

A DISCRETE-TIME STOCHASTIC MODEL OF JOB MATCHING

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Abstract

In this paper, an explicit micro scenario is developed which yields a well-defined aggregate job matching function. In particular, a stochastic model of job-matching behavior is constructed in which the system steady state is shown to be approximated by an exponential-type matching function, as the population becomes large. This steady-state approximation is first derived for fixed levels of both wages and search intensities, where it is shown (without using a free-entry condition) that there exists a unique equilibrium. It is then shown that if job searchers are allowed to choose their search intensities optimally, this model is again consistent with a unique steady state. Finally, the assumption of a fixed wage is relaxed, and an optimal ‘offer wage’ is derived for employers.

Keywords: discrete-time matching function, large population approximation, optimal search intensity, optimal offer wage.

JEL Classification: D83, J41, J61.

1. Introduction

In the last few years, the vision of the labor market has changed from a stock perspective to a flow perspective. Indeed, labor markets in both the United States and Europe are characterized by very important gross flows that are associated with high rates of job creation and job destruction (Blanchard and Diamond, 1989, Burda and Wyploz, 1994, Davis, Haltiwanger and Schuh, 1996). Because of these large flows, the labor market cannot perfectly *match* workers and jobs, so that vacant jobs and unemployed workers coexist.

Mortensen and Pissarides were the first to develop a unified theoretical framework for analyzing this complex matching process, involving mutual search by workers and firms (for a synthesis, see Pissarides, 1990, and Mortensen and Pissarides, 1999). This has become widely known as the Mortensen-Pissarides model (MP hereafter). A central feature of their analysis is the *aggregate matching function* which combines job vacancies and unemployed workers to yield new active jobs. As pointed out by Pissarides (1990) and Blanchard and Diamond (1989), this notion of a matching function hides a complex reality in which skill differences between workers and jobs have a major role to play.

In the present paper, there is assumed to exist a fixed population of workers with heterogeneous skills. Firms are similarly characterized by a flow of heterogeneous production possibilities, with newly profitable jobs being opened and unprofitable jobs being closed. Thus *job matching* is taken to constitute a process whereby heterogeneous workers compete for jobs with different skill requirements. Here heterogeneity of workers need not imply any superiority or inferiority among their abilities. Rather, all are assumed to possess the same level of general human capital, which is manifested in a variety of different skills (as for example college graduates with degrees in different fields).¹

In this context, our first objective is to *derive an explicit micro scenario that leads to a well-defined job matching function*. This scenario involves a day-to-day process in which currently vacant jobs are posted by firms, and currently active job seekers apply for these vacancies. A vacancy is filled on a given day if and only if from among all those currently active job seekers, at least one applies for the position who meets all the job requirements. From these conditions one may

¹This approach to modeling heterogeneity on both sides of the labor market is similar to that of Hamilton, Thisse and Zenou (1999) and to Sattinger (1993), although they focus on different issues since, in particular, there is no search in their analysis.

derive an explicit *job-filling probability* on any given day.² The corresponding daily *matching function* is then given by the product of this probability and the expected number of vacant jobs per worker on each day. When the population of workers is allowed to become large, it is shown (Section 2) that the asymptotic form of this matching function is of exponential type and has the standard properties, i.e., is concave, monotone increasing in both arguments, and homogeneous of degree one. However, a comparison of this matching function with its finite-population counterpart reveals that while the latter is also increasing and concave, it *fails to be linearly homogeneous*. Hence in the present case, linear homogeneity turns out to be more a property of *limits* than of the underlying behavioral process.

This exponential-type matching function (which is essentially a limiting form of the classical ‘urn model’ in discrete probability theory) was first employed in a market context by Butters (1977) to model contacts between buyers and sellers in commodity markets. The first application of this model to job matching was by Hall (1979). The present application is however most closely related to that of Blanchard and Diamond (1993, 1994), who developed a continuous-time version of the job-matching process. Our specific model can be most accurately described as a variation of the discrete-time model developed by Blanchard and Diamond (1993, Appendix A) for their numerical simulations. The key difference between this discrete-time model (BD hereafter) and ours is largely one of focus. BD is concerned with firm hiring behavior, and in particular, the effect of unemployment duration on hiring probabilities. Hence firms are assumed to have private information about the unemployment duration of jobs applicants, and to rank applicants on this basis. In the present paper, the focus is more on job-seeking behavior, and in particular the decision process by which unemployed workers choose their intensity (frequency) of job search. This decision process in turn requires a more explicit consideration of the actual *matching* between applicants and jobs.³

Given this micro-based matching function, we begin with a simple stochastic model of the labor market (section 2) in which both wage levels and search intensities are assumed to be given. Here the focus is primarily on the steady-state behavior arising from the interaction of a finite population of workers with a flow of job opportunities linked by this matching function. This model departs from

²Equivalently, one may also determine the *job-hiring probability* that a job seeker is employed on a given day.

³In particular, the overall job ‘application rate’ in BD is here modeled more explicitly to include both an expected ‘search intensity level’ for job seekers, and a ‘qualification probability’ for job applicants.

both the BD model and MP model in terms of the basic flow of job creations and destructions. Such flows are here modeled as a type of stochastic birth-death process in which profitable jobs are ‘born’ and unprofitable jobs ‘die’ at rates depending on the overall state of the economy. In BD it is assumed rather that there exists a fixed stock of potentially productive jobs which continually switch between ‘productive’ and ‘unproductive’ states, depending on the economy. A *steady state* in both these models then involves a balancing of this job flow with the alternating flow of workers between employment and unemployment. In contrast, such a balance is achieved in MP by assuming a ‘free entry’ condition in which firms continue to create new job vacancies until the expected gains from advertising new jobs fall to zero. But while this condition is quite natural within a given industry where jobs are similar in nature, it is more difficult to interpret in a stochastic environment where individual job creations and closings are treated as essentially independent events.⁴

As in both the MP and BD models, we then show that when the population of workers becomes large, there exists a unique steady-state for this labor market characterized by the mean job-vacancy rate and unemployment rate.⁵ In particular, the steady-state unemployment rate is shown to be *positive*. Hence the imperfect nature of this job-matching process always gives rise to permanent *frictional unemployment* regardless of how many jobs are available.

Given this basic steady-state model, our second objective is to relax the assumption of fixed search intensities, and *to derive an explicit optimal search intensity level for unemployed workers*. In particular, we focus on the utility-maximizing problem of an unemployed worker in deciding how much time to spend in job search (i.e., how many days per week to actively search for a job). Here it is shown (section 3) that the optimal strategy is to search up to the point where the marginal gain of additional search (in terms of decreased unemployment

⁴In particular, if jobs are similar in nature, then changes in the economy will tend to affect all jobs alike. Thus, in this context, it is more difficult to support the usual assumption that individual job openings and closings are independent ‘Poisson’ events.

⁵In both the MP and BD models, this argument is implicitly made by passing directly to the limit. For example, the BD model involves a system of deterministic difference equations which represent the transitions in mean unemployment and job vacancy levels from week to week. Here we begin with the actual stochastic steady states for each finite population size, and show that (under mild conditions) the corresponding steady-state unemployment and job-vacancy rates converge in probability to their mean values as the population size becomes large. This provides an explicit probabilistic foundation for the resulting steady-state equations in terms of mean unemployment and job-vacancy rates.

duration) equals its marginal cost (in terms of reduced leisure). We then show that this behavior is consistent with the steady-state model above in the sense that there now exists a unique steady state equilibrium with endogenous search.

Given these endogenous search intensity levels for unemployed workers, our final objective is *to derive an optimal wage level for firms which maximizes their expected profit stream*. In the standard search and job matching literature (such as MP and BD), it is generally assumed that wages are set according to strategic bilateral Nash bargaining. Mortensen (1989) and Mortensen and Pissarides (1999) have surveyed the range of alternative wage-determination models in a search framework. Aside from the Nash bargaining model, they describe different wage setting rules including the ‘market clearing’ wage (wage equals unemployment benefit), the ‘insider-outsider’ wage, and the ‘efficiency’ wage. In the present paper, we propose a Stackelberg setting in which wages set by firms are viewed as ‘signals’ to workers in choosing their optimal search intensities. Firms face the following trade off in setting wages: If wages are set too low then search intensities will also be low, so that jobs will tend to remain vacant for long periods with no profits earned. On the other hand, if wages are set too high, then only small profits will be earned while jobs are active.⁶ In this context we show (section 4) that an optimal wage level always exists, and in addition that if optimal wages are ‘sufficiently large’ then they must be unique.

2. The Basic Model

Consider a population of N workers who compete for jobs in a given labor market. All jobs are offered at the same prevailing *daily wage*, w , but are assumed to be completely specialized in terms of skill requirements.⁷ Similarly, workers are assumed to be heterogeneous in terms of their skill endowments. As stated in the introduction, *job matching* in the present context constitutes a process whereby heterogeneous workers allocate themselves to jobs with different skill requirements.

⁶This model of wage determination is somewhat similar to that of Sattinger (1990), where firms face a trade off between the wage they pay and the ratio of job seekers to vacancies. Here high wages imply more interviews and thus a lower duration for vacancies but, of course, at a higher cost. However, the central focus of this framework is very different since the goal is to find an efficient wage in a standard macroeconomic matching framework, whereas ours is to determine an equilibrium wage in a stochastic matching model with endogenous search intensities.

⁷The assumption of completely specialized jobs can also be found in Pissarides (1990) and Blanchard and Diamond (1994).

Heterogeneity of workers does not here imply any superiority or inferiority among their abilities. Rather, all are assumed to possess the same level of general human capital, which is manifested in a variety of different skills. Hence all workers are assumed to have the same chance of being qualified for any given job, as modeled by a common *qualification probability*, γ .

In this context, the actual *job matching process* can be described as follows. At any point in time, each worker is either employed or unemployed, and only unemployed workers are assumed to search for jobs. Since individual jobs are completely specialized, their creation and closing can be regarded as independent events. In particular, job creations and job closings are here modeled as a simple *birth-death process* in which ‘births’ are governed by a *job-creation rate*, β (denoting the mean number of jobs per worker created each day) and ‘deaths’ are governed by a *job-closure rate*, ρ (denoting the probability that any currently existing job will be closed on a given day). This process is taken to depend on the general state of economy, and hence is treated as exogenous to the labor market. Two basic parameters in the model are initially taken to be given: the *daily wage*, w , and the *search intensity*, s , which denotes the expected fraction of days on which each unemployed worker actively seeks a job. The following is a brief description of the behavioral day-to-day scenario for the job market model on a given day, t :

- At the beginning of day t those unemployed workers currently seeking work appear at the job market. All current job vacancies are posted, and are offered at the going wage w . Each searcher applies for a single job. No additional prior information about jobs is available, and there is no communication between searchers. Hence searchers choose jobs at random, and more than one searcher may apply for the same job.
- As mentioned above, each job applicant has the same probability, γ , of satisfying all qualifications for the given job. If more than one applicant is qualified for a job, the employer chooses a qualified applicant at random. Otherwise the job is not filled on day t .
- At the end of day t each successful applicant is notified, and is requested to start work on the following day. In addition, decisions are made by employers as to which jobs are no longer profitable and should be closed. For currently active jobs which are closed, layoff notices are distributed to workers. Moreover, for jobs which are filled that day and then closed,

the successful (but unlucky) applicants are also given notices. Finally, those currently vacant jobs which are closed are simply removed from the postings at the beginning of the next day. As mentioned above, all jobs (active or vacant) have the same chance, ρ , of being closed on day t .

- In addition, those new job opportunities which have arisen during the day (at rate, β , per worker) are added to the vacant job postings for the next day.

This process is governed by the following system of accounting equations:

$$V_{t+1}^N = \underbrace{V_t^N}_{\text{Beginning of Day } t} + \underbrace{(B_t^N - F_t^N - C_{vt}^N)}_{\text{End of Day } t} \quad (2.1)$$

$$U_{t+1}^N = U_t^N + (C_{at}^N - F_t^N) \quad (2.2)$$

Here the random variables, V_t^N and U_t^N , denote respectively the numbers of *job vacancies* and *unemployed workers* at the beginning of day t . Variable F_t^N denotes the number of *vacant jobs filled* by the end of day t . Variables C_{vt}^N and C_{at}^N denote respectively the numbers of *vacant jobs closed* and *active jobs closed* at the end of day t (removed from the postings on day $t + 1$). Finally, variable B_t^N denotes the number of *new job openings* announced at the end of day t . (The reason for the population superscript N will become clear later). Equation (2.1) then states that the change in total vacant jobs from day t to $t + 1$ is given by the difference between the new jobs created and the vacant jobs either filled or closed on day t . Similarly, equation (2.2) states that the change in unemployment from day t to $t + 1$ is given by the difference between the number of workers laid off (i.e., the number of active jobs closed) and the number of workers hired (i.e., the number of vacant jobs filled) on day t .

2.1. Job Matching Process

Within this overall accounting framework, we now focus on the key variables, F_t^N , which summarize job hiring activity on each day. To do so, we begin by fixing the number of vacant jobs and unemployed workers, say $V_t^N = m$ and $U_t^N = n$ (where

it is assumed that there is at least one vacant job, i.e. $m \geq 1$). In this context, if for each vacant job, $j = 1, \dots, m$, we define the indicator variable

$$F_t^j = \begin{cases} 1, & \text{if job } j \text{ is filled on day } t \\ 0, & \text{otherwise} \end{cases} \quad (2.3)$$

then by definition, the conditional value of F_t^N given m is of the form

$$F_t^N = \sum_{j=1}^m F_t^j. \quad (2.4)$$

Hence to model F_t^N we begin by considering the conditional distribution of each F_t^j given $V_t^N = m$ and $U_t^N = n$. To do so, recall first that not all unemployed workers necessarily seek work on any given day. Rather, there is a probability (*search intensity*), s , that any worker chooses to search on day t . Hence, assuming independent decision behavior by all individuals, the probability that l ($\leq n$) individuals apply for jobs on day t is given by the binomial probability:

$$P(l|n) = \binom{n}{l} s^l (1-s)^{n-l}. \quad (2.5)$$

Next, recalling that jobs are chosen at random by applicants, the chance that any worker applies for job j is given by $1/m$. Hence the chance of k ($\leq l$) applicants for job j is given by

$$P_m(k|l) = \binom{l}{k} \left(\frac{1}{m}\right)^k \left(1 - \frac{1}{m}\right)^{l-k}, \quad (2.6)$$

A straightforward calculation (see section A.1 of the Appendix) then shows that the probability of k applicants given m and n is of the form:

$$P(k|m, n) = \binom{n}{k} \left(\frac{s}{m}\right)^k \left(1 - \frac{s}{m}\right)^{n-k} \quad (2.7)$$

Finally, recalling that workers are qualified for any given job with probability, γ , and that a job is filled only if at least one qualified worker applies, it follows that the probability of filling job j given k applicants is

$$P(F_t^j = 1|k) = 1 - (1 - \gamma)^k. \quad (2.8)$$

Hence, combining (2.7) and (2.8), we have

$$\begin{aligned}
P(F_t^j = 1|m, n) &= \sum_{k=0}^n P(F_t^j = 1|k) P(k|m, n) \\
&= \sum_{k=0}^n \left[1 - (1 - \gamma)^k\right] \binom{n}{k} \left(\frac{s}{m}\right)^k \left(1 - \frac{s}{m}\right)^{n-k} \\
&= 1 - \sum_{k=0}^n (1 - \gamma)^k \binom{n}{k} \left(\frac{s}{m}\right)^k \left(1 - \frac{s}{m}\right)^{n-k}. \quad (2.9)
\end{aligned}$$

But recalling that the moment generating function of the binomial, $B(p, n)$, is of the form, $E(\theta^K) = (1 - p + \theta p)^n$, it follows that

$$\begin{aligned}
P(F_t^j = 1|m, n) &= 1 - E\left[(1 - \gamma)^K\right] \\
&= 1 - \left(1 - \frac{s}{m} + (1 - \gamma) \frac{s}{m}\right)^n \\
&= 1 - \left(1 - \frac{\gamma s}{m}\right)^n. \quad (2.10)
\end{aligned}$$

Finally, observing that this probability is precisely the conditional mean, $E(F_t^j|m, n)$, of the random variable F_t^j , it follows from (2.4) that the expected total number of jobs filled on any day for which there are m vacant jobs and n unemployed workers is given by

$$E(F_t^N|m, n) = \sum_{j=1}^m E(F_t^j|m, n) = m \left[1 - \left(1 - \frac{\gamma s}{m}\right)^n\right] \quad (2.11)$$

The function Φ_N defined by the right hand side, i.e. by

$$\Phi_N(m, n) = m \left[1 - \left(1 - \frac{\gamma s}{m}\right)^n\right] \quad (2.12)$$

is thus seen to summarize the key workings of this job market, and is designated as the *matching function* for the system.

By employing arguments similar to those above, one can in principle analyze the day-to-day stochastic behavior of this finite-population system. In particular, one may determine the explicit form of the discrete conditional distribution of F_t^N

given $(V_t^N = m, U_t^N = n)$ ⁸. This, together with the properties of the (discrete-time) birth-death model for job creations and closings, allows one to determine the conditional distribution of (V_{t+1}^N, U_{t+1}^N) given (V_t^N, U_t^N) [by employing the basic accounting equations (2.1) and (2.2)]. For our present purposes, it suffices to observe that the Markov chain defined by these conditional distributions is necessarily both positive recurrent and acyclic, so that for each population size N this process converges to a *unique steady-state distribution* [as for example in Kulkarni (1995, Theorem 5.3.15)].

While the exact form of this steady-state distribution can be exceedingly complex for small N , it turns out that under fairly mild conditions, this distribution concentrates in the neighborhood of its mean as N becomes large.⁹ Hence, following the standard approach adopted in most of the literature, we henceforth assume that *the system is in steady state*, and that N is sufficiently large to allow steady-state fluctuations in both unemployment levels and job vacancies per worker to be ignored. We now develop this nonstochastic approximation for large N .

2.2. Mean Steady-State Relations

If the system is in steady state, then the mean values of all random variables in (2.1) and (2.2) are constant over time. Hence, letting $(V^N, U^N, F^N, C_a^N, C_v^N)$, denote the steady-state counterparts of the random variables in these two equations, it then follows that $E(V_t^N) = E(V^N)$ for all t , and similarly for all other random variables. By replacing all random variables in (2.1) and (2.2) with their steady-state counterparts and taking expectations, we obtain the following (reduced) steady-state relations:

$$E(F^N) + E(C_v^N) = E(B^N) \quad (2.13)$$

$$E(F^N) = E(C_a^N) \quad (2.14)$$

Recall next from the basic model description that each of the $(V^N - F^N)$ vacant jobs remaining at the end of the day has the same chance, ρ , of being closed (removed from the job postings), so that the expected number $E(C_v^N)$ of vacant

⁸The case of $V_t^N = 0$ must be treated separately. By our assumptions, it follows that $P(F_t^N = 0 = C_{vt}^N | V_t^N = 0) = 1$. Hence $P(V_{t+1}^N = m | V_t^N = 0) = P(B_t^N = m | V_t^N = 0) = P(B_t^N = m)$ and $P(U_{t+1}^N = n | V_t^N = 0, U_t^N + C_{at}^N = n) = 1$.

⁹See for example the results of Brumelle and Gerchak (1980).

jobs closed is given by:

$$\begin{aligned}
E(C_v^N) &= \sum_k E(C_v^N | V^N - F^N = k) P(V^N - F^N = k) \\
&= \sum_k (\rho k) P(V^N - F^N = k) \\
&= \rho E(V^N - F^N) = \rho [E(V^N) - E(F^N)] \tag{2.15}
\end{aligned}$$

Similarly, each active job has the same chance, ρ , of being closed. But since active jobs include both those jobs active at the beginning of the day and those filled that day, the number of active jobs is identical to the number of currently employed workers, $N - U^N$, plus the number of vacancies filled, F^N . Hence it follows from the same argument as in (2.15) that the expected number, $E(C_a^N)$, of vacant jobs closed is given by:

$$E(C_a^N) = \rho [N - E(U^N) + E(F^N)] \tag{2.16}$$

Finally, by substituting (2.15) and (2.16) into (2.13) and (2.14) and rearranging terms, we may conclude that in the steady state:

$$(1 - \rho) E(F^N) + \rho E(V^N) = E(B^N) \tag{2.17}$$

$$(1 - \rho) E(F^N) = \rho [N - E(U^N)] \tag{2.18}$$

2.3. Large Population Approximations

Given this reduced system of relations, we next ask how the steady-state distributions of (B^N, U^N, V^N, F^N) behave as the population N becomes large. Each of these random variables will be considered in order.

2.3.1. Job Creation Rate.

It should be clear from the basic model described above that this economy is largely driven by the creation of new job opportunities. Moreover, it was implicitly assumed that the per-worker rate of job-creation, β , is *independent* of population size. In other words, increments in population (and hence in potential demand for goods and services) are assumed to generate proportional increments in mean job creations. This can be modeled formally by assuming that for each individual i there is a random variable, B_i , representing the daily jobs created by the presence

of i in the economy. This convention allows the total number of jobs created to be represented as a sum

$$B^N = \sum_{i=1}^N B_i \quad (2.19)$$

If it then assumed that these individual contributions are independently and identically distributed, with mean β and finite variance σ^2 , then by the Weak Law of Large Numbers it follows that B^N/N converges to β in probability, i.e., that

$$plim_{N \rightarrow \infty} \frac{B^N}{N} = \beta \quad (2.20)$$

Hence for large populations the daily rate of job creation per-worker can be treated as *nonstochastic*.¹⁰

2.3.2. Unemployment Rate

A similar argument can be made for the steady-state unemployment level in the economy. In particular, if the unemployment status of each worker i is represented by a random (indicator) variable, U_i (with $U_i = 1$ when i is unemployed and $U_i = 0$ otherwise), then the daily level of unemployment can be represented as in (2.19) by

$$U^N = \sum_{i=1}^N U_i \quad (2.21)$$

Hence if large populations of workers are treated as homogeneous collections of independent behaving units, then these employment status variables can be treated as independently and identically distributed with common mean, u [and finite variance $u(1-u)$]. It then follows, as in (2.20), that

$$plim_{N \rightarrow \infty} \frac{U^N}{N} = u \quad (2.22)$$

and thus that for large populations the steady-state *unemployment rate*, u , can also be regarded as nonstochastic.¹¹

¹⁰Note that our assumptions imply strong convergence in (2.20), i.e. that B^N/N converges almost surely to β . But since weak convergence ensures the validity of nonstochastic approximations for any (sufficiently large) *fixed* population size, N , this suffices for our purposes.

¹¹The assumption of statistically independent worker behavior can be relaxed to some degree. For example, if workers communicate only with a boundedly finite set of other workers, then

2.3.3. Job Vacancy rate

These nonstochastic approximations in turn imply that the steady-state job vacancy rate can be treated as nonstochastic. In particular, if the random variable J^N represents *total jobs* in steady state, so that by definition,

$$J^N = V^N + (N - U^N) \quad (2.23)$$

it is then shown in section A.2 of the Appendix that our assumptions imply

$$plim_{N \rightarrow \infty} \frac{J^N}{N} = \lim_{N \rightarrow \infty} \frac{E(J^N)}{N} = \frac{\beta}{\rho}. \quad (2.24)$$

Thus, by dividing through (2.23) and taking probability limits, we may conclude that

$$\begin{aligned} plim_{N \rightarrow \infty} \frac{J^N}{N} &= plim_{N \rightarrow \infty} \left(\frac{V^N}{N} + 1 - \frac{U^N}{N} \right) \\ &\Rightarrow \frac{\beta}{\rho} = plim_{N \rightarrow \infty} \frac{V^N}{N} + (1 - u) \\ &\Rightarrow plim_{N \rightarrow \infty} \frac{V^N}{N} = \frac{\beta}{\rho} - (1 - u) \end{aligned} \quad (2.25)$$

Hence the steady-state *job vacancy rate*, v , can also be treated as nonstochastic for large N , and is seen to be given by

$$v = u + a \quad (2.26)$$

where

$$a = \frac{\beta}{\rho} - 1 \quad (2.27)$$

a simple application of Chebyshev's Inequality shows that (2.22) still holds. More generally, the same arguments show that it is enough to require that the *average* (absolute) correlation, $(1/N) \sum_{j \neq i} \rho_{ij}$, of each worker i 's unemployment status with all other workers eventually vanish [see for example the proof of Theorem 4.4.2 of Renyi (1970)]. Similar (but more stringent) conditions for almost sure convergence can be found in Stout (1974, section 3.7).

2.3.4. Vacancy Filling Rate

Finally, these results taken together also imply that the steady-state rate at which vacant jobs are filled can also be treated as nonstochastic. To see this, observe first that if for each x, y, N we let

$$p_N(x, y) = 1 - \left(1 - \frac{\gamma s}{Ny}\right)^{Nx} \quad (2.28)$$

then the job-filling probability in (2.10) can be rewritten as follows:

$$\begin{aligned} P(F_t^j = 1 | n, m) &= 1 - \left(1 - \frac{\gamma s}{m}\right)^n \\ &= 1 - \left(1 - \frac{\gamma s}{N \left(\frac{m}{N}\right)}\right)^{N \left(\frac{n}{N}\right)} \\ &= p_N\left(\frac{n}{N}, \frac{m}{N}\right) \end{aligned} \quad (2.29)$$

In terms of this notation, observe next from (2.4) that the total number, F^N , of job vacancies filled must then be conditionally binomially distributed given $V^N = m$ and $U^N = n$, with mean

$$E(F^N | n, m) = p_N\left(\frac{n}{N}, \frac{m}{N}\right) \cdot m \quad (2.30)$$

Hence, the conditional expectation of the *vacancy-filling rate*, F^N/N , is

$$E\left(\frac{F^N}{N} \middle| n, m\right) = p_N\left(\frac{n}{N}, \frac{m}{N}\right) \cdot \frac{m}{N} \quad (2.31)$$

and it follows that the unconditional mean of F^N/N is given by

$$E\left(\frac{F^N}{N}\right) = E_{U^N, V^N} \left[p_N\left(\frac{U^N}{N}, \frac{V^N}{N}\right) \frac{V^N}{N} \right] \quad (2.32)$$

But if we now let

$$p(x, y) = 1 - e^{-(\gamma s x / y)} \quad (2.33)$$

and observe that

$$\begin{aligned}
\lim_{N \rightarrow \infty} p_N(x, y) &= \lim_{N \rightarrow \infty} \left\{ 1 - \left[\left(1 - \frac{\gamma s}{Ny} \right)^N \right]^x \right\} \\
&= 1 - \left[\lim_{N \rightarrow \infty} \left(1 - \frac{\gamma s}{Ny} \right)^N \right]^x \\
&= 1 - [e^{-(\gamma s/y)}]^x \\
&= p(x, y)
\end{aligned} \tag{2.34}$$

it can then be shown (see section A.3 of the Appendix) that this limiting relation, together with the probability limits in (2.22) and (2.26) imply that

$$\begin{aligned}
\text{plim}_{N \rightarrow \infty} \frac{F^N}{N} &= \lim_{N \rightarrow \infty} E \left(\frac{F^N}{N} \right) \\
&= \lim_{N \rightarrow \infty} E_{U^N, V^N} \left[p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right] \\
&= p(u, v) \cdot v \\
&= v [1 - e^{-(\gamma su/v)}] .
\end{aligned} \tag{2.35}$$

If the steady-state *vacancy-filling rate*, f , for the system is defined by

$$f = \text{plim}_{N \rightarrow \infty} \frac{F^N}{N} , \tag{2.36}$$

then it follows from (2.35) that f is related to v and u by

$$f = \phi(u, v) = v [1 - e^{-(\gamma su/v)}] . \tag{2.37}$$

The function, ϕ , is seen to be precisely the asymptotic (normalized) form of the functions, Φ_N , defined in (2.12) above, and hence is again designated as the *matching function* for the system. Notice also from (2.29) and (2.34) that the quantity

$$p_f(u, v) = 1 - e^{-(\gamma su/v)} \tag{2.38}$$

can be interpreted as the asymptotic *job-filling probability* for the system (i.e. the asymptotic probability that any given vacant job will be filled on a given day).

Finally we note that while the above analysis was developed in terms of job-filling probabilities, all results could equivalently have been formulated in terms of *hiring* probabilities. This follows from the obvious accounting identity between the number of jobs filled and the number of workers hired on any given day. For our later purposes, it will be convenient to treat the asymptotic hiring probability as well. This can be derived from the above results as follows. Observe first that, as a parallel to V^N and $p_N\left(\frac{n}{N}, \frac{m}{N}\right)$, we may let H^N denote the number of job searchers hired on any day for population N , and let $q_N\left(\frac{n}{N}, \frac{m}{N}\right)$ denote the conditional probability that any given job searcher is hired on a day with n unemployed workers and m vacant jobs. Then, observing that by definition the expected number of job searchers on any day given n is simply $s \cdot n$, it follows that the above accounting identity takes the form:

$$\begin{aligned}
E(F^N|n, m) &\equiv E(H^N|n, m) \\
\Rightarrow E\left(\frac{F^N}{N}|n, m\right) &\equiv E\left(\frac{H^N}{N}|n, m\right) \\
\Rightarrow p_N\left(\frac{n}{N}, \frac{m}{N}\right) \cdot \frac{m}{N} &\equiv q_N\left(\frac{n}{N}, \frac{m}{N}\right) \cdot s \cdot \frac{n}{N} \\
\Rightarrow p_N\left(\frac{U^N}{N}, \frac{V^N}{N}\right) \cdot \frac{V^N}{N} &\equiv q_N\left(\frac{U^N}{N}, \frac{V^N}{N}\right) \cdot s \cdot \frac{U^N}{N} \tag{2.39}
\end{aligned}$$

Hence, taking probability limits on both sides and applying the results above, we see that

$$\begin{aligned}
p(v, u) \cdot v &= \left\{ \text{plim}_{N \rightarrow \infty} q_N\left(\frac{U^N}{N}, \frac{V^N}{N}\right) \right\} \cdot s \cdot u \\
\Rightarrow \text{plim}_{N \rightarrow \infty} q_N\left(\frac{U^N}{N}, \frac{V^N}{N}\right) &= \frac{v}{s u} \cdot p(u, v) \tag{2.40}
\end{aligned}$$

Finally, denoting the desired asymptotic *hiring probability* for the system by

$$p_h(u, v) = \text{plim}_{N \rightarrow \infty} q_N\left(\frac{U^N}{N}, \frac{V^N}{N}\right) \tag{2.41}$$

it follows from (2.35) that

$$p_h(u, v) = \frac{v}{s u} [1 - e^{-(\gamma s u/v)}] . \tag{2.42}$$

2.4. Properties of the Matching Function

Before determining the steady state values of u and v , it is of interest to consider the matching function (2.37) in more detail. First, we show that ϕ is an instance of the general class of matching functions proposed by Pissarides (1990), i.e. that (see section A.4 of the Appendix for the proof):

Proposition 2.1. *The function, ϕ , in (2.37) is concave, linearly homogeneous, and monotone increasing in both arguments.*

Additional insight can be gained by employing the concept of *labor market tightness*, defined in Pissarides (1990) as $\theta \equiv \frac{v}{su}$. Our *job-filling probability* (2.38) can be written in terms of θ as:

$$p_f(u, v) \equiv q(\theta) = 1 - e^{-\gamma/\theta} \quad (2.43)$$

with associated *hiring probability* (2.42) given by:

$$p_h(u, v) \equiv \theta q(\theta) = \theta [1 - e^{-\gamma/\theta}] \quad (2.44)$$

For $0 < \theta < \infty$, it is easily verified that (i) $q'(\theta) < 0$, (ii) the elasticity of $q(\theta)$ is between 0 and -1 and (iii) the elasticity of $\theta q(\theta)$ is positive. This implies for example that if θ increases, say by an increase in the number of job vacancies, then it will be easier for workers to find a job (p_h increases) and more difficult for firms to fill their vacant jobs (p_f decreases). Thus, as observed by Pissarides (1990), such markets involve *search (or congestion) externalities* in which the probability of finding a job (or filling a vacancy) depends on the relative numbers of ‘traders’ in the labor market, as reflected by θ . In other words there exists ‘stochastic rationing’ which is independent of price adjustments.

Next we compare the matching function (2.37) with its finite-population counterpart in (2.12) above. To do so, it is convenient to consider the normalized form, ϕ_N , of Φ_N as defined in (2.31) [together with (2.28)] above for all $u = n/N$ and $v = m/N$ by

$$\begin{aligned} \phi_N(u, v) &= E \left(\frac{F^N}{N} \middle| n, m \right) = \frac{m}{N} \cdot p_N \left(\frac{n}{N}, \frac{m}{N} \right) \\ &= v \cdot p_N(u, v) = v \left[1 - \left(1 - \frac{\gamma s}{Nv} \right)^{Nu} \right] \end{aligned} \quad (2.45)$$

If u and v are treated as continuous variables, then partial differentiation of (2.45) again shows that $\phi_N(u, v)$ is increasing and concave in the relevant range of each variable (i.e., for all values of $u \geq 1/N$ and $v \geq 1/N$). However, it should be clear from the form of (2.45) that ϕ_N *fails* to be linearly homogeneous. Indeed, it is clear from the argument above that linear homogeneity is essentially a consequence of *limits*. To see this, first observe that ϕ_N is of the form

$$\phi_N(u, v) = v \cdot P(Nu, Nv) \quad (2.46)$$

where $P(x, y) = 1 - \left(1 - \frac{\gamma s}{y}\right)^x$. More generally, suppose that ϕ_N is of the form (2.46) for *any* choice of P for which a limiting value

$$\xi(u, v) = \lim_{N \rightarrow \infty} P(Nu, Nv) \quad (2.47)$$

exists, so that the desired asymptotic matching function can be written obtained as

$$\phi(u, v) = v \cdot \xi(u, v) = v \cdot \lim_{N \rightarrow \infty} P(Nu, Nv) \quad (2.48)$$

Then for any $\lambda > 0$ it follows from (2.47) that

$$\begin{aligned} \xi(\lambda u, \lambda v) &= \lim_{N \rightarrow \infty} P[N(\lambda u), N(\lambda v)] \\ &= \lim_{N \rightarrow \infty} P[(N\lambda)u, (N\lambda)v] \\ &= \lim_{(N\lambda) \rightarrow \infty} P[(N\lambda)u, (N\lambda)v] \\ &= \xi(u, v) \end{aligned} \quad (2.49)$$

Hence the limit probability function, ξ , must be homogeneous of degree zero, and it follows at once from (2.48) that ϕ must be homogenous of degree one, i.e. that

$$\phi(\lambda u, \lambda v) = \lambda v \cdot \xi(\lambda u, \lambda v) = \lambda v \cdot \xi(u, v) = \lambda \cdot \phi(u, v) \quad (2.50)$$

Note finally that this limit property also shows that any finite-population matching function of the form (2.46) must always be *approximately* linearly homogeneous as N becomes large. In the present case, it is well known that the limit in (2.34) is reached rather quickly. In fact, numerical examples show that for populations as small $N = 5$ the function ϕ_N in (2.45) is approximately linearly homogeneous for all $\lambda > 1$.

2.5. System Steady States

Given these asymptotic approximations, the appropriate equations describing steady states of the system can now be summarized as follows. Observe first from the limiting relations

$$(u, v, f) = plim_{N \rightarrow \infty} \left[E \left(\frac{U^N}{N} \right), E \left(\frac{V^N}{N} \right), E \left(\frac{F^N}{N} \right) \right] \quad (2.51)$$

that by dividing both (2.17) and (2.18) by N and taking limits, we obtain the steady-state relations

$$(1 - \rho) f + \rho v = \beta \quad (2.52)$$

$$(1 - \rho) f = \rho (1 - u) \quad (2.53)$$

which together with (2.37) can be transformed into the pair of *steady-state equations*¹²

$$v + (1 - u) = \beta / \rho \quad (2.54)$$

$$\rho (1 - u) = (1 - \rho) v [1 - e^{-(\gamma s u / v)}] \quad (2.55)$$

If for any given search intensity, $s \in [0, 1]$, we now designate each solution, $[u(s), v(s)]$, to [(2.54), (2.55)] {or [(2.26), (2.55)]} as a *steady state* for s , then the main result of this section is to establish the existence and uniqueness of steady states. To do so, note first that when $\beta < \rho$ [so that by (2.24) there is less than one job per person in the steady state], it is possible to obtain negative values for vacancy rates, v , in (2.54). But since steady states are only meaningful if $v \geq 0$, we now specify this nonnegativity condition in terms of unemployment rates, u , as follows. Recall from (2.26) that $v = u + a \geq 0$ iff $u \geq -a$. Hence letting

$$u_a = \max\{0, -a\}, \quad (2.56)$$

it follows that the only meaningful steady states are those with $u \in [u_a, 1]$. With these observations, our main result is to show that (the proof is given in section A.5 of the Appendix):

Theorem 2.2 (Fixed-Search Equilibrium). *For each search intensity, $s \in (0, 1]$, there exists a unique steady state, $[u(s), v(s)]$, with $u(s) \in (u_a, 1)$. In addition, both $u(s)$ and $v(s)$ are positive decreasing differentiable functions of s with $u(0) = 1$ and $v(0) = \beta / \rho$.*

¹²Notice that (2.54) is identical with the steady-state relation already obtained in (2.26).

This steady state can be compared to that of the MP (Mortensen-Pissarides) model, by noting that the Beveridge curve in MP is very similar to our steady-state condition (2.55). In contrast to the free entry condition in MP (mentioned in the Introduction), our steady-state condition (2.54) results from the underlying birth-death process on vacancies. This steady-state can also be compared to that of the BD (Blanchard-Diamond) model by observing that equations (2.54) and (2.55) are essentially a variation of the steady-state conditions derivable from their recursive equation system [(A.1) through (A.5)].¹³ The only real differences (aside from stochasticity) are that (i) their discrete time period is a *week* rather than a *day*, (ii) their job seekers are indexed by *unemployment duration*, and (iii) their stock of possible jobs is *finite* rather than *denumerable*, as in our case.¹⁴

Turning next to the comparative statics of this steady state, observe that variations with respect to parameter s were established in Theorem 2.2. The following additional comparative-static properties can also be established (see section A.6 of the Appendix):

$$\frac{\partial u}{\partial \gamma} < 0 \quad \frac{\partial u}{\partial \beta} < 0 \quad \frac{\partial u}{\partial \rho} > 0 \quad (2.57)$$

$$\frac{\partial v}{\partial \gamma} < 0 \quad \frac{\partial v}{\partial \beta} > 0 \quad \frac{\partial v}{\partial \rho} < 0 \quad (2.58)$$

These relations can be used to further clarify the basic role of each parameter in the model. From the unemployed worker's viewpoint (2.57) it is clear that he/she is more likely to get a job by searching more frequently ($s \uparrow$) or being qualified for a wider range of jobs ($\gamma \uparrow$). Similarly, hiring is more likely if new job opportunities are created at a faster rate ($\beta \uparrow$) or existing job opportunities are closed at a slower rate ($\rho \downarrow$). From the firm's viewpoint (2.58) it is equally clear that increased searching effort ($s \uparrow$) and broader worker qualifications ($\gamma \uparrow$) will increase the likelihood of a job being filled, and hence decrease vacancies. Also, vacancies are more likely to be filled if new vacancies are created at a slower rate

¹³In particular, if ρ is replaced by the transition probability π_0 in BD, then our condition (2.55) is a consequence of the steady-state conditions derivable from conditions [(A1),(A3),(A5)] in BD. Similarly, if our steady-state expected jobs per worker, β/ρ , is replaced by their steady-state expected jobs per worker, $\frac{\pi_1 - \frac{K}{N}}{\pi_0 + \pi_1}$, then our condition (2.54) is a consequence of the steady-state conditions derivable from conditions [(A1),(A2),(A3),(A5)] in BD.

¹⁴Recall that our basic states are of the form ($V_t^N = m, U_t^N = n$) with implicit domains $m \in \mathbf{Z}_+$ and $n \in \{0, 1, \dots, N\}$. Note also that the finite-stock model in BD requires that each job have *three* possible states: 'filled', 'vacant', and 'idle'. However this distinction is of little consequence in terms of the actual behavior of the two models.

($\beta \downarrow$) or if existing vacancies die at a faster rate ($\rho \uparrow$). Note in particular, that the qualification probability, γ , can now be viewed as a *matching index* in the sense that larger values increase both the likelihood of unemployed workers being hired and vacant jobs being filled.

To illustrate these results, let $\rho = .0027$ [so that expected life of a job is about one year] and let $\beta = .0054$ [so that by (2.24) there are about two jobs (vacant or active) per worker in steady state]. In addition, suppose that workers have a 50-50 chance of being qualified for any given job ($\gamma = .5$). If each day of the year is a working day, and if unemployed workers search for jobs every day ($s = 1$), then under these conditions the steady-state unemployment rate is 0.5% ($u = .005$). [In addition, the steady-state vacancy rate is $v = 1.005$ (about one job per worker).] Hence even in ‘good times’ with two jobs per worker, and with full search effort by the unemployed, there is a small level of residual unemployment due purely to frictional effects of job competition and qualification matching. But, as stated in Theorem 2.2, if search intensity decreases, then unemployment must increase. For example, if workers search only once a month ($s = .033$) then residual unemployment rises to about 14% ($u = .141$).

This situation may be contrasted with ‘bad times’, as represented by a drop in the job creation rate to $\beta = .00135$ (resulting in only one job for every two workers). In this case, even with full search intensity ($s = 1$), the unemployment rate in steady state exceeds 50% ($u = .5013$). [Note that $a = -.5$ implies $u_a = .5$, so that as in Theorem 2.2, $u \in (u_a, 1)$, and $v = .0013 > 0$.] It is also interesting to observe that unemployment rates are now quite insensitive to search intensities over a wide range of values. For example, a search intensity of only one day a year ($s = .0027$) has almost no perceptible effect on unemployment rates ($u = .5014$). The intuition here is that with so few jobs created, new jobs tend to be quickly filled even if only a few potential workers show up at the market (as evidenced by the low vacancy rate above). Hence unemployment in ‘bad times’ tends to be more structural than frictional. This also shows that diligent search efforts can sometimes be quite fruitless, and suggests that unemployed workers may wish to alter their search intensities depending on the situation.

3. Endogenous Search Intensities

These observations lead naturally to the question of how unemployment levels may be affected if unemployed workers are allowed to choose their own levels of search intensity. Basically, this choice involves a trade-off between the leisure

time lost and the expected income gained (from quicker job acquisition) by more frequent job search. To model this trade-off explicitly, we begin by considering the relative value of employment and unemployment in terms of the utility levels and discounted utility streams associated with each state. Here the respective classes of *unemployed workers* and *employed workers* are denoted by ‘0’ and ‘1’.

3.1. Effective-Income Utility

If individual daily *income* is denoted by y , and if the fraction of time spent each day in *leisure activities* is denoted by l , then the individual’s *utility* for (l, y) is postulated to be of the form

$$U(l, y) = l^\alpha y \tag{3.1}$$

with parameter $\alpha \in (0, 1)$. This utility can be regarded as the individual’s *effective daily income*, discounted by the fraction of leisure time available for consumption.¹⁵ As will become clear below, the assumption of *decreasing returns to leisure* (i.e., $0 < \alpha < 1$) is critical for our purposes, in that it allows meaningful optimal search intensities, s , other than the extremes 0 and 1. In particular, if the leisure time for unemployed workers is taken to include all time spent not searching for work, then the expected fraction of leisure time for an unemployed worker with search intensity, s , is simply $1 - s$. Hence if it is assumed that income for such individuals is given by a daily *unemployment benefit*, b , then the relevant *effective daily income* for each unemployed worker is given by

$$U_0(s) = (1 - s)^\alpha b . \tag{3.2}$$

Here search intensity, s , constitutes the only relevant decision variable for unemployed workers.

Similarly, if it is assumed that the income of employed workers is given by the daily wage, w , and that their fraction of leisure time, l_1 , is constant, then the *effective daily income* for all employed workers is given by

$$U_1 = l_1^\alpha w . \tag{3.3}$$

¹⁵Note also that this utility can be viewed as the *indirect utility* obtained from a standard log-linear function, $U(l, z) = l^\alpha z^\beta$, in leisure time, l , and composite good, z , subject to time and budget constraints. In particular, if the price of z is taken to be the numeraire, and l is treated as the fraction of time spent in leisure (so that total time is one), then the budget and time constraints can be written respectively as $z = y$ and $l + s = 1$ so that $0 \leq l \leq 1$. Hence the corresponding indirect utility (with respect to the budget constraint) is obtained by replacing z with y . The choice of $\beta = 1$ (together with the dimensionless nature of l) yields an indirect utility in (3.1) which is in *monetary units*, and hence is interpretable as ‘effective’ income.

Since the wage level is here taken to be fixed, there are no relevant decision variables for employed workers. Hence the value U_1 can be regarded as an exogenous parameter in the present model.

3.2. Lifetime Effective-Income Streams

Recall that our basic objective is to model the decision problem for an unemployed worker who is currently considering his/her choice of search intensity, s , (which for simplicity can be regarded as the choice of a roulette wheel to use each morning in deciding whether to search that day). To weigh alternative choices, the worker must evaluate the expected future effective-income streams resulting from each choice of s . [In the following development, we suppress dependence of all variables on s except when needed, so that for example we write U_0 for $U_0(s)$.] At each point of time in the future the worker will be in one of two states: unemployed (0) or employed (1). If the relevant *discount rate* for all workers is the same, and is denoted by $\sigma \in (0, 1)$ [representing the value today of a dollar received tomorrow], then the expected values, $E(I_0)$ and $E(I_1)$, of the *discounted effective income streams*, I_0 and I_1 , starting from each possible state can be determined as follows. Observe that if the *duration times* (number of consecutive days) in each state are denoted respectively by T_0 and T_1 , then by employing the identity, $\sum_{k=0}^{t-1} \sigma^k = (1 - \sigma^t) / (1 - \sigma)$, it follows that the conditional expectation of I_0 given a duration of t days in unemployment must be of the form:

$$\begin{aligned} E(I_0|T_0 = t) &= \sum_{k=0}^{t-1} \sigma^k U_0 + \sigma^t E(I_1) \\ &= \left(\frac{1 - \sigma^t}{1 - \sigma} \right) U_0 + \sigma^t E(I_1) , \quad t = 1, 2, \dots \end{aligned} \quad (3.4)$$

For example, workers hired on the first day do not start work until the next day (by assumption). This implies that on the first day, workers continue to receive unemployment benefit, b , and realize utility level, U_0 . From the next day onward, workers will enjoy the expected utility stream, $E(I_1)$, starting in the employed state, so that $E(I_0|T_0 = 1) = U_0 + \sigma E(I_1)$.

Similarly, when a worker is employed, the conditional expectation of I_1 given an employment duration of t days is easily seen to be of essentially the same form:

$$E(I_1|T_1 = t) = \left(\frac{1 - \sigma^t}{1 - \sigma} \right) U_1 + \sigma^t E(I_0) , \quad t = 0, 1, 2, \dots \quad (3.5)$$

Notice however that in this case, employment duration starts from *zero* rather than one. This is a consequence of our assumption that a worker may be hired and let go on the same day. In this particular case, the expected utility stream is not different from that of an unemployed worker, so that $E(I_1|T_1 = 0) = (0)U_1 + \sigma^0 E(I_0) = E(I_0)$. On the other hand, employed workers who are not layed off immediately will enjoy at least one day of effective employment income, U_1 .

If we now identify the *lifetime values*, V_0 and V_1 , of these states with the unconditional expectations, $E(I_0)$ and $E(I_1)$,¹⁶ then we may employ (3.4) and (3.5) to solve for these values in terms of the effective incomes, U_0 and U_1 , as follows. First observe that by definition,

$$\begin{aligned}
V_0 &= E(I_0) = E_{T_0} [E(I_0|T_0)] \\
&= E_{T_0} \left[\left(\frac{1 - \sigma^{T_0}}{1 - \sigma} \right) U_0 + \sigma^{T_0} V_1 \right] \\
&= \left(\frac{1 - E(\sigma^{T_0})}{1 - \sigma} \right) U_0 + E(\sigma^{T_0}) V_1 \\
&= \left(\frac{1 - e_0}{1 - \sigma} \right) U_0 + e_0 V_1
\end{aligned} \tag{3.6}$$

where $e_0 = E(\sigma^{T_0})$, and similarly that,

$$\begin{aligned}
V_1 &= E(I_1) = E_{T_1} [E(I_1|T_1)] \\
&= E_{T_1} \left[\left(\frac{1 - \sigma^{T_1}}{1 - \sigma} \right) U_1 + \sigma^{T_1} V_0 \right] \\
&= \left(\frac{1 - e_1}{1 - \sigma} \right) U_1 + e_1 V_0
\end{aligned} \tag{3.7}$$

where $e_1 = E(\sigma^{T_1})$. These equations may in turn be solved simultaneously to yield the following expressions for V_0 and V_1 in terms of U_0 and U_1 :

$$V_0 = \frac{1 - e_0}{1 - e_0 e_1} \left(\frac{U_0}{1 - \sigma} \right) + \frac{e_0 (1 - e_1)}{1 - e_0 e_1} \left(\frac{U_1}{1 - \sigma} \right) \tag{3.8}$$

$$V_1 = \frac{1 - e_1}{1 - e_0 e_1} \left(\frac{U_1}{1 - \sigma} \right) + \frac{e_1 (1 - e_0)}{1 - e_0 e_1} \left(\frac{U_0}{1 - \sigma} \right) \tag{3.9}$$

¹⁶As with all models of this type, workers are assumed to ignore their own mortality, and to behave as though their future were infinite.

What remains to be determined are the expected discount factors e_0 and e_1 . To do so, we begin by establishing the exact distributions of T_0 and T_1 . Turning first to T_0 , let the hiring probability in (2.42) be denoted by p_h , and observe that the probability of leaving unemployment on day $t = 1$ is by definition the joint probability, $s p_h$, of going to the labor market on that day and being hired. Moreover, since an unemployment duration of t days means precisely that this joint event first occurs on day t , it follows from our independence assumptions that T_0 must be geometrically distributed according to

$$P(T_0 = t) = (1 - s p_h)^{t-1} s p_h, \quad t = 1, 2, \dots \quad (3.10)$$

Next, to determine the distribution of T_1 , recall that the probability of job termination on any day is given by ρ . Hence it follows from our ‘birth-death’ assumptions on jobs that T_1 must also be geometrically distributed according to¹⁷

$$P(T_1 = t) = (1 - \rho)^t \rho, \quad t = 0, 1, 2, \dots \quad (3.11)$$

Given these two distributions, we may now compute the desired expectations as follows. First, by employing the identity, $\sum_{t=1}^{\infty} a^t = a/(1 - a)$ for all $a \in [0, 1)$, it follows from (3.10) that

$$\begin{aligned} e_0 &= E(\sigma^{T_0}) = \sum_{t=1}^{\infty} \sigma^t (1 - s p_h)^{t-1} s p_h \\ &= \frac{p_h s}{1 - p_h s} \sum_{t=1}^{\infty} [\sigma (1 - p_h s)]^t \\ &= \frac{p_h s}{1 - p_h s} \left(\frac{\sigma (1 - p_h s)}{1 - \sigma (1 - p_h s)} \right) \\ &= \frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \end{aligned} \quad (3.12)$$

Similarly, by employing the identity, $\sum_{t=0}^{\infty} a^t = 1/(1 - a)$ for all $a \in [0, 1)$, it

¹⁷In the *discrete time* version of standard birth-death processes, exponential lifetimes are replaced by their discrete geometric counterparts.

follows from (3.11) that

$$\begin{aligned}
e_1 &= E(\sigma^{T_1}) = \sum_{t=0}^{\infty} \sigma^t (1-\rho)^t \rho \\
&= \rho \sum_{t=0}^{\infty} [\sigma(1-\rho)]^t = \rho \left(\frac{1}{1-\sigma(1-\rho)} \right) \\
&= \frac{\rho}{1-\sigma+\sigma\rho}
\end{aligned} \tag{3.13}$$

3.3. Calculation of the Optimal Search Intensity

Returning to our basic decision problem, suppose that an unemployed worker is currently reconsidering his search intensity level, s . With an eye toward our ultimate goal of determining an equilibrium search intensity for the system, suppose also that the system is in a steady state where all workers are currently using the *same* search intensity level, say \bar{s} . Associated with this search intensity level (as in section 1 above) is a steady-state *hiring probability* (2.42) which we again denote by $p_h = p_h(\bar{s})$, and a steady-state *lifetime value* of employment, which we denote by $V_1 = V_1(\bar{s})$. Here it is assumed that perturbations in the search intensity, s , of a single unemployed worker cannot influence these steady state values, and hence that p_h and V_1 can be treated as constants in the worker's decision problem. To formulate this problem, observe next that the *expected discount value*, e_0 , in (3.12) now takes the form

$$e_0(s) = \frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \tag{3.14}$$

where the hiring probability, p_h , is again beyond the individual's control, but where his choice of s affects his unemployment duration, and thus his value of e_0 . Given these observations, we may now use (3.2), (3.6) and (3.12) to express his

lifetime value, V_0 , solely in terms of s as follows:

$$\begin{aligned}
V_0(s) &= \left(\frac{1 - e_0(s)}{1 - \sigma} \right) U_0(s) + e_0(s) V_1 \\
&= \left(1 - \frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \right) \frac{U_0(s)}{1 - \sigma} + \left(\frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \right) V_1 \\
&= \left(\frac{1 - \sigma}{1 - \sigma + \sigma p_h s} \right) \frac{(1 - s)^\alpha b}{1 - \sigma} + \left(\frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \right) V_1 \\
&= \frac{(1 - s)^\alpha b + \sigma p_h s V_1}{1 - \sigma + \sigma p_h s} \tag{3.15}
\end{aligned}$$

Thus the relevant decision problem is to choose a value of $s \in [0, 1]$ which maximizes (3.15). To solve this problem, we begin by differentiating (3.15),

$$\begin{aligned}
V_0'(s) &= \frac{\sigma p_h V_1 - \alpha (1 - s)^{\alpha-1} b}{1 - \sigma + \sigma p_h s} - \frac{\sigma p_h s V_1 + (1 - s)^\alpha b}{1 - \sigma + \sigma p_h s} \left(\frac{\sigma p_h}{1 - \sigma + \sigma p_h s} \right) \\
&= \frac{1}{1 - \sigma + \sigma p_h s} [\sigma p_h V_1 - \alpha (1 - s)^{\alpha-1} b - \sigma p_h V_0(s)] , \tag{3.16}
\end{aligned}$$

and observing that the first-order condition, $V_0'(s) = 0$, holds iff the bracketed term is zero, which can be rewritten as

$$\alpha (1 - s)^{\alpha-1} b = \sigma p_h [V_1 - V_0(s)] . \tag{3.17}$$

The interpretation of (3.17) is straightforward. The left hand side is the short-run utility loss from a marginal increase in search intensity, and the right hand side is the corresponding long-run utility gain from future employment. Thus, the level of search intensity is optimal when the marginal gain of searching (reduced unemployment duration) is equal to its marginal cost (reduced leisure time). By totally differentiating (3.17), we obtain the following comparative-statics results (where p_h and V_1 are here taken to be parameters):

$$\frac{\partial s}{\partial b} < 0 \quad \frac{\partial s}{\partial p_h} > 0 \quad \frac{\partial s}{\partial V_1} > 0 \quad \frac{\partial s}{\partial \sigma} > 0 \tag{3.18}$$

These lend support to the observations above. When there is an increase unemployment benefits ($b \uparrow$), working becomes less attractive and unemployed workers

are less motivated to search. On the other hand, when there is an increase in either the chance of being hired ($p_h \uparrow$), the value of being employed ($V_1 \uparrow$), or the importance of the future ($\sigma \uparrow$), unemployed workers are more motivated to search.

To establish the existence of solutions to (3.17) observe next from (3.16) that $\lim_{s \rightarrow 1} V_0'(s) = -\infty$. Hence a sufficient condition for at least one solution to (3.17) in $(0, 1)$ is that $V_0'(0) > 0$. Moreover, by differentiating (3.16) once more, we see that

$$V_0''(s) = \frac{-\alpha(1-\alpha)(1-s)^{\alpha-2} - 2\sigma p_h V_0'(s)}{1-\sigma + \sigma p_h s} \quad (3.19)$$

and hence that

$$V_0'(s) = 0 \Rightarrow V_0''(s) = \frac{-\alpha(1-\alpha)(1-s)^{\alpha-2}}{1-\sigma + \sigma p_h s} < 0 \quad (3.20)$$

Thus (3.15) can have at most one interior maximum, and we conclude that:

Proposition 3.1. *For any given population search intensity, \bar{s} , satisfying the condition that $V_0'(0) > 0$, there exists a unique individual-optimum search intensity, $s(\bar{s})$. Moreover, this optimal search intensity is always in the open interval $(0, 1)$, and is given by the unique solution to (3.17).*

3.4. Equilibrium with Endogenous Search Intensities

This solution to the optimal-search-intensity problem for unemployed workers leads directly to an equilibrium condition for the system as a whole. In particular, when search intensities are allowed to be endogenous, it is clear that a population search intensity level, \bar{s} , is a (Nash) equilibrium for the system iff \bar{s} is the optimal individual response to itself, i.e., iff $\bar{s} = s(\bar{s})$. Hence, if we now drop the ‘bar’ notation, and designate each common population choice of s as a *population search intensity*, then it follows from (3.17) that a population search intensity, s , is an *equilibrium* for the system iff s satisfies the following condition:

$$\alpha(1-s)^{\alpha-1} b = \sigma p_h(s) [V_1(s) - V_0(s)] \quad (3.21)$$

where the hiring probability, $p_h(s)$, and the lifetime value of employment, $V_1(s)$, are now written as functions of the population search intensity, s . Note however

that this equilibrium condition assumes that s is *positive*, which in view of Proposition 3.1 is equivalent to assuming that $V_0'(0) > 0$. Moreover, the functions $p_h(s)$ and $V_1(s)$ are not expressible in closed form, but rather are *implicit functions* of s which depend on many other equilibrium quantities [including the steady-state unemployment rate, $u(s)$, and the equilibrium expected discount rate, $e_0(s)$]. Hence our next objective is to give an *exact* formulation of the desired equilibrium.

First we give an appropriate parametric specification of the positivity condition, $V_0'(0) > 0$. In section A.7 of the Appendix, it is shown that this condition is equivalent to:

$$U_1 > \left[1 + \frac{\alpha(1 - \sigma + \sigma\rho)}{\sigma\gamma(1 - \rho)} \right] b, \quad (3.22)$$

which we now designate as the *positivity condition*. To interpret this condition, recall that since the unemployment benefit, b , is precisely the effective income of an unemployed worker, this positivity condition simply requires that the effective wage income of employment [in (3.3)] be sufficiently greater than the unemployment benefit to encourage some degree of job search. Note that the simpler condition that U_1 be greater than b is by itself *not* sufficient, precisely because the lifetime value of employment necessarily involves some time spent in unemployment. Note also that this basic positivity condition involves all parameters of the present model *except* for the job-creation rate, β . The key effect of this parameter is on the hiring probability, which (as we have seen) reduces to γ as s approaches zero. Hence the only condition on β needed to ensure positive equilibrium search intensity is that $\beta > 0$, i.e., that some vacant jobs be available to potential searchers.

Given this basic positivity condition, we now gather together all variables and conditions which define the desired equilibrium. For any positive parameter vector $(\alpha, \beta, \rho, \gamma, \sigma, b, U_1)$ satisfying condition (3.22) with $\alpha, \rho, \gamma, \sigma \in (0, 1)$, a positive vector $\mathcal{E} = (s, u, v, p_h, e_0, V_0, V_1)$ is designated as an *endogenous-search equilibrium* iff \mathcal{E} satisfies the following seven conditions with $s \in (0, 1)$ [where $a = (\beta/\rho) - 1$, $e_1 = \rho/(1 - \sigma + \sigma\rho)$ and $U_0(s) = (1 - s)^{\alpha b}$]:

$$v = u + a \quad (3.23)$$

$$p_h = \frac{v}{s u} \left(1 - e^{-\gamma s \frac{u}{v}} \right) \quad (3.24)$$

$$\rho(1 - u) = (1 - \rho) u s p_h \quad (3.25)$$

$$e_0 = \frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \quad (3.26)$$

$$V_0 = \left(\frac{1 - e_0}{1 - \sigma} \right) U_0(s) + e_0 V_1 \quad (3.27)$$

$$V_1 = \left(\frac{1 - e_1}{1 - \sigma} \right) U_1 + e_1 V_0 \quad (3.28)$$

$$\sigma p_h (V_1 - V_0) = \alpha (1 - s)^{\alpha-1} b \quad (3.29)$$

The first three conditions follow directly from the steady-state equations (2.54) and (2.55) together with the definition of the hiring probability in (2.42). Conditions (3.26), (3.27), and (3.28) are precisely the definitions of e_0 , V_0 , and V_1 in (3.14), (3.6), and (3.7), respectively. Finally, (3.29) is a restatement of the basic first-order condition in (3.21). Given this formal definition, our main result is to show that (the proof is given in section A.8 of the Appendix):

Theorem 3.2 (Endogenous-Search Equilibrium). *For each positive parameter vector $(\alpha, \beta, \rho, \gamma, \sigma, b, U_1)$ satisfying condition (3.22) with $\alpha, \rho, \gamma, \sigma \in (0, 1)$, there exists a unique endogenous-search equilibrium.*

The characterization of equilibrium search intensity, s , as the unique root of an explicitly defined function [see (A.111) in the Appendix] also facilitates a comparative-static analysis of s with respect to each parameter of the model. With respect to the key wage parameter, w , (which will be endogenized below) it is shown in section A.10 of the Appendix that higher wages always induce higher levels of search intensity, i.e. that

Corollary 3.3 (Search Monotonicity). *The unique equilibrium search intensity, $s(w)$, is a strictly increasing differentiable function of w .*

These results can be illustrated by an extension of the example following Theorem 2.2. Suppose that the leisure-time elasticity of effective income is $\alpha = .2$ [so that a five percent gain in (the fraction of) leisure time yields a one percent gain in effective income] and that the prevailing daily discount rate is $\sigma = .999739$ (yielding a compounded interest rate of ten percent per year). In addition, suppose that employed workers spend one third of their time working ($l_1 = .66$) and that unemployed workers receive daily unemployment benefits of $b = \$60$. Then it follows from (3.22) [with $U_1 = l_1^\alpha w$] that the minimum wage required to induce

any search activity is given by $w_0 = \$65.28$. For the ‘good times’ scenario in the example above ($\beta = .0054$ and $\rho = .0027$) suppose that wages are just enough to induce some search, say $w = \$65.30$. Then the equilibrium search intensity is naturally quite low ($s = .045$) and unemployment is about 12% ($u = .119$). On the other hand, if wages are raised, say to $w = \$80$, then (as in Corollary 3.3) the unemployed are now motivated to search on three out of every four days ($s = .757$) and unemployment all but vanishes ($u = .007$). More interesting from the individual’s viewpoint is the *duration* of unemployment, T_0 . To determine the expected unemployment duration, $D_u = E(T_0)$, observe from the properties of the geometric distribution in (3.10) together with the steady state condition (3.25) that

$$D_u = \frac{1}{s p_h} = \frac{(1 - \rho) u}{\rho(1 - u)} \quad (3.30)$$

In the present case, an increase in wages from $w = \$65.3$ to $w = \$80$ induces a decrease in unemployment duration D_u from 50 days to less than 3 days. Hence it is clear that the key motivation for increased search intensity is to maximize the fraction of days earning \$80 rather than \$60.

This is perhaps even more clear in the ‘bad times’ scenario (where β drops to .00135). In this case, an increase in wages from $w = \$65.3$ to $w = \$80$ continues to induce a sharp increase in search intensity from $s = .007$ to $s = .49$, even though there is only a small effect on unemployment duration (decreasing from 374 days to 372 days). As in the original example, this unemployment situation is primarily structural in nature and is not sensitive to frictional effects (decreasing only from $u = .503$ to $u = .501$ with the increase in s). However, in the presence of high wages, the prospect of cutting lengthy unemployment periods (of more than a year) by only two days is sufficiently attractive to increase search intensity by orders of magnitude. This responsiveness of search intensity to wage increases forms a key link between the decision behavior of workers and firms — to which we now turn.

4. Endogenous Wage Formation

In this final section, we relax the assumption that wages are fixed and allow profit-maximizing firms to set wages. To do so, recall that each job generates its own profit stream (while active). Hence we now assume that wages are chosen

by the firm to maximize the present value of this profit stream.¹⁸ Here there is a fundamental trade-off. If wages are set too low then (as in the Corollary 3.3 above) search intensities will also be low, so that jobs will remain vacant for long periods with no profits earned. On the other hand, if wages are set too high, then only small profits will be earned while the job is active. To model this trade-off, we begin by considering the profits earned from a job initially posted on day $t = 0$. Let the number of days until the job is filled be designated as the *filling time*, T_f , and similarly, let the number of days until the job is closed be designated as the *closing time*, T_c . Clearly no profits are earned if $T_c < T_f$. Moreover, if $T_c = T_f$, then (by our conventions) the job is closed before any production occurs, and no profits are earned. Hence if $T_f = t$, then the first day on which profits can be earned is day $t + 1$. If the present value of the profit stream realized is denoted by π , then by definition the expectation of π can be written as

$$E(\pi) = \sum_{t=0}^{\infty} \sum_{d=1}^{\infty} E(\pi | T_f = t, T_c = t + d) P(T_f = t, T_c = t + d) \quad (4.1)$$

If the *daily revenue* earned from each active job is assumed to be the same, and is denoted by y , and if the above discount rate, σ , is also used by firms, then for each choice of wage w , the conditional expectation $E(\pi | T_f = t, T_c = t + d)$ in (4.1) is completely deterministic and is given by

$$E(\pi | T_f = t, T_c = t + d) = \sum_{k=t+1}^{t+d} \sigma^k (y - w) = (y - w) \frac{\sigma^{t+1} (1 - \sigma^d)}{(1 - \sigma)} \quad (4.2)$$

Turning next to the joint probability $P(T_f = t, T_c = t + d)$, recall from our ‘birth-death’ assumptions on jobs that closings are caused by external market forces, and hence are independent of the job-filling process. Thus $P(T_f = t, T_c = t + d) = P(T_f = t) P(T_c = t + d)$, where the marginal distribution of T_c is given by (3.11), and where the marginal distribution of T_f is also geometric with sp_h in (3.10) now replaced by the *filling probability*, p_f , in (2.38), i.e.,

$$P(T_f = t) = (1 - p_f)^t p_f, \quad t = 0, 1, 2, \dots \quad (4.3)$$

¹⁸This implicitly assumes that wages are part of the job offer, and are not negotiable. In the present context, where jobs are completely specialized and are open to large numbers of potential applicants, it is not unreasonable to assume that bargaining power of workers is minimal.

Hence, substituting (4.2), (3.11) and (4.3) into (4.1), we see that

$$\begin{aligned}
E(\pi) &= \sum_{t=0}^{\infty} \sum_{d=1}^{\infty} \left[(y-w) \frac{\sigma^{t+1} (1-\sigma^d)}{(1-\sigma)} \right] (1-p_f)^t p_f (1-\rho)^{t+d} \rho \\
&= \frac{\sigma}{(1-\sigma)} (y-w) p_f \left(\sum_{t=0}^{\infty} [\sigma(1-\rho)(1-p_f)]^t \right) \left(\sum_{d=1}^{\infty} (1-\sigma^d) \rho (1-\rho)^d \right) \\
&= \frac{\sigma}{(1-\sigma)} (y-w) \left(\frac{p_f}{1-\sigma(1-\rho)(1-p_f)} \right) \left(\frac{(1-\rho)(1-\sigma)}{1-\sigma+\sigma\rho} \right) \\
&= \frac{y-w}{1-\sigma+\sigma\rho} \cdot \frac{\sigma(1-\rho)p_f}{(1-\sigma+\sigma\rho)+\sigma(1-\rho)p_f} \tag{4.4}
\end{aligned}$$

What is crucial here from the firm's viewpoint is the length of time during which a given job produces profits. Once the job is announced, it takes time to be filled (p_h) and only stays open as long as there are profits to be earned (ρ). Moreover, since the job-closure rate, ρ , is governed by general economic conditions, the firm's wage policy, w , can only influence p_f (through the search intensity, s , as in Corollary 3.3 above).

To express (4.4) in a more convenient form, we next recall from (2.54) and (3.25) that

$$(1-\rho) u s p_h = \rho(1-u) = \beta - \rho v \tag{4.5}$$

and from (2.38) and (2.42) that

$$u s p_h = v p_f \tag{4.6}$$

By combining these and substituting for $(1-\rho) p_f$ in (4.4), we can write the expectation in terms of the wage, w , and vacancy rate, v , as follows:

$$E(\pi) = \left(\frac{y-w}{1-\sigma+\sigma\rho} \right) \left(\frac{\sigma\beta - \sigma\rho v}{\sigma\beta + (1-\sigma)v} \right) \tag{4.7}$$

Hence expected discounted profits, $E(\pi)$, are seen to be a product of two factors. To interpret each factor, observe first that if the vacancy rate were zero then $E(\pi)$ reduces to the first factor. But since a zero vacancy rate implies that all new jobs are filled instantaneously, the first factor represents the expected discounted profit stream which could be earned if there were no employment-friction effects, and hence may be designated as the *friction-free profit factor*. The second factor (which is always between zero and one) represents the reduction in this

maximum-profit stream due to the presence of employment-friction effects, and thus is designated as the *employment-friction factor*.

To express this profit stream in terms of w alone, recall that the steady-state value of v is determined by the search intensity level, s , which in turn is determined by the prevailing wage level, w . Hence we may write $v = v[s(w)]$, and express (4.7) as a function of w , say

$$\Pi(w) = \left(\frac{y - w}{1 - \sigma + \sigma\rho} \right) \left(\frac{\sigma\beta - \sigma\rho v[s(w)]}{\sigma\beta + (1 - \sigma)v[s(w)]} \right) \quad (4.8)$$

In this form, the basic trade-off described above is now transparent. On the one hand, the friction-free profit factor is clearly *decreasing* in w , reflecting the loss in profits due to higher wage costs. On the other hand, since $v(s)$ is decreasing (Theorem 2.2), and $s(w)$ is increasing (Corollary 3.3), it follows that $v[s(w)]$ is decreasing. Hence the employment-friction factor is *increasing* in w , reflecting the gain in profits due to shorter expected filling times for vacant jobs.

It should also be emphasized that since wages affect *search intensity*, firms cannot simply set wages at $w = b$ and thereby extract all ‘rents’ earned by their workers. As a consequence, employment is strictly preferable to unemployment, and the state of being unemployed is necessarily *involuntary* (see the discussion of Mortensen, 1989, pp.352-354).

Observe next that the construction of $\Pi(w)$ is meaningful only if w is taken to be a *uniform wage* for all jobs. Given that all active jobs yield the same daily revenue, y , and that all workers have the same chance, γ , of being qualified for any given job, it is not unreasonable to assume that firms regard uniform wage levels as the only possible equilibrium states. If firms are also able to determine the steady-state vacancy rates generated by such wages, i.e., to estimate the composite function $v[s(w)]$, then $\Pi(w)$ can be taken as an appropriate objective function for firms to maximize in choosing an optimal wage level, w^* . Finally, if it is assumed to be common knowledge that all firms can make the same calculations, then under these conditions the mutual adoption of w^* may reasonably be said to constitute an equilibrium for the system. In so far as this equilibrium involves maximizing behavior by firms constrained by worker responses, it may be regarded as a type of *Stackelberg equilibrium* for firms and workers.¹⁹

¹⁹Note in particular that the type of implicit wage-coordination behavior required in the estimation of worker’s lifetime utility streams by firms closely parallels the ‘efficiency wage’ equilibrium of Shapiro and Stiglitz (1984).

To verify the existence of such an optimal wage, w^* , it is convenient to denote the friction-free profits factor and employment-friction factor respectively by

$$P(w) = \frac{y - w}{1 - \sigma + \sigma\rho}, \quad (4.9)$$

$$F(w) = \frac{\sigma \{\beta - \rho v[s(w)]\}}{\sigma\beta + (1 - \sigma)v[s(w)]}, \quad (4.10)$$

so that (4.8) can be written simply as

$$\Pi(w) = P(w) \cdot F(w). \quad (4.11)$$

In these terms, note first that since the optimal search intensity for all sufficiently small wage levels w must be given by $s(w) = 0$,²⁰ it follows from Theorem 2.2 that $v[s(w)] = \beta/\rho$, and hence from (4.10) that $F(w) = 0$ [so that $\Pi(w) = 0$] for all sufficiently small w . On the other hand, it is clear from (4.9) that $P(w) \leq 0$ [and hence that $\Pi(w) \leq 0$] for all $w \geq y$. Thus the continuity of $\Pi(w)$ is enough to ensure the existence of an optimum wage level, w^* , in the open interval $(0, y)$.

Finally, observe that the above notion of equilibrium implicitly assumes that w^* is unique.²¹ While definitive conditions for uniqueness are unfortunately difficult to establish (given the analytical complexity of the composite function, $v[s(w)]$), numerical examples show that w^* is indeed unique over a wide range of parameter values. Some analytical insight can be gained by observing that the second derivative of (4.11) is of the form

$$\Pi''(w) = 2P'(w)F'(w) + P(w)F''(w) \quad (4.12)$$

Since we have already seen that $P'(w) < 0$ and $F'(w) > 0$, the positivity of $P(w)$ on $(0, y)$ implies that $\Pi''(w) < 0$ whenever $F''(w) < 0$. Moreover, by letting $v(w) = v[s(w)]$, it is straightforward to show that

$$F''(w) = -\frac{\sigma\beta + (1 - \sigma)F(w)}{\sigma\beta + (1 - \sigma)v(w)} \left\{ v''(w) + \frac{(1 - \sigma)v'(w)^2}{\sigma\beta + (1 - \sigma)v(w)} \right\}, \quad (4.13)$$

²⁰This may be verified by observing from (3.3) and (3.13), together with (A.95) and (A.98), that $V'_0(0)$ in (A.94) of the Appendix must become negative as w approaches zero. Hence $s(w) = 0$ for all w at which $V'_0(0) < 0$.

²¹However, such equilibria are in principle possible even with multiple optima. For example, if the *lowest* optimal wage level is mutually recognized by firms as a 'prominent point' (Schelling, 1960), then they may be able to reach a tacit agreement with respect to this wage offer.

which together with the positivity of the second term in the braces on the right hand side (4.13) implies that $F''(w) < 0$ whenever $v''(w) > 0$. Thus the profit function $\Pi(w)$ must always be strictly concave in the region where $v(w)$ is strictly convex. But for w in the range of the positivity condition (3.22) [with $U_1 = l_1^\alpha w$], the function $v(w)$ must always decrease from β/ρ toward the positive asymptote corresponding to full search intensity, $s = 1$. In particular, this implies that $v(w)$ must (at least) be strictly convex for ‘sufficiently large’ wage levels, w . Moreover (as borne out by a range of numerical examples), $v(w)$ is generally strictly convex over the full range of w satisfying the positivity condition (3.22), so that $\Pi(w)$ is strictly concave on $(0, y)$.

These qualitative results, can be illustrated by a further extension of the examples following Theorems 2.2 and 3.2 above. To do so, suppose that active jobs earn a daily revenue of $y = \$100$. Under the ‘good times’ scenario above, the wage level which maximizes expected discounted profits is then given by $w = \$65.65$. Note that this is only slightly above the minimum wage, $w_0 = \$65.28$, required to induce positive search. However, the induced search intensity, $s = .225$, is substantial (about one day in four spent searching) and the corresponding unemployment level is quite low ($u = .024$). The intuition here is that effective-income utility in (3.2) is relatively insensitive to search intensities near $s = 0$, so that leisure time only becomes critical for individuals when it is in very short supply. Hence individuals are willing to give up a considerable amount of leisure time in order to gain the first few dollars of added income above permanent unemployment. This low-wage property turns out to be quite robust to *any* changes in the model parameters. In particular, if daily revenues are doubled to $y = \$200$, then the optimal wage level increases only to $w = \$66.19$.²²

In the ‘bad times’ scenario, the situation is naturally worse for everyone, especially workers. With a daily revenue of $y = \$100$, firms are now motivated to offer a wage of $w = \$65.34$, only a few pennies above the minimal wage w_0 . While this meager wage induces a seemingly low search intensity ($s = .0094$), this is enough in the present tight job market to drive vacancy rates almost to zero ($v = .002$). Hence from the firm’s viewpoint, there is no need to offer more.

²²The ability of firms to extract most of the revenue surplus is of course a consequence of our present job market structure, in which individual workers are assumed to be wage-takers who do not engage in strategic bargaining.

5. Concluding Remarks

In this paper, we have presented an explicit micro model of job matching in which heterogeneous workers allocate themselves to jobs with different skill requirements. Within this framework it was shown that when the population size is large, the asymptotic form of aggregate matching function is of an exponential type, and is an instance of the general class of ‘production-like’ matching functions described by Pissarides (1990). This function is of course only one among many possibilities, including the Cobb-Douglas matching function used most frequently in empirical research. Hence its major advantage is that it is derivable directly from an explicit micro scenario which captures many important aspects of job matching behavior. However, it is equally clear that this scenario is in many ways too simplistic, and for example assumes both identical wages and revenues for all jobs and identical information levels and matching (qualification) probabilities for all job seekers. Hence this model is perhaps best regarded as a benchmark for constructing more realistic behavioral scenarios.

In the second part of the paper, it was shown how this model can be used to analyze the decision behavior of unemployed workers in choosing their optimal search intensities, i.e., how many days per week to search. This focus was motivated by our desire to extend the present framework to urban spatial labor markets, where there a close connection between the choice of search intensity and the spatial location of job seekers. In particular, it is shown in a companion paper (Smith and Zenou, 1999) that the travel costs involved in job search can play an important role in determining the location patterns of both employed and unemployed workers with respect to jobs.

In the final section of the paper, it was shown how this model can be applied to study the decision behavior of firms in choosing wage offers to maximize their expected profit streams. Here again search intensities provided the key behavioral link. In particular, unemployed workers were assumed to be wage-takers who choose their levels of search intensity based on the prevailing wage level. This is in sharp contrast to the wage-bargaining models which are most frequently used in the literature. However, it is clear that elements of both are present in most job markets, and should be reflected in more comprehensive models of job-matching behavior.

References

- [1] Blanchard, O.J. and P. Diamond (1989), 'The Beveridge curve', *Brookings Papers on Economic Activity*, 1:1-60.
- [2] Blanchard, O.J. and P. Diamond (1993), 'Ranking, random hiring, unemployment duration, and wages', unpublished manuscript, MIT.
- [3] Blanchard, O.J. and P. Diamond (1994), 'Ranking, unemployment duration, and wages', *Review of Economic Studies*, 61:417-434.
- [4] Brumelle, S.L. and Y. Gerchak (1980), 'A stochastic model allowing interaction among individuals and its behavior for large populations', *Journal of Mathematical Sociology*, 7:73-90.
- [5] Burda, M. and C. Wyploz (1994), 'Gross worker and job flows in Europe', *European Economic Review*, 38:1287-1315.
- [6] Butters, G. (1977), 'Equilibrium distributions of sales and advertising prices', *Review of Economic Studies*, 44:465-491.
- [7] Hall, R.E. (1979), 'An aspect of the economic role of unemployment', in G.Harcourt (ed.), *Microeconomic Foundations of Macroeconomics*.
- [8] Davis, S.J., Haltiwanger, J. and S. Schuh (1996), *Job Creation and Destruction*, MIT Press, Cambridge (MA).
- [9] Hamilton, J., Thisse, J-F. and Y. Zenou (1999), 'Wage competition with heterogeneous workers and firms', *Journal of Labor Economics*, forthcoming.
- [10] Kulkarni, V.G. (1995), *Modeling and Analysis of Stochastic Processes*, Chapman & Hall, London.
- [11] Mortensen, D.T. and C.A. Pissarides (1999), 'New developments in models of search in the labor market', in Card, D. and O. Ashenfelter (eds.), *Handbook of Labor Economics*, Elsevier Science, Amsterdam, forthcoming.
- [12] Mortensen, D.T. (1989), 'The persistence and indeterminacy of unemployment in search equilibrium', *Scandinavian Journal of Economics*, 91:347-370.
- [13] Pissarides, C.A. (1990), *Equilibrium Unemployment Theory*, Basil Blackwell, Oxford.

- [14] Rao, C.R. (1973), *Linear Statistical Inference and its Applications*, Second edition, Wiley, New York.
- [15] Renyi, A. (1970), *Foundations of Probability*, Holden-Day, San Francisco.
- [16] Sattinger, M. (1990), ‘Unemployment, the market for interviews, and wage determination’, *Journal of Political Economy*, 98:356-371.
- [17] Sattinger, M. (1993), ‘Assignment models of the distribution of earnings’, *Journal of Economic Literature*, 31:831-880.
- [18] Schelling, T.C. (1960), *The Strategy of Conflict*, Oxford University Press, London.
- [19] Shapiro, C. and J. Stiglitz (1984), ‘Equilibrium unemployment as a worker discipline device’, *American Economic Review*, 74:433-444.
- [20] Smith, T.E. and Y. Zenou (1999) ‘A spatial labor market model with job-matching behavior’, CEPR Discussion Paper, London.
- [21] Stout, W.F. (1974), *Almost Sure Convergence*, Academic Press, New York.
- [22] Wilks, S. (1962), *Mathematical Statistics*, Wiley, New York..

A. Appendix

In this appendix, some of the more technical results in the text are established. Each result is listed as a separate subsection:

A.1. Derivation of Expression (2.7)

To establish this result, observe first that by definition,

$$\begin{aligned}
 P(k|m, n) &= \sum_{l=0}^{k-1} (0) + \sum_{l=k}^n P_m(k|l)P(l|n) \\
 &= \sum_{l=k}^n \left[\frac{l!}{k!(l-k)!} \left(\frac{1}{m}\right)^k \left(1 - \frac{1}{m}\right)^{l-k} \right] \cdot \left[\frac{n!}{l!(n-l)!} s^l (1-s)^{n-l} \right] \\
 &= \frac{n!}{l!(n-l)!} \sum_{l=k}^n \frac{(n-k)!}{(l-k)!(n-l)!} \left(1 - \frac{1}{m}\right)^{l-k} s^l (1-s)^{n-l}. \quad (\text{A.1})
 \end{aligned}$$

Hence, if we now let $h = l - k$ (so that $l = h + k$) then it may be concluded that:

$$\begin{aligned}
P(k|m, n) &= \binom{n}{k} \left(\frac{1}{m}\right)^k \sum_{h=0}^{n-k} \frac{(n-k)!}{h!(n-k-h)!} \left(1 - \frac{1}{m}\right)^h s^{h+k} (1-s)^{n-(h+k)} \\
&= \binom{n}{k} \left(\frac{s}{m}\right)^k \sum_{h=0}^{n-k} \binom{n-k}{h} \left(s - \frac{s}{m}\right)^h (1-s)^{(n-k)-h} \\
&= \binom{n}{k} \left(\frac{s}{m}\right)^k \left[\left(s - \frac{s}{m}\right) + (1-s)\right]^{n-k} \\
&= \binom{n}{k} \left(\frac{s}{m}\right)^k \left(1 - \frac{s}{m}\right)^{n-k}. \tag{A.2}
\end{aligned}$$

A.2. Derivation of Expression (2.24)

To establish this limiting result for total jobs per capita, it is convenient to begin by considering the underlying adjustment process more explicitly. If the total number of jobs in the system on day t is denoted by J_t^N , so that by definition, $J_t^N = V_t^N + (N - U_t^N)$, then it follows by adding (2.1) and (2.2) and collecting terms that

$$J_{t+1}^N = J_t^N + B_t^N - (C_{at}^N + C_{vt}^N) \tag{A.3}$$

But if for each job, $j = 1, \dots, J_t^N$, we let the indicator variable, δ_j , be zero if job j is closed on day t and one otherwise, then it follows by definition that

$$\sum_{j=1}^{J_t^N} \delta_j = J_t^N - (C_{at}^N + C_{vt}^N) \tag{A.4}$$

and hence that

$$J_{t+1}^N = B_t^N + \sum_{j=1}^{J_t^N} \delta_j \tag{A.5}$$

By our basic model assumptions, individual job closing are mutually independent, and also independent of new job creations. Hence

$$E(J_{t+1}^N | J_t^N) = \beta N + (1 - \rho) J_t^N$$

implies that the unconditional mean of J_{t+1}^N is given by

$$\begin{aligned}
E(J_{t+1}^N) &= E_{J_t^N} [E(J_{t+1}^N | J_t^N)] \\
&= \beta N + (1 - \rho) E(J_t^N) \tag{A.6}
\end{aligned}$$

and similarly,

$$\text{var} (J_{t+1}^N | J_t^N) = N\sigma^2 + (1 - \rho) \rho J_t^N$$

implies that the unconditional variance of J_{t+1}^N is given by

$$\begin{aligned} \text{var} (J_{t+1}^N) &= E_{J_t^N} [\text{var} (J_{t+1}^N | J_t^N)] \\ &= N\sigma^2 + (1 - \rho) \rho E (J_t^N) \end{aligned} \quad (\text{A.7})$$

Hence, letting J^N denote the steady-state level of employment [so that $E (J_t^N) = E (J^N)$ for all t], it follows from (A.6) that

$$\begin{aligned} E (J^N) &= \beta N + (1 - \rho) E (J^N) \\ \Rightarrow E (J^N) &= N \frac{\beta}{\rho} \\ \Rightarrow E \left(\frac{J^N}{N} \right) &= \frac{\beta}{\rho} \end{aligned} \quad (\text{A.8})$$

which together with (A.7) implies that

$$\begin{aligned} \text{var} \left(\frac{J^N}{N} \right) &= \frac{1}{N^2} [N\sigma^2 + (1 - \rho) \rho E (J^N)] \\ &= \frac{1}{N} [\sigma^2 + (1 - \rho) \beta] \end{aligned} \quad (\text{A.9})$$

Finally, since these equations imply both that $\lim_{N \rightarrow \infty} E (J^N/N) = \beta/\rho$ and $\lim_{N \rightarrow \infty} \text{var} (J^N/N) = 0$, it then follows from an application of Chebyshev's Inequality that (2.24) must hold.

A.3. Derivation of Expression (2.35)

To establish (2.35) we first observe that the variance of the conditional random variable in (2.30) has the binomial form:

$$\text{var} (F^N | m, n) = m \cdot p_N (1 - p_N) \quad (\text{A.10})$$

where $p_N = p_N \left(\frac{m}{N}, \frac{n}{N} \right)$, so that in a manner similar to (2.31),

$$\text{var} \left(\frac{F^N}{N} \middle| m, n \right) = \frac{1}{N^2} [m \cdot p_N (1 - p_N)] \quad (\text{A.11})$$

Hence, as a parallel to (2.32), we see that the variance of F^N/N is given by

$$\text{var} \left(\frac{F^N}{N} \right) = \left(\frac{1}{N} \right) E_{U^N, V^N} \left[p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right] \quad (\text{A.12})$$

But $0 \leq p_N \leq 1$ together with the nonnegativity of V^N and expression (A.8) then implies that

$$\text{var} \left(\frac{F^N}{N} \right) \leq \left(\frac{1}{N} \right) E \left(\frac{V^N}{N} \right) = \left(\frac{1}{N} \right) \frac{\beta}{\rho} \quad (\text{A.13})$$

Thus we see that

$$\lim_{N \rightarrow \infty} \text{var} \left(\frac{F^N}{N} \right) = 0 \quad (\text{A.14})$$

and may conclude from Chebyshev's inequality that

$$p\lim_{N \rightarrow \infty} \frac{F^N}{N} = \lim_{N \rightarrow \infty} E \left(\frac{F^N}{N} \right) \quad (\text{A.15})$$

whenever the limit on the right hand side exists. Hence by (2.32) and (2.33) it suffices to show that

$$\lim_{N \rightarrow \infty} E_{V^N, U^N} \left[p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right] = p(u, v) \cdot v \quad (\text{A.16})$$

We begin by showing that the variance of this random variables diminishes to zero. First observe that since $\text{var}(X \cdot Y) \leq 2 \cdot \text{var}(X) \text{var}(Y)$ and $0 \leq p_N \leq 1 \Rightarrow \text{var}(p_N) \leq 1$, it follows that

$$\text{var} \left[p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right] \leq 2 \cdot \text{var} \left(\frac{V^N}{N} \right) \quad (\text{A.17})$$

Moreover, the identity, $V^N = J^N + U^N - N$, (together with the Cauchy-Schwartz Inequality) implies that

$$\begin{aligned} \text{var} \left(\frac{V^N}{N} \right) &= \text{var} \left(\frac{J^N}{N} + \frac{U^N}{N} \right) \\ &\leq \text{var} \left(\frac{J^N}{N} \right) + 2 \left| \text{cov} \left(\frac{J^N}{N}, \frac{U^N}{N} \right) \right| + \text{var} \left(\frac{U^N}{N} \right) \\ &\leq \text{var} \left(\frac{J^N}{N} \right) + 2 \left[\text{var} \left(\frac{J^N}{N} \right) \text{var} \left(\frac{U^N}{N} \right) \right]^{1/2} + \text{var} \left(\frac{U^N}{N} \right) \end{aligned} \quad (\text{A.18})$$

Hence, observing that $\text{var}(U^N/N) = u(1-u)/N \Rightarrow \lim_{N \rightarrow \infty} \text{var}(U^N/N) = 0$ and that expression (A.9) implies $\lim_{N \rightarrow \infty} \text{var}(J^N/N) = 0$, we may conclude from (A.17) and (A.18) that

$$\lim_{N \rightarrow \infty} \text{var} \left[p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right] = 0 \quad (\text{A.19})$$

Thus, if it can be shown that

$$p \lim_{N \rightarrow \infty} p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} = p(u, v) \cdot v \quad (\text{A.20})$$

then it will follow from well-known properties of moment convergence [see for example Rao (1973, Theorem 2c.viii, p.121)] that (A.16) must hold.

To establish (A.20) it must be shown that for every $\epsilon > 0$,

$$\lim_{N \rightarrow \infty} \Pr \left[\left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p(u, v) \cdot v \right| > \epsilon \right] = 0 \quad (\text{A.21})$$

But since

$$\begin{aligned} & \Pr \left[\left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p(u, v) \cdot v \right| > \epsilon \right] \\ & \leq \Pr \left[\left| p_N \left(\frac{V^N}{N}, \frac{U^N}{N} \right) \frac{V^N}{N} - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right| + \right. \\ & \quad \left. \left| p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p(u, v) \cdot v \right| > \epsilon \right] \\ & \leq \Pr \left[\left\{ \left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right| > \frac{\epsilon}{2} \right\} \cup \right. \\ & \quad \left. \left\{ \left| p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p(u, v) \cdot v \right| > \frac{\epsilon}{2} \right\} \right] \\ & \leq \Pr \left\{ \left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right| > \frac{\epsilon}{2} \right\} + \\ & \quad \Pr \left\{ \left| p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p(u, v) \cdot v \right| > \frac{\epsilon}{2} \right\} \end{aligned} \quad (\text{A.22})$$

and since the continuity of the function, $p(x, y)$, implies that for all $\epsilon > 0$ [see for example Wilks (1962, Theorem 4.3.6, p.163)],

$$\lim_{N \rightarrow \infty} \Pr \left\{ \left| p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p(u, v) \cdot v \right| > \epsilon \right\} = 0, \quad (\text{A.23})$$

it suffices to show that for all $\epsilon > 0$,

$$\lim_{N \rightarrow \infty} \Pr \left\{ \left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \frac{V^N}{N} \right| > \epsilon \right\} = 0. \quad (\text{A.24})$$

Moreover, since the product function is continuous, and since the sequence, V^N/N converges in probability to a constant (2.25), it is enough to show that for all $\epsilon > 0$ [see for example Wilks (1962, Theorem 4.3.6)]:

$$\lim_{N \rightarrow \infty} \Pr \left\{ \left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \right| > \epsilon \right\} = 0. \quad (\text{A.25})$$

We do so by establishing bound on the rate of convergence of the sequence of functions in (2.34). First we show that the function $w_N(x)$ defined for all N and x by

$$w_N(x) = e^{-x} - \left(1 - \frac{x}{N}\right)^N \quad (\text{A.26})$$

satisfies the following inequality:

$$0 \leq x \leq N \Rightarrow 0 \leq w_N(x) \leq \frac{e^{-1}}{N} \quad (\text{A.27})$$

To see this, observe first that if $w_N(x) < 0$ for some $x \in [0, N]$, then since $w_N(0) = 0$ and $w_N(N) > 0$, it follows that w_N must achieve a negative minimum value in the open interval $(0, N)$. But since

$$\begin{aligned} 0 &= w'_N(x) = -e^{-x} + \left(1 - \frac{x}{N}\right)^{N-1} \\ &\Rightarrow \left(1 - \frac{x}{N}\right)^N = \left(1 - \frac{x}{N}\right) e^{-x} \\ &\Rightarrow w_N(x) = \frac{x}{N} e^{-x} > 0 \end{aligned} \quad (\text{A.28})$$

the minimum cannot be negative. Hence the first inequality on the right hand side of (A.27) must hold. Moreover, since the expression, $\frac{x}{N} e^{-x}$, is easily seen to

achieve its maximum value at $x = 1$, it also follows that $w_N(x) \leq e^{-1}/N$, and hence that (A.27) must hold. Next we show that for any x, y, z with $0 \leq x \leq 1$ and $y \geq z \geq 0$,

$$z \geq \frac{y}{2} \Rightarrow y^x - z^x \leq (y - z) x \left(\frac{y}{2}\right)^{x-1} \quad (\text{A.29})$$

To do so, observe that the (differentiable) function, $f(y) = y^x$, is concave for $0 \leq x \leq 1$, and hence that

$$\begin{aligned} y^x - z^x &= f(y) - f(z) \\ &\leq (y - z) f'(z) \\ &= (y - z) (x \cdot z^{x-1}) \end{aligned} \quad (\text{A.30})$$

But since $0 \leq x \leq 1$ and $z \geq (y/2)$ imply that $z^{x-1} \leq (y/2)^{x-1}$, it then follows that (A.29) must hold. Finally, by letting

$$\phi(x, y) = e^{-1} x \left(\frac{e^{-y}}{2}\right)^{x-1} \quad (\text{A.31})$$

and

$$r(y) = \gamma s / y \quad (\text{A.32})$$

we may now employ these results to show that for all $x \in [0, 1]$, $y > 0$ and $N \geq 2e^{r(y)}$, the difference between the functions in (2.28) and in (2.33) can be bounded by:

$$0 \leq p(x, y) - p_N(x, y) \leq \frac{\phi(x, r(y))}{N} \quad (\text{A.33})$$

To do so, observe first from (A.26) and (A.27) that $0 < r(y) < 2e^{r(y)} \leq N$ implies that

$$0 \leq e^{-r(y)} - \left(1 - \frac{r(y)}{N}\right)^N \leq \frac{e^{-1}}{N} \quad (\text{A.34})$$

But since $N \geq 2e^{r(y)} > (2e^{r(y)}) e^{-1} \Rightarrow e^{-1}/N \leq e^{-r(y)}/2$, it then follows from

(A.29) and (A.34) that

$$\begin{aligned}
0 &\leq e^{-r(y)} - \left(1 - \frac{r(y)}{N}\right)^N \leq \frac{e^{-r(y)}}{2} \\
&\Rightarrow \left(1 - \frac{r(y)}{N}\right)^N \geq \frac{e^{-r(y)}}{2} \\
&\Rightarrow 0 \leq [e^{-r(y)}]^x - \left[\left(1 - \frac{r(y)}{N}\right)^N\right]^x \\
&\quad \leq \left[e^{-r(y)} - \left(1 - \frac{r(y)}{N}\right)^N\right] \cdot x \left(\frac{e^{-r(y)}}{2}\right)^{x-1} \\
&\Rightarrow 0 \leq [e^{-r(y)}]^x - \left[\left(1 - \frac{r(y)}{N}\right)^N\right]^x \leq \left(\frac{e^{-1}}{N}\right) x \left(\frac{e^{-r(y)}}{2}\right)^{x-1} \\
&\Rightarrow 0 \leq e^{-\gamma s x/y} - \left(1 - \frac{\gamma s}{Ny}\right)^{Nx} \leq \frac{\phi(x, r(y))}{N} \\
&\Rightarrow 0 \leq p(x, y) - p_N(x, y) \leq \frac{\phi(x, r(y))}{N}.
\end{aligned} \tag{A.35}$$

To use this result, observe that if we restrict attention to values of $\epsilon > 0$ small enough to ensure that $v - \epsilon > 0$, and let

$$B_\epsilon(v, u) = \{(x, y) : |v - x| \leq \epsilon, |u - y| \leq \epsilon\}, \tag{A.37}$$

then function, $\phi(x, r(y))$, is seen to be continuous on the compact set, $B_\epsilon(u, v)$, so that the maximum value,

$$\phi_\epsilon = \max \{\phi(x, r(y)) : (x, y) \in B_\epsilon(u, v)\} \tag{A.38}$$

is well defined and finite. Hence, if for each $\delta > 0$ we choose $N_o = N_o(\epsilon, \delta)$ large

enough to ensure that the following three conditions hold for all $N \geq N_o$:

$$N \geq \max \left\{ \frac{\phi_\epsilon + 1}{\epsilon}, 2e^{r(v-\epsilon)} \right\} \quad (\text{A.39})$$

$$\Pr \left(\left| \frac{V^N}{N} - v \right| > \epsilon \right) < \frac{\delta}{2} \quad (\text{A.40})$$

$$\Pr \left(\left| \frac{U^N}{N} - u \right| > \epsilon \right) < \frac{\delta}{2} \quad (\text{A.41})$$

[where conditions (A.40) and (A.41) are possible in view of (2.25) and (2.22), respectively], then for all such N we can obtain a probability bound for (A.25) as follows. First, for any realized values, u_N ($\in [0, 1]$) and v_N (> 0), of U^N/N and V^N/N , respectively, observe that

$$\begin{aligned} & |p_N(u_N, v_N) - p(u_N, v_N)| > \epsilon \\ \Rightarrow & |p_N(u_N, v_N) - p(u_N, v_N)| > \min \left\{ \epsilon, \frac{\phi(u_N, r(v_N))}{N} \right\} \end{aligned} \quad (\text{A.42})$$

But if this minimum is not ϵ , then inequality, $|p_N(u_N, v_N) - p(u_N, v_N)| > \phi(u_N, r(v_N))/N$ implies from (A.33) that $N < 2e^{r(v_N)}$, which together with (A.39) implies that $2e^{r(v-\epsilon)} < 2e^{r(v_N)}$. Hence by (A.32) we must have $v_N < v - \epsilon$. On the other hand, if ϵ is the minimum, then by (A.39), $\phi(u_N, r(v_N)) \geq N\epsilon > \phi_\epsilon$. But since this in turn implies from (A.38) that either $|u_N - u| > \epsilon$ or $|v_N - v| > \epsilon$ must hold, we see that in all cases,

$$\begin{aligned} |p_N(u_N, v_N) - p(u_N, v_N)| & > \min \left\{ \epsilon, \frac{\phi(u_N, r(v_N))}{N} \right\} \\ \Rightarrow & \{|u_N - u| > \epsilon\} \text{ or } \{|v_N - v| > \epsilon\} \end{aligned} \quad (\text{A.43})$$

Hence, by combining (A.42) and (A.43), we may conclude that for all N satisfying

(A.39) through (A.41):

$$\begin{aligned}
& \Pr \left\{ \left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \right| > \epsilon \right\} \\
& \leq \Pr \left(\left| p_N \left(\frac{U^N}{N}, \frac{V^N}{N} \right) - p \left(\frac{U^N}{N}, \frac{V^N}{N} \right) \right| > \min \left\{ \epsilon, \frac{\phi(U_N, r(V_N))}{N} \right\} \right) \\
& \leq \Pr \left(\left\{ \left| \frac{U^N}{N} - u \right| > \epsilon \right\} \cup \left\{ \left| \frac{V^N}{N} - v \right| > \epsilon \right\} \right) \\
& \leq \Pr \left(\left| \frac{U^N}{N} - u \right| > \epsilon \right) + \Pr \left(\left| \frac{V^N}{N} - v \right| > \epsilon \right) \\
& < \delta
\end{aligned} \tag{A.44}$$

Finally, since the choice of δ was arbitrary, it may be concluded that (A.25) holds for every (sufficiently small) choice of $\epsilon > 0$.

A.4. Proof of Proposition 2.1

First observe that for all $\lambda > 0$,

$$\phi(\lambda u, \lambda v) = \lambda v [1 - e^{-(\gamma s \lambda u / \lambda v)}] = \lambda v [1 - e^{-(\gamma s u / v)}] = \lambda \cdot \phi(u, v)$$

and hence that ϕ is linearly homogeneous. Next observe that ϕ is twice continuously differentiable with:

$$\frac{\partial}{\partial u} \phi(u, v) = \gamma s e^{-(\gamma s u / v)} \tag{A.45}$$

$$\frac{\partial^2}{\partial u^2} \phi(u, v) = -\frac{(\gamma s)^2}{v} e^{-(\gamma s u / v)} \tag{A.46}$$

$$\frac{\partial}{\partial v} \phi(u, v) = 1 - \left(1 + \frac{\gamma s u}{v} \right) e^{-(\gamma s u / v)} \tag{A.47}$$

$$\frac{\partial^2}{\partial v^2} \phi(u, v) = -\frac{(\gamma s u)^2}{v^3} e^{-(\gamma s u / v)} \tag{A.48}$$

so that monotonicity follows from the positivity of (A.45) and (A.47).²³ Finally, to establish concavity, observe in addition that

$$\frac{\partial^2}{\partial v \partial u} \phi(u, v) = \left(\frac{\gamma s}{v} \right)^2 u e^{-(\gamma s u / v)} \tag{A.49}$$

²³To see that (A.47) is positive, let $z(x) = 1 - (1+x)e^{-x}$, and observe that $z(0) = 0$ and $z'(x) = xe^{-x} > 0$ for all $x > 0$.

and hence from [(A.46),(A.48),(A.49)], that the Hessian of ϕ is given by

$$H = (\gamma s)^2 e^{-(\gamma su/v)} \begin{pmatrix} -\frac{1}{v} & \frac{u}{v^2} \\ \frac{u}{v^2} & -\frac{u^2}{v^3} \end{pmatrix} \quad (\text{A.50})$$

Thus for any column vector, $z = (x, y)'$, it follows that

$$\begin{aligned} z' H z &= (\gamma s)^2 e^{-(\gamma su/v)} \left[(x, y) \begin{pmatrix} -\frac{1}{v} & \frac{u}{v^2} \\ \frac{u}{v^2} & -\frac{u^2}{v^3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] \\ &= (\gamma s)^2 e^{-(\gamma su/v)} \left(-\frac{1}{v} \right) \left[x^2 - 2 \left(\frac{u}{v} \right) x y + \left(\frac{u}{v} \right)^2 y^2 \right] \\ &= -(\gamma s)^2 e^{-(\gamma su/v)} \left(\frac{1}{v} \right) \left(x - \frac{u}{v} y \right)^2 \leq 0 \end{aligned} \quad (\text{A.51})$$

and we see that H is negative semidefinite. Moreover, (A.51) is strictly negative except for vectors z colinear with (u, v) , i.e., with $z = (\lambda u, \lambda v)$ for some λ . Hence ϕ is seen to be strictly concave except for the linearity-on-rays property implied by linear homogeneity.

A.5. Proof of Theorem 2.2

By substituting (2.26) into (2.55) and letting

$$G(u, s) = \rho (1 - u) - (1 - \rho) (u + a) \left(1 - e^{-\gamma s \frac{u}{u+a}} \right), \quad (\text{A.52})$$

we see that for each given $s \in (0, 1]$ the steady-state values of u are precisely the roots of the equation

$$G(u, s) = 0 \quad (\text{A.53})$$

To establish existence of solutions, observe first that for $u = 1$ we have

$$G(1, s) = - (1 - \rho) (1 + a) \left(1 - e^{-\gamma s \frac{1}{1+a}} \right) \quad (\text{A.54})$$

But since $\beta/\rho > 0$ implies from (2.27) that $1 + a > 0$, we see that

$$G(1, s) < 0 \quad (\text{A.55})$$

Next we show that

$$G(u_a, s) > 0 \quad (\text{A.56})$$

for u_a in (2.56). To do so observe first that if $a > 0$ then $u_a = 0$ and

$$G(u_a, s) = \rho > 0. \quad (\text{A.57})$$

Next, if $a = 0$ then $G(u, s) = \rho(1 - u) - (1 - \rho)u(1 - e^{-\gamma s})$ and $u_0 = 0$, so again

$$G(u_a, s) = \rho > 0. \quad (\text{A.58})$$

Finally, if $a < 0$ then $\beta > 0$ implies that $u_a = -a = |a| < 1$, so $\lim_{u \downarrow u_a} u/(u + a) = \infty$ implies that $\lim_{u \downarrow u_a} \left(1 - e^{-\gamma s \frac{u}{u+a}}\right) = 1$, and hence that

$$\lim_{u \downarrow u_a} G(u, s) = \rho(1 - |a|) > 0 \quad (\text{A.59})$$

Thus (A.56) holds in all cases, and we may conclude from the continuity of G that for each $s \in (0, 1]$ there exists a steady-state value, $u(s)$, in the *open* interval $(u_a, 1)$. In particular, this implies that $u(s) > 0$. Next, to establish uniqueness of this steady state, it suffices to show the partial derivative of G with respect to u is everywhere *negative* in the interval $(u_a, 1)$, i.e. that

$$\frac{\partial}{\partial u} G(u, s) < 0, \quad u \in (u_a, 1) \quad (\text{A.60})$$

For this will imply that $G(\cdot, s)$ can pass through zero no more than once in the interval $(u_a, 1)$. By using the identity

$$\frac{d}{du} \left(\frac{u}{u+a} \right) = \frac{a}{(u+a)^2} \quad (\text{A.61})$$

it may be verified that

$$\frac{\partial}{\partial u} G(u, s) = (1 - \rho) e^{-\gamma s \frac{u}{u+a}} \left(\frac{u+a - \gamma s a}{u+a} \right) - 1 \quad (\text{A.62})$$

which together with $u + a > 0$ for all $u > u_a$ implies that (A.62) is well defined on (u_a, ∞) , and in particular satisfies

$$\lim_{u \rightarrow \infty} \frac{\partial}{\partial u} G(u, s) = (1 - \rho) e^{-\gamma s} (1) - 1 < 0 \quad (\text{A.63})$$

Hence it suffices to show that the second partial derivative is positive for all $u > u_a$ [which will imply that (A.62) must be everywhere negative on (u_a, ∞)]. By direct calculation it follows from (A.62) and (A.61) that

$$\frac{\partial^2}{\partial u^2} G(u, s) = \frac{(1 - \rho)}{(u+a)^3} (\gamma s a)^2 e^{-\gamma s \frac{u}{u+a}} > 0 \quad (\text{A.64})$$

for all $u > u_a$. Thus (A.60) holds, and $u(s)$ must be the unique solution of (A.52) in the interval $[u_a, 1]$. These unique solutions define a function of s which can be analyzed by implicit differentiation of $G [u(s), s]$ as follows. Since $G [u(s), s] \equiv 0$, we see that

$$\begin{aligned} 0 &\equiv \frac{\partial}{\partial s} G [u(s), s] \equiv \frac{\partial G}{\partial u} u'(s) + \frac{\partial G}{\partial s} \\ \Rightarrow u'(s) &\equiv -\frac{\partial G / \partial s}{\partial G / \partial u} \end{aligned} \quad (\text{A.65})$$

for all $s \in [0, 1]$. But since

$$\frac{\partial}{\partial s} G (u, s) = - (1 - \rho) \gamma u e^{-\gamma s \frac{u}{u+a}} < 0 , \quad (\text{A.66})$$

we may then conclude from (A.60) and (A.65) that $u'(s) < 0$, and hence that $u(s)$ is strictly decreasing in s . Finally since (2.26) implies that

$$v(s) = u(s) + a \quad (\text{A.67})$$

for all s , and since $u(s) \in (u_a, 1)$ implies that $0 < u(s) + a = v(s)$, we see that $v(s)$ is also a decreasing positive differentiable function of s .

A.6. Verification of (2.57) and (2.58)

First observe that since γ appears only as a product with s in (2.55) it follows at once that changes with respect to γ and s are always in the same direction, and hence (by the proof of Theorem 2.2) that both $\partial u / \partial \gamma < 0$ and $\partial v / \partial \gamma < 0$. Thus we need only consider changes with respect to β and ρ . To establish the properties in (2.57) we begin by solving for v in (2.54) and substituting into (2.55) to obtain the relation:

$$H(u, \rho, \beta) = \rho(1 - u) - (1 - \rho) \left(u + \frac{\beta}{\rho} - 1 \right) [1 - e^{-X(u, \rho, \beta)}] \quad (\text{A.68})$$

where

$$X(u, \rho, \beta) = \frac{\gamma s u}{u + \frac{\beta}{\rho} - 1} \quad (\text{A.69})$$

with γ and s held fixed. Next recalling from (2.54) together with Theorem 2.2 that $u + \frac{\beta}{\rho} - 1 = v > 0$, we see that X is always positive and continuously differentiable

in each of its arguments. In particular we have

$$\frac{\partial X}{\partial u} = \frac{\gamma s}{u + \frac{\beta}{\rho} - 1} \left(1 - \frac{u}{u + \frac{\beta}{\rho} - 1} \right) \quad (\text{A.70})$$

$$\frac{\partial X}{\partial \rho} = - \left(\frac{\beta}{\rho^2} \right) \frac{X}{u + \frac{\beta}{\rho} - 1} \quad (\text{A.71})$$

$$\frac{\partial X}{\partial \beta} = - \left(\frac{1}{\rho} \right) \frac{X}{u + \frac{\beta}{\rho} - 1} \quad (\text{A.72})$$

Hence from (A.70) it follows that

$$\begin{aligned} \frac{\partial H}{\partial u} &= -\rho - (1 - \rho) \left\{ (1 - e^{-X}) + \left(u + \frac{\beta}{\rho} - 1 \right) \left(e^{-X} \cdot \frac{\partial X}{\partial u} \right) \right\} \\ &= -\rho - (1 - \rho) \left\{ (1 - e^{-X}) + \gamma s e^{-X} \left(1 - \frac{u}{u + \frac{\beta}{\rho} - 1} \right) \right\} \\ &= -\rho - (1 - \rho) \{ 1 + \gamma s e^{-X} - (1 + X) e^{-X} \} \end{aligned} \quad (\text{A.73})$$

But since $1 + X < e^X$ for all $X > 0$, and since

$$1 + X < e^X \Rightarrow (1 + X) e^{-X} < 1 \quad (\text{A.74})$$

it follows that the part of expression (A.73) in braces is positive, and hence that

$$\frac{\partial H}{\partial u} < 0 \quad (\text{A.75})$$

Next observe from (A.72) that

$$\begin{aligned} \frac{\partial H}{\partial \beta} &= -(1 - \rho) \left\{ \frac{1}{\rho} (1 - e^{-X}) + \left(u + \frac{\beta}{\rho} - 1 \right) \left(e^{-X} \cdot \frac{\partial X}{\partial \beta} \right) \right\} \\ &= - \left(\frac{1 - \rho}{\rho} \right) \{ 1 - e^{-X} - X e^{-X} \} \\ &= - \left(\frac{1 - \rho}{\rho} \right) \{ 1 - (1 + X) e^{-X} \} \end{aligned} \quad (\text{A.76})$$

and we may again conclude from (A.74) that

$$\frac{\partial H}{\partial \beta} < 0 \quad (\text{A.77})$$

But since $H(u, \rho, \beta) \equiv 0$ for all (u, β) with ρ held fixed, it then follows from (A.75) and (A.77) that

$$\begin{aligned} 0 &= \frac{\partial H}{\partial u} du + \frac{\partial H}{\partial \beta} d\beta \\ \Rightarrow \frac{du}{d\beta} &= -\frac{\partial H/\partial \beta}{\partial H/\partial u} < 0 \end{aligned} \quad (\text{A.78})$$

and hence that the second condition in (2.57) holds. To establish the third condition, observe from (A.71) that

$$\begin{aligned} \frac{\partial H}{\partial \rho} &= (1-u) - \left\{ -\left(u + \frac{\beta}{\rho} - 1\right) (1 - e^{-X}) \right. \\ &\quad \left. - (1-\rho) \left(\frac{\beta}{\rho^2}\right) (1 - e^{-X}) \right. \\ &\quad \left. + (1-\rho) \left(u + \frac{\beta}{\rho} - 1\right) \left(e^{-X} \cdot \frac{\partial X}{\partial \rho}\right) \right\} \\ &= (1-u) + \left(u + \frac{\beta}{\rho} - 1\right) (1 - e^{-X}) \\ &\quad + (1-\rho) \left(\frac{\beta}{\rho^2}\right) (1 - e^{-X} + X e^{-X}) \end{aligned} \quad (\text{A.79})$$

and hence from the positivity of $u + \frac{\beta}{\rho} - 1$ and X that

$$\frac{\partial H}{\partial \rho} > 0 \quad (\text{A.80})$$

Thus by (A.75) and (A.80) we may now conclude [in a manner similar to (A.78)] that with β held fixed,

$$\begin{aligned} 0 &= \frac{\partial H}{\partial u} du + \frac{\partial H}{\partial \rho} d\rho \\ \Rightarrow \frac{du}{d\rho} &= -\frac{\partial H/\partial \rho}{\partial H/\partial u} > 0 \end{aligned} \quad (\text{A.81})$$

so that the last condition in (2.57) holds.

Next to establish the conditions for β and ρ in (2.58), we now solve for u in (2.54) and again substitute into (2.55) to obtain the relation:

$$G(v, \rho, \beta) = \beta - \rho v - (1-\rho)v [1 - e^{-\gamma s} e^{-Y(v, \rho, \beta)}] \quad (\text{A.82})$$

where

$$Y(v, \rho, \beta) = \frac{\gamma s(\rho - \beta)}{\rho v} = \frac{\gamma s}{v} - \frac{\gamma s \beta}{\rho v} \quad (\text{A.83})$$

The function Y is continuously differentiable in its arguments with

$$\frac{\partial Y}{\partial v} = -\frac{Y}{v} \quad (\text{A.84})$$

$$\frac{\partial Y}{\partial \rho} = \frac{\gamma s \beta}{\rho^2 v} \quad (\text{A.85})$$

$$\frac{\partial Y}{\partial \beta} = -\frac{\gamma s}{\rho v} \quad (\text{A.86})$$

By using (A.84) we obtain

$$\begin{aligned} \frac{\partial G}{\partial v} &= -\rho - (1 - \rho) \left\{ (1 - e^{-\gamma s} e^{-Y}) + v e^{-\gamma s} \left(e^{-Y} \cdot \frac{\partial Y}{\partial v} \right) \right\} \\ &= -\rho - (1 - \rho) \left\{ (1 - e^{-\gamma s} e^{-Y}) - e^{-\gamma s} Y e^{-Y} \right\} \\ &= -\rho - (1 - \rho) \left\{ 1 - e^{-\gamma s} (1 + Y) e^{-Y} \right\} \end{aligned} \quad (\text{A.87})$$

which together with the fact that $(1 + Y)e^{-Y} \leq 1$ for *all* Y , implies that

$$\frac{\partial G}{\partial v} < 0 \quad (\text{A.88})$$

Moreover, since (A.86) shows that

$$\begin{aligned} \frac{\partial G}{\partial \beta} &= 1 - (1 - \rho)v \left(e^{-\gamma s} e^{-Y} \cdot \frac{\partial Y}{\partial \beta} \right) \\ &= 1 + (1 - \rho) \left(\frac{\gamma s}{\rho} \right) e^{-\gamma s} e^{-Y} > 0 \end{aligned} \quad (\text{A.89})$$

it now follows from (A.88) and (A.89) that with ρ held fixed,

$$\begin{aligned} 0 &= \frac{\partial G}{\partial v} dv + \frac{\partial G}{\partial \beta} d\beta \\ \Rightarrow \frac{dv}{d\beta} &= -\frac{\partial G / \partial \beta}{\partial G / \partial v} > 0 \end{aligned} \quad (\text{A.90})$$

so that the second condition in (2.58) holds. Finally, to establish the third condition, observe from (A.85) that

$$\begin{aligned}\frac{\partial G}{\partial \rho} &= -v - (1 - \rho) \left\{ 1 - e^{-\gamma s} e^{-Y} + v e^{-\gamma s} e^{-Y} \cdot \frac{\partial Y}{\partial \rho} \right\} \\ &= -v - (1 - \rho) \left\{ (1 - e^{-\gamma s} e^{-Y}) + e^{-\gamma s} e^{-Y} \cdot \frac{\gamma s \beta}{\rho^2} \right\}\end{aligned}\quad (\text{A.91})$$

But since the expression in braces is positive, it then follows that

$$\frac{\partial G}{\partial \rho} < 0 \quad (\text{A.92})$$

and we may conclude from (A.88) and (A.92) that with β held fixed,

$$\begin{aligned}0 &= \frac{\partial G}{\partial v} dv + \frac{\partial G}{\partial \rho} d\rho \\ \Rightarrow \frac{dv}{d\rho} &= -\frac{\partial G / \partial \rho}{\partial G / \partial v} < 0\end{aligned}\quad (\text{A.93})$$

Hence the third condition in (2.58) also holds, and the result is established.

A.7. Derivation of (3.22)

Observe from (3.16) that $V_0'(0)$ now takes the form:

$$V_0'(0) = \frac{1}{1 - \sigma} [\sigma p_h(0) V_1(0) - \alpha b - \sigma p_h(0) V_0(0)] \quad (\text{A.94})$$

To evaluate the relevant terms, observe next from (3.15) that

$$V_0(0) = \frac{b}{(1 - \sigma)} \quad (\text{A.95})$$

[which is seen from the identity, $\sum_{t=0}^{\infty} \sigma^t = 1/(1 - \sigma)$, to be precisely the discounted lifetime value of permanent unemployment resulting from zero job search].

Next, to evaluate the hiring probability, $p_h(0)$, observe from (2.42) that

$$\begin{aligned}p_h(s) &= \frac{v(s)}{s u(s)} \{1 - e^{-[\gamma s u(s)/v(s)]}\} \\ &= \frac{1}{w(s)} [1 - e^{-\gamma w(s)}]\end{aligned}\quad (\text{A.96})$$

where $w(s) = s u(s)/v(s)$. But by Theorem 2.2, $u(0) = 1$ and $v(0) = \beta/\rho > 0$, so that $\lim_{s \rightarrow 0} w(s) = 0$, and we may conclude (from an application of l'Hospital's rule) that

$$\begin{aligned} p_h(0) &= \lim_{s \rightarrow 0} p_h(s) = \lim_{s \rightarrow 0} \frac{1 - e^{-\gamma w(s)}}{w(s)} \\ &= \lim_{w \rightarrow 0} \frac{1 - e^{-\gamma w}}{w} = \lim_{w \rightarrow 0} \frac{\gamma e^{-\gamma w}}{1} = \gamma \end{aligned} \quad (\text{A.97})$$

[To see the economic meaning of this relation, observe that as population search intensity falls to zero, all competition for jobs vanishes. Hence the limiting probability of a being hired is simply the probability of being qualified for a given job]. Finally, noting from (3.7) together with (A.95) that

$$\begin{aligned} V_1(0) &= \left(\frac{1 - e_1}{1 - \sigma} \right) U_1 + e_1 V_0(0) \\ &= \left(\frac{1 - e_1}{1 - \sigma} \right) U_1 + e_1 \frac{b}{(1 - \sigma)}, \end{aligned} \quad (\text{A.98})$$

it follows by substituting (A.95), (A.96), and (A.97) into (A.94) that the desired condition [i.e., positivity of the bracketed term in (A.94)] is given by

$$\begin{aligned} 0 &< \sigma p_h(0) V_1(0) - \alpha (1 - s)^{\alpha-1} b - \sigma p_h(0) V_0(0) \\ &= \sigma \gamma \left[\left(\frac{1 - e_1}{1 - \sigma} \right) U_1 + e_1 \frac{b}{(1 - \sigma)} \right] - \alpha b - \sigma \gamma \frac{b}{(1 - \sigma)}. \end{aligned} \quad (\text{A.99})$$

By employing the definition of e_1 in (3.13) and rearranging terms, this inequality can be rewritten as (3.22).

A.8. Proof of Theorem 3.2

To establish existence of a solution to equations (3.23) through (3.29), we first show that this system can be reduced to a *single* equation in search intensity, s ,

as follows. First observe from (3.26) and (3.27) that

$$\begin{aligned}
V_0 &= (1 - e_0) \frac{U_0(s)}{1 - \sigma} + e_0 V_1 \\
&= \left(\frac{1 - \sigma}{1 - \sigma + \sigma p_h s} \right) \frac{U_0(s)}{1 - \sigma} + \left(\frac{\sigma p_h s}{1 - \sigma + \sigma p_h s} \right) V_1 \\
&\Rightarrow (1 - \sigma + \sigma p_h s) V_0 = U_0(s) + \sigma p_h s V_1 \\
&\Rightarrow \sigma p_h s (V_1 - V_0) = (1 - \sigma) V_0 - U_0(s)
\end{aligned} \tag{A.100}$$

Also by (3.27) and (3.28),

$$\begin{aligned}
V_0 &= (1 - e_0) \frac{U_0(s)}{1 - \sigma} + e_0 \left[\left(\frac{1 - e_1}{1 - \sigma} \right) U_1 + e_1 V_0 \right] \\
&\Rightarrow (1 - e_0 e_1) V_0 = (1 - e_0) \frac{U_0(s)}{1 - \sigma} + e_0 (1 - e_1) \frac{U_1}{1 - \sigma} \\
&\Rightarrow (1 - \sigma) V_0 = \left(\frac{1 - e_0}{1 - e_0 e_1} \right) U_0(s) + \left(\frac{e_0 (1 - e_1)}{1 - e_0 e_1} \right) U_1,
\end{aligned} \tag{A.101}$$

which together with (A.100) implies that

$$\begin{aligned}
\sigma p_h s (V_1 - V_0) &= \left[\left(\frac{1 - e_0}{1 - e_0 e_1} \right) U_0(s) + \left(\frac{e_0 (1 - e_1)}{1 - e_0 e_1} \right) U_1 \right] - U_0(s) \\
&= \left(\frac{e_0 (1 - e_1)}{1 - e_0 e_1} \right) [U_1 - U_0(s)].
\end{aligned} \tag{A.102}$$

Hence by (3.29) and (A.102),

$$\begin{aligned}
\left(\frac{e_0 (1 - e_1)}{1 - e_0 e_1} \right) [U_1 - U_0(s)] &= \sigma p_h s (V_1 - V_0) = s (1 - s)^{\alpha-1} b \alpha \\
&= \left(\frac{s}{1 - s} \right) [(1 - s)^\alpha b \alpha] = \left(\frac{s}{1 - s} \right) U_0(s) \alpha \\
&\Rightarrow \frac{1}{\alpha} \left(\frac{1 - s}{s} \right) \left[\frac{U_1 - U_0(s)}{U_0(s)} \right] = \frac{1 - e_0 e_1}{e_0 (1 - e_1)},
\end{aligned} \tag{A.103}$$

where the left hand side is now seen to be an *explicit* function of s . Next we show that the right hand side is an explicit function of the unemployment rate, u . To do so, observe from (3.25) that

$$s p_h = \left(\frac{\rho}{1 - \rho} \right) \frac{1 - u}{u} \tag{A.104}$$

which together with (3.26) yields

$$e_0 = \frac{\sigma \rho (1 - u)}{(1 - \sigma)(1 - \rho)u + \sigma \rho (1 - u)}. \quad (\text{A.105})$$

By substituting (A.105) into the right hand side of (A.103) and reducing, we obtain

$$\begin{aligned} \frac{1 - e_0 e_1}{e_0 (1 - e_1)} &= \frac{(1 - \sigma)(1 - \rho)u + \sigma \rho (1 - e_1)(1 - u)}{\sigma \rho (1 - e_1)(1 - u)} \\ &= 1 + \mu \left(\frac{u}{1 - u} \right) \end{aligned} \quad (\text{A.106})$$

where [recalling that $e_1 = \rho / (1 - \sigma + \sigma \rho)$] the constant μ is given by

$$\begin{aligned} \mu &= \left(\frac{1 - \sigma}{\sigma} \right) \left(\frac{1 - \rho}{\rho} \right) \frac{1}{1 - e_1} \\ &= \frac{1 - \sigma + \sigma \rho}{\sigma \rho} > 0. \end{aligned} \quad (\text{A.107})$$

Thus, letting $W(s)$ be defined for all $s \in (0, 1]$ by

$$\begin{aligned} W(s) &= \frac{1}{\alpha} \left(\frac{1 - s}{s} \right) \left[\frac{U_1 - U_0(s)}{U_0(s)} \right] - 1 \\ &= \frac{1}{\alpha} \left(\frac{1 - s}{s} \right) \left[\frac{U_1 - (1 - s)^{\alpha b}}{(1 - s)^{\alpha b}} \right] - 1 \end{aligned} \quad (\text{A.108})$$

we see from (A.106) and (A.108) that (A.103) now takes the form, $W(s) = \mu [u / (1 - u)]$, so that u can be written in terms of s as

$$u = \frac{W(s)}{\mu + W(s)}. \quad (\text{A.109})$$

Moreover, by substituting (3.23) and (3.24) into (3.25), we obtain the following additional equation in s and u :

$$\rho (1 - u) = (1 - \rho) \frac{u + a}{s u} \left(1 - e^{-\gamma s \frac{u}{u+a}} \right) \quad (\text{A.110})$$

Thus, letting $G(u, s)$ again be defined as in (A.52), it follows by substituting (A.109) into (A.110) that this system can be reduced to single equation in s of the form

$$G \left[\frac{W(s)}{\mu + W(s)}, s \right] = 0 \quad (\text{A.111})$$

To solve this equation, we next observe that (A.109) is only meaningful for values of s which yield *positive* values of u in (A.109) and v in (3.23). It can be shown (see Lemma A.1 in section A.9 of this Appendix) that for each possible value of a (i.e., $a > -1$) *there exists a unique search intensity, $s_a \in (0, 1)$, such that both these conditions hold iff $s \in (0, s_a)$.* With these observations, the key step in the proof (established in Lemma A.2 in section A.9 of this Appendix) is to show that for each $a > -1$ *there exists a unique solution to (A.111) in the open interval $(0, s_a)$.*

Given this solution, s , it remains only to show that s generates a unique set of values $[u(s), v(s), p_h(s), e_0(s), V_0(s), V_1(s)]$ which together with s constitute an endogenous-search equilibrium. To do so, let $W(s)$ be defined by (A.108) and observe that if $u(s) \in (0, 1)$ is in turn defined by (A.109), and if we let $v(s) = u(s) + a$, then it follows from the definition of G that (3.23), (3.24) and (3.25) must hold for these choices, with $p_h(s)$ defined by

$$p_h(s) = \frac{v(s)}{s u(s)} \left\{ 1 - e^{-[\gamma s \frac{u(s)}{v(s)}]} \right\} \quad (\text{A.112})$$

In particular, this is seen to imply that $[u(s), v(s)]$ constitutes a steady state *given* s , and hence (by Theorem 2.2) that $v(s) > 0$. Moreover, since $p_h(s)$ is of the form, $(1 - e^{-\gamma w})/w$, which is seen to satisfy $0 < (1 - e^{-\gamma w})/w < \gamma < 1$ for all $w > 0$, it follows that $p_h(s) \in (0, 1)$. Next let

$$e_0(s) = \frac{\sigma p_h(s) s}{1 - \sigma + \sigma p_h(s) s} \in (0, 1) \quad (\text{A.113})$$

[so that (3.26) holds] and let $V_0(s)$ and $V_1(s)$ be defined in terms of (3.8) and (3.9) by

$$V_0(s) = \frac{1 - e_0(s)}{1 - e_0(s) e_1} \left(\frac{U_0(s)}{1 - \sigma} \right) + \frac{e_0(s) (1 - e_1)}{1 - e_0(s) e_1} \left(\frac{U_1}{1 - \sigma} \right) > 0 \quad (\text{A.114})$$

$$V_1(s) = \frac{1 - e_1}{1 - e_0(s) e_1} \left(\frac{U_1}{1 - \sigma} \right) + \frac{e_1 (1 - e_0(s))}{1 - e_0(s) e_1} \left(\frac{U_0(s)}{1 - \sigma} \right) > 0 \quad (\text{A.115})$$

so that (3.27) and (3.28) must also hold [by the definitions of (3.8) and (3.9)]. Hence all values $[u(s), v(s), p_h(s), e_0(s), V_0(s), V_1(s)]$ are uniquely defined, have the correct domains, and satisfy conditions (3.23) through (3.28). To see that the positivity condition (3.29) also holds, observe that since the relations (A.100)

through (A.102) are independent of (3.29), it follows that these relations must continue to hold, so that

$$\sigma p_h(s) s [V_1(s) - V_0(s)] = \left(\frac{e_0(s)(1 - e_1)}{1 - e_0(s)e_1} \right) [U_1 - U_0(s)] . \quad (\text{A.116})$$

Finally, since the relations in (A.104) through (A.106) are also independent of (3.29), it follows from the definition of $W(s)$ that

$$\begin{aligned} \frac{1}{\alpha} \left(\frac{1-s}{s} \right) \left[\frac{U_1 - U_0(s)}{U_0(s)} \right] &= 1 + W(s) = 1 + \mu \left(\frac{u}{1-u} \right) \\ &= \frac{1 - e_0(s)e_1}{e_0(s)(1 - e_1)} \\ \Rightarrow \left(\frac{e_0(s)(1 - e_1)}{1 - e_0(s)e_1} \right) [U_1 - U_0(s)] &= \alpha \left(\frac{s}{1-s} \right) U_0(s) \\ &= \alpha s (1-s)^{\alpha-1} b , \end{aligned} \quad (\text{A.117})$$

which together with (A.116) [and the positivity of s] is seen to imply that (3.29) must hold. Thus the vector $[s, u(s), v(s), p_h(s), e_0(s), V_0(s), V_1(s)]$ constitutes the unique endogenous-search equilibrium for these parameters, and the result is established.

A.9. Proof of Lemmas A.1 and A.2

If the function g is defined for all $s \in (0, 1]$ by

$$g(s) = G \left[\frac{W(s)}{\mu + W(s)}, s \right] \quad (\text{A.118})$$

then to complete the proof of Theorem 3.2 it remains to be shown that g has a unique root in the open interval $(0, s_a)$. To do so, we first show that if

$$u(s) = \frac{W(s)}{\mu + W(s)} \quad (\text{A.119})$$

$$v(s) = u(s) + a \quad (\text{A.120})$$

for each $s \in (0, 1]$, then

Lemma A.1. For each $a > -1$ there exists a unique search intensity, $s_a \in (0, 1)$, such that

$$\min\{u(s), v(s)\} > 0 \Leftrightarrow s \in (0, s_a) \quad (\text{A.121})$$

Proof. Observe first that if $a \geq 0$, then by (A.119) and (A.120)

$$\min\{u(s), v(s)\} = u(s) > 0 \Leftrightarrow W(s) > 0. \quad (\text{A.122})$$

On the other hand, if $a < 0$, then

$$\begin{aligned} \min\{u(s), v(s)\} = v(s) > 0 &\Leftrightarrow u(s) + a > 0 \\ &\Leftrightarrow \frac{W(s)}{\mu + W(s)} + a > 0 \\ &\Leftrightarrow W(s) > -\frac{a\mu}{(1+a)} \end{aligned} \quad (\text{A.123})$$

[where $1 + a > 0$ by hypothesis]. Hence $\min\{u(s), v(s)\} > 0$ iff

$$W(s) > \max\left\{0, -\frac{a\mu}{(1+a)}\right\} \quad (\text{A.124})$$

But since (A.108) implies that

$$\lim_{s \rightarrow 0} W(s) = \lim_{s \rightarrow 0} \frac{1}{\alpha} \left(\frac{1-s}{s} \right) \left[\frac{U_1 - (1-s)^{\alpha b}}{(1-s)^{\alpha b}} \right] - 1 = \infty \quad (\text{A.125})$$

it follows that (A.124) always holds for values of s near zero. Moreover, since (A.108) also implies that

$$\begin{aligned} W(s) > 0 &\Leftrightarrow (1-s)[U_1 - U_0(s)] > \alpha s U_0(s) \\ &\Leftrightarrow (1-s)^{1-\alpha} (U_1/b) > 1 - s + \alpha s \end{aligned} \quad (\text{A.126})$$

we see from the positivity of α implies that $W(1) < 0$. Hence in the last inequality of (A.126), the strict concavity of the left-hand side and linearity of the right-hand side are easily seen to imply that these two terms must be equal at exactly one intermediate point, $s_0 \in (0, 1)$, and that (A.126) holds iff $s \in (0, s_0)$. Next observe that

$$W'(s) = \frac{1}{\alpha s^2} \left[1 + \frac{U_1}{U_0(s)} (\alpha s - 1) \right] \quad (\text{A.127})$$

implies

$$W'(s) < 0 \Leftrightarrow (1 - \alpha s) U_1 > U_0(s), \quad (\text{A.128})$$

Moreover, since

$$\begin{aligned} & (1 - s) [U_1 - U_0(s)] > \alpha s U_0(s) \\ \Rightarrow & (1 - s) U_1 > (1 - s + \alpha s) U_0(s) \\ \Rightarrow & (1 - \alpha s) U_1 > U_0(s) \end{aligned} \quad (\text{A.129})$$

[where the last line follows from the inequality, $(1 - \alpha s)(1 - s + \alpha s) > 1 - s$], we see from (A.126) through (A.129) that $W(s) > 0 \Rightarrow W'(s) < 0$. Hence W is strictly decreasing on the interval $(0, s_0)$, and it may be concluded that for each $a > -1$ there is a unique $s_a \in (0, s_0]$ satisfying $W(s_a) = \max\{0, -a\mu/(1+a)\} \geq 0$. Finally, since the decreasing monotonicity of W also implies that (A.124) holds iff $s \in (0, s_a)$, we see that s_a satisfies (A.121). ■

Given this admissible range on s , our main result is the following:

Lemma A.2. *For each $a > -1$, the function g has a unique root in the open interval $(0, s_a)$.*

Proof. To verify the existence of such root, we first show that it suffices to establish the following four properties of the function g :

$$g(0) = 0 \quad (\text{A.130})$$

$$g'(0) < 0 \quad (\text{A.131})$$

$$\lim_{s \uparrow s_a} g(s) > 0 \quad (\text{A.132})$$

$$\{s \in (0, s_a), g'(s) = 0\} \Rightarrow g''(s) > 0 \quad (\text{A.133})$$

For it will then follow from (A.130) and (A.131) that $g(s) < 0$ near $s = 0$, and from (A.132) that $g(s) > 0$ near $s = s_a$. Hence by continuity, g must pass upward through zero at some intermediate point, $s^* \in (0, s_a)$, so that in particular, $g'(s^*) \geq 0$. To see that this root must be unique, observe first that since (A.130) and (A.131) also imply that g is negative on a maximal open interval $(0, s_m)$, it may be assumed that $s^* = s_m$, i.e., that s^* is the smallest root of g in $(0, s_a)$. Next observe that since g is continuously differentiable in $(0, s_a)$, it follows from (A.133) that

$$g'(s^*) > 0 \quad (\text{A.134})$$

[for if $g'(s^*) = 0$, then $g''(s^*) > 0$ would imply that $g'(s) < 0$ holds for every s in some open interval $(s^* - \epsilon, s^*)$, and hence that g does not pass upward through zero at s^*]. Hence if g were to have an additional root, $s_* \in (s^*, s_a)$, then (A.134) [together with the continuity of g] would imply that must g achieve a (differentiable) maximum at some interior point $\bar{s} \in (s^*, s_*)$. Finally, since the maximality conditions $g'(\bar{s}) = 0$ and $g''(\bar{s}) \leq 0$ would then contradict (A.133), it follows that no such root can exist, and hence that s^* is unique. To establish conditions (A.130) through (A.133), we proceed in order:

Proof of (A.130). Observe from (A.119) and (A.125) that

$$\lim_{s \rightarrow 0} W(s) = \infty \Rightarrow \lim_{s \rightarrow 0} u(s) = 1$$

and hence that

$$\lim_{s \rightarrow 0} g(s) = G[1, 0] = \rho \cdot 0 - (1 - \rho)(1 + a)(1 - e^0) = 0. \quad (\text{A.135})$$

Thus it follows from the continuity of g that (A.130) must hold.

Proof of (A.131). To compute the derivative

$$g'(s) = \frac{\partial G}{\partial u} \cdot u'(s) + \frac{\partial G}{\partial s} \quad (\text{A.136})$$

observe first from (A.119) and (A.127) that

$$\begin{aligned} u'(s) &= \frac{\mu W'(s)}{[\mu + W(s)]^2} \\ &= \frac{\frac{\mu}{\alpha s^2} \left[1 + \frac{U_1}{U_0(s)} (\alpha s - 1) \right]}{\left[\mu + \frac{1}{\alpha} \left(\frac{1-s}{s} \right) \left(\frac{U_1 - (1-s)^{\alpha b}}{(1-s)^{\alpha b}} \right) - 1 \right]^2} \\ &= \frac{\frac{\mu}{\alpha} \left[1 + \frac{U_1}{U_0(s)} (\alpha s - 1) \right]}{\left[\mu s + \left(\frac{1-s}{\alpha} \right) \left(\frac{U_1 - (1-s)^{\alpha b}}{(1-s)^{\alpha b}} \right) - s \right]^2} \end{aligned} \quad (\text{A.137})$$

Hence $U_0(0) = b$ implies that

$$\lim_{s \rightarrow 0} u'(s) = \frac{\frac{\mu}{\alpha} \left[1 - \frac{U_1}{b} \right]}{\left[\frac{1}{\alpha} \left(\frac{U_1 - b}{b} \right) \right]^2} = -\mu \alpha \left(\frac{b}{U_1 - b} \right) \quad (\text{A.138})$$

Also from (A.62) and (A.66) we see that

$$\frac{\partial G}{\partial u} [u(0), 0] = (1 - \rho) e^0 \left(\frac{1 + a - 0}{1 + a} \right) - 1 = -\rho \quad (\text{A.139})$$

$$\frac{\partial G}{\partial s} [u(0), 0] = -(1 - \rho) \gamma e^0 = -(1 - \rho) \gamma \quad (\text{A.140})$$

Thus, by substituting (A.138) through (A.140) into (A.136), we see that

$$g'(0) = \alpha \rho \mu \left(\frac{b}{U_1 - b} \right) - (1 - \rho) \gamma, \quad (\text{A.141})$$

which together with (A.107) implies that

$$\begin{aligned} g'(0) < 0 &\Leftrightarrow \alpha \rho \mu \left(\frac{b}{U_1 - b} \right) < (1 - \rho) \gamma \\ &\Leftrightarrow \alpha \rho \left[\left(\frac{1 - \rho}{\sigma \rho} \right) (1 - \sigma + \sigma \rho) \right] \left(\frac{b}{U_1 - b} \right) < (1 - \rho) \gamma \\ &\Leftrightarrow \frac{\alpha}{\sigma} (1 - \sigma + \sigma \rho) \left(\frac{b}{U_1 - b} \right) < \gamma \\ &\Leftrightarrow U_1 - b > \frac{\alpha}{\sigma} (1 - \sigma + \sigma \rho) b \end{aligned} \quad (\text{A.142})$$

But by the positivity condition (3.22) it follows that

$$U_1 - b > \frac{\alpha}{\sigma} \left(\frac{1 - \sigma + \sigma \rho}{1 - \rho} \right) b > \frac{\alpha}{\sigma} (1 - \sigma + \sigma \rho) b, \quad (\text{A.143})$$

and thus that (A.131) must hold.

Proof of (A.132). Suppose first that $a \geq 0$ (so that by definition $s_a = s_0$), and observe from (A.108) and (A.119) that

$$\lim_{s \uparrow s_0} W(s) = 0 \Rightarrow \lim_{s \uparrow s_0} u(s) = 0 \quad (\text{A.144})$$

Hence from (A.118) together with (A.52) we see that

$$\lim_{s \uparrow s_0} g(s) = \rho - (1 - \rho) a (1 - e^0) = \rho > 0 \quad (\text{A.145})$$

Next suppose that $a < 0$. In this case, (A.108), (A.119), and (A.123) imply that

$$\begin{aligned} \lim_{s \uparrow s_a} W(s) &= -\frac{a\mu}{1+a} \\ &\Rightarrow \lim_{s \uparrow s_a} u(s) = u(s_a) = -a \\ &\Rightarrow \lim_{s \uparrow s_a} v(s) = 0, \end{aligned} \tag{A.146}$$

which together with $1 + a > 0$, allows us to conclude that

$$\begin{aligned} \lim_{s \uparrow s_a} g(s) &= \rho(1+a) - (1-\rho)(0)(1-e^{-\infty}) \\ &= \rho(1+a) > 0 \end{aligned} \tag{A.147}$$

Hence we see that (A.132) holds in all cases, and it remains only to establish (A.133).

Proof of (A.133). To establish this key property, we first introduce the following simplifying notation. For all $s \in (0, s_a)$ let

$$\begin{aligned} L(s) &= \frac{u(s)}{u(s)+a} = \frac{W(s)}{a(\mu+W(s))+W(s)} \\ &= \frac{W(s)}{(1+a)W(s)+a\mu} \end{aligned} \tag{A.148}$$

and [recalling (A.62) and (A.66)] let

$$\begin{aligned} M(s) &\equiv \frac{\partial G}{\partial u} [u(s), s] = (1-\rho) e^{-\gamma s \frac{u(s)}{u(s)+a}} \left(\frac{u(s)+a-\gamma sa}{u(s)+a} \right) - 1 \\ &= (1-\rho) e^{-\gamma s L(s)} [1-\gamma s + \gamma s L(s)] - 1 \end{aligned} \tag{A.149}$$

$$\begin{aligned} Q(s) &\equiv \frac{\partial G}{\partial s} [u(s), s] = -(1-\rho) \gamma u(s) e^{-\gamma s \frac{u(s)}{u(s)+a}} \\ &= -(1-\rho) \gamma u(s) e^{-\gamma s L(s)} \end{aligned} \tag{A.150}$$

With this notation, it follows (by dropping functional dependencies on s) that

$$g' = M u' + Q \tag{A.151}$$

and hence that

$$g'' = M u'' + M' u' + Q'. \tag{A.152}$$

To evaluate the first term of (A.152), observe first from (A.151) together with (A.137) and (A.150) that $g' = 0$ iff

$$\begin{aligned} M &= -Q/u' \\ &= \frac{(\mu + W)^2}{\mu W'} \left[(1 - \rho) \gamma \left(\frac{W}{\mu + W} \right) e^{-\gamma s L} \right] \end{aligned} \quad (\text{A.153})$$

Next observe from (A.137) that

$$\begin{aligned} u'' &= \frac{d}{ds} \left[\frac{\mu W'}{(\mu + W)^2} \right] \\ &= \frac{\mu}{(\mu + W)^2} \left(W'' - \frac{2(W')^2}{\mu + W} \right) \end{aligned} \quad (\text{A.154})$$

and hence (after some reduction) that

$$M u'' = (1 - \rho) \gamma e^{-\gamma s L} \left(\frac{W}{\mu + W} \right) \left[\frac{W''}{W'} - \frac{2W'}{\mu + W} \right] \quad (\text{A.155})$$

To evaluate the second term of (A.152), observe from (A.149) that

$$M' = - (1 - \rho) \gamma e^{-\gamma s L} [1 - \gamma s (1 - L) (L + s L')] \quad (\text{A.156})$$

which together with (A.137) yields

$$M' u' = - (1 - \rho) \gamma e^{-\gamma s L} [1 - \gamma s L (1 - L) (L + s L')] \left\{ \frac{\mu W'}{(\mu + W)^2} \right\} \quad (\text{A.157})$$

To evaluate the last term of (A.152), observe first from (A.150) that

$$\begin{aligned} Q' &= - (1 - \rho) \gamma \frac{d}{ds} \left[\frac{W}{\mu + W} e^{-\gamma s L} \right] \\ &= - (1 - \rho) \gamma e^{-\gamma s L} \left[\frac{\mu W'}{(\mu + W)^2} - \gamma \left(\frac{W}{\mu + W} \right) (L + s L') \right], \end{aligned}$$

But since (A.148) implies

$$L' = \frac{d}{ds} \left[\frac{W}{(1 + a)W + a\mu} \right] = \frac{a\mu W'}{[(1 + a)W + a\mu]^2}, \quad (\text{A.158})$$

it follows (after some reduction) that

$$Q' = -(1 - \rho) \gamma e^{-\gamma s L} \left\{ \frac{\mu W'}{(\mu + W)^2} [1 - \gamma s L (1 - L)] - \gamma \left(\frac{W}{\mu + W} \right) L \right\} \quad (\text{A.159})$$

Hence, combining (A.155), (A.157), (A.159), factoring out the (positive) common term, $(1 - \rho) \gamma e^{-\gamma s L} / (\mu + W)$, and simplifying, it follows from (A.152) that when $g' = 0$, we have $g'' > 0$ iff

$$0 < W \left[\frac{W''}{W'} - \frac{2W'}{\mu + W} \right] + \gamma W L - \left(\frac{\mu W'}{\mu + W} \right) \{ 2 [1 - \gamma s L (1 - L)] - \gamma s^2 (1 - L) L \} \quad (\text{A.160})$$

To reduce (A.160) further, we next use (A.127) together with the identity, $U'_0/U_0 = -b/(1 - s)$, to evaluate

$$\begin{aligned} W'' &= \frac{d}{ds} \left\{ \frac{1}{\alpha s^2} \left[1 + \frac{U_1}{U_0} (\alpha s - 1) \right] \right\} \\ &= \frac{1}{\alpha s^3} \left\{ -2 + \frac{U_1}{U_0} \left[2 - \alpha s \left(1 + \frac{1 - \alpha s}{1 - s} \right) \right] \right\} \end{aligned} \quad (\text{A.161})$$

Then, after some manipulation, the ratio, W''/W' , reduces to

$$\frac{W''}{W'} = \frac{\alpha s U_1 (1 - \alpha)}{(1 - s) [(1 - \alpha s) U_1 - U_0]} - \frac{2}{s}, \quad (\text{A.162})$$

Upon substituting (A.162) into (A.160), dividing through by $2W/s > 0$, and rearranging terms, we obtain the equivalent condition

$$\begin{aligned} 1 &< \left(-\frac{W'}{W} s \right) \left(\frac{W + \mu [1 - \gamma s L (1 - L)]}{W + \mu} \right) \\ &+ \frac{\gamma s L}{2} \left\{ \left(\frac{\mu}{\mu + W} \right) \left(\frac{W'}{W} \right) \left(\frac{L'}{L} \right) s^2 (1 - L) + 1 \right\} \\ &+ \frac{1}{2} \left(\frac{s}{1 - s} \right) \left[\frac{U_1 s \alpha (1 - \alpha)}{U_1 (1 - \alpha s) - U_0} \right] \end{aligned} \quad (\text{A.163})$$

To simplify this condition further, recall from (A.129) that $(1 - \alpha s) U_1 > U_0(s)$ for all $s \in (0, s_a)$, and hence that the last term in (A.163) is always positive.

Hence it suffices to show that

$$1 < \left(-\frac{W'}{W}s\right) \left(\frac{W + \mu[1 - \gamma s L (1 - L)]}{\mu + W}\right) + \frac{\gamma s L}{2} \left\{ \left(\frac{\mu}{\mu + W}\right) \left(\frac{W'}{W}\right) \left(\frac{L'}{L}\right) s^2 (1 - L) + 1 \right\} \quad (\text{A.164})$$

The second term in (A.164) can be simplified by observing from (A.148) that

$$L' = \frac{a \mu W'}{[(1 + a)W + a \mu]^2} \Rightarrow \frac{L'}{L} = \frac{a \mu}{(1 + a)W + a \mu} \left(\frac{W'}{W}\right) \quad (\text{A.165})$$

and in addition, from (A.108) and (A.127) that

$$\begin{aligned} \frac{W'}{W} &= \frac{\frac{1}{\alpha s^2} \left[1 + \frac{U_1}{U_0} (\alpha s - 1)\right]}{\frac{1}{\alpha} \left(\frac{1-s}{s}\right) \left(\frac{U_1 - U_0}{U_0}\right) - 1} \\ &= \frac{\frac{1}{s} [U_0 + U_1 (\alpha s - 1)]}{(1 - s) (U_1 - U_0) - \alpha s U_0} \end{aligned} \quad (\text{A.166})$$

By using these results, and letting

$$R = -s \frac{W'}{W} = \frac{(1 - \alpha s) U_1 - U_0}{(1 - s) (U_1 - U_0) - \alpha s U_0}, \quad (\text{A.167})$$

we can rewrite the right hand side of (A.164) as

$$\begin{aligned} &R \left[\frac{(W + \mu) - \mu \gamma s L (1 - L)}{W + \mu} \right] + \\ &\frac{\gamma s L}{2} \left\{ \left(\frac{\mu}{\mu + W}\right) R^2 \left(\frac{a \mu}{(1 + a)W + a \mu}\right) (1 - L) + 1 \right\} \\ &= R \left[1 - s \gamma \left(\frac{\mu}{W + \mu}\right) L (1 - L) \right] + \\ &\frac{\gamma s L}{2} \left(\frac{\mu}{\mu + W}\right) R^2 \left(\frac{a \mu}{(1 + a)W + a \mu}\right) (1 - L) + \frac{\gamma s L}{2} \end{aligned} \quad (\text{A.168})$$

so that condition (A.164) becomes

$$1 < R \left[1 - s \gamma \left(\frac{\mu}{\mu + W} \right) L (1 - L) \right] + \frac{\gamma s L}{2} \left(\frac{\mu}{\mu + W} \right) R^2 \left[\frac{a \mu}{(1 + a) W + a \mu} \right] (1 - L) + \frac{\gamma s L}{2} \quad (\text{A.169})$$

We next show that $R(s) > 1$ for all $s \in (0, s_a)$. To do so, recall from (A.129) that $(1 - \alpha s) U_1 - U_0 > 0$, and hence [again from the inequality $(1 - \alpha s) (1 - s + \alpha s) > 1 - s$ and positivity of $(1 - s + \alpha s)$] that

$$\begin{aligned} (1 - \alpha s) U_1 - U_0 &> (1 - s + \alpha s) [(1 - \alpha s) U_1 - U_0] \\ &> (1 - s) U_1 - (1 - s + \alpha s) U_0 \\ &= (1 - s) (U_1 - U_0) - \alpha s U_0 \end{aligned} \quad (\text{A.170})$$

Thus by (A.167) we see that $R > 1$ on $(0, s_a)$. To use this result, we next show that both the terms in on the right hand side of (A.169) involving R are always nonnegative. To establish nonnegativity of the first term, observe first that if $a \geq 0$, then by (A.148) it follows that $L(s) \in [0, 1]$, and hence that $L(1 - L) \in [0, 1/4]$. This together with $s \gamma [\mu / (\mu + W)] \in (0, 1)$ implies that the bracketed expression in the first term is positive. Next, if $a < 0$, observe again from (A.148) that $L(s) > 1$, and hence that $L(1 - L) < 0$, so that the bracketed expression is again positive. Hence $R > 1$ implies that the first term is in fact positive. Turning to the second term, we need only consider the product, $A \cdot (1 - L)$, with $A = a \mu / [(1 + a) W + a \mu]$. Again if $a \geq 0$, then both A and $(1 - L)$ are nonnegative, so that their product is also. Finally, if $a < 0$, then since the definition of s_a implies from (A.124) that $W(s) > -a \mu (1 - a)$ and hence that $(1 + a) W + a \mu > 0$, it now follows that both A and $(1 - L)$ are negative, and again have a nonnegative product. Given these nonnegativity properties, it thus suffices to establish (A.169) with R replaced by one, which (after regrouping terms) is equivalent to showing that

$$1 < 1 - s \gamma \left(\frac{\mu}{\mu + W} \right) L (1 - L) \left[1 - \frac{1}{2} \left(\frac{a \mu}{(1 + a) W + a \mu} \right) \right] + \frac{\gamma s L}{2} \quad (\text{A.171})$$

By subtracting one, then dividing by $-(\gamma s L)$, and using (A.148), this reduces to:

$$\begin{aligned} \frac{1}{2} &> \left(\frac{\mu}{\mu + W} \right) (1 - L) \left[1 - \frac{1}{2} \left(\frac{a \mu}{(1 + a) W + a \mu} \right) \right] \\ &= \left(\frac{a \mu}{(1 + a) W + a \mu} \right) \left[1 - \frac{1}{2} \left(\frac{a \mu}{(1 + a) W + a \mu} \right) \right] \end{aligned} \quad (\text{A.172})$$

Finally, letting $x = a \mu / [(1 + a) W + a \mu] < 1$, and observing that

$$\begin{aligned} (1 - x)^2 &= 1 - 2x + x^2 > 0 \\ &\Rightarrow 1 > 2x - x^2 \\ &\Rightarrow \frac{1}{2} > x \left(1 - \frac{1}{2}x \right) \end{aligned} \quad (\text{A.173})$$

we may conclude that (A.172) holds, and hence that the result is established. ■

A.10. Monotonicity of Search Intensity in Wages

To show that $s'(w) > 0$ for all wage levels, w , we first write g explicitly as a function of w , i.e., $g(s, w)$, and observe from the definition of $s(w)$ that

$$0 \equiv g[s(w), w] = G\{u[s(w), w], s(w)\} \quad (\text{A.174})$$

Hence we must have

$$\begin{aligned} 0 &\equiv \frac{d}{dw} g[s(w), w] = \frac{d}{dw} G\{u[s(w), w], s(w)\} \\ &= \frac{\partial G}{\partial u} \left\{ \frac{\partial u}{\partial s} s'(w) + \frac{\partial u}{\partial w} \right\} + \frac{\partial G}{\partial s} s'(w) \\ &= s'(w) \left[\frac{\partial G}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial G}{\partial s} \right]_{s(w)} + \left[\frac{\partial G}{\partial u} \frac{\partial u}{\partial w} \right]_{s(w)} \end{aligned} \quad (\text{A.175})$$

and may write $s'(w)$ as

$$s'(w) = - \frac{\left[\frac{\partial G}{\partial u} \frac{\partial u}{\partial w} \right]_{s(w)}}{\left[\frac{\partial G}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial G}{\partial s} \right]_{s(w)}} \quad (\text{A.176})$$

To establish positivity of (A.176) we first recall from (A.137) together with the proof of Lemma A.1 that

$$\begin{aligned} W[s(w)] > 0 &\Rightarrow W'[s(w)] > 0 \\ &\Rightarrow \left[\frac{\partial u}{\partial w} \right]_{s(w)} > 0 \end{aligned} \tag{A.177}$$

Moreover, since the proof of Theorem 2.2 showed that $u[s(w), w] \in (u_a, 1)$, it also follows from (A.60) that $[\partial G / \partial u]_{s(w)} < 0$, and hence from (A.177) that the numerator of (A.176) is negative. Thus it remains to show that the denominator of (A.177) is positive. But by setting $s^* = s(w)$ in proof of Lemma A.1, we see from (A.134) that

$$0 < g'(s^*) = g'[s(w)] = \left[\frac{\partial G}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial G}{\partial s} \right]_{s(w)} \tag{A.178}$$

and hence that $s'(w) > 0$.