data and scaling theories for the critical behavior of random resistor networks near the percolation threshold. We determine the critical exponents of a suitably defined resistance correlation function by a Padé analysis of low-concentration expansions as a function of dimensionality. We verify that $d=6$ is the critical dimensionality for the onset of mean-field behavior. We use the coherent-potential approximation to construct a mean-field scaling function for the critical region.

In this Letter we report some new ideas concerning the properties of random resistor networks near the percolation threshold.[1] The model we treat is that of an electrical network on a d-dimensional hypercubic lattice of L^d sites with conductances σ_{ij} connecting nearest neighboring pairs of lattice sites i and j. Each σ_{ij} is an independent random variable assuming the values $\sigma_<$ or $\sigma_>$ with respective probabilities $1-p$ and p. The macroscopic conductivity, Σ, is then defined to be the configurational average of σL^{2-d}, where $\sigma\equiv I/V$, where I is the current when the potential difference V is applied between two opposite $(d-1)$-dimensional faces of the hypercube. We may define clusters as being groups of sites which are connected with respect to the conductances $\sigma_>$. The statistics of cluster size and the associated pair connectedness correlation length, $\xi(p)$, were shown[2] to be related to the thermodynamics of the s-state Potts model in the limit $s\rightarrow 1$, if the identification $p=1-e^{-J/kT}$ is made, where J is the coupling constant for nearest-neighbor interaction in the Potts model. This relation indicates that the usual exponent description for phase transitions can be applied to the percolation threshold and that the various scaling relations and universality predictions can be expected to hold as well. It was later shown[3,4] that for $d>d_c=6$, mean-field theory gives correct values for cluster statistics near the percolation threshold: $\alpha=-1$, $\beta=1$, $\gamma=1$, and $\nu=\frac{1}{2}$. In view of scaling arguments which relate the resistor network and percolation problems, de Gennes[5] has suggested that $d_c=6$. Here we present numerical evidence which confirms that this suggestion is correct. We also discuss several new scaling relations.

A way to determine d_c without using the renormalization group (RG) is to analyze the high-temperature expansion for the susceptibility, χ, as a function of dimensionality, d.[6] At $d=4$ an analysis for the Ising model[7] based on a single power-law divergence yields $\gamma=1.09$. However, the RG predicts[8] corrections to scaling of the form

$$\chi=A[1+(1-B/\epsilon\Delta_1)(t^{\epsilon\Delta_1}-1)]^{\theta}t^{-\gamma}, \quad (1)$$

where $\epsilon=d_c-d$, $t=(T-T_c)/T_c$, A and B are nonuniversal constants, and Δ_1, θ, and γ are universal exponents. Fitting to a form of this type, Van Dyke and Camp[6] found $\gamma=1.00\pm0.02$ for the Ising model at $d=4$, thus confirming $d_c=4$ for this model. We have carried out the same program for the random resistor network by analyzing a low-concentration expansion for an analogous susceptibility. In order to do this, one must identify the order parameter. For arbitrary networks, Kasteleyn and Fortuin[2] show that the correlation function for the $s\to0$ state Potts model yields the resistance, R_{ij}, between lattice points i and j. This correlations function can also be obtained by averaging x_ix_j over the Gaussian density matrix, $\rho=\exp[-\frac{1}{2}\sum_{mn}\sigma_{mn}(x_m-x_n)^2]$. These observations suggest that R_{ij} plays the role of the correlation function and therefore we define the resistive susceptibility as $\chi^{(r)}=\sum_j\chi_{ij}$, where

$$\chi_{ij}=[R_{ij}\nu_{ij}]_{\rm av}, \quad (2)$$

where $\nu_{ij}=1$ if i and j are in the same cluster and $\nu_{ij}=0$ otherwise, and $[\ ]_{\rm av}$ denotes a configurational average.

We first consider the case $\sigma_<=0$ and relate $\gamma^{(r)}$, the exponent for $\chi^{(r)}$, to the conductivity exponent μ defined by $\Sigma\sim(p-p_c)^{\mu}$, $p\to p_c^+$. Consider the Green's function $G(\vec{r},\vec{r}\,')$ which gives the voltage at $\vec{r}\,'$ in response to a unit current source at $\vec{r}$. In the limit $q\to0$ the Fourier transform of this Green's function is of the form

$$G(q)=P^2(p)\{\Sigma(p)q^2F[\xi(p)q]\}^{-1}, \quad (3)$$

where $P(p)$ is the fraction of sites in the infinite cluster and $\xi(p)$ is the correlation length. This Green's function is identical to that for transverse excitations in a Heisenberg ferromagnet at zero temperature for which Eq. (3) holds.[9] In Eq. (3) $F(z)$ is analytic at $z=0$, so that if one writes

$$G(q)=\sum_{n=-2}^{\infty}G^{(n)}q^n,$$

then one has

$$G^{(2)}\sim P^2(p)\,\xi(p)^4/\Sigma(p). \quad (4)$$

Now consider the response of finite clusters for $p<p_c$, to a current source $I_0e^{i\vec{q}\cdot\vec{r}}$. In order to be able to solve Kirchoff's equations we must eliminate the $q=0$ component of current. Thus we take the current source to be $I_0[\exp(i\vec{q}\cdot\vec{r})-\exp(i\vec{q}\cdot\bar{\vec{r}})]$, where $\bar{\vec{r}}$ is approximately the center of gravity of the cluster. In the limit $q\to0$ the induced voltage $V(\vec{r}\,')$ is then

$$V(\vec{r}\,')=I_0\sum_{\vec{r}}G(\vec{r},\vec{r}\,')i\vec{q}\cdot(\vec{r}-\bar{\vec{r}}). \quad (5)$$

Here and below, the lattice sums are restricted to a single cluster. To get the response at wave vector $\vec{q}$ we need to evaluate

$$V(\vec{q})=\sum_{\vec{r}\,'}V(\vec{r}\,')[\exp(-i\vec{q}\cdot\vec{r}\,')-\exp(-i\vec{q}\cdot\bar{\vec{r}})]. \quad (6)$$

Here again we subtract the term $\exp(-i\vec{q}\cdot\bar{\vec{r}})$, because we can arbitrarily fix the average voltage of each cluster to be zero. Thus

$$V(\vec{q})/I_0=\sum_{\vec{r},\vec{r}\,'}G(\vec{r},\vec{r}\,')(\vec{q}\cdot\delta\vec{r})(\vec{q}\cdot\delta\vec{r}\,'), \quad (7)$$

where $\delta\vec{r}=\vec{r}-\bar{\vec{r}}$. In a cluster the distances are of order $\xi(p)$ and $G(\vec{r},\vec{r}\,')$ is related to $R_{\vec{r},\vec{r}\,'}$, so that

$$G^{(2)}\sim\xi^2(p)\sum_{\vec{r},\vec{r}\,'}R_{\vec{r},\vec{r}\,'}=\xi^2(p)\chi^{(r)}. \quad (8)$$

If we assume $G^{(2)}$ to be governed by the same exponent for $p<p_c$ as for $p>p_c$, we obtain from Eqs. (4) and (8) that[10]

$$\gamma^{(r)}=\mu-2\beta+2\nu=\mu-(d-2)\nu+\gamma, \quad (9)$$

where $P(p)\sim(p-p_c)^{\beta}$ and $\xi(p)\sim|p-p_c|^{-\nu}$. The second equality in Eq. (9) follows by using the scaling relation $2\beta=d\nu-\gamma$ for the percolation exponents.

We have developed a low-p expansion for $\chi^{(r)}$ as $\chi^{(r)}=\sum a_n(d)p^n$ and have obtained the polynomials[7] in d for $a_n(d)$ for $n\le10$. This was done as follows. For any cluster, Γ, we define $\chi^0(\Gamma)$ to be the value of $\chi^{(r)}$ for that cluster. The cumulant, $\chi^c(\Gamma)$, is defined recursively as

$$\chi^c(\Gamma)=\chi^0(\Gamma)-\sum_{\gamma\in\Gamma}\chi^c(\gamma),$$

where the sum is over all connected clusters contained within Γ. Then the result is $a_n(d)=\sum_{\Gamma(n)}b(\Gamma)\chi^c(\Gamma)$, where the sum is over all clusters with n bonds and $b(\Gamma)$ is the number of times the cluster Γ can be formed on a hypercubic lattice in d dimensions. The mean-field-theory value of $\gamma^{(r)}$, denoted $\gamma_{\rm MF}^{(r)}$, can be obtained by taking d large and keeping only self-avoiding walk diagrams. For n-step walks $b\sim(2d)(2d-1)^{n-1}$ and χ^c is equal to the resistance between the ends of the walk, viz. $n/\sigma_>$. Thus

$$\chi^{(r)}\sim\sum_n p^n(n/\sigma_>)(2d-1)^n, \quad (10)$$

TABLE I. Padé approximant results for p_c and $\gamma^{(r)}$.

d	p_c	$\gamma^{(r)}$
2	1/2	3.7[a]
3	0.2465	2.78
4	0.1600	2.46
5	0.1181	2.19 (2.22)[b]
5.5	0.1047	2.13 (2.10)[b]
6	0.0943	2.09 (2.00)[b]
6.5	0.0858	2.06
7	0.0787	2.04

[a]This value is rather uncertain (±0.3) (see Ref. 13). The errors for the other entries (p_c and $\gamma^{(r)}$) are of order one in the last decimal place quoted.

[b]Values obtained using corrections to scaling, Eq. (1).

which gives $\gamma_{\mathrm{MF}}^{(r)}=2$. For finite d one uses the a_n's to construct the various Padé approximants to $\partial \ln\chi^{(r)}/\partial p \sim \gamma^{(r)}/(p_c-p)$ to determine $\gamma^{(r)}$. The results of this and similar procedures are summarized in Table I. [The coefficients $a_n(d)$ will be published elsewhere.]

The effect for d near d_c of the logarithmic corrections of Eq. (1) [with $t+(p_c-p)/p_c$] is scaled by θ. For isotropic n-vector φ^4 and φ^3 models[11] one has $\theta=2(\partial\gamma^{(r)}/\partial\epsilon)|_{\epsilon=0}$ and $\Delta_1=\frac{1}{2}$. Without including logarithmic corrections we find $\gamma^{(r)}-\gamma_{\mathrm{MF}}^{(r)}=0.09$ for $d=6$, and $\gamma^{(r)}-\gamma_{\mathrm{MF}}^{(r)}\sim\epsilon/5$ for $\epsilon\leq 1$, which gives $\theta\approx 0.4$. These values are quite similar to those for the Ising model for $d=4$. Including logarithmic corrections as in Eq. (1) we find an acceptable fit to the series at $d=6$ for $\gamma^{(r)}=\gamma_{\mathrm{MF}}^{(r)}$ with $B=0.4$. This result is again similar to that found for the Ising model, and confirms that $d_c=6$ here.

The scaling relation, Eq. (9), is satisfied by the mean-field-theory values, viz. $\gamma^{(r)}=2$, $\beta=1$, $\nu=\frac{1}{2}$, and (see below) $\mu=3$. For $d=3$ one has[12] $\mu=1.6$, $\beta=0.4$, and $\nu=0.95$, which yields $\gamma^{(r)}=2.7$ from Eq. (9), in good agreement with our value $\gamma^{(r)}=2.78$. For $d=2$, the values[12] $\mu=1.1$, $\beta=0.15$, and $\nu=1.35$ yield $\gamma^{(r)}=3.5$, which is also consistent with our result, $\gamma^{(r)}=3.7\pm 0.3$.[13]

Finally, we propose scaling relations for the case $\sigma_<\neq 0$. We will use the coherent-potential approximation[14] (CPA) to generate mean-field values of the critical exponents. Within the CPA Kirkpatrick[15] finds that the network is described by an effective conductance σ_{eff} which is determined by

$$p=(\sigma_{\mathrm{eff}}-\sigma_<)\left[\frac{1-p}{\sigma_>-\sigma_{\mathrm{eff}}}+\frac{2}{z\,\sigma_{\mathrm{eff}}}\right]. \qquad (11)$$

In this approximation $p_c=p^*=2/z$. For $|t|\ll 1$, with $t=(p-p^*)/p^*$, and $\sigma_</\sigma_>\ll 1$, the dominant behavior of σ_{eff} is given by the scaling form

$$\sigma_{\mathrm{eff}}=\tfrac{1}{2}\sigma_>\{t+[t^2+2(\sigma_</\sigma_>)/(z-2)]^{1/2}\}. \qquad (12)$$

If we define the exponents[16] $\bar{\mu}$ by $\Sigma(p)\sim(p_c-p)^{-\bar{\mu}}$ for $p\to p_c^-$ with $\sigma_>=\infty$, and τ by $\Sigma(p_c)\sim\sigma_<^{\tau}\sigma_>^{1-\tau}$, then Eq. (12) implies $\bar{\mu}=1$ and $\tau=\frac{1}{2}$. To determine μ from Eq. (12) we must be able to relate σ_{eff} to Σ, when $\sigma_<=0$. Kirkpatrick[15] assumed σ_{eff} and Σ to be equivalent. However, it is more plausible to assume that the CPA provides an approximation to the Green's function of Eq. (3), in which case one would write $G^{-1}(q)\sim\sigma_{\mathrm{eff}}q^2$, for $q\xi(p)\ll 1$. Equation (3) would then imply that $\Sigma(p)\sim\sigma_{\mathrm{eff}}P^2(p)$, which would give $\mu=3$ as the mean-field result. These mean-field values of μ and $\bar{\mu}$ agree with those given, respectively, by de Gennes[5] and Straley.[16] In particular, our treatment and numerical values of $\gamma^{(r)}$ confirm de Gennes's argument[5] that Stinchcombe's result[17] ($\mu=2$) for the conductivity of the Bethe lattice is not the appropriate value for μ_{MF}. The CPA result for τ does not agree with Straley's value,[16] which, however, is based in part on the Bethe lattice value for μ.

Now we consider the regime $d<6$, where mean-field theory is not valid. For $p<p_c$ and $\sigma_>\to\infty$, we assume that $\Sigma(p)$ obeys a scaling form

$$\Sigma(p)=\sigma_<^{\tau}\sigma_>^{1-\tau}F((\sigma_>/\sigma_<)(p-p_c)^{\varphi}), \qquad (13)$$

where $F(0)=1$. [The CPA, Eq. (12), provides the mean-field-theory result, $\varphi=2$.] For $\sigma_>=\infty$, Eq. (13) implies that

$$\bar{\mu}=(1-\tau)\varphi. \qquad (14)$$

To develop another scaling relation we consider the regime $p<p_c$, $\sigma_<\to 0$ and study the behavior of $\chi^{(r)}$ as defined in Eq. (2). In the limit of large separation R_{ij} tends to a value proportional to $\Sigma(p)^{-1}$, since we take $\sigma_<\neq 0$. In this limit, then, one has $[R_{ij}\nu_{ij}]_{\mathrm{av}}=[\nu_{ij}]_{\mathrm{av}}/\Sigma(p)$, so that $\chi^{(r)}\sim\chi^{(p)}/\Sigma(p)$, where $\chi^{(p)}$ is the percolation susceptibility, i.e., the mean-square cluster size. We now assume a scaling form for $\chi^{(r)}$:

$$\chi^{(r)}=\sigma_>^{-1}(p-p_c)^{-\gamma^{(r)}}G((\sigma_</\sigma_>)(p_c-p)^{-\varphi}) \qquad (15)$$

with $G(0)=1$ and $G(x)\sim x^{-s}$ for large x. Then for $\sigma_</\sigma_>\gg(p-p_c)^{\varphi}$ we have

$$\chi^{(p)}/\Sigma(p)=(p-p_c)^{-\gamma^{(r)}+s\varphi}(\sigma_<^{-s}\sigma_>^{s-1}). \qquad (16)$$

For this to agree with the definition of τ, we must set $s=\tau$. Then we get

$$\chi^{(p)} \sim (p-p_c)^{-\gamma^{(r)}+\tau\varphi},$$

so that

$$\gamma^{(r)} - \tau\varphi = \gamma. \qquad (17)$$

Combining relations (9), (14), and (17) we get[18]

$$\tau = [\mu - (d-2)\nu]/[\mu + \bar{\mu} - (d-2)\nu] \qquad (18)$$

as contrasted to Straley's result,[16] $\tau = \mu/(\mu+\bar{\mu})$. For $d=2$, our result agrees with Straley's and with the exact results,[19] $\tau = \frac{1}{2}$, $\bar{\mu}=\mu$. For $d=6$, our mean-field-theory values of the exponents satisfy Eq. (18), whereas Straley's do not. The numerical evidence for $d=3$ is not yet conclusive, although Eq. (18) does not work very well for the current values, $\mu^{(r)}=1.6$, $\bar{\mu}^{(r)}=0.6$, and $\tau = 0.77$.[19] However, since Straley's values[19] of the exponents do not satisfy universality very well, they may not be sufficiently accurate for this comparison.

Our conclusions are summarized in the Abstract.

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$$(\sigma^{(r)})^2 = \lim_{r_{ij}\to\infty} [R_{ij}\nu_{ij}]_{\rm av},$$

for $p > p_c$. In this limit we have that $[\nu_{ij}]_{\rm av} = P^2(p)$ and $[R_{ij}] \sim L$, the path length between nodes, introduced by Shklovskii {B. I. Shklovskii, Fiz. Tekh. Poluprovodn. 8, 1586 (1974) [Sov. Phys. Semicond. 8, 1029 (1975)]} and de Gennes (Ref. 5), who find that $\Sigma(p) \sim L^{-1}\xi^{2-d}$. Thus, $(\sigma^{(r)})^2 \sim P^2(p)\xi^{2-d}/\Sigma(p)$, i.e., $2\beta^{(r)} = 2\beta + (d-2)\nu - \mu$, from which Eq. (9) follows. Even though there are doubts as to the validity of defining L, this argument may yield the correct answer, since L does not appear in the final result.

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