# PUNCTURED JSJ TORI AND TAUTOLOGICAL EXTENSIONS OF AZUMAYA ALGEBRAS

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*To my grandparents Ping Wang, Ling Liu, Meiying Ji, Baoqing Wang, and to my parents Xiaoying Wang and Dajin Wang, whose strength and sacrifices inspire me every day.*

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# ABSTRACT

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The $SL_2(\mathbb{C})$ character variety $X(M)$ has emerged as an important tool in studying the topology of hyperbolic 3-manifolds. Chinburg et al. (2022) constructed arithmetic invariants stemming from a canonical quaternion algebra over the normalization of an irreducible component of $X(M)$ containing a lift of the holonomy representation of $M$. We provide an explicit topological criterion for extending the canonical quaternion algebra over an ideal point, potentially leading to finer arithmetic invariants than those derived in Chinburg et al. (2022). This topological criterion involves Culler-Shalen theory (Culler and Shalen (1983)) and, in some cases, JSJ decompositions of toroidal Dehn fillings of knot complements in the three-sphere. Inspired by the work of Paoluzzi and Porti (2012) and Tillmann (2012), we provide examples of several cases where these refined invariants exist. Along the way, we show that certain families of once- and twice-punctured tori in hyperbolic knot complements can be associated with ideal points of character varieties.

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# INTRODUCTION

## 1.1. Overview

One of the central problems in topology is to distinguish manifolds from each other. The most common method to do this is to define *invariants*, which can take many forms. The work of Thurston (1979) revolutionized the classification problem for 3-manifolds, allowing the construction of invariants stemming from the geometry of 3-manifolds, particularly *hyperbolic* ones:

**Definition 1.1.1.** We will refer to a 3-manifold as *hyperbolic* if it admits a complete metric with constant sectional curvature -1 and admits a finite volume.

**Definition 1.1.2.** Let $K \subset M$ be a knot in any closed 3-manifold. The *knot complement* is the 3-manifold obtained by removing a tubular neighborhood $N(K)$ of $K$ and taking the closure of the resulting manifold, i.e. $\overline{M \setminus N(K)}$. A knot in $S^3$ is *hyperbolic* if its complement is a hyperbolic 3-manifold.

One foundational result is Thurston's *hyperbolic Dehn filling theorem* (Thurston (1979), Section 5.8), which roughly states that all but finitely many Dehn fillings on a hyperbolic knot complement will still be hyperbolic. Another way to put this is that the space of geometric structures on a hyperbolic knot complement largely consists of hyperbolic structures. Indeed, the proof of the hyperbolic Dehn filling theorem formalizes this idea.

Let $M$ be a hyperbolic 3-manifold with finitely generated fundamental group. Then $M$ is the quotient of 3-dimensional hyperbolic 3-space, whose orientation-preserving isometry group is isomorphic to $PSL_2(\mathbb{C})$. THusm $\pi_1(M)$ is a discrete subgroup of $PSL_2(\mathbb{C})$.

**Definition 1.1.3.** The discrete faithful representation

$$\rho_M : \pi_1(M) \hookrightarrow SL_2(\mathbb{C}) \tag{1.1}$$

is called the *holonomy representation of $M$*.

Because geodesics in $M$ correspond to conjugacy classes in $M$, the conjugacy class of holonomy representation contains much of the geometric information of $M$. For irreducible representations in $SL_2(\mathbb{C})$, the conjugacy class is determined by the character. For instance, lengths of geodesics in $M$ are determined by the trace of the holonomy representation on conjugacy classes in $\pi_1(M)$ corresponding to those geodesics. In addition, Culler (1986) showed that any discrete subgroup of $PSL_2(\mathbb{C})$ with no 2-torsion - in particular, a holonomy representation of a hyperbolic 3-manifold - can be lifted to $SL_2(\mathbb{C})$. Therefore, we consider the following space.

**Definition 1.1.4.** The space of traces of representations from $\pi_1(M) \to SL_2(\mathbb{C})$, denoted

$$X(M) = \{\chi_\rho \mid \rho : \pi_1(M) \to SL_2(\mathbb{C})\} \tag{1.2}$$

We refer to this as the *$SL_2(\mathbb{C})$-character variety of $M$*.

This will be a suitable space of geometric structures on $M$. The $SL_2(\mathbb{C})$-character variety is an affine variety cut out by polynomial equations. In the case of hyperbolic knot complements, it turns out that the irreducible component of $X(M)$ containing the trace of the holonomy representation is a curve, i.e. a complex one-dimensional space. The $SL_2(\mathbb{C})$-character variety has become an important tool in the study of the topology of hyperbolic knots. At its core, this thesis bridges two seemingly unrelated constructions stemming from $X(M)$ to study 3-manifolds.

The more classical of the two constructions stems from the work of Culler and Shalen (1983). One can take a “point at infinity" (*ideal point*) of $X(M)$ and associate an *essential surface* in $M$ to it. This is the first bridge connecting $X(M)$ with the topology of $M$, and has been used to prove several fundamental conjectures in 3-manifolds, such as the Smith conjecture (Culler and Shalen (1983)) and the cyclic surgery theorem (Culler et al. (1987)).

The second construction is the work of Chinburg et al. (2022). There are well-studied arithmetic invariants of hyperbolic 3-manifolds such as the *trace field* and the *quaternion algebra* associated to the holonomy representation. Chinburg et al. (2022) generalized these invariants, constructing what is known as an *Azumaya algebra* over $X(M)$. This unlocks more number-theoretic hyperbolic 3-manifold invariants lying in Brauer groups of components of $X(M)$ containing traces of holonomy representations.

One major question addressed in Chinburg et al. (2022) is whether or not their invariant can extend over ideal points of $X(M)$. It was shown that an extension exists, and each possible extension must lie a certain equivalence class of Azumaya algebras. The argument for this result is purely algebraic, relying on results from K-theory. The main results of this thesis provide an explicit extension of the Chinburg-Reid-Stover invariant over ideal points of $X(M)$ related to the Culler-Shalen construction of essential surfaces. This is a refinement of the existing invariant that could shed more light on open conjectures involving character varieties.

## 1.2. Statement of results

We state the main results of the thesis. The main two results will be restated shortly before they are proven in Section 5. In order to effectively state the main results, we include the relevant definitions here. They will be redefined and discussed in more detail in the expository chapters 2 and 3. Let $M$ be a hyperbolic 3-manifold with torus boundary $T$. The concepts in the next few definitions will be elaborated upon in Section 2.5.

**Definition 1.2.1.** A *slope* of $T$ is a choice of $\gamma \in \pi_1(M)$. Note that $\pi_1(T) \cong \mathbb{Z}^2$, generated by a choice of $\mu$ and $\ell$. Usually $\ell$ is a slope that is trivial in the first homology of $M$.

**Definition 1.2.2.** A $(p/q)$-*Dehn filling* of $M$ is a closed 3-manifold obtained by gluing a solid torus to the torus boundary $T$ such that the meridian $\mu$ is glued to $p\mu + q\ell$. This is denoted $M(p/q)$.

**Definition 1.2.3.** A *system of punctured JSJ tori* $\{T_i\}_{i=1}^n$ in $M$ is a disjoint union of $n_i$-punctured tori meeting the boundary torus at slope $\beta$ such that $M(\beta)$ admits a JSJ decomposition $\{\widehat{T}_i\}_{i=1}^n$ where $\widehat{T}_i \cap M = T_i$. A *system of once-punctured JSJ tori* is such that all $n_i = 1$, and a *system of twice-punctured JSJ tori* is such that all $n_i = 2$.

**Definition 1.2.4.** A *Seifert surface* of a knot is a surface in a knot complement with one boundary component and boundary slope 0.

We now state several definitions related to specific 3-manifolds and orbifolds. See Section 2.3 for more on these definitions.

**Definition 1.2.5.** A *2-orbifold* is the quotient of a Euclidean manifold by a properly discontinuous action. This can be described as a 2-manifold with cone points, i.e. points that are fixed under the quotient action.

*Example* 1.2.1. The 2-dimensional disk with cone points of order $p$ and $q$ is an orbifold, and is denoted $D^2(p, q)$. The 2-dimensional annulus with one cone point of order $p$ is another orbifold, and is denoted $A^2(p)$.

**Definition 1.2.6.** Given a 3-manifold $M$ (potentially with boundary), a *Seifert fibration* $p : M \to \mathbb{O}$ is a whose fiber is hoemomorphic to $S^1$. This can be described as a 3-manifold with a decomposition into a disjoint union or circles, where $\mathcal{O}$ is the space of such circles.

We finally define the following terms related to character varieties. More on these can be found in Sections 2.6 and 2.7.

**Definition 1.2.7.** Given a character variety $X(M)$, we let $\widetilde{X(M)}$ denote the normalization of a projective closure of $X(M)$ (see Hartshorne (1977)). The same notation will apply to an irreducible component $C$.

**Definition 1.2.8** (Boyer et al. (2002))**.** A *norm curve* is a component $C$ of the $SL_2(\mathbb{C})$ character variety such that given any nontrivial peripheral torus group element $g \in \pi_1(\partial M)$, $\mathrm{tr}(g)$ is nonconstant on $C$.

Finally, we define the canonical quaternion algebra $A_{k(C)}$ and tautological extension here, as they are heavily involved with the conclusions of the main theorems. For more on this, see Chapter 3.

**Definition 1.2.9.** The *canonical quaternion algebra* $A_{k(C)}$ is the $k(C)$-subalgebra of $M_2(F)$ generated by the elements of $P_C(\pi_1(M))$, i.e.

$$A_{k(C)} = \left\{ \sum_{i=1}^{n} \alpha_i P_C(\gamma_i) \mid \alpha_i \in k(C), \gamma_i \in \pi_1(M) \right\} \tag{1.3}$$

This is a quaternion algebra over $k(C)$.

**Definition 1.2.10.** An algebra $A$ over $R$ is an *Azumaya algebra* if $A$ is free of finite rank $r \geq 1$ as an $R$-module and $A \otimes k$ is a central simple algebra over $k$.

**Definition 1.2.11.** Let $x \subset C$ be a codimension one point. We say that $A_{k(C)}$ *tautologically extends over* $x$ if there exists $g, h \in \pi_1(M)$ such that the $\mathcal{O}_x$-span of $\{1, P_C(g), P_C(h), P_C(gh)\}$ is an Azumaya algebra over $\mathcal{O}_x$.

The first theorem addresses the case of essential once-punctured tori (aka *Seifert surfaces*) that cap off to a JSJ decomposition under 0-Dehn surgery on a hyperbolic knot. The content of this theorem is that under favorable circumstances, such a family of once-punctured tori is detected by an ideal point of $X(M)$ via Culler-Shalen theory, and an explicit extension of the Chinburg-Reid-Stover invariant over this ideal point can be identified.

**Theorem 1.2.1.** *Let $M = S^3 \setminus K$ be a hyperbolic knot complement, and let $\{T_i\}_{i=1}^n \subset M$ be a system of disjoint non-parallel once-punctured JSJ tori with slope 0. Suppose the following:*

- *All irreducible components of $X(M)$ containing irreducible characters are norm curves.*
- *No JSJ complementary region of $M(0)$ is Seifert fibered over the annulus.*
- *No JSJ torus in $M(0)$ bounds two hyperbolic components on both sides.*

*Then $\bigcup_{i=1}^n T_i$ is detected by an ideal point $x$ on a norm curve $\tilde{C} \subset \widetilde{X(M)}$, and $A_{k(C)}$ tautologically extends over $x$.*

The second theorem is similar to the first, but addresses the case of twice-punctured tori.

**Theorem 1.2.2.** *Let $M = S^3 \setminus K$ be a hyperbolic knot complement, and let $\bigcup_{i=1}^{n-1} T_i$ be a system* *of disjoint non-parallel separating twice-punctured JSJ tori with slope $\beta$. Suppose the following:*

1. *All components of $X(M)$ containing traces of irreducible representations are norm curves.*
2. *$M(\beta)$ has JSJ components that are either*
   (a) *Seifert-fibered over $D^2(p,q)$ with $p \geq 2, q > 2$*
   (b) *Seifert-fibered over $A^2(r)$ with $r \geq 2$*
   (c) *hyperbolic*
   (d) *the twisted $I$-bundle over the Klein bottle*
3. *If $n = 2$, the JSJ torus in $M(\beta)$ does not bound two twisted $I$-bundles of Klein bottles.*
4. *Let $\mathcal{O}$ be a Seifert-fibered JSJ component $\mathcal{O}$ of $M(\beta)$ meeting a twisted $I$-bundle over the Klein bottle, denoted $\tilde{K}$. Then the regular fiber of $\mathcal{O}$ is not glued to the regular fiber of $\tilde{K}$ when viewing $\tilde{K}$ as a 3-manifold Seifert-fibered over $D^2(2,2)$.*

*Then $\bigcup_{i=1}^{n-1} T_i$ is detected by an ideal point $x$ on a norm curve $\tilde{C} \subset \widetilde{X(M)}$, and $A_{k(C)}$ tautologically extends over $x$.*

Consider the following infinite examples of knots.

**Definition 1.2.12.** A *two-bridge knot* is a knot which has a diagram where the height function has two maxima and minima.

**Definition 1.2.13.** A $(p_1, \ldots, p_n)$-*pretzel link* is a link whose diagram consists of tangles with $p_i$ twists whose tangles are connected cyclically. A $(p_1, \ldots, p_n)$ pretzel link is a knot if and only if both $n$ and all the $p_i$ are odd or exactly one of the $p_i$ are even.

For a concrete infinite family of examples, we are able to show the following:

**Corollary 1.2.1.** *Any system of punctured JSJ tori in a two-bridge knot, with the exception of the* *minimal genus Seifert surface of the figure-eight knot, is detected by an ideal point $x$ on a norm curve $C$ in the character variety. In addition, $A_{k(C)}$ tautologically extends over $x$.*

**Corollary 1.2.2.** *Let $K$ be a $(-2, 3, 2n+1)$ pretzel knot with $n \not\equiv 1 \mod 3$, and let $M = S^3 \setminus K$ be its complement in the three-sphere. Then the twice-punctured torus of slope $2(2n+4)$ is detected by an ideal point $x$ on the canonical component of the character variety, and $A_{k(C)}$ extends over $x$.*

## 1.3. Outline of thesis

Chapter 2 covers relevant background on 3-manifolds, i.e. geometrization and hyperbolic 3-manifolds, essential surfaces, Seifert fibrations, and JSJ decompositions, and Dehn fillings. The chapter then summarizes some of the existing literature on $SL_2(\mathbb{C})$-character varieties of hyperbolic 3-manifolds (particularly Culler-Shalen theory). Chapter 3 builds the theory of Azumaya algebras over schemes, summarizes the Chinburg-Reid-Stover hyperbolic knot invariant while providing an interpretation via Brauer groups, and then motivates the main results in the thesis by presenting a potential refinement of the Chinburg-Reid-Stover invariant called the *tautological extension*. Chapter 4 covers a technical condition called $SL_2(\mathbb{C})$-compatibility which is needed for the proof of the main results. Chapter 5 states and proves the main results of the thesis surrounding detection of punctured tori in hyperbolic knot complements. Chapter 6 demonstrates examples, related results, and future directions this research could lead to.

# CHAPTER 2

# 3-MANIFOLDS

## 2.1. 3-manifolds and geometrization

Roughly speaking, a *smooth manifold* is a topological space locally homeomorphic to $\mathbb{R}^n$ with smooth coordinate changes. In what follows, all smooth manifolds will be assumed connected unless otherwise stated. Smooth manifolds are some of the central objects of mathematics, and a central aim of research in geometry and topology is to classify smooth manifolds. Three-dimensional smooth manifolds (which we will henceforth refer to as "3-manifolds") became of particular interest in the twentieth century, motivated in part by the famous Poincare conjecture, stated as follows:

**Conjecture 2.1.1** (Poincare conjecture)**.** *Any closed, connected, orientable, simply-connected 3-manifold is homeomorphic to the three-sphere.*

This conjecture was proven as a consequence of Thurston's revolutionary Geometrization conjecture, which was proven by Perelman (2003). We first state some definitions in order to state the conjecture.

**Definition 2.1.1.** A *prime* 3-manifold is one that cannot be expressed as the connected sum of manifolds that are not $S^3$.

**Definition 2.1.2.** A *geometric structure* is a model simply connected space $X$ and a group action of $G$ on $X$ such that coordinate change maps are in $G$.

*Remark* 2.1.1. Often, a model space is a universal cover, and the group $G$ is the isometry group. This is the case for hyperbolic 3-manifolds.

We are now ready to state the geometrization theorem for 3-manifolds.

**Theorem 2.1.1** (Perelman (2003))**.** *Every prime 3-manifold can be cut along embedded tori so that the interior of each complementary region has one of the following eight geometric structures:*

1. *spherical geometry, modeled on* $S^3$
2. *Euclidean geometry, modeled on* $\mathbb{E}^3$
3. *hyperbolic geometry, modeled on* $\mathbb{H}^3$
4. $S^2 \times \mathbb{R}$ *geometry*
5. $\mathbb{H}^2 \times \mathbb{R}$ *geometry*
6. *The universal cover* $\widetilde{SL_2(\mathbb{R})}$
7. *Nil geometry, modeled on the Heisenberg group*
8. *Solv geometry, fibering over* $\mathbb{R}$ *with* $\mathbb{R}^2$ *fiber*

This theorem implies that the geometry and topology of three-manifolds are intimately connected. Of the eight geometries, this connection bore the most fruit for hyperbolic three-manifolds, whose geometric invariants continue to be an active area of study. One manifestation of the connection between geometry and topology of hyperbolic 3-manifolds is the following fundamental result:

**Theorem 2.1.2** (Mostow (1968))**.** *Suppose $M$ and $N$ are complete finite-volume hyperbolic 3-manifolds. If there is an isomorphism $f : \pi_1(M) \to \pi_1(N)$, then it is induced by a unique isometry $\hat{f} : M \to N$.*

For our purposes, Mostow's rigidity theorem implies that fundamental groups of finite volume hyperbolic 3-manifolds determine their geometry and topology. It is thus fruitful to study various aspects of fundamental groups of hyperbolic 3-manifolds in order to study their geometry and topology.

The rest of this introduction will discuss various constructions and aspects of 3-manifolds and their fundamental groups, all of which will become central in the main results of the thesis.

## 2.2. Essential surfaces

Surfaces in 3-manifolds are fundamental tools to study the geometry and topology of 3-manifolds. To name a few, minimal surfaces, totally geodesic surfaces, normal surfaces, and essential surfaces have all been studied in depth. We will focus on essential surfaces in 3-manifolds.

**Definition 2.2.1.** Let $S$ be a surface in a three-manifold $M$. A *trivial compressing disk* is a disk $D \subset M$ such that $D \cap S = \partial D$ such that $\partial D$ does bounds a disk in $S$.

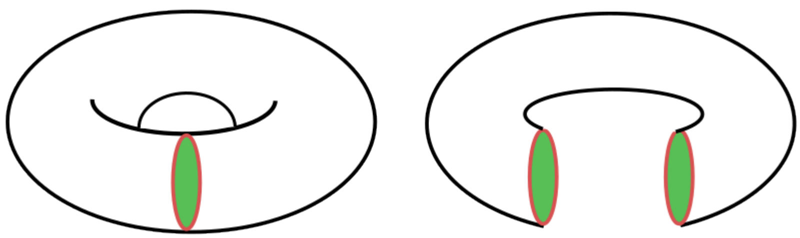

Figure 2.1: A compressing disk and its compression.

**Definition 2.2.2.** Let $M$ be a 3-manifold. An embedded surface $S \subset M$ is *incompressible* if $S$ admits no nontrivial compressing disks.

**Definition 2.2.3.** The *complementary regions* of an incompressible surface $S \subset M$ are the connected components of $M \setminus S$.

*Remark* 2.2.1. One characterizing property of orientable incompressible surfaces $S \subset M$ is that the induced map $\pi_1(S) \hookrightarrow \pi_1(M)$ is an injection (Lemma III.8 in Jaco (1980)). This means that $\pi_1(M)$ is either an $HNN$-extension or a free product with amalgamation of the fundamental groups of the complementary regions.

One way of describing incompressible surfaces is that they are “seen" in the topology, i.e. the fundamental group, of a 3-manifold.

**Definition 2.2.4.** Let $M$ be a compact orientable 3-manifold with nonempty boundary. Given a surface $S \subset M$, a *boundary-compressing disk* is a disk $D \subset M$ such that $D \cap S = \partial D \cap S = \alpha$ is an arc in $S$ and $D \cap \partial M = \partial D - \alpha$. We say that $S$ is *boundary-incompressible* if there does not exist a boundary-compressing disk.

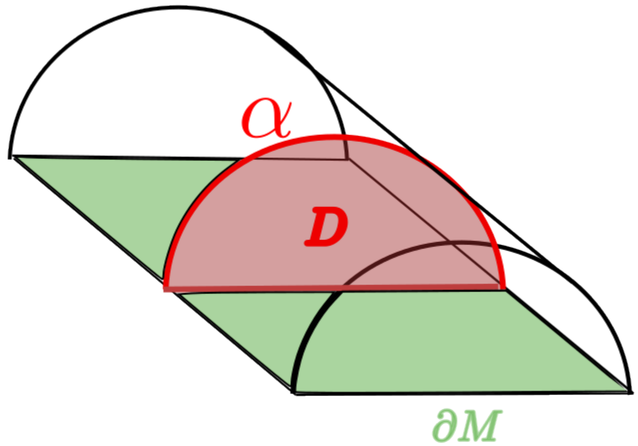


Figure 2.2: A boundary-compressing disk.

**Definition 2.2.5.** Given a 3-manifold $M$ with nonempty boundary, an embedded surface $S \subset M$ is *boundary-parallel* if it can be isotoped to lie in the boundary of $M$.

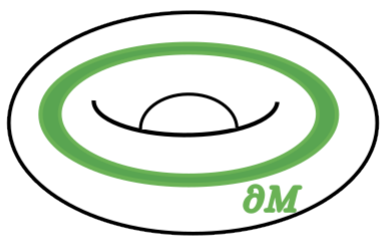


Figure 2.3: A boundary-parallel torus.

**Definition 2.2.6.** An embedded surface $S \subset M$ is *essential* if it is incompressible, boundary-incompressible, and not boundary-parallel.

In this thesis, we will mainly be concerned with essential surfaces in hyperbolic knot complements.

## 2.3. Seifert fibrations

We will discuss an important topological property of 3-manifolds called *Seifert fibration*.

**Definition 2.3.1.** A *2-orbifold* is the quotient of a Euclidean manifold by a properly discontinuous action. This can be described as a 2-manifold with cone points, i.e. points that are fixed under the quotient action.

*Example* 2.3.1. The *pillowcase orbifold* is obtained as a quotient of the torus by the "flipping" homeomorphism pictured below. It is a sphere with four cone points of order two.

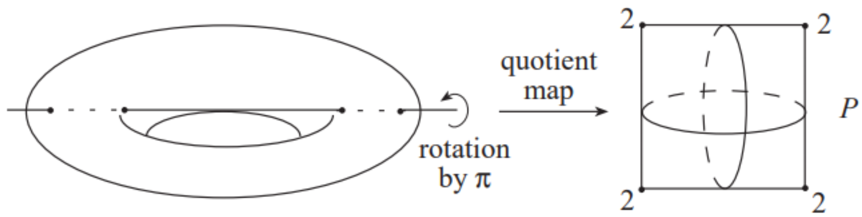


Figure 2.4: Source: Example 2.1 in "Three-dimensional orbifolds and cone manifolds", Cooper et al. (2000)

*Example* 2.3.2. The *modular orbifold* is obtained as a quotient of the hyperbolic plane by the action of $PSL_2(\mathbb{Z})$. It is a disk with one cone point of order two and one cone point of order three. Another name for this orbifold is the "$(2, 3, \infty)$-turnover".

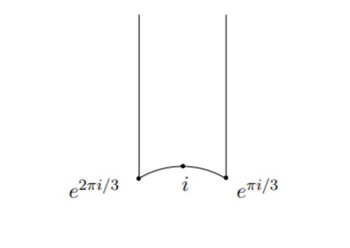


Figure 2.5: The fundamental domain of the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}^2$. Source: Chapter 12 of "A Primer on Mapping Class Groups", Farb and Margalit (2012)

Orbifolds, like 2-manifolds, have geometry that is either Euclidean, spherical, or hyperbolic. (See Chapter 13 of Thurston (1979).) Notably, the isometry group of $\mathbb{H}^2$ is isomorphic to $PSL_2(\mathbb{R})$.

**Definition 2.3.2.** Given a 3-manifold $M$ (potentially with boundary), a *Seifert fibration* $p : M \to \mathbb{O}$ is a map whose fiber is homeomorphic to $S^1$. This can be described as a 3-manifold with a decomposition into a disjoint union or circles, where $\mathcal{O}$ is the space of such circles.

*Example* 2.3.3. It is well-known (see, for instance, Chapter 10 of Milnor (1972)) that the trefoil knot complement is Seifert-fibered over the modular orbifold.

A Seifert fibration induces the fundamental group sequence

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(\mathcal{O}) \to 1 \tag{2.1}$$

**Definition 2.3.3.** A *regular fiber* of a Seifert-fibered 3-manifold $M$ is a representative in $\pi_1(M)$ of the generator of $\mathbb{Z}$ in the above exact sequence.

## 2.4. JSJ decompositions

A central concept in this thesis is the *JSJ decomposition*. The following result is closely related to geometrization of 3-manifolds.

**Theorem 2.4.1** (Jaco and Shalen (1979))**.** *Any prime orientable closed 3-manifold has, up to isotopy, a unique minimal collection of disjointly embedded incompressible tori such that each complementary region obtained by cutting along these tori is either atoroidal or Seifert-fibered.*

Note that the JSJ decomposition is slightly different from the torus decomposition from the geometrization conjecture of 3-manifolds, since components with Solv geometry are not Seifert-fibered but also not hyperbolic. This theorem establishes Seifert-fibered spaces as the other prevalent type of 3-manifolds (aside from hyperbolic 3-manifolds). JSJ decompositions have the following important properties.

**Theorem 2.4.2** (Jaco and Shalen (1979))**.** *Let $N$ be a compact irreducible orientable 3-manifold with JSJ tori $\{T_i\}$. Then:*

1. *Each complementary region of $\overline{N \setminus \bigcup\{T_i\}}$ is atoroidal or Seifert-fibered.*

2. *If $T_i$ cobounds Seifert fibered connected components $N_j, N_k$ of the complementary region, with matching boundary tori $T_j \subset \partial N_j, T_k \subset \partial N_k$, then their regular fibers do not match. That is, if $[h_j] \in \pi_1(N_j), [h_k] \in \pi_1(N_k)$ are the regular fibers, then the gluing map $\phi : T_j \to \partial T_k$ does*

*not send $[h_j]$ to $[h_k]$.*

3. *If $T^2 \times I$ is a complementary region, then $N$ is a torus bundle over $S^1$.*

## 2.5. Dehn fillings

Due to the nature of JSJ decompositions, it is very common to study 3-manifolds with boundary consisting of disjoint tori. Let $M$ be such a manifold with torus boundary components $T_1, \ldots, T_n$. One can obtain other 3-manifolds from $M$ as follows.

**Definition 2.5.1.** A $(p_1/q_1, \ldots, p_n/q_n)$-*Dehn filling* of $M$ is a closed 3-manifold obtained by gluing a solid torus to the torus boundary $T_i$ such that the meridian $\mu$ of the solid torus is glued to $q_i m_i + p_i \ell_i$, where $m, \ell$ are the meridian and longitude of $T_i$. Here $(p_1/q_1, \ldots, p_n/q_n)$ is referred to as the *slope* of the Dehn filling.

Dehn filling is an important way to change 3-manifolds from one to the other. One of the central results about hyperbolic 3-manifolds is Thurston's hyperbolic Dehn filling theorem:

**Theorem 2.5.1** (Thurston (1979))**.** *Let $M$ be a complete hyperbolic 3-manifold homeomorphic to a compact 3-manifold with finitely many torus boundary components $T_1, \ldots, T_n$. Then for all but finite values of $(p_1/q_1, \ldots, p_n/q_n)$, the Dehn filling $M(p_1/q_1, \ldots, p_n/q_n)$ is also hyperbolic.*

In a rough sense, Dehn filling of 3-manifolds largely preserves hyperbolic geometry. The finitely many slopes whose associated Dehn fillings are non-hyperbolic are called *exceptional Dehn fillings*.

One way for a Dehn filling to be exceptional is the presence of an essential torus.

**Proposition 2.5.1.** *A manifold that contains an embedded essential torus cannot be hyperbolic.*

*Proof.* Suppose $M$ contains an essential torus. Then $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z}^2$, which is generated by two parabolic group elements fixing the same point of $\partial_\infty \mathbb{H}^3$. The thick-thin decomposition (see Section 5.10 of Thurston (1979)) implies that the torus is boundary-parallel, creating a contradiction. $\square$

Let $M$ be a one-cusped hyperbolic 3-manifold, e.g. a hyperbolic knot complement.

**Definition 2.5.2.** A *toroidal Dehn filling* of $M$ is one that contains an essential torus.

Let $N = M(\beta)$ be a toroidal Dehn filling of $M$. Then $N$ admits a JSJ decomposition. The fact that $M$ is atoroidal and the proof of Lemma 3 in Hatcher and Thurston (1985) imply that there must be a system of punctured tori in $M$ which cap off to the JSJ tori in $N$. This leads us to the following definition:

**Definition 2.5.3.** A *system of punctured JSJ tori* $\{T_i\}_{i=1}^n$ in $M$ is a disjoint union of $n_i$-punctured tori meeting the boundary torus at slope $\beta$ such that $M(\beta)$ admits a JSJ decomposition $\{\widehat{T}_i\}_{i=1}^n$ where $\widehat{T}_i \cap M = T_i$.

Systems of punctured JSJ tori in hyperbolic knot complements are one of the central objects in the main results of this thesis.

The proof of the hyperbolic Dehn filling theorem involves several key steps, as detailed in Appendix B of Boileau and Porti (2001):

1. Construct an algebraic deformation space of geometric structures on $M$.
2. Associate a space of “generalized" Dehn fillings of $M$ to points in this deformation space.
3. Show that the image of the map associating points in the deformation space to generalized Dehn fillings is a neighborhood of the hyperbolic structure of the hyperbolic 3-manifold.

The key motivating concept of this proof is a deformation space of geometric structures on $M$. To construct a deformation space of “generalized" Dehn fillings, we introduce the concept of $SL_2(\mathbb{C})$-character varieties of hyperbolic 3-manifolds.

## 2.6. Character varieties

Let $\mathbb{H}^3$ be the hyperbolic upper half space, i.e.

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} \tag{2.2}$$

with metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2} \tag{2.3}$$

This is the unique complete simply connected 3-manifold with constant sectional curvature -1, by the Cartan-Hadamard theorem. Thus, an orientable hyperbolic 3-manifold $M$ is universally covered by $\mathbb{H}^3$. Thus, $M$ is homeomorphic to $\mathbb{H}^3/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a discrete subgroup of the group $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$.

**Definition 2.6.1.** The discrete faithful embedding $\rho_0 : \pi_1(M) \to PSL_2(\mathbb{C})$ is the *holonomy representation* of $\pi_1(M)$.

The holonomy representation captures the hyperbolic-geometric information of $M$ via a construction called the *developing map* (see Thurston (1979), Section 3.5).

It is often convenient to work with $SL_2(\mathbb{C})$ as opposed to $PSL_2(\mathbb{C})$. One reason is that $SL_2(\mathbb{C})$ actually consists of matrices (as opposed to equivalence classes of matrices), which allows one to work with algebraic concepts such as traces and quaternion algebras, as we will later see. Thurston showed that the holonomy representation of any hyperbolic 3-manifold can be lifted to $SL_2(\mathbb{C})$. This argument was detailed in Culler and Shalen (1983). Any $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ which is a lift of the holonomy representation will also be referred to as a holonomy representation of $\pi_1(M)$.

Since geodesics in $M$ correspond to conjugacy classes in $\pi_1(M)$, we utilize the *holonomy character*

$$\chi_0 : g \mapsto \mathrm{tr}(\rho_0(g)) \tag{2.4}$$

Since the holonomy representation $\rho_0$ is *irreducible* (i.e. not conjugate to upper triangular matrices), its conjugacy class is entirely determined by the holonomy character. Thus, an appropriate deformation space of geometric structures on $M$ would be the space of traces of representations $\pi_1(M) \to SL_2(\mathbb{C})$. This is denoted $X(M)$ and is an affine algebraic set.

**Definition 2.6.2.** The $SL_2(\mathbb{C})$ *character variety of* $M$, denoted $X(M)$, is given by

$$X(M) = \{\chi_\rho \mid \rho : \pi_1(M) \to SL_2(\mathbb{C})\} \tag{2.5}$$

Here $\chi_\rho$ denotes the character of the representation $\rho$.

Locally around $\rho_0$, this variety can be viewed as the space of *deformations* of the holonomy representation. Note that in general $X(M)$ is not necessarily irreducible as an algebraic set.

**Definition 2.6.3.** Let $X_0(M)$ denote an irreducible component of the $SL_2(\mathbb{C})$ character variety containing a holonomy representation, called a $SL_2(\mathbb{C})$*-canonical component.*

For a hyperbolic 3-manifold with boundary the union of tori, the complex dimension of $X_0(M)$ is equal to the number of torus boundary components by Thurston (1979), Section 5.5.

It is well-known that the lift of the holonomy representation from $PSL_2(\mathbb{C})$ to $SL_2(\mathbb{C})$ is not necessarily unique; there are various interpretations of the number of such lifts. The one relevant to this thesis is the following:

**Theorem 2.6.1** (Menal-Ferrer and Porti (2010), Section 3.2)**.** *Given a group $G$ with no 2-torsion and a representation $\rho : G \to PSL_2(\mathbb{C})$, the number of lifts into $SL_2(\mathbb{C})$ are in bijection with the first group cohomology $H^1(G; \mathbb{Z}/2\mathbb{Z})$, i.e. homomorphisms $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$.*

These homomorphisms signify how one can flip signs of lifts to $SL_2(\mathbb{C})$ of $PSL_2(\mathbb{C})$ coming from $G$. The (co)-homological interpretation of these lifts will be pivotal to the discussion of $SL_2(\mathbb{C})$-compatibility in Chapter 4.

There are other irreducible components of $X(M)$ which contain topological information. First, let

$\chi_g : X(M) \to \mathbb{C}$ map $\chi \mapsto \chi(g)$.

**Definition 2.6.4** (Boyer et al. (2002))**.** A *norm curve* is an irreducible component $C$ of the $SL_2(\mathbb{C})$ character variety such that given any component $T \subset \partial M$ and nontrivial $g \in \pi_1(T) \hookrightarrow \pi_1(M)$, $\chi_g$ is nonconstant on $C$.

Norm curves were used by Boyer-Zhang in the proof of the finite filling conjecture Boyer and Zhang (2001). In general, norm curves possess desirable properties which allow them to behave similarly to canonical components.

## 2.7. Culler-Shalen theory

Another area that connects the theory of character varieties with 3-manifolds is *Culler-Shalen theory*, a process which takes "points at infinity" on character varieties and associates them to essential surfaces in 3-manifolds. We provide a brief summary here. For the next few definitions, see Chapters I-II of Hartshorne (1977).

**Definition 2.7.1.** Given an algebraic variety $Y$ over $k$, the *function field* $k(C)$ is the field of rational functions over $Y$.

**Definition 2.7.2.** An affine variety $Y$ is *normal* if for all points $P \in Y$, $\mathcal{O}_P$ is integrally closed in its fraction field. For any affine variety $Y$, there is a variety $Y^{\#}$ with morphism $\pi : Y^{\#} \to Y$ such that when $Z$ is a normal variety and $\varphi : Z \to Y$ has Zariski-dense image in $Y$, there is a unique morphism $\theta : Z \to Y^{\#}$ such that $\varphi = \pi \circ \theta$. Then $Y^{\#}$ is the *normalization* of $Y$.

**Definition 2.7.3.** Given an affine curve $C \subset \mathbb{A}^2$, we say that the *projective completion* is a curve in $\mathbb{P}^2$ that restricts to $C$ under an affine chart.

Let $C$ be a complex 1-dimensional curve in $X(M)$ and let $\tilde{C}$ be the projective completion of $C^{\#}$.

**Definition 2.7.4.** An *ideal point* of $C$ is an element $x \in \tilde{C}$ that is birationally identified via $\pi$ with a point in $\overline{C} \setminus C$.

Informally, an ideal point can informally be described as a “point at infinity". We will also need the following:

**Definition 2.7.5.** Let $M$ be a hyperbolic 3-manifold with finitely generated fundamental group. The $SL_2(\mathbb{C})$*-representation variety* $R(M)$ is the set of representations $\pi_1(M) \to SL_2(\mathbb{C})$.

*Remark* 2.7.1. Notice that $R(M)$ is also an affine algebraic set whose equations come directly from the relations of $\pi_1(M)$. There is also a map $t : R(M) \to X(M)$ taking $\rho$ to $\chi_\rho$.

Sections 2 and 3 of Culler and Shalen (1983) describe the following process to attribute an essential surface to any ideal point $x \in \widetilde{C}$.

- Let $D$ be a curve lying in the $SL_2(\mathbb{C})$ representation variety that maps to the component $C$ via $t$, and let $\tilde{D}$ be the projective completion of its normalization. Section 1.3 of Culler and Shalen (1983) defined $P_C : \pi_1(M) \to SL_2(F)$, for $F = k(D)$, to be the *tautological representation*

$$P_C(g) = \begin{pmatrix} f_{11}(g) & f_{12}(g) \\ f_{21}(g) & f_{22}(g) \end{pmatrix} \in SL_2(F) \tag{2.6}$$

  where $f_{ij}(g) : D \to \mathbb{C}$ is a function that, evaluated at $\rho \in D$, outputs the $ij$th entry of $\rho(g)$, for $\rho \in D$.

- An ideal point $x$ gives rise to a discrete valuation $v_x$ on the function field of $C$ (defined by the degree of the zero of the function at $x$), which extends to a valuation $v_{\tilde{x}}$ on $F$. Here $\tilde{x} \in \tilde{D}$ is an ideal point of $D$ lying over $x$ via the projective map $\tilde{t}$. Here $\tilde{t}$ is a regular map for which the inverse image of an ideal point in $\tilde{C}$ consists of ideal points of $\tilde{D}$. (See the proof of Theorem 2.2.1 in Culler and Shalen (1983).)

- The valuation $v_{\tilde{x}}$ defines a tree called the *Bass-Serre tree*, as defined in Section 2.1 of Culler and Shalen (1983). The vertices of the tree are lattices inside $F^2$ which are integral with respect to $v_{\tilde{x}}$ up to scaling, and two vertices are connected by an edge if there exist lattice representatives $L$ and $L'$ such that $\pi L \subsetneq L' \subsetneq L$, where $\pi$ generates the valuation ring of $v_{\tilde{x}}$, denoted $\mathcal{O}_{v_{\tilde{x}}}$.

- Culler and Shalen (1983), Theorem 2.2.1: The tautological representation $P_C$ gives rise to a group action of $\pi_1(M)$ on the Bass-Serre tree. This action satisfies the property that no edges are reversed and that stabilizers of vertices are conjugate to a subgroup of $SL_2(\mathcal{O}_{v_{\tilde{x}}})$.

- Culler and Shalen (1983), Proposition 2.3.1: Consider an equivariant map from the universal cover of $M$ to the tree; an essential surface is obtained by taking the inverse image of the midpoints of edges of the tree and taking an appropriate surface in the isotopy class.

**Definition 2.7.6.** We call an essential surface arising from this construction a *detected essential surface.*

It is a subtle problem to determine which essential surfaces and boundary slopes are detected by ideal points of character varieties. For instance, Chesebro and Tillmann (2007) showed that there exist surfaces in hyperbolic knot complements that are not detected by ideal points in the character variety. Paoluzzi and Porti (2012) showed that certain families of essential Conway spheres in hyperbolic knot complements are always detected by some ideal point in the character variety. Our main results are of a similar type to the results of Paoluzzi and Porti (2012), demonstrating that certain families of once- and twice-punctured tori in hyperbolic knot complements are detected.

# CHAPTER 3

## AZUMAYA ALGEBRAS AND CANONICAL COMPONENTS

### 3.1. Canonical quaternion algebras

The following exposition is based on Maclachlan and Reid (2002). The latter parts of this chapter are adapted from Yi Wang. Punctured JSJ tori and tautological extensions of Azumaya algebras. 2023. URL https://arxiv.org/abs/2309.07848.

We define some arithmetic invariants on non-elementary subgroups $H < SL_2(\mathbb{C})$. Most often, $H$ will be a *finite-volume Kleinian group*, i.e. a group isomorphic to the fundamental group of a complete orientable finite-volume hyperbolic 3-manifold.

**Definition 3.1.1.** A *quaternion algebra* $A$ over $F$ is a four-dimensional $F$-algebra with basis vectors 1, $i$, $j$, $k$ where multiplication on $A$ is such that 1 is an identify element, and

$$i^2 = a1 \qquad j^2 = b1 \qquad ij = -ji = k \tag{3.1}$$

for some $a, b \in F^*$. We denote $A$ by the *Hilbert symbol*

$$\left(\frac{a, b}{F}\right) \tag{3.2}$$

**Definition 3.1.2** (Chapter 3 of Maclachlan and Reid (2002))**.** The *trace field* of $H$ is the field

$$k_H = \mathbb{Q}(\operatorname{tr}(\gamma) \mid \gamma \in H) \tag{3.3}$$

The *quaternion algebra* associated to $H$ is

$$A_H = \left\{ \sum_{i=1}^{n} \alpha_i \gamma_i \mid \alpha_i \in k_H, \gamma_i \in H \right\} \tag{3.4}$$

This can be described as the $k_H$ span of $H$ in the space of 2x2 matrices with $\mathbb{C}$-entries.

The Hilbert symbol for $A_H$ can be described as follows. Suppose $g$ and $h$ are hyperbolic elements of $H$ that do not commute. Then a Hilbert symbol for $A_H$ is given by

$$\left(\frac{\mathrm{tr}^2(g) - 4, \mathrm{tr}([g,h]) - 2}{k_H}\right) \tag{3.5}$$

Now let $C$ be a norm curve of $X(M)$, and let $F = k(D)$ be the function field of a curve in the reprentation variety $D$ lying above $C$ via the trace map $t : R(M) \to X(M)$.

**Definition 3.1.3.** The *canonical quaternion algebra* $A_{k(C)}$ is the $k(C)$-subalgebra of $M_2(F)$ generated by the elements of $P_C(\pi_1(M))$, i.e.

$$A_{k(C)} = \left\{\sum_{i=1}^{n} \alpha_i P_C(\gamma_i) \mid \alpha_i \in k(C), \gamma_i \in \pi_1(M)\right\} \tag{3.6}$$

This is a quaternion algebra over $k(C)$ by Lemmas 2.8 and 2.9 of Chinburg et al. (2022).

**Lemma 3.1.1** (Chinburg et al. (2022), Lemma 2.9)**.** *Let $g, h \in \pi_1(M)$ be two elements such that there exists a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ with character $\chi_\rho \in C$ such that $\rho|_{\langle g,h\rangle}$ is irreducible. Then $A_{k(C)}$ is described by the Hilbert symbol*

$$\left(\frac{I_g^2 - 4, I_{[g,h]} - 2}{k(C)}\right) \tag{3.7}$$

Equipped with an associated quaternion algebra $A_{k(C)}$, Chinburg et al. (2022) address the issue of generalizing the quaternion algebra to an *Azumaya algebra* over a normalized projectivization of $C$.

## 3.2. Azumaya algebras

The following exposition is based on Section 3 of Chinburg et al. (2022) and Chapter IV of Milne (1980).

An quaternion Azumaya algebra generalizes the concept of quaternion algebras to lie over rings, as opposed to just fields. Let $R$ be a commutative local ring with residue field $k$.

**Definition 3.2.1.** An algebra $A$ over $R$ is an *Azumaya algebra* if $A$ is free of finite rank $r \geq 1$ as an $R$-module and $A \otimes k$ is a central simple algebra over $k$.

We can also define Azumaya algebras over curves.

**Definition 3.2.2.** An *Azumaya algebra* $\mathcal{A}$ on a Noetherian scheme $X$ is a locally free sheaf of $\mathcal{O}_X$-algebras such that $\mathcal{A}_x$ is an Azumaya algebra over the stalk $\mathcal{O}_{X,x}$ for every $x \in X$.

**Definition 3.2.3.** A *quaternion Azumaya algebra* is an Azumaya algebra that is rank 4 as a locally free $\mathcal{O}_X$-module.

A fundamental definition is the following:

**Definition 3.2.4.** Let $x$ be a codimension-one point of $X$, and $R = \mathcal{O}_{X,x}$ a discrete valuation ring with fraction field $K$. An Azumaya algebra $A_K$ over $K$ *extends over* $x$ if there exists an Azumaya algebra $\mathcal{A}_R$ over $R$ such that $A_K \cong A_{\mathcal{R}} \otimes_R K$. We say $A_K$ *extends to an Azumaya algebra over* $X$ if $A_K$ extends over $x$ for all codimension one points $x \in X$.

Let $x$ be the generic point of a norm curve $C \subset X(M)$. Then $K = k(C)$, and $A_K = A_{k(C)}$.

**Question 3.2.1.** *When does $A_{k(C)}$ extend to an Azumaya algebra $\mathcal{A}_{\tilde{C}}$ over the smooth projective model $\tilde{C}$ of $C$?*

Chinburg et al. (2022) first shows that $A_{k(C)}$ extends over $\chi \in C$ which are characters of irreducible representations.

**Lemma 3.2.1** (Proposition 4.1a in Chinburg et al. (2022))**.** *The canonical quaternion algebra $A_{k(C)}$ extends to a quaternion Azumaya algebra over the set of $\chi \in \tilde{C}$ where $\chi$ is the character of an irreducible representation.*

*Proof.* Let $\rho$ be an irreducible representation whose character is $\chi$, corresponding to the point $x \in C$.

Choose $g', h' \in \pi_1(M)$ such that $\{1, \rho(g'), \rho(h'), \rho(g'h')\}$ is a $k_\rho$-basis for $A_\rho$. (Here $k_\rho$ denotes the trace field and $A_\rho$ denotes the quaternion algebra associated to $\rho$.) Then $\{1, P_C(g'), P_C(h'), P_C(g'h')\}$ is a basis for $A_{k(C)}$. Let $A_x$ be the $\mathcal{O}_x$-span of this basis. Then the reduction of $A_x$ modulo the maximal ideal of $\mathcal{O}_x$ is equal to $A_\rho$, the quaternion algebra over $k_\rho$. Thus, $A_x \otimes k_\rho = A_x \otimes k(x) = A_\rho$ is a central simple algebra. It is immediate from our constructions that $A_x \otimes k(C) \cong A_{k(C)}$, so by definition, $A_{k(C)}$ extends over $x$. $\square$

This proof directly utilizes the tautological representation to produce an explicit canonical Azumaya algebra over $\chi \in \tilde{C}$. In contrast, we have the proof of the following lemma:

**Lemma 3.2.2** (Proposition 4.1b in Chinburg et al. (2022))**.** *The canonical quaternion algebra $A_{k(C)}$ extends to Azumaya algebra over ideal points of $\tilde{C}$.*

The proof of this lemma relies on the following K-theoretic criterion for extending Azumaya algebras over codimension one points:

**Definition 3.2.5.** Let $x \in X$ be a codimension one point and $A_K$ be an Azumaya algebra over the fraction field $K$ of the stalk at $x$, with Hilbert symbol $\left(\frac{\alpha,\beta}{K}\right)$. The *tame symbol* $\{\alpha, \beta\}_x$ of $A_K$ at $x$ is the class of

$$(-1)^{\mathrm{ord}_x(\alpha)\mathrm{ord}_x(\beta)}\beta^{\mathrm{ord}_x(\alpha)}/\alpha^{\mathrm{ord}_x(\beta)} \in k(x)^*/(k(x)^*)^2 \tag{3.8}$$

**Lemma 3.2.3.** *If the characteristic of $k(x)$ is not 2, the tame symbol is trivial if and only if $A_K$ extends over $x$.*

*Remark* 3.2.1. One important distinction between the extension of $A_{k(C)}$ over irreducible characters and ideal points is that the former provides an *explicit* quaternion Azumaya algebra over the local ring of $\chi \in \tilde{C}$ coming from the tautological representation, while the latter shows *existence* of an Azumaya algebra over the local ring at the ideal point. The main aim of this thesis is to bridge this gap, i.e. find situations where an explicit quaternion Azumaya algebra coming from the tautological representation can be identified over the local ring of an ideal point of $\tilde{C}$, mimicking the proof of

Lemma 3.2.1.

The only type of points left in $\tilde{C}$ are $\chi \in \tilde{C}$ corresponding to reducible characters. Chinburg et al. (2022) provide the following criterion:

**Proposition 3.2.1.** *Let $\Delta_K(t)$ be the Alexander polynomial of the hyperbolic knot $K$. Suppose that for any $w$ such that $\Delta_K(w^2) = 0$, $\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$. Then $A_{k(C)}$ extends over all reducible characters in $\tilde{C}$.*

*Remark* 3.2.2. The proof of this proposition also relies on the tame symbol criterion. In addition, there cannot be a canonical extension coming from the tautological extension over reducible characters, since the dimension of such an algebra would be 3, not 4. It is still open whether or not one can find a canonical Azumaya algebra over the local ring at a reducible character.

## 3.3. Brauer groups

Azumaya algebras over a scheme, under a certain equivalence relation, form a group called the *Brauer group*.

**Definition 3.3.1.** Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if there exist locally free sheaves of $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{F}) \tag{3.9}$$

Here $\mathrm{End}_{\mathcal{O}_X}(\mathcal{H})$ is the sheaf of $\mathcal{O}_X$-module endomorphisms of an $\mathcal{O}_X$ module $\mathcal{H}$. The group of equivalence classes of Azumaya algebras under tensor product is called the *Brauer group* of $X$, denoted $\mathrm{Br}(X)$.

Here are some facts about Brauer groups whose proof can be found in Milne (1980).

**Theorem 3.3.1** (Chapters III and IV of Milne (1980))**.** *Suppose $X$ is a regular integral scheme of dimension at most 2 with function field $K$ that is quasi-projective over a field or a Dedekind ring.*

1. *There is a bijection between the set of isomorphism class of Azumaya algebras $A_X$ of rank $n^2$ over $X$ and the elements of the etale Cech cohomology group $\hat{H}_{\text{ét}}(X, PGL_n)$.*

2. *There is a bijection between the set of isomorphism classes of rank $n$ locally free $\mathcal{O}_X$-modules and elements of $\hat{H}^1_{\text{ét}}(X, GL_n)$.*

3. $Br(X) = \hat{H}^2_{\text{ét}}(X, \mathbb{G}_m)$

4. *There is an exact sequences of etale sheaves of groups*

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1 \tag{3.10}$$

*on $X$. The cohomology of this sequences gives us*

$$\hat{H}^1_{\text{ét}}(X, GL_n) \to \hat{H}^1_{\text{ét}}(X, PGL_n) \to \hat{H}^2_{\text{ét}}(X, \mathbb{G}_m) \tag{3.11}$$

*where $E \in \hat{H}^1_{\text{ét}}(X, GL_n)$ is identified with $End_{\mathcal{O}_X}(E) \in \hat{H}^1_{\text{ét}}(X, PGL_n)$, and the isomorphism class of $A_X \in \hat{H}^1_{\text{ét}}(X, PGL_n)$ is sent to $[A_X] \in Br(X)$.*

5. *Every element of $Br(X)$ has finite order $n$, each of which is represented by an Azumaya algebra of rank $n^2$ over $X$.*

Given a hyperbolic knot $K$, knot complement $M = S^3 \setminus K$ and a norm curve $C \subset X(M)$, Lemmas 3.2.1, 3.2.2 and Proposition 3.2.1 show the following.

**Theorem 3.3.2** (Chinburg et al. (2022))**.** *Suppose that for all $w \in \mathbb{C}$ such that $\Delta_K(w^2) = 0$, $\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$. Then $A_{k(C)}$ extends to Brauer class of $\tilde{C}$.*

One motivating problem for the results in this thesis is to find an explicit canonical extension of $A_{k(C)}$ over ideal points, as opposed to showing an equivalence class exists. In addition, we would like this canonical extension to be related to the geometry and topology of $M$. This would be a refinement of the invariant constructed in Chinburg et al. (2022), in the following sense. Let $X$

be a scheme. By Part 1 of Theorem 3.3.1, the set of isomorphism classes of quaternion Azumaya algebras over $X$ is in bijection with $\check{H}^1(X_{\text{ét}}, PSL_2)$. From the exact sequence of sheaves

$$1 \to \mu_2 \to SL_2 \to PSL_2 \to 1 \tag{3.12}$$

we get the cohomology exact sequence

$$\check{H}^1_{\text{ét}}(X, SL_2) \to \check{H}^1_{\text{ét}}(X, PSL_2) \to \check{H}^2_{\text{ét}}(X, \mu_2) \tag{3.13}$$

Let $C$ be a norm curve in a character variety $X(M)$, and let $U \subset \tilde{C}$ be the Zariski-open subset consisting of points not associated to traces of reducible representations. Our goal is to produce an explicit cohomology element $\check{H}^1_{\text{ét}}(U, PSL_2)$.

## 3.4. Tautological extension

The proof that $A_{k(C)}$ extends over points corresponding to irreducible representations, which is part of the proof of Lemma 3.2.1, directly uses the image of the tautological representation to produce an extension of $A_{k(C)}$ over that point. We would like to use the same construction over ideal points. This leads to the following definition:

**Definition 3.4.1.** Let $x \subset C$ be a codimension one point. We say that $A_{k(C)}$ *tautologically extends over* $x$ if there exists $g, h \in \pi_1(M)$ such that the $\mathcal{O}_x$-span of $\{1, P_C(g), P_C(h), P_C(gh)\}$ is an Azumaya algebra over $\mathcal{O}_x$.

Tautological extension is an explicit and more refined extension of $A_{k(C)}$ over codimension-one points of the canonical component than the tame symbol arguments of Chinburg et al. (2022). The proof of Lemma 3.2.1 shows that $A_{k(C)}$ tautologically extends over any $\chi = \text{tr}(\rho) \in C$ such that $\rho$ is irreducible. If $\rho$ is reducible, we have the following.

**Proposition 3.4.1.** *Let $x_w \in C$ be a point corresponding to a reducible representation $\rho_w$. Then $A_{k(C)}$ does not tautologically extend over $x_w$.*

*Proof.* Suppose $A_{k(C)}$ has a tautological extension over $x_w$. Let $\mathcal{O}_{x_w}$ be the valuation ring associated to $x_w$, with residue field $k(x_w)$, and let $A_{x_w}$ be the $\mathcal{O}_{x_w}$-span of the basis $\{1, P_C(g), P_C(h), P_C(gh)\}$ for some $g, h$ which is an Azumaya algebra. Recall that

$$A_{k(C)} = \left\{ \sum_{i=1}^{n} \alpha_i P_C(\gamma_i) \mid \alpha_i \in k(C), \gamma_i \in \pi_1(M) \right\} \tag{3.14}$$

Then $A_{x_w} \otimes k(x_w)$ must be the $k(x_w)$-span of the set $\{1, \rho_w(g), \rho_w(h), \rho_w(gh)\}$. Since $\rho_w$ is reducible, by Lemma 1.2.4 in Maclachlan and Reid (2002), these vectors are dependent, and hence cannot generate a 4-dimensional algebra over $k_w$. Thus this span is not a 4-dimensional central simple algebra over $k_w$. This leads to a contradiction, and thus $A_{k(C)}$ cannot tautologically extend over $x_w$. $\square$

In light of this, the ideal points $x \in \widetilde{C}$ are the only points where existence of a tautological extension of $A_{k(C)}$ is yet to be determined. The main question this thesis addresses is the following:

**Question 3.4.1.** *Given an ideal point $x$ in $\widetilde{X(M)}$, when does $A_{k(C)}$ tautologically extend over $x$?*

## 3.5. The limiting character

Let $D \subset R(M)$ be a curve such that $t(D) = C$, and let $\tilde{D}$ be the projective completion of the normalization of $D$. Recall the tautological representation $P_C : \pi_1(M) \to SL_2(F)$, with $F = k(D)$. Now suppose $S$ is an essential surface detected by an ideal point $x \in \widetilde{C}$, with associated valuation ring $\mathcal{O}_x$. Let $\tilde{x}$ be an ideal point in $D$ lying over $x$. Note that since $x$ is an ideal point, there will exist some $h \in \pi_1(M)$ such that $P_c(h)$ cannot be conjugated into $SL_2(\mathcal{O}_{\tilde{x}})$, i.e. some trace functions will have poles at the ideal point. Let $M_1, \ldots, M_n$ be the components of $\overline{M \setminus S}$, i.e. the complementary regions. Since $S$ is essential, we have natural maps $\pi_1(M_i) \to \pi_1(M)$. Let $G_1, \ldots, G_n$ be the images of each such map. By Theorem 2.1.2 in Culler and Shalen (1983), $P_c(G_i)$ is $GL_2(F)$-conjugate to a subgroup of $SL_2(\mathcal{O}_{\tilde{x}})$. Thus, for $g \in G_i$, $\chi_g$ does not have a pole at $\tilde{x}$. We construct a representation

$$\rho_{\infty,i} : \pi_1(M_i) \to SL_2(\mathbb{C}) \tag{3.15}$$

as follows. Given $g \in \pi_1(M_i)$, map $g$ into $G_i$, then take the matrix $P_c(g) \in SL_2(F)$, conjugate into $SL_2(\mathcal{O}_{\tilde{x}})$, and compose with the quotient map $\mathcal{O}_{\tilde{x}} \to k_{\tilde{x}} = \mathcal{O}_{\tilde{x}}/\mathfrak{m}_{\mathcal{O}_{\tilde{x}}}$ to obtain an element of $SL_2(k_{\tilde{x}}) \subset SL_2(\mathbb{C})$. Here $\mathfrak{m}_{\mathcal{O}_{\tilde{x}}}$ is the maximal ideal of the valuation ring. Since conjugation preserves trace, the trace of this element is well-defined.

**Definition 3.5.1.** The *limiting character on $M_i$*, denoted $\chi_{\infty,i} = \mathrm{tr}(\rho_{\infty,i})$, is the character of $\rho_{\infty,i}$.

*Remark* 3.5.1. Let $\{\chi_j\} \subset C$ be a sequence of characters that approach $x$. One can see that

$$\chi_{\infty,i} = \lim_{\chi_j \to x} \chi_j|_{G_i} \tag{3.16}$$

In other words, the limiting character is exactly what it sounds like.

The limiting character takes on finite values on each connected component of $S$, since $\pi_1(S_i) \subset \pi_1(M_j)$ for each connected component $S_i$, and for some $M_j$. We have the following:

**Proposition 3.5.1.** *The limiting character is equal to the trace of a reducible representation on a detected essential surface at the ideal point.*

*Proof.* We wish to show that under the Bass-Serre theory from Section 2 of Culler and Shalen (1983), the edge stabilizer is conjugate to the group of matrices in $SL_2(\mathcal{O}_{\tilde{x}})$ consisting of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, d \in \mathcal{O}_{\tilde{x}}, c \in \pi\mathcal{O}_{\tilde{x}} \tag{3.17}$$

where $\pi$ is the uniformizer of $\mathcal{O}_{\tilde{x}}$. Notice that a link of a vertex $[\Lambda_0]$ in a Bass-Serre tree consists of lattices $\Lambda$ such that $\pi\Lambda_0 \subset \Lambda \subset \Lambda_0$, which are in bijective correspondence with submodules of $V = \Lambda_0/\pi\Lambda_0$. This is a 2-dimensional vector space, and thus the link of $[\Lambda_0]$ is bijection with 1-dimensional subspaces of $V$ over the residue field $k_{\tilde{x}}$ of $\mathcal{O}_{\tilde{x}}$. This is the projective line over $k_{\tilde{x}}$. The stabilizer of an edge is the intersection of the stabilizer of $[\Lambda_0]$ within the stabilizer of $[\Lambda_1]$. This can be thought of as the stabilizer of a point in the projective line of $k_{\tilde{x}}$ under of the action of $SL_2(k_{\tilde{x}})$, which is conjugate to the upper triangular matrices in $SL_2(k_{\tilde{x}})$. This completes the proof. $\square$

We have already seen that tautological extensions cannot come from reducible representations associated to any point, and so the basis for a tautological extension of $A_{k(C)}$ over an ideal point cannot come from the fundamental group of a detected essential surface. Instead, we will look toward its complement.

**Proposition 3.5.2.** *Let $S$ be an essential surface detected by the ideal point $x \in \widetilde{C}$. Then if there exists a component $M' \subset \overline{M \setminus S}$ such that the limiting character $\chi_\infty : \pi_1(M') \to \mathbb{C}$ is the trace of an irreducible representation, then $A_{k(C)}$ tautologically extends over $x$.*

*Proof.* Suppose that $\chi_\infty$ is the trace of an irreducible representation $\rho_\infty : \pi_1(M') \to SL_2(\mathbb{C})$. Let $k_\infty$ be the field generated by traces of elements in $\pi_1(M')$ under $\rho_\infty$. Since $\chi_g \mid g \in \pi_1(M')$ is contained in the affine ring in a neighborhood around $x$ he residue field $k(x)$ of $\widetilde{C}$ at $x$ contains $k_\infty$. Let

$$A_{\rho_\infty} = \left\{ \sum_{i=1}^{n} \alpha_i \rho_\infty(\gamma_i) \mid \alpha_i \in k_\infty, \gamma_i \in \pi_1(M') \right\} \tag{3.18}$$

Note that $A_{\rho_\infty}$ is a quaternion algebra over $k_\infty$ because $\rho_\infty$ is irreducible. Since $\rho_\infty$ is irreducible, it also follows that $\pi_1(M')$ is nonabelian. Choose noncommuting $g, h \in \pi_1(M')$. By Section 3.6 in Maclachlan and Reid (2002), $\{1, \rho_\infty(g), \rho_\infty(h), \rho_\infty(gh)\}$ is a $k_\infty$-basis for $A_{\rho_\infty}$. Then $\{1, P_C(g), P_C(h), P_C(gh)\}$ forms a basis for $A_{k(C)}$. Let $A_x$ be the $\mathcal{O}_x$-span of this basis. Then the reduction of $A_x$ modulo the maximal ideal $\mathfrak{m}_{\mathcal{O}_x}$ is equal to $A_{\rho_\infty}$, which is a quaternion algebra over $k_\infty$. Thus, $A_x \otimes k_\infty = A_x \otimes k(x) = A_{\rho_\infty}$ is a central simple algebra. It is immediate from our constructions that $A_x \otimes k(C) \cong A_{k(C)}$, so by definition, $A_{k(C)}$ tautologically extends over $x$. $\square$

By the above proposition, our main question can be interpreted as:

**Question 3.5.1.** *When is the limiting character $\chi_\infty$ irreducible at an ideal point of a norm curve?*

The limiting character is connected to Culler-Shalen theory via the following lemma.

**Lemma 3.5.1** (Tillmann (2003))**.** *Let $S$ be an essential surface in an orientable irreducible 3-manifold $M$ with complementary regions $M_1, \dots, M_n$. Let $x_\infty$ be an ideal point on a curve inside the character variety $X(M)$, and the limiting character $\chi_\infty$ at $x_\infty$. There is a natural restriction map*

$$r : X(M) \to X(M_1) \times \cdots \times X(M_n) \tag{3.19}$$

*Then $S$ is detected by $x_\infty$ if and only if the limiting character $(\chi_1, \dots, \chi_n) \in X(M_1) \times \cdots \times X(M_n)$ has the following properties:*

1. *$\chi_i$ is finite for each $i = 1, \dots, n$, i.e. the values of $\chi_i$ on group elements have no poles at $x_\infty$ when restricted to $X(M_i)$.*
2. *For any proper essential subsurface $S'$ of $S$, there exists a complementary region of $\overline{M \setminus S'}$ on which the limiting character is not finite.*
3. *$\chi_i$ match via gluing homeomorphisms on the boundaries of $M_1, \dots, M_n$.*
4. *$\chi_i$ restricted to any boundary component of $M_1, \dots, M_n$ is reducible.*
5. *There is a connected open neighborhood $U$ of $x_\infty$ in $X(M)$ such that $r(U)$ contains an open neighborhood of $(\chi_1, \dots, \chi_n)$ but not $(\chi_1, \dots, \chi_n)$ itself.*

*Remark* 3.5.2. It was alluded to in Tillmann (2003) that Conditions 1, 3, 4, and 5 were sufficient if $S$ was connected, but not if $S$ has multiple components, since condition 3 for surface detection would not be satisfied; it is also stated that Condition 2 rectifies this discrepancy.

# CHAPTER 4

# $SL_2(\mathbb{C})$-COMPATIBILITY

## 4.1. $SL_2(\mathbb{C})$-compatibility

This chapter is adapted from parts of Yi Wang. Punctured JSJ tori and tautological extensions of Azumaya algebras. 2023. URL https://arxiv.org/abs/2309.07848. and Yi Wang. Detection of twice-punctured tori in hyperbolic knot complements 2024. URL https://arxiv.org/abs/2404.06388.

To prove the main theorem, we must establish a property for JSJ decompositions, defined as follows:

**Definition 4.1.1.** Suppose we are given a 3-manifold $N$ with JSJ composition given by tori $T_1, \ldots, T_n$, and connected components $N_1, \ldots, N_r$. We say that $\{T_i\}$ is a $SL_2(\mathbb{C})$-*compatible JSJ decomposition* if there exist representations $\rho_i : \pi_1(N_i) \to SL_2\mathbb{C}$ such that the following hold.

1. For any $g \in \pi_1(\partial N_j)$, $\mathrm{tr}(\rho_j(g)) = \pm 2$.

2. Let $T_i', T_j'$ be torus boundary components of $N_i$ and $N_j$. If $\varphi : T_i' \to T_j'$ is a gluing homeomorphism, then for all $g \in \pi_1(\partial T_i')$, $\mathrm{tr}(\rho_i(g)) = \mathrm{tr}(\rho_j(\varphi(g)))$.

Informally, $SL_2(\mathbb{C})$-compatibility means that traces can be “glued" together along their torus boundary components. The goal of this chapter is to prove that the JSJ decompositions that appear in toroidal Dehn fillings are $SL_2(\mathbb{C})$-compatible.

## 4.2. Cable spaces

The main theorems of this thesis are about JSJ decompositions of toroidal surgeries. Here we set up a discussion of $SL_2(\mathbb{C})$-compatiblity of the JSJ decompositions of these toridal Dehn fillings, which will be needed for the proof of the main theorems..

**Definition 4.2.1.** Let $A^2(q)$ be a 2-orbifold with base space an annulus and a cone point of order $q$ for $q \geq 2$. A *cable space* is a 3-manifold $N$ Seifert-fibered over $A^2(q)$.

Cable spaces will be the most common JSJ-components of toroidal Dehn fillings of knot complements with essential once-punctured tori. In order to prove the main theorems, we must analyze the character varieties of these spaces.

**Lemma 4.2.1.** *Let $N$ be a cable space. Then $X(N)$ is two-dimensional.*

*Proof.* By Lemma 3.1 in Kitano (1994), every irreducible representation of $\pi_1(N)$ into $SL_2(\mathbb{C})$ sends the regular fiber to $\pm I$. It follows that the dimension of $X(N)$ is equal to the dimension of $X(A^2(q))$. We have the fundamental group presentation

$$\pi_1(A^2(q)) = \langle a, b \mid (ab)^q = 1 \rangle \tag{4.1}$$

Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Knowing that $X(F_2) \cong \mathbb{C}^3$, it follows that $X(A^2(q)) = \{(x, y, z) \in \mathbb{C}^3 \mid z = \zeta_q + \zeta_q^{-1}\} \cong \mathbb{C}^2$, where $\zeta_q$ is any $q$th root of unity. This is two-dimensional, so we are done. $\square$

**Lemma 4.2.2.** *The only hyperbolic 2-orbifolds with two geodesic boundary components whose character variety has dimension two are $A^2(q)$, where $A^2$ is the annulus and $q \geq 2$ is the order of the cone point.*

*Proof.* First, suppose the base space of a Seifert-fibered space is orientable, i.e. a twice-punctured orientable surface of genus $g$. Suppose we are in the situation of an orbifold whose base space is a twice-punctured surface $S_g$ with $c$ cone points. Call the orbifold $S_g'$. Let $S_{g,n}$ be the $n$-punctured surface; we have $\dim(X(S_g')) \geq \dim(S_{g,c+2}) - c$, since there are $c$ equations defining the relations induced by the cone points. Thus, it suffices to show that for all $S_{g,n}$ with $n \geq 2$ and not a twice or thrice-punctured sphere, $\dim(X(S_{g,n})) > n$. Note that $S_{g,n}$ is homotopy equivalent to a wedge of $n + 2g - 1$ circles, and so $\pi_1(S_{g,n}) \cong F_{n+2g-1}$, i.e. the free group on $n + 2g - 1$ generators. By Section 2 of Ashley et al. (2017), $\dim(X(F_{n+2g-1})) = 3n + 6g - 6$, and if $n \geq 2$ and $g > 0$ this is greater than $n$. If $g = 0$, $n > 3$, and so $3n + 6g - 6 > n$ still. So we are done if the orbifold is orientable.

If the orbifold is non-orientable, the base orbifold is a twice-punctured connected sum of $g$ copies of $\mathbb{R}P^2$ with some cone points; let the base surface with $n$ punctures be denoted $S_{g,n}$ as above. By the same logic as above, it suffices to show that $\dim(X(S_{g,n})) > n$. Indeed, since $S_{g,n}$ is homotopy equivalent to a wedge of $n + g - 1$ circles, we have that $\pi_1(S_{g,n}) = F_{n+g-1}$, and from Ashley et al. (2017) we have $\dim(X(F_{n+g-1})) = 3n + 3g - 6 > n$ if $n \geq 2$ and $g > 0$. So we are done in both the orientable and non-orientable cases now. $\square$

Combining with Lemma 3.1 in Kitano (1994) gives us the following:

**Corollary 4.2.1.** *If an orientable 3-manifold with two torus boundary components is not Seifert-fibered over the annulus and has character variety of dimension two, it is either hyperbolic or a cable space.*

## 4.3. The twisted $I$-bundle over the Klein bottle

One space that often appears in JSJ decompositions of Dehn fillings capping of twice-punctured tori in knot complements is the *twisted $I$-bundle over the Klein bottle*, denoted $\tilde{K}$. This is an orientable 3-manifold with torus boundary. For an in-depth discussion of $\tilde{K}$, we refer the reader to Lidman and Watson (2014).

We will study representations of $\pi_1(\tilde{K})$. It is known that

$$\pi_1(\tilde{K}) = \langle s, t \mid s^2t^2 = 1 \rangle \tag{4.2}$$

The incompressible torus boundary has fundamental group generated by the meridian $st$ and longitude $s^2$. It will be useful to compute the representation and character varieties of this group. Utilizing the trace relation

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}) \tag{4.3}$$

we can determine that the character variety is a complex cubic surface:

$$X(\tilde{K}) = \left\{(x, y, z) \in \mathbb{C}^3 \mid xyz - x^2 - y^2 = 0\right\} \tag{4.4}$$

with coordinates are $x = \mathrm{tr}(s), y = \mathrm{tr}(t), z = \mathrm{tr}(st)$. We now determine all the irreducible $SL_2(\mathbb{C})$-representations of $\pi_1(\tilde{K})$. Any such representation is conjugate to

$$\rho(s) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \qquad \rho(t) = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix} \tag{4.5}$$

Determining $\rho(s^2t^2)$ and setting the entries equal to the identity, we obtain that any irreducible representation must satisfy

$$\rho(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho(t) = \begin{pmatrix} \pm i & 0 \\ r & \mp i \end{pmatrix} \qquad \rho(st) = \begin{pmatrix} r-1 & -i \\ -ri & -1 \end{pmatrix} \tag{4.6}$$

which correspond to the point $(0, 0, r-2) \in X(\tilde{K})$. The reducible representations can be conjugated into

$$\rho(s) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \qquad \rho(t) = \begin{pmatrix} y & r \\ 0 & y^{-1} \end{pmatrix} \tag{4.7}$$

Once again solving for $\rho(s^2t^2) = I$, we find that the reducible representations can be given by two families.

1. The representations that satisfy

$$\rho(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho(t) = \begin{pmatrix} \pm i & r \\ 0 & \mp i \end{pmatrix} \tag{4.8}$$

   which correspond to the points $(0, 0, \pm 2) \in X(\tilde{K})$.

2. The representations that satisfy

$$\rho(s) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \qquad \rho(t) = \begin{pmatrix} \pm x^{-1} & \mp 1 \\ 0 & \pm x \end{pmatrix} \tag{4.9}$$

which correspond to the points $(a, \pm a, \pm 2) \in X(\tilde{K})$. Note that the traces of these points contain the traces of the previous type of reducible representation.

We will be interested in representations of $\pi_1(\tilde{K})$ where $\mathrm{tr}(st) = z = \pm 2$ and $\mathrm{tr}(s^2) = x^2 - 2 = \pm 2$. From the character variety, we see that the only points in $X(\tilde{K})$ satisfying this condition are $(0, 0, \pm 2)$ or $(\pm 2, \pm 2, \pm 2)$ (with an even number of negatives) i.e. the representations

$$\rho_A(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho_A(t) = \begin{pmatrix} \pm i & 0 \\ 4 & \mp i \end{pmatrix} \tag{4.10}$$

$$\rho_B(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho_B(t) = \begin{pmatrix} \pm i & r \\ 0 & \mp i \end{pmatrix} \tag{4.11}$$

$$\rho_C(s) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \qquad \rho_C(t) = \begin{pmatrix} \pm 1 & \mp 1 \\ 0 & \pm 1 \end{pmatrix} \tag{4.12}$$

Notice that all of these representations are reducible, since the coordinates always satisfy $x^2 + y^2 + z^2 - xyz - 4 = 0$. We will use this in the proof of the main theorem. To summarize, we state the following lemma:

**Lemma 4.3.1.** *Let $\tilde{K}$ be the twisted $I$-bundle over the Klein bottle. Then the following are the representations of $\pi_1(\tilde{K})$ such that tr$(st) = \pm 2$ and tr$(s^2) = \pm 2$:*

$$\rho_A(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho_A(t) = \begin{pmatrix} \pm i & 0 \\ 4 & \mp i \end{pmatrix} \tag{4.13}$$

$$\rho_B(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho_B(t) = \begin{pmatrix} \pm i & r \\ 0 & \mp i \end{pmatrix} \tag{4.14}$$

$$\rho_C(s) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \qquad \rho_C(t) = \begin{pmatrix} \pm 1 & \mp 1 \\ 0 & \pm 1 \end{pmatrix} \tag{4.15}$$

## 4.4. Negative-flexibility

Suppose $M = S^3 \setminus K$ is a hyperbolic knot complement, with $\mathbb{T}$ a system of punctured JSJ tori with slope $\beta$. We must analyze the behavior of the signs of the traces of the boundary components of $M(\beta)$. In particular, the proof of the main theorems require a choice of a set of characters on the JSJ complementary regions $M(\beta)$ such that the traces on the boundary tori match. This property is known as $SL_2(\mathbb{C})$-*compatibility*. Proving $SL_2(\mathbb{C})$ makes heavy use of a related property called *negative-flexibility*, which we define here:

**Definition 4.4.1.** A pair $(M, \rho)$ consisting of a 3-manifold $M$ with torus boundary components $T_1, \ldots, T_m$ and a representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ is called *negative-flexible* if for any $(\gamma_1, \ldots, \gamma_m) \in \pi_1(T_1) \times \cdots \times \pi_1(T_m)$ with $\gamma_i$ simple and nontrivial, there exists a lift $\hat{\rho} : \pi_1(M) \to SL_2(\mathbb{C})$ such that $\mathrm{tr}(\hat{\rho}(\gamma_i)) = -2$ for all $i = 1, \ldots, m$.

For hyperbolic 3-manifolds, we have the following lemma from Menal-Ferrer and Porti (2010):

**Lemma 4.4.1** (Lemma 3.9 of Menal-Ferrer and Porti (2010))**.** *Hyperbolic 3-manifolds with finitely many torus boundary components are negative-flexible.*

*Remark* 4.4.1. It is known that lifts of $PSL_2(\mathbb{C})$ representations are in bijection with $H^1(\pi_1(M); \mathbb{Z}/2\mathbb{Z})$, i.e. homomorphisms $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$. It is exactly those elements that map to the identity under all homomorphisms whose traces are independent of the lift. (These elements turn out to be squares modulo the commutator subgroup of $\pi_1(M)$.) By the universal coefficient theorem this is isomorphic to $H_1(M; \mathbb{Z}/2\mathbb{Z})$, and so lifts of the holonomy representation into $PSL_2(\mathbb{C})$ to $SL_2(\mathbb{C})$ are in bijection with the singular $\mathbb{Z}/2\mathbb{Z}$ homology. Thus, Lemma 4.4.1 implies that any $\mathbb{Z}/2\mathbb{Z}$-homologically trivial group element of $\pi_1(M)$ must lift to trace -2 in the holonomy representation of a hyperbolic 3-manifold. In addition the traces of any elements of $\pi_1(M)$ that are $\mathbb{Z}/2\mathbb{Z}$-homologous are the same for any lift of the holonomy representation.

Our goal is to show that cable spaces and spaces fibered over $D^2(p,q)$ with $p > q \geq 2$ are also negative-flexible. We will use the $\mathbb{Z}/2\mathbb{Z}$ coefficients version; this is Lemma 5.3 in Schleimer (2018).

**Theorem 4.4.1** (Schleimer (2018), "half lives, half dies"). *Suppose $M$ is a compact oriented connected 3-manifold. Let $i : \partial M \to M$ be the inclusion, inducing $i_* : H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$. Then*

$$\frac{1}{2}\dim(H_1(\partial M)) = \dim(\ker(i_*)) = \dim(im(i_*)) \tag{4.16}$$

**Lemma 4.4.2.** *Suppose $M$ is a cable space fibered over $A^2(q)$ with torus boundary components $T_1, T_2$. Then $M$ is negative-flexible.*

*Proof.* The holonomy representation of the thrice-punctured sphere, whose fundamental group is given by $\langle a, b, c \mid c = ab\rangle$, is

$$\rho(a) = \left[\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right] \qquad \rho(b) = \left[\pm\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right] \qquad \rho(c) = \left[\pm\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\right] \tag{4.17}$$

In particular, for any lift $\widehat{\rho}$ of this representation to $SL_2(\mathbb{C})$, if the signs of $\widehat{\rho}(a)$ and $\widehat{\rho}(b)$ are the same, then the sign of $\widehat{\rho}(c)$ is negative, and if the traces of $a$ and $b$ have different sign, then $c$ has positive sign. The holonomy representation of $A^2(q)$ is given by

$$\rho(a) = \left[\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right] \qquad \rho(b) = \left[\pm\begin{pmatrix} 1 & 0 \\ x_q & 1 \end{pmatrix}\right] \qquad \rho(c) = \left[\pm\begin{pmatrix} 1+2x_q & 2 \\ x_q & 1 \end{pmatrix}\right] \tag{4.18}$$

where $\xi_{2q}$ is a primitive $2q$th root of unity and $x_q$ is such that

$$2 + 2x_q = -(\xi_{2q} + \xi_{2q}^{-1}) \tag{4.19}$$

Since the holonomy of $A^2(q)$ is a continuous deformation of the holonomy of the thrice-punctured sphere, it follows that the signs of the traces of a lift of $a$ and $b$ are equal if and only if $c$ lifts to a negative trace, as is true for the thrice-punctured sphere.

We divide into three cases. Let $\pi_1(A^2(q)) = \langle a, b, c \mid c = ab, c^q = 1\rangle$. The strategy will be to determine the kernel of $H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$, and use the homological interpretation of the kernel in Remark 4.4.1 to reach our conclusion. Let $h$ be the regular fiber.

Case 1: $q$ is even. In this case, we have $c^q h^p = 1$ for some $p$ which must be odd. The holonomy representation $\rho$ satisfies

$$\rho(c) = \left[\pm \begin{pmatrix} \xi_{2q} & 0 \\ 0 & \xi_{2q}^{-1} \end{pmatrix}\right] \tag{4.20}$$

where $\xi_{2q}$ denotes a primitive $2q$th root of unity. In this case, since $q$ is even, given any lift $\widehat{\rho} : \pi_1(M) \to SL_2(\mathbb{C})$, $\widehat{\rho}(c^q) = -I$. By Lemma 3.1 in Kitano (1994), we know that $\rho(h) = [\pm I]$, and the relation $c^q h^p = 1$ implies that for all lifts $\widehat{\rho}$, $\widehat{\rho}(h) = -I$ as well. Let $h_j \in H_1(T_j; \mathbb{Z}/2\mathbb{Z})$ be the singular homology classes of $h$ in the boundary tori of $M$. Since the sign of $\mathrm{tr}(\widehat{\rho}(h))$ is fixed for all lifts $\widehat{\rho}$, both $h_j$ lie in the kernels of $H_1(T_j; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$. By the "half lives, half dies" theorem, $h_1, h_2$ generate the kernel of $H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$. In particular, the signs of the traces of pairs of other elements in $\pi_1(T_j)$ can be changed independently, which proves the lemma in this case.

Case 2: $q$ is odd, $c^q h^p = 1$ with $p$ even. In this case, the sign of the trace of $c$ must be negative for all lifts $\widehat{\rho}$, i.e.

$$\widehat{\rho}(c) = \begin{pmatrix} -\xi^{2q} & 0 \\ 0 & -\xi_{2q}^{-1} \end{pmatrix} \tag{4.21}$$

This means that the sign of the traces of $a$ and $b$ are equal for all lifts. In addition, the traces of $h$ as an element of $\pi_1(T_1)$ and $h$ as an element of $\pi_1(T_2)$ are also equal. View $H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \cong H_1(T_1; \mathbb{Z}/2\mathbb{Z}) \oplus H_1(T_2; \mathbb{Z}/2\mathbb{Z})$. By the "half lives, half dies" theorem, the kernel of $H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$ is generated by $[a] \oplus [b]$ and $[h] \oplus [h]$. In particular, there exist lifts of $\rho$ such that any two of $[a], [h], [ah]$ have trace $-2$, meaning that the same respective two of $[b], [h], [bh]$ also have trace $-2$, proving the lemma in this case.

Case 3: $q$ is odd, $c^q h^p = 1$ with $p$ odd. In this case, the signs of the traces of $c$ and $h$ are opposite for all lifts $\widehat{\rho}$. In particular, for all lifts, the trace of $ah$ and the trace of $b$ have the same sign, for if $h$ has positive sign, then $c$ has negative sign, and hence the traces of $a$ and $b$ have the same sign, and if $h$ has negative sign, then $c$ has positive sign, and the traces of $a$ and $b$ have opposite sign. By the "half lives, half dies" theorem, the kernel of $H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$ is generated by $[ah] \oplus [b]$ and $[h] \oplus [h]$. In particular, there exist lifts of $\rho$ such that any two of $[a], [h], [ah]$ have trace $-2$, meaning that the same respective two of $[bh], [h], [b]$ also have trace $-2$, proving the lemma in this case. $\square$

*Remark* 4.4.2. Lemmas 4.4.1 and 4.4.2 show that for any $M$ hyperbolic or a cable space, any $\gamma \in \pi_1(\partial M)$ whose trace is independent of the chosen lift $\hat{\rho}$ has trace -2. Otherwise, one may change the sign of the trace of $\gamma$ independent of the signs of the traces of the other $\mathbb{Z}/2\mathbb{Z}$ homology classes.

**Lemma 4.4.3.** *Suppose $M$ is Seifert-fibered over $D(p,q)$ with $p \geq 2, q > 2$. Let $\rho$ be the holonomy representation of $D(p,q)$. Then $(M,\rho)$ is negative-flexible.*

*Proof.* The holonomy representation of the thrice-punctured sphere, whose fundamental group is given by $\langle a, b, c \mid c = ab \rangle$, is

$$\rho(a) = \left[\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right] \qquad \rho(b) = \left[\pm\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right] \qquad \rho(c) = \left[\pm\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\right] \tag{4.22}$$

In particular, for any lift $\hat{\rho}$ of this representation to $SL_2(\mathbb{C})$, if the signs of $\hat{\rho}(a)$ and $\hat{\rho}(b)$ are the same, then the sign of $\hat{\rho}(c)$ is negative, and if the traces of $\hat{\rho}(a)$ and $\hat{\rho}(b)$ are different, then the sign of $\hat{\rho}(c)$ is positive. The holonomy representation of $D^2(p,q)$, $p \geq 2, q > 2$ is thus given by

$$\rho(a) = \left[\pm\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\right] \qquad \rho(b) = \left[\pm\begin{pmatrix}\xi_{2p} & 0\\ x_q & \xi_{2p}^{-1}\end{pmatrix}\right] \qquad \rho(c) = \left[\pm\begin{pmatrix}\xi_{2p}+2x_q & 2\xi_{2p}^{-1}\\ x_q & \xi_{2p}^{-1}\end{pmatrix}\right] \tag{4.23}$$

where

$$\xi_{2p} + \xi_{2p}^{-1} + 2x_q = \xi_{2q} + \xi_{2q}^{-1} \tag{4.24}$$

Since the holonomy of $D^2(p,q)$ is a continuous deformation of the holonomy of the thrice-punctured sphere, it follows that the signs of the traces of a lift of $a$ and $b$ are equal if and only if $c$ lifts to a negative trace, as is true for the thrice-punctured sphere. We will show that in all cases, the $\mathbb{Z}/2\mathbb{Z}$-homologically trivial element of $\pi_1(\partial M)$, i.e. the one with trace sign fixed in all lifts, must lift to a negative trace. Let $h$ be the regular fiber.

<u>Case 1: Either $p$ or $q$ are even.</u> Without loss of generality, let $q$ be even. In this case, $c^q h^r = 1$ for $r$ odd. Since $q$ is even, given any lift $\hat{\rho} : \pi_1(M) \to SL_2(\mathbb{C})$, $\operatorname{tr}(\hat{\rho}(c^q)) = -2$. By Lemma 3.1 in Kitano (1994), for all lifts, $\hat{\rho}$, $\hat{\rho}(h) = \pm I$. Since $r$ is odd and $c^q h^r = 1$, this means that $\operatorname{tr}(\hat{\rho}(h)) = -2$. Since $\operatorname{tr}(\hat{\rho}(h)) = -2$ for all lifts $\hat{\rho}$, this means that $[h] \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ is trivial. By the half-lives, half-dies theorem, $[h]$ must generate the kernel of the induced map $H^1(\partial M; \mathbb{Z}/2\mathbb{Z}) \to H^1(M; \mathbb{Z}/2\mathbb{Z})$. Since $\operatorname{tr}(\hat{\rho}(h)) = -2$, this shows that $(M, \rho)$ is negative-flexible in this case.

<u>Case 2: $p$ and $q$ are both odd.</u> In this case, $b^p h^{r_1} = c^q h^{r_2} = 1$.

<u>Subcase 2.1</u>: $r_1$ and $r_2$ are both even. In this case, for all lifts $\hat{\rho}$, $\hat{\rho}(h^{r_i}) = I$, $i = 1, 2$. This means that $\operatorname{tr}(\hat{\rho}(b^p)) = \operatorname{tr}(\hat{\rho}(c^q)) = 2$, and since $p$ and $q$ are both odd, this means that $\operatorname{tr}(\hat{\rho}(b)) = -(\xi_{2p} + \xi_{2p}^{-1})$ and $\operatorname{tr}(\hat{\rho}(c)) = -(\xi_{2q} + \xi_{2q}^{-1})$. Since $c$ lifts to a negative trace, the signs of the traces of $a$ and $b$ are equal. Thus $\operatorname{tr}(\hat{\rho}(a)) = -2$ for all lifts $\hat{\rho}$, and as in the previous case, $[a]$ generates the kernel of the boundary-induced $\mathbb{Z}/2\mathbb{Z}$-homology map. Thus $(M, \rho)$ is negative-flexible.

<u>Subcase 2.2</u>: $r_1$ is even and $r_2$ is odd. Then for all lifts $\hat{\rho}$, $\hat{\rho}(h^{r_1}) = 1$, and so $\operatorname{tr}(\hat{\rho}(b^p)) = 2$, hence $\operatorname{tr}(\hat{\rho}(b)) = -(\xi_{2p} + \xi_{2p}^{-1})$. In addition, if $\operatorname{tr}(\hat{\rho}(c)) = -(\xi_{2q} + \xi_{2q}^{-1})$, then $\operatorname{tr}(\hat{\rho}(c^q)) = 2$ and

thus $\mathrm{tr}(\hat{\rho}(h^{r_1})) = \mathrm{tr}(\hat{\rho}(h)) = 2$. If $\mathrm{tr}(\hat{\rho}(c)) = \xi_{2q} + \xi_{2q}^{-1}$, then $\mathrm{tr}(\hat{\rho}(c^q)) = -2$ and $\mathrm{tr}(\hat{\rho}(h^{r_1})) = \mathrm{tr}(\hat{\rho}(h)) = -2$, i.e. $c$ and $h$ have opposite sign traces under every lift. In particular, if $\mathrm{tr}(\hat{\rho}(c))$ is negative, then $\mathrm{tr}(\hat{\rho}(a))$ and $\mathrm{tr}(\hat{\rho}(b))$ have the same sign. It follows that $\mathrm{tr}(\hat{\rho}(a))$ is negative while $\mathrm{tr}(\hat{\rho}(h))$ is positive. If $\mathrm{tr}(\hat{\rho}(c))$ is positive , then $\mathrm{tr}(\hat{\rho}(a))$ and $\mathrm{tr}(\hat{\rho}(b))$ have opposite signs. It follows that $\mathrm{tr}(\hat{\rho}(a))$ is positive and $\mathrm{tr}(\hat{\rho}(h))$ is negative. In either case, $a$ and $h$ have negative traces, so $ah$ always has a negative trace. Thus $[ah]$ generates the kernel of the boundary-induced $\mathbb{Z}/2\mathbb{Z}$-homology map. Thus $(M, \rho)$ is negative-flexible.

<u>Subcase 2.3</u>: $r_1$ is odd and $r_2$ is even. A symmetric argument to the previous case shows that $(M, \rho)$ is also negative-flexible in this case.

<u>Subcase 2.4</u>: $r_1$ and $r_2$ are odd. Then by the argument involving $\mathrm{tr}(\hat{\rho}(c))$ in Subcase 2.2, both $\mathrm{tr}(\hat{\rho}(c))$ and $\mathrm{tr}(\hat{\rho}(b))$ have the opposite sign trace as $\mathrm{tr}(\hat{\rho}(h))$; this means that $\mathrm{tr}(\hat{\rho}(c))$ and $\mathrm{tr}(\hat{\rho}(b))$ have the same sign. If $\mathrm{tr}(\hat{\rho}(c))$ is negative, then the traces of $a$ and $b$ under the same lift are equal. But since $\mathrm{tr}(\hat{\rho}(b))$ is also negative, $\mathrm{tr}(\hat{\rho}(a))$ is negative as well. If $\mathrm{tr}(\hat{\rho}(c))$ is positive, then the traces of $a$ and $b$ under that lift are different. But since $\mathrm{tr}(\hat{\rho}(b)$ is also positive, it follows that $\mathrm{tr}(\hat{\rho}(a))$ is once again negative. Thus $a$ has negative trace for all lifts, and jence $[a]$ generates the kernel of the boundary-induced $\mathbb{Z}/2\mathbb{Z}$-homology map. Thus $(M, \rho)$ is negative-flexible. □

*Remark* 4.4.3. In light of Lemmas 4.4.1, 4.4.2, and 4.4.3, a reasonable conjecture is the following. For any 3-manifold $M$ Seifert-fibered over a hyperbolic 2-orbifold $\mathcal{O}$, $p : M \to \mathcal{O}$ the fibering map, and $\rho$ the holonomy representation of $\mathcal{O}$, $(M, \rho \circ p)$ is negative-flexible. A proof sketch mimics the proof of Lemma 4.4.1, though there are many details left to be filled in.

**Lemma 4.4.4.** *Suppose $M$ is the twisted $I$-bundle over the Klein bottle $\tilde{K}$. Let $\rho_A, \rho_B, \rho_C$ : $\pi_1(\tilde{K})$ be as in Lemma 4.3.1. Then $(M, \rho_A)$ and $(M, \rho_B)$ are negative-flexible, while $(M, \rho_C)$ is not negative-flexible.*

*Proof.* Recall that

$$\rho_A(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho_A(t) = \begin{pmatrix} \pm i & 0 \\ 4 & \mp i \end{pmatrix} \tag{4.25}$$

$$\rho_B(s) = \begin{pmatrix} \pm i & 1 \\ 0 & \mp i \end{pmatrix} \qquad \rho_B(t) = \begin{pmatrix} \pm i & r \\ 0 & \mp i \end{pmatrix} \tag{4.26}$$

$$\rho_C(s) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \qquad \rho_C(t) = \begin{pmatrix} \mp 1 & \pm 1 \\ 0 & \mp 1 \end{pmatrix} \tag{4.27}$$

The behavior of the representations on the boundary torus is as follows:

$$\rho_A(st) = \begin{pmatrix} 3 & \mp i \\ \mp 4i & -1 \end{pmatrix} \qquad \rho_A(s^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.28}$$

$$\rho_B(st) = \pm \begin{pmatrix} -1 & \pm i(r-1) \\ 0 & -1 \end{pmatrix} \qquad \rho_B(s^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.29}$$

$$\rho_C(st) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \qquad \rho_C(s^2) = \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix} \tag{4.30}$$

Notice that $\rho_A(s^2)$ and $\rho_B(s^2)$ are fixed at $-I$, while $\rho_C(s^2)$ has trace 2. This proves the lemma. □

## 4.5. Results on $SL_2(\mathbb{C})$-compatibility

**Theorem 4.5.1.** *Let $M$ be an irreducible closed 3-manifold with a JSJ decomposition with JSJ tori $\{T_i\}_{i=1}^m$, gluing homeomorphisms $\{\varphi_i\}_{i=1}^m$, and complementary components $\{M_i\}_{i=1}^n$ such that:*

1. *Each $M_i$ is either hyperbolic or a cable space.*

2. *Each $M_i$ has exactly two torus boundary components $T_{i_1}, T_{i_2}$, and the JSJ complementary components are arranged in a circle.*

*Then the JSJ decomposition is $SL_2(\mathbb{C})$-compatible.*

*Proof.* Take any $M_i$, and let $\rho_i$ be either the holonomy representation or the composition of the Seifert fibration map with the holonomy of $A^2(q)$. Let $i_* : H_1(\partial M_i; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}^4 \to H_1(M_i; \mathbb{Z}/2\mathbb{Z})$ be the induced map of the boundary inclusion. By the "half lives, half dies" theorem, the kernel is 2-dimensional, i.e.

$$\ker(i_*) = \langle \gamma_1, \gamma_2 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset H_1(\partial M_i; \mathbb{Z}/2\mathbb{Z}) \tag{4.31}$$

We now analyze the possibilities for $\gamma_1, \gamma_2$. First, note that there are two possibilities for $\gamma_k$:

1. $\gamma_k \in H_1(T_{i_j}; \mathbb{Z}/2\mathbb{Z})$ for some $j \in \{1, 2\}$. We say $\gamma_k$ is *contained in* $T_{i_j}$ in this case. By Lemmas 4.4.1 and 4.4.2, the trace of any lift of $\rho_i$ evaluated on a representative of $\gamma_k$ must be -2.

2. $\langle \gamma_k \rangle \cap H_1(T_{i_j}; \mathbb{Z}/2\mathbb{Z}) \neq 0$ for $j = 1, 2$. We say $\gamma_k$ is *spanning* in this case. The trace of any lift of $\rho_i$ evaluated on representatives of $\langle \gamma_k \rangle \cap H_1(T_{i_1}; \mathbb{Z}/2\mathbb{Z})$ and $\langle \gamma_k \rangle \cap H_1(T_{i_2}; \mathbb{Z}/2\mathbb{Z})$ are equal, since they are $\mathbb{Z}/2\mathbb{Z}$-homologous.

Here are the cases for how $\gamma_1, \gamma_2$ can interact with each other.

- Type 0: $\gamma_1$ and $\gamma_2$ are contained in the same torus boundary component. If $\langle \gamma_1, \gamma_2 \rangle = H_1(T_{i_j}; \mathbb{Z}/2\mathbb{Z})$ for $j \in \{1, 2\}$, we have a generating set of representatives $(\widehat{\gamma_1}, \widehat{\gamma_2}) = \pi_1(T_{i_j})$. By Lemmas 4.4.1 and 4.4.2, for any lift $\widehat{\rho_i}$ to $SL_2(\mathbb{C})$, $\mathrm{tr}(\widehat{\rho_i}(\widehat{\gamma_1})) = \mathrm{tr}(\widehat{\rho_i}(\widehat{\gamma_2})) = -2$, and since $\widehat{\gamma_1}$ and $\widehat{\gamma_2}$ commute, it follows that for all $\widehat{\rho_i}$, $\mathrm{tr}(\widehat{\rho_i}(\widehat{\gamma_1 \gamma_2})) = 2$, which contradicts either Lemma 4.4.1 or Lemma 4.4.2.

- Type 1: $\gamma_1$ and $\gamma_2$ are contained in $T_{i_1}$ and $T_{i_2}$, respectively. Then let $a_j \in \pi_1(\partial T_{i_j})$ be a representative of $\gamma_j$, and let $b_j \in \pi_1(\partial T_{i_j})$ be a representative of an element of $H_1(\partial T_{i_j}; \mathbb{Z}/2\mathbb{Z})$ not in the kernel of $i_*$. By Case 1 in the proof of Lemma 4.4.2, cable spaces with base orbifold $A^2(q)$ where $q$ is even fall into this type.

- Type 2: $\gamma_1$ is contained in $T_{i_1}$ and $\gamma_2 = (\gamma_2^1, \gamma_2^2)$ is spanning, with $\gamma_2^k \in H_1(T_{i_k}; \mathbb{Z}/2\mathbb{Z})$. Let $a \in \pi_1(\partial T_{i_1})$ be a representative of $\gamma_1$, and let $g^k \in \pi_1(T_{i_k})$ be a representative of $\gamma_2^k$. This means that for any lift $\widehat{\rho_i}$ of $\rho_i$, $\mathrm{tr}(\widehat{\rho_i}(a)) = -2$ and $\mathrm{tr}(\widehat{\rho_i}(g^1)) = \mathrm{tr}(\widehat{\rho_i}(g^2))$. Let $b \in T_{i_1}$ be a representative of the $\mathbb{Z}/2\mathbb{Z}$-homology class that is not $\gamma_1$ or $\gamma_2^1$. Then for all lifts $\widehat{\rho_i}$,

$\text{tr}(\widehat{\rho}_i(b)) = -\text{tr}(\widehat{\rho}_i(g^1)) = -\text{tr}(\widehat{\rho}_i(g^2))$, which is a contradiction of either Lemma 4.4.1 or Lemma 4.4.2. If $\gamma_1 = \gamma_2^1$, then both $\gamma_1$ and $\gamma_2^1$ are -2 for all lifts, are we are in the situation of Type 1. So this case can be subsumed into Type 1.

- Type 3: $\gamma_1 = (\gamma_1^1, \gamma_2^2)$ and $\gamma_2 = (\gamma_2^1, \gamma_2^2)$ are both spanning, with $\gamma_j^k \in H_1(T_{i_k}; \mathbb{Z}/2\mathbb{Z})$. Let $g_j^k \in \pi_1(T_{i_k})$ be a representative of $\gamma_j^k$. Then for any lift $\widehat{\rho}_i$ of $\rho_i$, $\text{tr}(\widehat{\rho}_i(g_j^1) = \text{tr}(\widehat{\rho}_i(g_j^2))$. By cases 2 and 3 in the proof of Lemma 4.4.2, cable spaces with base orbifold $A^2(q)$ where $q$ is odd fall into this type.

Since components of Type 0 or Type 2 cause contradictions, they do not exist. So we can label the components $M_i$ to be of Type 1 or 3, depending on how the map $i_*$ behaves. We now deal with the possibilities for the JSJ decomposition of $M$.

- All the $M_i$ are of Type 1. If we have components $M_i$ with boundary torus $T_{i_1}$ and $M_{i+1}$ with boundary torus $T_{i+1_2}$, let $\varphi : \pi_1(T_{i_1}) \to \pi_1(T_{i+1_2})$ be the gluing isomorphism. Since both $M_i$ and $M_{i+1}$ are of Type 1, there exist generating sets $a_1, b_1 \in \pi_1(T_{i_1})$ and $a_2, b_2 \in \pi_1(T_{i+1_2})$ such that for all lifts $\widehat{\rho}_i$ and $\widehat{\rho_{i+1}}$, $\text{tr}(\widehat{\rho}_i(a_1)) = \text{tr}(\widehat{\rho_{i+1}}(a_2)) = -2$. If $\varphi(a_1)$ is $\mathbb{Z}/2\mathbb{Z}$-homologous to $a_2$, then we can pick $\widehat{\rho}_i, \widehat{\rho_{i+1}}$ so that $\text{tr}(\widehat{\rho}_i(b_1)) = \text{tr}(\rho_{i+1}(\varphi(b_2))) = -2$, and the traces are equal on all of $\pi_1(T_{i_1})$. If $\varphi(a_1)$ is not $\mathbb{Z}/2\mathbb{Z}$-homologous to $a_2$, we can pick a lift $\widehat{\rho_{i+1}}$ so that $\text{tr}(\widehat{\rho_{i+1}}(\varphi(a_1))) = -2$, and a lift $\widehat{\rho}_i$ such that $\text{tr}(\widehat{\rho}_i(\varphi^{-1}(a_2))) = -2$. Note that in components of type 1, the choices we make on signs of traces on one boundary torus are independent of choices we make on the other. Therefore, we can make such choices for all JSJ tori, and so in this case the JSJ decomposition of $M$ is $SL_2(\mathbb{C})$-compatible.

- All of the $M_i$ are of Type 3. Then we may choose lifts $\widehat{\rho}_i$ such that all peripheral elements have trace 2. This satisfies the $SL_2(\mathbb{C})$-compatibility condition, and we are done in this case.

- There are both $M_i$ of Type 1 and $M_i$ of Type 3. Due to the assumption that the JSJ complementary regions are arranged in a circle, any chain of Type 3 complementary components must end in Type 1 components on both sides. Notice that a chain of Type 3 complementary components has the effect of composing a gluing homeomorphism between two Type 1 components. So we are in the same situation as when all the $M_i$ are of Type 1, where the gluing homomorphisms go through chains of Type 3 components. The same proof as that case goes through. So in this case, $M$ is also $SL_2(\mathbb{C})$-compatible.

So in all possible cases, the JSJ decomposition of $M$ is $SL_2(\mathbb{C})$-compatible, as desired. □

**Theorem 4.5.2.** *Let $M$ be a closed irreducible 3-manifold with JSJ decomposition with JSJ tori $\{T_i\}_{i=1}^{n-1}$, gluing homomorphisms $\{\varphi_i : \partial M_i \to \partial M_{i+1}\}_{i=1}^{n-1}$, and complementary regions $\{M_i\}_{i=1}^{n}$, such that:*

1. *The JSJ graph is a line, with endpoints $M_1, M_n$.*
2. *$M_1, M_n$ are either Seifert-fibered over $D(p, q)$ with $p \geq 2, q > 2$ or the twisted $I$-bundle over the Klein bottle.*
3. *The other $M_i$ are either hyperbolic or cable spaces.*
4. *If $n = 2$, then $M_1$ and $M_2$ cannot both be the twisted $I$-bundle over the Klein bottle.*

*Then the JSJ decomposition of $M$ is $SL_2(\mathbb{C})$-compatible. Moreover, there exist $SL_2(\mathbb{C})$-compatible choices $\rho_i$ of representations on the $\pi_1(M_i)$ such that if $M_1$ or $M_n$ is the twisted $I$-bundle over the Klein bottle, $\rho_i$ is one of $\rho_A$ or $\rho_B$ from Lemma 4.3.1.*

*Proof.* Take any $M_i$ that is not the twisted $I$-bundle over the Klein bottle, and let $\rho_i$ be the associated hyperbolic holonomy representation. If $M_i$ is the twisted $I$-bundle of the Klein bottle, then let $\rho_i$ be either $\rho_A$ or $\rho_B$ from Lemma 4.3.1. By Lemmas 4.4.1, 4.4.3, and 4.4.4, all the $(M_i, \rho_i)$ are negative-flexible.

We know that the $M_i$, $i \neq 1, n$ (i.e. the JSJ components with two torus boundaries) can be categorized into two types: Type 1 and Type 3 (see the previous proof for the definition).

For $M_i$, $i = 1, n$, by the half-lives half-dies theorem and negative-flexibility, the generator of the kernel of the induced homology map must have trace -2 for any lift $\widehat{\rho_i}$ or $\rho_i$ to $SL_2(\mathbb{C})$. We now have the following possibilities for the $M_i, i \neq 1, n$:

- There are no such $M_i$, i.e. $n = 2$. In this case, two spaces with one torus boundary component are glued together. Let $\pi_1(\partial M_i) = \langle m_i, \ell_i \rangle$ where $\ell_i$ is the $\mathbb{Z}/2\mathbb{Z}$-homologically trivial element of $\pi_1(M_i)$. If $\varphi_1 : \partial M_1 \to \partial M_2$ takes $[\ell_1]$ to $[\ell_2]$, then we simply make sure the other negatives match. If not, then the two $\mathbb{Z}/2\mathbb{Z}$-homology classes will be locked into trace -2. This realizes $SL_2(\mathbb{C})$-compatibility.

- All the $M_i$ are of Type 1. Repeat the above argument for each $\varphi_i$. This will realize $SL_2(\mathbb{C})$-compatibility.

- All the $M_i$ are of Type 3. Note that both $M_1$ and $M_n$ have $\mathrm{tr}(\widehat{\rho_i}(\ell_i)) = -2$ for $i = 1, 2$, while the traces of the $\mathbb{Z}/2\mathbb{Z}$-homology class of $m$ and $m\ell$ must have the opposite sign under any lift. For each type 3 component, let the homology class glued to $\ell_1$ and $\ell_n$ have trace -2 under the chosen lift. This will satisfy the $SL_2(\mathbb{C})$-compatibility condition.

- The $M_i$ are mixed type. Note that type 3 components can be ignored in this setting, so we are in the same situation as the case where the $M_i$ are of Type 1. So the same argument suffices.

$\square$

# CHAPTER 5

# DETECTION OF PUNCTURED TORI AND TAUTOLOGICAL EXTENSION

## 5.1. The method of Paoluzzi-Porti

The content of this chapter is adapted from Yi Wang. Punctured JSJ tori and tautological extensions of Azumaya algebras. 2023. URL https://arxiv.org/abs/2309.07848. and Yi Wang. Detection of twice-punctured tori in hyperbolic knot complements 2024. URL https://arxiv.org/abs/2404.06388..

The techniques utilized to prove the main theorems of this thesis are inspired by the techniques in Paoluzzi and Porti (2012), particularly the proof of the following theorem:

**Theorem 5.1.1** (Paoluzzi and Porti (2012))**.** *Let $C$ be a system of essential Conway spheres in a hyperbolic knot complement $M = S^3 \setminus K$ such that $M(2,0)$ admits a geometric decomposition by pillowcase orbifolds stemming from $C$. Then $C$ is detected by an ideal point of the character variety.*

We summarize their proof technique, which will guide the philosophy of the proofs of the main theorems.

1. Let $M_1, \ldots, M_n$ be the connected components of $M \setminus C$. Define a restriction map
$$r : X(M) \to \mathcal{X} = \prod_{i=1}^{n} X(M_i) \tag{5.1}$$
which restricts characters of $\pi_1(M)$ to their subgroups $\pi_1(M_i)$.

2. Show that the image of $X(M)$ is one-dimensional in $\mathcal{X}$.

3. Let $\mathcal{O}_i$ be the pillowcase complementary regions of $M(2,0)$. Show that $\mathcal{O}_i$ are all either hyperbolic or Seifert-fibered over hyperbolic 2-orbifolds.

4. Pick lifts of holonomy representations $\rho_i : \pi_1(\mathcal{O}_i) \to SL_2(\mathbb{C})$ so that the traces match on their

boundaries. Conclude that the trace $(\chi_1, \ldots, \chi_n)$ lies in the subvariety of traces that can be glued on $C$, lying inside $\mathcal{X}$.

5. Prove that $(\chi_1, \ldots, \chi_n)$ restricted to the boundaries are reducible but not conjugate. Conclude that $(\chi_1, \ldots, \chi_n) \notin \mathrm{Im}(r)$, but its surrounding points lie in $\mathrm{Im}(r)$.

6. This means that $(\chi_1, \ldots, \chi_n)$ must corresponding to an ideal point of $X(M)$. Use Culler-Shalen theory to show that $C$ is detected by that ideal point.

Our main theorem is analogous to the theorem of Paoluzzi and Porti (2012). In lieu of the pillowcase orbifold decomposition, we use the JSJ decomposition.

## 5.2. Complementary regions of once and twice punctured tori

As a crucial component of our proof method, we must analyze complementary regions of essential once- and twice-punctured tori in hyperbolic knot complements.

Let $M = S^3 \setminus K$ be a hyperbolic knot complement containing a disjoint union of non-isotopic separating once- or twice-punctured tori $\mathbb{T} = \bigcup_{i=1}^{n-1} T_i$. Suppose that $\mathbb{T}$ meets the boundary torus of $M$ at slope $\beta$.

We first consider the case where $\mathbb{T}$ consists of once-punctured tori. Then $\mathbb{T}$ is a disjoint family of non-isotopic Seifert surfaces, and $\beta$ is the canonical longitude of the knot.

**Lemma 5.2.1.** *Then the number of complementary regions of $\overline{M \setminus \mathbb{T}}$, the number of punctured JSJ tori $T_i$, the number of JSJ complementary regions of $M(0)$, and the number of JSJ tori of $M(0)$ are all equal to $n$. The JSJ complementary regions of $M(0)$ have two boundary tori. In particular, the JSJ complementary regions of $M(0)$ are arranged in a circle.*

We use the following observations / notations, coming from Lemma 5.2.1.

1. Let $\{H_i\}_{i=1}^n$ be the components of $\overline{M \setminus \bigcup_{i=1}^n T_i}$, and let $\{\mathcal{O}_i\}_{i=1}^n$ be the JSJ components of $M(0)$. Note that the number of complementary regions is equal to the number of JSJ tori,

and both are equal to $n$.

2. The JSJ component $\mathcal{O}_i$ is bounded by the tori $\widehat{T}_{i_1}, \widehat{T}_{i_2}$.

3. $T_{i_j} \subset \partial H_i$ are the incompressible punctured tori which cap off to $\widehat{T}_{i_j}$, and the commutators of $\pi_1(T_{i_1})$ and $\pi_1(T_{i_2})$ are equal in $\pi_1(M)$ (they are represented by the curve $\beta$)

4. In the JSJ decomposition of $M(0)$, let $\widehat{T}_{i_1}$ be glued to $\widehat{T}_{(i-1)_2}$ and $\widehat{T}_{i_2}$ be glued to $\widehat{T}_{(i-1)_1}$. (Here $i \in \mathbb{Z}/n\mathbb{Z}$.)

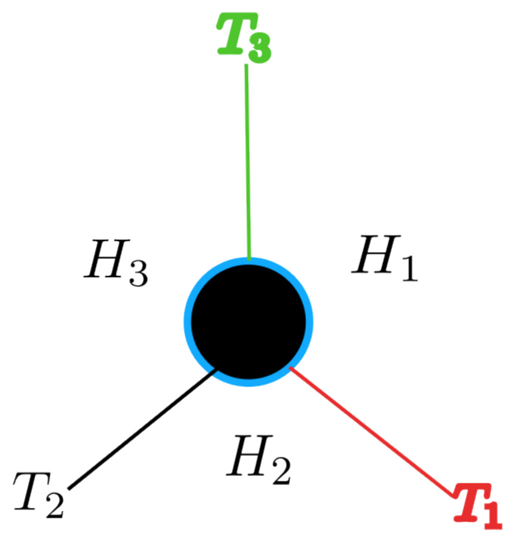


Figure 5.1: A schematic of how the once-punctured tori and complementary regions are arranged in the knot complement.

The picture is as follows: Each $\mathcal{O}_i$ is Seifert-fibered or hyperbolic. We also have that $\partial H_i = T_i \cup A_i \cup T_{i+1}$ is a closed genus 2 surface, which can be viewed as two once-punctured tori glued together via an annulus. The 3-manifold $\mathcal{O}_i$ is obtained by capping off $H_i$ with an annulus (representing a portion of the glued-in torus when perfoming 0-surgery on $M$).

We now consider the case where $\mathbb{T}$ consists of twice-punctured tori. Since each $T_i$ is separating, there are $n$ complementary regions $M_i \subset M \setminus \mathbb{T}$. Notice that each $M_i$ has $\partial M_i$ either one or two genus two surfaces, as depicted below:

Under Dehn filling, each boundary component of $M_i$ has an annulus glued to one of the handles of

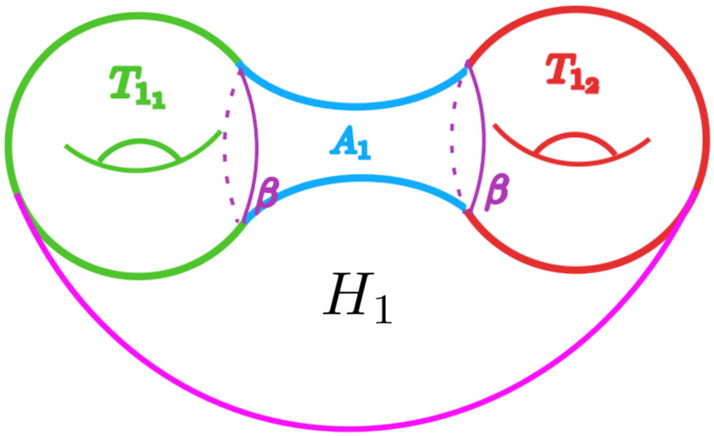


Figure 5.2: Rough drawing of what each complementary region looks like. Colors correspond to the previous figure.

the genus two surface, as shown below. This becomes $\mathcal{O}_i$, a JSJ complementary region of $M(\beta)$. This is depicted in the below figure.

In particular, the complementary regions at the two end vertices have one torus boundary component, while the rest of the complementary regions have two torus boundary components. In summary, we have the following lemma:

**Lemma 5.2.2.** *Given $n-1$ essential separating twice-punctured tori $T_i$ inside a knot complement $M = S^3 \setminus K$ with boundary slope $\beta$, there are $n$ complementary regions $M_i \subset M$, two of which have boundary a single genus 2 surface, and the rest of which have two genus 2 surfaces as their boundaries. In $M(\beta)$, there are $n-1$ JSJ tori $\widetilde{T}_i$ with $n$ complementary regions $\mathcal{O}_i$, two of which have one torus as their boundary, and the rest of which have two tori as their boundary.*

## 5.3. Detection of once-punctured tori and tautological extension

For the convenience of the reader, we restate the first main result here:

**Theorem 5.3.1.** *Let $M = S^3 \setminus K$ be a hyperbolic knot complement, and let $\{T_i\}_{i=1}^n \subset M$ be a system of disjoint non-parallel once-punctured JSJ tori with slope 0. Suppose the following:*

- *All irreducible components of $X(M)$ containing irreducible characters are norm curves.*

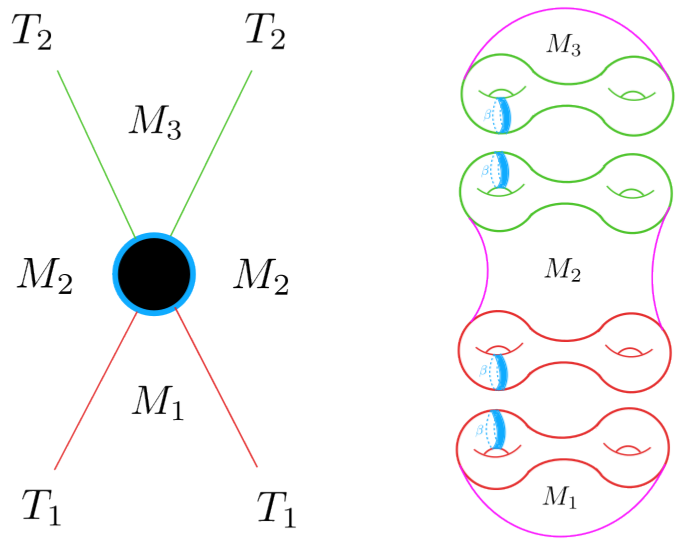


Figure 5.3: Left: Schematic of how $T_1$ and $T_2$ are arranged in $M$. Right: The complementary regions $M_i$ when cutting $M$ via $T_1$ and $T_2$. Note the blue annuli marked $\beta$, which come from a tubular neighborhood of the knot.

- *No JSJ complementary region of $M(0)$ is Seifert fibered over the annulus.*
- *No JSJ torus in $M(0)$ bounds two hyperbolic components on both sides.*

*Then $\bigcup_{i=1}^n T_i$ is detected by an ideal point $x$ on a norm curve $\tilde{C} \subset \widetilde{X(M)}$, and $A_{k(C)}$ tautologically extends over $x$.*

*Proof.* Suppose we are in the situation in the theorem statement. Let $[\beta] \in \pi_1(M)$ be the canonical longitude of the torus boundary, i.e. the boundary slope of the once-punctured tori. If some $\mathcal{O}_i$ is Seifert-fibered over a 2-orbifold $S_i$, then $S_i$ must have two circle boundary components. By Theorem 13.3.6 in Thurston (1979), if $S_i$ is non-hyperbolic it must be the annulus, which is ruled out by assumption. So we can assume that all the JSJ components $\mathcal{O}_i$ are either hyperbolic or Seifert-fibered over a hyperbolic 2-orbifold.

We now examine the dimensions of $X(H_i)$ and $X(\mathcal{O}_i)$. Recall that $\partial H_i$ is a closed genus 2 surface. By Theorem 5.6 in Thurston (1979), $\dim_{\mathbb{C}}(X(H_i)) \geq -\frac{3}{2}\chi(\partial H_i) = 3$. Recall that $\mathcal{O}_i$ is a 3-manifold with two torus boundary components. By the same theorem, we know that $\dim_{\mathbb{C}}(X(\mathcal{O}_i)) \geq 2$.

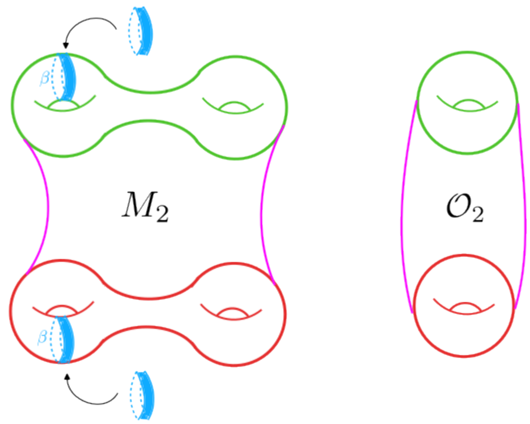


Figure 5.4: Left: Demonstrating where the Dehn filling annuli get glued. Right: After the gluing, the boundary becomes two tori, forming the JSJ complementary region $\mathcal{O}_i$.

We have the inclusions $\pi_1(H_i) \hookrightarrow \pi_1(M)$, inducing the restriction maps

$$r_i : X(M) \to X(H_i) \tag{5.2}$$

Take the product of all these maps to get

$$r : X(M) \to X\left(\bigcup H_i\right) = \prod_{i=1}^{n} X(H_i) = \mathcal{X} \tag{5.3}$$

Notice that $\mathcal{X}$ is an affine variety of dimension at least $3n$. We will now show that $\mathrm{Im}(r)$ is a one-dimensional subvariety in $\mathcal{X}$. Let $V'$ be the algebraic set inside $\mathcal{X}$ defined by the following equations:

1. Each $H_i$ has a coordinate associated with the mutual commutator of the generators of $\pi_1(T_{i_1})$ and $\pi_1(T_{i_2})$. This commutator lifts to $\beta \in \pi_1(M)$, so all the commutators must have the same trace in order to be in the image of $r$. Thus, there are $n-1$ equations that equate the trace polynomials of these commutators.

2. There are $n$ gluings between $T_{i_2}$ and $T_{(i+1)_1}$, with $i \in \mathbb{Z}/n\mathbb{Z}$. There are three equations per gluing here. If $\pi_1(T_{i_2}) = \langle a_2, b_2 \rangle$, $\pi_1(T_{(i+1)_1}) = \langle a_1, b_2 \rangle$, and the gluing map is $\varphi : \pi_1(T_{i_2}) \to$

$\pi_1(T_{(i+1)_1})$, then the three equations are

$$\operatorname{tr}(a_2) = \operatorname{tr}(\varphi(a_2)) \qquad \operatorname{tr}(b_2) = \operatorname{tr}(\varphi(b_2)) \qquad \operatorname{tr}(a_2 b_2) = \operatorname{tr}(\varphi(a_2 b_2)) \tag{5.4}$$

However, only two of these equations are necessary. Let $x = \operatorname{tr}(a_2)$ and $x' = \operatorname{tr}(\varphi(a_2))$, with similar notation for $y, y'$ for $b_2$ and $z, z'$ for $a_2 b_2$. Without loss of generality, suppose that the first two equations apply. Write them as $x = x'$ and $y = y'$. By the $n-1$ equations that equate the commutators of all the trace functions, we also have that

$$x^2 + y^2 + z^2 - xyz - 2 = x'^2 + y'^2 + z'^2 - x'y'z' - 2 \tag{5.5}$$

By the other two equations, we see that

$$z^2 - xyz = z'^2 - xyz' \Longrightarrow (z^2 - z'^2) - (xyz - xyz') = (z - z')(z + z' - xy) = 0 \tag{5.6}$$

Thus, the algebraic set $V'$ includes $2n$ equations, which are of the form $x = x'$ and $y = y'$.

Let $V$ be the subvariety of $V'$ which takes the $z = z'$ components of the equations of the second type, as described above. Notice also that $\operatorname{Im}(r) \subset V$, since any element in the image or $r$ must satisfy the gluing equations. Let $X^{irr}(M)$ be the Zariski open subset consisting of traces of irreducible representations in $X(M)$, and let $V^{irr}$ be the Zariski open subset of $V$ defined by $\operatorname{tr}(\beta) \neq 2$. Note that $V^{irr} \subset \operatorname{Im}(X^{irr}(M))$, since the traces on $X(H_i)$ restrict to matching irreducible traces on the once-punctured tori in their boundary, so the lifts of the conjugacy classes of irreducible representations on these once-punctured tori can be glued, giving an irreducible trace on $X(M)$. (See, for instance, the argument in Lemma 6 in Paoluzzi and Porti (2012), which goes through for HNN-extensions.) Since $V$ is a component of an algebraic set defined by $3n-1$ equations inside $\mathcal{X}$, $\dim(V) \geq 1$, and since by assumption $X^{irr}(M)$ is one-dimensional, $\dim(V^{irr}) \leq 1$. If the dimension of $V^{irr}$ is zero, then $V^{irr} \subset V$ is a Zariski open subset such that $\dim(V^{irr}) < \dim(V)$, meaning that $V^{irr}$ is empty; this is a contradiction since by assumption, there exist representations with $\operatorname{tr}(\beta) \neq 2$. So $\dim(V) = \dim(V^{irr}) = 1$. This also means that $\dim(\mathcal{X}) = 3n$, and so $\dim(X(H_i)) = 3$

for all $i$. The subvariety $X(\mathcal{O}_i) \subset X(H_i)$ is defined by setting the trace of $\beta \in \pi_1(\partial H_i)$ equal to 2, but by assumption, the trace of $\beta$ is nonconstant on $X(M)$, and hence nonconstant in $X(H_i)$ as well. Therefore, $\dim(\mathcal{O}_i) = 2$.

By Lemma 4.2.2, all $\mathcal{O}_i$ must be either cable spaces or hyperbolic. By Theorem 4.5.1, $\{\widehat{T}_i\}_{i=1}^n$ is an $SL_2(\mathbb{C})$-compatible JSJ decomposition, so there exist representations $\rho_i : \pi_1(\mathcal{O}_i) \to SL_2(\mathbb{C})$ so that the cusp group elements have matching traces. In fact, by the proof of Theorem 4.5.1, the $\rho_i$ can be chosen as lifts of holonomy representations of either $\mathcal{O}_i$ or their underlying 2-orbifolds $S_i = A^2(p_i)$. There are quotient maps $q_i : \pi_1(H_i) \to \pi_1(\mathcal{O}_i)$ that kill the loop corresponding to the punctures on the incompressible once-punctured tori embedded in the boundary of $H_i$; in other words, $q_i$ sends a representative of the core of $A_i$ to the identity. Let $\chi_i' = \mathrm{tr}(\rho_i \circ q_i)$, and let $\chi_0 = (\chi_1', \ldots, \chi_n')$.

Note that $V$ contains $\chi_0$, since we chose the coordinates of $\chi_0$ to be $SL_2(\mathbb{C})$-compatible characters, but $\chi_0 \notin V^{irr}$. If $\chi_0$ was isolated, $\{\chi_0\}$ would be a zero-dimensional component of $V$, which contradicts the fact that $V$ was defined with $3n-1$ equations in a $3n$-dimensional variety. In addition, since the coordinate functions on $\chi_0$ are irreducible on each $X(H_i)$, $\chi_0$ cannot be surrounded by restrictions of reducible representations of $\pi_1(M)$. These observations combined with the fact that $V^{irr}$ is Zariski-open in $V$ shows that we may pick a sequence of points $\{\chi_j\}_{j=1}^\infty \subset V^{irr} \subset \mathrm{Im}(r)$ approaching $\chi_0$. Let $\{\alpha_j\} \in X(M)$ be such that $r(\alpha_j) = \chi_j$. Suppose for contradiction that up to subsequence, $\{\alpha_j\}$ converges to a character $\alpha_\infty \in X(M)$. Then $\alpha_\infty$ must be the trace of some $\rho_\infty : \pi_1(M) \to SL_2\mathbb{C}$. Since $r(\alpha_\infty) = \chi_0$, it follows that for all $i$, $\rho_\infty|_{\pi_1(H_i)}$ is conjugate to $\rho_i \circ q_i$, since the trace of $\rho_i \circ q_i$ is $\chi_i'$. In particular, this means that for all $i \in \mathbb{Z}/n\mathbb{Z}$, $\rho_i|_{\pi_1(\widehat{T}_{i_2})} \circ q_i|_{\pi_1(T_{i_2})}$ and $\rho_{i+1}|_{\pi_1(\widehat{T}_{(i+1)_1})} \circ q_i|_{\pi_1(T_{(i+1)_1})}$ are conjugate representations of the fundamental group of the once-punctured torus, which is the free group on two generators. We are left with the following three cases: $T_{i_2}$ and $T_{(i+1)_1}$ glue to form the incompressible once-punctured torus $T_k \subset M$ that caps off to $\widehat{T}_k \subset M(0)$ such that:

1. $\widehat{T}_k$ bounds a Seifert-fibered and a Seifert-fibered space. Let $\pi_1(\widehat{T}_{i_1}) = \langle m_1, \ell_1 \rangle$ and $\pi_1(\widehat{T}_{i_2}) = \langle m_2, \ell_2 \rangle$ be the copies of $\widehat{T}_j$ in the two Seifert-fibered spaces with holonomies $\rho_1', \rho_2'$. Here $\ell_k$

is the regular fiber. Then up to conjugacy,

$$\rho_k'(m_k) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho_k'(\ell_k) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.7}$$

for $k = 1, 2$, and some canonical choice of longitude $\ell_k$ and a choice of sign determined by $SL_2(\mathbb{C})$-compatibility. By Theorem 2.4.2, the fibers of the Seifert fibrations do not match, i.e. if $\varphi : \langle m_1, \ell_1 \rangle \to \langle m_2, \ell_2 \rangle$ is the gluing map, $\varphi(\ell_1) \neq \ell_2$. Thus, $\varphi^{-1}(\ell_2)$ is a nontrivial simple closed curve in $\pi_1(\widehat{T}_{i_1})$ that is not $\ell_1$; this means it must be of the form $m_1^p \ell_1^q$ with $p \neq 0$. However,

$$\rho_1'(m_1^p \ell_1^q) = \pm \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.8}$$

which means that the two representations are not conjugate under the gluing map $\varphi$. This contradicts the earlier assertion.

2. $\widehat{T}_k$ bounds a Seifert-fibered and a hyperbolic space. The hyperbolic space will have a holonomy representation $\rho_h$ that restricts to the torus cusp $\langle m_h, \ell_h \rangle$ as follows:

$$\rho_h(m_h) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho_h(\ell_h) = \pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \tag{5.9}$$

where $\mathrm{Im}(\tau) > 0$, and the choice of sign is determined by $SL_2(\mathbb{C})$-compatibility. We call $\tau$ the *cusp shape* of the representation $\rho_h$. As seen previously, the cusp shape for a Seifert-fibered manifold is 0, but it is non-zero for a hyperbolic manifold, so this leads to a contradiction.

3. $\widehat{T}_k$ bounds a hyperbolic and a hyperbolic space. This is excluded by hypothesis.

In all cases, we get a contradiction. Thus, a subsequence of $\{\alpha_j\}$ converges to an ideal point $\alpha_\infty$ of $X(M)$. By construction, $\lim_{j\to\infty} \mathrm{tr}(\alpha_j)|_{\pi_1(H_i)} = \chi_i'$ is the limiting character of the ideal point, and the component whose projectivization contains $\alpha_\infty$ must contain irreducible representations. By the assumptions, it follows that $\alpha_\infty$ is an ideal point on a norm curve $C$. Since the limiting characters $(\chi_1', \ldots, \chi_n')$ satisfy the conditions in Lemma 3.5.1, $\alpha_\infty \in \widetilde{X(M)}$ detects $\bigcup T_i$. Since the limiting character on each $H_i$ is given by the trace of $\rho_i \circ q : \pi_1(H_i) \to SL_2(\mathbb{C})$, which is irreducible on all components, by Proposition 3.5.2 $A_{k(C)}$ tautologically extends over $\alpha_\infty$, and so we are done. □

## 5.4. Detection of twice-punctured tori and tautological extension

For the convenience of the reader, we restate the second main result here:

**Theorem 5.4.1.** *Let $M = S^3 \setminus K$ be a hyperbolic knot complement, and let $\bigcup_{i=1}^{n-1} T_i$ be a system of disjoint non-parallel separating twice-punctured JSJ tori with slope $\beta$. Suppose the following:*

1. *All components of $X(M)$ containing traces of irreducible representations are norm curves.*
2. *$M(\beta)$ has JSJ components that are either*
   (a) *Seifert-fibered over $D^2(p,q)$ with $p \geq 2, q > 2$*
   (b) *Seifert-fibered over $A^2(r)$ with $r \geq 2$*
   (c) *hyperbolic*
   (d) *the twisted $I$-bundle over the Klein bottle*
3. *If $n = 2$, the JSJ torus in $M(\beta)$ does not bound two twisted $I$-bundles of Klein bottles.*
4. *Let $\mathcal{O}$ be a Seifert-fibered JSJ component $\mathcal{O}$ of $M(\beta)$ meeting a twisted $I$-bundle over the Klein bottle, denoted $\tilde{K}$. Then the regular fiber of $\mathcal{O}$ is not glued to the regular fiber of $\tilde{K}$ when viewing $\tilde{K}$ as a 3-manifold Seifert-fibered over $D^2(2,2)$.*

*Then $\bigcup_{i=1}^{n-1} T_i$ is detected by an ideal point $x$ on a norm curve $\tilde{C} \subset \widetilde{X(M)}$, and $A_{k(C)}$ tautologically extends over $x$.*

*Proof.* Let $H_i$ be the complementary regions of $\overline{M \setminus \mathbb{T}}$ for $i = 1, \ldots, n$, and let $\widehat{\mathbb{T}} = \bigcup_{i=1}^{n-1} \hat{T}_i$ denote the JSJ tori in $M(\beta)$. We have the restriction map

$$r : X(M) \to \mathcal{X} = \prod X(H_i) \tag{5.10}$$

By the Lemma 5.2.2, $H_1, H_n$ have a single genus 2 surface as their boundaries, while any other complementary regions $H_i$ (if they exist) have two genus 2 surfaces as their boundaries. Note that by Theorem 13.3.6 in Thurston (1979), $\dim(X(H_i)) \geq 3$ for $i = 1, n$, and $\dim(X(H_i)) \geq 6$ for any other $H_i$. Thus $\dim(\mathcal{X}) \geq 6(n-1)$. Let $V$ be the variety inside $\mathcal{X}$ defined by the following equations.

1. In viewing $H_i$ as the complementary regions of the knot complement $M$, there are $n-1$ gluing maps between twice-punctured tori in $H_i$. For each gluing, there are seven matching equations. If the first twice-punctured torus has $\pi_1$ a free group generated by $a_1, b_1, c_1$ and the other one is a free group generated $a_2, b_2, c_2$, the equations are

$$\mathrm{tr}(a_1) = \mathrm{tr}(a_2) \qquad \mathrm{tr}(b_1) = \mathrm{tr}(b_2) \qquad \mathrm{tr}(c_1) = \mathrm{tr}(c_2) \qquad \mathrm{tr}(a_1b_1) = \mathrm{tr}(a_2b_2) \tag{5.11}$$

$$\mathrm{tr}(a_1c_1) = \mathrm{tr}(a_2c_2) \qquad \mathrm{tr}(b_1c_1) = \mathrm{tr}(b_2c_2) \qquad \mathrm{tr}(a_1b_1c_1) = \mathrm{tr}(a_2b_2c_2) \qquad \mathrm{tr}(a_1c_1b_1) = \mathrm{tr}(a_2c_2b_2) \tag{5.12}$$

We make the following variable assignments.

$$\mathrm{tr}(a_1) = x_1 \qquad \mathrm{tr}(b_1) = x_2 \qquad \mathrm{tr}(c_1) = x_3 \qquad \mathrm{tr}(a_1b_1) = x_{12} \tag{5.13}$$

$$\mathrm{tr}(a_1c_1) = x_{13} \qquad \mathrm{tr}(b_1c_1) = x_{23} \qquad \mathrm{tr}(a_1b_1c_1) = x_{123} \qquad \mathrm{tr}(a_1c_1b_1) = x_{132} \tag{5.14}$$

Similarly, we assign $\mathrm{tr}(a_2) = x_1'$, and so on. We now show that only five of these equations suffice to define $V$. Suppose we have the equations

$$x_1 = x_1' \qquad x_2 = x_2' \qquad x_3 = x_3' \qquad x_{12} = x_{12}' \qquad x_{13} = x_{13}' \tag{5.15}$$

On each twice-punctured torus, since the punctures correspond to the same loop $[\beta]$ in $M$, the two punctures are known to be have equal trace. We thus have the trace equation

$$\mathrm{tr}(bc^{-1}) = \mathrm{tr}(aba^{-1}c^{-1}) \tag{5.16}$$

which translates into the equations

$$x_2x_3 - x_3 = x_{12}x_{13} - x_1x_{123} + x_{23} \qquad x_2'x_3' - x_3' = x_{12}'x_{13}' - x_1'x_{123}' + x_{23}' \tag{5.17}$$

Hence we can use our existing six equations to conclude that $x_{23} = x_{23}'$. By Section 5.1 in Goldman (2009), $(x_{123}, x_{132})$ and $(x_{123}', x_{132}')$ are pairs of solutions to quadratics with coefficients in the other six coordinates, which we already know are equal. Thus, either $(x_{123}, x_{132}) = (x_{132}', x_{123}')$ or $(x_{123}, x_{132}) = (x_{123}', x_{132}')$; each option represents one component of the variety $V$. Pick the component with $(x_{123}, x_{132}) = (x_{123}', x_{132}')$; we now have an irreducible component defined with $5(n-1)$ equations (5 equations for $n-1$ gluings).

2. The other equations defining $V$ are as follows: there are $n-2$ complementary regions $H_i$ that have two twice-punctured tori inside the two genus 2 surface boundary components $S_1, S_2$. For each such component, there is one extra equation which dictates that the trace of the gluing annulus (i.e. the annulus coming from the knot complement) on $S_1$ is equal to the corresponding trace on $S_2$.

Thus $V$ is defined by $(5n-5)+(n-2) = 6n-7$ equations in $\mathcal{X}$, whose dimension is at least $6n-6$. Hence $\dim(V) \geq 1$. We now show that $V$ is in fact one-dimensional. Notice that $\mathrm{Im}(r) \subset V$, since any restriction of a character in $X(M)$ satisfies the gluing equations. Note that $\beta^2$ is the product of two punctures in $T$, hence a commutator in $\pi_1(T)$. Let $V^{irr} \subset V$ be the Zariski-open subset of $V$ defined by $\mathrm{tr}(\beta^2) \neq 2$, i.e. $\mathrm{tr}(\beta) \notin \{-2, 2\}$. This means that $V^{irr}$ is a subset of traces that match on the boundary and are irreducible on $T$, which glue to give traces of $\pi_1(M)$ by Lemma 6 in Paoluzzi and Porti (2012). Hence $V^{irr} \subset \mathrm{Im}(r)$. By assumption, $\dim(X(M)) = 1$, hence $\dim(\mathrm{Im}(r)) \leq 1$. The dimension of $V^{irr}$ cannot be zero, since this would mean that $\dim(V^{irr}) < \dim(V)$ and it Zariski open, hence $V^{irr}$ is empty, contradicting the assumptions. Thus

$\dim(V) = \dim(V^{irr}) = 1$. This also means that $\dim(X(H_1)) = \dim(X(H_n)) = 3$ and for all other $H_i$, $\dim(X(H_i)) = 6$.

Let $\mathcal{O}_i$ denote the JSJ complementary regions of $M(\beta)$ corresponding to a glued $H_i$. By Theorem 4.5.1, $M(\beta)$ is $SL_2(\mathbb{C})$-compatible. let $\chi_i$ be the corresponding traces realizing $SL_2(\mathbb{C})$-compatibility, and let $\chi_\infty = (\chi_1, \ldots, \chi_n) \in V$. Note that $\chi_\infty$ cannot be an isolated point of $V$, since $V$ is defined by $6n-7$ equations in a $6n-6$-dimensional algebraic set. Furthermore, by assumption, there exists at least one $\chi_i$ that is irreducible, and by Zariski-openness of irreducibility, $\chi_i$ is surrounded by irreducible representations. Therefore $\chi_\infty$ cannot be surrounded by restrictions of reducible representations of $\pi_1(M)$, and from this we have a sequence of points $\{\chi_j\}_{j=1}^\infty \subset V^{irr}$ approaching $\chi_\infty$. Let $\{\alpha_j\} \in X(M)$ be such that $r(\alpha_j) = \chi_j$. Suppose for contradiction that up to subsequence, $\{\alpha_j\}$ converges to a character $\alpha_\infty \in X(M)$. Then $\alpha_\infty$ must be the trace of some $\rho_\infty : \pi_1(M) \to SL_2(\mathbb{C})$. Let $q_i : \pi_1(H_i) \to \pi_1(\mathcal{O}_i)$ be the quotient map induced by the gluing annulus. Since $r(\alpha_\infty) = \chi_\infty$, it follows that $\rho_\infty|_{\pi_1(H_i)}$ has the same trace as $\rho_i \circ q_i$. In the case where $\mathcal{O}_i$ is not the twisted $I$-bundle over the Klein bottle, this means that $\rho_i \circ q_i$ is irreducible, so $\rho_\infty|_{\pi_1(H_i)}$ is conjugate to $\rho_i \circ q_i$. In particular, $\rho_i|_{T_i} \circ q_i|_{\partial H_i}$ is the restriction of the holonomy representation to the boundary tori. When $\mathcal{O}_i$ is the twisted $I$-bundle over the Klein bottle, by the $SL_2(\mathbb{C})$-compatibility conditions from Theorem 4.5.1, this means that $\rho_i|_{T_i} \circ q_i|_{\partial H_i} = \rho_j$ for $j = A, B$. We have the following cases for the incompressible twice-punctured torus $T_k \subset M$ that caps off to $\widehat{T_k} \subset M(\beta)$. Let $\widehat{T_{k_1}}, \widehat{T_{k_2}}$ be the corresponding boundary tori in the complementary JSJ components, with $\pi_1(\widehat{T_{k_1}}) = \langle m_1, \ell_1 \rangle, \pi_1(\widehat{T_{k_2}}) = \langle m_2, \ell_2 \rangle$.

1. $\widehat{T_k}$ bounds two Seifert-fibered spaces. In this case, $\ell_1, \ell_2$ are the regular fibers of the complementary regions, which have holonomy regions $\rho_1', \rho_2'$. Then up to conjugacy,

$$\rho_k'(m_k) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho_k'(\ell_k') = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.18}$$

for $k = 1, 2$ and a choice of sign determined by $SL_2(\mathbb{C})$-compatibility. By Theorem 2.4.2, the regular fibers don't match, i.e. if $\varphi : \langle m_1, \ell_2 \rangle \to \langle m_2, \ell_2 \rangle$ is the gluing map, then $\varphi(\ell_1) \neq \ell_2$.

Thus, $\varphi^{-1}(\ell_2)$ is a nontrivial simple closed curve in $\pi_1(\widehat{T}_{k_1})$ that is not $\ell_1$; this means it must be of the form $m_1^p \ell_1^q$ with $p \neq 0$. However,

$$\rho_1'(m_1^p \ell_1^q) = \pm \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.19}$$

which means that the two representations are not conjugate under the gluing map $\varphi$. This contradicts the assertion that $\alpha_\infty$ is the trace of some $\rho_\infty : \pi_1(M) \to SL_2(\mathbb{C})$.

2. $\widehat{T_k}$ bounds a Seifert-fibered and a hyperbolic space. The hyperbolic space will have a holonomy representation $\rho_h$ that restricts to the torus cusp $\langle m_h, \ell_h \rangle$ as follows.

$$\rho_h(m_h) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho_h(\ell_h) = \pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \tag{5.20}$$

where $\mathrm{Im}(\tau) > 0$, and the choice of sign is determined by $SL_2(\mathbb{C})$-compatibility. This is the *cusp shape* of $\rho_h$. As seen previously, the cusp shape of the holonomy representation of a Seifert-fibered manifold is 0, but it is non-zero for a hyperbolic manifold, so the two torus representations are nonconjugate, leading to a contradiction.

3. $\widehat{T_k}$ bounds two hyperbolic spaces. Since the gluing map is orientation-reversing, the cusp shape of one of the holonomy representations has positive imaginary part, while the other has negative imaginary part. Since the cusp shapes are different, the holonomy representations restricted to the glued boundary tori cannot be conjugate, leading to a contradiction.

4. $\widehat{T_k}$ bounds a twisted $I$-bundle over the Klein bottle $\tilde{K}$ and a Seifert-fibered space. Let $\rho_{\tilde{K}}$ be $\rho_\infty$ restricted to $\tilde{K}$. We know that $\rho_{\tilde{K}} = \rho_A$ or $\rho_B$. In either case, the image of the regular fiber under the induced representation is negative the identity matrix, behaving the same way as a manifold Seifert-fibered over a hyperbolic orbifold. Thus, the argument from Case 1 leads to another contradiction.

5. $\widehat{T_k}$ bounds a twisted $I$-bundle over the Klein bottle and a hyperbolic component. Since no

element of the holonomy of the hyperbolic component restricted to the boundary torus is plus or minus the identity matrix, that representation cannot be conjugate to the twisted Klein bottle holonomy restricted to the boundary torus. We thus have a contradiction.

Thus, $\{\chi_j\}$ approaches an ideal point $x$ on a norm curve of $X(M)$, and the limiting character restricted to $H_i$ is $\chi_\infty$. By Lemma 3.5.1, $x$ detects $T$. Since the limiting character is irreducible on some $H_i$, $A_{k(C)}$ tautologically extends over $x$, so we are done. $\square$

# CHAPTER 6

# EXAMPLES, CONJECTURES, AND FUTURE DIRECTIONS

## 6.1. Toroidal Dehn fillings in alternating knots

The content of this chapter is adapted from Yi Wang. Punctured JSJ tori and tautological extensions of Azumaya algebras. 2023. URL https://arxiv.org/abs/2309.07848. and Yi Wang. Detection of twice-punctured tori in hyperbolic knot complements 2024. URL https://arxiv.org/abs/2404.06388.

The main family of examples on which we apply our main results are the punctured JSJ tori in alternating knots. The following theorem classifies such punctured JSJ tori:

**Theorem 6.1.1** (Ichihara and Masai (2013)). *Let $M = S^3 \setminus K$ be a hyperbolic alternating knot complement. Suppose $M(r)$ is toroidal but not Seifert-fibered. Then $K$ is equivalent to one of:*

1. *the figure-eight knot, with $r = 0, \pm 4$*
2. *a two-bridge knot $K_{(b_1,b_2)}$ with $|b_1|, |b_2| > 2$, with $r = 0$ if $b_1, b_2$ are even and $r = 2b_2$ if $b_1$ is odd and $b_2$ is even*
3. *a twist knot $K_{(2n,\pm 2)}$ with $|n| > 1$ and $r = 0, \pm 4$*
4. *a pretzel knot $P(q_1, q_2, q_3)$ with $q_i \neq 0, \pm 1$ for $i = 1, 2, 3$ and $r = 0$ if $q_i$ are all odd, and $r = 2(q_2 + q_3)$ if $q_1$ is even and $q_2, q_3$ are odd*

## 6.2. The figure-eight knot

We review a calculation of the limiting character of the figure-eight knot contained in the final section of Tillmann (2012), which will serve as an explicit example of one case where Theorem 5.4.1 holds, i.e. a twice-punctured torus detected by an ideal point.

Let $M$ denote the figure-eight knot complement. We have the group presentation

$$\pi_1(M) = \langle a, b \mid ab^{-1}a^{-1}bab^{-1}aba^{-1}b^{-1} \rangle \tag{6.1}$$

The character variety of the figure-eight knot only has one component containing irreducible representations, which is the canonical component $X_0(M)$. $X_0(M)$ can be described with the following equation:

$$X(M) = \{x^2(2-y) + y^2 - y - 1 = 0\} \tag{6.2}$$

where

$$x = \operatorname{tr}(a) \qquad y = \operatorname{tr}(ab) \tag{6.3}$$

The equation for the projectivization $\widetilde{X(M)}$ is thus

$$\widetilde{X(M)} = \{[x : y : z] \mid 2x^2z - x^2y + y^2z - yz^2 - z^3 = 0\} \tag{6.4}$$

and the ideal points are at the points where $z = 0$, i.e. $yx^2 = 0$, and so we have ideal points $[1 : 0 : 0]$ and $[0 : 1 : 0]$.

It is well-known (see, e.g. Cooper et al. (1994)) that the following surface, which is homeomorphic to a twice-punctured torus, is a surface detected by some ideal point $x$ in a projectivization of $X(M)$. Let $S \subset M$ be the essential twice-punctured torus with slope $(4, 1)$.

The $(4, 1)$ Dehn filling on the figure-eight knot is a closed manifold which admits a nontrivial JSJ decomposition with one incompressible torus. The Dehn filling fills both punctures on $S$, giving the incompressible torus $S' \subset M(4, 1)$. The two components of $M - S$ are homeomorphic to the trefoil knot complement and a nontrivial twisted $I$-bundle over the Klein bottle. Let $C_1$ be the trefoil knot complement, and let $C_2$ be the twisted $I$-bundle over the Klein bottle.

*Remark* 6.2.1. The figure-eight knot complement also admits an incompressible once-punctured Klein bottle, whose tubular neighborhood boundary is $S$. In $M(4, 1)$, the Klein bottle caps off to

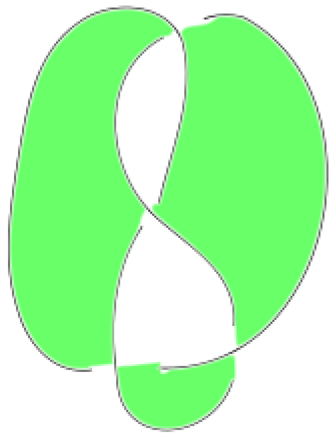

Figure 6.1: The once-pnctured Klein bottle in the figure-eight knot. $S$ is the boundary of its tubular neighborhood.

an incompressible Klein bottle, the $I$-bundle of which forms $C_2$.

*Remark* 6.2.2. There is another incompressible Klein bottle, whose tubular neighborhood boundary is another twice-punctured torus in $M$. The boundary slope of both of these surfaces is $(-4, 1)$. This case will be symmetric to the one we analyze here.

The fundamental group of the trefoil knot complement $C_1$ is given by

$$\pi_1(C_1) = \langle u, v \mid u^3 = v^2 \rangle \tag{6.5}$$

The boundary torus of $C_1$ is generated by the meridian $u^{-1}v$ and longitude $u^3$. Note that $C_1$ is Seifert fibered over the hyperbolic 2-orbifold $D^2(2,3) = S^2(2,3,\infty)$. To realize $\pi_1(C_1)$ as a subgroup of $\pi_1(M(4,1))$, we write:

$$u = a^{-1}bab^{-1}a^{-1}b \qquad v = a^{-1}bab^{-1}a^{-1}ba^{-1}bab^{-1}a^{-1}bab \tag{6.6}$$

The boundary torus is then given by

$$m_{C_1} = u^{-1}v = a^{-1}bab^{-1}a^{-1}bab \qquad l_{C_1} = u^3 = a^{-1}bab^{-1}a^{-1}ba^{-1}bab^{-1}a^{-1}ba^{-1}bab^{-1}a^{-1}b \tag{6.7}$$

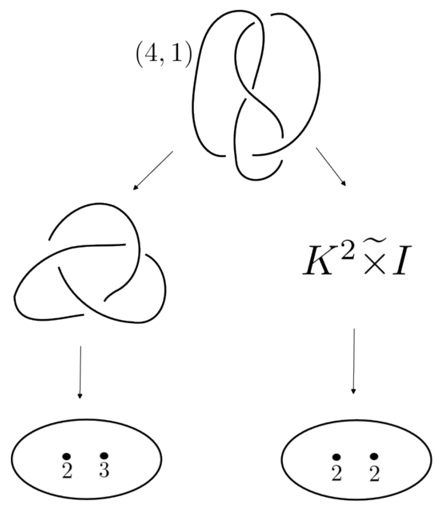


Figure 6.2: The splitting of $(4,1)$ surgery on the figure-eight knot, where the Seifert fiberings are shown. The limiting character reflects this information.

The fundamental group of $C_2$, the nontrivial $I$-bundle over the Klein bottle, is given by

$$\pi_1(C_2) = \langle s, t \mid s^2 t^2 = 1 \rangle \tag{6.8}$$

The boundary torus is generated by the meridian $st$ and the longitude $s^2$. We note that $C_2$ fibers over a Euclidean 2-orbifold $D(2,2)$, and $C^2$ itself has a Euclidean structure. So there is no hyperbolic holonomy representation into $PSL_2(\mathbb{C})$ to speak of, as we will observe in the behavior of the limiting character on $\pi_1(C_2) \subset \pi_1(M(4,1))$. To realize $\pi_1(C_2)$ as a subgroup of $\pi_1(M(4,1))$ we write:

$$s = b^{-1}a^2ba^{-1}b^{-1}a \qquad t = a^{-1}bab^{-1}a^{-2}ba^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1}ab \tag{6.9}$$

The boundary torus is given by

$$m_{C_2} = st = a^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1}ab \qquad l_{C_2} = s^2 = b^{-1}a^2ba^{-1}b^{-1}ab^{-1}a^2ba^{-1}b^{-1}a \tag{6.10}$$

The amalgamated free product $\pi_1(C_1) *_{\sim} \pi_1(C_2)$ is given by the equivalence

$$m_{C_1} = m_{C_2}^{-1} \qquad l_{C_1} = l_{C_2}^{-1} m_{C_2} \tag{6.11}$$

The representation of $\pi_1(C_1)$ corresponding to the holonomy of $D(2,3)$ is given by

$$\rho_1(u) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho_1(v) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.12}$$

On the boundary torus, this restricts to

$$\rho_1(m_{C_1}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \rho_1(l_{C_1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.13}$$

Tillmann (2012) computed that the limiting character is the trace of a representation $\rho_\infty$ which restricts to $\rho_1$ on $\pi_1(C_1)$. Meanwhile, on $C_2$, the limiting character tends to the trace of

$$\rho_\infty(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \rho_\infty(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.14}$$

So on the boundary torus, the limiting character tends to the trace of

$$\rho_\infty(m_{C_2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_\infty(l_{C_2}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.15}$$

We see that under the identification $m_{C_1} = m_{C_2}^{-1}$, the matrices $\rho_\infty(m_{C_1})$ and $\rho_\infty(m_{C_2})$ cannot possibly be conjugate, like in Paoluzzi and Porti (2012). The fact that $\chi_\infty|_{\pi_1(C_1)}$ is the trace of an irreducible holonomy representation, combined with Proposition 2.2 and a symmetric argument for the other twice-punctured torus whose slope is the negative of $S$, gives us

**Corollary 6.2.1.** *$A_{k(C)}$ tautologically extends over both ideal points of $X(M)$.*

*Remark* 6.2.3. The tautological extension comes from the limiting character on $\pi_1(C_1)$, which is homeomorphic to the trefoil knot complement; in particular, the extension comes from the holonomy structure on the underlying 2-orbifold of the Seifert fibration on the trefoil. The limiting character on $\pi_1(C_2)$ is the character of a reducible representation, and does not contribute to the tautological extension.

## 6.3. Two-bridge knots

We now analyze punctured JSJ tori in two-bridge knots. The JSJ structure of toroidal Dehn fillings of two-bridge knots are determined in Theorem 3.10 of Patton (1995) and and Proposition 4.1 of Sekino (2023). We also include the cases of the twice-punctured tori in the figure-eight knot, which was stated in Section 7 of Tillmann (2012). By Corollary 1.2 in Ichihara and Masai (2013), this is complete account of JSJ decompositions of toroidal Dehn fillings of two-bridge knots.

**Theorem 6.3.1** (Patton (1995), Tillmann (2012), Sekino (2023))**.** *Let $M = S^3 \setminus K_{p/q}$ be a two-bridge knot complement associated to $p/q \in \mathbb{Q}$, and*

$$p/q = r + \frac{1}{b_1 - \frac{1}{b_2}} \qquad r, b_1, b_2 \in \mathbb{Z} \tag{6.16}$$

*Then:*

1. *If $|b_1|, |b_2| > 2$ are even then $M$ admits two once-punctured JSJ tori with slope 0, and the JSJ complementary regions of $M(0)$ are cable spaces.*

2. *If $|b_1| = 2$ and $|b_2| > 2$ is even, then $M$ admits one once-punctured JSJ torus with slope 0, and the JSJ complementary region of $M(0)$ is a cable space.*

3. *If $b_1 = -2, b_2 = 2$ then $M$ is the figure-eight knot complement, with one once-punctured JSJ torus with slope 0 such that the JSJ complementary region of $M(0)$ is $T^2 \times I$.*

4. *The figure-eight knot complement also contains two twice-punctured JSJ tori with slopes $\pm 4$ such that the JSJ complementary region of $M(\pm 4)$ are the twisted $I$-bundle of the Klein bottle and the trefoil knot complement.*

5. *If $b_1$ is odd and $|b_2| > 2$ is even, then $M$ admits two twice-punctured JSJ tori with slope $2b_2$ such that the JSJ complementary regions of $M(2b_2)$ are the twisted $I$-bundle of the Klein bottle, a $(2, 2m + 1)$ torus knot complement for some $m \neq 0, -1$, and a cable space.*

6. *If $b_1$ is odd and $|b_2| = 2$, then $M$ is the $m$-twist knot complement. In this case, $M$ admits one once-punctured JSJ torus with slope 0, and the JSJ complementary region of $M(0)$ is a cable space.*

7. *The $m$-twist knot complement also contains one twice-punctured JSJ torus with slope 4. The JSJ complementary regions of $M(4)$ are the twisted $I$-bundle of the Klein bottle and a $(2, 2m+1)$*

We will show that all of these families of punctured JSJ tori are detected by an ideal point on the character variety, and $A_{k(C)}$ tautologically extends over those ideal points.

**Corollary 6.3.1.** *The families of Seifert surfaces in cases 1, 2, and 6 of Theorem 6.3.1 are detected by an ideal point of a norm curve $C$ of the character variety. In addition, $A_{k(C)}$ tautologically extends over this ideal point.*

*Proof.* Theorem 1.2 in Macasieb et al. (2009) combined with Theorem 7.3 in Petersen and Reid (2014) implies that all components of the character varieties of the two-bridge knots in Theorem 6.3.1 containing traces of irreducible components are norm curves. Then Cases 1, 2, and 6 of Theorem 6.3.1 combined with Theorem 5.3.1 implies the conclusion. □

In order to satisfy the hypotheses of Theorem 5.4.1, it only remains to study the JSJ decompositions of the toroidal Dehn fillings associated to the essential twice-punctured tori in two-bridge knots. For the case of the twice-punctured tori in twist knots, we use Proposition 2.2 from Teragaito (2013):

**Lemma 6.3.1** (Teragaito (2013))**.** *Let $M = S^3 \setminus K$ be a hyperbolic twist knot. Then $M(4,1)$ is a graph manifold whose JSJ regions are the twisted $I$-bundle over the Klein bottle and a torus knot exterior. In addition, the regular fiber of the torus knot exterior is not identified with the regular fiber of the twisted $I$-bundle over the Klein bottle viewed as a Seifert-fibered space over $D^2(2,2)$.*

For the case of the other two-bridge knots with twice-punctured tori, we use Lemma 3.1 from

Clay and Teragaito (2013):

**Lemma 6.3.2** (Clay and Teragaito (2013))**.** *Let $M = S^3 \setminus K$ be the complement of the two bridge knot $K_{b_1,b_2}$ with $b_1$ odd and $b_2$ even. Then $M(2b_2)$ is a graph manifold whose JSJ regions are the twisted $I$-bundle over the Klein bottle, a space Seifert-fibered over $A^2(b_2)$, and a torus knot exterior. In addition, the regular fiber of the space Seifert-fibered over $A^2(b_2)$ is not identified with the regular fiber of the twisted $I$-bundle over the Klein bottle viewed as a Seifert-fibered space over $D^2(2,2)$.*

*Remark* 6.3.1. The lemma actually shows that the regular fiber of the cable space is identified with the regular fiber of $\tilde{K}$ viewed as a Seifert-fibered space over the Mobius strip.

These two results show that all two-bridge knots with essential twice-punctured JSJ tori in their knot complements satisfy the hypotheses of the main theorems. We thus have the following corollary.

**Theorem 6.3.2.** *Any system of punctured JSJ tori in a two-bridge knot, with the exception of the minimal genus Seifert surface of the figure-eight knot, is detected by an ideal point $x$ on a norm curve $C$ in the character variety. In addition, $A_{k(C)}$ tautologically extends over $x$.*

## 6.4. Pretzel knots

Less is generally known regarding character varieties and toroidal surgeries of pretzel knots. We have the following result from Ichihara et al. (2012) which determines the JSJ decomposition of the toroidal Dehn fillings of $(-2,p,q)$ pretzel knots with $3 \leq p \leq q$, $p,q$ odd:

**Theorem 6.4.1** (Ichihara et al. (2012))**.** *Consider the toroidal manifold $M$ obtained by $2(p+q)$-surgery on the hyperbolic $(-2,p,q)$ pretzel knot with odd integers $3 \leq p \leq q$. Then $M$ admits one JSJ torus which splits $M$ into the twisted $I$-bundle over the Klein bottle and $M_{p,q}$, which is either Seifert-fibered over $D^2(p,q)$, $p > 2, q \geq 2$, or hyperbolic.*

All of these pretzel knots satisfy the hypotheses of Theorem 5.4.1 that involve the JSJ decompositions of the toroidal Dehn fillings. Furthermore, Mattman Mattman (2002) showed the following:

**Theorem 6.4.2** (Theorem 1.6, Claim 6.3 in Mattman (2002))**.** *The $SL_2(\mathbb{C})$-character variety of the $(-2, 3, 2n+1)$ pretzel knot complement with $n \not\equiv 1 \mod 3$ consists of the canonical component and the curve of reducible components. In addition, the regular fiber of the twisted $I$-bundle over the Klein bottle viewed as a fibering over $D^2(2, 2)$ does not match with the regular fiber of the other component.*

These two results combined gives

**Corollary 6.4.1.** *Let $K$ be a $(-2, 3, 2n+1)$ pretzel knot with $n \not\equiv 1 \mod 3$, and let $M = S^3 \setminus K$ be its complement in the three-sphere. Then the twice-punctured torus of slope $2(2n+4)$ is detected by an ideal point $x$ on the canonical component of the character variety, and $A_{k(C)}$ extends over $x$.*

As for once-punctured tori, the following example should be highlighted in particular:

**Proposition 6.4.1** (Baldwin and Sivek (2022))**.** *Let $K_n$ be the $(-3, 3, 2n+1)$-pretzel knot, and $M_n$ its complement in $S^3$. Then $M_n$ admits a genus one Seifert surface which caps off to a JSJ torus in $M_n(0)$. The complementary region of this JSJ torus in $M_n(0)$ is homeomorphic to the torus link $T_{2,4}$, which is Seifert-fibered over the annulus.*

This example violates the second hypothesis of Theorem 5.3.1. It would be interesting to determine whether or not the Seifert surface of $M_n$ is detected by an ideal point of a norm curve of $X(M_n)$. If so, then the limiting character at that ideal point must be reducible, providing the first example of an ideal point over which $A_{k(C)}$ does not tautologically extend. If not, this would be a rare example of a non-fiber Seifert surface not detected by an ideal point of a norm curve. Either way, Seifert surfaces of $M_n$ are expected to be examples of rare phenomena in Culler-Shalen theory.

## 6.5. Roots of unity phenomenon

Recall the “roots of unity phenomenon" first proven in Cooper et al. (1994) and elaborated upon in Dunfield (1998):

**Theorem 6.5.1** (Cooper et al. (1994))**.** *Suppose $S$ is an essential surface in $M$ with nonempty* *boundary detected by the ideal point $x$. Let $b = [\partial S] \in \pi_1(\partial M)$ be the fundamental group element associated to the boundary of $S$, which is an element of the peripheral torus. Then the limiting character at $b$ approaches $\lambda + \lambda^{-1}$ at $x$, and all other peripheral elements approach infinity. Here $\lambda = \xi_n$, a primitive $n$th root of unity. In addition, $n$ divides the number of boundary components of any connected component of $S$.*

**Conjecture 6.5.1.** *Let $M$ be a one-cusped hyperbolic 3-manifold, and let $x \in \widetilde{C}$ be an ideal point detecting the essential surface $S \subset M$ with limiting eigenvalue a primitive $n$th root of unity and boundary slope $\beta$. Let*

$$\beta' = \begin{cases} n\beta & n \text{ odd} \\ \frac{n}{2}\beta & n \text{ even} \end{cases} \tag{6.17}$$

*Let $S(\beta')$ be the surface $S$ but with cone points of order $n$ or $\frac{n}{2}$ (as prescribed above) instead of punctures. Suppose that $S(\beta')$ is a Euclidean orbifold, and there exists a connected component $M' \subset M(\beta') \setminus S(\beta')$ that is either hyperbolic or Seifert-fibered over a hyperbolic 3-manifold. Then the limiting character at $x$ restricted to $M'$ is equal to the trace of the holonomy representation of $M'$, and $A_{k(C)}$ tautologically extends over $x$.*

This conjecture is inspired by the roots of unity phenomenon, which implies that the limiting character at the ideal point descends to a character on $M'$. Stemming from the discussion in this paper, the natural candidate for the limiting character is the trace of the irreducible holonomy representation on $M'$, which would provide a basis for tautological extension over that ideal point.

We now discuss one such situation where Conjecture 6.5.1 holds, with $n = 4$ and $M$ is a knot complement in $S^3$. In Paoluzzi and Porti (2012), it is shown that ideal points exist that detect systems of Conway spheres. We summarize these results here:

**Theorem 6.5.2** (Paoluzzi and Porti (2012))**.** *Let $K$ be a hyperbolic knot complement in $S^3$ with a Bonahon-Siebenmann system of essential Conway spheres $\{C_i\}$. Let $\{M_j\}$ be the components of the complement of $\{C_i\}$ in $K$. Then the following hold:*

1. *$\bigcup_i C_i$ is detected by an ideal point of $X(K)$. Let $\chi_\infty$ be the limiting character at $x$.*

2. *Let $m \in \pi_1(K)$ be a meridian of $K$. Then $\chi_\infty = 0$, and in fact, the representation the limiting character comes from is conjugate to*

$$\rho_\infty(m) = \pm \begin{pmatrix} i & * \\ 0 & -i \end{pmatrix} \tag{6.18}$$

3. *Each $\mathcal{O}_i$ is either hyperbolic or Seifert fibered over a hyperbolic 2-orbifold.*

4. *$\chi_\infty$ corresponds to the trace of the hyperbolic holonomy representation of $\pi_1(\mathcal{O}_i)$, where $\mathcal{O}_i$ is an orbifold obtained from $M_i$ by endowing each knot component with ramification index 2. Note that $\mathcal{O}_i$ has cusps homeomorphic to the pillowcase orbifold.*

Combining the above results with Proposition 3.5.2 yields

**Corollary 6.5.1.** *Let $x$ be an ideal point of $X(K)$ that detects a system of Conway spheres. If $x$ lies on a norm curve $C$ of $X(K)$, then $A_{k(C)}$ tautologically extends over $x$.*

In Dunfield (1998), the author found examples of non-trivial roots of unity at ideal points, such as $m137$, which has limiting eigenvalue a sixth root of unity. Tillmann Tillmann (2003) also computed that the limiting character at this ideal point is irreducible. In light of these examples, we make the following conjecture, which is the $n = 6$ case of Conjecture 6.5.1.

**Conjecture 6.5.2.** *Let $M$ be a one-cusped hyperbolic 3-manifold such that $X(M)$ satisfies hypotheses of Theorem 5.3.1, and let $\{P_i\}_{i=1}^n$ be a family of non-isotopic thrice-punctured spheres that cap off to a Bonahon-Siebenmann decomposition of $M(3,0)$ consisting of $(3,3,3)$ turnovers. Then $\bigcup_{i=1}^n P_i$ is detected by an ideal point of a norm curve, and $A_{k(C)}$ extends over that ideal point.*

The manifold $m137$ is also notable because its canonical component contains no traces of reducible representations. The same is also true for knots with Alexander polynomial 1.

## 6.6. Future directions

We provide a list of related problems that bear further investigation.

1. When are essential thrice-punctured spheres detected by an ideal point? Does $A_{k(C)}$ tautologically extend over that ideal point?

2. When are non-fiber Seifert surfaces detected by an ideal point? Does $A_{k(C)}$ tautologically extend over that ideal point?

3. Determine JSJ decompositions of toroidal Dehn fillings of pretzel knots.

4. Are the Seifert surfaces of the $(-3, 3, 2n+1)$ pretzel knots detected by an ideal point?

5. Do there exist $M$ and components $C \subset X(M)$ such that $A_{k(C)}$ extends tautologically over all of $X(M)$? In particular, is this true of the canonical component of $m137$?

6. Does the refinement of the Chinburg-Reid-Stover invariant give a refinement of the Tate-Shafarevich invariants discussed in section 5 of Chinburg et al. (2022)? Does this shed any light on distinguishing arcs of $SU(2)$ representations, as the Tate-Shafarevich invariants do?

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