The Equity Premium Implied by Production

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Abstract

This paper studies the determinants of the equity premium as implied by producers' first-order conditions. A closed form expression is presented for the Sharpe ratio at steady-state as a function of investment volatility and adjustment cost curvature. Calibrated to the U.S. postwar economy, the model can generate a sizeable equity premium, with reasonable volatility for market returns and risk free rates. The market's Sharpe ratio and the market price of risk are very volatile. Contrary to most models, the model generates a negative correlation between conditional means and standard deviations of aggregate excess returns.

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In the twenty years since Mehra and Prescott's paper on the equity premium puzzle many studies have proposed and evaluated utility functions for their ability to explain the most salient aggregate asset pricing facts. Several specifications have demonstrated their ability to improve considerably over a basic time-separable constant relative risk aversion setup. Despite the progress, however, it seems that we have not yet reached the state where there would be a widely accepted replacement for the standard time-separable utility specification.

Contrary to the consumption side, the production side of asset pricing has received considerably less attention. Focusing on the production side shifts the burden towards representing production technologies and interpreting production data. While a number of asset pricing studies have considered nontrivial production sectors, these have generally been studied jointly with some specific preference specification. Thus, the analysis could not escape the constraints imposed by

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the preference side. A *pure* production asset pricing literature has emerged from the Q-theory of investment. However, typically, these studies consider a limited set of implications for the links between investment and stock returns, but not the equity premium.[1]

In this paper I am interested in studying the macroeconomic determinants of asset prices given by a multi-input aggregate production technology. The focus is exclusively on the producers' first-order conditions that link production variables and state prices, with sectorial investment playing the key role. Two sets of questions are considered. First, what properties of investment and production technologies are important for the first and second moments of risk free rates and aggregate equity returns? Second, does a model plausibly calibrated to the U.S. economy have the ability to replicate first and second moments of risk free rates and aggregate equity returns?

The work most closely related to mine are Cochrane's contributions on production-based asset pricing (1988, 1991). Some of the features that differentiate my work are that I focus explicitly on the equity premium, use more general functional forms for adjustment cost, and base the empirical evaluation on the two main sectorial aggregates of U.S. capital investment, namely equipment & software as well as structures. Cochrane (1993) derives a set of asset pricing implications of a production function where the productivity level can be selected in a state-contingent way.

My model pictures the problem of a representative producer that selects multiple fixed input factors. In order to be able to recover the state-price process, the setup needs to have two related properties. First, markets need to be complete and the producer has to face a full set of state-prices. Second, there needs to be as many predetermined factors of production as there are states of nature. This assumption of "complete technologies" is necessary in order to be able to recover the full set of state-contingent prices from the production side. In most studies with nontrivial production sectors this property is not satisfied; of course, in a general equilibrium environment it doesn't usually play such an important role.

The model is calibrated to a two sector representation. I use U.S. data on investment for equipment & software, as well as for structures. These two types of investments sum to the total of U.S. fixed business investment. With this representation the two sectors also have some natural asymmetries. As becomes clear below, asymmetries across sectors play an important role in the analysis.

[1] An incomplete list of contributions include: for successful utility functions, Abel (1990), Campbell and Cochrane (1999), Constantinides (1990); for models with nontrivial production sectors Jermann (1998) and Rouwenhorst (1995); for production asset pricing studies, Cochrane (1988, 1991), Li, Vassalou and Xing (2003), and Gomes, Yaron and Zhang (2002). Other examples of related asset pricing studies with rich production structures are Berk, Green and Naik (1999), Carlson, Fisher and Giammarino (2003), Hugonnier, Morellec and Sundaresan (2005), and Tuzel (2004).

One of the paper's main contributions is to characterize the determinants of the equity premium. Specifically, a closed form expression is presented for the Sharpe ratio at steady-state as a function of investment volatility and adjustment cost curvature. The key quantitative findings are the following. For unconditional moments, the model can plausibly generate an equity premium of several percentage points with risk free rates having a reasonable mean and volatility. For conditional moments, the expected excess equity return is quite volatile—more volatile than the risk free rate. Also, concerning aggregate excess returns, the correlation between conditional means and volatilities is negative.

The paper is organized as follows. Section 1 presents the model and section 2 some general asset pricing implications. Section 3 introduces functional forms. Section 4 characterizes theoretical links between asset prices and investment. Section 5 contains the calibration and section 6 the quantitative analysis.

1 Model

The model represents the producer's choice of capital inputs for a given state price process. Key ingredients are capital adjustment cost and stochastic productivity.

Assume an environment where uncertainty is modelled as the realization of s, one out of a finite set $S = (\mathfrak{s}_1, \mathfrak{s}_2, ...\mathfrak{s}_N)$, with s_t the current period realization and $s^t \equiv (s_0, s_1, ...s_t)$ the history up to and including t. Probabilities of s^t are denoted by $\pi\left(s^t\right)$. Assume an aggregate production function

$$Y\left(s^t\right) = F\left(\left\{K_j\left(s^{t-1}\right)\right\}_{j\in J}, s^t, N\left(s^t\right)\right),$$

where s^t allows for a technology shock, K_j is the j-th capital stock, and N labor. Note that in the analysis of the model, labor will not play an active role. Capital accumulation for capital good of type j is represented by

$$K_j\left(s^t\right) = K_j\left(s^{t-1}\right)(1-\delta_j) + Z_j\left(s^t\right) I_j\left(s^t\right),$$

where δ_j is the depreciation rate and $Z_j\left(s^t\right)$ the technology for producing capital goods. Assume $Z_j\left(s^t\right) = Z_j\left(s^{t-1}\right)\cdot\lambda^{Z_j}(s_t)$, with $\lambda^{Z_j}(s_t)$ following a N-state Markov process. The total cost of investing in capital good of type j is given by

$$H_j\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right).$$

This specification will be further specialized below.

Taking as given state prices $P(s^t)$, the representative firm solves the following problem

$$\max_{\{I,K',N\}} \sum_{t=0}^{\infty} \sum_{s^t} P(s^t) \left[\begin{array}{c} F\left(\{K_j(s^{t-1})\}_{j\in J}, s^t, N(s^t) \right) - w(s^t) N(s^t) \\ -\sum_j H_j\left(K_j(s^{t-1}), I_j(s^t), Z_j(s^t)\right) \end{array} \right]$$

$$\text{s.t. } \left[P(s^t) q_j(s^t)\right] : K_j(s^{t-1})(1-\delta_j) + Z_j(s^t) I_j(s^t) - K_j(s^t) \geq 0, \ \forall s^t, j$$

with s_0 and $K_j(s_{-1})$ given, and $P(s_0) = 1$ without loss of generality.

The scaling of the multipliers is chosen so as to generate intuitive labels. Indeed, q represents the marginal value of one unit of installed capital in terms of the numeraire of the same period. In equilibrium, it is also the cost of installing one unit of capital including adjustment cost. Given the homogeneity assumptions made below qZ is the ratio of the market value over the book value of capital, that is, Tobin's Q. Indeed, $1/Z$ is equal to the price of a unit of capital in terms of the final good. The book value (or replacement cost) of the capital stock is then K/Z. The introduction of the investment specific technology Z allows the model to capture the historical downward trend observed in U.S. equipment prices.

First-order conditions are summarized by

$$0 = -H_{j,2}\left(K_j(s^{t-1}), I_j(s^t), Z_j(s^t)\right) + Z_j(s^t) q_j(s^t),$$

$$q_j(s^t) = \sum_{s_{t+1}} \frac{P(s^t, s_{t+1})}{P(s^t)} \left[\begin{array}{c} F_{K_j}\left(\{K_i(s^t)\}_{i\in J}, s^t, s_{t+1}, N(s^t, s_{t+1}) \right) \\ -H_{j,1}\left(K_j(s^t), I_j(s^t, s_{t+1}), Z_j(s^t, s_{t+1})\right) + (1-\delta_j) q_j(s^t, s_{t+1}) \end{array} \right],$$

and

$$F_N\left(\{K_j(s^{t-1})\}, s^t, N(s^t) \right) - w(s^t) = 0.$$

Substituting out shadow prices, we have

$$\sum_{s_{t+1}} P(s_{t+1}|s^t) \left[\begin{array}{c} F_{K_j}\left(\{K_j(s^t)\}, s^t, s_{t+1}, N(s^t, s_{t+1}) \right) \\ -H_{j,1}\left(K_j(s^t), I_j(s^t, s_{t+1}), Z_j(s^t, s_{t+1})\right) \\ +(1-\delta_j) \frac{H_{j,2}\left(K_j(s^t), I_j(s^t, s_{t+1}), Z_j(s^t, s_{t+1})\right)}{Z_j(s^t, s_{t+1})} \end{array} \right] \left(\frac{Z_j(s^t)}{H_{j,2}\left(K_j(s^{t-1}), I_j(s^t), Z_j(s^t)\right)} \right) = 1$$

for each j, where the notation $P(s_{t+1}|s^t)$ shows the price of the numeraire in s_{t+1} conditional on s^t and in units of the numeraire at s^t. From this condition, define the investment return $R_j^I(s^t, s_{t+1})$ implicitly through $\sum_{s_{t+1}} P(s_{t+1}|s^t) R_j^I(s^t, s_{t+1}) = 1$. $R_j^I(s^t, s_{t+1})$ is the return realized in s_{t+1} from adding one (marginal) unit of capital of type j in state s^t. The first-order condition shows that in equilibrium adding one marginal unit of a given type of capital produces a change in the profit plan that is worth one unit.[2]

[2]Based on the details given below, strict concavity will be assumed, so that first-order and transversality conditions (given below) are sufficient for a maximum.

2 From investment returns to state prices and asset returns

In order to recover state prices uniquely from the producers first-order conditions it is necessary to have as many types of capital inputs as there are states of nature. This "complete technologies" requirement represents the producers ability to move resources across all states of nature. Representing the first-order conditions in matrix form yields for the case with two states of nature and two capital inputs

$$\begin{bmatrix} R_1^I\left(s^t,\mathfrak{s}_1\right) & R_1^I\left(s^t,\mathfrak{s}_2\right) \\ R_2^I\left(s^t,\mathfrak{s}_1\right) & R_2^I\left(s^t,\mathfrak{s}_2\right) \end{bmatrix} \begin{bmatrix} P\left(\mathfrak{s}_1|s^t\right) \\ P\left(\mathfrak{s}_2|s^t\right) \end{bmatrix} = \mathbf{1}, \tag{1}$$

or more compactly $R^I\left(s^t\right)\cdot p\left(s^t\right) = \mathbf{1}$. The state price vector is obtained by the matrix inversion

$$p\left(s^t\right) = \left(R^I\left(s^t\right)\right)^{-1}\mathbf{1}.$$

Clearly, it isn't necessarily the case that this matrix inversion is feasible nor that state prices are necessarily positive for any chosen set of returns. As further discussed below, the requirement for positive state prices will constrain the empirical implementation.

In this environment, the risk free return is given by

$$1/R^f\left(s^t\right) = \mathbf{1}p\left(s^t\right) = P\left(\mathfrak{s}_1|s^t\right) + P\left(\mathfrak{s}_2|s^t\right).$$

Starting with equation 1, it is easy to check that if one of the investment returns is not state-contingent, that is $R_j^I\left(s^t,s_{t+1}\right) = R_j^I\left(s^t\right)$, then, as is implied by no-arbitrage, it equals the risk free rate, $R_j^I\left(s^t\right) = R^f\left(s^t\right)$.

Consider aggregate capital returns

$$R\left(s^t,s_{t+1}\right) \equiv \frac{D\left(s^t,s_{t+1}\right) + V\left(s^t,s_{t+1}\right)}{V\left(s^t\right)},$$

where $D\left(s^t,s_{t+1}\right) = F\left(\left\{K_j\left(s^{t-1}\right)\right\},s^t,N\left(s^t\right)\right) - w\left(s^t\right)N\left(s^t\right) - \sum_j H_j\left(K_j\left(s^{t-1}\right),I_j\left(s^t\right),Z_j\left(s^t\right)\right)$ represents the dividends paid by the firm and $V\left(s^t,s_{t+1}\right)$ the ex-dividend value of the firm. Assuming constant returns to scale in $F(.)$ and $H_j(.)$, Hayashi's (1982) result applies, and this return will be equal to a weighted average of the investment returns:

$$R\left(s^t,s_{t+1}\right) = \sum_j \frac{q_j\left(s^t\right)K_j\left(s^t\right)}{\sum_i q_i\left(s^t\right)K_i\left(s^t\right)} \cdot R_j^I\left(s^t,s_{t+1}\right). \tag{2}$$

The market price of risk, aka the highest Sharpe ratio, also has a simple expression. Let us, introduce the stochastic discount factor $m\left(s_{t+1}|s^t\right)$ by dividing and multiplying through by the probabilities $\pi\left(s_{t+1}|s^t\right)$, so that

$$P\left(s_{t+1}|s^t\right) = \left(\frac{P\left(s_{t+1}|s^t\right)}{\pi\left(s_{t+1}|s^t\right)}\right)\pi\left(s_{t+1}|s^t\right) = m\left(s_{t+1}|s^t\right)\pi\left(s_{t+1}|s^t\right).$$

Ruling out arbitrage implies $E_t\left(m\left(s_{t+1}|s^t\right)R^e\left(s^t,s_{t+1}\right)\right) = 0$, for $\forall R^e\left(s^t,s_{t+1}\right)$ defined as an excess return. It can then be shown that

$$\max\frac{E\left[R^e\left(s^t,s_{t+1}\right)|s^t\right]}{Std\left[R^e\left(s^t,s_{t+1}\right)|s^t\right]} = \sqrt{\frac{\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)^2/\pi\left(s_{t+1}|s^t\right)}{\left[\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)\right]^2}-1}.$$

3 Functional Forms

This section presents the functional forms and the simulation strategies.

3.1 Investment cost function

The investment cost function plays a crucial role in the analysis. Its form is chosen to satisfy two criteria. First, I require investment returns to be stationary. This is achieved through a particular type of homogeneity. Second, I want the curvature of the cost function to be slightly more general than the standard quadratic specification.

A simple functional form that satisfies these criteria is

$$H\left(K,I,Z\right)=\left\{\frac{b}{\nu}\left(ZI/K\right)^{\nu}+c\right\}\left(K/Z\right),$$

with $b,c>0$, $\nu>1$. For each capital stock, different parameter values will be allowed. For compactness, the notation doesn't express that. As can easily be seen, this function is convex in I for $\nu>1$. Adjustment cost and the direct cost for additional capital goods are separable, trivially so because $H\left(K,I,Z\right)=\left[H\left(1,ZI/K\right)-ZI/K+ZI/K\right]\cdot\left(K/Z\right)=\left[H\left(1,ZI/K\right)-ZI/K\right]\cdot\left(K/Z\right)+I\equiv C\left(1,ZI/K\right)\cdot\left(K/Z\right)+I$. I impose restrictions on the parameters of $H\left(.\right)$ so that $C\left(1,ZI/K\right)\geq 0$, that is, the pure adjustment cost is nonnegative.

The cost function is homogenous of degree 1 in I and K/Z. This is required for balanced growth. Indeed, given the capital accumulation equation, IZ and K are cointegrated, and so are I and K/Z. With this homogeneity assumption, the investment cost $H(.)$ will share the same trend as I and K/Z. As further discussed below, some balanced growth requirement will contribute to making investment returns stationary.

For a given investment process, the curvature parameter ν determines the volatility of the market price of capital. This parameter will be a crucial contributor to return volatility and risk premiums. From the first-order conditions the following relationship between the investment rate and the marginal cost of capital is obtained

$$qZ=b\left(IZ/K\right)^{\nu-1}.$$

The elasticity of qZ with respect to IZ/K is

$$\frac{\partial qZ}{\partial(IZ/K)}\frac{IZ/K}{qZ}=\nu-1.$$

In addition to having the degree of freedom to chose this curvature parameter, the ability to have different curvatures across sectors will be important.

The parameters b and c are less important for asset pricing implications. They provide the flexibility to center the adjustment cost function and to minimize the amount of resources lost due to adjustment cost. It is easy to see that by setting $\nu=b=1$, and $c=0$, the case without adjustment cost is obtained

$$H(K,I,Z)=I.$$

3.2 Production function

I chose a production function that is consistent with stationary investment returns and that is easily tractable. Specifically, output, after payments to labor, is a linearly separable function of the capital stocks

$$F\left(\left\{K_j\left(s^t\right)\right\}_{j\in J},s^t,s_{t+1},N\left(s^{t+1}\right)\right)-w\left(s^{t+1}\right)N\left(s^{t+1}\right)=\sum_j\frac{A_j\left(s_{t+1}\right)}{Z_j\left(s^t\right)}K_j\left(s^t\right).$$

Marginal products of capital are then

$$F_{K_j}\left(\left\{K_j\left(s^t\right)\right\}_{j\in J},s^t,s_{t+1},N\left(s^t,s_{t+1}\right)\right)=\frac{A_j\left(s_{t+1}\right)}{Z_j\left(s^t\right)}.$$

The term Z_j is introduced to guarantee stationary returns. It implies, for instance, that as a given capital gets cheaper to produce, that is as Z increases, it also becomes less productive. This is related to one of the properties implied by Greenwood, Hercowitz and Krusell's (1997) balanced growth path. $A_j(s_{t+1})$ can be thought of as a productivity shock.

3.3 Simulation strategy and stationarity of returns

The spirit of the quantitative analysis is to assume that the optimal investment process is known. The implied investment returns and state prices can then easily be derived. As mentioned above, I want returns to be stationary. This also imposes additional restrictions on technologies and the assumed optimal investment process. These issues are discussed here in detail.

Assume that sectorial investment growth rates follow finite element Markov chains, that is, $I_j\left(s^t,s_{t+1}\right)=I_j\left(s^t\right)\lambda^{I_j}\left(s_{t+1}\right)$. Under the assumed functional forms, investment returns can then be written as

$$R_j^I\left(s^t,s_{t+1}\right) = \frac{A_{j,t+1}}{b\left(Z_{jt}I_{j,t}/K_{j,t}\right)^{\nu-1}} + \left(1/\lambda_{t+1}^{Z_j}\right)\cdot\frac{b\left(1-\frac{1}{\nu}\right)\left(Z_{j+1t}I_{jt+1}/K_{jt+1}\right)^{\nu}-c}{b\left(Z_{jt}I_{j,t}/K_{j,t}\right)^{\nu-1}} + \left(1/\lambda_{t+1}^{Z_j}\right)\cdot\left(1-\delta_j\right)\cdot\frac{b\left(Z_{jt+1}I_{j,t+1}/K_{j,t+1}\right)^{\nu-1}}{b\left(Z_{jt}I_{j,t}/K_{j,t}\right)^{\nu-1}}, \tag{3}$$

with,

$$Z_{jt+1}I_{j,t+1}/K_{j,t+1} = \left(Z_{jt}I_{jt}/K_{j,t+1}\right)\lambda_{t+1}^{I_j}\lambda_{t+1}^{Z_j},$$

where for compactness the state-dependence is not explicit.

Inspection of equation (3) reveals that given the various assumptions made on the exogenous processes and functional forms, investment returns are stationary. However, stationarity of sectorial investment returns is not sufficient for the stationarity of aggregate asset returns. Indeed, as shown in equation (2), the aggregate return equals a weighted average of the sectorial returns. For stationarity, the weights need to be stationary too. Aggregate returns are given by

$$R\left(s^t,s_{t+1}\right) = \sum_j \frac{\frac{b(Z_{jt}I_{j,t}/K_{j,t})^{\nu-1}}{Z_{j,t}}K_{j,t+1}}{\sum_i \frac{b(Z_{it}I_{i,t}/K_{i,t})^{\nu-1}}{Z_{i,t}}K_{i,t+1}} R_j^I\left(s^t,s_{t+1}\right).$$

A sufficient (and necessary) condition for stationarity, given the previous assumptions, is that $K_{1,t+1}/Z_{1,t}$ and $K_{2,t+1}/Z_{2,t}$ are cointegrated. Given that the investment capital ratios $Z_{jt}I_{j,t}/K_{j,t}$ are stationary, this is equivalent to $I_{1,t}$ and $I_{2,t}$ being cointegrated. Setting investment expenditure growth rates equal across sectors, that is $\lambda^{I1}\left(s_{t+1}\right) = \lambda^{I2}\left(s_{t+1}\right)$, guarantees that $I_{1,t}$ and $I_{2,t}$ are cointegrated. Thus, because individual quantities have stochastic trends, I end up choosing identical investment expenditure growth realizations across sectors to guarantee stationarity of aggregate equity returns. However, I am free to choose the realizations for λ_t^{Z1} and λ_t^{Z2} independently. This is less restrictive than it might appear for several reasons. As seen above, what matters for the investment returns is the behavior of the product $\lambda_t^{I1}\lambda_t^{Z1}$, and not λ_t^{I1} individually. That is to say that the important element in the calibration is to fit the process of real investment growth rather than the growth in investment expenditure. Moreover, for the considered empirical counterparts, as shown below, the historical volatilities of $\lambda^{I1}$and λ^{I2} are nearly identical, and realizations of the two growth rates are strongly positively correlated. Alternatively, one could introduce additional components for each process that have purely transitory effects and would thus not need to be restricted to ensure balanced growth. However, given the requirement to keep the number of states small, the additional flexibility introduced in this way would be rather limited.

To summarize the dynamic structure, realized investment returns are given as functions of the following

$$R_j^I\left(s^t, s_{t+1}\right) = R_j^I\left(Z_j\left(s^t\right) I_j\left(s^t\right) / K_j\left(s^{t-1}\right); \lambda^I\left(s_{t+1}\right), \lambda^{Z_j}\left(s_{t+1}\right), A_j\left(s_{t+1}\right)\right) \text{ for } j = 1, 2.$$

For the simulations, I can generate realizations of all quantities of interest based on a probability matrix describing the law of motion for the exogenous state s_{t+1}. The law of motion for the rest of the variables follows as

$$\begin{aligned} I_j\left(s^t, s_{t+1}\right) &= I_j\left(s^t\right) \lambda^I\left(s_{t+1}\right) \text{ for } j = 1, 2, \\ Z_j\left(s^t, s_{t+1}\right) &= Z_j\left(s^t\right) \lambda^{Z_j}\left(s_{t+1}\right) \text{ for } j = 1, 2, \text{ and} \\ K_j\left(s^t\right) &= K_j\left(s^{t-1}\right)\left(1 - \delta_j\right) + Z_j\left(s^t\right) I_j\left(s^t\right) \text{ for } j = 1, 2. \end{aligned}$$

Seven variables are a sufficient statistic for the current state of the economy s^t, namely s_t, $K_1\left(s^{t-1}\right)$, $K_2\left(s^{t-1}\right)$, $I_1\left(s^t\right)$, $I_2\left(s^t\right)$, $Z_1\left(s^t\right)$, $Z_2\left(s^t\right)$. Clearly $K_j\left(s^t\right)$ matters too, but it is a function of the state variables. The probability distribution of the shocks is summarized by s_t, the realization of the return does not depend on s_t. As initial conditions, I set $K_2\left(s^{-1}\right) = Z_1\left(s^0\right) = Z_2\left(s^0\right) = 1$, and $K_1\left(s^{-1}\right)$ is set equal to the historical average of the ratio of the value of capital in this sector relative to the other sector. Initial investment levels are assumed at their implied steady state values.

4 Theoretical analysis

This section contains a series of theoretical results that explain key model mechanisms. The first issue concerns the determinants of the equity premium. I present a simple expression for the Sharpe ratio that depends only on the investment cost curvature and the investment volatility. Second, the issue of what constitutes an admissible investment process for a given technology specification is examined. One finding is that sectorial differences in the adjustment cost parameters ν_j are crucial for generating admissible state prices from investment. Two additional results are included. First, an upper bound for the Sharpe ratio is derived. This will illustrate some of the quantitative findings. Finally, it is shown that in a model without technology shocks interest rates cannot be constant if one is interested in recovering state prices from producers first-order conditions.

4.1 What determines the equity premium?

I consider here the relationship between investment and asset prices, and in particular the equity premium. For this analysis, a continuous-time representation is more convenient than the discrete-time model used sofar. The analysis proceeds in two steps. First, I show that in order to have a positive equity premium, the investment return that is expected to be higher needs to be the more volatile. Second, I show that under some conditions, asymmetries in the investment cost curvature ν can generate this property, and a simple expression for the Sharpe ratio is presented.

As a counterpart to the two-state representation in discrete time, consider a one-dimensional Brownian motion. Investment returns are given by

$$\frac{dR_j}{R_j} = \mu_j\left(.\right)dt + \sigma_j\left(.\right)dz, \text{ for } j = 1, 2, \tag{4}$$

and the state-price process also has this form

$$\frac{d\Lambda}{\Lambda} = -r^f\left(.\right)dt + \sigma\left(.\right)dz. \tag{5}$$

Assume that the two returns are positively (perfectly) correlated so that $sign\left(\sigma_1\right) = sign\left(\sigma_2\right)$. The drift and diffusion coefficients are allowed to change with the state of the economy. For compactness, from now on, the notation will not explicitly acknowledge this.

The objective is to derive the drift and diffusion terms of the state-price process, $-r^f$ and σ, from the given return processes, that is from the four values μ_j and σ_j for $j = 1, 2$. Remember, in this environment, the absence of arbitrage implies that

$$0 = E_t\left(\frac{d\Lambda_t}{\Lambda_t}\right) + E_t\left(\frac{dR_{jt}}{R_{jt}}\right) + E_t\left(\frac{d\Lambda_t}{\Lambda_t}\frac{dR_{jt}}{R_{jt}}\right), \tag{6}$$

so that

$$0 = -r^f dt + \mu_i dt + \sigma_i \sigma dt,$$

and thus there are 2 equations and 2 unknowns. The solution of this system is

$$\begin{aligned} r^f &= \frac{\sigma_2\mu_1 - \sigma_1\mu_2}{\sigma_2 - \sigma_1} \\ -\sigma &= \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}. \end{aligned}$$

Clearly, in order to be able to recover the state-price process from the two returns, the two volatility terms have to be different across sectors, that is $\sigma_2 - \sigma_1 \neq 0$. This is an invertibility requirement similar to the one for the discrete time case. However, there is no issue here about possibly negative state prices. Indeed, a process such as (5) cannot become negative if it is initially positive.

From the pricing equation (6), the volatility term equals the Sharpe ratio

$$-\sigma = \frac{\mu_1 - r^f}{\sigma_1} = \frac{\mu_2 - r^f}{\sigma_2},$$

and using the solutions derived above

$$\mu_1 - r^f = -\sigma\sigma_1 = \sigma_1\left[\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right]. \tag{7}$$

Clearly, with positively correlated returns, that is $sign\left(\sigma_1\right) = sign\left(\sigma_2\right)$, the signs of both sectorial risk premiums are identical, and thus the sign of the aggregate equity premium, a weighted average of the sectorial premiums, will be the same as for the two sectorial premiums. From equation (7) it is easy to see that there is a positive equity premium in the aggregate if, and only if, the return with the higher risk premium is more volatile.[3]

Let us now make the link to the production side of the model. I consider a model without technology shocks, where the only source of uncertainty are the state prices. Technology shocks could be added for this analysis, but given their relatively minor quantitative impact, as shown later in the paper, keeping the expressions simple seems preferable. As shown in the appendix, the realized return to a given capital stock equals

$$\left\{\frac{A-c}{b\left(\frac{I_t}{K_t}\right)^{\nu-1}} + \left(1-\frac{1}{\nu}\right) I_t/K_t - \delta + (\nu-1)\left[\left(\lambda^I - 1\right) - \left(I_t/K_t - \delta\right) + \frac{1}{2}(\nu-2)\sigma_I^2\right]\right\} dt + (\nu-1)\sigma_I dz, \tag{8}$$

where $\left(\lambda^I - 1\right)$ and σ_I are drift and diffusion terms of investment. Consider this return when $\left(\lambda^I - 1\right) - \left(I_t/K_t - \delta\right) = 0$. This holds at the deterministic steady state for given $\left(\lambda^I - 1\right)$ and δ, assuming $\left(\lambda^I - 1\right) + \delta > 0$. The return then simplifies to

$$\left\{\left(\bar{R}-1\right) + \frac{1}{2}(\nu-1)(\nu-2)\sigma_I^2\right\} dt + (\nu-1)\sigma_I dz. \tag{9}$$

Where $\bar{R}$ is the return in a deterministic model at the steady state with the same technology parameters and where investment growth equals λ^I.[4] As discussed further below, it is convenient to calibrate the model by picking a value for $\bar{R}$, the steady-state return, independently from other parameters, which implicitly sets A at a given level. Focusing on the return at this steady state point should be informative about average model behavior. There is an example at the end of the quantitative analysis that confirms this.

Consider now how ν and δ contribute to the sign and magnitude of the equity premium, given that these are the main asymmetries between equipment and structures that are considered in the quantitative analysis. As is clear from equation (9), for a given $\bar{R}$, there is no separate role for depreciation rates at steady state. Let's normalize $\nu_2 > \nu_1$. Because $(\nu - 1)$ multiplies $\sigma_I dz$, given the normalization, the volatility term of sector 2 is larger, that is $\sigma_2 > \sigma_1$, having assumed equal investment volatility σ_I across sectors.[5] Whether this asymmetry can generate a positive

[3] Indeed, if $\sigma_1, \sigma_2 > 0$, this implies that if $\mu_2 - \mu_1 > 0$, one needs $\sigma_2 - \sigma_1 > 0$, and it can be seen that $\mu_1 - r^f > 0$. Alternatively, if $\sigma_1, \sigma_2 < 0$, this condition implies that if $\mu_2 - \mu_1 > 0$ one needs $\sigma_2 - \sigma_1 < 0$, (sector 2 is more volatile), and then again $\mu_1 - r^f > 0$.

[4] $\bar{R} = \frac{A-c}{b\left(\lambda^I - (1-\delta)\right)^{v-1}} + \left(1 - \frac{1}{v}\right)\lambda^I + \frac{1}{v}(1-\delta)$.

[5] As will be shown in the quantitative analysis, historical investment growth volatilities in the two sectors are roughly identical.

equity premium depends then on the effect of ν on the drift. It is easy to see that

$$\frac{\partial(\nu-1)(\nu-2)}{\partial\nu}=2\left(\nu-\frac{3}{2}\right)\rightarrow \text{ if } \nu>\frac{3}{2} \text{ then} \frac{\partial(\nu-1)(\nu-2)}{\partial\nu}>0.$$

That is, starting from a common curvature parameter $\nu > \frac{3}{2}$, and increasing the curvature in one sector, the sector with the higher curvature will have a higher drift, everything else equal. Thus, if $\nu > 1.5$, and if there are no sectorial asymmetries except the difference in ν, $\sigma_2 > \sigma_1$and $\mu_2 > \mu_1$, that is, as shown in equation 7 above, the equity premium is positive.

Some algebra shows that the Sharpe ratio at steady state, again assuming $\sigma_{Ij} = \sigma_I$, is given as

$$\frac{\mu_j-r^f}{\sigma_j}|_{ss}=\frac{\bar{R}_2-\bar{R}_1}{(v_2-v_1)\,\sigma_I}+\frac{\nu_1+\nu_2-3}{2}\sigma_I. \tag{10}$$

The first term shows how a difference in the deterministic returns $\bar{R}_j$ contributes to an increase in the Sharpe ratio if the higher deterministic return corresponds to the more volatile return. Unfortunately, there seems to be little direct empirical evidence about the levels of $\bar{R}_j$, in particular about the exact level of the marginal product terms A_j. For this reason I will later focus the quantitative analysis on the case $\bar{R}_j = R$.[6]

For the case $\bar{R}_j = R$, because, σ_j and σ_I have the same sign (given $\nu_j > 1$), a necessary and sufficient condition for a positive equity premium is that $\nu_1 + \nu_2 > 3$. Clearly, the equity premium is then increasing in the sum of the curvature parameters. The equation suggests that the curvature parameters ν have a similar role as the risk aversion coefficient in the basic consumption-based model. The equation for the Sharpe ratio, together with the return equations (8) and (9), highlight a fundamental trade-off in the model's ability to explain asset returns. Increasing the curvature parameters ν increases the equity premium. However, this also makes returns more volatile. Therefore, asset prices alone will impose a clear limit on how much curvature can be used to generate large risk premiums. In standard consumption-based asset pricing models this trade-off is much less present. In fact, in a basic constant relative risk aversion environment, for the benchmark case with IID consumption growth, increasing risk aversion increases the equity premium without affecting return volatility.

Under the assumptions made here, the instantaneous interest rate at steady state, assuming $\bar{R}_j = \bar{R}$ and $\sigma_{Ij} = \sigma_I$, is given as

$$r^f|_{ss}=\left(\bar{R}-1\right)-(v_1-1)\,(v_2-1)\,\frac{\sigma_I^2}{2}.$$

This expression shows how investment uncertainty contributes to a lower steady state interest rate

[6] Clearly, in a deterministic model, $\bar{R}_j = \bar{R}$ would be required to rule out arbitrage. However, in a model with uncertainty, sectorial differences in $\bar{R}_j$ are possible.

by an extent that is determined by the amount of the adjustment cost curvature. This parallels the precautionary saving effect on interest rates in standard consumption based models.

Sofar, $\bar{R}_2$ and $\bar{R}_1$ have been set independently from other parameters as a way of implicitly selecting values for A_j. As mentioned above, this will also be the approach chosen in the quantitative analysis. Alternatively, one could take all coefficients determining $\bar{R}$ as given, and consider the partial derivatives with respect to ν and δ. In this case, the depreciation rate also matters because it affects $\bar{R}$. For the calibration with two sectors representing equipment and structures, respectively, structures are harder to adjust and depreciate more slowly than equipment. That is, $\nu_S > \nu_E$, and $\delta_S < \delta_E$. Let us therefore consider the case where the capital with the higher adjustment cost curvature depreciates more slowly. With a few steps of algebra, under the standard assumptions that $\nu > 1$, $\lambda - 1 - \delta > 0$, and $(A - c) / b (\lambda - (1 - \delta))^{\nu - 1} > 0$, one finds that $\frac{\partial \bar{R}}{\partial \nu} > 0$, and $\frac{\partial \bar{R}}{\partial \delta} < 0$. Thus, given the two dimensions of sectorial asymmetries assumed here, both would contribute to $\mu_2 > \mu_1$, and because $\sigma_2 > \sigma_1$, both would contribute positively to the equity premium.

4.2 What is an admissible investment process?

In this section I consider the requirements for an investment process to be admissible, in the sense that it has to represent a solution to the firm's problem for the implied state-price process, and that this price process is itself well behaved. The two key requirements are that the derived state prices have to be positive and that the implied firm value has to be finite. While a large set of investment processes are admissible, these requirements nevertheless impose constraints on the simulations. For this reason, this section also provides the motivations for some of the choices made in the empirical analysis.

4.2.1 Positive state prices

The first key requirement is that state prices have to be positive. It is easy to see from equation (1) that the relative state prices in the two-state case are given by

$$\frac{P\left(s^t, \mathfrak{s}_1\right)}{P\left(s^t, \mathfrak{s}_2\right)} = \frac{R_2^I\left(s^t, \mathfrak{s}_2\right) - R_1^I\left(s^t, \mathfrak{s}_2\right)}{R_1^I\left(s^t, \mathfrak{s}_1\right) - R_2^I\left(s^t, \mathfrak{s}_1\right)}. \tag{11}$$

For nonnegative investment returns, that is $R > 0$, state prices are positive if and only if the numerator and the denominator have the same sign. This requirement implies that each capital stock has to have a higher return than the other capital stock in one of the two states. Indeed, if one type of investment were to generate a higher return in both states, then resources would be moved into this type of capital from the other, meaning that this would not be an equilibrium outcome.[7]

To see what properties are needed to satisfy this requirement, consider now a second-order Taylor-series approximation of the investment return around the deterministic steady state. To focus on the quantitatively important channels, I again consider a model without technology shocks where the only source of uncertainty are the state prices. Without technology shocks, the investment return from equation (3) simplifies to

$$R^{I}_{t,t+1}\left(s^{t},\mathfrak{s}_{j}\right)=$$

$$\frac{A-c}{b\left(\frac{I_t(s^t)}{K_t(s^{t-1})}\right)^{\nu-1}}+\left\{(1-\delta)+\left(1-\frac{1}{\nu}\right)\left[\frac{\frac{I_t(s^t)}{K_t(s^{t-1})}}{\frac{I_t(s^t)}{K_t(s^{t-1})}+(1-\delta)}\right]\lambda\left(s_{t+1}\right)\right\}\left(\frac{1}{\frac{I_t(s^t)}{K_t(s^{t-1})}+(1-\delta)}\lambda\left(s_{t+1}\right)\right)^{\nu-1}. \tag{12}$$

Return realizations are now driven by the investment growth rate $\lambda\left(s_{t+1}\right)$, while the only relevant state variable is the current investment-capital ratio $I_t\left(s^t\right)/K_t\left(s^{t-1}\right)$. A second-order Taylor approximation is obtained by assuming that the investment-capital ratio is at its steady state $I_t\left(s^t\right)/K_t\left(s^{t-1}\right)=\bar{\lambda}-1+\delta$, for a given steady state growth rate $\bar{\lambda}$, so that

$$R^{I}_{t,t+1}=\bar{R}+(\nu-1)\,\Delta\lambda'+\frac{B}{2}\left(\Delta\lambda'\right)^{2}+o\left(\left(\Delta\lambda'\right)^{2}\right) \tag{13}$$

where $\Delta\lambda'=\lambda'-\bar{\lambda}$ and

$$B=\frac{\nu-1}{\lambda}\left\{\nu-1-\frac{1-\delta}{\lambda}\right\}.$$ [8]

Assume equally sized up and down movements in a two-state setting so that

$$\Delta\lambda_j\left(\mathfrak{s}_2\right)=-\Delta\lambda_j\left(\mathfrak{s}_1\right)\equiv\overline{\Delta\lambda_j},\text{ for each } j\in(1,2)\,.$$

Assume also, like in subsection 4.1, that the investment growth volatilities are equal in the two sectors and positively correlated, so that

$$\overline{\Delta\lambda_1}=\overline{\Delta\lambda_2}=\overline{\Delta\lambda}.$$

With this approximation, the ratio determining relative state prices is given as

$$\frac{P\left(.,\mathfrak{s}_1\right)}{P\left(.,\mathfrak{s}_2\right)}=\frac{\left[\nu_2-\nu_1\right]\overline{\Delta\lambda}+\left[\left(\bar{R}_2-\bar{R}_1\right)+\frac{1}{2}\left(B_2-B_1\right)\left(\overline{\Delta\lambda}\right)^{2}\right]+o\left(\left(\overline{\Delta\lambda}\right)^{2}\right)}{\left[\nu_2-\nu_1\right]\overline{\Delta\lambda}-\left[\left(\bar{R}_2-\bar{R}_1\right)+\frac{1}{2}\left(B_2-B_1\right)\left(\overline{\Delta\lambda}\right)^{2}\right]+o\left(\left(\overline{\Delta\lambda}\right)^{2}\right)} \tag{14}$$

[7] A related requirement for the ability to recover state prices is that the matrix of the investment returns R has to be invertible.

[8] The only difference compared to the continuous-time equation derived above is the second-order term. With $(1-\delta)=\lambda=1$, we would have $B=(v-1)(v-2)$, which is the term in the continuous time counterpart.

As shown by equation (14), in order to have positive prices at steady state, the first term in the fraction needs to be far away from zero, that is $\left|[\nu_2 - \nu_1]\overline{\Delta\lambda}\right| >> 0$. Thus, clearly, asymmetry in the curvature parameters ν_j is needed to generate positive state prices.

State prices are also required to be positive away from steady state. Indeed, one of the issues faced in the numerical experiments is that the investment capital ratio can reach very low levels. In this case, the coefficients of a second-order approximation around the current investment-capital ratio can differ substantially from their steady state values. Specifically, as can be seen from equation (12), investment returns can get arbitrarily large as $I_t\left(s^t\right)/K_t\left(s^{t-1}\right)$ gets close to 0. Indeed, the first term $(A-c)/b\left(\frac{I_t\left(s^t\right)}{K_t\left(s^{t-1}\right)}\right)^{\nu-1}$ can get arbitrarily large. Under these conditions, the requirement for positive state prices can become hard to satisfy for all possible paths. What makes this condition hard to satisfy is that it has to hold with probability one. In order to deal with this in the simulations, the marginal product term, A, is made state contingent and its value in the low growth state is set specifically so that state prices are positive for the lowest possible $I_t\left(s^t\right)/K_t\left(s^{t-1}\right)$. As shown below, shocks to A have only second-order effects on asset price implications in general. This is because the level of A is so small relative to the other terms in the return equation (12).

4.2.1.1 An upper bound for the Sharpe ratio The requirement for nonnegative state prices has an implication for two-state environments in general. Namely, there is an upper bound to the Sharpe ratio. In particular, if the two states are equally likely, the Sharpe ratio is bounded by 1. Because the simulations will never deviate much from this case, this constraint typically matters.

In a two-state environment, no arbitrage requires that for the excess return on the market $R^M(s) - R^f$,

$$P(\mathfrak{s}_1)\left(R^M(\mathfrak{s}_1) - R^f\right) + P(\mathfrak{s}_2)\left(R^M(\mathfrak{s}_2) - R^f\right) = 0.$$

This implies that, in line with equation (11), the ratio of the state prices is given by

$$\frac{P(\mathfrak{s}_1)}{P(\mathfrak{s}_2)} = \frac{\left(R^M(\mathfrak{s}_2) - R^f\right)}{-\left(R^M(\mathfrak{s}_1) - R^f\right)} = \frac{\left[R^M(\mathfrak{s}_2) - ER^M\right] + \left[ER^M - R^f\right]}{-\left[R^M(\mathfrak{s}_1) - ER^M\right] - \left[ER^M - R^f\right]}.$$

It is easy to see that if both states are equally likely then $\left[R^M(\mathfrak{s}_2) - ER^M\right] = -\left[R^M(\mathfrak{s}_1) - ER^M\right]$ $= Std\left(R^M\right)$, so that

$$\frac{P(\mathfrak{s}_1)}{P(\mathfrak{s}_2)} = \frac{Std\left(R^M\right) + \left[ER^M - R^f\right]}{Std\left(R^M\right) - \left[ER^M - R^f\right]}.$$ [9]

Because of the requirement of positive state prices, the denominator has to be positive, and

[9] The implicit normalization that $R^M(\mathfrak{s}_2) > R^M(\mathfrak{s}_1)$ is without loss of generality.

therefore the Sharpe ratio, $\frac{ER^M - R^f}{Std(R^M)}$, is bounded,

$$\begin{aligned} Std\left(R^M\right) &> ER^M - R^f \\ 1 &> \frac{ER^M - R^f}{Std\left(R^M\right)}. \end{aligned}$$

Clearly, this result applies to all models without arbitrage in a two-state environment. For instance, it applies to the classic model used by Mehra and Prescott (1985). Based on the data of sectorial investment growth I consider in this study, a reasonable calibration cannot deviate much from the case where up and down movements are equally likely. Thus, the result derived here is relevant. Because the requirement of positive state prices has to hold with probability one, the constraints on the technology and the investment process end up reducing average Sharpe ratios in the simulations to a level substantially below 1.[10]

4.2.2 Other requirements

Another requirement for simulations is that the value of the firm implied by an investment process and a state price process has to be finite. If not, the equivalence between investment returns and returns to the firm breaks down. A related condition that guarantees optimality of the path satisfying the first-order condition is the transversality condition

$$\lim_{t\to\infty} \sum_{s^t} \frac{P\left(s^t\right)}{P\left(s_0\right)} \left\{A\left(s^t\right) + H_I\left(s^t\right)(1-\delta) - H_K\left(s^t\right)\right\} K_t\left(s^{t-1}\right) = 0.$$

Both conditions are checked in the simulations. I also make sure that gross investment returns are nonnegative, $R \geq 0$. This limited liability requirement is not necessarily needed. On the other hand, it doesn't impact any quantitative conclusions.

4.2.3 Models with no technology shocks: Interest rates cannot be constant

I consider here the benchmark environment where the only source of uncertainty that firms face are stochastic state prices. That is to say, there are no shocks to the production technology. I have one result: even without technology shocks, investment returns can be optimally state-contingent as long as interest rates are not constant. In an environment where interest rates are constant forever, investment returns are constant too. Thus, with constant interest rates it is not possible to recover the state price process from producers' first-order conditions. The basic economic idea in

[10] For the case with unequal probabilities, through a similar argument, the bound equals,

$$\left(\frac{1-\pi}{\pi}\right)^{0.5} > \frac{ER^M - R^f}{std\left(R^M\right)},$$

where π is the probability of the state with the low return realization. Thus, with postively skewed returns the bound is tighter, with negatively skewed returns it is looser.

this section is that if a firm is subject to convex capital adjustment costs, it will not find it optimal to choose a volatile investment plan unless forced by changing prospects in future valuations.

Consider the discrete-time model with no technology shocks. Assuming a general two-state environment where state prices do not necessarily have a Markov representation. The firm's problem is given by

$$\max_{\{I(s^t),K(s^{t+1})\}} \sum_{t=0}^{T-1} \sum_{s^t} \left[AK_t\left(s_{t-1}\right) - \left\{ \frac{b}{\nu}\left(I\left(s^t\right)/K_t\left(s^{t-1}\right)\right)^{\nu} + c \right\} K_t\left(s^{t-1}\right) \right] \times \left(\prod_{j=0}^{t-1} P\left(s_{j+1}|s^j\right) \right)$$
$$+ \sum_{s^T} \Psi K_T\left(s^{T-1}\right) \left(\prod_{j=1}^{T-1} P\left(s_{j+1}|s^j\right) \right),$$

subject to

$$0 = (1-\delta) K_t\left(s^{t-1}\right) + I\left(s^t\right) - K_{t+1}\left(s^t\right),$$

with $K(s_0)$ given, $P\left(s_0|s^0\right) = 1$, and $\Psi > 0$ a parameter; assuming that $s_t \in (\mathfrak{s}_1, \mathfrak{s}_2)$.

The solution to this problem for $T \to \infty$ is equivalent to the solution of the general version of the problem with enough regularity so that the firm value is finite. However, it is easy to see in this model why interest rate volatility is needed. Indeed, from $T-1$ to T, without technology shocks, the return to capital equals the risk free rate. For the second-to-last return-period, that is, from $T-2$ to $T-1$, it can be checked that the return is given by

$$R_{T-2,T-1}\left(s^{T-2}, \mathfrak{s}_j\right) = \frac{\alpha\left(\frac{\Psi}{R^f_{T-1,T}\left(s^{T-2},\mathfrak{s}_j\right)}\right)}{\alpha\left(\frac{\Psi}{R^f_{T-1,T}\left(s^{T-2},\mathfrak{s}_1\right)}\right) P\left(\mathfrak{s}_1|s^{T-2}\right) + \alpha\left(\frac{\Psi}{R^f_{T-1,T}\left(s^{T-2},\mathfrak{s}_2\right)}\right) P\left(\mathfrak{s}_2|s^{T-2}\right)}, \text{ for } j = 1, 2.$$

with $\alpha(x) = (A-c) + (1-\delta)x + \left(1-\frac{1}{\nu}\right)\left(\frac{1}{b}\right)^{\frac{1}{\nu-1}} x^{\frac{\nu}{\nu-1}}$. Clearly, if the interest rate $R^f_{T-1,T}\left(s^{T-2}, \mathfrak{s}_j\right)$ is constant, that is if it does not depend on $\mathfrak{s}_j$, then, $R_{T-2,T-1}\left(s^{T-2}, \mathfrak{s}_j\right) = R_{T-2,T-1}\left(s^{T-2}\right) = R^f_{T-2,T-1}\left(s^{T-2}\right)$. However, to the extent that one-period interest rates are state-contingent at $T-1$, the return to the firm from $T-2$ to $T-1$ will be state-contingent, and it will depend on the technology of the firm, in particular, the parameters of the adjustment cost function. Going backwards in time, this same argument can be made if all future one-period interest rates are constant.[11] The following proposition summarizes these derivations.

Proposition 1 *If one-period interest rates are constant in every period, without technology shocks, the returns to the firm (and the investment returns) are equal to the one-period interest rate.*

A consequence of this result is that, without technology shocks, if investment returns for one capital stock are state-contingent, then one-period interest rates cannot be constant. The

[11] As shown in the appendix, the continuous time version of the model admits a more compact proof for this property.

importance of this result is that one cannot work with a "nice" benchmark environment with constant interest rates, in general.[12] On the other hand, as shown in the quantitative applications below, interest rate volatility doesn't have to be excessively high, even when investment returns are quite volatile.

5 Calibration

Parameter values are assigned based on 3 types of criteria. First, a set of parameter values are picked to match direct empirical counterparts. Second, some parameters are chosen to yield the best implications for key asset pricing moments. Third, some parameters are chosen to make sure the derived state-prices are admissible. I first present a short summary of the baseline calibration. The details and the specification with shocks to the investment technology are given thereafter.

5.1 Summary

Table 1 lists the main parameters chosen for the baseline case,

Table 1: Parameter values

ρ	$=$	$(0.2, 0)$
$\lambda^I(\mathfrak{s}_1), \lambda^I(\mathfrak{s}_2)$	$=$	$0.9712, 1.0954$
δ_e, δ_s	$=$	$0.112, 0.031$
$\overline{(K_e/Z_e)/(K_s/Z_s)}$	$=$	0.6
ν_e, ν_s	$=$	$2.5, 5$
b_e, b_s, c_e, c_s so that $\overline{qZ}$	$=$	1.5
A_e, A_s so that $\bar{R}$	$=$	1.03
$A_j(\mathfrak{s}_1), A_j(\mathfrak{s}_2)$	$=$	$A_j(1-.236), A_j(1+.236)$

where ρ stands for the first-order serial correlation of investment growth. A set of parameters is chosen based on direct empirical counterparts; namely, ρ, (δ_e, δ_s), and $\overline{(K_e/Z_e)/(K_s/Z_s)}$. The investment realizations, $\lambda^I(\mathfrak{s}_j)$, are chosen to mimic first and second moments of postwar U.S. investment growth, with the additional restriction that the lowest investment-capital ratio is above 0. In order to replicate steady-state values for qZ, (b_e, b_s) are selected; (c_e, c_s) are then determined to generate the lowest possible total cost. For the curvature parameters, it is assumed that $\nu_e < \nu_s$, with the exact values picked to maximize the model's fit. $\bar{R}$, and thus A_e and A_s, primarily affect

[12] Note, setting $v = 1 = b$, and $c = 0$ for one of the firms in our analysis would seem to imply constant interest rates. However, this is not an admissible specification, because the first-order conditions do not describe optimal firm behavior in general, as this problem is linear.

the risk free rate. Finally, the volatility of $A_j(s)$ is chosen so that state prices and gross investment returns are always positive.

5.2 Details of calibration

This section discusses the details of the calibration.

5.2.1 Investment and productivity processes

I consider the Bureau of Economic Analysis' (BEA) quantity indexes of investment for equipment & software as well as for structures as the empirical counterparts to investment in units of capital goods, IZ. Because Z measures the number of capital goods that can be produced from one unit of the final good, ruling out arbitrage implies that $1/Z$ is the price of the capital good in terms of the final good. Equivalently, $1/Z$ is the replacement cost for capital (not including adjustment cost), or the bookvalue of capital. For each of the two sectors, Z is computed as the deflator for nondurable consumption and services divided by the deflator of the sectorial investment good. Investment expenditure, I, can then be obtained by combining the series for IZ and Z. Based on annual data covering 1947-2003, the properties of the growth rates of these series are shown in Table 2.

Table 2: U.S. Investment 1947-2003 (Growth rates)

		Mean	Standard Deviation	1^{st} Autocorrelation
Investment expenditure	$I_{E\&S}$	3.81%	6.98%	.08
	I_S	2.85%	7.94%	.27
Investment	$IZ_{E\&S}$	5.71%	7.81%	.13
	IZ_S	2.29%	6.86%	.28
Investment technology	$Z_{E\&S}$	1.82%	2.56%	.66
	Z_S	$-.44\%$	2.35%	.31

As is well known, the price of equipment and software has been decreasing over time. The 1.82% annual increase in Z shows that in Table 2. Table 2 also shows that the volatilities of investment, and investment expenditure, are very similar across the two sectors.

The calibration of the investment growth process proceeds in two steps. First, the probability matrix is determined to match the serial correlation and the frequency of low and high growth states. These two moments do not depend on the shock values themselves but only on the probabilities. Specifically, the two diagonal elements of the probability matrix are given as

$$\pi_{11} = \frac{\rho + fr}{1 + fr};\ \pi_{22} = \frac{1 + fr \cdot \rho}{1 + fr},$$

where fr is the relative frequency of state 1, the recession state. The numbers of realizations of investment growth rates above and below the mean are almost the same, thus I set $fr = 1$. As shown in Table 2, the first-order serial correlations of the growth rates of investment are 0.13 and 0.28, respectively, and 0.08 and 0.27 for investment expenditure. The common ρ is set at the average for investment expenditure of 0.2; the case where $\rho = 0$ is also considered.

For the baseline calibration, I abstract from shocks to the investment technology, Z. Due to the balanced growth requirement, the growth rates of investment expenditures are equalized across sectors. The mean of $\lambda^I - 1$ is set at 3.33% per year, which is the average of the historical investment growth rates across the two sectors. The implied standard deviation is 6.21%. This is 20% lower than the historic average across the two sectors. With this reduction in volatility, the investment-capital ratio for structures at the steady state corresponding to the low growth state is $\lambda^I(\mathfrak{s}_1) - 1 + \delta_s = 0.9712 - 1 + 0.031 = 0.0022$. This is sufficient to make sure that the investment-capital ratio is bounded away from zero.[13] As explained in the previous section, this is a way to guarantee positive state prices. Note that the perfect positive sectorial correlation in the model is not that far from the historical reality. Indeed, the historical sample correlations for investment across the two sectors are 0.61 and 0.64, for investment and investment expenditure, respectively.

For the case where the investment specific technology Z is allowed to vary in both sectors, the 6 values for the realized growth rates of investment expenditure (2) and the sector specific investment technologies (4) are set so as to match as closely as possible the 8 means and standard deviations of the growth rates of IZ_E, IZ_S, $Z_{E\&S}$ and Z_S. This objective can be achieved quite well. The standard deviations are again reduced, here by 30%, for the reason explained in the previous paragraph. The empirical correlations of sectorial investment with its specific technological growth are 0.43 and -0.32, while the correlations of the technological growth across sectors is 0.3. Clearly, due the limited degrees of freedom, the two-state process cannot match all these correlations. As shown below, for most quantities of interest, the Z shocks don't turn out to matter that much.

5.2.2 Depreciation rates

The depreciation rates for equipment and software as well as for structures, (δ_e, δ_s) are based on time series averages of the depreciation rates reported in the Fixed Assets tables form the

[13] This is not an issue for equipment & software because the depreciation rate is higher.

BEA. These are 13.06% and 2.7%, respectively, for the period 1947-2002. Because the BEA's depreciation includes physical wear as well as economic obsolescence, the data is adjusted to take into account that depreciation in the model covers only physical depreciation. To do this the price increase in the capital good is added, so that

$$\delta_t = \frac{D_t}{K_t} + \left(Z_{t-1}/Z_t - 1\right),$$

with D_t depreciation according to the BEA. This adjustment decreases depreciation by 1.82% for equipment and -0.44% for structures, so that $(\delta_e, \delta_s) = (.112, .031)$.

5.2.3 Relative size of capital stocks

The average ratio of the capital stocks, $\overline{(K_{e,t}/Z_{e,t})\,/\,(K_{s,t}/Z_{s,t})}$, is needed only for computing aggregate returns, which, as shown earlier, are value weighted averages of the sectorial returns. Based on the Current-Cost Net Stocks of Fixed Assets from the BEA, for the period 1947-2002, the average of $(K_{e,t}/Z_{e,t})\,/\,(K_{s,t}/Z_{s,t})$ is 0.6. Note that given the balanced growth assumptions, in the model, the ratio of the physical capital stocks $K_{e,t}/K_{s,t}$ is trending, while the ratio of the book values of the capital stocks $(K_{e,t}/Z_{e,t})\,/\,(K_{s,t}/Z_{s,t})$ is stationary.

5.2.4 Adjustment cost and marginal product

The parameters of the adjustment cost functions are set through the following procedure.

1) Select ν's to get good results for asset prices under the restriction that $\nu_e < \nu_s$. Specifically, the selected values generate roughly the highest equity premium with the lowest reasonable return volatility, by also guaranteeing positive state prices.

2) Pick b's so that qZ is consistent with average values reported in the literature.

3) The c's are then picked to minimize the overall amount of output lost due to adjustment cost.

In addition to casual empiricism, there is also more direct evidence that suggests that the adjustment cost curvature is larger for structures than for equipment & software. For example, as shown in Table 2, the fact that the serial correlation of the growth rates is somewhat higher for structures than for equipment can be interpreted as an expression of the desire to smooth investment over time due to the high adjustment cost. As another example, Guiso and Parigi (1999) examine investment behavior for equipment and structures with Italian data on investment and sales. Their findings are also consistent with the notion that structures are more costly to adjust than equipment.

There are many examples of studies that estimate qZ. Lindenberg and Ross (1981) report averages for two-digit sectors for the period 1960-77 between .85 and 3.08. Lewellen and Badrinath

(1997) report an average of 1.4 across all sectors for the period 1975-91. Gomes (1999) reports an average of 1.56. Based on this, I use a steady-state target value for qZ, $\overline{qZ}$, of 1.5 for both sectors. One problem with using empirical studies to infer the required heterogeneity in the sectorial costs is that most studies consider adjustment costs by sector of activity. For the analysis here, I would need information about the adjustment costs by type of capital.

One way to gauge whether the adjustment cost parameters are reasonable is to consider the amount of resources lost due to the adjustment process. For the baseline calibration, the mean average adjustment cost (obtained in simulations) is 7% and 9% of investment for equipment & software and structures, respectively. These values depend primarily on the target value for qZ, which itself does not affect much the model's asset pricing implications.

The marginal product coefficients A_e and A_s are set implicitly so as to have the steady-state return $\bar{R}_j$ equalized in the two sectors, to replicate the mean risk free rate, and to make sure the firm value is finite and the transversality conditions are satisfied. The implied values are $(A_1, A_2) = (0.1492, 0.0612)$.

Finally, the variability of the marginal product terms, x in $A_j(1 \pm x)$, is chosen so that for all paths the implied state-prices are positive, as explained in section 4.2.1, and so that the gross returns are positive. While these shocks are useful in insuring that the implied state-prices are admissible, they have only second-order effects on key asset pricing moments. This is because the marginal product component A represents a small part of the return. The implied correlation between productivity shocks and investment is positive, which seems reasonable.

6 Quantitative findings

Table 3 presents model implications for the baseline calibration as well as empirical counterparts for a set of moments. Model results are based on sample moments of very long simulated time series. For unconditional moments, the key finding is that the model is able to generate an equity premium of several percentage points with reasonable volatility for the equity return as well as for the risk free rate. The model's mean Sharpe ratio is about one third of the one that is implied by the historic equity premium. Consistent with the analysis in subsection 4.1, given the higher adjustment cost curvature for structures relative to equipment, structures have a higher return volatility and a higher risk premium than equipment.

The model is able to generate considerable time-variation in conditional risk premiums. Indeed, the standard deviation of the one-period ahead conditional equity premium is 5.2%, which is considerably higher than the standard deviation of the risk free rate at 2.24%. There is a variety of empirical studies measuring return predictability. For example, Campbell and Cochrane (1999) report $R^{2\prime}s$ of 0.18 and 0.04 for regressions of excess returns on lagged price-dividend ratios at a one-year horizon for the periods $1947-95$ and $1871-1993$, respectively. Combining the R^2 with the volatility of the excess returns, $\sqrt{R^2} std\left(R-R^f\right)$ provides an estimate of the volatility of the conditional equity premium. Setting $R^2 = 0.1$ this would be $\sqrt{0.1} \times 0.17 = 5.27\%$. Thus, the model's value of 5.2% is close.

What is driving expected excess returns? In general, assuming the absence of arbitrage, we have that

$$E_t\left(R_{t+1}-R_t^f\right) = -\frac{\sigma_t\left(m_{t+1}\right)}{E_t m_{t+1}} \sigma_t\left(R_{t+1}\right) \rho_t\left(m_{t+1}, R_{t+1}\right).$$

Possibly, return volatility $\sigma_t\left(R_{t+1}\right)$ can drive risk premiums. However, according to Lettau and Ludvigson (2004) this is not the case for the U.S. postwar period. They find negative correlations between conditional means and volatilities. The model here is consistent with this fact. For the baseline calibration the correlation between conditional means and volatilities is -0.56. This negative correlation is very robust to parameter changes.

Most standard models cannot replicate this finding of a negative correlation between conditional means and volatilities. With CRRA utility and lognormal consumption, expected returns are given by

$$E_t\left(R_{t+1}-R_t^f\right) \cong -\gamma \cdot \sigma_t\left(\ln C'/C\right) \cdot \sigma_t\left(R_{t+1}\right) \cdot \rho_t\left(m_{t+1}, R_{t+1}\right).^{14}$$

In the Mehra-Prescott setup, all terms in the equation are roughly constant, with the correlation, $\rho_t\left(m_{t+1}, R_{t+1}\right)$, roughly equal to one. In Campbell and Cochrane's (1999) model, $\frac{\sigma_t\left(m_{t+1}\right)}{E_t m_{t+1}}$ displays considerable variation. However, as is clear from their Figures 4 and 5, conditional means and volatilities are positively correlated.

What is driving the negative correlation between the conditional mean and volatility in the model? It can be shown under fairly general assumptions that this correlation is actually positive for individual (sectorial) investment returns at steady state levels of investment-capital ratios. And it is positive for sectorial returns in all simulations. The negative correlation displayed for the aggregate returns is generated by movements in the sectorial weights. For instance, when investment capital ratios are low in both sectors, the value of the more volatile sector declines by relatively more. And, because conditional volatilities of the sectorial returns are relatively stable, the shift in sectorial weights dominates.

Let us focus now directly on the Sharpe ratio

$$\frac{E_t\left(R_{t+1}-R_t^f\right)}{\sigma_t\left(R_{t+1}\right)} = -\frac{\sigma_t\left(m_{t+1}\right)}{E_t m_{t+1}} \rho_t\left(m_{t+1}, R_{t+1}\right). \tag{15}$$

[14] The approximation comes from replacing $\sqrt{\exp\left[\gamma^2 var_t\left(\ln C'/C\right)\right]-1}$ by $\gamma \cdot \sigma_t\left(\ln C'/C\right)$

Given the volatile conditional means and the negative correlation between conditional means and volatilities, Sharpe ratios are very volatile. According to Lettau and Ludvigson (2004), for quarterly data, market implied Sharpe ratios have a mean of 0.39 and a standard deviation of 0.448, which implies a coefficient of variation of $0.448/0.39 = 1.15$. In the model here, for the baseline calibration, this ratio equals, $0.26/0.18 = 1.44$. That is, considering that the model generates average Sharpe ratios of roughly 1/3 of the ones implied by the aggregate market, it nevertheless has the ability to generate considerable volatility in Sharpe ratios

What drives the volatility of the Sharpe ratio? Both parts on the right hand side of equation (15) contribute. As shown in Table 3, the market price of risk is moving, but its mean and standard deviation differ from those of the market's Sharpe ratio. The mean of the market price of risk is (obviously) larger, while the volatility is lower. Given the two-state setup, the conditional correlation $\rho_t (m_{t+1}, R_{t+1})$ can only be 1 or -1. Therefore, $\rho_t (m_{t+1}, R_{t+1})$ switches between values of 1 and -1 as a function of the state of the economy. The case with IID investment growth rates, presented in Table 4, displays similar properties.

Tables 5 and 6 show results for the calibrations with investment specific technology shocks Z. In Table 5 the correlation of Z with the same sector's investment growth equals, 1, in Table 6, it is -1. While there are some quantitative differences compared to the baseline case and between the two cases considered here, none of the main conclusions are affected. Note, for maximum comparability, I only recalibrated $\bar{R}$ and x to make sure implied state-prices are admissible.

To further illustrate model properties, I consider the implications from feeding through the investment realizations for the U.S. for the period 1947-2003. Given that investment growth in the model has only two values, the fit of the driving process is not perfect. Nevertheless, as shown in Figure 1, the fit can be very good, with correlations between the model and the data of 0.78 and 0.71 for equipment and structures, respectively. Figure 2 shows that the model's generated returns are indeed related to actually realized stock returns, with a correlation of 0.48. Figure 3 shows conditional moments. The two panels on the left show that conditional volatility is more persistent than expected returns. The right hand side panel shows the market price of risk and the market's Sharpe ratio. Considering the 1990s, through the series of 8 high realizations in investment growth, expected returns and Sharpe ratios are declining over time. The figure also shows that with a low investment growth realization, the market's Sharpe ratio becomes negative, and thus the conditional correlation $\rho_t (m_{t+1}, R_{t+1})$ becomes positive. It is interesting here to consider again the calibration with IID investment growth to further highlight the persistent component driving risk premiums. Figure 3b, presents the realized conditional moments corresponding to the IID case presented in Table 4. In this case, the relevant state of the economy is summarized by the two investment-capital ratios, $\left(I_j\left(s^t\right)/K_j\left(s^{t-1}\right)\right)_{j=1,2}$. The sequence of positive investment growth realization in the 90s, pushes up these ratios, leading to lower Sharpe ratios. Only three times in the postwar period does the market's Sharpe ratio become negative. In the 1990s, it is at the 6^{th} realization of a high investment rate that the market's Sharpe ratio becomes negative. The story told by the model is that throughout the 90's firms continued to invest heavily, despite declining expected returns, because investment returns were considered less and less risky.

6.1 Discussion and sensitivity

I provide here some additional information about the factors driving quantitative results.

Let us reconsider equation (10) of the Sharpe ratio at steady state in the continuous-time model for the baseline parameterization

$$\frac{\mu_j - r^f}{\sigma_j}|_{ss} = \frac{\nu_1 + \nu_2 - 3}{2}\sigma_I. \tag{16}$$

Using the values from the baseline calibration, $(\nu_1, \nu_2) = (2.5, 5)$ and $\sigma_I = 6.21\%$, the Sharpe ratio computed from (16) is 0.14. As shown in Table 3 and 4, average Sharpe ratios obtained from the simulations in the discrete-time model are at 0.18 for the baseline case. Thus, the continuous-time approximation at steady state gets slightly less than 80% of the simulated average Sharpe ratios. To get a sense of how much of the difference is due to the approximation and how much to the off steady-state realizations, note that in the baseline model, the Sharpe ratio is at 0.148 at the steady-state. Thus, the somewhat higher Sharpe ratios reported from the simulations are mainly a product of the off steady-state behavior. Of course, the baseline model also is subject to stochastic marginal products, that is, shocks to A. But not surprisingly, this has only minor effects. Table 7 reports simulation results that further confirm this point. Here the shocks to marginal products in the equipment sectors are turned off, while they are maintained for structures so as to assure that implied state prices are always positive. The main effect of this is to reduce the volatility for returns on equipment by about 2.5 percentage points. The Sharpe ratio gets to 0.16, compared to 0.18 in the benchmark case. As suggested by equation (16), turning of the A shocks in the equipment sector has an effect on the Sharpe ratio primarily because of the asymmetry introduced across the sectors, rather than because of the volatility in A itself.

7 Conclusions

The paper has examined the implications of producers' first-order conditions for asset prices in a model where convex adjustment cost play a major role. One lesson of this analysis is that some asymmetries across sectors are crucial. One reason for this is that the considered technology does not allow a firm to make state-contingent investment decisions for each capital stock individually. State-contingent investment decisions are possible through the combination of the two stocks. If both sectorial technologies were identical, optimal decision would generate identical returns, which would make it impossible to recover the state price process.

The analysis demonstrates the ability of a simple investment cost representation to explain a number of features of aggregate asset prices. Investment cost curvature and investment volatility are the main ingredients to explain return volatility and risk premiums.

The quantitative asset pricing implications from a basic representation of the production side are encouraging. With reasonable assumptions on the quantities, risk premiums and interest rates come close to explaining observed empirical counterparts. Despite being very stylized, the model has rich implications for time-varying moments. Specifically, the negative correlation between means and volatilities is noteworthy.

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Appendix: Continuous-time model

This appendix presents a continuous-time investment model that replicates the setup of the discrete-time environment. The technology side of the model follows Abel and Eberly (1994) but without shocks. The main difference is that here the firm faces changing state prices, while in their case pricing is risk neutral with constant interest rates. The steps needed to derive the return equation (8) are also presented.

The capital stock evolves as $dK_t = (I_t - \delta K_t)\, dt$, and the investment cost is given by

$$H(I_t, K_t) = \left\{ \frac{b}{\nu} (I_t/K_t)^{\nu} + c \right\} K_t,$$

which is homogenous of degree one in I and K. The gross profit is given as

$$AK_t.$$

Assume that the state-price process is given as

$$d\Lambda_t = -\Lambda_t r(x_t)\, dt + \Lambda_t \sigma(x_t)\, dz_t,$$

where dz_t is a one-dimensional Brownian motion, and

$$dx_t = \mu_x(x_t) dt + \sigma_x(x_t) dz_t.$$

Assume that the functions $\mu_x(x_t), \sigma_x(x_t), r(x_t)$ and $\sigma(x_t)$ satisfy the regular conditions such that there are solutions for the above two SDEs.

The firm maximizes its value

$$V = \max_{\{I_{t+s}\}} E_t \left\{ \int_0^{\infty} [AK_{t+s} - H(I_{t+s}, K_{,t+s})] \frac{\Lambda_{t+s}}{\Lambda_t} ds \right\}.$$

Given the dynamics of Λ_t, it is obvious that the firm's value function V is independent of Λ_t. Following from the Markov property of the state variable x_t, the firm's value function would be a function of (K_t, x_t). The HJB equation is

$$rV = \max_{\{I_t\}} \left\{ [AK_t - H(I_t, K_t)] + (I_t - \delta K_t) V_K + \mu_x V_x + \frac{1}{2}\sigma_x^2 V_{xx} + \sigma \sigma_x V_x \right\}.$$

The first-order condition is

$$H_I(I_t, K_t) = V_K \equiv q_t$$

That is,

$$\begin{aligned} V_K &= b (I_t/K_t)^{\nu - 1} \\ I_t &= \left(\frac{V_K}{b} \right)^{\frac{1}{\nu - 1}} K_t \end{aligned}$$

Because of constant returns to scale in K_t, following Hayashi, it is easy to see that $V(K_t, x_t) = K_t V_K(x_t)$. Thus, it is clear that optimal investment follows an Ito process, $dI_t/I_t = \mu_I(K_t, x_t)\, dt + \sigma_I(K_t, x_t)\, dz_t$.

Define realized returns to the firm as

$$\frac{AK_t - H(I_t, K_{,t})}{V_t} dt + \frac{dV_t}{V_t}.$$

Given Hayashi's result and the first-order conditions

$$\frac{AK_t - H(I_t, K_{,t})}{V_t} dt + \frac{dV_t}{V_t} = \frac{AK_t - H(I_t, K_{,t})}{q_t K_t} dt + \frac{dK_t}{K_t} + \frac{dq_t}{q_t}.$$

Using $q_t = b(I_t/K_t)^{\nu-1}$ and Ito's lemma, the return equation 8 given in the main text can be derived.

Proposition 1 in the text shows that for the model without technology shocks, constant interest rates imply constant investment returns. The continous time model admits a more compact proof for this property. Indeed, changing to the risk-neutral measure $\mathbb{Q}$, the firm's problem becomes

$$V = \max_{\{I_{t+s}\}} E_t^{\mathbb{Q}} \left\{ \int_0^{\infty} e^{-\int_t^{t+s} r_u du} \left[AK_{t+s} - H(I_{t+s}, K_{,t+s})\right] ds \right\},$$

with

$$dx_t = (\mu_x(x_t) + \sigma(x_t)\sigma_x(x_t))\, dt + \sigma_x(x_t) dz_t^{\mathbb{Q}}$$

and

$$dK_t = (I_t - \delta K_t)\, dt.$$

Written in this form, it is obvious that if the interest rate r_t is constant, the firm faces no uncertainty, and thus, it will not introduce any uncertainty into an optimal investment plan.

Table 3

Asset Pricing Implications: Baseline calibration

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		3.59%	1.43%	0.26	0.18		2.14%		5.09%
Std	20.52%		2.24%	0.19	0.26	12.06%		27.23%	

Std[E(R^M-R^f|t)]	5.20%	Corr(E(R^M-R^f|t), Std(R^M-R^f|t))	-0.56
Std[Std(R^M-R^f|t)]	0.51%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , MPR)
E&S	-0.03	-0.68	-0.58
S	-0.34	-0.32	-0.82

	Corr(λ^{IZ}, R^M)
E&S, S	0.98

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe Market
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

(v_1=2.5, v_2=5 , R=1.03 , x=0.23643 , reduction $\sigma_{\Delta I}$ =20%)

Table 4
Asset Pricing Implications: IID case; (no serial correlation)

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		3.52%	1.43%	0.20	0.18		2.13%		4.94%
Std	20.75%		1.91%	0.13	0.15	12.21%		27.09%	

Std[E(R^M-R^f|t)]	3.04%	Corr(E(R^M-R^f|t), Std(R^M-R^f|t))	-0.95
Std[Std(R^M-R^f|t)]	0.45%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , MPR)
E&S	-0.62	-0.70	-0.54
S	-0.89	-0.35	-0.78

	Corr(λ^{IZ}, R^M)
E&S, S	0.99

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe Market
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

(v_1=2.5, v_2=5 , R=1.03 , x=0.23643 , reduction $\sigma_{\Delta I}$ =20%)

Table 5

Asset Pricing Implications: with shocks to investment technology, positive correlation λ^{I} and λ^{Z}

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		2.84%	2.01%	0.29	0.17		1.73%		4.04%
Std	17.14%		2.95%	0.20	0.30	10.53%		22.59%	

Std[E(R^M-R^f|t)]	5.01%	Corr(E(R^M-R^f|t), Std(R^M-R^f|t))	-0.70
Std[Std(R^M-R^f|t)]	0.29%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , MPR)
E&S	0.11	-0.74	0.14
S	-0.33	-0.29	-0.16

	Corr(λ^{IZ}, R^M)
E&S, S	0.98

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

(v_1=2.5, v_2=5 , R=1.033 , x=0.28245 , reduction $\sigma_{\Delta I}$ =30%)

Table 6

Asset Pricing Implications: with shocks to investment technology, negative correlation λI and λZ

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		4.13%	1.19%	0.30	0.20		2.77%		5.49%
Std	20.97%		3.17%	0.21	0.30	14.06%		26.45%	

Std[E(R^M-R^f|t)]	6.07%	Corr(E(R^M-R^f|t), Std(R^M-R^f|t))	-0.66
Std[Std(R^M-R^f|t)]	0.29%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , MPR)
E&S	0.11	-0.57	0.15
S	-0.32	-0.08	-0.16

	Corr(λ^{IZ}, R^M)
E&S, S	0.98

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

(v_1=2.5, v_2=5 , R=1.033 , x=.30846 , reduction $\sigma_{\Delta I}$ =30%)

Table 7
Asset Pricing Implications: Baseline calibration; No A shocks in sector 1 (Equipment)

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		2.89%	2.19%	0.24	0.16		1.45%		4.35%
Std	19.47%		2.39%	0.17	0.25	9.69%		27.24%	

Std[E(R^M-R^f|t)]	4.65%	Corr(E(R^M-R^f|t), Std(R^M-R^f|t))	-0.33
Std[Std(R^M-R^f|t)]	1.01%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , MPR)
E&S	0.06	-0.87	-0.58
S	-0.25	-0.60	-0.83

	Corr(λ^{IZ}, R^M)
E&S, S	0.98

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe Market
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

(v_1=2.5, v_2=5 , R=1.03 , x=0.23643 , reduction $\sigma_{\Delta I}$ =20%)

Figure 1

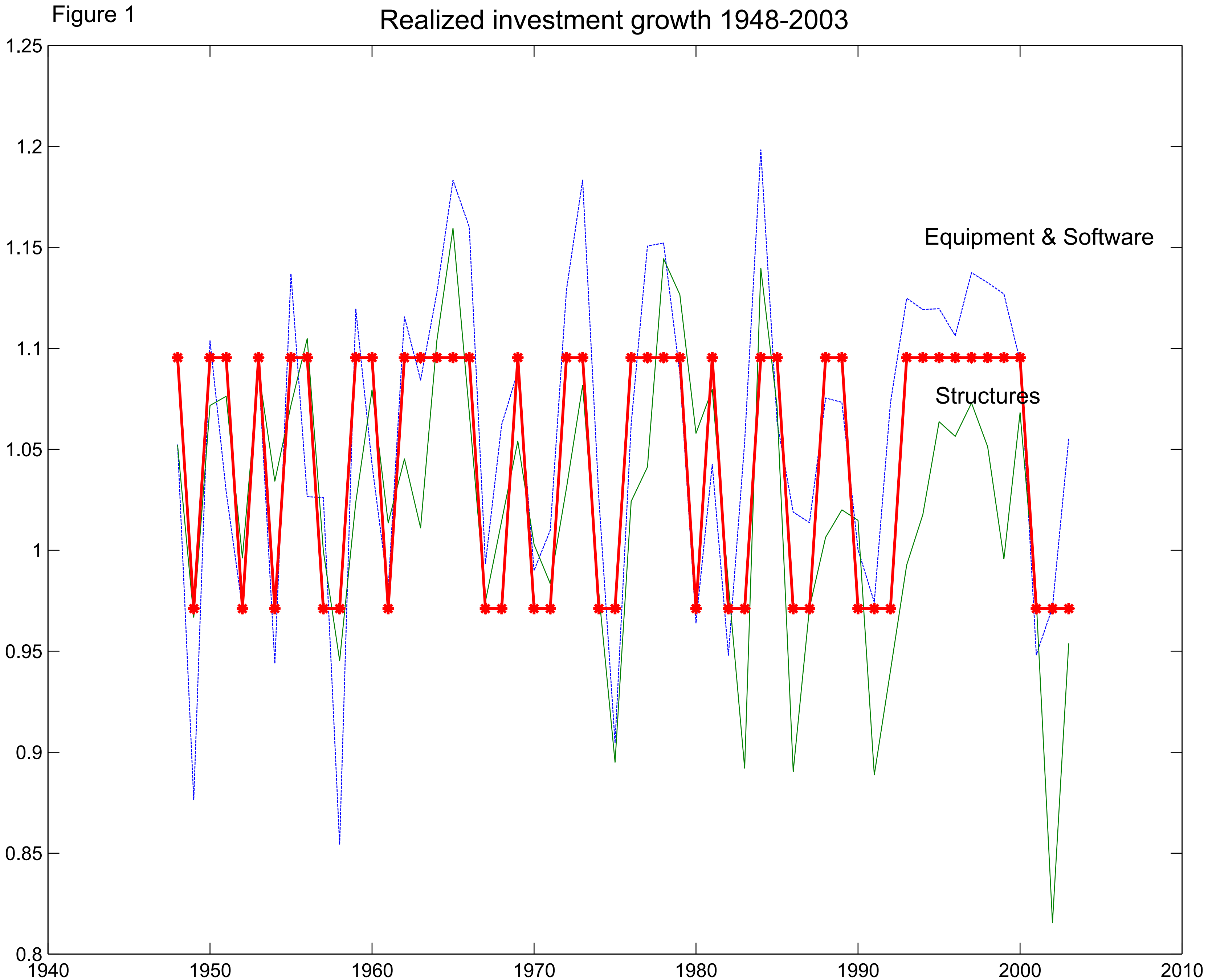

Realized investment growth 1948-2003
Equipment & Software
Structures
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
1940
1950
1960
1970
1980
1990
2000
2010

Figure 2

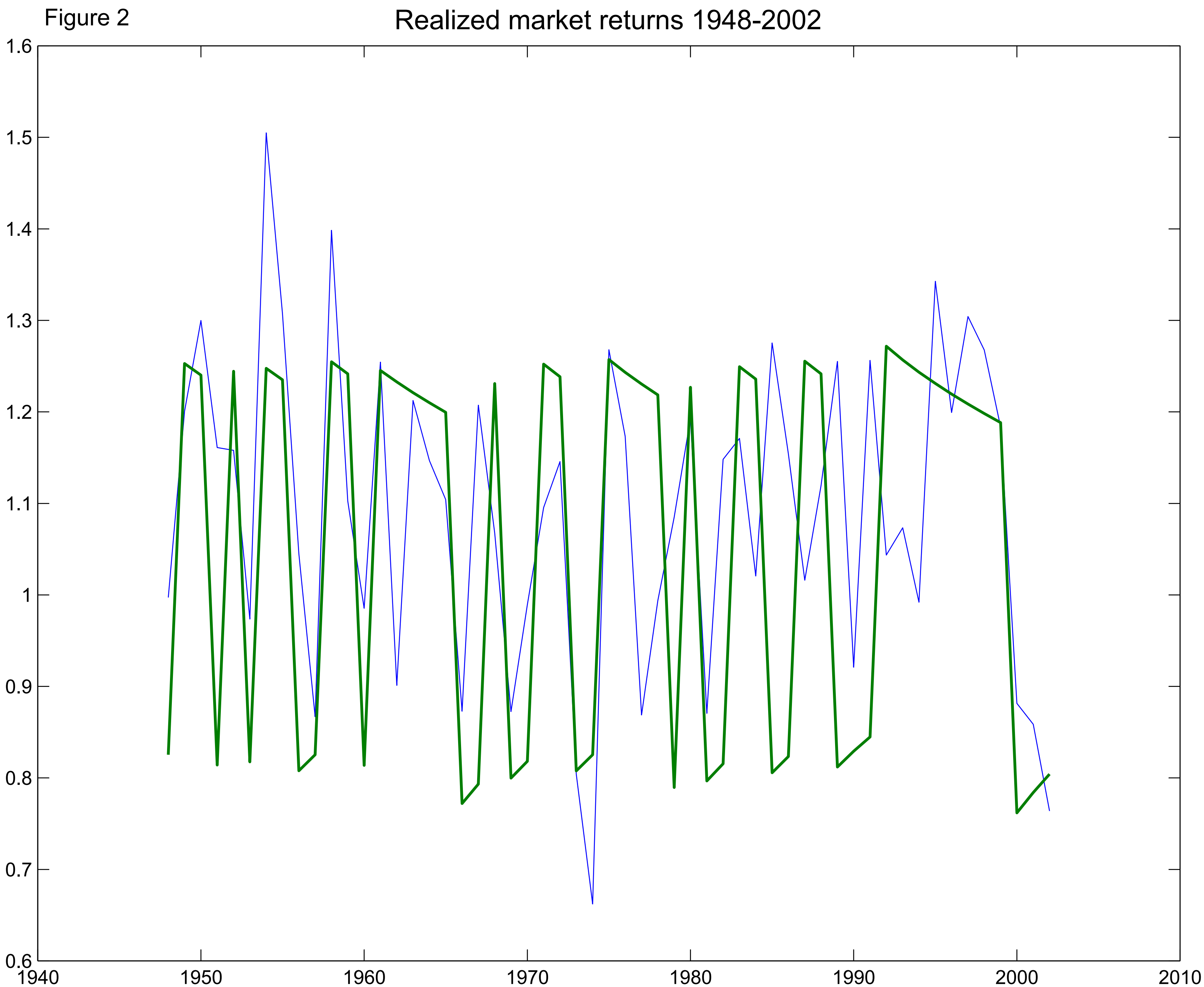
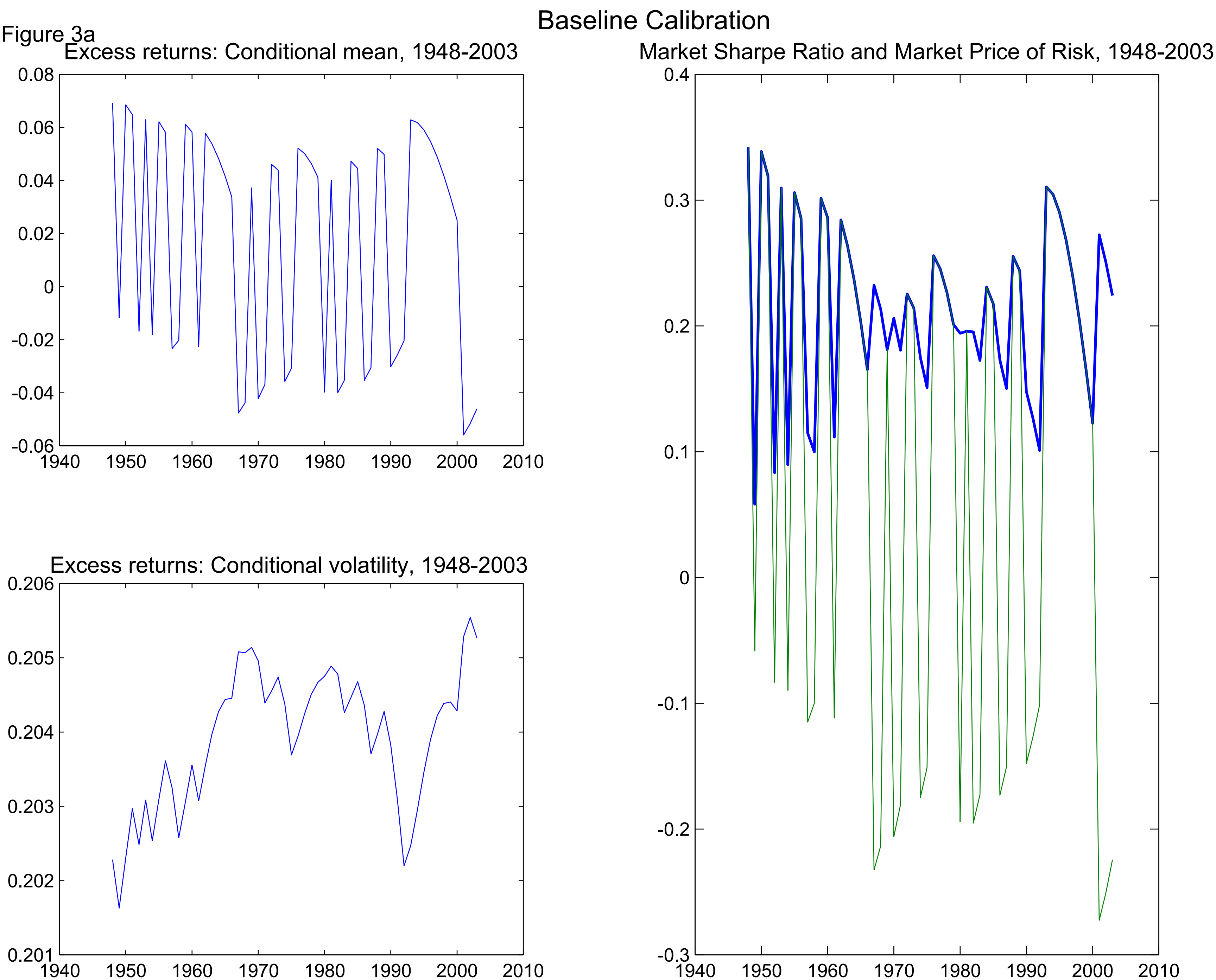

Realized market returns 1948-2002
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
1940
1950
1960
1970
1980
1990
2000
2010

Figure 3a


Baseline Calibration
Excess returns: Conditional mean, 1948-2003
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
1940
1950
1960
1970
1980
1990
2000
2010
Excess returns: Conditional volatility, 1948-2003
0.206
0.205
0.204
0.203
0.202
0.201
Market Sharpe Ratio and Market Price of Risk, 1948-2003
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3

IID Calibration

Figure 3b