Coupled-mode theory for chirowaveguides

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In this paper, electromagnetic wave propagation and mode coupling in a chirowaveguide are treated using the coupled-mode theory. A chirowaveguide, as defined in our previous work, is a conventional cylindrical waveguide filled with homogeneous chiral materials. A set of coupled linear differential equations is derived for various mode amplitudes in the waveguide. We then show that, in any single chirowaveguide, owing to the handed properties of chiral materials filling the waveguide, energy coupling occurs from one mode to the other. We also demonstrate that in a parallel-plate chirowaveguide a TE mode can be completely converted into a TM mode and vice versa as they propagate in the guide. Thus a chirowaveguide can indeed be used as a mode converter. Selected results are compared with those reported in the literature. Applications of such mode coupling in the design of novel microwave, millimeter-wave, and optical devices and components are mentioned.

I. INTRODUCTION

A *chiral* object is formally defined to be a three-dimensional body that is not superimposable with its mirror image by translation and rotation. Such an object has the handed property and must be either left handed or right handed. The fundamental concept of chirality has been investigated in a large number of fields and has been actually a subject of interest since the early part of the 19th century with the works of Arago,[1] Biot,[2-4] Fresnel,[5] and Pasteur.[6] These researchers were mainly interested in the rotation of plane of polarization of optical waves in a certain class of crystals and liquids, and discovered a new phenomenon called *optical activity*. Several years later, in the early and midpart of the 20th century, chiral materials started to generate a great deal of interest in the electromagnetics community with the microwave experiments of Lindman[7,8] and Pickering,[9] which were analogous to the optical experiments performed in the 19th century. More recently, a wide variety of problems related to chiral media, and, in general, gyrotropic materials, have been investigated and reported in the literature.[10-23] For instance, it has been shown[10] that for time-harmonic electromagnetic fields with $\exp(-i\omega t)$ excitation, a homogeneous, lossless, isotropic, chiral medium can be described by the following constitutive relations:

$$\mathbf{D} = \epsilon\mathbf{E} + i\xi_c\mathbf{B}, \tag{1}$$

$$\mathbf{H} = i\xi_c\mathbf{E} + \mathbf{B}/\mu, \tag{2}$$

where ϵ, μ, and ξ_c represent, respectively, the permittivity, permeability, and chirality admittance of the chiral medium. This set of constitutive relations is a subset of the more general constitutive relations used to describe bianisotropic media.[24] Bianisotropic media and their electromagnetic properties have been studied by Kong[16,24,25] and Cheng and Kong.[26]

In the past few years, electromagnetic chirality[27] and chiral materials have been extensively investigated in a large number of applications. Among those is the effect of electromagnetic chirality in guided-wave structures. In previous works,[11,12] we introduced the idea of *chirowaveguide*: a conventional cylindrical waveguide filled with homogeneous chiral materials. The detailed analysis of chirowaveguides revealed some notable features such as bifurcation of modes, impossibility to support individual TE, TM, or TEM modes, etc.[11,12] In other words, we demonstrated that such waveguides always support hybrid modes. The hybrid structure of modes in these waveguides may potentially be used as a mechanism for mode coupling. More specifically, if a TE wave enters into such a chirowaveguide, since this wave is not one of the eigenmodes of the chirowaveguide, it will be coupled into those guided modes supported by the waveguide. Chien, Kim, and Grebel,[19] showed experimentally that TE↔TM conversion is indeed possible in optically active and isotropic waveguides. They also studied theoretically the problem of coupling of a pair of lowest TE and TM modes in a weakly guided slab dielectric waveguide filled with optically active materials. However, in their analysis the weakly guiding approximation was assumed and fields were expressed in terms of only two modes. It is the purpose of the present study to generalize the analysis of mode coupling in chirowaveguides and to use the coupled-mode theory to obtain a complete set of coupled-mode equations for amplitudes of all modes available in such waveguides. We will show that results of our general analysis will reduce to what Chien and co-workers obtained for the weakly guided dielectric waveguides.[19]

Since in most cases, chirality of the medium is weak and thus ξ_c is small compared with $(\epsilon/\mu)^{1/2}$, this parameter is treated as a small perturbation. The results of the present investigation emphasize that a transfer of energy between the modes can occur inside a general chirowaveguide. It is well known that many conventional mode converters use materials such as ferroelectric or ferromagnetic media which require biasing fields in order to operate properly. Furthermore, these converters are generally nonreciprocal elements. Chirowaveguides, however, have the advantage of operating with no such biasing fields, and moreover, they are reciprocal devices. Such features, along with the mode coupling properties of chirowaveguides, may be used to design simple and efficient TE↔TM converters in the microwave, millimeter-wave, and optical regimes.

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II. PROBLEM FORMULATION

Let us consider a cylindrical structure of arbitrary cross-sectional shape with its axis in the direction of z axis and filled with a homogeneous, lossless, isotropic chiral material described by Eqs. (1) and (2), where ϵ, μ, and ξ_c are now assumed to be real scalar quantities. As was noted earlier, ξ_c is taken to be much smaller than $(\epsilon/\mu)^{1/2}$. It is also assumed that in this structure the electromagnetic energy is guided along the z axis. Such a waveguide can be either a closed or an open structure. Since $\xi_c \ll (\epsilon/\mu)^{1/2}$, the present problem can be considered as a small perturbation of the problem of wave guiding through a conventional waveguide with the same geometry filled with a simple nonchiral lossless dielectric material characterized by ϵ and μ. Following this assumption, the electromagnetic fields in the perturbed waveguide, i.e., $\mathbf{E}'$ and $\mathbf{H}'$ in the chirowaveguide, can be expressed in terms of a superposition of fields of guided modes in the unperturbed waveguide, i.e., $\mathbf{E}$ and $\mathbf{H}$ of the conventional guide filled with the simple nonchiral dielectric. To obtain the relationship between the perturbed and unperturbed field quantities, the Lorentz reciprocity theorem is needed. The fields $\mathbf{E}$ and $\mathbf{H}$ of the unperturbed waveguide satisfy the Maxwell equations with $\xi_c = 0$:

$$\nabla\times\mathbf{E} = i\omega\mu\mathbf{H}, \tag{3}$$

$$\nabla\times\mathbf{H} = -i\omega\epsilon\mathbf{E}, \tag{4}$$

and the fields $\mathbf{E}'$ and $\mathbf{H}'$ inside the chirowaveguide satisfy the following Maxwell equations with $\xi_c \neq 0$:

$$\nabla\times\mathbf{E}' = i\omega\mu\mathbf{H}' + \omega\mu\xi_c\mathbf{E}', \tag{5}$$

$$\nabla\times\mathbf{H}' = \omega\mu\xi_c\mathbf{H}' - i\omega\epsilon\left(1 + \frac{\mu}{\epsilon}\xi_c^2\right)\mathbf{E}'. \tag{6}$$

Following the standard technique used in the Lorentz reciprocity theorem,[28,29] and ignoring the second- and higher-order terms of ξ_c, we obtain, from (3)–(6), the following equation:

$$\nabla\cdot(\mathbf{E}^*\times\mathbf{H}' + \mathbf{E}'\times\mathbf{H}^*) = \omega\mu\xi_c(\mathbf{H}^*\cdot\mathbf{E}' - \mathbf{E}^*\cdot\mathbf{H}'), \tag{7}$$

where superscript * denotes the complex conjugation. Integrating both sides of Eq. (7) over a volume V and making use of Gauss' theorem yields

$$\int\int_{S_c}(\mathbf{E}^*\times\mathbf{H}' + \mathbf{E}'\times\mathbf{H}^*)\cdot d\mathbf{S} = \int\int_V \omega\mu\xi_c(\mathbf{H}^*\cdot\mathbf{E}' - \mathbf{E}^*\cdot\mathbf{H}')dV, \tag{8}$$

where the closed surface S_c surrounds the volume V. This relation is, indeed, in the form of the Lorentz reciprocity theorem. The closed surface S_c can be chosen as a circular cylinder of infinitely large radius centered about the z axis of the waveguide and of infinitesimal width Δz along the z axis (see Fig. 1). It can be easily shown that as Δz approaches zero, in the limit, Eq. (8) may be rewritten as follows:

$$\hat{\mathbf{z}}\cdot\int\int_S \frac{\partial}{\partial z}(\mathbf{E}_t^*\times\mathbf{H}_t' + \mathbf{E}_t'\times\mathbf{H}_t^*)dS = \int\int_S \omega\mu\xi_c(\mathbf{H}^*\cdot\mathbf{E}' - \mathbf{E}^*\cdot\mathbf{H}')dV, \tag{9}$$

where S, as shown in Fig. 1, is one of the circular bases of the

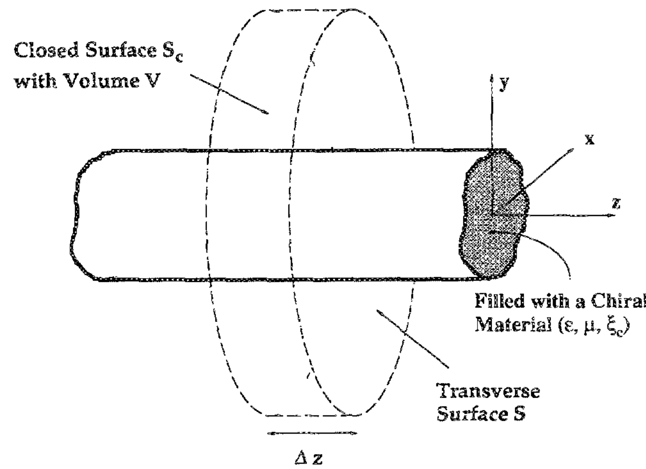

FIG. 1. A sketch of the geometry of the problem: a cylindrical chirowaveguide which is filled with a homogeneous isotropic lossless chiral material. The closed mathematical surface S_c is used in the mathematical derivation of the coupled-mode equations in the chirowaveguide.

cylinder S_c transverse to the z axis, $\hat{\mathbf{z}}$ is the unit vector along the z axis, and subscript t indicates the transverse part of a vector. It must be noted that if the waveguide under study is an open structure; i.e., if the guide's walls are not perfect conductors, the surface S for the integral of the left side of (9) shall extend to infinity and the integration on the right side shall be carried out over those regions of surface S with nonzero perturbation, i.e., those areas on S where $\xi_c \neq 0$. For a closed chirowaveguide with perfectly conductive walls, however, the surface S shall be taken to be the finite cross section of the guide.

Let us now assume that the unperturbed fields $\mathbf{E}$ and $\mathbf{H}$ are the solutions for the nth mode in the unperturbed waveguide, i.e.,

$$\mathbf{E} = \mathbf{e}_n e^{i\beta_n z} = (\mathbf{e}_{nt} + e_{nz}\hat{\mathbf{z}})e^{i\beta_n z}, \tag{10}$$

$$\mathbf{H} = \mathbf{h}_n e^{i\beta_n z} = (\mathbf{h}_{nt} + h_{nz}\hat{\mathbf{z}})e^{i\beta_n z}, \tag{11}$$

where $\mathbf{e}_n$ and $\mathbf{h}_n$ depend only on the transverse coordinates, with $\mathbf{e}_{nt}$ and $\mathbf{h}_{nt}$ being the transverse parts and e_{nz} and h_{nz} the longitudinal components. Without loss of generality, these modes are assumed to be normalized to a power flow of unity in the z direction. β_n is the guide wave number of the nth mode propagating along z axis. It is known that transverse components of these modes form a complete set of orthonormal bases on the transverse plane S. Therefore, the transverse components of the perturbed fields $\mathbf{E}'$ and $\mathbf{H}'$, i.e., $\mathbf{E}_t'$ and $\mathbf{H}_t'$ can be represented in terms of superposition of these orthonormal bases as

$$\mathbf{E}_t' = \sum_m a_m(z)\mathbf{e}_{mt}e^{i\beta_m z}, \tag{12}$$

$$\mathbf{H}_t' = \sum_m a_m(z)\mathbf{h}_{mt}e^{i\beta_m z}, \tag{13}$$

where coefficients $a_m(z)$ are functions of z, and the summation is over all possible modes.[30] From Maxwell's equations and the constitutive relations, and ignoring the second-order terms of ξ_c, we obtain the following expressions for the longitudinal components of perturbed field, E'_z and H'_z:

$$E'_z\hat{\mathbf{z}} = \frac{i}{\omega\epsilon}\nabla_t\times\mathbf{H}'_t - \frac{\xi_c}{\omega\epsilon}\nabla_t\times\mathbf{E}'_t, \tag{14}$$

$$H'_z\hat{\mathbf{z}} = -\frac{\xi_c}{\omega\epsilon}\nabla_t\times\mathbf{H}'_t - \frac{i}{\omega\mu}\nabla_t\times\mathbf{E}'_t, \tag{15}$$

where $\nabla_t = \nabla - \hat{\mathbf{z}}(\partial/\partial z)$. Substituting Eqs. (12) and (13) into (14) and (15), we obtain

$$E'_z\hat{\mathbf{z}} = \sum_m a_m(z)\left(e_{mz}\hat{\mathbf{z}} - i\frac{\mu\xi_c}{\epsilon}h_{mz}\hat{\mathbf{z}}\right)e^{i\beta_m z}, \tag{16}$$

$$H'_z\hat{\mathbf{z}} = \sum_m a_m(z)(h_{mz}\hat{\mathbf{z}} + i\xi_c e_{mz}\hat{\mathbf{z}})e^{i\beta_m z}. \tag{17}$$

Since the chirality admittance is assumed to be small, from (12), (13), (16), and (17), we can express, approximately, the total perturbed fields in terms of the total unperturbed fields. That is,

$$\mathbf{E}' \approx \sum_m a_m(z)\mathbf{e}_m e^{i\beta_m z}, \tag{18}$$

$$\mathbf{H}' \approx \sum_m a_m(z)\mathbf{h}_m e^{i\beta_m z}. \tag{19}$$

Let us now apply the Lorentz reciprocity theorem in (9) with **E**, **H**, **E**′, and **H**′ given, respectively, by Eqs. (10), (11), (18), and (19). After some mathematical manipulations, we obtain

$$\sum_m\left(\frac{da_m(z)}{dz} + i(\beta_m - \beta_n)a_m(z)\right)e^{i(\beta_m-\beta_n)z} \times \int\int_S(\mathbf{e}_{mt}\times\mathbf{h}^*_{nt} + \mathbf{e}^*_{nt}\times\mathbf{h}_{mt})\cdot\hat{\mathbf{z}}\,dS = \sum_m a_m(z)e^{i(\beta_m-\beta_n)z}\int\int_S\omega\mu\xi_c(\mathbf{e}_m\cdot\mathbf{h}^*_n - \mathbf{e}^*_n\cdot\mathbf{h}_m)dS. \tag{20}$$

For the unperturbed fields, the orthogonality relation indicates

$$\int\int_S(\mathbf{e}_{mt}\times\mathbf{h}^*_{nt} + \mathbf{e}^*_{nt}\times\mathbf{h}_{mt})\cdot\hat{\mathbf{z}}\,dS = 4\,\mathrm{sgn}(n)\delta_{|m||n|}, \tag{21}$$

where sgn(n) indicates sign of n, and $\delta_{|m||n|}$ is the Kronecker delta.[31] Substituting (21) into (20), we get

$$\frac{da_n(z)}{dz} = \frac{1}{4\,\mathrm{sgn}(n)}\sum_m a_m(z)e^{i(\beta_m-\beta_n)z} \times\int\int_S\omega\mu\xi_c(\mathbf{e}_m\cdot\mathbf{h}^*_n - \mathbf{e}^*_n\cdot\mathbf{h}_m)dS, \tag{22}$$

or, equivalently,

$$\frac{da_n(z)}{dz} = \sum_m a_m(z)e^{i(\beta_m-\beta_n)z}C_{mn}, \tag{23}$$

with the coupling coefficients defined as

$$C_{mn} \equiv \frac{1}{4\,\mathrm{sgn}(n)}\int\int_S\omega\mu\xi_c(\mathbf{e}_m\cdot\mathbf{h}^*_n - \mathbf{e}^*_n\cdot\mathbf{h}_m)dS. \tag{24}$$

As can be seen from (24), these coupling coefficients are proportional to the chirality admittance ξ_c. It can also be shown that the self-terms C_{mm} are identically zero. From the above analysis, it becomes evident that the electromagnetic fields guided by a chirowaveguide can be expanded in terms of the modes of unperturbed conventional waveguides, and the coefficients of expansion $a_m(z)$ satisfy a set of coupled linear differential equations given in (23).

III. CHIROWAVEGUIDE AS A MODE CONVERTER

The set of differential equations given in (23) is particularly useful in analyzing how energy can be converted from one mode to the other inside a single chirowaveguide. Consider, for example, a simple waveguide filled with nonchiral material such that only the first two unperturbed orthonormal modes can propagate (modes 1 and 2), and the guide is below cutoff for other modes. Let us then introduce some chirality ξ_c in the waveguide; we assume ξ_c small enough to be treated as a small perturbation. Due to the chirality, the unperturbed orthonormal modes are no longer orthogonal in the sense of (21), and there will be a transfer of energy from one mode to the other. The wave numbers of these two modes (modes 1 and 2) along z are denoted by β_1 and β_2, respectively.

Considering the first two terms $m = 1$ and 2 in Eq. (23), and ignoring the other terms corresponding evanescent modes, we obtain

$$\frac{da_1(z)}{dz} = a_2(z)e^{i(\beta_2-\beta_1)z}C_{21}, \tag{25}$$

$$\frac{da_2(z)}{dz} = a_1(z)e^{-i(\beta_2-\beta_1)z}C_{12}, \tag{26}$$

with C_{21} and C_{12} defined in (24), and $C_{11} = C_{22} = 0$. We also note that $C^*_{21} = -C_{12}$. Suppose that at $z = 0$, all the energy is in mode 2, i.e., that $a_1(0) = 0$ and $a_2(0) = 1$. Then, the solutions for $a_1(z)$ and $a_2(z)$ are

$$a_1(z) = \frac{C_{21}}{\Gamma}e^{i(\beta_2-\beta_1)z/2}\sin(\Gamma z), \tag{27}$$

$$a_2(z) = \frac{1}{\Gamma}e^{i(\beta_1-\beta_2)z/2}\left(\Gamma\cos(\Gamma z) + i\frac{\beta_2-\beta_1}{2}\sin(\Gamma z)\right), \tag{28}$$

where $\Gamma = \sqrt{[(\beta_2-\beta_1)^2/4] + |C_{21}|^2}$. These results are similar to that obtained by Chien and co-workers[19] in their theoretical study of mode conversion in weakly guided slab dielectric waveguides made from optically active and isotropic polymers. Figure 2 presents plots of the power $|a_1(z)|^2$ and $|a_2(z)|^2$ carried by each individual modes (modes 1 and 2), as a function of z with $\beta_1 \neq \beta_2$. In this case, since propagation constants of the two unperturbed modes are unequal, i.e., $\beta_1 \neq \beta_2$, the coupling of energy from one mode to the other is not complete, and only a fraction of modal power of mode 2 transfers into mode 1. However, if the phase-matched condition is fulfilled, i.e., $\beta_1 = \beta_2$, a complete power transfer between the two modes will occur periodically along the z axis. For example, consider a parallel-plate waveguide where the two unperturbed modes TE_{10} and TM_{10} have the equal propagation constants ($\beta_1 = \beta_2$) for any frequency. If we introduce some chirality in the material filling the guide and then apply a TE_{10} mode at $z = 0$, since $\beta_1 = \beta_2$, a complete power transfer between TE_{10} and

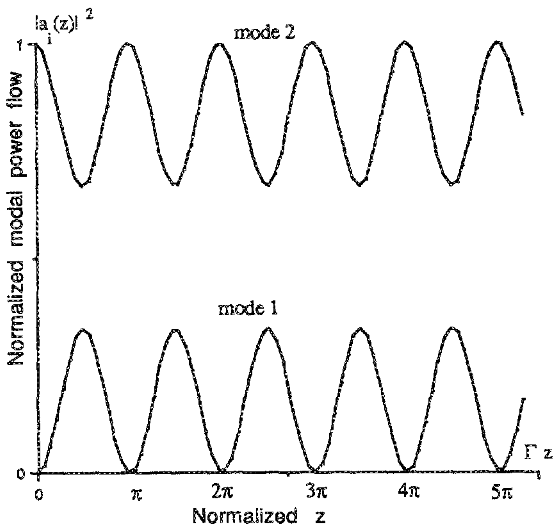

FIG. 2. Modal power variation along the z direction for modes 1 and 2. This figure is depicted for the case where the propagation constants along the z direction of the two unperturbed modes β_1 and β_2 are unequal.

TM_{10} will occur periodically along z axis. The lengths required for a complete mode conversion from TE_{10} to TM_{10} is given by

$$z = k\pi/\Gamma = k\pi/|C_{21}|. \tag{29}$$

It is worth noting that the larger the chirality admittance ξ_c is, the bigger the coupling coefficient C_{21} becomes, and the shorter distance is required for complete mode conversion.

The mode coupling afforded by chirality of materials have potential applications in a variety of novel microwave, infrared, and optical devices and components. Examples given in this paper can be used in the design of efficient TE↔TM converters without using conventional microwave materials which require biasing fields. It must be noted that the coupled-mode theory given in this report can be easily extended to the case of mode coupling between two and more chirowaveguides and similar results will be obtained.

IV. CONCLUSIONS

In this paper, we have analyzed the problem of wave propagation and mode coupling in chirowaveguides which are defined as cylindrical waveguides filled with homogeneous chiral materials. The chirality admittance is assumed to be small enough to be treated as a small perturbation. We have studied the coupled-mode theory for these waveguides and derived coupled-mode equations for different unperturbed modes of the waveguide. These equations were solved for a particular case where only the first two unperturbed modes were considered, and the results were compared with those reported in the literature. In this case, using the coupled-mode theory, we demonstrated that in a chirowaveguide there exists a periodic transfer of power from one mode to the other. The coupling coefficient is shown to be proportional to the chirality admittance ξ_c. We have also shown that for the phase-matched conditions, where propagation constants of unperturbed modes are equal, a complete power transfer from TE to TM mode is possible. This feature has potential application to novel microwave and optical devices such as TE↔TM converters without employing materials which need biasing fields.

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[30]In this expansion, only the discrete guided modes are considered, and continuous modes, which correspond to radiation modes in open waveguides, are neglected. In so doing, we are implicitly assuming that, if the guide is an open structure, no radiation is either coupled or scattered into or out of the waveguide by some perturbing mechanism. However, if such mechanism exists, the present analysis can be extended to include radiation modes.
[31]Positive integer n corresponds to forward-travelling waves and negative n to backward-travelling waves.