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ϵ Expansion for the Conductivity of a Random Resistor Network

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We present a reanalysis of the renormalization-group calculation to first order in $\epsilon = 6-d$, where d is the spatial dimensionality, of the exponent, t, which describes the behavior of the conductivity of a percolating network at the percolation threshold. If we set $t=(d-2)\nu_p+\zeta$, where ν_p is the correlation-length exponent, then our result is $\zeta = 1+(\epsilon/42)$. This result clarifies several previously paradoxical results concerning resistor networks and shows that the Alexander-Orbach relation breaks down at order ϵ.

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The static and dynamic properties of percolating networks have been the object of many studies in recent years.[1] Of particularly enduring interest is the problem of the conductivity of randomly diluted resistor networks.[2] In this problem, each bond between nearest-neighboring sites on a lattice in d spatial dimensions is occupied with a resistor of conductance σ with probability p and vacant (open) with probability $1-p$. Of interest are the properties of the infinite connected network which appears for $p > p_c$, where p_c is the threshold value of p. The exponents describing the threshold behavior of (a) the probability $P(p)$ that a site be in the infinite cluster, (b) the correlation length ξ_p associated with the pair-connectedness correlation function $\chi_p(\vec{x}, \vec{x}')$ indicating whether or not two sites are in the same cluster, (c) the pair-connectedness function $\chi_p = \sum_{\vec{x}} \chi_p(\vec{x}, \vec{x}')$, and (d) the macroscopic conductivity Σ have been extensively discussed.[1,2] The respective associated critical exponents are defined for $p \to p_c$ by $P(p) \sim (p-p_c)^{\beta_p}$, $\xi_p \sim (p-p_c)^{-\gamma_p}$, $\chi_p \sim |p-p_c|^{-\nu_p}$, and $\Sigma \sim |p-p_c|^t$. For $d > 6$ mean-field theory holds, in which case $\beta_p = \gamma_p = 2\nu_p = 1$. For $d < 6$, an expansion in powers of $\epsilon = 6-d$ gave[3] $\beta_p = 1-\epsilon/7$, $\gamma_p = 1+\epsilon/7$, and $\nu_p = (1/2)+(5\epsilon/84)$. Results to order ϵ^2 have been given[4] but are not needed here.

Simple scaling arguments[5-9] imply that the conductivity exponent can be expressed as

$$t=(d-2)\nu_p+\zeta, \tag{1}$$

where ζ is a crossover exponent which physically describes the length L of strands connecting nodes in the node-link picture,[5,8] $L \sim |p-p_c|^{-\zeta}$. In mean-field theory, $\zeta = 1$. In an ϵ expansion about six dimensions, one expects $\zeta = 1 + a\epsilon + b\epsilon^2 + \ldots$. Dasgupta, Harris, and Lubensky[7] (DHL) calculated ζ using the zero-state limit of a diluted s-state Potts model and found $a = 0$. Dasgupta, in an unpublished thesis, found $b = 0$. Stephen[9] reported $a = 0$ using a totally different approach based on a phase-fluctuation, i.e., x-y, model. Finally, Wallace and Young[10] (WY) presented a "proof" that $\zeta = 1$ to all orders in perturbation theory.

Ever since this work, efforts have continued to reconcile this apparently firm result with numerical evidence, which has become progressively more compelling[11] that $t \neq 1$ for $d = 2$, as Eq. (1) with $\zeta = 1$ would imply. In addition, a conjecture by Alexander and Orbach,[12] which implies that[13]

$$\zeta = \tfrac{1}{2}(\gamma_p + \beta_p), \tag{2}$$

for all $d < 6$, agrees extremely well with numerical calculations for $2 < d < 6$, causing many to wonder if it were in fact exact, even though it disagrees with the results of Dasgupta and WY at order ϵ^2. In view of the important ramifications of the result $\zeta = 1$ and because the calculation of DHL has been criticized,[14] we decided to reexamine the calculations of DHL and Stephen.

This reexamination has led us to develop techniques which make possible calculations of many hitherto inaccessible crossover phenomena at the percolation threshold. In this Letter we focus on the calculation of ζ. We find that some terms neglected in DHL must be retained and lead to order-ϵ corrections to ζ:

$$\zeta = 1 + \frac{\epsilon}{42} \neq \frac{1}{2}(\gamma_p + \beta_p), \quad (3)$$

so that the Alexander-Orbach conjecture breaks down to first rather than second order in ϵ. In addition, we find that the diluted s-state Potts model with coupling σJ in the limit $s \to 0$ is characterized by an *infinite number of crossover exponents*,

$$\phi_k = 1 + a_k \epsilon + O(\epsilon^2), \quad (4)$$

so that the order-parameter susceptibility in the vicinity of the percolation threshold satisfies the scaling relation

$$\chi(\vec{x}, \vec{x}') = |p - p_c|^{2\beta_p} f(\{\tilde{w}_k |p - p_c|^{-\phi_k}\}, (\vec{x} - \vec{x}')/\xi), \quad (5)$$

where $\{\}$ refers to the set of variables with $k = 1, 2, \ldots$. In this equation, the variable $\tilde{w}_k$ is proportional to J^{-k} in the limit $J \to \infty$. Thus, as discussed in DHL, we have

$$\chi(\vec{x}, \vec{x}') \to \chi_p(\vec{x}, \vec{x}') - (1/\sigma J)[R(\vec{x}, \vec{x}')]_{\mathrm{av}}, \quad (6)$$

as $J \to \infty$, where $[]_{\mathrm{av}}$ denotes an average over all configurations θ of occupied and unoccupied bonds, $\chi_p(\vec{x}, \vec{x}')$ is the pair-connectedness susceptibility of the percolation problem,[1] and $R(\vec{x}, \vec{x}')$ is the resistance between sites $\vec{x}$ and $\vec{x}'$ if they are connected in the configuration θ and zero otherwise. One has[6,7,9] $\sum_{\vec{x}'}[R(\vec{x}, \vec{x}')]_{\mathrm{av}} \sim |p - p_c|^{-\gamma_p - \zeta}$, so that $\zeta = \phi_1$. Alternatively, ϕ_1 measures the power-law decay of the average resistance at $p = p_c$: $[R(0,x)]_{\mathrm{av}} \sim x^{(-2\beta_p + \phi_1)/\nu}$. It can be shown that the other crossover exponents ϕ_k measure the power-law decay of the kth cumulant of the resistance at $p = p_c$, e.g.,

$$[R(0,x)^2]_{\mathrm{av}} - [R(0,x)]_{\mathrm{av}}^2 \sim x^{(-2\beta_p + \phi_2)/\nu}.$$

Our calculation follows that of DHL but with some differences which we note. The calculation uses an effective Hamiltonian, H, for an s-state Potts model replicated n times (i.e., having s^n states) in the limit $n, s \to 0$:

$$-H/kT = \sum_{\langle \vec{x}, \vec{x}' \rangle} \sum_{t=1}^{n} A_t (s-1)^t \sum_{1 \le \alpha_1 < \alpha_2 \ldots < \alpha_t \le n} \prod_{i=1}^{t} [\vec{v}_{\alpha_i}(\vec{x}) \cdot \vec{v}_{\alpha_i}(\vec{x}')], \quad (7)$$

where $\langle \vec{x}, \vec{x}' \rangle$ denotes a sum over pairs of nearest neighbors and in each replica α, $\vec{v}_\alpha(\vec{x})$ is the Potts vector for site $\vec{x}$ which can point to any of s vertices of an $s-1$ dimensional simplex. The A_t are given by

$$A_t = s^{-n} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left[\frac{p}{1-p}\right]^m \frac{[1 - e^{-sJ\sigma m}]^t}{[1 + (s-1)e^{-sJ\sigma m}]^{t-n}}. \quad (8)$$

$$\to \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\frac{p}{1-p}\right)^m \left(1 + \frac{1}{\sigma J m}\right)^{-t}, \quad \text{as } n, s \to 0. \quad (9)$$

It can be shown that if $\langle\rangle$ denotes an average with relative weight $\exp(-H/kT)$, then one has

$$\lim_{s \to 0} \lim_{n \to 0} \langle \vec{v}_\alpha(\vec{x}) \cdot \vec{v}_\alpha(\vec{x}') \rangle = \chi(\vec{x}, \vec{x}'). \quad (10)$$

A field theory can be obtained from the Hamiltonian of Eq. (7) using the Hubbard-Stratanovich transformation. There is one field for each distinct product of t replicas of $\vec{v}_\alpha$ with bare propagator

$$[G_t^0(\vec{q})]^{-1} = [A_t \gamma(\vec{q})]^{-1} - 1 \sim r_t + q^2, \quad (11)$$

where $\gamma(\vec{q}) = \sum_{\vec{\delta}} e^{i\vec{q}\cdot\vec{\delta}}$ is the sum over nearest-neighbor vectors $\vec{\delta}$. One can see from Eqs. (9) and (11) that

$$r_t = r + \sum_{k=1}^{\infty} w_k t^k, \quad (12)$$

where $w_k \sim J^{-k}$ for large J. As a test of the field theory, we used it to generate an expansion for $\chi(\vec{x}, \vec{x}')$ to order p^3. This calculation obviously involves keeping track of all the nonuniversal constants and cutoffs which normally are of little interest in field theories. This calculation for an arbitrary Bravais lattice reproduced the exact result for $\chi(\vec{x}, \vec{x}')$ to order p^3, but only if the full series in Eq. (9) was used. Since loops can occur in this order, this check tests that the field theory does handle simple series and parallel circuits correctly.

Turning now to the ϵ expansion, we consider the momentum-shell recursion relations[15] for r_t:

$$dr_t/dl = (2 - \eta_p) r_t - u^2 \Sigma_t, \quad (13)$$

where η_p is the critical-point exponent equal to $(-\epsilon/21)+O(\epsilon^2)$ and u is the third-order coupling potential of the field theory. The self energy Σ_t for $n \to 0$ is obtained from the diagram of Fig. 1 as

$$\Sigma_t = -2G_t(1)G_0(1) + G_o^2(1) + \delta\Sigma_t, \tag{14}$$

where $G_0(1)$ is $G_t(q)$ evaluated at $t=0$ and $q=1$ and

$$\delta\Sigma_t = -G_0(1)^2 + \sum_{j,k,p}(1-s)^p(s-2)^{t-j-k}\frac{(t+p-1)!}{p!(t-1)!}\frac{t!}{j!k!(t-j-k)!}G_{t+p-k}(1)G_{t+p-j}(1). \tag{15a}$$

An equivalent but more useful form for $\delta\Sigma_t$ is

$$\delta\Sigma_t = \sum_{k=1}^{t} C_k^t(-1)^k \sum_{m=0}^{\infty} C_m^{m+k-1}[\Delta^{(k)}G_m(1)]^2(1-s)^m, \tag{15b}$$

where $C_m^n = n![m!(n-m)!]^{-1}$ and $\Delta^{(k)}$ is the kth-order finite difference operator,

$$\Delta^{(k)}G_m(1) = \sum_l C_l^k(-1)^l G_{m+1}(1). \tag{16}$$

Apart from $\delta\Sigma_t$, Eq. (14) agrees with DHL. For $J=\infty$, r_t and G_t are indepedent of t, and using Eq. (15b) one sees that $\delta\Sigma_t = 0$ so that the recursion relations reduce to those of percolation. We now need to show the splitting of r_t to order J^{-1}. Since $\Delta^{(k)}G_p(1) \sim J^{-k}$, it might seem that Σ_t is of order J^{-2} and could be neglected for the conductivity calculation. This was tacitly assumed in the calculation of DHL and the others. For $0 < s < 1$, when the sum over m converges rapidly, this reasoning is correct. However, for $s=0$ the situation is more complicated. To analyze this case we start by setting $r_t = r + w_1 t$. We see that to leading order in J^{-1} ($w_1 \sim J^{-1}$, recall)

$$\delta\Sigma_t \sim -t\sum_m [G_{m+1}(1) - G_m(1)]^2 \tag{17a}$$

$$\sim -t\int_0^\infty w_1^2[1+r+mw_1]^{-4}dm \tag{17b}$$

$$\sim -(tw_1/3)G_0(1)^3. \tag{17c}$$

In other words, the sum over m in Eq. (17) is dominated by values of m which are of order $\omega_1^{-1} \sim J$ so that $\delta\Sigma_t$ is of order J^{-1} and contributes to the recursion relation for w_1.

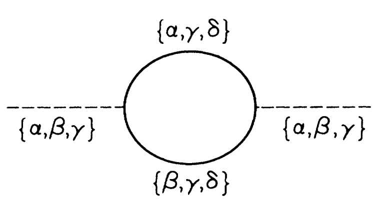

FIG. 1. Diagram for Σ_t of Eq. (14). The external lines carry t replica indices which are partitioned into sets $\{\alpha\}$, $\{\beta\}$, and $\{\gamma\}$ which appear, respectively, on the top, bottom and both internal lines. From the remaining $n-t$ replicas a set $\{\delta\}$ can be chosen to cover the internal lines. The sum over the choices of these sets gives rise to the sums in Eq. (15).

When all the w's are simultaneously nonzero, it is convenient to introduce the variables $v_k = (w_k)^{1/k}$ whose bare values are of order J^{-1} for all k. The linearized recursion relations in terms of these variables obtained from Eqs. (14) and (15b) are

$$k\,dv_k/dl = \sum_{k'} M_{k,k'}v_{k'} + O(v^2), \tag{18}$$

where M is tridiagonal: $M_{k,k'} = 0$ for $k > k'$. Thus the eigenvalues of M needed to determine the crossover exponents for the w_k are simply the diagonal elements $M_{k,k}$ given by $M_{k,k} = 2 - \eta_p + u^2(2+c_k)$, where

$$c_k = \frac{1}{k!}\frac{(-1)^k}{(k-1)!}\int_0^\infty m^{k-1}\left[\left(\frac{d}{dm}\right)^k \frac{1}{1+m^k}\right]^2 dm, \tag{19}$$

so that $c_1 = -\frac{1}{3}$, $c_2 = \frac{1}{5}$, and $c_3 = -\frac{71}{105}$. At the percolation fixed point, $u_2 = \epsilon/7$ so that the crossover exponents for w_k are

$$\phi_k = [2 - (5+3c_k)\epsilon/21]\nu_p = 1 - c_k\epsilon/14. \tag{20}$$

A consequence of Eq. (18) is that the scaling fields $\tilde{w}_k$ are no longer strictly proportional to a single w_k. For example, $\tilde{w}_1 = w_1 + aw_2^{1/2} + bw_3^{1/2}\ldots$ and so forth. Since the conductivity is determined by the scaling behavior of w_1, we have $\zeta = \phi_1 = 1 + \epsilon/42$.

We make a few comments about the result. First of all, since this result violates Eq. (2) at first order in ϵ, we conclude that the Alexander-Orbach conjecture cannot be exact for general spatial dimension, although as said above, it may be numerically quite accurate. It is thus very similar to the Flory approximation[16] for the exponent describing the radius of gyration of a polymer in a good solvent. Secondly, since the anomalous correction term of Eq. (15b) only comes into play for $s \to 0$, we expect the "proof" of Wallace and Young[10] breaks down in this limit. Thirdly, if one discards the result

$\zeta = 1$, several difficulties are immediately overcome. Now we guess that ζ is a weak function of d such that for $d = 2$, $\zeta = t \approx 1.28$. It is no longer necessary to postulate[17] the existence of a critical dimension at which ζ breaks away from $\zeta = 1$ to reach its value at $d = 2$. Fourthly, we have identified an infinite sequence of operators that are relevant at the percolation critical point, each of which has an independent crossover exponent that reduces to unity at six dimensions. We know of no other problem that has such a large number of relevant fields with crossover exponents that are of all the same order as the upper critical dimension is approached. Fifthly, since the anomalous corrections found here only enter for $s \to 0$ they do not affect the original calculation of Stephen and Grest[17] for the dilute Ising model after which the calculation of DHL was patterned. Finally, we note that the technique used here can be used to study other similar models. For example, we have obtained[18] the same set of exponents reported here for the randomly diluted x-y model and for the random-resistor network using Stephen's formalism.[9] For the x-y model, we also find additional exponents measuring angle correlations at p_c.

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