ON THE STABILITY OF STEADY-STATE SOLUTIONS OF A TWO-PHASE STOKES PROBLEM WITH SURFACE TENSION

Jae Ho Choi

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Supervisor of Dissertation

Robert Strain

Professor of Mathematics

Co-Supervisor of Dissertation

Yoichiro Mori

Professor of Mathematics and Biology

Graduate Group Chairperson

Ron Donagi, Professor of Mathematics

Dissertation Committee

Robert Strain, Professor of Mathematics
Yoichiro Mori, Professor of Mathematics and Biology
Scott Weinstein, Professor of Philosophy

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To my family, with love.

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ABSTRACT

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Jae Ho Choi

Robert Strain

Yoichiro Mori

In this work, we study the well-posedness of a system of partial differential equations (PDE) that model the dynamics of a two-dimensional Stokes bubble immersed in two-dimensional ambient Stokes fluid of the same viscosity that extends to infinity under the effect of surface tension. We assume that the two fluids are immiscible and incompressible and that there is no interfacial jump in the fluid velocity. For this PDE system, a circular fluid bubble is a steady-state solution. Given an initial contour for the fluid bubble which is sufficiently close to a circle, we show that there exists a unique, global-in-time fluid bubble satisfying the given initial contour and the PDE. This unique solution decays to a circle exponentially fast, which means that circular fluid bubbles are stable steady-state solutions. We also obtain a result concerning the regularity of the unique solution, that although the initial perturbation around a circular contour is assumed to be of low regularity, any later perturbation becomes real analytic, hence smooth. Lastly, we devise a boundary-integral type numerical scheme to computationally verify that the fluid bubble does indeed decay to a circle at the exponential rate predicted by the analytical results.

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CHAPTER 1

Introduction

1.1. Relevant Literature

A system of partial differential equations (PDE) constrained by an initial condition gives rise to the natural yet fundamental question of whether there exists a solution and, if so, whether it is unique. This ubiquitous question is referred to as that of "well-posedness" of the PDE. In this work, we study the well-posedness of a PDE system arising in fluid mechanics, which models the dynamics of a two-dimensional fluid bubble immersed in two-dimensional ambient fluid of the same viscosity that extends to infinity under the effect of surface tension. We assume that the two fluids are immiscible and incompressible and that there is no interfacial jump in the fluid velocity. The fluids are driven internally by the Stokes equation and interact with one another via surface tension around their interface. As there are two fluids that constitute the system, such a system is commonly referred to as a two-phase Stokes problem with surface tension in fluid mechanics. The Stokes equation is an approximation of the Navier-Stokes equation in the limit of the diminishing low Reynolds number, in which the fluid is exceptionally slow, exceedingly viscous, or minuscule in size. Mathematically, the Stokes equation is obtained from the Navier-Stokes equation by dropping terms that account for inertial effects due to fluid motion while keeping the other terms, which include terms that account for fluid viscosity. The Stokes problem is sometimes called a quasi-stationary approximation of the Navier-Stokes problem because the Stokes equation, which is blind to inertial effects due to fluid motion, is being employed to describe the motion of slow yet non-stationary fluids.

For our two-phase Stokes problem with surface tension, a circular fluid bubble is a steady-state solution. Given an initial contour for the fluid bubble which is sufficiently close to a circle, we show that there exists a unique, global-in-time fluid bubble satisfying the given initial contour and the PDE system. This unique solution decays to a circle exponentially fast, which means that circular fluid bubbles are stable steady-state solutions. We also obtain a result concerning the regularity of the unique solution, that although the initial perturbation around a circular contour is assumed to be of low regularity, any later perturbation becomes real analytic, hence smooth.

The Navier-Stokes problem with surface tension has attracted mathematicians' attention since the 1980s, starting with the one-phase problem in which an isolated liquid is driven by capillary forces acting on its boundary. The one-phase problem was pioneered in a series of papers published between 1984 and 2003 by Solonnikov (Solonnikov, 1987b, 1986, 1987a,c, 1989, 1991; Ambrosio et al., 2003; Solonnikov, 2003) and by Mogilevskii and Solonnikov (Mogilevskii and Solonnikov, 1992), in which short-time existence for arbitrary data and long-time existence for small data were established in Hölder and anisotropic Sobolev-Slobodetskii spaces. Since then, well-posedness for the one-phase problem has been established in a multitude of settings, such as the case in which the fluid domain is either bounded, a perturbed infinite layer, or a perturbed half-space (Shibata and Shimizu, 2007, 2008, 2011); and the case in which an infinite viscous incompressible fluid layer is bounded below and above by a solid surface and a free surface, respectively, experiencing surface tension and gravity (Allain, 1987; Beale, 1984; Beale and Nishida, 1985; Tani, 1996; Tani and Tanaka, 1995).

The two-phase problem gained traction in the 1990s. The first well-posedness results were established by Denisova (Denisova, 1990, 1994) and Denisova and Solonnikov (Denisova and Solonnikov, 1994, 1995). Since then, well-posedness for the two-phase problem has been established in a number of settings, such as the case in which the system is driven by thermo-capillary convection in bounded domains (Tanaka, 1995); and the case in which the free boundary is given as the graph of a function on a hyperplane (Prüss and Simonett, 2010; Anger and Simonett, 2010; Prüss and Simonett, 2011), sometimes with gravity (Prüss and Simonett, 2010, 2011).

As for the quasi-stationary approximation of the Navier-Stokes problem, the first well-posedness results for one-phase Stokes flow were established by Günther and Prokert (Günther and Prokert, 1997) and Prokert (Prokert, 1999). A handful of results concerning the regularity of solutions exist. Escher and Prokert (Escher and Prokert, 2006) obtained joint spatial and temporal analyticity of the moving boundary for one-phase Stokes flow with surface tension. Günther and Prokert (Günther and Prokert, 1997) proved short-time existence and uniqueness of a solution for one-phase Stokes flow with a free boundary driven by surface tension in Sobolev spaces of sufficiently high order. Friedman and Reitich (Friedman and Reitich, 2002) proved joint analyticity of solution for three-dimensional one-phase Stokes flow.

In this literary backdrop, our work makes novel contributions on two fronts. First of all, instantaneous analyticity of solutions is established for two-phase Stokes flow, which is the first regularity result of its kind for two-phase Stokes flow. Secondly, the analytical framework used to establish this work had not been used before to study the Navier-Stokes problem with surface tension or the quasi-stationary approximation thereof. The majority of aforementioned studies of these problems make use of the so-called direct mapping method, where the original free boundary problem, i.e., an initial value problem on an a priori unknown domain, is transformed into an abstract PDE problem on a fixed manifold.

1.2. Connections to Muskat and Peskin Problems

For our two-phase Stokes problem, the force driving the system is surface tension, i.e., a Newtonian stress imbalance across the interface which depends exclusively on its geometry via curvature. Since it has a low Reynolds number, it can serve as a rudimentary model to study the behavior of an oil droplet inside water, which is often added as an emulsifier to oil fields to reduce the viscosity of crude oil to facilitate its extraction. However, the Muskat model is a more refined and established model in this setting (Gancedo et al., 2023a,b).

Having its roots in petrochemical engineering, the Muskat model is a PDE system describing the dynamics of incompressible fluids of different nature (e.g., oil and water) permeating porous media (e.g., tar sands) under gravity. The fluids' motions are governed by a momentum equation called Darcy's law, which relates the fluid velocity and the pressure like the Stokes equation.

Also closely related to my model is the Peskin model, which is a fluid-structure interaction (FSI) model describing the dynamics of a one-dimensional closed elastic string immersed in two-dimensional Stokes fluid. Originally, it emerged as a model for blood flow through heart valves (Cameron and Strain, 2024). Being one of the simplest FSI models, it has since been used for other kinds of physical modelling and for building numerical algorithms.

Both the Muskat and Peskin problems have interesting connections to our two-phase Stokes problem with surface tension, which will be explained in depth in Subsections 1.2.1 and 1.2.2.

1.2.1. Spectral Decomposition of Linearized Operator

Recently, there has been a flurry of mathematical activity on the Muskat model studying its well-posedness. During this process, a multitude of techniques have been devised and employed. Of particular interest to our Stokes problem is called spectral decomposition of the linearized operator, which has been applied by Gancedo, García-Juárez, Patel, and Strain to establish global regularity of a two-dimensional Muskat bubble which is unstable under gravity (Gancedo et al., 2023a). The same technique has also been employed to show global-in-time well-posedness of the Peskin model (García-Juárez et al., 2023). The main idea behind this technique is to linearize the dynamics equation of interest around a steady state solution, which separates the equation into a linear part, which in principle is a Fourier analytically well-understood operator, and the remainder part, which is "small" in some appropriate sense that depends on a clever selection of the solution space. This linearization is valid only for a small neighborhood around the steady state. For example, the dynamics equation considered by Gancedo, García-Juárez, Patel, and Strain is written in the form

$$\partial_t g + (-\Delta)^{1/2} g = \mathfrak{R}, \tag{1.1}$$

where $\mathfrak{R}$ denotes the part of the equation consisting of terms that are superlinear in g. We note that the principal linear part, $(-\Delta)^{1/2}g$, is the Hilbert transform acting on the spatial derivative of g. In the Fourier space, this equation becomes

$$\partial_t \hat{g}(k) = -|k|\,\hat{g}(k) + \hat{\mathfrak{R}}(k),$$

which clearly reveals that the principal linear part is "diagonalized." This explains why the technique is often called spectral decomposition of the linearized operator.

The family of Banach spaces used by Gancedo, García-Juárez, Patel, and Strain that witness the remainder part $\mathfrak{R}$ to be "small" are

$$\dot{\mathcal{F}}^{s,1}_{\nu(t)} = \left\{ f : \mathbb{T} \to \mathbb{R} \mid \sum_{k\neq 0} e^{\nu(t)|k|} |k|^s \left| \hat{f}(k) \right| < \infty \right\}, \tag{1.2}$$

where $\nu(t) = \frac{t}{1+t}\nu_0$ for some $\nu_0 > 0$. The second superscript, 1, is simply to indicate that the l^1 norm is taken with respect to the wave number k. The first superscript, s, measures the space's critical-ness with respect to an intrinsic scale invariance of the equation for g. Let us illustrate the meaning of this statement for the Peskin model. Suppose that $\boldsymbol{X}(t, \theta)$ is a solution to the Peskin model, where $\boldsymbol{X}(t, \theta)$ is the closed elastic string at time t in the Lagrangian coordinate θ. Then

$$\boldsymbol{X}_\lambda(t, \theta) = \lambda^{-1}\boldsymbol{X}(\lambda t, \lambda\theta)$$

is also a solution to the Peskin model. We say that, with respect to $\boldsymbol{X}$, the space with $s = 1$ is critical; the space with $s > 1$ is sub-critical; and the space with $s < 1$ is super-critical.

Given a sufficiently small initial datum of low regularity describing the initial perturbation of the interface from a circle, which is a steady-state solution, we used spectral decomposition of the linearized operator to establish global-in-time existence and uniqueness of a two-dimensional bubble that satisfies the initial datum and our Stokes problem. The result is formulated using a sub-critical member of the family of spaces in (1.2). The dynamics equation for our Stokes problem is written in the form (1.1), as in the Muskat model studied by Gancedo, García-Juárez, Patel, and Strain. The Peskin model, which bears much similarity with ours, can also be written in that form (Cameron and Strain, 2024).

The time-dependent exponential weight in the norm associated with (1.2) leads to the remarkable property that even though the initial perturbation from a circle is of low regularity, it becomes instantaneously analytic. This is the first regularity result of its kind for two-phase Stokes flow driven by surface tension.

1.2.2. Unconventional Parametrization

At face value, the mathematical formulation of the Peskin model looks similar to that of ours. After all, the only difference is the nature of the force driving the system. The Peskin model is driven by the elasticity of the string, which obeys the following general law of elasticity:

$$\partial_\theta \left(T(|\partial_\theta \boldsymbol{X}|) \cdot \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right) \cdot |\partial_\theta \boldsymbol{X}|^{-1} . \tag{1.3}$$

If we let $T(\alpha) = \alpha$, then this law reduces to Hooke's law, which is commonly adopted for the analytical study of well-posedness for the Peskin problem. Unlike the Peskin model, our PDE model is driven by surface tension. This difference, however, begets an important analytical consequence.

In the Peskin model, the closed elastic string is parametrized using the Lagrangian coordinate. As the elastic force is critically dependent on this parametrization of the string, it is impossible to choose an arbitrary parametrization to aid in the analysis without fundamentally altering the physical system. This is a major point of difference for the Peskin model from both the Muskat model and our PDE model. In the Muskat model, the normal velocity at the free boundary is well-defined, while the tangential velocity is ill-defined. As the dynamics of the boundary are completely determined by the normal velocity, one can take advantage of the degree of freedom "in the tangential direction" and choose a parametrization that yields nicely to one's analytical framework. In our PDE model, the sole force driving the system is surface tension, which depends exclusively on the geometry of the interface. Therefore, one can employ any convenient parametrization for the interface without affecting its actual dynamics.

To prove our results, we deployed a particular parametrization (Hou et al., 1994) of the fluids' interface that yields nicely to spectral decomposition of the linearized operator. This parametrization is unusual in the sense that the boundary of the fluid bubble is parametrized not by its x- and y-positions, but by the direction of its tangent vector and the length of the boundary. We adopted a certain change of frame in this parametrization that gives way to our analytical framework, in which the tangent vector is independent of the spatial variable and depends only on time. The same parametrization and change of frame had also been used for a Muskat problem (Gancedo et al., 2023a).

Intriguingly, this particular frame emerged out of a strictly numerical context. Roughly twenty-five years before this frame found its use for the analytical study of well-posedness of the Muskat problem (Gancedo et al., 2023a), Hou, Lowengrub, and Shelley devised it to improve numerical simulation of the motion of the free boundary driven by surface tension between two-dimensional, irrotational, incompressible fluids. Using their novel numerical scheme in which the tangent vector's lack of dependence on the spatial variable removed "numerical stiffness," they computed flows that had been unobtainable, such as the motion of the Hele-Shaw interface moving under the competing effects of gravity and surface tension, and discovered new singularity formations, such as the roll-up and collision of vortex sheets with surface tension in two-dimensional Euler flow.

As a matter for thought, it is worth mentioning that, in fact, it is possible to cast our Stokes problem as a Peskin model whose force satisfies the general law of elasticity in (1.3) with $T(\alpha) = 1$. However, since the most general setting in which well-posedness has been established for the Peskin problem is when $T(\alpha) > 0$ and $T'(\alpha) > 0$, our Stokes problem corresponds to a degenerate case for which no well-posedness results are available. This implies that none of the techniques that have been successfully used as of now to establish well-posedness of the Peskin model can be used for our Stokes problem, which highlights the significance of our analytical framework.

1.2.3. Problem Formulation

Let Γ be a time-dependent simple closed curve in $\mathbb{R}^2$ that represents the interface between two immiscible fluids. Then the model is given by

$$\mu\Delta \boldsymbol{u} - \nabla p = \boldsymbol{0} \quad \text{on } \mathbb{R}^2 \setminus \Gamma, \tag{1.4}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{on } \mathbb{R}^2 \setminus \Gamma, \tag{1.5}$$

$$[\boldsymbol{u}] = \boldsymbol{0}, \tag{1.6}$$

$$[\Sigma(\boldsymbol{u}, p)\boldsymbol{n}] = -\gamma\kappa\boldsymbol{n}, \tag{1.7}$$

where $\boldsymbol{u}$ and p denote the fluid velocity and the fluid pressure, respectively; μ is the fluid viscosity, which is a constant within each fluid but may differ across the two fluids; $\Sigma(\boldsymbol{u}, p)$ represents the stress tensor for a Newtonian fluid of viscosity μ; $\boldsymbol{n}$ is the outward-pointing unit normal vector to the interface Γ; γ is the surface tension coefficient which is a constant; κ is the signed curvature of the interface; and the notation $[\cdot]$ means the limit value for the boundary as it is approached in the normal direction from the interior fluid minus the limit value for the boundary as it is approached in the normal direction from the exterior fluid. We assume that the two fluids share the same viscosity μ, which we normalize to 1.

In words, this model says that the interior and exterior fluids are incompressible Stokes fluids with no interfacial jump in the fluid velocity and that they are driven by a stress imbalance along the interface given by $-\gamma\kappa\boldsymbol{n}$. The observation that the interfacial force depends exclusively on the geometry of the interface via curvature κ is important, because it allows us to introduce a convenient parametrization for the interface without affecting the physical dynamics of the system.

In this model, there are two unknown variables to solve for: the two-dimensional fluid velocity $\boldsymbol{u}$ and the scalar pressure p. In the remainder of this work, we study the well-posedness of this model in terms of the fluid velocity by imposing a certain ansatz on it satisfying the specified model. Using the ansatz reduces the original problem to that of well-posedness for the PDE system for the interface dynamics. The latter is summarized in the main theorem of this work stated in Section 3. Throughout the rest of this work, we may suppress certain expressions' dependence on time t for readability.

CHAPTER 2

Preliminary Work

2.1. Key Function Spaces

In an analytical study of well-posedness, function spaces provide an essential framework to formulate results and can induce interesting properties of solutions by imposing sufficiently strong constraints on its functions, such as analyticity. In this Section, we introduce families of function spaces for our work. For a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, its Fourier transform is defined as

$$\mathcal{F}(f)(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) e^{-ik\alpha} d\alpha.$$

We may sometimes write $\hat{f}(k)$ to denote the Fourier transform of f with no intended difference in meaning. Then

$$f(\alpha) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik\alpha}. \tag{2.1}$$

In our work, we use families of Banach spaces $\mathcal{F}_\nu^{0,1}$ and $\dot{\mathcal{F}}_\nu^{s,1}$, $s \geq 0$, equipped respectively with norms

$$\|f\|_{\mathcal{F}_\nu^{0,1}} = \sum_{k \in \mathbb{Z}} e^{\nu(t)|k|} \left| \hat{f}(k) \right|,$$
$$\|f\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k \neq 0} e^{\nu(t)|k|} |k|^s \left| \hat{f}(k) \right|,$$

where

$$\nu(t) = \frac{t}{1+t} \nu_0. \tag{2.2}$$

Observe that if $\nu_0 > 0$, then $0 < \nu'(t) \leq \nu_0$. We also use Banach spaces $\mathcal{F}^{0,1}$ and $\dot{\mathcal{F}}^{s,1}$, $s \geq 0$, equipped respectively with norms

$$
\begin{aligned}
\|f\|_{\mathcal{F}^{0,1}} &= \sum_{k\in\mathbb{Z}} \left|\hat{f}(k)\right|, \\
\|f\|_{\dot{\mathcal{F}}^{s,1}} &= \sum_{k\neq 0} |k|^s \left|\hat{f}(k)\right|.
\end{aligned}
\tag{2.3}
$$

The space $\mathcal{F}^{0,1}$ equipped with the norm in (2.3) is the classical Wiener algebra, i.e., the space of absolutely convergent Fourier series.

Proposition 1. *(Embeddings.) For $0 < s_1 \leq s_2$,*

$$\|f\|_{\dot{\mathcal{F}}^{s_1,1}_\nu} \leq \|f\|_{\dot{\mathcal{F}}^{s_2,1}_\nu}.$$

Proposition 2. *(Estimates.) Let $n \geq 1$. Then*

$$\|f_1 f_2 \cdots f_n\|_{\mathcal{F}^{0,1}_\nu} \leq \Pi_{k=1}^n \|f_k\|_{\mathcal{F}^{0,1}_\nu}.$$

For $s > 0$,

$$\|f_1 f_2 \cdots f_n\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq b(n,s) \sum_{j=1}^n \|f_j\|_{\dot{\mathcal{F}}^{s,1}_\nu} \Pi_{k=1,k\neq j}^n \|f_k\|_{\mathcal{F}^{0,1}_\nu},$$

where

$$b(n,s) = \begin{cases} 1 & 0 \leq s \leq 1, \\ n^{s-1} & s > 1. \end{cases}$$

Remark 3. *The estimates in Proposition 2 hold with $\mathcal{F}^{0,1}_\nu$ and $\dot{\mathcal{F}}^{s,1}_\nu$ replaced by $\mathcal{F}^{0,1}$ and $\dot{\mathcal{F}}^{s,1}$, respectively. For proof of Proposition 2, see Lemma 5.1 of Gancedo et al. (2023a).*

Proposition 4. *For $s \geq 0$,*

$$\|g_1 g_2\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq b(2,s)\Big(\|g_1\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|g_2\|_{\mathcal{F}^{0,1}_\nu} + \|g_1\|_{\mathcal{F}^{0,1}_\nu} \|g_2\|_{\dot{\mathcal{F}}^{s,1}_\nu}\Big),$$

where

$$b(n,s) = \begin{cases} 1 & 0 \leq s \leq 1, \\ n^{s-1} & s > 1. \end{cases}$$

Proof. The case in which $s > 0$ follows from Proposition 2. Let us consider the case $s = 0$.

$$\begin{aligned}
\|g_1 g_2\|_{\dot{\mathcal{F}}^{0,1}_\nu} &= \sum_{k \neq 0} e^{\nu(t)|k|} |\mathcal{F}(g_1 g_2)(k)| \\
&= \sum_{k \neq 0} e^{\nu(t)|k|} \left| \sum_{j \in \mathbb{Z}} \hat{g}_1(k-j) \hat{g}_2(j) \right| \\
&\leq \sum_{k \neq 0} \sum_{j \in \mathbb{Z}} e^{\nu(t)|k|} |\hat{g}_1(k-j)| \, |\hat{g}_2(j)| \\
&\leq \sum_{k \neq 0} \sum_{j \in \mathbb{Z}} e^{\nu(t)|k-j|} e^{\nu(t)|j|} |\hat{g}_1(k-j)| \, |\hat{g}_2(j)| \qquad (2.4) \\
&\leq \|g_1\|_{\dot{\mathcal{F}}^{0,1}_\nu} \|g_2\|_{\mathcal{F}^{0,1}_\nu} + \|g_1\|_{\mathcal{F}^{0,1}_\nu} \|g_2\|_{\dot{\mathcal{F}}^{0,1}_\nu} .
\end{aligned}$$

The last inequality holds because

$$\begin{aligned}
&\|g_1\|_{\dot{\mathcal{F}}^{0,1}_\nu} \|g_2\|_{\mathcal{F}^{0,1}_\nu} + \|g_1\|_{\mathcal{F}^{0,1}_\nu} \|g_2\|_{\dot{\mathcal{F}}^{0,1}_\nu} \\
&= \sum_{k \neq 0} e^{\nu(t)|k|} |\hat{g}_1(k)| \cdot \sum_{j \in \mathbb{Z}} e^{\nu(t)|j|} |\hat{g}_2(j)| + \sum_{k \in \mathbb{Z}} e^{\nu(t)|k|} |\hat{g}_1(k)| \cdot \sum_{j \neq 0} e^{\nu(t)|j|} |\hat{g}_2(j)| \\
&= \sum_{k \neq 0} \sum_{j \in \mathbb{Z}} e^{\nu(t)|k|} e^{\nu(t)|j|} |\hat{g}_1(k)| \, |\hat{g}_2(j)| + \sum_{k \in \mathbb{Z}} \sum_{j \neq 0} e^{\nu(t)|k|} e^{\nu(t)|j|} |\hat{g}_1(k)| \, |\hat{g}_2(j)| \\
&= \sum_{k \neq j} \sum_{j \in \mathbb{Z}} e^{\nu(t)|k-j|} e^{\nu(t)|j|} |\hat{g}_1(k-j)| \, |\hat{g}_2(j)| + \sum_{k \in \mathbb{Z}} \sum_{j \neq k} e^{\nu(t)|k|} e^{\nu(t)|k-j|} |\hat{g}_1(k)| \, |\hat{g}_2(k-j)| . \qquad (2.5)
\end{aligned}$$

The first term in (2.5) contains all but terms of the form

$$e^{\nu(t)|j|}\,|\hat{g}_1(0)|\,|\hat{g}_2(j)|\,,\; j\in\mathbb{Z}$$

while the second term in (2.5) contains terms of the form

$$e^{\nu(t)|-j|}\,|\hat{g}_1(0)|\,|\hat{g}_2(-j)|\,,\; j\neq 0.$$

The only term that is not covered between these two terms is $|\hat{g}_1(0)|\,|\hat{g}_2(0)|$. However, this term is not covered by the sum in (2.4), either. This completes the proof. ■

We define the following frequently used operator

$$\mathcal{M}(f)(\alpha)=\int_0^{\alpha} f(\eta)d\eta-\frac{\alpha}{2\pi}\int_{-\pi}^{\pi} f(\eta)d\eta. \tag{2.6}$$

We note that

$$\mathcal{F}(\mathcal{M}(f))(k)=\begin{cases}-\frac{i}{k}\hat{f}(k) & k\neq 0\\ \sum_{j\neq 0}\frac{i}{j}\hat{f}(j) & k=0.\end{cases} \tag{2.7}$$

For $N\geq 0$, we also define high frequency cut-off operators $\mathcal{J}_N$ and $\mathcal{J}_N^1$ as

$$\mathcal{F}(\mathcal{J}_N f)(k)=1_{|k|\leq N}\mathcal{F}(f)(k), \tag{2.8}$$

$$\mathcal{F}(\mathcal{J}_N^1 f)(k)=1_{|k|\neq 1}1_{|k|\leq N}\mathcal{F}(f)(k). \tag{2.9}$$

2.2. Boundary Integral Formulation

We mentioned at the end of Section 1.2.3 that a certain ansatz that satisfies our model will be imposed on the fluid velocity. We adopt

$$u_j(\boldsymbol{x})=\frac{1}{4\pi}\int_\Gamma(-\gamma\kappa(s)\boldsymbol{n}(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds,\quad \boldsymbol{x}\in\mathbb{R}^2, \tag{2.10}$$

where $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ and $G = (G_{ij})$ given by

$$G_{ij}(\boldsymbol{w}) = -\delta_{ij} \log |\boldsymbol{w}| + \frac{w_i w_j}{|\boldsymbol{w}|^2}$$

is the Green's function for two-dimensional unbounded incompressible Stokes flow (Pozrikidis, 1992). Being a Green's function, G can be used to represent a solution of two-dimensional incompressible Stokes flow driven by a concentrated point force of some strength in the plane. In our model, there is a force density $-\gamma\kappa\boldsymbol{n}$ along the interface as opposed to a concentrated force at a single point. In this case, the solution can be represented via (2.10), which will henceforth be referred to as the single-layer potential. In general, the Green's function for two-dimensional unbounded incompressible Stokes flow suffers from the so-called Stokes' paradox of logarithmic growth of the fluid velocity at infinity. However, the fluid velocity in our model does not suffer from this paradox because the force density $-\gamma\kappa\boldsymbol{n}$ along the interface integrates to 0. The single-layer potential ensures that the fluid velocity satisfies equations (1.4) through (1.7). In particular, its analytical form guarantees continuity across the interface. The representation of the fluid velocity as a single-layer potential provides a convenient framework to study well-posedness of our model both analytically and numerically.

2.3. Interface Parametrization

In our model, the fluids are driven exclusively by a stress imbalance along the interface given by $-\gamma\kappa\boldsymbol{n}$, which can be derived explicitly from first principles of physics by assuming that surface tension along the interface be proportional to the unit tangent vector to the interface. The fact that this force differential depends exclusively on the geometry of the interface, via curvature κ, ensures that whatever parametrization we choose for the interface will have no bearing on the physical dynamics of the system. We will adopt a parametrization in which the unit tangent vector and the interface length provide coordinates for the interface instead of its x- and y-coordinates. A detailed derivation of this parametrization is in order.

Due to continuity of the fluid velocity across the interface as stated in (1.6), the interfacial fluid velocity is well-defined. We note that the interface's shape is determined entirely by its normal velocity; the tangential velocity can only alter the frame of parametrization. This means that the tangential velocity can be entered into the equations without affecting the interface's shape. Let us write the interfacial fluid velocity as

$$\boldsymbol{u} = -U\boldsymbol{n} + T\boldsymbol{\tau}, \tag{2.11}$$

where $\boldsymbol{\tau}$ is the unit tangent vector. There is a minus sign in front of the normal term, because $\boldsymbol{n}$ is by definition the outward-pointing unit normal vector to the interface. We first represent the interface with some parametrization $z(\alpha, t)$ where $\alpha \in [-\pi, \pi)$. Let us define a tangential angle variable θ by writing the tangent vector $z_\alpha(\alpha, t)$ in complex variable notation

$$z_\alpha(\alpha, t) = |z_\alpha(\alpha, t)|\, e^{i(\alpha+\theta(\alpha,t))}. \tag{2.12}$$

Using the parametrization, we can rewrite (2.11) as

$$z_t(\alpha, t) = -U(\alpha, t)\boldsymbol{n}(\alpha, t) + T(\alpha, t)\boldsymbol{\tau}(\alpha, t), \tag{2.13}$$

which in complex variable notation becomes

$$z_t(\alpha, t) = U(\alpha, t) \cdot ie^{i(\alpha+\theta(\alpha,t))} + T(\alpha, t) \cdot e^{i(\alpha+\theta(\alpha,t))}, \tag{2.14}$$

keeping in mind that in complex variable notation

$$\begin{aligned} \boldsymbol{\tau}(\alpha, t) &= e^{i(\alpha+\theta(\alpha,t))}, \\ \boldsymbol{n}(\alpha, t) &= -ie^{i(\alpha+\theta(\alpha,t))}. \end{aligned}$$

After differentiating (2.12) with respect to t and then differentiating (2.14) with respect to α, we equate their real and imaginary parts to derive evolution equations for the interface in terms of θ

and $|z_\alpha(\alpha,t)|$:

$$
\begin{aligned}
|z_\alpha(\alpha,t)|_t &= -U(\alpha,t) - U(\alpha,t)\theta_\alpha(\alpha,t) + T_\alpha(\alpha,t), && (2.15)\\
\theta_t(\alpha,t) &= \frac{1}{|z_\alpha(\alpha,t)|}\Big(U_\alpha(\alpha,t) + T(\alpha,t) + T(\alpha,t)\theta_\alpha(\alpha,t)\Big). && (2.16)
\end{aligned}
$$

Of all possible frames of parametrization, a particularly useful one can be selected by requiring the tangential speed $T(\alpha,t)$ to be of the form

$$
T(\alpha,t) = \int_0^\alpha (1+\theta_\eta(\eta,t))U(\eta,t)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta,t))U(\eta,t)d\eta + T(0,t), \tag{2.17}
$$

where $T(0,t)$ is a number that depends on t, which allows for a change of frame. The frame chosen by the imposition of (2.17) ensures that $|z_\alpha(\alpha,t)|$ is independent of α, i.e.,

$$
|z_\alpha(\alpha,t)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|z_\alpha(\eta,t)|\,d\eta = \frac{L(t)}{2\pi},
$$

where $L(t)$ is the length of the interface at time t. This can be checked by integrating (2.15) with respect to time from 0 to t and then differentiating with respect to α. Using this tangential speed formula, (2.15) and (2.16) can be rewritten as

$$
\begin{aligned}
L_t(t) &= -\int_{-\pi}^{\pi}(1+\theta_\alpha(\alpha))U(\alpha)d\alpha && (2.18)\\
\theta_t(\alpha,t) &= \frac{2\pi}{L(t)}U_\alpha(\alpha) + \frac{2\pi}{L(t)}T(\alpha)(1+\theta_\alpha(\alpha)). && (2.19)
\end{aligned}
$$

The use of this particular frame of parametrization for a fluid interface was pioneered by Hou et al. (1994) in the context of removing numerical stiffness from interfacial flows with surface tension. In their honor, we will refer to it as Hou-Lowengrub-Shelley (HLS) parametrization from now on. For our analysis, HLS parametrization is useful because it lays a natural foundation for a powerful analytical and numerical principle for solving interfacial fluid problems called small-scale decomposition. Under this principle, the principal linear operator of the evolution equation of θ is extracted and the remainder terms are shown to be of lower order in some sense under the choice of an appropriate function space (Mori et al., 2019). In the Fourier analytic viewpoint, the principal linear operator acting on θ corresponds to the part of the operator for the evolution equation of θ in which every Fourier mode of θ is multiplied by a multiplier of the form $-|k|^{\alpha}$, where α is the highest possible power on $|k|$. The idea behind this principle is to have a "good" analytical control over the principal linear operator, as it helps to control the terms with the most penchant for unbounded growth. A good analytical control entails a proper analytical treatment for large values of k, i.e., "high-frequency" Fourier modes of the principal linear operator. In the physical space, such Fourier modes encode the finest detail of the interface which is close in shape to the steady state about which the evolution equation of θ is linearized. This explains why this principle is called small-scale decomposition. Gancedo et al. (2023a) contains an application of this principle for an analytical study of the two-dimensional Muskat problem with two immiscible fluids under gravity in which one fluid is completely surrounded by the other. While Mori et al. (2019) does not use HLS parametrization, it employs small-scale decomposition to address the well-posedness of the Peskin problem in which the model is set up identically to our own except the force differential driving the system is of elastic nature, not surface tension.

2.4. The Interface Length $L(t)$

We can derive an analytical expression for $L(t)$ from the incompressibility of the internal fluid. In fact, this analytical expression and (2.18) are equivalent provided that $L(t) > 0$ for all time t. The following proposition, whose proof can be garnered from Gancedo et al. (2023a), summarizes these observations. As we shall see later in the paper, a careful estimate of the analytical expression itself shows that it is bounded above and below by certain expressions in terms of an appropriate norm of a tangential angle variable, which is useful for deriving a key *a priori* estimate for the tangent angle variable.

Proposition 5. *Let $V_0 = \pi R^2$ be the initial volume of the internal fluid. For any $t \geq 0$ such that $L(t) > 0$,*

$$\left(\frac{L(t)}{2\pi}\right)^2 = R^2\left(1 + \frac{1}{2\pi} Im \int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!}(\theta(\alpha) - \theta(\eta))^n d\eta d\alpha\right)^{-1} \tag{2.20}$$

implies

$$L_t(t) = -\int_{-\pi}^{\pi} (1+\theta_\alpha(\alpha))U(\alpha)d\alpha.$$

Remark 6. *That $V_0 = \pi R^2$ is not to say that the internal fluid is initially a circle of radius R.*

We can derive (2.20) from the incompressibility condition on the internal fluid. To see this, let $\mathcal{D}$ be the region enclosed by the fluid boundary Γ. Then the volume of the region $\mathcal{D}$ is given by

$$\begin{aligned} V &= \int_{\mathcal{D}} dx \wedge dy && (2.21)\\ &= \frac{1}{2}\int_{\mathcal{D}} d(-ydx + xdy) && (2.22)\\ &= \frac{1}{2}\int_{\Gamma} -ydx + xdy && (2.23)\\ &= \frac{1}{2}\int_{-\pi}^{\pi} (-z_2(\alpha), z_1(\alpha)) \cdot z_\alpha(\alpha)d\alpha, && (2.24) \end{aligned}$$

where $\wedge$ in (2.21) is the wedge product of differential forms; (2.22) results from the exterior derivative of the differential form; (2.23) is due to the generalized Stokes' theorem; and (2.24) follows from the definition of the line integral. Taking $z(\alpha)$ and $z_\alpha(\alpha)$ to be complex numbers instead of vectors, we can express the volume in complex-variable notation

$$V = \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{Im}\Big(\overline{z(\alpha)}z_\alpha(\alpha)\Big)d\alpha = \frac{1}{2}\mathrm{Im}\int_{-\pi}^{\pi} \overline{z(\alpha)}z_\alpha(\alpha)d\alpha.$$

Using that

$$\begin{aligned} z_\alpha(\alpha) &= \frac{L(t)}{2\pi}e^{i(\alpha+\theta(\alpha))}\\ z(\alpha) &= z(0) + \int_0^\alpha z_\eta(\eta)d\eta, \end{aligned}$$

we can write

$$
\begin{aligned}
V &= \frac{1}{2}\mathrm{Im}\int_{-\pi}^{\pi}\overline{z(\alpha)}z_\alpha(\alpha)d\alpha \\
&= \frac{1}{2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)}e^{i(\theta(\alpha)-\theta(\eta))}d\eta d\alpha \\
&= \frac{1}{2}\left(\frac{L(t)}{2\pi}\right)^2 2\pi\cdot\mathrm{Im}\left(i+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)}\sum_{n=1}^\infty\frac{i^n}{n!}(\theta(\alpha)-\theta(\eta))^n d\eta d\alpha\right) \\
&= \pi\left(\frac{L(t)}{2\pi}\right)^2\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)}\sum_{n=1}^\infty\frac{i^n}{n!}(\theta(\alpha)-\theta(\eta))^n d\eta d\alpha\right).
\end{aligned}
\tag{2.25}
$$

Since the internal fluid is incompressible,

$$V_0 = \pi R^2 = V, \tag{2.26}$$

which implies

$$\left(\frac{L(t)}{2\pi}\right)^2 = R^2\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)}\sum_{n=1}^\infty\frac{i^n}{n!}(\theta(\alpha)-\theta(\eta))^n d\eta d\alpha\right)^{-1}.$$

This reveals that the converse to Proposition 5 holds without the condition $L(t) > 0$. Now, we commence the proof of Proposition 5.

Proof. Setting (2.25) and (2.26) equal to each other, we obtain

$$\pi R^2 = \frac{1}{2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)}e^{i(\theta(\alpha)-\theta(\eta))}d\eta d\alpha.$$

After differentiating this equation with respect to t and then using $L(t) > 0$, we can rearrange the equation to obtain

$$
\begin{aligned}
L'(t) =& -\frac{1}{2R^2}\left(\frac{L(t)}{2\pi}\right)^3 \mathrm{Im}\bigg(\int_{-\pi}^{\pi}\int_0^{\alpha} ie^{i(\alpha-\eta)}e^{i(\theta(\alpha)-\theta(\eta))}(\theta_t(\alpha)-\theta_t(\eta))d\eta d\alpha\bigg)\\
=& -\frac{1}{2R^2}\left(\frac{L(t)}{2\pi}\right)^3 \bigg(\mathrm{Im}\bigg(\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha\bigg)\\
& - \mathrm{Im}\bigg(\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta d\alpha\bigg)\bigg).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha\\
=&i\int_{-\pi}^{\pi} e^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha\\
=&i\int_{-\pi}^{\pi} \frac{\partial}{\partial\alpha}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha\\
=&i\int_{-\pi}^{\pi}\bigg(\frac{\partial}{\partial\alpha}\bigg(\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta\bigg)\\
&-\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\frac{\partial}{\partial\alpha}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta\bigg)d\alpha\\
=&i\bigg(\int_0^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta-\int_0^{-\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\int_0^{-\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta\\
&-\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha\bigg).
\end{aligned}
$$

Using that

$$
\int_{-\pi}^{\pi} e^{i(\eta+\theta(\eta))}d\eta = 0, \tag{2.27}
$$

we can write

$$
\begin{aligned}
&\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha\\
=&i\bigg(\int_0^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta+\int_{-\pi}^{0} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot\int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta\\
&-\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha\bigg)\\
=&i\bigg(\int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta\cdot\int_{-\pi}^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta-\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha\bigg).
\end{aligned}
$$

Due to (2.27),

$$
\int_{-\pi}^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta=-i\int_{-\pi}^{\pi} e^{i\eta}\frac{\partial}{\partial t}e^{i\theta(\eta)}d\eta=-i\frac{\partial}{\partial t}\int_{-\pi}^{\pi} e^{i\eta}e^{i\theta(\eta)}d\eta=0.
$$

Hence,

$$
\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha=-i\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha.
$$

Therefore,

$$
L'(t)=\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^3\mathrm{Im}\bigg(\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta d\alpha\bigg).
$$

Using (2.19), we obtain

$$
\begin{aligned}
&\frac{L(t)}{2\pi}\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta\\
=&\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}\Big(U_\eta(\eta)+T(\eta)(1+\theta_\eta(\eta))\Big)d\eta\\
=&\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}U_\eta(\eta)d\eta+\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}T(\eta)(1+\theta_\eta(\eta))d\eta\\
=&\int_0^\alpha \frac{\partial}{\partial\eta}\Big(e^{-i\eta}e^{-i\theta(\eta)}U(\eta)\Big)-\frac{\partial}{\partial\eta}\Big(e^{-i\eta}e^{-i\theta(\eta)}\Big)U(\eta)d\eta+i\int_0^\alpha T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta\\
=&e^{-i\alpha}e^{-i\theta(\alpha)}U(\alpha)-e^{-i\theta(0)}U(0)+i\int_0^\alpha e^{-i(\eta+\theta(\eta))}(1+\theta_\eta(\eta))U(\eta)d\eta\\
&+i\int_0^\alpha T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta.
\end{aligned}
$$

Using that

$$
T_\eta(\eta)=(1+\theta_\eta(\eta))U(\eta)-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\xi))U(\xi)d\xi,
$$

we obtain

$$
\begin{aligned}
&\int_0^\alpha T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta\\
=&\int_0^\alpha \frac{\partial}{\partial\eta}\Big(T(\eta)e^{-i(\eta+\theta(\eta))}\Big)-T_\eta(\eta)e^{-i(\eta+\theta(\eta))}d\eta\\
=&T(\alpha)e^{-i(\alpha+\theta(\alpha))}-T(0)e^{-i\theta(0)}-\int_0^\alpha T_\eta(\eta)e^{-i(\eta+\theta(\eta))}d\eta\\
=&T(\alpha)e^{-i(\alpha+\theta(\alpha))}-T(0)e^{-i\theta(0)}\\
&-\int_0^\alpha\Big((1+\theta_\eta(\eta))U(\eta)-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\xi(\xi))U(\xi)d\xi\Big)e^{-i(\eta+\theta(\eta))}d\eta.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\int_0^\alpha T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta\\
=&T(\alpha)e^{-i(\alpha+\theta(\alpha))}-T(0)e^{-i\theta(0)}-\int_0^\alpha(1+\theta_\eta(\eta))U(\eta)e^{-i(\eta+\theta(\eta))}d\eta\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{L(t)}{2\pi}\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta\\
=&e^{-i\alpha}e^{-i\theta(\alpha)}U(\alpha)-e^{-i\theta(0)}U(0)+T(\alpha)ie^{-i(\alpha+\theta(\alpha))}-T(0)ie^{-i\theta(0)}\\
&+\frac{i}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\cdot\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta.
\end{aligned}$$

Then, using (2.27), we obtain

$$\begin{aligned}
L'(t)=&\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^3\mathrm{Im}\bigg(\int_{-\pi}^{\pi}ie^{i\alpha}e^{i\theta(\alpha)}\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta d\alpha\bigg)\\
=&\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\bigg(\int_{-\pi}^{\pi}ie^{i\alpha}e^{i\theta(\alpha)}\Big(e^{-i\alpha}e^{-i\theta(\alpha)}U(\alpha)-e^{-i\theta(0)}U(0)\\
&+T(\alpha)ie^{-i(\alpha+\theta(\alpha))}-T(0)ie^{-i\theta(0)}\\
&+\frac{i}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\cdot\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta\Big)d\alpha\bigg)\\
=&\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\bigg(i\int_{-\pi}^{\pi}U(\alpha)d\alpha-ie^{-i\theta(0)}U(0)\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}d\alpha-\int_{-\pi}^{\pi}T(\alpha)d\alpha\\
&+T(0)e^{-i\theta(0)}\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}d\alpha\\
&-\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\cdot\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta d\alpha\bigg)\\
=&\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\bigg(i\int_{-\pi}^{\pi}U(\alpha)d\alpha-\int_{-\pi}^{\pi}T(\alpha)d\alpha\\
&-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta d\alpha\bigg).
\end{aligned}$$

By the divergence theorem,

$$\int_{\mathcal{D}} \nabla \cdot \boldsymbol{u} = \int_{\Gamma} \boldsymbol{u} \cdot \boldsymbol{n} = -\int_{-\pi}^{\pi} U(\alpha)\,|z_\alpha(\alpha)|\,d\alpha = -\frac{L(t)}{2\pi}\int_{-\pi}^{\pi} U(\alpha)d\alpha.$$

Using the incompressibility of the internal fluid, $\nabla \cdot \boldsymbol{u} = 0$, we obtain

$$\int_{-\pi}^{\pi} U(\alpha)d\alpha = 0.$$

Hence,

$$\begin{aligned}
L'(t) =& \frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2 \mathrm{Im}\Bigg(-\int_{-\pi}^{\pi} T(\alpha)d\alpha \\
&- \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta \int_{-\pi}^{\pi} e^{i\alpha}e^{i\theta(\alpha)}\int_0^{\alpha} e^{-i(\eta+\theta(\eta))}d\eta d\alpha\Bigg) \\
=& -\frac{1}{2\pi R^2}\left(\frac{L(t)}{2\pi}\right)^2 \mathrm{Im}\Bigg(\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta \int_{-\pi}^{\pi} e^{i\alpha}e^{i\theta(\alpha)}\int_0^{\alpha} e^{-i(\eta+\theta(\eta))}d\eta d\alpha\Bigg) \\
=& -\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta,
\end{aligned}$$

as needed. ■

Remark 7. *The incompressibility of the interior fluid combined with the isoperimetric inequality ensures that $L(t) > 0$ is satisfied for all $t \geq 0$.*

2.5. The Circular Interface under HLS Parametrization

The following proposition characterizes the circular interface under HLS parametrization.

Proposition 8. *Let $R > 0$. The interface at time t is a circle of radius R if and only if*

$$(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R),$$

where the parametrization is HLS.

Proof. First, we check that $(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R)$ is a circle of radius R for fixed t. It suffices to show that the curve has a constant curvature $\left|\frac{d^2z}{ds^2}\right|$ of $1/R$. Observe that

$$\frac{d^2z}{ds^2} = \frac{d}{d\alpha}\left(\frac{dz}{d\alpha}\cdot\frac{d\alpha}{ds}\right)\frac{d\alpha}{ds} = \frac{d^2z}{d\alpha^2}\cdot|z_\alpha(\alpha,t)|^{-2} = \frac{ie^{i(\alpha+\hat{\theta}(0,t))}}{R}.$$

Since $\hat{\theta}(0,t)$ is a real number,

$$\left|\frac{d^2z}{ds^2}\right| = \frac{1}{R},$$

as needed. To prove the converse, suppose that the interface at time t is a circle of radius R. Then $L(t) = 2\pi R$. That $\left|\frac{d^2z}{ds^2}\right| = \frac{1}{R}$ shows that $|1+\theta_\alpha(\alpha,t)| = 1$. Due to the periodicity of θ, we have $\theta_\alpha(\alpha,t) = 0$, i.e., $\theta(\alpha,t)$ depends only on time t. Then $\hat{\theta}(0,t) = \theta(\alpha,t)$, as needed. ■

In Section 5, we remark on whether circular interfaces can solve our model.

CHAPTER 3

Statement of the Main Theorem

We are now ready to state the main theorem of our work. To study the simple two-dimensional model given by (1.4) through (1.7), we have adopted the single-layer potential form (2.10) for the fluid velocity. As a result, the fluid velocity anywhere in the plane can be obtained by convolving the interfacial stress imbalance against the Green's function for two-dimensional unbounded incompressible Stokes flow along the interface. To completely describe the dynamics of the fluid velocity, it is therefore sufficient to study the dynamics of the interface itself. To that end, we take HLS parametrization of the interface to obtain a pair of dynamics equations, (2.18) and (2.19), for the interface. Lastly, we have reformulated the dynamics equation (2.18) for the length of the interface into (2.20). The main theorem of our work is that the equations (2.20) and (2.19) for the dynamics of the interface have a unique solution that is global in time, provided that the initial datum is sufficiently small as measured by the norm of $\dot{\mathcal{F}}^{1,1}$. The unique solution also decays exponentially in time in the norm of $\dot{\mathcal{F}}^{1,1}_{\nu}$, where ν is given in (2.2) and $\nu_0 > 0$ is dependent on the initial datum. In view of Proposition 8, this implies that the initial perturbed interface decays exponentially to a circular shape.

Theorem 9. *Fix $\gamma > 0$. If the initial datum $\theta^0 \in \dot{\mathcal{F}}^{1,1}$ such that $\left|\mathcal{F}(\theta^0)(0)\right|$ and $\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}$ are sufficiently small, then for any $T \in (0,\infty)$ there exists a unique solution*

$$\theta(\alpha,t) \in C([0,T];\dot{\mathcal{F}}^{1,1}_{\nu}) \cap L^1([0,T];\dot{\mathcal{F}}^{2,1}_{\nu})$$

to the equations (2.20) and (2.19), where ν is given in (2.2) and $\nu_0 > 0$ is dependent on θ^0. The solution becomes instantaneously analytic. In particular, for any $t \in [0,T]$

$$\|\theta(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\theta(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}\, d\tau \leq \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}},$$

where $\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})$ is given in (12.31). Moreover, $\|\theta(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$ decays exponentially in time.

Remark 10. *The assumption that the initial datum be "sufficiently small" can be made explicit in the sense that for any $\gamma > 0$, there is an analytical constraint that places an upper bound on the magnitudes of $\left|\mathcal{F}(\theta^0)(0)\right|$ and $\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}$.*

CHAPTER 4

The Interfacial Fluid Velocity

To even speak of the interfacial fluid velocity, we need to ensure that it is well-defined. Fortunately, the single-layer potential form imposed on the fluid velocity satisfies the stipulation (1.6) that the fluid velocity be continuous across the interface, making the interfacial fluid velocity a well-defined quantity.

4.1. Formulation in Complex Variable Notation

We set out to rewrite (2.10) using complex variable notation, which is more conducive to calculation than vector notation. The signed curvature κ that appears in the single-layer potential is defined by, in vector notation,

$$\boldsymbol{\tau}'(s) = -\kappa(s)\boldsymbol{n}(s), \tag{4.1}$$

where s denotes arclength parametrization. Letting $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and $z = (z_1, z_2)$, we use the Jacobian between arclength parametrization and HLS parametrization to obtain

$$\tau_i'(s) = \frac{d\tau_i}{ds} = \frac{d}{ds}\left(\frac{dz_i}{ds}\right) = \frac{d}{d\beta}\left(\frac{dz_i}{d\beta} \cdot \frac{d\beta}{ds}\right) \cdot \frac{d\beta}{ds} = \frac{d^2 z_i}{d\beta^2} \cdot |z_\beta(\beta, t)|^{-2},$$

which yields, in vector notation,

$$
\begin{aligned}
u_j(\boldsymbol{x}) &= \frac{1}{4\pi}\int_\Gamma (-\gamma\kappa(s)\boldsymbol{n}(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds \\
&= \frac{\gamma}{4\pi}\int_\Gamma (\boldsymbol{\tau}'(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds \\
&= \frac{\gamma}{4\pi}\sum_{i=1}^{2}\int_\Gamma \tau_i'(s) G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds \\
&= \frac{\gamma}{4\pi}\sum_{i=1}^{2}\int_{-\pi}^{\pi} z_i''(\beta) G_{ij}(\boldsymbol{x}-z(\beta))\left|z_\beta(\beta,t)\right|^{-1} d\beta \\
&= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\sum_{i=1}^{2}\int_{-\pi}^{\pi} z_i''(\beta) G_{ij}(\boldsymbol{x}-z(\beta))d\beta \\
&= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} z''(\beta)\cdot G_{\cdot j}(\boldsymbol{x}-z(\beta))d\beta,
\end{aligned}
$$

where

$$
G_{\cdot j}(\boldsymbol{x}-z(\beta)) = (G_{1j}(\boldsymbol{x}-z(\beta)), G_{2j}(\boldsymbol{x}-z(\beta))).
$$

Let $\boldsymbol{x} = z(\alpha) \in \Gamma$. To rewrite the current expression for $u_j(\boldsymbol{x}) = u_j(z(\alpha))$ in complex variable notation, we use the following complex variable expressions

$$
\begin{aligned}
G_{\cdot j}(z(\alpha)-z(\beta)) &= G_{1j}(z(\alpha)-z(\beta)) + iG_{2j}(z(\alpha)-z(\beta)) \\
z'(\beta) &= \frac{L(t)}{2\pi}e^{i(\beta+\theta(\beta))},
\end{aligned}
$$

which yields, in complex variable notation,

$$
\begin{aligned}
u_j(z(\alpha)) =& \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\Big(\overline{z''(\beta)}G_{\cdot j}(z(\alpha)-z(\beta))\Big)d\beta \\
=& \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\bigg(\frac{d}{d\beta}\Big(\overline{z'(\beta)}\Big)G_{\cdot j}(z(\alpha)-z(\beta))\bigg)d\beta \\
=& \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\bigg(\frac{d}{d\beta}\Big(\overline{z'(\beta)}G_{\cdot j}(z(\alpha)-z(\beta))\Big) - \overline{z'(\beta)}\frac{d}{d\beta}\Big(G_{\cdot j}(z(\alpha)-z(\beta))\Big)\bigg)d\beta \\
=& -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\bigg(\overline{z'(\beta)}\frac{d}{d\beta}\Big(G_{\cdot j}(z(\alpha)-z(\beta))\Big)\bigg)d\beta,
\end{aligned}
$$

where

$$
\begin{aligned}
&\operatorname{Re}\left(\overline{z'(\beta)}\frac{d}{d\beta}\left(G_{\cdot j}(z(\alpha)-z(\beta))\right)\right)\\
&=\frac{L(t)}{2\pi}\left(\cos(\beta+\theta(\beta))\frac{d}{d\beta}\left(G_{1j}(z(\alpha)-z(\beta))\right)+\sin(\beta+\theta(\beta))\frac{d}{d\beta}\left(G_{2j}(z(\alpha)-z(\beta))\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_j(z(\alpha)) = -\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\cos(\beta+\theta(\beta))\frac{d}{d\beta}\left(G_{1j}(z(\alpha)-z(\beta))\right)\\
+\sin(\beta+\theta(\beta))\frac{d}{d\beta}\left(G_{2j}(z(\alpha)-z(\beta))\right)d\beta.
\end{aligned}
$$

By changing the variable of integration from β to $\beta' = \alpha-\beta$ and rewriting the sine and cosine in complex variable notation, we obtain

$$
\begin{aligned}
u_j(z(\alpha)) &= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\cos(\alpha-\beta'+\theta(\alpha-\beta'))\frac{d}{d\beta'}\left(G_{1j}(z(\alpha)-z(\alpha-\beta'))\right)\\
&+\sin(\alpha-\beta'+\theta(\alpha-\beta'))\frac{d}{d\beta'}\left(G_{2j}(z(\alpha)-z(\alpha-\beta'))\right)d\beta'\\
&=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}+e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{d}{d\beta}\left(G_{1j}(z(\alpha)-z(\alpha-\beta))\right) &(4.2)\\
&+\frac{1}{2i}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}-e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{d}{d\beta}\left(G_{2j}(z(\alpha)-z(\alpha-\beta))\right)d\beta. &(4.3)
\end{aligned}
$$

4.2. The Normal Speed U

To obtain the normal speed in complex variable notation, we take the dot product of (2.11) and $-\boldsymbol{u}$ to get

$$
U = \boldsymbol{u}\cdot(-\boldsymbol{n}),
$$

which can be rewritten in complex variable notation as

$$
U(\alpha) = \operatorname{Re}\left((u_1(\alpha)-iu_2(\alpha))ie^{i(\alpha+\theta(\alpha))}\right).
$$

To obtain an analytical expression for $U(\alpha)$ in complex variable notation, we first simplify (4.2) and (4.3). We note that

$$\begin{aligned}
G_{11}(z(\alpha)-z(\alpha-\beta)) &= -\log|z(\alpha)-z(\alpha-\beta)| + \frac{(z_1(\alpha)-z_1(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2},\\
G_{12}(z(\alpha)-z(\alpha-\beta)) &= \frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2},\\
G_{21}(z(\alpha)-z(\alpha-\beta)) &= \frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2},\\
G_{22}(z(\alpha)-z(\alpha-\beta)) &= -\log|z(\alpha)-z(\alpha-\beta)| + \frac{(z_2(\alpha)-z_2(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}.
\end{aligned}$$

Letting

$$w(\alpha,\beta) = \int_0^1 e^{i(\alpha+(s-1)\beta+\theta(\alpha+(s-1)\beta))}ds,$$

we can write

$$z(\alpha)-z(\alpha-\beta) = \beta\int_0^1 z_\alpha(\alpha+(s-1)\beta)ds = \frac{\beta L(t)}{2\pi}w(\alpha,\beta).$$

Denoting the complex conjugate of w by $\overline{w}$, we then obtain

$$\begin{aligned}
&\frac{\partial}{\partial\beta}\Big(-\log|z(\alpha)-z(\alpha-\beta)|\Big)\\
&= -\frac{1}{2}\cdot\frac{\partial}{\partial\beta}\log|z(\alpha)-z(\alpha-\beta)|^2\\
&= -\frac{1}{2}\cdot\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2}\cdot\frac{\partial}{\partial\beta}\Big(|z(\alpha)-z(\alpha-\beta)|^2\Big)\\
&= -\frac{1}{2}\cdot\frac{1}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\cdot\frac{\partial}{\partial\beta}\left(\left(\frac{L(t)}{2\pi}\right)^2\beta^2 w\overline{w}\right)\\
&= -\frac{1}{2}\cdot\frac{1}{\beta^2 w\overline{w}}\cdot\frac{\partial}{\partial\beta}\Big(\beta^2 w\overline{w}\Big)\\
&= -\frac{1}{2\beta^2 w\overline{w}}\Big(2\beta w\overline{w} + \beta^2(w_\beta\overline{w} + w\overline{w}_\beta)\Big)\\
&= -\frac{1}{\beta} - \frac{w_\beta\overline{w}+w\overline{w}_\beta}{2w\overline{w}}\\
&= -\frac{1}{\beta} - \frac{w_\beta}{2w} - \frac{\overline{w}_\beta}{2\overline{w}}.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\\
=&\frac{\partial}{\partial\beta}\left(\frac{\left(\frac{1}{2}\left(\frac{\beta L(t)}{2\pi}w+\frac{\beta L(t)}{2\pi}\overline{w}\right)\right)^2}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\right)\\
=&\frac{\partial}{\partial\beta}\left(\frac{(w+\overline{w})^2}{4w\overline{w}}\right)\\
=&\frac{1}{4}\cdot\frac{\partial}{\partial\beta}\left((w+\overline{w})^2(w\overline{w})^{-1}\right)\\
=&\frac{1}{4}\left(2(w+\overline{w})(w_\beta+\overline{w}_\beta)(w\overline{w})^{-1}-(w+\overline{w})^2(w\overline{w})^{-2}(w_\beta\overline{w}+w\overline{w}_\beta)\right)\\
=&\frac{(w+\overline{w})(w_\beta+\overline{w}_\beta)}{2w\overline{w}}-\frac{(w+\overline{w})^2(w_\beta\overline{w}+w\overline{w}_\beta)}{4(w\overline{w})^2}\\
=&\frac{1}{2}\cdot\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)(w_\beta+\overline{w}_\beta)-\frac{1}{4}\cdot\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\frac{(z_2(\alpha)-z_2(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\\
=&\frac{\partial}{\partial\beta}\left(\frac{\left(\frac{1}{2i}\left(\frac{\beta L(t)}{2\pi}w-\frac{\beta L(t)}{2\pi}\overline{w}\right)\right)^2}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\right)\\
=&-\frac{1}{4}\cdot\frac{\partial}{\partial\beta}\left((w-\overline{w})^2(w\overline{w})^{-1}\right)\\
=&-\frac{1}{4}\left(2(w-\overline{w})(w_\beta-\overline{w}_\beta)(w\overline{w})^{-1}-(w-\overline{w})^2(w\overline{w})^{-2}(w_\beta\overline{w}+w\overline{w}_\beta)\right)\\
=&-\frac{1}{2}\cdot\frac{w-\overline{w}}{w\overline{w}}(w_\beta-\overline{w}_\beta)+\frac{1}{4}\left(\frac{w-\overline{w}}{w\overline{w}}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta)\\
=&-\frac{1}{2}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)(w_\beta-\overline{w}_\beta)+\frac{1}{4}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta).
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\frac{\partial}{\partial \beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\\
=&\frac{\partial}{\partial \beta}\left(\frac{\frac{1}{2}\left(\frac{\beta L(t)}{2\pi}w+\frac{\beta L(t)}{2\pi}\overline{w}\right)\frac{1}{2i}\left(\frac{\beta L(t)}{2\pi}w-\frac{\beta L(t)}{2\pi}\overline{w}\right)}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\right)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial \beta}\left(\frac{(w+\overline{w})(w-\overline{w})}{w\overline{w}}\right)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial \beta}\left(\frac{w^2-\overline{w}^2}{w\overline{w}}\right)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial \beta}\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial \beta}\left(w\overline{w}^{-1}-\overline{w}w^{-1}\right)\\
=&\frac{1}{4i}\left(w_\beta\overline{w}^{-1}-w\overline{w}^{-2}\overline{w}_\beta-\overline{w}_\beta w^{-1}+\overline{w}w^{-2}w_\beta\right)\\
=&\frac{1}{4i}\left(\frac{w_\beta}{\overline{w}}-\frac{w\overline{w}_\beta}{\overline{w}^2}-\frac{\overline{w}_\beta}{w}+\frac{\overline{w}w_\beta}{w^2}\right)\\
=&\frac{1}{4i}\left(2\left(\frac{w_\beta}{\overline{w}}-\frac{\overline{w}_\beta}{w}\right)-\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\left(\frac{w_\beta}{w}+\frac{\overline{w}_\beta}{\overline{w}}\right)\right)\\
=&\frac{1}{2i}\left(\frac{w_\beta}{\overline{w}}-\frac{\overline{w}_\beta}{w}\right)-\frac{1}{4i}\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\left(\frac{w_\beta}{w}+\frac{\overline{w}_\beta}{\overline{w}}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{\partial}{\partial \beta}\left(G_{11}(z(\alpha)-z(\alpha-\beta))\right)\\
=&\frac{\partial}{\partial \beta}\left(-\log|z(\alpha)-z(\alpha-\beta)|\right)+\frac{\partial}{\partial \beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\\
=&-\frac{1}{\beta}-\frac{w_\beta}{2w}-\frac{\overline{w}_\beta}{2\overline{w}}+\frac{1}{2}\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)(w_\beta+\overline{w}_\beta)-\frac{1}{4}\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\Big(G_{22}(z(\alpha)-z(\alpha-\beta))\Big)\\
=&\frac{\partial}{\partial\beta}\Big(-\log|z(\alpha)-z(\alpha-\beta)|\Big)+\frac{\partial}{\partial\beta}\left(\frac{(z_2(\alpha)-z_2(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\\
=&-\frac{1}{\beta}-\frac{w_\beta}{2w}-\frac{\overline{w}_\beta}{2\overline{w}}-\frac{1}{2}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)(w_\beta-\overline{w}_\beta)+\frac{1}{4}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\Big(G_{12}(z(\alpha)-z(\alpha-\beta))\Big)=\frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big)\\
=&\frac{\partial}{\partial\beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\\
=&\frac{1}{2i}\left(\frac{w_\beta}{\overline{w}}-\frac{\overline{w}_\beta}{w}\right)-\frac{1}{4i}\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\left(\frac{w_\beta}{w}+\frac{\overline{w}_\beta}{\overline{w}}\right).
\end{aligned}
$$

For notational convenience, let us write

$$
\begin{aligned}
w &= C_1+L_1+N_1,\\
w^{-1} &= C_2+L_2+N_2,\\
w_\beta &= C_\beta+L_\beta+N_\beta,
\end{aligned}
$$

where C_1, L_1, and N_1 are the parts of w which are constant, linear, and superlinear in the variable $\phi=\theta-\hat{\theta}(0)$, respectively; C_2, L_2, and N_2 are the parts of w^{-1} which are constant, linear, and superlinear in the variable ϕ, respectively; lastly, C_β, L_β, and N_β are the parts of w_β which are constant, linear, and superlinear in the variable ϕ. We note that

$$\begin{aligned}
C_1 =& \frac{-e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}i(-1+e^{i\beta})}{\beta} \\
L_1 =& ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds \\
N_1 =& e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}\sum_{n=2}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^n}{n!}ds \\
=& e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)} \\
&\cdot\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds - i\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds + \frac{i(-1+e^{i\beta})}{\beta}\right),
\end{aligned}$$

$$\begin{aligned}
C_2 =& \frac{e^{-i\hat{\theta}(0)}e^{-i\alpha}i\beta}{1-e^{-i\beta}} \\
L_2 =& \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha+\beta)}i\beta^2}{(1-e^{-i\beta})^2}\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds \\
N_2 =& \frac{e^{-i\hat{\theta}(0)}e^{-i\alpha}\beta^2}{(1-e^{-i\beta})^2}\int_0^1 e^{i(s-1)\beta}\sum_{m=2}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds \\
&+ e^{-i\hat{\theta}(0)}e^{-i\alpha}\sum_{n=2}^{\infty}(-1)^n\frac{(i\beta)^{n+1}}{(1-e^{-i\beta})^{n+1}}\left(\int_0^1 e^{i(s-1)\beta}\sum_{m=1}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \\
=& \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha+\beta)}\beta^2}{(1-e^{-i\beta})^2} \\
&\cdot\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds - i\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds + \frac{i(-1+e^{i\beta})}{\beta}\right) \\
&+ \frac{e^{-i\hat{\theta}(0)}e^{-i\alpha}e^{-2i\beta}(i\beta)^3}{(1-e^{-i\beta})^3}\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds + \frac{i(-1+e^{i\beta})}{\beta}\right)^2 \\
&\cdot\left(1 - \frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds + \frac{i(-1+e^{i\beta})}{\beta}\right)\right)^{-1},
\end{aligned}$$

and

$$
\begin{aligned}
C_\beta =& \frac{ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}(e^{i\beta}-i\beta-1)}{\beta^2}\\
L_\beta =& -e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\phi(\alpha+(s-1)\beta)ds\\
&+ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\phi_\alpha(\alpha+(s-1)\beta)ds\\
N_\beta =& ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\sum_{n=2}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^n}{n!}ds\\
&+ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^n}{n!}\phi_\alpha(\alpha+(s-1)\beta)ds\\
=& ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\Bigg(\int_0^1 e^{is\beta}(s-1)e^{i\phi(\alpha+(s-1)\beta)}ds-i\int_0^1 e^{is\beta}(s-1)\phi(\alpha+(s-1)\beta)ds\\
&-\frac{e^{i\beta}-i\beta-1}{\beta^2}\Bigg)\\
&+ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\Bigg(\int_0^1 e^{is\beta}(s-1)e^{i\phi(\alpha+(s-1)\beta)}\phi_\alpha(\alpha+(s-1)\beta)ds\\
&-\int_0^1 e^{is\beta}(s-1)\phi_\alpha(\alpha+(s-1)\beta)ds\Bigg).
\end{aligned}
$$

Similarly, let us write

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big)=C_{11}+L_{11}+N_{11},\\
&\frac{\partial}{\partial\beta}\Big(G_{12}(z(\alpha)-z(\alpha-\beta))\Big)=\frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big)=C_{12}+L_{12}+N_{12},\\
&\frac{\partial}{\partial\beta}\Big(G_{22}(z(\alpha)-z(\alpha-\beta))\Big)=C_{22}+L_{22}+N_{22},
\end{aligned}
$$

where C_{11}, L_{11}, and N_{11} are the parts of $\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big)$ which are constant, linear, and superlinear in the variable ϕ; C_{12}, L_{12}, and N_{12} are the parts of $\frac{\partial}{\partial\beta}\Big(G_{12}(z(\alpha)-z(\alpha-\beta))\Big)$ which are constant, linear, and superlinear in the variable ϕ; lastly, C_{22}, L_{22}, and N_{22} are the parts of $\frac{\partial}{\partial\beta}\Big(G_{22}(z(\alpha)-z(\alpha-\beta))\Big)$ which are constant, linear, and superlinear in the variable ϕ. We note that

$$\begin{aligned}
C_{11} =& -\frac{1}{\beta} - \frac{1}{2}C_2C_\beta - \frac{1}{2}\overline{C_2C_\beta} + \frac{1}{2}(C_2+\overline{C_2})(C_\beta+\overline{C_\beta}) \\
& - \frac{1}{4}(C_2+\overline{C_2})^2(C_\beta\overline{C_1}+C_1\overline{C_\beta}), \\
L_{11} =& -\frac{1}{2}(C_2L_\beta+C_\beta L_2) - \frac{1}{2}(\overline{C_2L_\beta}+\overline{L_2C_\beta}) \\
& + \frac{1}{2}\bigg((C_2+\overline{C_2})(L_\beta+\overline{L_\beta}) + (L_2+\overline{L_2})(C_\beta+\overline{C_\beta})\bigg) \\
& - \frac{1}{4}\bigg((C_2+\overline{C_2})^2(C_\beta\overline{L_1}+L_\beta\overline{C_1}+C_1\overline{L_\beta}+L_1\overline{C_\beta}) \\
& + 2(C_2+\overline{C_2})(L_2+\overline{L_2})(C_\beta\overline{C_1}+C_1\overline{C_\beta})\bigg),
\end{aligned}$$

$$
\begin{aligned}
N_{11} =& -\frac{1}{2}\Big(C_2N_\beta + L_2(L_\beta + N_\beta) + N_2(C_\beta + L_\beta + N_\beta)\Big)\\
& -\frac{1}{2}\Big(\overline{C_2N_\beta} + \overline{L_2}(\overline{L_\beta} + \overline{N_\beta}) + \overline{N_2}(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\Big)\\
& +\frac{1}{2}\Big((C_2 + \overline{C_2})(N_\beta + \overline{N_\beta}) + (L_2 + \overline{L_2})(L_\beta + \overline{L_\beta} + N_\beta + \overline{N_\beta})\\
& + (N_2 + \overline{N_2})(C_\beta + \overline{C_\beta} + L_\beta + \overline{L_\beta} + N_\beta + \overline{N_\beta})\Big)\\
& -\frac{1}{4}\Big((C_2 + \overline{C_2})^2\Big(C_\beta\overline{N_1} + L_\beta(\overline{L_1} + \overline{N_1}) + N_\beta(\overline{C_1} + \overline{L_1} + \overline{N_1})\\
& + C_1\overline{N_\beta} + L_1(\overline{L_\beta} + \overline{N_\beta}) + N_1(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\Big)\\
& + 2(C_2 + \overline{C_2})(L_2 + \overline{L_2})\Big(C_\beta\overline{L_1} + L_\beta\overline{C_1} + C_1\overline{L_\beta} + L_1\overline{C_\beta} + C_\beta\overline{N_1}\\
& + L_\beta(\overline{L_1} + \overline{N_1}) + N_\beta(\overline{C_1} + \overline{L_1} + \overline{N_1}) + C_1\overline{N_\beta} + L_1(\overline{L_\beta} + \overline{N_\beta})\\
& + N_1(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\Big)\\
& + \Big((C_2 + \overline{C_2})(N_2 + \overline{N_2}) + (L_2 + \overline{L_2})(L_2 + \overline{L_2} + N_2 + \overline{N_2})\\
& + (N_2 + \overline{N_2})(C_2 + \overline{C_2} + L_2 + \overline{L_2} + N_2 + \overline{N_2})\Big)\\
& \cdot\Big(C_\beta\overline{C_1} + C_1\overline{C_\beta} + C_\beta\overline{L_1} + L_\beta\overline{C_1} + C_1\overline{L_\beta} + L_1\overline{C_\beta} + C_\beta\overline{N_1} + L_\beta(\overline{L_1} + \overline{N_1})\\
& + N_\beta(\overline{C_1} + \overline{L_1} + \overline{N_1}) + C_1\overline{N_\beta} + L_1(\overline{L_\beta} + \overline{N_\beta}) + N_1(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{12} =& \frac{1}{2i}\Big(C_\beta\overline{C_2} - \overline{C_\beta}C_2\Big) - \frac{1}{4i}\Big(C_1\overline{C_2} - \overline{C_1}C_2\Big)\Big(C_\beta C_2 + \overline{C_\beta C_2}\Big),\\
L_{12} =& \frac{1}{2i}\Big(C_\beta\overline{L_2} + L_\beta\overline{C_2} - \overline{C_\beta}L_2 - \overline{L_\beta}C_2\Big) - \frac{1}{4i}\Big((C_1\overline{C_2} - \overline{C_1}C_2)(C_\beta L_2 + L_\beta C_2 + \overline{C_\beta L_2} + \overline{L_\beta C_2})\\
& + (C_1\overline{L_2} + L_1\overline{C_2} - \overline{C_1}L_2 - \overline{L_1}C_2)(C_\beta C_2 + \overline{C_\beta C_2})\Big),
\end{aligned}
$$

$$
\begin{aligned}
N_{12} =& \frac{1}{2i}\Bigg(C_\beta\overline{N_2} + L_\beta(\overline{L_2}+\overline{N_2}) + N_\beta(\overline{C_2}+\overline{L_2}+\overline{N_2}) \\
& - \Bigg(\overline{C_\beta}N_2 + \overline{L_\beta}(L_2+N_2) + \overline{N_\beta}(C_2+L_2+N_2)\Bigg)\Bigg) \\
& - \frac{1}{4i}\Bigg((C_1\overline{C_2} - \overline{C_1}C_2)\Bigg(C_\beta N_2 + L_\beta(L_2+N_2) + N_\beta(C_2+L_2+N_2) \\
& + \overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2}+\overline{N_2}) + \overline{N_\beta}(\overline{C_2}+\overline{L_2}+\overline{N_2})\Bigg) \\
& + \Bigg(C_1\overline{L_2} + L_1\overline{C_2} - (\overline{C_1}L_2 + \overline{L_1}C_2)\Bigg) \\
& \cdot \Bigg(C_\beta L_2 + L_\beta C_2 + \overline{C_\beta L_2} + \overline{L_\beta C_2} + C_\beta N_2 + L_\beta(L_2+N_2) \\
& + N_\beta(C_2+L_2+N_2) + \overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2}+\overline{N_2}) + \overline{N_\beta}(\overline{C_2}+\overline{L_2}+\overline{N_2})\Bigg) \\
& + \Bigg(C_1\overline{N_2} + L_1(\overline{L_2}+\overline{N_2}) + N_1(\overline{C_2}+\overline{L_2}+\overline{N_2}) \\
& - \Bigg(\overline{C_1}N_2 + \overline{L_1}(L_2+N_2) + \overline{N_1}(C_2+L_2+N_2)\Bigg)\Bigg) \\
& \cdot \Bigg(C_\beta C_2 + \overline{C_\beta C_2} + C_\beta L_2 + L_\beta C_2 + \overline{C_\beta L_2} + \overline{L_\beta C_2} + C_\beta N_2 + L_\beta(L_2+N_2) \\
& + N_\beta(C_2+L_2+N_2) + \overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2}+\overline{N_2}) + \overline{N_\beta}(\overline{C_2}+\overline{L_2}+\overline{N_2})\Bigg)\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
C_{22} =& -\frac{1}{\beta} - \frac{1}{2}C_\beta C_2 - \frac{1}{2}\overline{C_\beta C_2} - \frac{1}{2}(\overline{C_2}-C_2)(C_\beta - \overline{C_\beta}) \\
& + \frac{1}{4}(\overline{C_2}-C_2)^2(C_\beta\overline{C_1} + C_1\overline{C_\beta}), \\
L_{22} =& -\frac{1}{2}(C_\beta L_2 + L_\beta C_2) - \frac{1}{2}(\overline{C_\beta L_2} + \overline{L_\beta C_2}) \\
& - \frac{1}{2}\Bigg((\overline{C_2}-C_2)(L_\beta - \overline{L_\beta}) + (\overline{L_2}-L_2)(C_\beta - \overline{C_\beta})\Bigg) \\
& + \frac{1}{4}\Bigg((\overline{C_2}-C_2)^2(C_\beta\overline{L_1} + L_\beta\overline{C_1} + C_1\overline{L_\beta} + L_1\overline{C_\beta}) \\
& + 2(\overline{C_2}-C_2)(\overline{L_2}-L_2)(C_\beta\overline{C_1} + C_1\overline{C_\beta})\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
N_{22} = &-\frac{1}{2}\Big(C_\beta N_2 + L_\beta(L_2+N_2) + N_\beta(C_2+L_2+N_2)\Big)\\
&-\frac{1}{2}\Big(\overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2}+\overline{N_2}) + \overline{N_\beta}(\overline{C_2}+\overline{L_2}+\overline{N_2})\Big)\\
&-\frac{1}{2}\Big((\overline{C_2}-C_2)(N_\beta-\overline{N_\beta}) + (\overline{L_2}-L_2)(L_\beta-\overline{L_\beta}+N_\beta-\overline{N_\beta})\\
&+(\overline{N_2}-N_2)(C_\beta-\overline{C_\beta}+L_\beta-\overline{L_\beta}+N_\beta-\overline{N_\beta})\Big)\\
&+\frac{1}{4}\Big((\overline{C_2}-C_2)^2\Big(C_\beta\overline{N_1}+L_\beta(\overline{L_1}+\overline{N_1})+N_\beta(\overline{C_1}+\overline{L_1}+\overline{N_1})+C_1\overline{N_\beta}\\
&+L_1(\overline{L_\beta}+\overline{N_\beta})+N_1(\overline{C_\beta}+\overline{L_\beta}+\overline{N_\beta})\Big)\\
&+2(\overline{C_2}-C_2)(\overline{L_2}-L_2)\Big(C_\beta\overline{L_1}+L_\beta\overline{C_1}+C_1\overline{L_\beta}+L_1\overline{C_\beta}+C_\beta\overline{N_1}\\
&+L_\beta(\overline{L_1}+\overline{N_1})+N_\beta(\overline{C_1}+\overline{L_1}+\overline{N_1})\\
&+C_1\overline{N_\beta}+L_1(\overline{L_\beta}+\overline{N_\beta})+N_1(\overline{C_\beta}+\overline{L_\beta}+\overline{N_\beta})\Big)\\
&+\Big((\overline{C_2}-C_2)(\overline{N_2}-N_2)+(\overline{L_2}-L_2)(\overline{L_2}-L_2+\overline{N_2}-N_2)\\
&+(\overline{N_2}-N_2)(\overline{C_2}-C_2+\overline{L_2}-L_2+\overline{N_2}-N_2)\Big)\\
&\cdot\Big(C_\beta\overline{C_1}+C_1\overline{C_\beta}+C_\beta\overline{L_1}+L_\beta\overline{C_1}+C_1\overline{L_\beta}+L_1\overline{C_\beta}+C_\beta\overline{N_1}+L_\beta(\overline{L_1}+\overline{N_1})\\
&+N_\beta(\overline{C_1}+\overline{L_1}+\overline{N_1})+C_1\overline{N_\beta}+L_1(\overline{L_\beta}+\overline{N_\beta})+N_1(\overline{C_\beta}+\overline{L_\beta}+\overline{N_\beta})\Big)\Big),
\end{aligned}
$$

where $\overline{X}$ denotes the complex conjugate of X. It is clear from these expressions that C_{11}, L_{11}, N_{11}, C_{12}, L_{12}, N_{12}, C_{22}, L_{22}, and N_{22} are all real. Using these expressions, we can write

$$
\begin{aligned}
&u_1(z(\alpha)) \\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\Big(e^{i(\alpha-\beta+\theta(\alpha-\beta))}+e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\Big)\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big) \\
&+\frac{1}{2i}\Big(e^{i(\alpha-\beta+\theta(\alpha-\beta))}-e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\Big)\frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big)d\beta \\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\Big(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}+e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\Big) \\
&\cdot\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big) \\
&+\frac{1}{2i}\Big(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}-e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\Big) \\
&\cdot\frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big)d\beta \\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}}{2}\bigg(\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big) \\
&+\frac{1}{i}\frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big)\bigg) \\
&+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}}{2}\bigg(\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big) \\
&-\frac{1}{i}\frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big)\bigg)d\beta \\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}}e^{i(\alpha-\beta)}}{2}\bigg(1+i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!}\bigg) \\
&\cdot(C_{11}+L_{11}+N_{11}-i(C_{21}+L_{21}+N_{21})) \\
&+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg(1-i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}\bigg) \\
&\cdot(C_{11}+L_{11}+N_{11}+i(C_{21}+L_{21}+N_{21}))d\beta,
\end{aligned}
$$

and

$$
\begin{aligned}
&u_2(z(\alpha))\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}+e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right)\\
&+\frac{1}{2i}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}-e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}+e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\right)\\
&\cdot\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right)\\
&+\frac{1}{2i}\left(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}-e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\right)\\
&\cdot\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}}{2}\left(\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right)\right.\\
&\left.+\frac{1}{i}\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)\right)\\
&+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}}{2}\left(\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right)\right.\\
&\left.-\frac{1}{i}\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)\right)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\left(1+i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!}\right)\\
&\cdot(C_{12}+L_{12}+N_{12}-i(C_{22}+L_{22}+N_{22}))\\
&+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\left(1-i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}\right)\\
&\cdot(C_{12}+L_{12}+N_{12}+i(C_{22}+L_{22}+N_{22}))d\beta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&u_1(z(\alpha)) - iu_2(z(\alpha)) \qquad (4.4)\\
&= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\left(1 + i\phi(\alpha-\beta) + \sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!}\right)\\
&\cdot\left((C_{11}+L_{11}+N_{11}) - (C_{22}+L_{22}+N_{22}) - 2i(C_{12}+L_{12}+N_{12})\right)\\
&+ \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\left(1 - i\phi(\alpha-\beta) + \sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}\right)\\
&\cdot\left((C_{11}+L_{11}+N_{11}) + (C_{22}+L_{22}+N_{22})\right)d\beta.
\end{aligned}
$$

Let

$$u_1(\alpha) - iu_2(\alpha) = \mathfrak{C}(\alpha) + \mathfrak{L}(\alpha) + \mathfrak{N}(\alpha),$$

where $\mathfrak{C}$, $\mathfrak{L}$, and $\mathfrak{N}$ are the parts of $u_1 - iu_2$ which are constant, linear, and superlinear in the variable ϕ, respectively. Then

$$\begin{aligned}
\mathfrak{C}(\alpha) &= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(C_{11} - C_{22} - 2iC_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(C_{11} + C_{22})d\beta, \\
\mathfrak{L}(\alpha) &= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11} - C_{22} - 2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11} + C_{22})\right) \\
&\quad \cdot \phi(\alpha-\beta)d\beta \qquad (4.5) \\
&\quad + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11} - L_{22} - 2iL_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11} + L_{22})\right)d\beta, \\
\mathfrak{N}(\alpha) &= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\Bigg((N_{11} - N_{22} - 2iN_{12}) \qquad (4.6) \\
&\quad + i\phi(\alpha-\beta)(L_{11} - L_{22} - 2iL_{12} + N_{11} - N_{22} - 2iN_{12}) \\
&\quad + \sum_{n=2}^{\infty} \frac{(i\phi(\alpha-\beta))^n}{n!} \\
&\quad \cdot (C_{11} - C_{22} - 2iC_{12} + L_{11} - L_{22} - 2iL_{12} + N_{11} - N_{22} - 2iN_{12})\Bigg) \\
&\quad + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\Bigg((N_{11} + N_{22}) - i\phi(\alpha-\beta)(L_{11} + L_{22} + N_{11} + N_{22}) \\
&\quad + \sum_{n=2}^{\infty} \frac{(-i\phi(\alpha-\beta))^n}{n!}(C_{11} + C_{22} + L_{11} + L_{22} + N_{11} + N_{22})\Bigg)d\beta.
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathfrak{C}(\alpha) &= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(C_{11} - C_{22} - 2iC_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(C_{11} + C_{22})d\beta \\
&= 0.
\end{aligned}$$

Let $U = U_0 + U_1 + U_{\geq 2}$, where U_0, U_1, and $U_{\geq 2}$ are the parts of U which are constant, linear, and superlinear in the variable ϕ, respectively. Then

$$U_0(\alpha) = \mathrm{Re}\left(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{C}(\alpha)\right) = 0. \qquad (4.7)$$

To find expressions for U_1 and $U_{\geq 2}$, we rewrite

$$\begin{aligned}
U(\alpha) &= \mathrm{Re}\Big((u_1(\alpha) - iu_2(\alpha))ie^{i(\alpha+\theta(\alpha))}\Big) \\
&= \mathrm{Re}\Big((\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha))ie^{i(\alpha+\phi(\alpha)+\hat{\theta}(0))}\Big) \\
&= \mathrm{Re}\Big((\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha))ie^{i\alpha}e^{i\hat{\theta}(0)}e^{i\phi(\alpha)}\Big) \\
&= \mathrm{Re}\Bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}(\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha))\Bigg(1 + \sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!}\Bigg)\Bigg) \\
&= \mathrm{Re}\Bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\Bigg(\mathfrak{L}(\alpha) + \mathfrak{L}(\alpha)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!} + \mathfrak{N}(\alpha)\Bigg(1 + \sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!}\Bigg)\Bigg)\Bigg) \\
&= \mathrm{Re}\Bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\Bigg(\mathfrak{L}(\alpha) + \mathfrak{L}(\alpha)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!} + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\Bigg)\Bigg) \\
&= \mathrm{Re}\Bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha) + ie^{i\alpha}e^{i\hat{\theta}(0)}\Bigg(\mathfrak{L}(\alpha)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!} + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\Bigg)\Bigg) \\
&= \mathrm{Re}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha)\Big) + \mathrm{Re}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\Big(\mathfrak{L}(\alpha)(e^{i\phi(\alpha)} - 1) + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\Big)\Big).
\end{aligned}$$

Then it is clear that

$$U_1(\alpha) = \mathrm{Re}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha)\Big), \tag{4.8}$$

$$U_{\geq 2}(\alpha) = \mathrm{Re}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\Big(\mathfrak{L}(\alpha)(e^{i\phi(\alpha)} - 1) + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\Big)\Big). \tag{4.9}$$

4.3. The Tangential Speed T

Let us rewrite the right hand side of (2.19) as

$$\frac{2\pi}{L(t)}\Big(U_\alpha(\alpha) + T(\alpha)(1 + \phi_\alpha(\alpha))\Big) = \mathcal{C}(\alpha) + \mathcal{L}(\alpha) + \mathcal{N}(\alpha),$$

where $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{N}$ are the parts of the right hand side of the evolution equation for θ which are constant, linear, and superlinear in the variable $\phi = \theta - \hat{\theta}(0)$, respectively. We will completely determine the frame of parametrization by specifying the analytical expression for $T(\alpha)$ such that

$\mathcal{C} = 0$. To begin, let us rewrite the right hand side of (2.17) as

$$\begin{aligned}&\int_0^\alpha (1+\phi_\alpha(\eta))U(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}(1+\phi_\alpha(\eta))U(\eta)d\eta + T(0)\\ =&T_0(\alpha) + T_1(\alpha) + T_{\geq 2}(\alpha),\end{aligned}$$

where T_0, T_1, and $T_{\geq 2}$ are the parts of T which are constant, linear, and superlinear in the variable ϕ, respectively. We note that

$$\begin{aligned}T_0(\alpha) =& \int_0^\alpha U_0(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_0(\eta)d\eta + T(0) = T(0),\\ T_1(\alpha) =& \int_0^\alpha U_1(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_0(\eta)d\eta\\ &- \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_1(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_0(\eta)d\eta,\\ T_{\geq 2}(\alpha) =& \int_0^\alpha U_{\geq 2}(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta\\ &- \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_{\geq 2}(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta,\end{aligned} \tag{4.10}$$

where we define $U_{\geq 1} = U_1 + U_{\geq 2}$. Let $T(0) = 0$. Then using (4.7), we obtain

$$\mathcal{C}(\alpha) = \frac{2\pi}{L(t)}\Big((U_0)_\alpha(\alpha) + T_0(\alpha)\Big) = \frac{2\pi}{L(t)}T_0(\alpha) = \frac{2\pi}{L(t)}T(0) = 0.$$

It is important for our analysis that $\mathcal{C} = 0$ because we want the leading order term of the evolution equation for θ to be $\mathcal{L}$, which we show in Chapter 6 to be the Hilbert transform of the first spatial derivative of θ up to the ± 1 Fourier modes.

CHAPTER 5

Steady-State Solutions

In Section 2.5, we characterized the circular interface under HLS parametrization. In particular, we know from Proposition 8 that $(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R)$ corresponds to a circle of radius R. We note that

$$\int_0^\alpha z_\eta(\eta,t)d\eta = \int_0^\alpha |z_\eta(\eta,t)|\, e^{i(\eta+\theta(\eta,t))}d\eta = \frac{L(t)}{2\pi}\int_0^\alpha e^{i(\eta+\theta(\eta,t))}d\eta$$

under HLS parametrization. Plugging in $(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R)$, we obtain

$$z(\alpha,t) - z(0,t) = R\int_0^\alpha e^{i(\eta+\hat{\theta}(0,t))}d\eta = Re^{i\hat{\theta}(0,t)}\int_0^\alpha e^{i\eta}d\eta = -iRe^{i\hat{\theta}(0,t)}(e^{i\alpha}-1).$$

Rearranging the equation, we obtain

$$z(\alpha,t) = Re^{i\left(\alpha+\hat{\theta}(0,t)-\frac{\pi}{2}\right)} + \left(z(0,t) + Re^{i\left(\hat{\theta}(0,t)+\frac{\pi}{2}\right)}\right). \tag{5.1}$$

As expected, this expression reveals that the interface is a circle of radius R for any fixed time t. Since $\phi(\alpha,t) = \theta(\alpha,t) - \hat{\theta}(0,t) = 0$, it follows that $\mathfrak{L}(\alpha,t) = \mathfrak{N}(\alpha,t) = 0$. Then, $U_1 = U_{\geq 2} = 0$ by (4.8) and (4.9). Combined with (4.7), they imply $U(\alpha,t) = 0$. Due to the analytical expression chosen for $T(\alpha)$ in Section 4.3, $T(\alpha,t) = 0$. Then

$$z_t(\alpha,t) = -U(\alpha,t)\boldsymbol{n}(\alpha,t) + T(\alpha,t)\boldsymbol{\tau}(\alpha,t) = 0.$$

This means that $z(0,t)$ appearing in (5.1) is in fact identically equal to a constant. We can then rewrite (5.1) as

$$z(\alpha,t) = Re^{i\left(\alpha+\hat{\theta}(0,t)-\frac{\pi}{2}\right)} + \left(z(0,0) + Re^{i\left(\hat{\theta}(0,t)+\frac{\pi}{2}\right)}\right), \tag{5.2}$$

which describes a circle of radius R whose center is bounded in time. It turns out that the circular interface becomes a solution to (2.18) and (2.19) if $\hat{\theta}(0,t)$ is constant in time. In this case, the interface is stationary. The following proposition summarizes the existence of steady-state solutions to (2.18) and (2.19).

Proposition 11. *For any constant c, the circle defined by*

$$(\theta(\alpha,t),L(t)) = (c, 2\pi R)$$

is a time-independent solution of (2.18) and (2.19) in which $T(\alpha,t)$ is given by (2.17) and $U(\alpha,t)$ is given by

$$U(\alpha,t) = Re\Bigg((u_1(\alpha,t) - iu_2(\alpha,t))ie^{i(\alpha+\theta(\alpha,t))}\Bigg)$$

with $u_1(\alpha,t) - iu_2(\alpha,t)$ given by (4.4). This solution corresponds to a stationary circle of radius R.

Proof. Let $(\theta(\alpha,t),L(t)) = (\hat{\theta}(0,t), 2\pi R)$ be a circle of radius R such that $\hat{\theta}(0,t) = c$ for some constant c. Since $U(\alpha,t) = T(\alpha,t) = 0$, the right hand sides of (2.18) and (2.19) vanish. Since $(\theta_t(\alpha,t), L_t(t)) = (0,0)$, (2.18) and (2.19) are indeed satisfied by $(\theta(\alpha,t),L(t)) = (c, 2\pi R)$. That this solution is stationary follows from the fact that $\hat{\theta}(0,t) = c$ makes the right hand side of (5.2) independent of t, i.e., the circle is stationary, as needed. ∎

CHAPTER 6

The Principal Linear Operator for the θ Equation

In Section 4.3, we completely determined the frame of parametrization by specifying the analytical expression for $T(\alpha)$ such that $\mathcal{C} = 0$ to ensure that the linear operator $\mathcal{L}$ appearing in the evolution equation for θ, which acts on $\phi = \theta - \hat{\theta}(0)$, is the Hilbert transform of the first spatial derivative of θ up to the ± 1 Fourier modes. In this Subsection, we prove this claim about the operator $\mathcal{L}$ through explicit calculation in the Fourier space. We note that

$$\mathcal{L}(\alpha) = \frac{2\pi}{L(t)}\Big((U_1)_\alpha(\alpha) + T_0(\alpha)\phi_\alpha(\alpha) + T_1(\alpha)\Big) = \frac{2\pi}{L(t)}\Big((U_1)_\alpha(\alpha) + T_1(\alpha)\Big)$$

by (4.10). By (4.7),

$$\begin{aligned} T_1(\alpha) &= \int_0^\alpha U_1(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_0(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_1(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_0(\eta)d\eta \\ &= \int_0^\alpha U_1(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_1(\eta)d\eta. \end{aligned}$$

Using (2.6), we can write

$$\mathcal{L}(\alpha) = \frac{2\pi}{L(t)}\Big((U_1)_\alpha(\alpha) + \mathcal{M}(U_1)(\alpha)\Big). \tag{6.1}$$

6.1. The Fourier Modes of $\mathcal{L}$

In this Section, we confirm that $\mathcal{L}$ is the Hilbert transform of the first spatial derivative of θ up to the ± 1 Fourier modes by checking that its Fourier multiplier is $|k|$ for $|k| > 1$. Ultimately, we will calculate $\mathcal{F}(\mathcal{L})(k)$, the kth Fourier mode of $\mathcal{L}(\alpha)$, for all $k \in \mathbb{Z}\setminus\{0\}$. Using (2.7), we obtain that for $k \neq 0$,

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)}\Big(\mathcal{F}((U_1)_\alpha)(k) - \frac{i}{k}\mathcal{F}(U_1)(k)\Big). \tag{6.2}$$

First, we set out to find the expressions for U_1 and $(U_1)_\alpha$. From (4.8), we obtain

$$U_1(\alpha) = \mathrm{Re}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\Big)\mathrm{Re}\mathfrak{L}(\alpha) - \mathrm{Im}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\Big)\mathrm{Im}\mathfrak{L}(\alpha), \tag{6.3}$$

where $\mathfrak{L}(\alpha)$ is given by (4.5). In the expression for $\mathfrak{L}(\alpha)$, we have inside the first integral

$$\begin{aligned}
&\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11} - C_{22} - 2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11} + C_{22}) \\
=&\left(\mathrm{Re}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right) + i\mathrm{Im}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)\right)(C_{11} - C_{22} - 2iC_{12}) \\
&+\left(\mathrm{Re}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right) + i\mathrm{Im}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)\right)(C_{11} + C_{22}).
\end{aligned}$$

Since C_{11}, L_{11}, C_{12}, L_{12}, C_{22}, and L_{22} are all real, we obtain

$$\begin{aligned}
&\mathrm{Re}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11} - C_{22} - 2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11} + C_{22})\right) \\
=&\mathrm{Re}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(C_{11} - C_{22}) + \mathrm{Im}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)2C_{12} \\
&+\mathrm{Re}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)(C_{11} + C_{22})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11} - C_{22} - 2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11} + C_{22})\right) \\
=&\mathrm{Re}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(-2C_{12}) + \mathrm{Im}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(C_{11} - C_{22}) \\
&+\mathrm{Im}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)(C_{11} + C_{22}).
\end{aligned}$$

Similarly, we have inside the second integral

$$\begin{aligned}
&\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\\
=&\left(\mathrm{Re}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)+i\mathrm{Im}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)\right)(L_{11}-L_{22}-2iL_{12})\\
&+\left(\mathrm{Re}\left(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)+i\mathrm{Im}\left(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)\right)(L_{11}+L_{22}).
\end{aligned}$$

Since C_{11}, L_{11}, C_{12}, L_{12}, C_{22}, and L_{22} are all real,

$$\begin{aligned}
&\mathrm{Re}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\right)\\
=&\mathrm{Re}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(L_{11}-L_{22})+\mathrm{Im}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)2L_{12}\\
&+\mathrm{Re}\left(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)(L_{11}+L_{22})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\right)\\
=&\mathrm{Re}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(-2L_{12})+\mathrm{Im}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(L_{11}-L_{22})\\
&+\mathrm{Im}\left(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)(L_{11}+L_{22}).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mathrm{Re}\mathfrak{L}(\alpha) \qquad (6.4)\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11}-C_{22}-2iC_{12})-\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11}+C_{22})\bigg)\\
&\cdot\phi(\alpha-\beta)d\beta\\
&+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\bigg)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}-C_{22})+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2C_{12}\\
&+\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}+C_{22})\bigg)d\beta\\
&+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}-L_{22})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{12}\\
&+\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}+L_{22})\bigg)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2C_{11}+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2C_{12}\bigg)d\beta\\
&+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{11}+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{12}\bigg)d\beta
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Im}\mathfrak{L}(\alpha) \qquad\qquad (6.5)\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11}-C_{22}-2iC_{12})-\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11}+C_{22})\bigg)\\
&\cdot\phi(\alpha-\beta)d\beta\\
&+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\bigg)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{12})\\
&+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}-C_{22})+\mathrm{Im}\bigg(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(C_{11}+C_{22})\bigg)d\beta\\
&+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{12})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}-L_{22})\\
&+\mathrm{Im}\bigg(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(L_{11}+L_{22})\bigg)d\beta\\
=&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{12})+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{22})\bigg)d\beta\\
&+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{12})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{22})\bigg)d\beta.
\end{aligned}
$$

Plugging (6.4) and (6.5) back into (6.3) and then simplifying, we obtain

$$
\begin{aligned}
&U_1(\alpha) \qquad\qquad (6.6)\\
= &\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\Bigg(\\
&\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds\cdot\frac{-(-i+(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\\
&+\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds\cdot\frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2}\\
&+\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds\cdot\frac{-(-1+e^{i\beta})\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\\
&+\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds\cdot\frac{-e^{-i\beta}(-1+e^{i\beta})\beta(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\\
&+\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds\cdot\frac{-(-1+e^{i\beta})i\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\\
&+\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds\cdot\frac{e^{-i\beta}(-1+e^{i\beta})i\beta(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\\
&+\phi(\alpha-\beta)\cdot\frac{e^{-i\beta}(-1+e^{i\beta})(-i)(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})^2}\Bigg)d\beta.
\end{aligned}
$$

Differentiating (6.6) with respect to α, we obtain

$$
\begin{aligned}
&(U_1)_\alpha(\alpha) \\
= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\Bigg(& \\
&\int_0^1 e^{-i\beta s}\phi'(\alpha+\beta(-1+s))ds \cdot \frac{-(-i+(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \\
&+\int_0^1 e^{i\beta s}\phi'(\alpha+\beta(-1+s))ds \cdot \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2} \\
&+\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \cdot \frac{-(-1+e^{i\beta})\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \\
&+\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \cdot \frac{-e^{-i\beta}(-1+e^{i\beta})\beta(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \\
&+\int_0^1 e^{-i\beta s}(-1+s)\phi''(\alpha+\beta(-1+s))ds \cdot \frac{-(-1+e^{i\beta})i\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \\
&+\int_0^1 e^{i\beta s}(-1+s)\phi''(\alpha+\beta(-1+s))ds \\
&\quad\cdot\frac{-e^{-i\beta}(-1+e^{i\beta})i(-\beta)(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \\
&+\phi'(\alpha-\beta)\cdot\frac{-e^{-i\beta}(-1+e^{i\beta})i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})^2}\Bigg)d\beta.
\end{aligned}
$$

Now, taking the Fourier modes of U_1 and $(U_1)_\alpha$ and plugging them into (6.2), we obtain that for $k \notin \{0, \pm 1\}$,

$$
\begin{aligned}
&\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)} \frac{\gamma}{4\pi} \mathcal{F}(\phi)(k) \Bigg(\\
&\int_{-\pi}^{\pi} \frac{(i-(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \frac{e^{-i\beta}-e^{-i\beta k}}{\beta} d\beta \left(\frac{k}{k-1} + \frac{1}{k(1-k)} \right) \\
&+ \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2} \frac{e^{i\beta}-e^{-i\beta k}}{\beta} d\beta \left(\frac{k}{1+k} - \frac{1}{k(1+k)} \right) \\
&+ \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta} d\beta \left(\frac{-k^2}{(-1+k)^2} + \frac{1}{(-1+k)^2} \right) \\
&+ \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{ik^2}{-1+k} - \frac{i}{-1+k} \right) \\
&+ \int_{-\pi}^{\pi} \frac{e^{-i\beta}i(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta} d\beta \left(\frac{-k^2}{(1+k)^2} + \frac{1}{(1+k)^2} \right) \\
&+ \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{-k^2}{1+k} + \frac{1}{1+k} \right) \\
&+ \int_{-\pi}^{\pi} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})} \frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta} d\beta \left(\frac{ik}{(-1+k)^2} - \frac{i}{k(-1+k)^2} \right) \\
&+ \int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{k}{-1+k} - \frac{1}{k(-1+k)} \right) \\
&+ \int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta} d\beta \left(\frac{ik}{(1+k)^2} - \frac{i}{k(1+k)^2} \right) \\
&+ \int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{k}{1+k} - \frac{1}{k(1+k)} \right) \\
&+ \int_{-\pi}^{\pi} \frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})} d\beta \left(ik - \frac{i}{k} \right) \Bigg).
\end{aligned}
$$

For $k \notin \{0, \pm 1\}$, we define

$$
\begin{aligned}
&J_1(k) = \qquad\qquad (6.7)\\
&\int_{-\pi}^{\pi} \frac{(i-(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\frac{e^{-i\beta}-e^{-i\beta k}}{\beta}d\beta\bigg(\frac{k}{k-1}+\frac{1}{k(1-k)}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2}\frac{e^{i\beta}-e^{-i\beta k}}{\beta}d\beta\bigg(\frac{k}{1+k}-\frac{1}{k(1+k)}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta}d\beta\bigg(\frac{-k^2}{(-1+k)^2}+\frac{1}{(-1+k)^2}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\bigg(\frac{ik^2}{-1+k}-\frac{i}{-1+k}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{e^{-i\beta}i(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta}d\beta\bigg(\frac{-k^2}{(1+k)^2}+\frac{1}{(1+k)^2}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\bigg(\frac{-k^2}{1+k}+\frac{1}{1+k}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})}\frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta}d\beta\bigg(\frac{ik}{(-1+k)^2}-\frac{i}{k(-1+k)^2}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\bigg(\frac{k}{-1+k}-\frac{1}{k(-1+k)}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta}d\beta\bigg(\frac{ik}{(1+k)^2}-\frac{i}{k(1+k)^2}\bigg)\\
&+\int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\bigg(\frac{k}{1+k}-\frac{1}{k(1+k)}\bigg)
\end{aligned}
$$

and

$$J_2(k) = \int_{-\pi}^{\pi} \frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})}d\beta\bigg(ik-\frac{i}{k}\bigg).$$

Then for $|k| > 1$ we can write the kth Fourier mode of $\mathcal{L}$ as

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\bigg(J_1(k)+J_2(k)\bigg). \qquad (6.8)$$

Since (6.1) is real, for $k \in \mathbb{Z}^+$,

$$\mathcal{F}(\mathcal{L})(-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{L}(\alpha)e^{ik\alpha}d\alpha = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{L}(\alpha)e^{-ik\alpha}d\alpha} = \overline{\mathcal{F}(\mathcal{L})(k)}. \qquad (6.9)$$

Hence, it suffices to compute $\mathcal{F}(\mathcal{L})(k)$ only for $k > 1$.

6.1.1. Computing $J_2(k)$

For $J_2(k)$, it suffices to calculate the integral

$$ik \int_{-\pi}^{\pi} \frac{-e^{-i\beta} i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})} d\beta.$$

Using that

$$\frac{1}{-1+re^{i\beta}} = -\frac{1}{1-re^{i\beta}} = -\sum_{n=0}^{\infty}(re^{i\beta})^n,$$

we obtain

$$\begin{aligned}
&ik \int_{-\pi}^{\pi} \frac{-e^{-i\beta} i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})} d\beta \\
=&k \cdot \mathrm{pv} \int_{-\pi}^{\pi} \frac{e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})} d\beta \\
=&k \lim_{\epsilon \to 0^+} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \frac{e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})} d\beta \\
=&\frac{k}{4} \lim_{\epsilon \to 0^+} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \lim_{r \to 1^-} -e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta}) \sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \\
=&\frac{k}{4} \lim_{\epsilon \to 0^+} \lim_{r \to 1^-} \sum_{n=0}^{\infty} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} -e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})(re^{i\beta})^n d\beta \\
=&-\frac{k}{4} \lim_{\epsilon \to 0^+} \lim_{r \to 1^-} \sum_{n=0}^{\infty} r^n \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n} d\beta.
\end{aligned}$$

To calculate the outer integral, we note that

$$\begin{aligned}
&\int_{-\pi}^{\pi} e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta \\
=&\int_{-\pi}^{\pi} e^{-i\beta(k+1-n)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})d\beta \\
=&\begin{cases} 0 & \text{if } n \notin \{k+1,k,k-1,k-2\}, \\ 2\pi & \text{otherwise.} \end{cases}
\end{aligned}$$

Then

$$\begin{aligned}
&-\frac{k}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta \\
=&-\frac{k}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n 2\pi 1_{\{k-2,k-1,k,k+1\}}(n) \\
=&\begin{cases} 0 & \text{if } k<-1, \\ -2\pi k & \text{if } k>1. \end{cases}
\end{aligned}$$

Moreover,

$$-\frac{k}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta = k\pi.$$

Adding these two integrals together, we obtain

$$ik\int_{-\pi}^{\pi}\frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})}d\beta = \begin{cases} \pi k & \text{if } k<-1, \\ -\pi k & \text{if } k>1. \end{cases}$$

Then

$$-\frac{i}{k}\cdot \mathrm{pv}\int_{-\pi}^{\pi}\frac{e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})}d\beta = \begin{cases} -\frac{\pi}{k} & \text{if } k<-1, \\ \frac{\pi}{k} & \text{if } k>1. \end{cases}$$

Adding these two integrals together, we obtain

$$J_2(k) = \begin{cases} \pi\left(k - \frac{1}{k}\right) & \text{if } k < -1, \\ -\pi\left(k - \frac{1}{k}\right) & \text{if } k > 1. \end{cases} \tag{6.10}$$

6.1.2. Computing $J_1(k)$

In view of (6.9), we assume that $k > 1$. In this Subsection, we adopt the notational convention that any summation $\sum$ in which the upper bound is strictly less than the lower bound is defined to be 0. For example, if $k = 2$, then (6.13) vanishes. To begin, we simplify the first two integrals in (6.7). The first integral can be written as

$$\begin{aligned} &\int_{-\pi}^{\pi} \frac{(i - (i+\beta)e^{i\beta})(-1 - 2e^{i\beta} + e^{2i\beta})}{4(-1+e^{i\beta})^2} \frac{e^{-i\beta} - e^{-i\beta k}}{\beta} d\beta \\ =&\int_{-\pi}^{\pi} \frac{-i(-1 - 2e^{i\beta} + e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta} - e^{-i\beta k}}{\beta} d\beta \\ &+ \int_{-\pi}^{\pi} \frac{-\beta e^{i\beta}(-1 - 2e^{i\beta} + e^{2i\beta})}{4(-1+e^{i\beta})^2} \frac{e^{-i\beta} - e^{-i\beta k}}{\beta} d\beta, \end{aligned}$$

while the second integral can be written as

$$\begin{aligned} &\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1 + 2e^{i\beta} + e^{2i\beta})(\beta + i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2} \frac{e^{i\beta} - e^{-i\beta k}}{\beta} d\beta \\ =&\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1 + 2e^{i\beta} + e^{2i\beta})\beta}{4(-1+e^{i\beta})^2} \frac{e^{i\beta} - e^{-i\beta k}}{\beta} d\beta \\ &+ \int_{-\pi}^{\pi} \frac{(-1 + 2e^{i\beta} + e^{2i\beta})i}{4(-1+e^{i\beta})} \frac{1 - e^{-i\beta(k+1)}}{\beta} d\beta. \end{aligned}$$

For ease of notation, let us define

$$
\begin{aligned}
g_1(k) &= \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta}-e^{-i\beta k}}{\beta} d\beta, \\
g_2(k) &= \int_{-\pi}^{\pi} \frac{-\beta e^{i\beta}(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \frac{e^{-i\beta}-e^{-i\beta k}}{\beta} d\beta, \\
g_3(k) &= \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})\beta}{4(-1+e^{i\beta})^2} \frac{e^{i\beta}-e^{-i\beta k}}{\beta} d\beta, \\
g_4(k) &= \int_{-\pi}^{\pi} \frac{(-1+2e^{i\beta}+e^{2i\beta})i}{4(-1+e^{i\beta})} \frac{1-e^{-i\beta(k+1)}}{\beta} d\beta, \\
g_5(k) &= \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta, \\
g_6(k) &= \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta, \\
g_7(k) &= \int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta, \\
g_8(k) &= \int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta.
\end{aligned}
$$

Then we can rewrite

$$
\begin{aligned}
J_1(k) =&(g_1(k) + g_2(k))\left(\frac{k}{k-1} + \frac{1}{k(1-k)}\right) \\
&+ (g_3(k) + g_4(k))\left(\frac{k}{1+k} - \frac{1}{k(1+k)}\right) \\
&+ g_1(k)\left(\frac{-k^2}{(-1+k)^2} + \frac{1}{(-1+k)^2}\right) \\
&+ g_5(k)\left(\frac{ik^2}{-1+k} - \frac{i}{-1+k}\right) \\
&+ g_4(k)\left(\frac{-k^2}{(1+k)^2} + \frac{1}{(1+k)^2}\right) \\
&+ g_6(k)\left(\frac{-k^2}{1+k} + \frac{1}{1+k}\right) \\
&+ (-ig_1(k))\left(\frac{ik}{(-1+k)^2} - \frac{i}{k(-1+k)^2}\right) \\
&+ g_7(k)\left(\frac{k}{-1+k} - \frac{1}{k(-1+k)}\right) \\
&+ (ig_4(k))\left(\frac{ik}{(1+k)^2} - \frac{i}{k(1+k)^2}\right) \\
&+ g_8(k)\left(\frac{k}{1+k} - \frac{1}{k(1+k)}\right).
\end{aligned} \tag{6.11}
$$

Simplifying this expression, we obtain

$$
\begin{aligned}
J_1(k) =&\frac{k+1}{k}g_2(k) + \frac{k-1}{k}g_3(k) \\
&+ i(k+1)g_5(k) + (1-k)g_6(k) + \frac{k+1}{k}g_7(k) + \frac{k-1}{k}g_8(k).
\end{aligned} \tag{6.12}
$$

Let us first compute $g_2(k)$. Using that

$$
\begin{aligned}
1 - e^{-i\beta(k-1)} &= i\beta(k-1)\int_0^1 e^{-i\beta(k-1)s}ds \\
\frac{1}{(-1+re^{i\beta})^2} &= \sum_{n=0}^{\infty}(1+n)(re^{i\beta})^n,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
g_2(k) =&\text{pv}\int_{-\pi}^{\pi} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})^2}(1-e^{-i\beta(k-1)})d\beta \\
=&\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})^2} i\beta(k-1)\int_0^1 e^{-i\beta(k-1)s}dsd\beta \\
=&\frac{i(k-1)}{4} \\
&\cdot \lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}\int_0^1 (1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}\sum_{n=0}^{\infty}(1+n)(re^{i\beta})^n dsd\beta \\
=&\frac{i(k-1)}{4} \\
&\cdot \lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds.
\end{aligned}
$$

To simplify this expression, we first calculate the integral

$$
\begin{aligned}
&\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta \\
=&\frac{e^{-i\pi(n+s(1-k))}}{(n+s(1-k))^2}\Big(-1-i\pi(n+s(1-k))\Big) \\
&+\frac{e^{i\pi(n+s(1-k))}}{(n+s(1-k))^2}\Big(1-i\pi(n+s(1-k))\Big) \\
&+\frac{2e^{-i\pi(1+n+s(1-k))}}{(1+n+s(1-k))^2}\Big(-1-i\pi(1+n+s(1-k))\Big) \\
&+\frac{2e^{i\pi(1+n+s(1-k))}}{(1+n+s(1-k))^2}\Big(1-i\pi(1+n+s(1-k))\Big) \\
&+\frac{e^{-i\pi(2+n+s(1-k))}}{(2+n+s(1-k))^2}\Big(1+i\pi(2+n+s(1-k))\Big) \\
&+\frac{e^{i\pi(2+n+s(1-k))}}{(2+n+s(1-k))^2}\Big(-1+i\pi(2+n+s(1-k))\Big).
\end{aligned}
$$

For $t\in\{1,2\}$, we note that

$$
\begin{aligned}
&\int_0^1 \frac{e^{i\pi(n+s(1-k))}}{(n+s(1-k))^2}(1-i\pi(n+s(1-k)))ds = \frac{1}{k-1}\int_{n-(k-1)}^{n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds \\
&\int_0^1 \frac{e^{i\pi(t+n+s(1-k))}}{(t+n+s(1-k))^2}(1-i\pi(t+n+s(1-k)))ds = \frac{1}{k-1}\int_{t+n-(k-1)}^{t+n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds.
\end{aligned}
$$

Using these identities, we obtain

$$
\begin{aligned}
&\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
=&\frac{i(k-1)}{4}\cdot\Bigg[\Bigg(\sum_{n=0}^{k-1}+\sum_{n=k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
&+\Bigg(\sum_{n=0}^{k-2}+\sum_{n=k-1}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
&+\Bigg(\sum_{n=0}^{k-3}+\sum_{n=k-2}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}-e^{2i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\Bigg]\\
=&\frac{i(k-1)}{4}\cdot\Bigg[\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}\beta e^{i\beta n}\int_0^1 e^{-i\beta(k-1)s}dsd\beta\\
&+\sum_{n=k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
&+\sum_{n=0}^{k-2}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(k-1)s}dsd\beta\\
&+\sum_{n=k-1}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
&+\sum_{n=0}^{k-3}(1+n)r^n\int_{-\pi}^{\pi}-e^{2i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(k-1)s}dsd\beta\\
&+\sum_{n=k-2}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}-e^{2i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\Bigg].
\end{aligned}
\tag{6.13}
$$

After further simplification, we obtain

$$
\begin{aligned}
&\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
=&\frac{1}{4}\Big(\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\\
&+\sum_{n=0}^{k-2}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\\
&+\sum_{n=0}^{k-3}(1+n)r^n\int_{-\pi}^{\pi}-e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\Big)\\
&+\frac{i(k-1)}{4}\cdot\Big[\sum_{n=k}^{\infty}(1+n)r^n\Big(\frac{1}{k-1}\int_{n-(k-1)}^{n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds\\
&-\frac{1}{k-1}\int_{n-(k-1)}^{n}\frac{e^{-i\pi s}(1+i\pi s)}{s^2}ds\Big)\\
&+\sum_{n=k-1}^{\infty}(1+n)r^n\Big(\frac{2}{k-1}\int_{1+n-(k-1)}^{1+n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds\\
&-\frac{2}{k-1}\int_{1+n-(k-1)}^{1+n}\frac{e^{-i\pi s}(1+i\pi s)}{s^2}ds\Big)\\
&+\sum_{n=k-2}^{\infty}(1+n)r^n\Big(-\frac{1}{k-1}\int_{2+n-(k-1)}^{2+n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds\\
&+\frac{1}{k-1}\int_{2+n-(k-1)}^{2+n}\frac{e^{-i\pi s}(1+i\pi s)}{s^2}ds\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{4}\Bigg(\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \\
&+\sum_{n=0}^{k-2}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \\
&+\sum_{n=0}^{k-3}(1+n)r^n\int_{-\pi}^{\pi}-e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\Bigg) \\
&+\frac{1}{4}\Bigg(\sum_{n=k}^{\infty}(1+n)r^n\int_{-\infty}^{\infty}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)1_{[n-(k-1),n]}(s)ds \qquad (6.14)\\
&+\sum_{n=k-1}^{\infty}(1+n)r^n\cdot 2\int_{-\infty}^{\infty}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)1_{[1+n-(k-1),1+n]}(s)ds \qquad (6.15)\\
&+\sum_{n=k-2}^{\infty}(1+n)r^n(-1) \\
&\cdot\int_{-\infty}^{\infty}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)1_{[2+n-(k-1),2+n]}(s)ds\Bigg). \qquad (6.16)
\end{aligned}
$$

We will further simplify the terms in (6.14), (6.15), and (6.16). The term in (6.14) becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{\infty}(1+n)r^n 1_{[n-(k-1),n]}(s)ds\\
=&\bigg(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k-2}^{k-1}+\int_{k-1}^{\infty}\bigg)i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\\
&\cdot\sum_{n=k}^{\infty}(1+n)r^n 1_{[n-(k-1),n]}(s)ds\\
=&\int_{1}^{2} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k}(1+n)r^n ds\\
&+\int_{2}^{3} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k+1}(1+n)r^n ds\\
&+\cdots+\int_{k-2}^{k-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{2k-3}(1+n)r^n ds\\
&+\int_{k-1}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{\infty}(1+n)r^n 1_{[n-(k-1),n]}(s)ds\\
=&\sum_{j=1}^{k-2}\int_{j}^{1+j} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k+j-1}(1+n)r^n ds\\
&+\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k+(j-1)}^{2k-2+(j-1)}(1+n)r^n ds.
\end{aligned}
$$

Next, the term in (6.15) becomes

$$
\begin{aligned}
&2\int_{-\infty}^{\infty} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{\infty}(1+n)r^n 1_{[1+n-(k-1),1+n]}(s)ds\\
=&2\Big(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k-2}^{k-1}+\int_{k-1}^{\infty}\Big)i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\\
&\cdot\sum_{n=k-1}^{\infty}(1+n)r^n 1_{[1+n-(k-1),1+n]}(s)ds\\
=&2\int_{1}^{2} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{k-1}(1+n)r^n ds\\
&+2\int_{2}^{3} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{k}(1+n)r^n ds\\
&+\cdots+2\int_{k-2}^{k-1} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{2k-4}(1+n)r^n ds\\
&+2\int_{k-1}^{\infty} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{\infty}(1+n)r^n 1_{[1+n-(k-1),1+n]}(s)ds\\
=&\sum_{j=1}^{k-2}2\int_{j}^{1+j} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{k+j-2}(1+n)r^n ds\\
&+2\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1} i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1+(j-1)}^{2k-3+(j-1)}(1+n)r^n ds.
\end{aligned}
$$

Lastly, the term in (6.16) becomes

$$
\begin{aligned}
&-\int_{-\infty}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{\infty}(1+n)r^n 1_{[2+n-(k-1),2+n]}(s)ds\\
=&-\bigg(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k-2}^{k-1}+\int_{k-1}^{\infty}\bigg)i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\\
&\cdot\sum_{n=k-2}^{\infty}(1+n)r^n 1_{[2+n-(k-1),2+n]}(s)ds\\
=&-\int_{1}^{2} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{k-2}(1+n)r^n ds\\
&-\int_{2}^{3} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{k-1}(1+n)r^n ds\\
&-\cdots-\int_{k-2}^{k-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{2k-5}(1+n)r^n ds\\
&-\int_{k-1}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{\infty}(1+n)r^n 1_{[2+n-(k-1),2+n]}(s)ds\\
=&-\sum_{j=1}^{k-2}\int_{j}^{1+j} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{k+j-3}(1+n)r^n ds\\
&-\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2+(j-1)}^{2k-4+(j-1)}(1+n)r^n ds.
\end{aligned}
$$

Using that

$$
\begin{aligned}
&\frac{1}{4}\bigg(\sum_{n=0}^{k-1}(1+n)\int_{-\pi}^{\pi} e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\\
&+\sum_{n=0}^{k-2}(1+n)\int_{-\pi}^{\pi} 2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\\
&+\sum_{n=0}^{k-3}(1+n)\int_{-\pi}^{\pi} -e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\bigg)=\frac{\pi}{2}-\pi k,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\
=&\frac{1}{4}\Bigg(\sum_{j=1}^{k-2}\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k}^{k+j-1}(1+n)ds\\
&+\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k+(j-1)}^{2k-2+(j-1)}(1+n)ds\\
&+\sum_{j=1}^{k-2}2\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1}^{k+j-2}(1+n)ds\\
&+2\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-1+(j-1)}^{2k-3+(j-1)}(1+n)ds\\
&+\sum_{j=1}^{k-2}-\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-2}^{k+j-3}(1+n)ds\\
&-\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)\sum_{n=k-2+(j-1)}^{2k-4+(j-1)}(1+n)ds\Bigg)\\
&+\frac{\pi}{2}-\pi k\\
=&\frac{1}{4}\Bigg((2k+1)\sum_{j=1}^{k-2}j\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\\
&+\sum_{j=1}^{k-2}j^2\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\\
&+(-1+k)(-2+3k)\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\\
&+2(-1+k)\sum_{j=1}^{\infty}j\int_{k+j-2}^{k+j-1}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\Bigg)\\
&+\frac{\pi}{2}-\pi k.
\end{aligned}
$$

Now, we calculate the integral

$$\int_{\epsilon}^{-\epsilon}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta.$$

We can use the same procedure to obtain that

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\\=&\frac{\pi}{2}(k-1).\end{aligned}$$

Therefore,

$$\begin{aligned}g_2(k)=&-\frac{\pi}{2}k+\frac{1}{4}\Bigg((2k+1)\sum_{j=1}^{k-2}j\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds\\&+\sum_{j=1}^{k-2}j^2\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds\\&+(-1+k)(-2+3k)\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds\\&+2(-1+k)\sum_{j=1}^{\infty}j\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds\Bigg).\end{aligned}$$

Next, we calculate $g_3(k)$. Using that

$$\begin{aligned}1-e^{-i\beta(1+k)}&=i\beta(1+k)\int_0^1 e^{-i\beta(1+k)s}ds,\\\frac{1}{(-1+re^{i\beta})^2}&=\sum_{n=0}^{\infty}(1+n)(re^{i\beta})^n,\end{aligned}$$

we obtain

$$
\begin{aligned}
g_3(k) =& \mathrm{pv} \int_{-\pi}^{\pi} \frac{-1+2e^{i\beta}+e^{2i\beta}}{4(-1+e^{i\beta})^2}(1-e^{-i\beta(1+k)})d\beta \\
=& \lim_{\epsilon\to 0^+} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \frac{-1+2e^{i\beta}+e^{2i\beta}}{4(-1+e^{i\beta})^2} i\beta(1+k) \int_0^1 e^{-i\beta(1+k)s} ds d\beta \\
=& \frac{i(1+k)}{4} \lim_{\epsilon\to 0^+} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \int_0^1 \frac{(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}}{(-1+e^{i\beta})^2} ds d\beta \\
=& \frac{i(1+k)}{4} \\
& \cdot \lim_{\epsilon\to 0^+} \lim_{r\to 1^-} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \int_0^1 (-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s} \sum_{n=0}^{\infty}(1+n)(re^{i\beta})^n ds d\beta \\
=& \frac{i(1+k)}{4} \\
& \cdot \lim_{\epsilon\to 0^+} \lim_{r\to 1^-} \sum_{n=0}^{\infty}(1+n)r^n \int_0^1 \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} (-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s} e^{i\beta n} d\beta ds.
\end{aligned}
$$

To simplify this expression, we first calculate the expression

$$
\begin{aligned}
&\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
=&\frac{i(1+k)}{4}\Bigg[\bigg(\sum_{n=0}^{1+k}+\sum_{n=2+k}^{\infty}\bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}-\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&+\bigg(\sum_{n=0}^{k}+\sum_{n=1+k}^{\infty}\bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&+\bigg(\sum_{n=0}^{k-1}+\sum_{n=k}^{\infty}\bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}e^{2i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\Bigg]\\
=&\frac{i(1+k)}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-\beta e^{i\beta n}\int_0^1 e^{-i\beta(1+k)s}ds d\beta\\
&+\sum_{n=2+k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}-\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(1+k)s}ds d\beta\\
&+\sum_{n=1+k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(1+k)s}ds d\beta\\
&+\sum_{n=k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}e^{2i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\Bigg].
\end{aligned}
$$

Using the identity

$$
\int_0^1 e^{-i\beta(1+k)s}ds=\frac{1-e^{-i\beta(1+k)}}{i\beta(1+k)},
$$

we obtain

$$
\begin{aligned}
&\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
=&\frac{1}{4}\Big[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Big]\\
&+\frac{1}{4}\Big[\sum_{n=2+k}^{\infty}(1+n)r^n\Big(-\pi\int_{-\infty}^{\infty}\Big(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\Big)1_{[n-(1+k),n]}(s)ds \qquad (6.17)\\
&-i\int_{-\infty}^{\infty}\Big(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\Big)1_{[n-(1+k),n]}(s)ds\Big) \qquad (6.18)\\
&+\sum_{n=1+k}^{\infty}(1+n)r^n\Big(2i\int_{-\infty}^{\infty}\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)1_{[1+n-(1+k),1+n]}(s)ds\Big) \qquad (6.19)\\
&+\sum_{n=k}^{\infty}(1+n)r^n\Big(i\int_{-\infty}^{\infty}\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)1_{[2+n-(1+k),2+n]}(s)ds\Big)\Big]. \qquad (6.20)
\end{aligned}
$$

We will further simplify the terms in (6.17), (6.18), (6.19), and (6.20). The term in (6.17) becomes

$$
\begin{aligned}
&-\pi\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&-\pi\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\right)\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\\
&\cdot\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&-\pi\int_{1}^{2}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{2+k}(1+n)r^n ds-\pi\int_{2}^{3}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\\
&\cdot\sum_{n=2+k}^{3+k}(1+n)r^n ds\\
&-\cdots-\pi\int_{k}^{1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+2k}(1+n)r^n ds\\
&-\pi\int_{1+k}^{\infty}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&\sum_{n=1}^{k}-\pi\int_{j}^{1+j}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds\\
&-\pi\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&\sum_{j=1}^{k}-\pi\int_{j}^{1+j}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds\\
&-\pi\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds.
\end{aligned}
$$

Next, the term in (6.18) becomes

$$
\begin{aligned}
&-i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&-i\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\right)\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\\
&\cdot\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&-i\int_{1}^{2}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{2+k}(1+n)r^n ds-i\int_{2}^{3}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{3+k}(1+n)r^n ds\\
&-\cdots-i\int_{k}^{1+k}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{1+2k}(1+n)r^n ds\\
&-i\int_{1+k}^{\infty}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds\\
=&\sum_{j=1}^{k}-i\int_{j}^{1+j}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds\\
&-i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds.
\end{aligned}
$$

Next, the term in (6.19) becomes

$$
\begin{aligned}
& 2i\int_{-\infty}^{\infty}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds \\
=& 2i\bigg(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\bigg)\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg) \\
& \cdot\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds \\
=& 2i\int_{1}^{2}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=1+k}^{1+k}(1+n)r^n ds \\
& + 2i\int_{2}^{3}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=1+k}^{2+k}(1+n)r^n ds \\
& + \cdots + 2i\int_{k}^{1+k}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=1+k}^{2k}(1+n)r^n ds \\
& + 2i\int_{1+k}^{\infty}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds \\
=& \sum_{j=1}^{k} 2i\int_{j}^{1+j}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=1+k}^{k+j}(1+n)r^n ds \\
& + 2i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=j+k}^{j+2k}(1+n)r^n ds.
\end{aligned}
$$

Lastly, the term in (6.20) becomes

$$
\begin{aligned}
& i\int_{-\infty}^{\infty}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds \\
=& i\bigg(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\bigg)\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg) \\
& \cdot\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds \\
=& i\int_{1}^{2}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k}(1+n)r^n ds \\
& +i\int_{2}^{3}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k+1}(1+n)r^n ds \\
& +\cdots+i\int_{k}^{1+k}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{2k-1}(1+n)r^n ds \\
& +i\int_{1+k}^{\infty}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds \\
=& \sum_{j=1}^{k} i\int_{j}^{1+j}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{j-1+k}(1+n)r^n ds \\
& +i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=(j-1)+k}^{(j-1)+2k}(1+n)r^n ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
=&\frac{1}{4}\Big[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Big]\\
&+\frac{1}{4}\Big[\sum_{j=1}^{k}-\pi\int_j^{1+j}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds\\
&-\pi\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds\\
&+\sum_{j=1}^{k}-i\int_j^{1+j}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds\\
&-i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds\\
&+\sum_{j=1}^{k}2i\int_j^{1+j}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{k+j}(1+n)r^n ds\\
&+2i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=j+k}^{j+2k}(1+n)r^n ds\\
&+\sum_{j=1}^{k}i\int_j^{1+j}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{j-1+k}(1+n)r^n ds\\
&+i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=(j-1)+k}^{(j-1)+2k}(1+n)r^n ds\Big].
\end{aligned}
$$

Using that

$$
\begin{aligned}
&\frac{1}{4}\Big[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Big]=-\frac{\pi}{2}-\pi k,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&=-\frac{\pi}{2}-\pi k+\frac{1}{4}\Big[(2k+1)\sum_{j=1}^{k}j\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\\
&+\sum_{j=1}^{k}j^2\int_j^{1+j}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\\
&+3k(1+k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\\
&+2(1+k)\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}i\Big(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Big)ds\Big].
\end{aligned}
$$

Now, we calculate the expression

$$\begin{aligned}
&\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
=&\frac{i(1+k)}{4}\Bigg[\Bigg(\sum_{n=0}^{1+k}+\sum_{n=2+k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}-\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&+\Bigg(\sum_{n=0}^{k}+\sum_{n=1+k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}2e^{i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\\
&+\Bigg(\sum_{n=0}^{k-1}+\sum_{n=k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}e^{2i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\Bigg]\\
=&\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{\epsilon}^{-\epsilon}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg]\\
&+\frac{1}{4}\Bigg[\sum_{n=2+k}^{\infty}(1+n)r^n\Bigg(i\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\Bigg)1_{[n-(1+k),n]}(s)ds \qquad (6.21)\\
&+\epsilon\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\Bigg)1_{[n-(1+k),n]}(s)ds\Bigg) \qquad (6.22)\\
&+\sum_{n=1+k}^{\infty}(1+n)r^n\Bigg(-2\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\Bigg)1_{[1+n-(1+k),1+n]}(s)ds\Bigg) \qquad (6.23)\\
&+\sum_{n=k}^{\infty}(1+n)r^n\Bigg(-\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\Bigg)1_{[2+n-(1+k),2+n]}(s)ds\Bigg)\Bigg]. \qquad (6.24)
\end{aligned}$$

We are interested in how the terms in (6.21), (6.22), (6.23), and (6.24) behave as first r goes to 1 from below and then ϵ goes to 0 from above. For the term in (6.21),

$$
\begin{aligned}
& i\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \\
=& \sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \\
& + i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds \\
\xrightarrow{r\to 1^-}& \sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\frac{j(5+j+2k)}{2}ds \\
& + i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\frac{(1+k)(4+2j+3k)}{2}ds \\
=& \sum_{j=1}^{k}\frac{j(5+j+2k)}{2}\int_{j}^{1+j} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds \\
& + \frac{(1+k)(4+3k)}{2}\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds \\
& + (1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds \\
\xrightarrow{\epsilon\to 0^+}& (1+k)(-\pi).
\end{aligned}
$$

For the term in (6.22),

$$
\begin{aligned}
&\epsilon \int_{-\infty}^{\infty} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) \sum_{n=2+k}^{\infty} (1+n) r^n 1_{[n-(1+k),n]}(s) ds \\
=& \sum_{j=1}^{k} \epsilon \int_{j}^{1+j} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) \sum_{n=2+k}^{1+k+j} (1+n) r^n ds \\
&+ \epsilon \sum_{j=1}^{\infty} \int_{j+k}^{j+1+k} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) \sum_{n=(j+1)+k}^{(j+1)+2k} (1+n) r^n ds \\
\xrightarrow{r \to 1^-}& \epsilon \sum_{j=1}^{k} \frac{j(5+j+2k)}{2} \int_{j}^{1+j} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) ds \\
&+ \epsilon \frac{(1+k)(4+3k)}{2} \sum_{j=1}^{\infty} \int_{j+k}^{j+1+k} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) ds \\
&+ \epsilon (1+k) \sum_{j=1}^{\infty} j \int_{j+k}^{j+1+k} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) ds \\
\xrightarrow{\epsilon \to 0^+}& 0.
\end{aligned}
$$

For the term in (6.23),

$$
\begin{aligned}
&-2\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\right)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds\\
=&-2i\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds\\
=&-\sum_{j=1}^{k}2i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=1+k}^{k+j}(1+n)r^n ds\\
&-2i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=j+k}^{j+2k}(1+n)r^n ds\\
\xrightarrow{r\to 1^-}&-\sum_{j=1}^{k}j(3+j+2k)\int_{j}^{1+j}i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds\\
&-\epsilon\sum_{j=1}^{k}j(3+j+2k)\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds\\
&-(1+k)(2+3k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds\\
&-2(1+k)\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds\\
&-\epsilon(1+k)(2+3k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds\\
&-2(1+k)\epsilon\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds\\
\xrightarrow{\epsilon\to 0^+}&-2(1+k)(-\pi).
\end{aligned}
$$

Lastly, for the term in (6.24),

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\right)\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds\\
&=-\sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=k}^{j-1+k}(1+n)r^n ds\\
&\quad-i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=(j-1)+k}^{(j-1)+2k}(1+n)r^n ds\\
&\xrightarrow{r\to 1^-}-\sum_{j=1}^{k}\frac{j(1+j+2k)}{2}\int_{j}^{1+j} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds\\
&\quad-\epsilon\sum_{j=1}^{k}\frac{j(1+j+2k)}{2}\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds\\
&\quad-\frac{(1+k)3k}{2}\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds\\
&\quad-\epsilon\frac{(1+k)3k}{2}\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds\\
&\quad-(1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds\\
&\quad-\epsilon(1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds\\
&\xrightarrow{\epsilon\to 0^+}-(1+k)(-\pi).
\end{aligned}
$$

Using that

$$
\begin{aligned}
&\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{\epsilon}^{-\epsilon}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg]=0,
\end{aligned}
$$

we obtain

$$\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds$$
$$=\frac{\pi}{2}(1+k).$$

Therefore,

$$\begin{aligned}
g_3(k) = -\frac{\pi}{2}k+\frac{1}{4}\Bigg[&(2k+1)\sum_{j=1}^{k}j\int_j^{1+j}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)ds\\
&+\sum_{j=1}^{k}j^2\int_j^{1+j}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)ds\\
&+3k(1+k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)ds\\
&+2(1+k)\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)ds\Bigg].
\end{aligned}$$

To simplify the expressions for $g_2(k)$ and $g_3(k)$, we note that for $j\neq\{-1,0\}$,

$$\begin{aligned}
&\int_j^{j+1}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)ds\\
=&i\int_j^{j+1}\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}-i\pi\bigg(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\bigg)ds\\
=&i\int_j^{j+1}\frac{2i\sin(\pi s)}{s^2}-i\pi\frac{2\cos(\pi s)}{s}ds\\
=&-2\int_j^{j+1}\frac{\sin(\pi s)}{s^2}ds+2\pi\int_j^{j+1}\frac{\cos(\pi s)}{s}ds\\
=&-2\pi\int_j^{j+1}\frac{\cos(\pi s)}{s}ds+2\pi\int_j^{j+1}\frac{\cos(\pi s)}{s}ds\\
=&0.
\end{aligned}$$

Using this simplification, we obtain

$$g_2(k) = -\frac{\pi}{2}k,$$
$$g_3(k) = -\frac{\pi}{2}k.$$

Next, let us calculate $g_5(k)$. We have

$$\begin{aligned}
g_5(k) &= \mathrm{pv}\int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta \\
&= \frac{i}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \\
&= \frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta.
\end{aligned}$$

To simplify this expression, we first calculate the expression

$$
\begin{aligned}
&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} (-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta \\
=&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k)}d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} -2e^{i\beta}e^{i\beta(n-k)}d\beta \\
&+\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} e^{2i\beta}e^{i\beta(n-k)}d\beta\Bigg) \\
=&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k)}d\beta + \sum_{n=k+1}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k)}d\beta + r^k(-2\pi) \\
&+\sum_{n=0}^{k-2} r^n \int_{-\pi}^{\pi} -2e^{i\beta(n-k+1)}d\beta + \sum_{n=k}^{\infty} r^n \int_{-\pi}^{\pi} -2e^{i\beta(n-k+1)}d\beta + r^{k-1}(-4\pi) \\
&+\sum_{n=0}^{k-3} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+2)}d\beta + \sum_{n=k-1}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+2)}d\beta + r^{k-2}(2\pi)\Bigg) \\
=&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \frac{i}{k-n}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) \\
&+\sum_{n=k+1}^{\infty} r^n \frac{i}{k-n}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) + r^k(-2\pi) \\
&+\sum_{n=0}^{k-2} r^n \frac{-2i}{k-n-1}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) \\
&+\sum_{n=k}^{\infty} r^n \frac{-2i}{k-n-1}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) + r^{k-1}(-4\pi) \\
&+\sum_{n=0}^{k-3} r^n \frac{-i}{k-n-2}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) \\
&+\sum_{n=k-1}^{\infty} r^n \frac{-i}{k-n-2}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) + r^{k-2}(2\pi)\Bigg).
\end{aligned}
$$

Using that

$$\sum_{n=0}^{k-1} r^n \frac{i}{k-n}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) = 0$$

and

$$\sum_{n=k+1}^{\infty} r^n \frac{i}{k-n} \left(e^{i(k-n)\pi} - e^{-i(k-n)\pi} \right) = 0,$$

we obtain

$$\frac{i}{4} \lim_{\epsilon \to 0^+} \lim_{r \to 1^-} \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} (-1 - 2e^{i\beta} + e^{2i\beta}) e^{-i\beta k} e^{i\beta n} d\beta = -i\pi.$$

Next, we calculate the expression

$$\begin{aligned}
&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta\\
=&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}-2e^{i\beta}e^{i\beta(n-k)}d\beta\\
&+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}e^{2i\beta}e^{i\beta(n-k)}d\beta\Bigg)\\
=&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty} r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k)}d\beta+r^k\cdot 2\epsilon\\
&+\sum_{n=0}^{k-2} r^n\int_{\epsilon}^{-\epsilon}-2e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty} r^n\int_{\epsilon}^{-\epsilon}-2e^{i\beta(n-k+1)}d\beta+r^{k-1}4\epsilon\\
&+\sum_{n=0}^{k-3} r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k+2)}d\beta+\sum_{n=k-1}^{\infty} r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k+2)}d\beta+r^{k-2}(-2\epsilon)\Bigg)\\
=&\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n\frac{-i}{k-n}(e^{i(k-n)\epsilon}-e^{-i(k-n)\epsilon})\\
&+\sum_{n=k+1}^{\infty} r^n\frac{-i}{k-n}(e^{i(k-n)\epsilon}-e^{-i(k-n)\epsilon})+r^k\cdot 2\epsilon\\
&+\sum_{n=0}^{k-2} r^n\frac{2i}{k-n-1}(e^{-i(k-n-1)\epsilon}-e^{i(k-n-1)\epsilon})\\
&+\sum_{n=k}^{\infty} r^n\frac{2i}{k-n-1}(e^{-i(k-n-1)\epsilon}-e^{i(k-n-1)\epsilon})+r^{k-1}4\epsilon\\
&+\sum_{n=0}^{k-3} r^n\frac{i}{k-n-2}(e^{i(k-n-2)\epsilon}-e^{-i(k-n-2)\epsilon})\\
&+\sum_{n=k-1}^{\infty} r^n\frac{i}{k-n-2}(e^{i(k-n-2)\epsilon}-e^{-i(k-n-2)\epsilon})\\
&+r^{k-2}(-2\epsilon)\Bigg).
\end{aligned}$$

Using that

$$\sum_{n=0}^{k-1} r^n \frac{-i}{k-n}(e^{i(k-n)\epsilon} - e^{-i(k-n)\epsilon}) \xrightarrow{r\to 1^-} -i\sum_{n=1}^{k} \frac{(e^{-i\epsilon})^{-n}}{n} + i\sum_{n=1}^{k} \frac{(e^{i\epsilon})^{-n}}{n}$$
$$\xrightarrow{\epsilon\to 0^+} 0$$

and

$$\begin{aligned}
&\sum_{n=k+1}^{\infty} r^n \frac{-i}{k-n}(e^{i(k-n)\epsilon} - e^{-i(k-n)\epsilon}) \\
=&ie^{ik\epsilon}(re^{-i\epsilon})^k(-\mathrm{Log}(1-re^{-i\epsilon})) - ie^{-ik\epsilon}(re^{i\epsilon})^k(-\mathrm{Log}(1-re^{i\epsilon})) \\
\xrightarrow{r\to 1^-}& - i\mathrm{Log}(1-e^{-i\epsilon}) + i\mathrm{Log}(1-e^{i\epsilon}) \\
=& - i(\log\left|1-e^{-i\epsilon}\right| + i\mathrm{Arg}(1-e^{-i\epsilon})) + i(\log\left|1-e^{i\epsilon}\right| + i\mathrm{Arg}(1-e^{i\epsilon})) \\
=&\mathrm{Arg}(1-e^{-i\epsilon}) - \mathrm{Arg}(1-e^{i\epsilon}) \\
\xrightarrow{\epsilon\to 0^+}& \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi,
\end{aligned}$$

where Log denotes the principal branch of the complex logarithm, we obtain

$$\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta = \frac{i\pi}{2}.$$

Therefore,

$$g_5(k) = -\frac{i\pi}{2}.$$

Next, let us calculate $g_6(k)$. We have

$$
\begin{aligned}
g_6(k) &= \mathrm{pv}\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \\
&= -\frac{1}{4}\lim_{\epsilon\to 0^+} \int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k} \lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \\
&= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} (2-e^{-i\beta}+e^{i\beta})e^{-i\beta k}e^{i\beta n} d\beta.
\end{aligned}
$$

To simplify this expression, we first calculate the expression

$$
\begin{aligned}
&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(2-e^{-i\beta}+e^{i\beta})e^{-i\beta k}e^{i\beta n} d\beta \\
=& -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)} d\beta \\
&+\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(-e^{-i\beta})e^{i\beta(n-k)} d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta}e^{i\beta(n-k)} d\beta\Bigg) \\
=& -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)} d\beta + \sum_{n=k+1}^{\infty} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)} d\beta + r^k(4\pi) \\
&+\sum_{n=0}^{k} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)} d\beta + \sum_{n=k+2}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)} d\beta + r^{k+1}(-2\pi) \\
&+\sum_{n=0}^{k-2} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)} d\beta + \sum_{n=k}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)} d\beta + r^{k-2}(2\pi)\Bigg) \\
=& -\pi.
\end{aligned}
$$

Next, we calculate the expression

$$
\begin{aligned}
&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}(2-e^{-i\beta}+e^{i\beta})e^{-i\beta k}e^{i\beta n}d\beta\\
=&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta\\
&+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta\Bigg)\\
=&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n\int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty} r^n\int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta+r^k\cdot(-4\epsilon)\\
&+\sum_{n=0}^{k} r^n\int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta+\sum_{n=k+2}^{\infty} r^n\int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta+r^{k+1}2\epsilon\\
&+\sum_{n=0}^{k-2} r^n\int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty} r^n\int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta+r^{k-1}(-2\epsilon)\Bigg)\\
=&\frac{\pi}{2}.
\end{aligned}
$$

Therefore,

$$
g_6(k)=-\frac{\pi}{2}.
$$

Next, let us calculate $g_7(k)$. We have

$$
\begin{aligned}
g_7(k)&=\mathrm{pv}\int_{-\pi}^{\pi}\frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\\
&=-\frac{1}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta\\
&=-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(1+2e^{i\beta}-e^{2i\beta})e^{i\beta(n-k)}d\beta.
\end{aligned}
$$

To simplify this expression, we first calculate the expression

$$
\begin{aligned}
&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})e^{i\beta(n-k)}d\beta\\
=&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi} e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi} 2e^{i\beta(n-k+1)}d\beta+\sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi} -e^{i\beta(n-k+2)}d\beta\right)\\
=&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{k-1} r^n\int_{-\pi}^{\pi} e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty} r^n\int_{-\pi}^{\pi} e^{i\beta(n-k)}d\beta+r^k(2\pi)\right.\\
&+\sum_{n=0}^{k-2} r^n\int_{-\pi}^{\pi} 2e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty} r^n\int_{-\pi}^{\pi} 2e^{i\beta(n-k+1)}d\beta+r^{k-1}(4\pi)\\
&\left.+\sum_{n=0}^{k-3} r^n\int_{-\pi}^{\pi} -e^{i\beta(n-k+2)}d\beta+\sum_{n=k-1}^{\infty} r^n\int_{-\pi}^{\pi} -e^{i\beta(n-k+2)}d\beta+r^{k-2}(-2\pi)\right)\\
=&-\pi.
\end{aligned}
$$

Next, we calculate the expression

$$
\begin{aligned}
&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}(1+2e^{i\beta}-e^{2i\beta})e^{i\beta(n-k)}d\beta\\
=&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon} e^{i\beta(n-k)}d\beta\right.\\
&\left.+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon} 2e^{i\beta}e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon} -e^{2i\beta}e^{i\beta(n-k)}d\beta\right)\\
=&-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{k-1} r^n\int_{\epsilon}^{-\epsilon} e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty} r^n\int_{\epsilon}^{-\epsilon} e^{i\beta(n-k)}d\beta+r^k\cdot(-2\epsilon)\right.\\
&+\sum_{n=0}^{k-2} r^n\int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty} r^n\int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k+1)}d\beta+r^{k-1}(-4\epsilon)\\
&\left.+\sum_{n=0}^{k-3} r^n\int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k+2)}d\beta+\sum_{n=k-1}^{\infty} r^n\int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k+2)}d\beta+r^{k-2}(2\epsilon)\right)\\
=&\frac{\pi}{2}.
\end{aligned}
$$

Therefore,

$$g_7(k) = -\frac{\pi}{2}.$$

Lastly, let us calculate $g_8(k)$. We have

$$\begin{aligned}
g_8(k) &= \mathrm{pv}\int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \\
&= \frac{1}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \backslash(-\epsilon,\epsilon)}} e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \\
&= \frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\substack{[-\pi,\pi]\\ \backslash(-\epsilon,\epsilon)}} (2-e^{-i\beta}+e^{i\beta})e^{i\beta(n-k)} d\beta.
\end{aligned}$$

To simplify this expression, we first calculate the expression

$$\begin{aligned}
&\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(2-e^{-i\beta}+e^{i\beta})e^{i\beta(n-k)}d\beta \\
=&\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi} 2e^{i\beta(n-k)}d\beta \\
&+\sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi} -e^{i\beta(n-k-1)}d\beta + \sum_{n=0}^{\infty} r^n\int_{-\pi}^{\pi} e^{i\beta(n-k+1)}d\beta\Bigg) \\
=&\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n\int_{-\pi}^{\pi} 2e^{i\beta(n-k)}d\beta + \sum_{n=k+1}^{\infty} r^n\int_{-\pi}^{\pi} 2e^{i\beta(n-k)}d\beta + r^k(4\pi) \\
&+\sum_{n=0}^{k} r^n\int_{-\pi}^{\pi} -e^{i\beta(n-k-1)}d\beta + \sum_{n=k+2}^{\infty} r^n\int_{-\pi}^{\pi} -e^{i\beta(n-k-1)}d\beta + r^{k+1}(-2\pi) \\
&+\sum_{n=0}^{k-2} r^n\int_{-\pi}^{\pi} e^{i\beta(n-k+1)}d\beta + \sum_{n=k}^{\infty} r^n\int_{-\pi}^{\pi} e^{i\beta(n-k+1)}d\beta + r^{k-1}(2\pi)\Bigg) \\
=&\pi.
\end{aligned}$$

Next, we calculate the expression

$$
\begin{aligned}
&\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}(2-e^{-i\beta}+e^{i\beta})e^{i\beta(n-k)}d\beta\\
=&\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta(n-k)}d\beta\\
&+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k-1)}d\beta+\sum_{n=0}^{\infty} r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k+1)}d\beta\Bigg)\\
=&\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty} r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta(n-k)}d\beta+r^k\cdot(-4\epsilon)\\
&+\sum_{n=0}^{k} r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k-1)}d\beta+\sum_{n=k+2}^{\infty} r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k-1)}d\beta+r^{k+1}(2\epsilon)\\
&+\sum_{n=0}^{k-2} r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty} r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k+1)}d\beta+r^{k-1}(-2\epsilon)\Bigg)\\
=&-\frac{\pi}{2}.
\end{aligned}
$$

Therefore,

$$g_8(k)=\frac{\pi}{2}.$$

Plugging the calculated values of $g_2(k)$, $g_3(k)$, $g_5(k)$, $g_6(k)$, $g_7(k)$, and $g_8(k)$ into (6.12), we obtain

$$J_1(k)=-\frac{\pi}{k}.$$

Using (6.9) and (6.10), we deduce that

$$J_1(k)=\begin{cases}-\frac{\pi}{k} & k>1,\\ \frac{\pi}{k} & k<-1.\end{cases}\tag{6.25}$$

Plugging the results of Sections 6.1.1 and 6.1.2 into (6.8), we obtain that for $k > 1$,

$$\mathcal{F}(\mathcal{L})(k) = -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\pi k = -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\pi\,|k|\,.$$

Since for $k > 1$

$$\begin{aligned}\mathcal{F}(\mathcal{L})(-k) &= \overline{\mathcal{F}(\mathcal{L})(k)}\\ &= -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\overline{\mathcal{F}(\phi)(k)}\pi k\\ &= -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(-k)\pi\,|k|\,,\end{aligned}$$

we conclude that for $|k| > 1$,

$$\mathcal{F}(\mathcal{L})(k) = -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\pi\,|k|\,. \tag{6.26}$$

This concludes that proof that $\mathcal{L}$ is the Hilbert transform of the first spatial derivative of θ up to the ± 1 Fourier modes. To calculate $\mathcal{F}(\mathcal{L})(k)$ for $|k| = 1$, we use that for $k \in \mathbb{Z}$

$$\mathcal{F}((U_1)_\alpha)(k) = ik\mathcal{F}(U_1)(k)$$

to rewrite (6.2) as

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)}\left(ik\mathcal{F}(U_1)(k) - \frac{i}{k}\mathcal{F}(U_1)(k)\right),$$

where $k \neq 0$. Then $\mathcal{F}(\mathcal{L})(\pm 1) = 0$. That the ± 1 Fourier modes of $\mathcal{L}$ are zero poses a technical challenge in dealing with the term in the evolution equation for θ that induces an exponential decay in time of the initial perturbation of the interface. This challenge can be resolved by observing that the identity

$$\int_{-\pi}^{\pi} z_\alpha(\alpha, t)d\alpha = 0 \tag{6.27}$$

provides a means to control the ± 1 Fourier modes of $\mathcal{L}$ using the other nonzero Fourier modes.

CHAPTER 7

Derivation of an *a priori* Estimate

Before embarking on the derivation of a key *a priori* estimate for $\phi = \theta - \hat{\theta}(0)$, we first derive certain upper and lower bounds of $L(t)$, which tightly control it as long as $\|\phi(t)\|_{\mathcal{F}^{0,1}}$ is sufficiently small for all $t \geq 0$.

Proposition 12. *If $\|\phi(t)\|_{\mathcal{F}^{0,1}}$ is sufficiently small for all $t \geq 0$, then*

$$\frac{R^2}{1 + \frac{\pi}{2}(e^{2\|\phi(t)\|_{\mathcal{F}^{0,1}}} - 1)} \leq \left(\frac{L(t)}{2\pi}\right)^2 \leq \frac{R^2}{1 - \frac{\pi}{2}(e^{2\|\phi(t)\|_{\mathcal{F}^{0,1}}} - 1)}. \tag{7.1}$$

Proof. By the definition of the Fourier transform,

$$\begin{aligned}
&\mathcal{F}\left(\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta\right)(-1) \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta \cdot e^{i\alpha} d\alpha \\
=&\frac{1}{i} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta \cdot \frac{\partial}{\partial\alpha} e^{i\alpha} d\alpha.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
&\frac{1}{i} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta \cdot \frac{\partial}{\partial\alpha} e^{i\alpha} d\alpha \\
=&\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\partial}{\partial\alpha}\left(\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta \cdot e^{i\alpha}\right) d\alpha \\
=&\frac{1}{2\pi i}\left(-\int_0^\pi e^{-i\eta}(\phi(\pi) - \phi(\eta))^n d\eta - \int_{-\pi}^0 e^{-i\eta}(\phi(\pi) - \phi(\eta))^n d\eta\right) \\
=&-\frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{-i\eta}(\phi(\pi) - \phi(\eta))^n d\eta \\
=&i\mathcal{F}((\phi(\pi) - \phi(\eta))^n)(1).
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\\
=&\sum_{n\geq 1}\frac{i^n}{n!}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\\
=&2\pi i\sum_{n\geq 1}\frac{i^n}{n!}\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{Im}\bigg(\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\bigg)\\
=&\frac{1}{2i}\bigg(2\pi i\sum_{n\geq 1}\frac{i^n}{n!}\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)+2\pi i\sum_{n\geq 1}\frac{(-i)^n}{n!}\overline{\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)}\bigg)\\
=&\pi\bigg(\sum_{n\geq 1}\frac{i^n}{n!}\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)+\sum_{n\geq 1}\frac{(-i)^n}{n!}\overline{\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)}\bigg).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\mathrm{Im}\bigg(\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\bigg)\right|\\
\leq&2\pi\sum_{n\geq 1}\frac{|\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)|}{n!}\\
\leq&2\pi\sum_{n\geq 1}\frac{\|(\phi(\pi)-\phi(\cdot))^n\|_{\mathcal{F}^{0,1}}}{n!}.
\end{aligned}$$

By Proposition 2,

$$\|(\phi(\pi)-\phi(\cdot))^n\|_{\mathcal{F}^{0,1}}\leq\|\phi(\pi)-\phi(\cdot)\|_{\mathcal{F}^{0,1}}^n.$$

Then

$$
\begin{aligned}
\left|\mathrm{Im}\left(\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\right)\right| &\leq 2\pi\sum_{n\geq 1}\frac{\|\phi(\pi)-\phi(\cdot)\|_{\mathcal{F}^{0,1}}^n}{n!} \\
&= 2\pi\left(e^{\|\phi(\pi)-\phi(\cdot)\|_{\mathcal{F}^{0,1}}}-1\right) \\
&\leq 2\pi(e^{\|\phi(\pi)\|_{\mathcal{F}^{0,1}}}e^{\|\phi\|_{\mathcal{F}^{0,1}}}-1) \\
&= 2\pi(e^{\phi(\pi)}e^{\|\phi\|_{\mathcal{F}^{0,1}}}-1).
\end{aligned}
$$

By (2.1),

$$
|\phi(\pi)| \leq \sum_{k\in\mathbb{Z}}\left|\hat{\phi}(k)\right| = \|\phi\|_{\mathcal{F}^{0,1}}.
$$

Therefore,

$$
\left|\mathrm{Im}\left(\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\right)\right| \leq \pi^2(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1). \tag{7.2}
$$

This estimate shows that

$$
\frac{R^2}{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)} \leq \left(\frac{L(t)}{2\pi}\right)^2 \leq \frac{R^2}{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)},
$$

as needed. ∎

Using Proposition 12, we can also prove the following useful estimate.

Proposition 13. *For sufficiently small* $\|\phi\|_{\mathcal{F}^{0,1}}$*,*

$$
\left|R\frac{2\pi}{L(t)}-1\right| \leq 1-\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}.
$$

Proof. From Proposition 12, we obtain

$$\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1 \le \frac{2\pi R}{L(t)}-1 \le \sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1.$$

Then

$$\begin{aligned}\left|\frac{2\pi R}{L(t)}-1\right| &\le \max\left\{\left|\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1\right|, \left|\sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1\right|\right\}\\ &= \left|\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1\right|\\ &= 1-\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)},\end{aligned}$$

as needed. ■

We now derive a key *a priori* estimate for $\phi = \theta - \hat{\theta}(0)$. In Chapter 6, we have shown that

$$\mathcal{F}(\mathcal{L})(k) = \begin{cases} 0 & \text{if } |k| = 1,\\ \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k)+J_2(k)) & \text{if } |k| > 1,\end{cases}$$

where J_1 and J_2 are given in (6.25) and (6.10). Let

$$\begin{aligned}\widetilde{\mathcal{L}}(\alpha) &= \frac{L(t)}{2\pi}\mathcal{L}(\alpha),\\ \widetilde{\mathcal{N}}(\alpha) &= \frac{L(t)}{2\pi}\mathcal{N}(\alpha).\end{aligned}$$

Then for $|k| \geq 1$,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{F}(\phi)(k) &= \frac{2\pi}{L(t)}\bigg(\mathcal{F}(\widetilde{\mathcal{L}})(k) + \mathcal{F}(\widetilde{\mathcal{N}})(k)\bigg) \\
&= \frac{2\pi}{L(t)}\bigg(\frac{L(t)}{2\pi}\mathcal{F}(\mathcal{L})(k) + \mathcal{F}(\widetilde{\mathcal{N}})(k)\bigg) \\
&= \mathcal{F}(\mathcal{L})(k) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) \\
&= \begin{cases} \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) & \text{if } |k| = 1, \\ \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) & \text{if } |k| > 1. \end{cases}
\end{aligned}$$

For convenience of notation, define $J_1(k) = J_2(k) = 0$ for $|k| = 1$ so that for $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{\partial}{\partial t}\mathcal{F}(\phi)(k) = \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k). \tag{7.3}$$

We observe that the principal linear term, i.e., the first term on the right hand side, has a time-dependent coefficient. This dependence occurs, however, only through $L(t)$. We choose an initial circular interface of radius R to perturb around and make the principal linear term independent of time by replacing $L(t)$ with $2\pi R$ and keeping an error term. That is, we rewrite (7.3) as

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{F}(\phi)(k) = \frac{1}{R}\cdot\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) \\
+ \frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k))\bigg(-\frac{1}{R} + \frac{2\pi}{L(t)}\bigg).
\end{aligned} \tag{7.4}$$

We note that for $k > 0$,

$$\left|\hat{\phi}(-k)\right| = \left|\overline{\hat{\phi}(k)}\right| = \left|\hat{\phi}(k)\right| \tag{7.5}$$

since ϕ is real-valued. Then for $s > 0$,

$$\|\phi\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} = \sum_{k \neq 0} e^{\nu(t)|k|} |k|^s \left|\hat{\phi}(k)\right| = 2\sum_{k \geq 1} e^{\nu(t)k} k^s \left|\hat{\phi}(k)\right|.$$

Differentiating this equation with respect to t, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k\cdot k^s\left|\hat{\phi}(k)\right| + e^{\nu(t)k}k^s\frac{\partial}{\partial t}\left|\hat{\phi}(k)\right|\\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + e^{\nu(t)k}k^s\frac{1}{\left|\hat{\phi}(k)\right|}\frac{1}{2}\left(\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)\right)\\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + 2\sum_{k\geq 1} e^{\nu(t)k}k^s\frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{2\left|\hat{\phi}(k)\right|}.
\end{aligned}
$$

Let us simplify the second term. Using (7.4) and that J_1 and J_2 are real for $k \geq 1$, we obtain

$$
\begin{aligned}
&\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)\\
=&\frac{1}{R}\frac{\gamma}{4\pi}(J_1+J_2)(k)\left|\hat{\phi}(k)\right|^2 + \frac{2\pi}{L(t)}\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)\\
&+\frac{\gamma}{4\pi}(J_1+J_2)(k)\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\left|\hat{\phi}(k)\right|^2\\
&+\frac{1}{R}\frac{\gamma}{4\pi}(J_1+J_2)(k)\left|\hat{\phi}(k)\right|^2 + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)}\\
&+\frac{\gamma}{4\pi}(J_1+J_2)(k)\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\left|\hat{\phi}(k)\right|^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&2\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{2\left|\hat{\phi}(k)\right|} \\
=&\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \\
=&\sum_{k\geq 1} e^{\nu(t)k}k^s \Bigg(\frac{2}{R}\frac{\gamma}{4\pi}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \\
&+ \frac{2\pi}{L(t)} \frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)} + \overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \\
&+ 2\frac{\gamma}{4\pi}(J_1+J_2)(k)\left(-\frac{1}{R} + \frac{2\pi}{L(t)}\right)\left|\hat{\phi}(k)\right|\Bigg) \\
=&\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^s (J_1+J_2)(k)\left|\hat{\phi}(k)\right| \\
&+ \frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\theta}(k)} + \overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \\
&+ 2\frac{\gamma}{4\pi}\left(-\frac{1}{R} + \frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^s (J_1+J_2)(k)\left|\hat{\phi}(k)\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} =&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^s (J_1+J_2)(k)\left|\hat{\phi}(k)\right| && (7.6) \\
&+ \frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)} + \overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} && (7.7) \\
&+ 2\frac{\gamma}{4\pi}\left(-\frac{1}{R} + \frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^s (J_1+J_2)(k)\left|\hat{\phi}(k)\right|. && (7.8)
\end{aligned}
$$

First, let us estimate (7.8). Observe that

$$\begin{aligned}
&2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1}e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \\
=&2\frac{\gamma}{4\pi}\frac{1}{R}\left(-1+R\frac{2\pi}{L(t)}\right)\sum_{k\geq 2}e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \\
=&-\pi\cdot 2\frac{\gamma}{4\pi}\frac{1}{R}\left(R\frac{2\pi}{L(t)}-1\right)\sum_{k\geq 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|.
\end{aligned}$$

Using Proposition 13, we obtain

$$\left|R\frac{2\pi}{L(t)}-1\right|\leq A\left\|\phi\right\|_{\mathcal{F}^{0,1}},$$

where we define

$$A=A(\|\phi\|_{\mathcal{F}^{0,1}})=\frac{1-\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}}{\|\phi\|_{\mathcal{F}^{0,1}}}.$$

Then

$$\begin{aligned}
&\left|2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1}e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right|\right| \\
=&\left|-\pi\cdot 2\frac{\gamma}{4\pi}\frac{1}{R}\left(R\frac{2\pi}{L(t)}-1\right)\sum_{k\geq 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|\right| \\
\leq&2\pi\frac{\gamma}{4\pi}\frac{1}{R}\left|R\frac{2\pi}{L(t)}-1\right|\sum_{k\geq 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| \\
\leq&2\pi\frac{\gamma}{4\pi}\frac{1}{R}A\left\|\phi\right\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|. \qquad (7.9)
\end{aligned}$$

Next, let us estimate (7.6) and (7.7).

$$
\begin{aligned}
&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^{s}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \\
&+\frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^{s}\frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)}+\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| \\
&+\frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^{s}\frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)}+\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \\
\leq&\nu'(t)\left\|\phi\right\|_{\dot{\mathcal{F}}^{s+1,1}_{\nu}} - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^{s}2\left|\mathcal{F}(\widetilde{\mathcal{N}})(k)\right| \\
\leq&\nu'(t)\left\|\phi\right\|_{\dot{\mathcal{F}}^{s+1,1}_{\nu}} - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}. && (7.10)
\end{aligned}
$$

Plugging (7.9) and (7.10) into (7.6), we obtain

$$
\begin{aligned}
\frac{d}{dt}\left\|\phi\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} =&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^{s}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| && (7.11) \\
&+\frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^{s}\frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)}+\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \\
&+2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^{s}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \\
\leq&\nu'(t)\left\|\phi\right\|_{\dot{\mathcal{F}}^{s+1,1}_{\nu}} - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} && (7.12) \\
&+2\frac{\gamma}{4\pi}\frac{1}{R}A\left\|\phi\right\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|.
\end{aligned}
$$

With the minus sign in front, the second term in (7.12) is associated with dissipation of the initial interfacial perturbation. It is clear that the ± 1 Fourier modes of ϕ play no part in dissipation. This presents a technical difficulty, because the norm of the function space that we intend to use involves all nonzero Fourier modes of ϕ. To resolve this issue, we note that (6.27) and $\hat{\phi}(0) = 0$ imply

$$0 = \int_{-\pi}^{\pi} e^{i(\alpha + \hat{\phi}(1)e^{i\alpha} + \hat{\phi}(-1)e^{-i\alpha} + \sum_{|k|>1} \hat{\phi}(k)e^{ik\alpha})} d\alpha.$$

This identity provides an implicit relation between the ± 1 Fourier modes and the other nonzero Fourier modes of ϕ, which allows us to control the former in terms of the latter. This observation is summarized in Proposition 4.1 of Gancedo et al. (2023a). In particular, we use the following result contained in the proposition.

Proposition 14. *Let $r \in (0, \frac{1}{2}\log\frac{5}{4})$. Consider $\|\phi\|_{\mathcal{F}^{0,1}} < r$. Then*

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \le C_I(r) r \sum_{|k|\ge 2} \left|\hat{\phi}(k)\right|,$$

where

$$C_I(r) = \frac{1}{r} \cdot \frac{2e^r(e^r - 1)}{1 - 4(e^{2r} - 1)}.$$

Here, $C_I(r) > 0$ is a strictly increasing function of r where

$$\lim_{r \to 0^+} C_I(r) = 2,$$

$$\lim_{r \to \log\frac{5}{4}^-} C_I(r) = \infty.$$

Suppose that $\|\phi\|_{\mathcal{F}^{0,1}} \in (0, \frac{1}{2}\log\frac{5}{4})$. By Proposition 14, for all $r \in (\|\phi\|_{\mathcal{F}^{0,1}}, \frac{1}{2}\log\frac{5}{4})$,

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \le C_I(r) r \sum_{|k|\ge 2} \left|\hat{\phi}(k)\right|.$$

Then

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \le C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} \sum_{|k|\ge 2} \left|\hat{\phi}(k)\right|.$$

By (7.5), this simplifies to

$$2\left|\hat{\phi}(1)\right| \leq 2C_I(\|\phi\|_{\mathcal{F}^{0,1}})\, \|\phi\|_{\mathcal{F}^{0,1}} \sum_{k\geq 2}\left|\hat{\phi}(k)\right|.$$

Hence, for $s > 0$,

$$\begin{aligned}
\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} =& 2\sum_{k\geq 1} e^{\nu(t)k}k^s\left|\hat{\phi}(k)\right| \\
=& 2\Bigg(e^{\nu(t)}\left|\hat{\phi}(1)\right| + \sum_{k\geq 2} e^{\nu(t)k}k^s\left|\hat{\phi}(k)\right|\Bigg) \\
\leq& 2C_I(\|\phi\|_{\mathcal{F}^{0,1}})\, \|\phi\|_{\mathcal{F}^{0,1}} \sum_{k\geq 2}\left|\hat{\phi}(k)\right| e^{\nu(t)} + 2\sum_{k\geq 2} e^{\nu(t)k}k^s\left|\hat{\phi}(k)\right| \\
\leq& 2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\, \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg)\sum_{k\geq 2} e^{\nu(t)k}k^s\left|\hat{\phi}(k)\right|.
\end{aligned}$$

Replacing s with $s+1$, we obtain

$$\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \leq 2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\, \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg)\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|,$$

which, when rearranged, yields

$$-\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| \leq -\frac{1}{2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\, \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg)}\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}.$$

Using this estimate in (7.11), we obtain

$$\begin{aligned}
\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq& \nu'(t)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \frac{1}{2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\, \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg)}\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\pi\frac{2}{R}\frac{\gamma}{4\pi} \\
&+ \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} + 2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|.
\end{aligned} \tag{7.13}$$

From Proposition 12, we have

$$2\pi R A_1 \leq L(t) \leq 2\pi R A_2,$$

where we define

$$A_1 = \frac{1}{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}},$$
$$A_2 = \frac{1}{\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}}.$$

Using this estimate in (7.13), we obtain

$$\begin{aligned}\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq& \nu'(t)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \frac{1}{2\Big(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\|\phi\|_{\mathcal{F}^{0,1}}+1\Big)}\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\pi\frac{2}{R}\frac{\gamma}{4\pi} \\ &+ \frac{1}{R}\frac{1}{A_1}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} + 2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|.\end{aligned}$$

By Proposition 1,

$$\begin{aligned}&\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\ \leq&\left(\nu'(t) - \frac{1}{2\Big(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\|\phi\|_{\mathcal{F}^{0,1}}+1\Big)}\pi\frac{2}{R}\frac{\gamma}{4\pi} + \frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\right)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \\ &+ \frac{1}{R}\frac{1}{A_1}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}.\end{aligned} \tag{7.14}$$

CHAPTER 8

Estimating $\widetilde{\mathcal{N}}$

In Chapter 7, we derived an *a priori* estimate containing the $\dot{\mathcal{F}}^{s,1}_{\nu}$ norm of $\widetilde{\mathcal{N}}$, where

$$\widetilde{\mathcal{N}}(\alpha) = (U_{\geq 2})_{\alpha}(\alpha) + T_{\geq 2}(\alpha)(1+\phi_{\alpha}(\alpha)) + T_1(\alpha)\phi_{\alpha}(\alpha). \tag{8.1}$$

We consider the each of the three terms separately. In Sections 8.1 and 8.2, we will see that the bounds for the second and third terms depend on the $\dot{\mathcal{F}}^{s,1}_{\nu}$ and $\mathcal{F}^{0,1}_{\nu}$ norms of U_1 and $U_{\geq 2}$. In Chapters 9 and 10, respectively, we will estimate these norms in terms of the corresponding norms of ϕ. Although the first term in (8.1) can be bounded above by the $\dot{\mathcal{F}}^{s+1,1}_{\nu}$ norm of $U_{\geq 2}$, the resulting estimate is not strong enough for the purposes of our study. For this reason, we will estimate it more carefully in Chapter 11.

8.1. Estimating $T_{\geq 2}(\alpha)(1+\phi_{\alpha}(\alpha))$

We prove the following estimate.

Lemma 15. *For $s \geq 1$,*

$$\begin{aligned}
&\left\| T_{\geq 2}(1+\phi_{\alpha}) \right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} \\
\leq &\left(1 + b(2,s)\left\|\phi\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&\cdot \left(\left\|U_{\geq 2}\right\|_{\dot{\mathcal{F}}^{s-1,1}_{\nu}} + b(2,s-1)\left(\left\|\phi\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} \left\|U_{\geq 1}\right\|_{\mathcal{F}^{0,1}_{\nu}} + \left\|U_{\geq 1}\right\|_{\dot{\mathcal{F}}^{s-1,1}_{\nu}} \left\|\phi\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \right)\right) \\
&+ b(2,s)\left\|\phi\right\|_{\dot{\mathcal{F}}^{s+1,1}_{\nu}} \left(2\left\|U_{\geq 2}\right\|_{\dot{\mathcal{F}}^{0,1}_{\nu}} + 2\left(\left\|\phi\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \left\|U_{\geq 1}\right\|_{\mathcal{F}^{0,1}_{\nu}} + \left\|\phi\right\|_{\mathcal{F}^{1,1}_{\nu}} \left\|U_{\geq 1}\right\|_{\dot{\mathcal{F}}^{0,1}_{\nu}} \right)\right).
\end{aligned}$$

For $0 \leq s < 1$*,*

$$\begin{aligned}
&\|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}^{s,1}_\nu}\\
\leq &\left(1 + b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\\
&\cdot\Bigg(\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\Bigg(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\Bigg)\Bigg)\\
&+ b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\left(2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2\left(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi\|_{\mathcal{F}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\right)\right).
\end{aligned}$$

Proof. Using Proposition 4, we obtain that for $s \geq 0$,

$$\begin{aligned}
\|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}^{s,1}_\nu} &\leq \|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} + \|T_{\geq 2}\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu}\\
&\leq \|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} + b(2,s)\Bigg(\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu} + \|\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu}\Bigg)\\
&\leq \|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} + b(2,s)\Bigg(\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu}\Bigg).
\end{aligned}$$

Note that

$$\begin{aligned}
T_{\geq 2}(\alpha) = &\int_0^\alpha U_{\geq 2}(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta\\
&- \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_{\geq 2}(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta\\
=&\mathcal{M}(U_{\geq 2})(\alpha) + \mathcal{M}(\phi_\alpha U_{\geq 1})(\alpha).
\end{aligned}$$

Hence for $s \geq 1$, using Proposition 4, we obtain

$$\begin{aligned}
\|T_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{s,1}} &= \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^s\,|\mathcal{F}(T_{\geq 2})(k)| \\
&\leq \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^s\,|\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)| \\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^s\,|\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)| \\
&= \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^{s-1}\,|\mathcal{F}(U_{\geq 2})(k)| \\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^{s-1}\,|\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| \\
&= \|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{s-1,1}} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}_\nu^{s-1,1}} \\
&\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{s-1,1}} + b(2,s-1)(\|\phi_\alpha\|_{\dot{\mathcal{F}}_\nu^{s-1,1}}\,\|U_{\geq 1}\|_{\mathcal{F}_\nu^{0,1}} + \|U_{\geq 1}\|_{\dot{\mathcal{F}}_\nu^{s-1,1}}\,\|\phi_\alpha\|_{\mathcal{F}_\nu^{0,1}}) \\
&= \|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{s-1,1}} + b(2,s-1)(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\,\|U_{\geq 1}\|_{\mathcal{F}_\nu^{0,1}} + \|U_{\geq 1}\|_{\dot{\mathcal{F}}_\nu^{s-1,1}}\,\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} \left|\mathcal{F}(T_{\geq 2})(k)\right| \\
=& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} \left|\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)\right| + \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} \left|\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)\right| \\
=& \left|\mathcal{F}(\mathcal{M}(U_{\geq 2}))(0)\right| + \left|\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(0)\right| \\
&+ \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)\right| + \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)\right| \\
\leq& \left|\mathcal{F}(\mathcal{M}(U_{\geq 2}))(0)\right| + \left|\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(0)\right| \\
&+ \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(U_{\geq 2})(k)\right| + \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(\phi_\alpha U_{\geq 1})(k)\right| \\
\leq& \sum_{j\neq 0} |j|^{-1} \left|\mathcal{F}(U_{\geq 2})(j)\right| + \sum_{j\neq 0} |j|^{-1} \left|\mathcal{F}(\phi_\alpha U_{\geq 1})(j)\right| \\
&+ \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(U_{\geq 2})(k)\right| + \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(\phi_\alpha U_{\geq 1})(k)\right| \\
\leq& \sum_{j\neq 0} e^{\nu(t)|j|} \left|\mathcal{F}(U_{\geq 2})(j)\right| + \sum_{j\neq 0} e^{\nu(t)|j|} \left|\mathcal{F}(\phi_\alpha U_{\geq 1})(j)\right| \\
&+ \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(U_{\geq 2})(k)\right| + \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(\phi_\alpha U_{\geq 1})(k)\right| \\
=& 2\left\|U_{\geq 2}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2\left\|\phi_\alpha U_{\geq 1}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu}.
\end{aligned}
$$

Using Proposition 4, we obtain that

$$
\begin{aligned}
\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} &\leq 2\left\|U_{\geq 2}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2\left\|\phi_\alpha U_{\geq 1}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} \\
&\leq 2\left\|U_{\geq 2}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2(\left\|\phi_\alpha\right\|_{\dot{\mathcal{F}}^{0,1}_\nu}\left\|U_{\geq 1}\right\|_{\mathcal{F}^{0,1}_\nu} + \left\|\phi_\alpha\right\|_{\mathcal{F}^{0,1}_\nu}\left\|U_{\geq 1}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu}) \\
&= 2\left\|U_{\geq 2}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2(\left\|\phi\right\|_{\dot{\mathcal{F}}^{1,1}_\nu}\left\|U_{\geq 1}\right\|_{\mathcal{F}^{0,1}_\nu} + \left\|\phi\right\|_{\mathcal{F}^{1,1}_\nu}\left\|U_{\geq 1}\right\|_{\dot{\mathcal{F}}^{0,1}_\nu}).
\end{aligned}
$$

Now, let us consider the case in which $0 \le s < 1$. Then

$$\begin{aligned}
\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} &= \sum_{k\neq 0} e^{\nu(t)|k|} \, |k|^s \, |\mathcal{F}(T_{\geq 2})(k)| \\
&\leq \sum_{k\neq 0} e^{\nu(t)|k|} \, |k|^s \, |\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)| \\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} \, |k|^s \, |\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)| \\
&= \sum_{k\neq 0} e^{\nu(t)|k|} \, |k|^{s-1} \, |\mathcal{F}(U_{\geq 2})(k)| \\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} \, |k|^{s-1} \, |\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| \\
&\leq \sum_{k\neq 0} e^{\nu(t)|k|} \, |\mathcal{F}(U_{\geq 2})(k)| \\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} \, |\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| \\
&= \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} .
\end{aligned}$$

Using Proposition 4, we obtain that

$$\begin{aligned}
\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} &\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \\
&\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\Big(\|\phi_\alpha\|_{\dot{\mathcal{F}}^{0,1}_\nu} \|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu} \|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \Big) \\
&\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\Big(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \Big).
\end{aligned}$$

■

8.2. Estimating $T_1(\alpha)\phi_\alpha(\alpha)$

We prove the following estimate.

Lemma 16. *For $s \geq 1$,*

$$\|T_1 \phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq b(2,s) \, \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|U_1\|_{\dot{\mathcal{F}}^{s-1,1}_\nu} + b(2,s) \, \|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \, 2 \, \|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu} .$$

For $0 \leq s < 1$*,*

$$\|T_1\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq b(2,s)\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\,\|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\,\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\,2\,\|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu}.$$

Proof. Using Proposition 4, we obtain that for $s \geq 0$,

$$\begin{aligned}
\|T_1\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu} &\leq b(2,s)\Big(\|T_1\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,\|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu} + \|\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,\|T_1\|_{\mathcal{F}^{0,1}_\nu}\Big)\\
&= b(2,s)\Big(\|T_1\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\,\|T_1\|_{\mathcal{F}^{0,1}_\nu}\Big).
\end{aligned}$$

Recall that $T_1(\alpha) = \mathcal{M}(U_1)(\alpha)$. Then for $s \geq 1$,

$$\begin{aligned}
\|T_1\|_{\dot{\mathcal{F}}^{s,1}_\nu} &= \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^s\,|\mathcal{F}(T_1)(k)|\\
&= \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^s\,|\mathcal{F}(\mathcal{M}(U_1))(k)|\\
&= \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^{s-1}\,|\mathcal{F}(U_1)(k)|\\
&= \|U_1\|_{\dot{\mathcal{F}}^{s-1,1}_\nu}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|T_1\|_{\mathcal{F}^{0,1}_\nu} &= \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\,|\mathcal{F}(T_1)(k)|\\
&= \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\,|\mathcal{F}(\mathcal{M}(U_1))(k)|\\
&= |\mathcal{F}(\mathcal{M}(U_1))(0)| + \sum_{k\neq 0} e^{\nu(t)|k|}\,|\mathcal{F}(\mathcal{M}(U_1))(k)|\\
&= \left|\sum_{j\neq 0} \frac{i}{j}\mathcal{F}(U_1)(j)\right| + \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^{-1}\,|\mathcal{F}(U_1)(k)|\\
&\leq \sum_{j\neq 0} e^{\nu(t)|j|}\,|\mathcal{F}(U_1)(j)| + \sum_{k\neq 0} e^{\nu(t)|k|}\,|\mathcal{F}(U_1)(k)|\\
&= 2\,\|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu}.
\end{aligned}$$

Now, let us consider the case in which $0 \leq s < 1$. Then

$$\begin{aligned}
\|T_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} &= \sum_{k\neq 0} e^{\nu(t)|k|} |k|^s |\mathcal{F}(T_1)(k)| \\
&= \sum_{k\neq 0} e^{\nu(t)|k|} |k|^s |\mathcal{F}(\mathcal{M}(U_1))(k)| \\
&= \sum_{k\neq 0} e^{\nu(t)|k|} |k|^{s-1} |\mathcal{F}(U_1)(k)| \\
&\leq \|U_1\|_{\dot{\mathcal{F}}_\nu^{0,1}} .
\end{aligned}$$

■

CHAPTER 9

Estimating U_1

To estimate the $\dot{\mathcal{F}}^{s,1}_\nu$ and $\mathcal{F}^{0,1}_\nu$ norms of U_1, we first estimate the Fourier modes of U_1.

9.1. Estimating Fourier Modes of U_1

For any norm $\|\cdot\|$, we can estimate (4.8) as

$$\|U_1\| \leq \left\| ie^{i\alpha} e^{i\hat{\theta}(0)} \mathfrak{L}(\alpha) \right\|. \tag{9.1}$$

To estimate the $\dot{\mathcal{F}}^{s,1}_\nu$ and $\mathcal{F}^{0,1}_\nu$ norms of (9.1), we can write

$$ie^{i\alpha} e^{i\hat{\theta}(0)} \mathfrak{L}(\alpha) = \sum_{j=1}^{7} \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_j(\alpha,\beta) d\beta, \tag{9.2}$$

where

$$\begin{aligned}
E_1(\alpha,\beta) &= \frac{-e^{i\beta}(-1+e^{i\beta})(i(-1+e^{i\beta}) + \beta(1+e^{i\beta}))}{2(-1+e^{i\beta})^2} \int_0^1 e^{-i\beta s} \phi(\alpha+\beta(-1+s)) ds, \qquad (9.3)\\
E_2(\alpha,\beta) &= \frac{i(-1-2i\beta+e^{2i\beta})}{2(-1+e^{i\beta})^2} \int_0^1 e^{i\beta s} \phi(\alpha+\beta(-1+s)) ds,\\
E_3(\alpha,\beta) &= \frac{-(-1+e^{i\beta})\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2} \int_0^1 e^{-i\beta s}(-1+s) \phi(\alpha+\beta(-1+s)) ds,\\
E_4(\alpha,\beta) &= \frac{-(-1+e^{i\beta})\beta(1+e^{i\beta})}{2(-1+e^{i\beta})^2} \int_0^1 e^{i\beta s}(-1+s) \phi(\alpha+\beta(-1+s)) ds,\\
E_5(\alpha,\beta) &= \frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2} \int_0^1 e^{-i\beta s}(-1+s) \phi'(\alpha+\beta(-1+s)) ds,\\
E_6(\alpha,\beta) &= \frac{-(-1+e^{i\beta})i(-\beta)(1+e^{i\beta})}{2(-1+e^{i\beta})^2} \int_0^1 e^{i\beta s}(-1+s) \phi'(\alpha+\beta(-1+s)) ds,\\
E_7(\alpha,\beta) &= \frac{-(-1+e^{i\beta})i(-1+2e^{i\beta}+e^{2i\beta})}{2(-1+e^{i\beta})^2} \phi(\alpha-\beta).
\end{aligned}$$

First, we calculate the Fourier modes of $E_1(\alpha,\beta)$.

$$\mathcal{F}(E_1)(k,\beta) = \frac{-e^{i\beta}(i(-1+e^{i\beta}) + \beta(1+e^{i\beta}))}{2(-1+e^{i\beta})} \cdot \int_0^1 e^{-i\beta s} e^{ik\beta(-1+s)} ds \cdot \mathcal{F}(\phi)(k).$$

Since

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-e^{i\beta}}{2(-1+e^{i\beta})}(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \qquad (9.4)\\
=&\left|\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\frac{-ie^{i\beta}}{2}\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}dsd\beta\right.\right.\\
&\left.\left.+\int_{-\pi}^{\pi}\frac{-\beta e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}dsd\beta\right)\right|\\
=&\left|\frac{\gamma}{4\pi}\left(\int_0^1\int_{-\pi}^{\pi}\frac{-ie^{i\beta}}{2}e^{-i\beta s}e^{ik\beta(-1+s)}d\beta ds\right.\right.\\
&\left.\left.+\int_0^1\int_{-\pi}^{\pi}\frac{-\beta e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}e^{-i\beta s}e^{ik\beta(-1+s)}d\beta ds\right)\right|\\
\leq&\frac{\gamma}{4\pi}\left(\int_0^1\int_{-\pi}^{\pi}\frac{1}{2}d\beta ds\right.\\
&\left.+\left|\int_0^1\int_{-\pi}^{\pi}\left(\frac{i\beta}{-1+e^{i\beta}}-1+1\right)\frac{ie^{i\beta}(1+e^{i\beta})}{2}e^{-i\beta s}e^{ik\beta(-1+s)}d\beta ds\right|\right)\\
\leq&\frac{\gamma}{4\pi}\left(\pi+\int_0^1\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1\right|\cdot\frac{1}{2}d\beta ds+\int_0^1\int_{-\pi}^{\pi}\frac{1}{2}d\beta ds\right)\\
=&\frac{\gamma}{4\pi}\left(\pi+\int_0^1\int_{-\pi}^{\pi}|\beta|\,\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}d\beta ds+\pi\right)\\
=&\frac{\gamma}{4\pi}\left(2\pi+\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}\pi^2\right),
\end{aligned}
$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_1)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(2\pi+\frac{\pi^2}{4}\sqrt{1+\frac{\pi^2}{4}}\right)|\mathcal{F}(\phi)(k)|\,.$$

Next, we calculate the Fourier modes of $E_2(\alpha,\beta)$.

$$\mathcal{F}(E_2)(k,\beta)=\frac{i(-1-2i\beta+e^{2i\beta})}{2(-1+e^{i\beta})^2}\mathcal{F}(\phi)(k)\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}ds.$$

Since

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-1-2i\beta+e^{2i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-1+e^{2i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right. \\
&\left.+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-2i\beta)}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(e^{i\beta}+1)}{2(-1+e^{i\beta})}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right. \\
&\left.+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-2i\beta)\beta}{2(-1+e^{i\beta})^2}\frac{1}{\beta}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i\beta(e^{i\beta}+1)}{2(-1+e^{i\beta})}\frac{1}{\beta}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right. \\
&\left.+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{i\beta}{1-e^{i\beta}}\right)^2\frac{-1}{\beta}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \\
\le&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1+1\right|\left|\frac{e^{i\beta}+1}{-2}\right|\frac{1}{|\beta|}d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\left(\frac{i\beta}{1-e^{i\beta}}\right)^2-1+1\right|\frac{1}{|\beta|}d\beta \\
\le&\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\int_{-\pi}^{\pi}\frac{1}{|\beta|}d\beta\right) \\
&+\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\int_{-\pi}^{\pi}\frac{1}{|\beta|}d\beta\right) \\
=&\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)+\frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_2)(k,\beta)d\beta\right| \\
&\le\left(\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)+\frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)\right)\left|\mathcal{F}(\phi)(k)\right|.
\end{aligned}
$$

Next, we calculate the Fourier modes of $E_3(\alpha,\beta)$.

$$
\mathcal{F}(E_3)(k,\beta)=\frac{-(-1+e^{i\beta})\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot\mathcal{F}(\phi)(k).
$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right|\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{|\beta|}{2}d\beta=\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2},
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_3)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\left|\mathcal{F}(\phi)(k)\right|.$$

Next, we calculate the Fourier modes of $E_4(\alpha,\beta)$.

$$\mathcal{F}(E_4)(k,\beta)=\frac{-(-1+e^{i\beta})\beta(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot\mathcal{F}(\phi)(k).$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})\beta(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right|\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1+1\right|\left|\frac{i(1+e^{i\beta})}{-2}\right|d\beta\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}|\beta|\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}d\beta\\
=&\frac{\gamma}{4\pi}\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+\frac{\gamma}{4\pi}\cdot 2\pi,
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_4)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\left|\mathcal{F}(\phi)(k)\right|.$$

Next, we calculate the Fourier modes of $E_5(\alpha,\beta)$.

$$\begin{aligned}
&\mathcal{F}(E_5)(k,\beta)\\
=&\frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot ik\mathcal{F}(\phi)(k).
\end{aligned}$$

Since

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right|\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{|\beta|}{2}d\beta=\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2},
\end{aligned}
$$

we obtain

$$
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_5)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\left|k\right|\cdot\left|\mathcal{F}(\phi)(k)\right|.
$$

Next, we calculate the Fourier modes of $E_6(\alpha,\beta)$.

$$
\begin{aligned}
&\mathcal{F}(E_6)(k,\beta)\\
=&\frac{-(-1+e^{i\beta})i(-\beta)(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot ik\mathcal{F}(\phi)(k).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})i(-\beta)(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right|\\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{i\beta}{1-e^{i\beta}}-1+1\right)\frac{1+e^{i\beta}}{-2}\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right|\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1+1\right|d\beta\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}|\beta|\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}d\beta\\
=&\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right),
\end{aligned}
$$

we obtain

$$
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_6)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)|k|\left|\mathcal{F}(\phi)(k)\right|.
$$

Lastly, we calculate the Fourier modes of $E_7(\alpha,\beta)$.

$$\mathcal{F}(E_7)(k,\beta)=\frac{-(-1+e^{i\beta})i(-1+2e^{i\beta}+e^{2i\beta})}{2(-1+e^{i\beta})^2}e^{-ik\beta}\mathcal{F}(\phi)(k).$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})i(-1+2e^{i\beta}+e^{2i\beta})}{2(-1+e^{i\beta})^2}e^{-ik\beta}d\beta\right|\\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{i\beta}{1-e^{i\beta}}-1+1\right)\frac{(-1+2e^{i\beta}+e^{2i\beta})e^{-ik\beta}}{2\beta}d\beta\right|\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}|\beta|\,\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2\,|\beta|}d\beta+\frac{\gamma}{4\pi}\left|\int_{-\pi}^{\pi}\frac{e^{-ik\beta}(-1+2e^{i\beta}+e^{2i\beta})}{2\beta}d\beta\right|\\
\leq&\frac{\gamma}{4\pi}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot 2\pi+\frac{\gamma}{4\pi}\cdot\frac{1}{2}\left|\int_{-\pi}^{\pi}\frac{e^{-ik\beta}(-1+2e^{i\beta}+e^{2i\beta})}{\beta}d\beta\right|\\
\leq&\frac{\gamma}{4\pi}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot 2\pi+\frac{\gamma}{4\pi}\cdot\frac{1}{2}\cdot 4\cdot 5,
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_7)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot 2\pi+\frac{1}{2}\cdot 4\cdot 5\right)|\mathcal{F}(\phi)(k)|\,.$$

9.2. Estimating $\|U_1\|_{\mathcal{F}_\nu^{0,1}}$

In Section 9.1, we observed that

$$\|U_1\|_{\mathcal{F}_\nu^{0,1}}\leq\sum_{j=1}^{7}\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}}.$$

Since

$$
\begin{aligned}
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_1(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_1)(k,\beta)d\beta\right| \\
\leq& \frac{\gamma}{4\pi}\left(2\pi+\frac{\pi^2}{4}\sqrt{1+\frac{\pi^2}{4}}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}} \\
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_2(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_2)(k,\beta)d\beta\right| \\
\leq& \left(\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)\right. \\
&\left. +\frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}} \\
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_3(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_3)(k,\beta)d\beta\right| \\
\leq& \frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\|\phi\|_{\mathcal{F}_\nu^{0,1}} \\
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_4(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_4)(k,\beta)d\beta\right| \\
\leq& \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}} \\
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_5(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_5)(k,\beta)d\beta\right| \\
\leq& \frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\|\phi\|_{\mathcal{F}_\nu^{1,1}} \\
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_6(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_6)(k,\beta)d\beta\right| \\
\leq& \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\|\phi\|_{\mathcal{F}_\nu^{1,1}} \\
\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_7(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =& \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_7)(k,\beta)d\beta\right| \\
\leq& \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot 2\pi+\frac{1}{2}\cdot 4\cdot 5\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}},
\end{aligned}
$$

we obtain

$$
\|U_1\|_{\mathcal{F}_\nu^{0,1}} \leq H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\mathcal{F}_\nu^{1,1}},
$$

where H_3 and H_4 are constants.

9.3. Estimating $\|U_1\|_{\dot{\mathcal{F}}_\nu^{s,1}}$

In Section 9.1, we observed that

$$\|U_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq \sum_{j=1}^{7} \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_j(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} .$$

Since

$$\begin{aligned}
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \frac{\gamma}{4\pi} \left(2\pi + \frac{\pi^2}{4} \sqrt{1+\frac{\pi^2}{4}} \right) \sum_{k\neq 0} e^{\nu(t)|k|} |k|^s |\mathcal{F}(\phi)(k)| \\
\leq& \frac{\gamma}{4\pi} \left(2\pi + \frac{\pi^2}{4} \sqrt{1+\frac{\pi^2}{4}} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \left(\frac{\gamma}{4\pi} \left(\frac{1}{2} \sqrt{1+\frac{\pi^2}{4}} \cdot 2\pi + \pi^2 \right) \right. \\
&\left. + \frac{\gamma}{4\pi} \left(2 \cdot \frac{\pi}{2} \cdot \frac{1}{2} \sqrt{1+\frac{\pi^2}{4}} \cdot 2\pi + \pi^2 \right) \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \frac{\gamma}{4\pi} \cdot \frac{\pi^2}{2} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_4(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \frac{\gamma}{4\pi} \left(\frac{1}{2} \sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_5(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \frac{\gamma}{4\pi} \cdot \frac{\pi^2}{2} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} \\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_6(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \frac{\gamma}{4\pi} \left(\frac{1}{2} \sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} \\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_7(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq& \frac{\gamma}{4\pi} \left(\frac{1}{2} \sqrt{1+\frac{\pi^2}{4}} \cdot \frac{3}{2} \cdot 2\pi + \frac{1}{2} \cdot 4 \cdot 5 \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} ,
\end{aligned}$$

we obtain

$$\|U_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq H_1 \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + H_2 \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} ,$$

where H_1 and H_2 are constants.

CHAPTER 10

Estimating $U_{\geq 2}$

For any norm $\|\cdot\|$, we can estimate (4.9) as

$$\begin{aligned}\|U_{\geq 2}\| &\leq \left\| ie^{i\alpha}e^{i\hat{\theta}(0)}\left(\mathfrak{L}(\alpha)(e^{i\phi(\alpha)}-1)+\mathfrak{N}(\alpha)e^{i\phi(\alpha)}\right)\right\| \\ &\leq \left\| ie^{i\alpha}e^{i\hat{\theta}(0)}\left(e^{i\phi(\alpha)}(\mathfrak{L}(\alpha)+\mathfrak{N}(\alpha))-\mathfrak{L}(\alpha)\right)\right\|. \end{aligned} \tag{10.1}$$

To estimate the $\dot{\mathcal{F}}^{s,1}_{\nu}$ and $\mathcal{F}^{0,1}_{\nu}$ norms of (10.1), we can write

$$ie^{i\alpha}e^{i\hat{\theta}(0)}\left(e^{i\phi(\alpha)}(\mathfrak{L}(\alpha)+\mathfrak{N}(\alpha))-\mathfrak{L}(\alpha)\right) = \sum_{j=1}^{16}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_j(\alpha,\beta)d\beta,$$

where

$$\begin{aligned}
B_1(\alpha,\beta) =& -\frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds}\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \\
B_2(\alpha,\beta) =& -\frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \\
&\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
B_3(\alpha,\beta) =& \frac{e^{i(\beta+\phi(\alpha)+\phi(\alpha-\beta))}}{2\left(\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds\right)^2}\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds \\
&\cdot\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \\
B_4(\alpha,\beta) =& \frac{e^{i(\beta+\phi(\alpha)+\phi(\alpha-\beta))}}{2\left(\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds\right)^2}\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds \\
&\cdot\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
B_5(\alpha,\beta) =& \frac{e^{i(\beta+\phi(\alpha))}e^{i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds}\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds
\end{aligned}$$

$$
\begin{aligned}
B_6(\alpha,\beta) =& \frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \cdot \int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \\
B_7(\alpha,\beta) =& \frac{e^{i(\beta+\phi(\alpha))}e^{i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \\
& \cdot \int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
B_8(\alpha,\beta) =& \frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \\
& \cdot \int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
B_9(\alpha,\beta) =& \frac{e^{i\beta}}{2(-1+e^{i\beta})} \cdot (i(-1+e^{i\beta})+\beta(1+e^{i\beta})) \cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds \\
B_{10}(\alpha,\beta) =& -\frac{2\beta+i(-1+e^{2i\beta})}{2(-1+e^{i\beta})^2} \cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds \\
B_{11}(\alpha,\beta) =& \frac{\beta e^{i\beta}}{2} \cdot \int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds \\
B_{12}(\alpha,\beta) =& \frac{\beta(1+e^{i\beta})}{2(-1+e^{i\beta})} \cdot \int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds \\
B_{13}(\alpha,\beta) =& \frac{-i(-2e^{i\beta}+2e^{2i\beta})e^{i\phi(\alpha)}e^{-i\phi(\alpha-\beta)}}{2\beta(-1+e^{i\beta})} \\
B_{14}(\alpha,\beta) =& \frac{i\beta e^{i\beta}}{2} \cdot \int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
B_{15}(\alpha,\beta) =& \frac{-i\beta(1+e^{i\beta})}{2(-1+e^{i\beta})} \cdot \int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
B_{16}(\alpha,\beta) =& \frac{-i(\beta-2\beta e^{i\beta}-\beta e^{2i\beta})}{2\beta(-1+e^{i\beta})}\phi(\alpha-\beta).
\end{aligned}
$$

Using the Taylor expansion, we write

$$
\begin{aligned}
B_1(\alpha,\beta) = -\sum_{j_1,j_2,j_3,n\geq 0} & \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
& \cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
& \cdot \Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Big)^n
\end{aligned}
$$

$$B_2(\alpha,\beta) =$$

$$-\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n$$

$$B_3(\alpha,\beta) =$$

$$\frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n$$

$$B_4(\alpha,\beta) =$$

$$\frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n$$

$$B_5(\alpha,\beta) =$$

$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n$$

$$B_6(\alpha,\beta) =$$
$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds0$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n$$
$$B_7(\alpha,\beta) =$$
$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}0$$
$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n$$
$$B_8(\alpha,\beta) =$$
$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n$$
$$B_{13}(\alpha,\beta) =$$
$$\frac{-i(-2e^{i\beta}+2e^{2i\beta})}{2\beta(-1+e^{i\beta})}\sum_{j_1,j_2\geq 0}\frac{i^{j_1+j_2}(-1)^{j_2}}{j_1!j_2!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}.$$

For ease of notation, let $B(\alpha,\beta) = \sum_{j=1}^{16} B_j(\alpha,\beta)$. We now show that the part of $B(\alpha,\beta)$ which is constant or linear in ϕ is zero. We observe that for $i \in \{9,10,11,12,14,15,16\}$, $B_i(\alpha,\beta)$ is an expression linear in ϕ. To prove that $B(\alpha,\beta)$ has no part linear in ϕ, we first extract terms from $B_i(\alpha,\beta)$ for $i \in \{1,2,3,4,5,6,7,8,13\}$ which contain the integrals that appear in $B_i(\alpha,\beta)$ for $i \in \{9,10,11,12,14,15\}$. We first collect all terms containing $\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$.

In B_4, when $j_1 = j_2 = j_3 = j_4 = n = 0$, we have

$$\frac{1}{2}\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds.$$

In B_8, when $j_1 = j_2 = j_3 = n = 0$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds.$$

Next, we collect all terms containing $\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$. In B_2, when $j_1 = j_2 = j_3 = n = 0$, we have

$$-\frac{1}{2}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds.$$

In B_7, when $j_1 = j_2 = j_3 = n = 0$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds.$$

Next, we collect all terms containing $\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$. In B_3, when $j_1 = j_2 = j_3 = n = 0$ and $j_4 = 1$, we have

$$\frac{1}{2}\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i}{1}\int_0^1\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds.$$

In B_6, when $j_1 = j_2 = n = 0$ and $j_3 = 1$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\frac{i}{1}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds.$$

Next, we collect all terms containing $\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$. In B_1, when $j_1 = j_2 = n = 0$ and $j_3 = 1$, we have

$$-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)i}{2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds.$$

Inside B_5, when $j_1 = j_2 = n = 0$ and $j_3 = 1$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\frac{i(-1)}{1}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds.$$

Next, we collect all terms containing $\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds$. In B_3, when $j_1 = j_2 = j_3 = j_4 = 0$ and $n = 1$, we have

$$\begin{aligned}&\frac{1}{2}\cdot 2\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\cdot\\ &\Bigg(-ie^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)ds\\ &+\sum_{m=2}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\Bigg).\end{aligned}$$

In B_5, when $j_1 = j_2 = j_3 = 0$ and $n = 1$, we have

$$\begin{aligned}&\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg(\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}i\phi(\alpha+(s-1)\beta)ds\\ &+\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg).\end{aligned}$$

Inside B_6, when $j_1 = j_2 = j_3 = 0$ and $n = 1$, we have

$$\begin{aligned}&\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds\Bigg(\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}i\phi(\alpha+(s-1)\beta)ds\\ &+\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s+1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg).\end{aligned}$$

Lastly, we collect all terms containing $\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds$. In B_1, when $j_1 = j_2 = j_3 = 0$ and $n = 1$, we have

$$\begin{aligned}&-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg(\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(-i)\phi(\alpha+(s-1)\beta)d\beta\\ &+\sum_{m=2}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg).\end{aligned}$$

In B_3, when $j_1 = j_2 = j_4 = n = 0$ and $j_3 = 1$, we have

$$\frac{1}{2}\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i(-1)}{1}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds\int_0^1 e^{i\beta s}(-1+s)ds.$$

When added with $B_i(\alpha,\beta)$ for $i \in \{9, 10, 11, 12, 14, 15\}$, the terms extracted above containing $\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds$, and $\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds$ vanish. To complete the proof that $B(\alpha,\beta)$ has no part that is constant or linear in ϕ, we collect all terms constant or linear in $\phi(\alpha)$ or $\phi(\alpha-\beta)$. From B_1, we have

$$\begin{aligned}&-\Bigg(\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds\\&+\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)i}{2}\cdot\phi(\alpha-\beta)\int_0^1 e^{-i\beta s}(-1+s)ds+\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{i}{2}\cdot\phi(\alpha)\cdot\\&\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg).\end{aligned}$$

From B_3, we have

$$\begin{aligned}&\frac{1}{2}\Bigg(\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds+\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i}{1}\cdot\phi(\alpha)\cdot\\&\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\\&+\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i}{1}\cdot\phi(\alpha-\beta)\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\Bigg).\end{aligned}$$

From B_5, we have

$$\begin{aligned}&\frac{1}{2}\Bigg(\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i}{1}\cdot\phi(\alpha)\int_0^1 e^{-i\beta s}(-1+s)ds\\&+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i}{1}\cdot\phi(\alpha-\beta)\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg).\end{aligned}$$

From B_6, we have

$$\frac{1}{2}\Bigg(\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds + \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i}{1}\cdot\phi(\alpha)\int_0^1 e^{i\beta s}(-1+s)ds$$
$$+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i(-1)}{1}\cdot\phi(\alpha-\beta)\int_0^1 e^{i\beta s}(-1+s)ds\Bigg).$$

From B_{13}, we have

$$\frac{-i(-2e^{i\beta}+2e^{2i\beta})}{2\beta(-1+e^{i\beta})}\bigg(1+\frac{i}{1}\phi(\alpha)+\frac{i(-1)}{1}\phi(\alpha-\beta)\bigg).$$

From B_{16}, we have

$$\frac{-i(\beta-2\beta e^{i\beta}-\beta e^{2i\beta})}{2\beta(-1+e^{i\beta})}\cdot\phi(\alpha-\beta).$$

Of these terms, those linear in $\phi(\alpha-\beta)$ add up to 0. When integrated with respect to β, the terms which are constant and linear in $\phi(\alpha)$ become 0. Setting zero all but summation variables j_1 and j_2 in $B_i(\alpha,\beta)$ for $i\in\{1,3,5,6,13,16\}$, we obtain a smaller sum $\sum_{j_1+j_2\geq 0}$. Each of the above terms that are constant or linear in $\phi(\alpha)$ or $\phi(\alpha-\beta)$ belongs to one of these smaller sums. From these smaller sums, we take out these terms and add them up to obtain

$$\sum_{j_1+j_2\geq 2}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\frac{i^{j_1+j_2}}{j_1!j_2!}\bigg((-1)^{j_2}\cdot\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\bigg). \tag{10.2}$$

Observe that

$$B_1 = -\sum_{j_1,j_2,j_3,n\geq 0} = -\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} - \sum_{\substack{j_1,j_2\geq 0\\ j_3+n\geq 1}} = -\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} - \sum_{\substack{j_1,j_2\geq 0\\ j_3=1\\ n\geq 0}} - \sum_{\substack{j_1,j_2\geq 0\\ n=1\\ j_3\geq 0}}$$
$$= -\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} - \sum_{\substack{j_1=j_2=n=0\\ j_3=1}} - \sum_{\substack{j_1+j_2+n\geq 1\\ j_3=1}} - \sum_{\substack{j_1=j_2=j_3=0\\ n=1}} - \sum_{\substack{j_1+j_2+j_3\geq 1\\ n=1}},$$

$$
\begin{aligned}
B_2 =& -\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = -\frac{1}{2}\sum_{j_1=j_2=j_3=n=0} -\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1},\\
B_3 =& \frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0}\\
=& \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=j_4=n=0}} +\frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3+j_4+n\geq 1}} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=j_4=n=0}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 1}} +\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}\\
=& \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=j_4=n=0}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n=1}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 2}} +\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}},\\
B_4 =& \frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0} = \frac{1}{2}\sum_{j_1=j_2=j_3=j_4=n=0} +\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1},\\
B_5 =& \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} +\frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3+n\geq 1}} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 1}} +\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\\
=& \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n=1}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} +\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}},\\
B_6 =& \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} +\frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3+n\geq 1}} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 1}} +\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\\
=& \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n=1}} +\frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} +\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}},\\
B_7 =& \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{j_1=j_2=j_3=n=0} +\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1},\\
B_8 =& \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{j_1=j_2=j_3=n=0} +\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1},\\
B_{13} =& \sum_{j_1,j_2\geq 0} = \sum_{j_1+j_2=0} + \sum_{j_1+j_2=1} + \sum_{j_1+j_2\geq 2}.
\end{aligned}
$$

From these expressions for B_i for $i \in \{1,2,3,4,5,6,7,8,13\}$, we take out all the terms that are constant or linear in $\phi(\alpha)$ or $\phi(\alpha-\beta)$, or contain the integrals involving ϕ that had been identified earlier because they have been shown to vanish. Once they are taken out, we add all the smaller sums of the form $\sum_{j_1+j_2\geq 2}$ with that of B_{13}, which is equal to (10.2). Then we can write

$$ie^{i\alpha}e^{i\hat{\theta}(0)}\left(e^{i\phi(\alpha)}(\mathfrak{L}(\alpha)+\mathfrak{N}(\alpha))-\mathfrak{L}(\alpha)\right) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B(\alpha,\beta)d\beta, \tag{10.3}$$

where $B(\alpha,\beta)=\sum_{j=1}^{8}\widetilde{B_j}(\alpha,\beta)+\widetilde{B_{13}}(\alpha,\beta)$, in which

$$\begin{aligned}
\widetilde{B_1}(\alpha,\beta)=&-\sum_{\substack{j_1+j_2+n\geq 1\\ j_3=1}}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&-\sum_{\substack{j_1+j_2+j_3\geq 1\\ n=1}}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds\\
&\cdot\sum_{m=2}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\\
\widetilde{B_2}(\alpha,\beta)=&-\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n
\end{aligned}$$

$$
\begin{aligned}
\widetilde{B_3}(\alpha,\beta) =& \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 2}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n\\
&+\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n\\
&+\frac{1}{2}\cdot 2\cdot\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\sum_{m=2}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\\
&\cdot\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\\
\widetilde{B_4}(\alpha,\beta) =& \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B_5}(\alpha,\beta) =& \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} \frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&+\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&+\frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds\\
&\cdot\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\\
\widetilde{B_6}(\alpha,\beta) =& \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} \frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&+\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&+\frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds\\
&\cdot\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B_7}(\alpha,\beta) =& \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
\widetilde{B_8}(\alpha,\beta) =& \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
\widetilde{B_{13}}(\alpha,\beta) =& \sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right).
\end{aligned}
$$

10.1. Estimating Fourier Modes of $U_{\geq 2}$

In our calculations, we adopt the notational convention that any product $\prod$ in which the upper bound is strictly less than the lower bound is defined to be 1. To calculate the Fourier modes of $U_{\geq 2}$, we frequently use the identity

$$
\mathcal{F}(g_1g_2\cdots g_n)(k_1) = \sum_{k_2,\dots,k_n\in\mathbb{Z}}\left(\prod_{d=1}^{n-1}\mathcal{F}(g_d)(k_d-k_{d+1})\right)\mathcal{F}(g_n)(k_n). \tag{10.4}
$$

We define

$$
\begin{aligned}
P(k) &= \sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k),\\
\widetilde{P}(k) &= \sum_{m=2}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k),\\
Q(k) &= \sum_{m=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k),\\
\widetilde{Q}(k) &= \sum_{k=2}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k).
\end{aligned}
$$

For $n \geq 0$, let

$$\begin{aligned}
&I_n(k_1, k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n}, k_{j_1+j_2+n+1}, \beta) \\
=&\prod_{d=1}^{n}\left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\right)\cdot e^{-i\beta(k_1-k_{j_1+1})} \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds
\end{aligned}$$

and

$$C_n = \frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right).$$

The following estimate is used frequently.

Lemma 17. *For $n \geq 0$,*

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_n(k_1, k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n}, k_{j_1+j_2+n+1}, \beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \leq C_n.$$

Proof. We note that

$$\int_0^1 e^{-is\beta}e^{i(s-1)\beta k}ds = \begin{cases}\frac{i(e^{-i\beta}-e^{-i\beta k})}{\beta(1-k)} & \text{if } k \neq 1,\\ e^{-i\beta} & \text{if } k = 1.\end{cases}$$

First let $n \geq 1$. Suppose that $0 \leq l \leq n$ and l elements of $\{k_{j_1+j_2+d} - k_{j_1+j_2+d+1}\}_{d=1}^{n}$ satisfy $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} = 1$. Reordering the subscripts such that $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} \neq 1$ for $d = 1, \ldots, n-l$, we obtain

$$\begin{aligned}
I_n =& e^{-i\beta(k_1-k_{j_1+1})}\prod_{d=1}^{n-l}\frac{-(1-e^{-i\beta(-1+k_{j_1+j_2+d}-k_{j_1+j_2+d+1})})}{(1-e^{i\beta})(1-k_{j_1+j_2+d}+k_{j_1+j_2+d+1})}\left(\frac{i\beta}{1-e^{i\beta}}\right)^l \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds.
\end{aligned}$$

If $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} > 1$, then

$$\frac{-(1-e^{-i\beta(-1+k_{j_1+j_2+d}-k_{j_1+j_2+d+1})})}{1-e^{i\beta}} = e^{-i\beta} \sum_{r_d=0}^{-2+k_{j_1+j_2+d}-k_{j_1+j_2+d+1}} (e^{-i\beta})^{r_d}.$$

If $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} < 1$, then

$$\frac{-(1-e^{-i\beta(-1+k_{j_1+j_2+d}-k_{j_1+j_2+d+1})})}{1-e^{i\beta}} = - \sum_{r_d=0}^{-(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})} (e^{i\beta})^{r_d}.$$

Suppose that $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} < 1$ only for $d = w, \ldots, n-l$. Then

$$\begin{aligned}
&\prod_{d=1}^{n-l} \frac{-(1-e^{-i\beta(-1+k_{j_1+j_2+d}-k_{j_1+j_2+d+1})})}{1-e^{i\beta}} \\
=&e^{-i\beta} \sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}} (e^{-i\beta})^{r_1} \cdot \cdots \cdot e^{-i\beta} \sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} (e^{-i\beta})^{r_{w-1}} \\
&\cdot (-1) \sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})} (e^{i\beta})^{r_w} \cdot \cdots \cdot (-1) \sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})} (e^{i\beta})^{r_{n-l}} \\
=& \sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}} \cdots \sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} \\
& \sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})} \cdots \sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})} \\
& (e^{-i\beta})^{w-1}(-1)^{n-l-w+1}(e^{-i\beta})^{r_1+\cdots+r_{w-1}}(e^{i\beta})^{r_w+\cdots+r_{n-l}}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
I_n =& \prod_{d=1}^{n-l} \frac{1}{1-k_{j_1+j_2+d}+k_{j_1+j_2+d+1}} \cdot \\
& \sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}} \cdots \sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} \\
& \sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})} \cdots \sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})} \\
& (-1)^{n-l-w+1}(e^{-i\beta})^{w-1}(e^{-i\beta})^{r_1+\cdots+r_{w-1}}(e^{i\beta})^{r_w+\cdots+r_{n-l}} \\
& \cdot \left(\frac{i\beta}{1-e^{i\beta}}\right)^l \cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds \cdot e^{-i\beta(k_1-k_{j_1+1})}.
\end{aligned}
$$

Let

$$
\begin{aligned}
J_n =& \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (e^{-i\beta})^{w-1+r_1+\cdots+r_{w-1}-(r_w+\cdots+r_{n-l})} \left(\frac{i\beta}{1-e^{i\beta}}\right)^l e^{-i\beta(k_1-k_{j_1+1})} \\
& \cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta.
\end{aligned}
$$

For all $l \geq 0$,

$$
\left| \left(\frac{i\beta}{1-e^{i\beta}}\right)^l - 1 \right| \leq |\beta| \cdot l \cdot \left(\frac{\pi}{2}\right)^{l-1} \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}.
$$

Then

$$
\begin{aligned}
|J_n| &\leq \frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi} \left| \left(\frac{i\beta}{1-e^{i\beta}}\right)^{l+1} - 1 \right| d\beta + 2\pi\right) \\
&\leq \frac{\gamma}{4\pi}\left((l+1) \cdot \left(\frac{\pi}{2}\right)^l \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi\right) \\
&\leq \frac{\gamma}{4\pi}\left((n+1) \cdot \left(\frac{\pi}{2}\right)^n \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi\right) \\
&= C_n.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_n(k_1, k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n}, k_{j_1+j_2+n+1}, \beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \\
\leq&\prod_{d=1}^{n-l}\frac{1}{|1-k_{j_1+j_2+d}+k_{j_1+j_2+d+1}|}\cdot\sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}}\cdots\sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} \\
&\cdot\sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})}\cdots\sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})}|J_n| \\
\leq& C_n.
\end{aligned}
$$

If $n = 0$, then

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_0(k_1, k_{j_1+1}, k_{j_1+j_2+1}, \beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \\
\leq&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} e^{-i\beta(k_1-k_{j_1+1})}\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot(-e^{2i\beta})\left(\frac{i\beta}{1-e^{i\beta}}-1+1\right)d\beta\right| \\
\leq&\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}|\beta|\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+2\pi\right) \\
=&C_0,
\end{aligned}
$$

where

$$
C_0 = \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}+2\pi\right).
$$

■

Let us calculate the Fourier modes of $\widetilde{B_1}(\alpha,\beta)$. Let $\widetilde{B_1} = \sum_{j=1}^{3} \widetilde{B_{1,j}}$, where

$$\begin{aligned}
\widetilde{B_{1,1}}(\alpha,\beta) = -&\sum_{\substack{j_1+j_2+n\geq 1\\ j_3=1}} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \Bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \\
\widetilde{B_{1,2}}(\alpha,\beta) = -&\sum_{\substack{j_1+j_2+j_3\geq 1\\ n=1}} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \Bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \\
\widetilde{B_{1,3}}(\alpha,\beta) = -&\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{1}{2} \int_0^1 e^{-i\beta s}(-1+s)ds \\
&\cdot \sum_{m=2}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds.
\end{aligned}$$

First, we calculate the Fourier modes of $\widetilde{B_{1,1}}(\alpha,\beta)$.

$$\begin{aligned}
&\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \cdot \\
&\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \cdot \\
&\Bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n\Bigg)(k_1).
\end{aligned}$$

We can write

$$\begin{aligned}
&\mathcal{F}\bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\bigg)(k_1) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\bigg(\prod_{d=1}^{j_1}\mathcal{F}(\phi(\alpha-\beta))(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi)(k_{j_1+d}-k_{j_1+d+1}) \\
&\cdot\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg) \\
&(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\bigg) \\
&\cdot\mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} e^{-i\beta(k_1-k_{j_1+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\bigg(\frac{i\beta e^{i\beta}}{1-e^{i\beta}} \\
&\cdot\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\cdot\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\bigg) \\
&\cdot\mathcal{F}(\phi)(k_{j_1+j_2+n+1})\int_0^1 e^{-i\beta s}(-1+s)e^{ik_{j_1+j_2+n+1}\beta(-1+s)}ds \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2+n+1}) \\
&\cdot\prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot I_n(k_1,k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n},k_{j_1+j_2+n+1},\beta).
\end{aligned}$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta)d\beta \\
=&-\sum_{j_1+j_2+n\geq 1}\frac{(-1)^{j_1+1}i^{j_1+j_2+1}}{2j_1!j_2!} \\
&\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2+n+1}) \\
&\cdot\prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \\
&\cdot I_n(k_1,k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n},k_{j_1+j_2+n+1},\beta)d\beta.
\end{aligned}
$$

By Lemma 17,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta)d\beta\right| \\
\leq&\sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})| \\
&\cdot\prod_{d=1}^{n}|P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})|\,.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_{1,2}}(\alpha,\beta)$.

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta) \\
=&-\sum_{j_1+j_2+j_3\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \\
&\cdot\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1) \\
=&\sum_{k_2,\dots,k_{j_1+j_2+2}\in\mathbb{Z}}\Bigg(\prod_{d=1}^{j_1}\mathcal{F}(\phi(\alpha-\beta))(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi)(k_{j_1+d}-k_{j_1+d+1}) \\
&\cdot\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+1}-k_{j_1+j_2+2})\Bigg) \\
&\cdot\mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\Bigg)(k_{j_1+j_2+2}) \\
=&\sum_{k_2,\dots,k_{j_1+j_2+2}\in\mathbb{Z}} I_1(k_1,k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+2},\beta) \\
&\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})P(k_{j_1+j_2+1}-k_{j_1+j_2+2}).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta \\
=-&\sum_{j_1+j_2+j_3\geq 1}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \\
&\cdot\sum_{k_2,\dots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})P(k_{j_1+j_2+1}-k_{j_1+j_2+2}) \\
&\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}I_1(k_1,k_{j_1+1},k_{j_1+j_2+1,k_{j_1+j_2+2}},\beta)d\beta.
\end{aligned}
$$

By Lemma 17,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta\right| \\
\leq&\sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right| \\
&\cdot|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_{1,3}}(\alpha, \beta)$. We can write

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_{1,3}})(k_1, \beta) \\
=&\frac{i\beta e^{2i\beta}}{1 - e^{i\beta}} \cdot \frac{1}{2} \int_0^1 e^{-i\beta s}(-1 + s) ds \\
&\cdot \mathcal{F}\bigg(\sum_{m=2}^{\infty} \frac{i\beta}{1 - e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha + (s-1)\beta))^m}{m!} ds \bigg)(k_1) \\
=&\frac{i\beta e^{2i\beta}}{1 - e^{i\beta}} \cdot \frac{1}{2} \int_0^1 e^{-i\beta s}(-1 + s) ds \cdot \sum_{m=2}^{\infty} \frac{i\beta e^{i\beta}}{1 - e^{i\beta}} \frac{(-i)^m}{m!} \mathcal{F}(\phi^m)(k_1) \cdot \int_0^1 e^{-is\beta} e^{i(s-1)\beta k_1} ds \\
=&\frac{i\beta e^{2i\beta}}{1 - e^{i\beta}} \cdot \frac{1}{2} \int_0^1 e^{-i\beta s}(-1 + s) ds \cdot \frac{i\beta e^{i\beta}}{1 - e^{i\beta}} \cdot \int_0^1 e^{-is\beta} e^{i(s-1)\beta k_1} ds \cdot \widetilde{P}(k_1).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,3}})(k_1, \beta) d\beta \\
=&\widetilde{P}(k_1) \cdot \frac{1}{2} \cdot \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} e^{3i\beta} \bigg(\frac{i\beta}{1 - e^{i\beta}} \bigg)^2 \cdot \int_0^1 e^{-i\beta s}(-1 + s) ds \cdot \int_0^1 e^{-is\beta} e^{i(s-1)\beta k_1} ds d\beta.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\bigg| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,3}})(k_1, \beta) d\beta \bigg| \\
\leq&\frac{\big|\widetilde{P}(k_1)\big|}{2} \cdot \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \bigg| \bigg(\frac{i\beta}{1 - e^{i\beta}} \bigg)^2 - 1 + 1 \bigg| d\beta \\
\leq&\frac{1}{2} \cdot \frac{\gamma}{4\pi} \bigg(\frac{\pi}{2} \sqrt{1 + \frac{\pi^2}{4}} \cdot \pi^2 + 2\pi \bigg) \left| \widetilde{P}(k_1) \right|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B}_2(\alpha, \beta)$.

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_2})(k_1, \beta) \\
=&-\frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{j_1! j_2! j_3!} \\
&\cdot \mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1} \phi(\alpha)^{j_2} \int_0^1 e^{-i\beta s} \phi(\alpha+\beta(-1+s))^{j_3} (-1+s) \phi'(\alpha+\beta(-1+s)) ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)^n \Bigg)(k_1) \\
=&-\frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{j_1! j_2! j_3!} \\
&\cdot \sum_{k_2, \ldots, k_{j_1+j_2+n+1} \in \mathbb{Z}} \prod_{d=1}^{j_1} \mathcal{F}(\phi(\alpha-\beta))(k_d - k_{d+1}) \\
&\cdot \prod_{d=1}^{j_2} \mathcal{F}(\phi)(k_{j_1+d} - k_{j_1+d+1}) \\
&\cdot \prod_{d=1}^{n} \mathcal{F}\left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)(k_{j_1+j_2+d} - k_{j_1+j_2+d+1}) \\
&\cdot \mathcal{F}\left(\int_0^1 e^{-i\beta s} \phi(\alpha+\beta(-1+s))^{j_3} (-1+s) \phi'(\alpha+\beta(-1+s)) ds \right)(k_{j_1+j_2+n+1}).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_2})(k_1, \beta) \\
=&-\frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{j_1! j_2! j_3!} \\
&\cdot \sum_{k_2, \ldots, k_{j_1+j_2+n+1} \in \mathbb{Z}} \prod_{d=1}^{j_1+j_2} \mathcal{F}(\phi)(k_d - k_{d+1}) \\
&\cdot \prod_{d=1}^{n} \left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}} \int_0^1 e^{-is\beta} e^{i(s-1)\beta(k_{j_1+j_2+d} - k_{j_1+j_2+d+1})} ds \right) \\
&\cdot \prod_{d=1}^{n} P(k_{j_1+j_2+d} - k_{j_1+j_2+d+1}) \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot e^{-i\beta(k_1 - k_{j_1+1})} \\
&\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s) k_{j_1+j_2+n+1}} (-1+s) ds \cdot \mathcal{F}(\phi^{j_3} \phi')(k_{j_1+j_2+n+1}).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta\\
=&-\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\\
&\cdot\prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\\
&\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\prod_{d=1}^{n}\left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\right)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}e^{-i\beta(k_1-k_{j_1+1})}\\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)dsd\beta.
\end{aligned}
$$

By Lemma 17,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta\right|\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\\
&\cdot\prod_{d=1}^{n}\left|P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\right|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_3}(\alpha,\beta)$. Let $\widetilde{B_3} = \sum_{j=1}^{3} \widetilde{B_{3,j}}$, where

$$\widetilde{B_{3,1}}(\alpha,\beta) = \frac{1}{2} \sum_{\substack{j_1=j_2=0 \\ j_3+j_4+n\geq 2}} (n+1) \frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s) ds$$
$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i^m}{m!} e^{-i\alpha} \frac{i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n$$
$$\widetilde{B_{3,2}}(\alpha,\beta) = \frac{1}{2} \sum_{\substack{j_1+j_2\geq 1 \\ j_3+j_4+n\geq 1}} (n+1) \frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s) ds$$
$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i^m}{m!} e^{-i\alpha} \frac{i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n$$
$$\widetilde{B_{3,3}}(\alpha,\beta) = \frac{1}{2}\cdot 2\cdot \frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \int_0^1 e^{-i\beta s} ds \int_0^1 e^{i\beta s}(-1+s) ds \sum_{m=2}^{\infty} \frac{-i^m}{m!} e^{-i\alpha} \frac{i\beta}{1-e^{-i\beta}}$$
$$\cdot \int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds.$$

First, let us calculate the Fourier modes of $\widetilde{B_{3,1}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta) = \frac{1}{2} \sum_{j_3+j_4+n\geq 2} (n+1) \frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \frac{i^{j_3+j_4}(-1)^{j_3}}{j_3!j_4!}$$
$$\cdot \mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s) ds$$
$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i\beta(s-1)} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n\Bigg)(k_1).$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Big(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n\Big)(k_1)\\
=&\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}\mathcal{F}\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)(k_d-k_{d+1})\\
&\cdot\mathcal{F}\Big(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\Big)(k_{n+1}-k_{n+2})\\
&\cdot\mathcal{F}\Big(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\Big)(k_{n+2})\\
=&\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}\Big(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\sum_{m=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_d-k_{d+1})\Big)\\
&\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\int_0^1 e^{-i\beta s}e^{i(k_{n+1}-k_{n+2})\beta(-1+s)}ds\\
&\cdot\mathcal{F}(\phi^{j_4})(k_{n+2})\int_0^1 e^{i\beta s}(-1+s)e^{ik_{n+2}\beta(-1+s)}ds\\
=&\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}\Big(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Big)\prod_{d=1}^{n}Q(k_d-k_{d+1})\\
&\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\mathcal{F}(\phi^{j_4})(k_{n+2})\cdot\int_0^1 e^{-i\beta s}e^{i(k_{n+1}-k_{n+2})\beta(-1+s)}ds\\
&\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik_{n+2}\beta(-1+s)}ds\\
=&\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\widetilde{I}_n(k_1,\dots,k_{n+2},\beta)\prod_{d=1}^{n}Q(k_d-k_{d+1})\\
&\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\mathcal{F}(\phi^{j_4})(k_{n+2}),
\end{aligned}
$$

where

$$
\begin{aligned}
&\widetilde{I}_n(k_1,\dots,k_{n+2},\beta)\\
=&\prod_{d=1}^{n}\Big(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Big)\cdot e^{i\beta p}\\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)(k_{n+1}-k_{n+2})}ds\int_0^1 e^{i\beta s}e^{ik_{n+2}\beta(-1+s)}(-1+s)ds.
\end{aligned}
\tag{10.5}
$$

Then

$$\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta)d\beta\\
=&\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{i^{j_3+j_4}(-1)^{j_3}}{j_3!j_4!}\sum_{k_2,\ldots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\\
&\cdot\mathcal{F}(\phi^{j_4})(k_{n+2})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{I}_n(k_1,\ldots,k_{n+2},\beta)\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}d\beta.
\end{aligned}$$

Using an argument similar to Lemma 17, we obtain

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}\\
&\cdot\sum_{k_2,\ldots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\right|\left|\mathcal{F}(\phi^{j_4})(k_{n+2})\right|.
\end{aligned}$$

Next, let us calculate the Fourier modes of $\widetilde{B_{3,2}}(\alpha,\beta)$.

$$\begin{aligned}
&\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\\
&\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n\Bigg)(k_1).
\end{aligned}$$

We can write

$$
\begin{aligned}
&\mathcal{F}\bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\bigg)^n\bigg)(k_1)\\
=&\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1})\\
&\cdot\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\\
&\cdot\mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\bigg)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\\
&\cdot\mathcal{F}\bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\bigg)(k_{j_1+j_2+n+2})\\
=&\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\\
&\cdot\sum_{m=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\bigg)\\
&\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})}ds\\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+2}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2})\\
=&\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\widetilde{I}_n(k_1,\dots,k_{n+2},\beta)\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d})\\
&\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\cdot\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2}),
\end{aligned}
$$

where $\widetilde{I_n}$ is defined in (10.5). Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}$$

$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d})$$

$$\cdot\,\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\cdot\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2})$$

$$\cdot\,\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\widetilde{I_n}(k_1,\ldots,k_{n+2},\beta)d\beta.$$

Using an argument similar to Lemma 17, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}$$

$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d})|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2})\right|.$$

Next, let us calculate the Fourier modes of $\widetilde{B_{3,3}}(\alpha,\beta)$.

$$\begin{aligned}\mathcal{F}(\widetilde{B_{3,3}})(k_1,\beta) =&\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\\ &\cdot\mathcal{F}\Bigg(\sum_{k=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1)\\ =&\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\\ &\cdot\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\cdot\sum_{m=2}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_1).\end{aligned}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})d\beta = \widetilde{Q}(k_1)\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}e^{-2i\beta}\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^3\int_0^1 e^{-i\beta s}ds \cdot\int_0^1 e^{i\beta s}(-1+s)ds\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}dsd\beta.$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})d\beta\right| \le \left|\widetilde{Q}(k_1)\right|\cdot\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\left|\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^3-1\right|d\beta+2\pi\right) \le \left|\widetilde{Q}(k_1)\right|\cdot\frac{\gamma}{4\pi}\left(3\left(\frac{\pi}{2}\right)^2\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right).$$

Next, let us calculate the Fourier modes of $\widetilde{B_4}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_4})(k_1,\beta) = \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\ge1}(n+1)\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\phi(\alpha+(s-1)\beta)^m ds\right)^n\Bigg)(k_1).$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i^m}{m!}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\phi(\alpha+(s-1)\beta)^m ds\Bigg)^n\Bigg)(k_1) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \\
&\cdot\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\Bigg)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \\
&\cdot\mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\Bigg)(k_{j_1+j_2+n+2}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}} e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \\
&\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg)\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \\
&\cdot\int_0^1 e^{-i\beta s}e^{i(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\beta(-1+s)}ds\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+2}}(-1+s)ds \\
&\cdot\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\widetilde{I}_n(k_1,\ldots,k_{n+2},\beta)\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \\
&\cdot\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \\
&\cdot\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2}).
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta &= \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \\
&\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \\
&\cdot \prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \\
&\cdot \mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\widetilde{I_n}(k_1,\ldots,k_{n+2},\beta)d\beta,
\end{aligned}
$$

where $\widetilde{I_n}$ is defined in (10.5). Using an argument similar to Lemma 17, we obtain

$$
\begin{aligned}
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta\right| &\leq \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \\
&\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \\
&\cdot \left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\right|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_5}(\alpha,\beta)$. We will write $\widetilde{B_5} = \sum_{j=1}^{3} \widetilde{B_{5,j}}$, where

$$\widetilde{B_{5,1}}(\alpha,\beta) = \frac{1}{2} \sum_{\substack{j_1=j_2=0 \\ j_3+n\geq 2}} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds$$
$$\cdot \Big(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Big)^n$$
$$\widetilde{B_{5,2}}(\alpha,\beta) = \frac{1}{2} \sum_{\substack{j_1+j_2\geq 1 \\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds$$
$$\cdot \Big(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Big)^n$$
$$\widetilde{B_{5,3}}(\alpha,\beta) = \frac{1}{2} \cdot \frac{i\beta}{1-e^{-i\beta}} \int_0^1 e^{-i\beta s}(-1+s)ds$$
$$\cdot \sum_{m=2}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds.$$

First, let us calculate the Fourier modes of $\widetilde{B_{5,1}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta) = \frac{1}{2} \sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_3}(-1)^{j_3}}{j_3!}$$
$$\cdot \mathcal{F}\Big(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds$$
$$\cdot \Big(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Big)^n\Big)(k_1).$$

We can write

$$
\begin{aligned}
&\mathcal{F}\biggl(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\biggl(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\biggr)^n\biggr)(k_1) \\
=&\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\mathcal{F}\biggl(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\biggr)(k_d-k_{d+1}) \\
&\cdot\mathcal{F}\biggl(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\biggr)(k_{n+1}) \\
=&\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\biggl(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_d-k_{d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\biggr) \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}) \\
=&\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1})\prod_{d=1}^{n}\biggl(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\biggr) \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds \\
=&\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}I_{n,1}(k_1,\ldots,k_{n+1},\beta)\cdot\prod_{d=1}^{n}Q(k_d-k_{d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{n+1})
\end{aligned}
$$

where

$$
\begin{aligned}
I_{n,1}(k_1,\ldots,k_{n+1},\beta)=&\prod_{d=1}^{n}\biggl(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\biggr) \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta=\frac{1}{2}\sum_{j_3+n\geq 2}\frac{i^{j_3}(-1)^{j_3}}{j_3!} \\
&\cdot\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{n+1})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,1}(k_1,\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta.
\end{aligned}
$$

Using an argument similar to Lemma 17, we obtain

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_{n,1}(k_1,\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \\
\leq&\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \\
=&C_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_{5,2}}(\alpha,\beta)$.

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta)=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!} \\
&\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\quad\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\Bigg)(k_1) \\
&=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \\
&\quad\cdot\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\quad\cdot\mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \\
&=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \\
&\quad\cdot\prod_{d=1}^{n}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\quad\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{k_1+k_2+d+1})}ds\bigg) \\
&\quad\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \\
&=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\quad\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \\
&\quad\cdot e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\bigg) \\
&\quad\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds \\
&=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\quad\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\cdot I_{n,2}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
&I_{n,2}(k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n+1}, \beta) \\
=&e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \prod_{d=1}^{n} \left(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} \int_0^1 e^{is\beta} e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})} ds \right) \\
&\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}} (-1+s) ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,2}})(k_1, \beta) d\beta = \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1! j_2! j_3!} \\
&\cdot \sum_{k_2, \ldots, k_{j_1+j_2+n+1} \in \mathbb{Z}} \prod_{d=1}^{j_1+j_2} \mathcal{F}(\phi)(k_d - k_{d+1}) \prod_{d=1}^{n} Q(k_{j_1+j_2+d} - k_{j_1+j_2+d+1}) \\
&\cdot \mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \\
&\cdot \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} I_{n,2}(k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n+1}, \beta) \frac{i\beta}{1-e^{-i\beta}} d\beta.
\end{aligned}
$$

Using an argument similar to Lemma 17, we obtain

$$
\begin{aligned}
&\left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} I_{n,2}(k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n+1}, \beta) \frac{i\beta}{1-e^{-i\beta}} d\beta \right| \\
\leq& \frac{\gamma}{4\pi} \left((n+1) \left(\frac{\pi}{2} \right)^n \cdot \frac{1}{2} \sqrt{1 + \frac{\pi^2}{4}} \cdot \pi^2 + 2\pi \right) \\
=& C_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,2}})(k_1, \beta) d\beta \right| \leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} \\
&\cdot \sum_{k_2, \ldots, k_{j_1+j_2+n+1} \in \mathbb{Z}} \prod_{d=1}^{j_1+j_2} |\mathcal{F}(\phi)(k_d - k_{d+1})| \prod_{d=1}^{n} |Q(k_{j_1+j_2+d} - k_{j_1+j_2+d+1})| \\
&\cdot \left| \mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \right|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_{5,3}}(\alpha,\beta)$. We can write

$$\begin{aligned}
\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta) &= \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds \\
&\cdot\mathcal{F}\Big(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)(k_1) \\
=&\frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds\cdot\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\cdot\widetilde{Q}(k_1).
\end{aligned}$$

Then

$$\begin{aligned}
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta)d\beta\right| &\le \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\Big(\frac{-i\beta}{1-e^{-i\beta}}\Big)^2-1+1\right|d\beta \\
&\le \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\frac{\gamma}{4\pi}\Big(2\Big(\frac{\pi}{2}\Big)\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\Big).
\end{aligned}$$

Next, let us calculate the Fourier modes of $\widetilde{B_6}(\alpha,\beta)$. We will write $\widetilde{B_6}=\sum_{j=1}^{3}\widetilde{B_{6,j}}$, where

$$\begin{aligned}
\widetilde{B_{6,1}}(\alpha,\beta) &= \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\ge 2}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n \\
\widetilde{B_{6,2}}(\alpha,\beta) &= \frac{1}{2}\sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n \\
\widetilde{B_{6,3}}(\alpha,\beta) &= \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds \\
&\cdot\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds.
\end{aligned}$$

First, let us calculate the Fourier modes of $\widetilde{B_{6,1}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)=\frac{1}{2}\sum_{j_3+n\geq 2}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_3}}{j_3!}\mathcal{F}\bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds$$
$$\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\bigg)(k_1).$$

We can write

$$\begin{aligned}
&\mathcal{F}\bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\bigg)(k_1)\\
=&\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_d-k_{d+1})\\
&\cdot\mathcal{F}\bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\bigg)(k_{n+1})\\
=&\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\frac{i^m}{m!}\cdot\mathcal{F}(\phi^m)(k_d-k_{d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\bigg)\\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{n+1})\\
=&\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}Q(k_d-k_{d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\bigg)\\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{n+1})\\
=&\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1})I_{n,3}(k_1\dots,k_{n+1},\beta),
\end{aligned}$$

where

$$\begin{aligned}
&I_{n,3}(k_1\dots,k_{n+1},\beta)\\
=&\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\bigg)\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds.
\end{aligned}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)d\beta=\frac{1}{2}\sum_{j_3+n\geq 2}\frac{i^{j_3}}{j_3!}$$
$$\cdot\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,3}(k_1\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta.$$

Using an argument similar to Lemma 17, we obtain

$$\begin{aligned}&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,3}(k_1\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right|\\ \leq&\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\\ =&C_n.\end{aligned}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}$$
$$\cdot\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|.$$

Next, let us calculate the Fourier modes of $\widetilde{B_{6,2}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}$$
$$\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1).$$

We can write

$$
\begin{aligned}
&\mathcal{F}\bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\bigg)(k_1) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \\
&\cdot\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\mathcal{F}\bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \\
&\cdot\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds \\
&\cdot\sum_{d=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\bigg) \\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\cdot I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
&I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta)=e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \\
&\cdot\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\bigg) \\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\\
&\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\\
&\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\dots,k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta.
\end{aligned}
$$

Using an argument similar to Lemma 17, we obtain

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\dots,k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right|\\
\leq&\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\\
=&C_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\\
&\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})|\\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_{6,3}}(\alpha,\beta)$.

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)=\frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds\\
&\cdot\mathcal{F}\left(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&\mathcal{F}\bigg(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_1)\\
=&\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\sum_{m=2}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_1)\\
=&\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\cdot\widetilde{Q}(k_1).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)d\beta\right| &\leq \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^2-1+1\right|d\beta\\
&\leq \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\frac{\gamma}{4\pi}\left(2\cdot\left(\frac{\pi}{2}\right)\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right).
\end{aligned}
$$

Next, let us calculate the Fourier modes of $\widetilde{B_7}(\alpha,\beta)$.

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_7})(k_1,\beta)=\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\\
&\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\Bigg)(k_1).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n\Bigg)(k_1) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \\
&\cdot\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\Bigg)(k_{j_1+j_2+n+1}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \\
&\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\frac{i^m}{m!}\cdot\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1}) \\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
&\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})I_{n,5}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
&I_{n,5}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta) \\
=&e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) \\
&\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds.
\end{aligned}
$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_7})(k_1,\beta)d\beta=\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}$$
$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})$$
$$\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,5}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta.$$

Using an argument similar to Lemma 17, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,5}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right|$$
$$\leq\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)$$
$$=C_n.$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_7})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}$$
$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Next, let us calculate the Fourier modes of $\widetilde{B_8}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_8})(k_1,\beta)=\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}$$
$$\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1).$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n\Bigg)(k_1)\\
=&\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1})\\
&\cdot\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\\
&\cdot\mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\Bigg)(k_{j_1+j_2+n+1})\\
=&\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\\
&\cdot\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\frac{i^m}{m!}\cdot\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg)\\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\\
=&\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\\
&\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\cdot I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
&I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta)=e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\\
&\cdot\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg)\\
&\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds.
\end{aligned}
$$

Then

$$\begin{aligned}
\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta = & \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!} \\
& \cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \\
& \cdot \mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta.
\end{aligned}$$

Using an argument similar to Lemma 17, we obtain

$$\begin{aligned}
& \left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \\
\leq & \frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \\
= & C_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta\right| \leq & \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!} \\
& \cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \\
& \cdot \left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.
\end{aligned}$$

Next, let us calculate the Fourier modes of $\widetilde{B_{13}}(\alpha,\beta)$. We can write

$$\mathcal{F}(\widetilde{B_{13}})(k_1,\beta) = \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)\cdot\mathcal{F}(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2})(k_1)$$

$$= \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}} \prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2-1}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1})$$

$$\cdot\,\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+j_2})$$

$$= \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}$$

$$\cdot \prod_{d=1}^{j_1+j_2-1}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2})$$

$$= \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}} I_{n,7}(k_{j_1+1},\ldots,k_{j_1+j_2},\beta)\cdot\prod_{d=1}^{j_1+j_2-1}\mathcal{F}(\phi)(k_d-k_{d+1})$$

$$\cdot\,\mathcal{F}(\phi)(k_{j_1+j_2}),$$

where

$$I_{n,7}(k_{j_1+1},\ldots,k_{j_1+j_2},\beta) = \left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}.$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta = \sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}} \prod_{d=1}^{j_1+j_2-1}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,7}(k_{j_1+1},\ldots,k_{j_1+j_2},\beta)d\beta.$$

We note that for $l\in\mathbb{Z}$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{-i\beta\cdot l}}{1-e^{-i\beta}}d\beta = 1_{l\leq 0}(l)-1_{l\geq 1}(l).$$

For proof, see (5.9) in Gancedo et al. (2023a). Then

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_{n,7}(k_{j_1+1},\dots,k_{j_1+j_2},\beta)d\beta\right|\\
\le&\frac{\gamma}{4\pi}\left|\int_{-\pi}^{\pi}\frac{(-1)^{j_2}e^{i\beta}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}(1+e^{i\beta})}{2(-1+e^{i\beta})}d\beta\right|\\
&+\frac{1}{2}\frac{\gamma}{4\pi}\left|\int_{-\pi}^{\pi}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}d\beta\right|\\
\le&\frac{\gamma}{4\pi}\cdot\frac{1}{2}\left|\int_{-\pi}^{\pi}\frac{e^{i\beta(1-(k_{j_1+1}-k_{j_1+j_2})-k_{j_1+j_2})}}{1-e^{i\beta}}d\beta+\int_{-\pi}^{\pi}\frac{e^{i\beta(2-(k_{j_1+1}-k_{j_1+j_2})-k_{j_1+j_2})}}{1-e^{i\beta}}d\beta\right|\\
&+\frac{\gamma}{4\pi}\cdot\pi\\
\le&\frac{\gamma}{4\pi}(1+\pi).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right|\\
\le&\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\ge 2}\frac{1}{j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi)(k_{j_1+j_2})|\,.
\end{aligned}$$

10.2. Estimating $\|U_{\ge 2}\|_{\mathcal{F}^{0,1}_\nu}$

We prove the following estimate for $\|U_{\ge 2}\|_{\mathcal{F}^{0,1}_\nu}$.

Lemma 18.

$$\|U_{\ge 2}\|_{\mathcal{F}^{0,1}_\nu}\le D_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\,\|\phi\|^2_{\mathcal{F}^{0,1}_\nu}+D_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\,\|\phi\|_{\mathcal{F}^{0,1}_\nu}\,\|\phi'\|_{\mathcal{F}^{0,1}_\nu}\,,$$

where D_1 and D_2 are monotone increasing functions of $\|\phi\|_{\mathcal{F}^{0,1}_\nu}$.

Before commencing the proof of Lemma 18, let us introduce the setup for the proof. For ease of notation, we define the l^1_ν norm of a sequence $a = a(k)$ defined on $\mathbb{Z}$ by

$$\|a\|_{l^1_\nu} = \sum_{k \in \mathbb{Z}} e^{\nu(t)|k|} |a(k)| .$$

The following estimate of the l^1_ν norm of the convolution is frequently used.

Proposition 19. *If $a_1, \dots, a_n$ are sequences on $\mathbb{Z}$ whose l^1_ν norms are finite, then*

$$\|a_1 * \cdots * a_n\|_{l^1_\nu} \leq \prod_{j=1}^{n} \|a_j\|_{l^1_\nu} .$$

Proof. It suffices to show the case of $n = 2$ because the general case follows from repeated applications of this case. Indeed, we have

$$\begin{aligned}
\|a * b\|_{l^1_\nu} &= \sum_{k \in \mathbb{Z}} e^{\nu(t)|k|} |(a * b)(k)| \\
&\leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} e^{\nu(t)|k-j|} e^{\nu(t)|j|} |a(k-j)| |b(j)| \\
&= \sum_{j \in \mathbb{Z}} e^{\nu(t)|j|} |b(j)| \sum_{k \in \mathbb{Z}} e^{\nu(t)|k-j|} |a(k-j)| \\
&= \|a\|_{l^1_\nu} \|b\|_{l^1_\nu} ,
\end{aligned}$$

as needed. ■

We note that

$$\|P\|_{l^1_\nu} \leq \sum_{m=1}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1, \tag{10.6}$$

$$\left\|\widetilde{P}\right\|_{l^1_\nu} \leq \sum_{m=2}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1, \tag{10.7}$$

$$\|Q\|_{l^1_\nu} \leq \sum_{m=1}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1, \tag{10.8}$$

$$\left\|\widetilde{Q}\right\|_{l^1_\nu} \leq \sum_{m=2}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1. \tag{10.9}$$

To begin the proof of Lemma 18, we observe that

$$\|U_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} \leq \sum_{j=1}^{8} \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_j}(\alpha,\beta) d\beta \right\|_{\mathcal{F}^{0,1}_\nu} + \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{13}}(\alpha,\beta) d\beta \right\|_{\mathcal{F}^{0,1}_\nu}.$$

This means that it suffices to estimate each of the $\mathcal{F}^{0,1}_\nu$ norms on the right hand side. By Proposition 19 and (10.6), we obtain

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta) d\beta \right| \\
&\leq \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1!j_2!} (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |\mathcal{F}(\phi)|)(k_1) \\
&= \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1!j_2!} \| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |\mathcal{F}(\phi)| \|_{l^1_\nu} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1!j_2!} \|\phi\|^{j_1+j_2+1}_{\mathcal{F}^{0,1}_\nu} \|P\|^n_{l^1_\nu} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1!j_2!} \|\phi\|^{j_1+j_2+1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n.
\end{aligned}$$

By Propositions 2 and 19 and (10.6), we obtain

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta) d\beta \right| \\
&\leq \sum_{j_1+j_2+j_3\geq 1} \frac{C_1}{2j_1!j_2!j_3!} \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \\
&= \sum_{j_1+j_2+j_3\geq 1} \frac{C_1}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^1_\nu} \\
&\leq \sum_{j_1+j_2+j_3\geq 1} \frac{C_1}{2j_1 j_2!j_3!} \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1).
\end{aligned}$$

By (10.7), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,3}})(k_1,\beta) d\beta \right| \leq \frac{C_2}{2} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1).$$

By Propositions 2 and 19 and (10.6), we obtain

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!} \\
&\cdot\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l^1_\nu} \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}.
\end{aligned}
$$

By Propositions 2 and 19 and (10.8), we obtain

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta)d\beta\right| \\
\leq&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\cdot\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}(|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|)(k_1) \\
\leq&\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}\left\|Q\right\|^n_{l^1_\nu}\left\|\phi\right\|^{j_3+j_4}_{\mathcal{F}^{0,1}_\nu} \\
\leq&\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\left\|\phi\right\|^{j_3+j_4}_{\mathcal{F}^{0,1}_\nu}.
\end{aligned}
$$

Next, recalling that

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{\widetilde{C_n}}{j_1!j_2!j_3!j_4!} \\
&\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\prod_{d=1}^{n}\left|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2})\right|,
\end{aligned}
$$

we obtain by Propositions 2 and 19 and (10.8) that

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right| \\
\leq &\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\cdot\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \\
&\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|)(k_1) \\
\leq &\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \\
&\cdot\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|\right\|_{l^1_\nu} \\
\leq &\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\|\phi\|^{j_1+j_2+j_3+j_4}_{\mathcal{F}^{0,1}_\nu}\|Q\|^n_{l^1_\nu} \\
\leq &\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}\|\phi\|^{j_1+j_2+j_3+j_4}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n.
\end{aligned}
$$

By (10.9), we have

$$
\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})(k_1,\beta)d\beta\right|\leq C_3(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-\|\phi\|_{\mathcal{F}^{0,1}_\nu}-1).
$$

By Propositions 2 and 19 and (10.8), we obtain

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \\
&\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|)(k_1) \\
=&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \\
&\cdot\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|\right\|_{l^1_\nu} \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3+j_4}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}\left\|Q\right\|_{l^1_\nu}^n \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3+j_4}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-1)^n.
\end{aligned}
$$

By Propositions 2 and 19 and (10.8), we obtain

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\left\|Q\right\|_{l^1_\nu}^n\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_3} \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_3}.
\end{aligned}
$$

By Propositions 2 and 19 and (10.8), we obtain

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!} \\
&\cdot\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\left\|Q\right\|_{l^1_\nu}^n\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3} \\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}.
\end{aligned}$$

By (10.9), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta)d\beta\right|\leq\frac{C_1}{2}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}-1).$$

By Propositions 2 and 19 and (10.8), we have

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\left\|Q\right\|_{l^1_\nu}^n\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_3} \\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_3}.
\end{aligned}$$

By Propositions 2 and 19 and (10.8), we obtain

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta\right| \\
\le&\frac{1}{2}\sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \\
\le&\frac{1}{2}\sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}\left\|Q\right\|_{l^1_\nu}^{n} \\
\le&\frac{1}{2}\sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n}.
\end{aligned}$$

By (10.9), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)d\beta\right|\le\frac{C_1}{2}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-\|\phi\|_{\mathcal{F}^{0,1}_\nu}-1).$$

By Propositions 2 and 19 and (10.8), we obtain

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{7}})(k_1,\beta)d\beta\right| \\
\le&\frac{1}{2}\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{j_1!j_2!j_3!} \\
&\cdot\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \\
\le&\frac{1}{2}\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}\left\|Q\right\|_{l^1_\nu}^{n}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \\
\le&\frac{1}{2}\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}.
\end{aligned}$$

By Propositions 2 and 19 and (10.8), we obtain

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta\right| \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!} \\
&\cdot\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}\left\|Q\right\|_{l^1_\nu}^{n}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}.
\end{aligned}$$

Lastly, recalling that

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right| \\
\leq&\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi)(k_{j_1+j_2})\right|,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right| \\
\leq&\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|)(k_1) \\
\leq&\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2}.
\end{aligned}$$

This completes the proof of Lemma 18.

10.3. Estimating $\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}$

We prove the following estimate for $\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}$, $s>0$.

Lemma 20. *For $s > 0$,*

$$\begin{aligned}\|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & F_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + F_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|^2_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\ & + F_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + F_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}},\end{aligned}$$

where F_1, F_2, F_3, and F_4 are monotone increasing functions of $\|\phi\|_{\mathcal{F}_\nu^{0,1}}$.

Before commencing the proof of Lemma 20, let us introduce the setup for the proof. For ease of notation, we define the l_ν^s norm of a sequence $a = a(k)$ defined on $\mathbb{Z}$ by

$$\|a\|_{l_\nu^s} = \sum_{k \neq 0} e^{\nu(t)|k|} |k|^s |a_k|.$$

The following estimate of the l_ν^s norm of the convolution is frequently used.

Proposition 21. *Let $s > 0$. If $a_1, \ldots, a_n$ are sequences on $\mathbb{Z}$ whose l_ν^s norms are finite, then*

$$\|a_1 * \cdots * a_n\|_{l_\nu^s} \leq b(n,s) \sum_{j=1}^n \|a_j\|_{l_\nu^s} \prod_{\substack{k=1 \\ k \neq j}}^n \|a_k\|_{l_\nu^1}.$$

Proof. We note that for any $k_1, \ldots, k_n \in \mathbb{Z}$,

$$|k_1|^s \leq b(n,s)(|k_1 - k_2|^s + |k_2 - k_3|^s + \cdots + |k_{n-1} - k_n|^s + |k_n|^s),$$

which follows from convexity of the function $|\cdot|^s$ for $s \geq 1$ and the triangle inequality for $0 < s < 1$.

Then, using (10.4), we obtain

$$\begin{aligned}
\|a_1 * \cdots * a_n\|_{l^s_\nu} &= \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} |k_1|^s |(a_1 * \cdots * a_n)(k_1)| \\
&\leq \sum_{k_1\in\mathbb{Z}} \sum_{k_2,\ldots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} |k_1|^s \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| \, |a_n(k_n)| \\
&\leq \sum_{j=2}^{n} \sum_{k_1\in\mathbb{Z}} \sum_{k_2,\ldots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s) \, |k_{j-1} - k_j|^s \\
&\quad \cdot \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| \, |a_n(k_n)| \\
&\quad + \sum_{k_1\in\mathbb{Z}} \sum_{k_2,\ldots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s) \, |k_n|^s \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| \, |a_n(k_n)| \, .
\end{aligned}$$

For $j \in \{2, \ldots, n\}$, we have

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} \sum_{k_2,\ldots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s) \, |k_{j-1} - k_j|^s \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| \, |a_n(k_n)| \\
\leq& b(n,s) \sum_{k_n\in\mathbb{Z}} |a_n(k_n)| \, e^{\nu(t)|k_n|} \sum_{k_{n-1}\in\mathbb{Z}} |a_{n-1}(k_{n-1} - k_n)| \, e^{\nu(t)|k_{n-1}-k_n|} \\
&\cdots\cdot \sum_{k_{j-1}\in\mathbb{Z}} |k_{j-1} - k_j|^s \, |a_{j-1}(k_{j-1} - k_j)| \, e^{\nu(t)|k_{j-1}-k_j|} \\
&\cdots\cdot \sum_{k_2\in\mathbb{Z}} |a_2(k_2 - k_3)| \, e^{\nu(t)|k_2-k_3|} \sum_{k_1\in\mathbb{Z}} |a_1(k_1 - k_2)| \, e^{\nu(t)|k_1-k_2|}.
\end{aligned}$$

Changing the summation variables

$$\begin{aligned}
k_1' &= k_1 - k_2 \\
k_2' &= k_2 - k_3 \\
&\vdots \\
k_{n-1}' &= k_{n-1} - k_n
\end{aligned}$$

in that order, we obtain

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}}\sum_{k_2,\ldots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s)\left|k_{j-1}-k_j\right|^s \prod_{d=1}^{n-1}\left|a_d(k_d-k_{d+1})\right|\left|a_n(k_n)\right| \\
\leq & b(n,s)\left\|a_{j-1}\right\|_{l^s_\nu} \prod_{\substack{k=1\\k\neq j}}^{n}\left\|a_k\right\|_{l^1_\nu}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}}\sum_{k_2,\ldots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s)\left|k_n\right|^s \prod_{d=1}^{n-1}\left|a_d(k_d-k_{d+1})\right|\left|a_n(k_n)\right| \\
\leq & b(n,s)\left\|a_n\right\|_{l^s_\nu} \prod_{k=1}^{n-1}\left\|a_k\right\|_{l^1_\nu}.
\end{aligned}
$$

This completes the proof. ■

We note that

$$
\begin{aligned}
\|P\|_{l^s_\nu} &= \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s|P(k_1)| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \sum_{m=1}^{\infty}\frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \\
&= \sum_{m=1}^{\infty}\frac{1}{m!}\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}, && (10.10) \\
\left\|\widetilde{P}\right\|_{l^s_\nu} &= \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\left|\widetilde{P}(k_1)\right| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \sum_{m=2}^{\infty}\frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \\
&= \sum_{m=2}^{\infty}\frac{1}{m!}\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}, && (10.11) \\
\|Q\|_{l^s_\nu} &= \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s|Q(k_1)| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \sum_{m=1}^{\infty}\frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \\
&= \sum_{m=1}^{\infty}\frac{1}{m!}\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}, && (10.12) \\
\left\|\widetilde{Q}\right\|_{l^s_\nu} &= \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\left|\widetilde{Q}(k_1)\right| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \sum_{m=2}^{\infty}\frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \\
&= \sum_{m=2}^{\infty}\frac{1}{m!}\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}. && (10.13)
\end{aligned}
$$

To begin the proof of Lemma 20, we observe that

$$\|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq \sum_{j=1}^{8} \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_j}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} + \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{13}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}}.$$

This means that it suffices to estimate each of the $\dot{\mathcal{F}}_\nu^{s,1}$ norms on the right hand side. By Proposition 21 and (10.10), we obtain

$$\begin{aligned}
&\sum_{k_1\neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta) d\beta \right| \\
&\leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|} |k_1|^s \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1! j_2!} (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P|)(k_1) \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1! j_2!} \| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| \|_{l_\nu^s} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1! j_2!} b(j_1+j_2+n+1, s) \\
&\quad \cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|P\|_{l_\nu^1}^n (j_1+j_2+1) + \|P\|_{l_\nu^s} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+1} \|P\|_{l_\nu^1}^{n-1} \cdot n \right) \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1! j_2!} b(j_1+j_2+n+1, s) \cdot \\
&\quad \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n (j_1+j_2+1) \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \cdot n \Bigg).
\end{aligned}$$

By Proposition 21 and (10.10), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{1,2}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta\right| \\
\le &\sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \sum_{j_1+j_2+j_3\ge 1}\frac{C_1}{2j_1!j_2!j_3!}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \\
\le &\sum_{j_1+j_2+j_3\ge 1}\frac{C_1}{2j_1!j_2!j_3!}b(j_1+j_2+2,s)\Big(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\|P\|_{l_\nu^1}\cdot(j_1+j_2) \\
&+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|P\|_{l_\nu^1}+\|P\|_{l_\nu^s}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\Big) \\
\le &\sum_{j_1+j_2+j_3\ge 1}\frac{C_1}{2j_1!j_2!j_3!}b(j_1+j_2+2,s)\Big(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)\cdot(j_1+j_2) \\
&+b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}\cdot j_3\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}\Big).
\end{aligned}$$

By (10.11), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{1,3}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,3}})(k_1,\beta)d\beta\right| \\
\le &\frac{C_1}{2}\left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}.
\end{aligned}$$

By Proposition 21 and (10.10), we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\left|k_1\right|^s \left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta\right| \\
&\leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\left|k_1\right|^s \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \\
&\quad \prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\cdot\prod_{d=1}^{n}\left|P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\right|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right| \\
&\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\|\left|\mathcal{F}(\phi)\right|*\cdots*\left|\mathcal{F}(\phi)\right|*\left|P\right|*\cdots*\left|P\right|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l^s_{\nu}} \\
&\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s) \\
&\quad\cdot\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}\left\|\phi\right\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_{\nu}}\left\|P\right\|^{n}_{l^1_{\nu}}\left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}^{0,1}_{\nu}}(j_1+j_2) \\
&\quad+\left\|P\right\|_{l^s_{\nu}}\left\|\phi\right\|^{j_1+j_2}_{\mathcal{F}^{0,1}_{\nu}}\left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}^{0,1}_{\nu}}\left\|P\right\|^{n-1}_{l^1_{\nu}}\cdot n \\
&\quad+\left\|\phi^{j_3}\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}\left\|\phi\right\|^{j_1+j_2}_{\mathcal{F}^{0,1}_{\nu}}\left\|P\right\|^{n}_{l^1_{\nu}}\Bigg) \\
&\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s) \\
&\quad\cdot\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}\left\|\phi\right\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_{\nu}}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_{\nu}}}-1)^n\left\|\phi'\right\|_{\mathcal{F}^{0,1}_{\nu}}(j_1+j_2) \\
&\quad+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|^{m-1}_{\mathcal{F}^{0,1}_{\nu}}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}\left\|\phi\right\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_{\nu}}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_{\nu}}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_{\nu}}}-1)^{n-1}\cdot n \\
&\quad+b(j_3+1,s)(\left\|\phi\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}\left\|\phi\right\|^{j_3-1}_{\mathcal{F}^{0,1}_{\nu}}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_{\nu}}\cdot j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}\left\|\phi\right\|^{j_3}_{\mathcal{F}^{0,1}_{\nu}}) \\
&\quad\cdot\left\|\phi\right\|^{j_1+j_2}_{\mathcal{F}^{0,1}_{\nu}}(e^{\left\|\phi\right\|_{\mathcal{F}^{0,1}_{\nu}}}-1)^n\Bigg).
\end{aligned}
$$

By Proposition 21 and (10.12), we obtain

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{3,1}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq &\sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta) d\beta \right| \\
\leq &\sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{j_3+j_4+n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} \\
&\cdot (|Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4})\right|)(k_1) \\
\leq &\frac{1}{2} \sum_{j_3+j_4+n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} \left\| |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4})\right| \right\|_{l_\nu^s} \\
\leq &\frac{1}{2} \sum_{j_3+j_4+n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} b(n+2,s) \Bigg(\|Q\|_{l_\nu^s} \|Q\|_{l_\nu^1}^{n-1} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n \\
&+ \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^{n} \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} + \left\|\phi^{j_4}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^{n} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \Bigg) \\
\leq &\frac{1}{2} \sum_{j_3+j_4+n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} b(n+2,s) \\
&\cdot \Bigg(\Bigg(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \Bigg) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3+j_4} \cdot n \\
&+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \\
&+ b(j_4,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \cdot j_4 \cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \Bigg).
\end{aligned}
$$

By Proposition 21 and (10.12), we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{3,2}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{k_1\neq 0} e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right| \\
\leq &\sum_{k_1\neq 0} e^{\nu(t)|k_1|}\left|k_1\right|^s\cdot\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \\
&\cdot\left(\left|\mathcal{F}(\phi)\right|*\cdots*\left|\mathcal{F}(\phi)\right|*\left|Q\right|*\cdots*\left|Q\right|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|\right)(k_1) \\
\leq &\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} \\
&\cdot\left\|\left|\mathcal{F}(\phi)\right|*\cdots*\left|\mathcal{F}(\phi)\right|*\left|Q\right|*\cdots*\left|Q\right|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|\right\|_{l_\nu^s} \\
\leq &\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}\cdot b(j_1+j_2+n+2,s) \\
&\cdot\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|Q\right\|_{l_\nu^1}^{n}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}}(j_1+j_2) \\
&+\left\|Q\right\|_{l_\nu^s}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}}\left\|Q\right\|_{l_\nu^1}^{n-1}\cdot n \\
&+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\left\|Q\right\|_{l_\nu^1}^{n}\left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} \\
&+\left\|\phi^{j_4}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\left\|Q\right\|_{l_\nu^1}^{n}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\Bigg) \\
\leq &\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}\cdot b(j_1+j_2+n+2,s) \\
&\cdot\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1}\left(e^{\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}}-1\right)^n(j_1+j_2) \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4}\left(e^{\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}}-1\right)^{n-1}\cdot n \\
&+b(j_3,s)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\cdot j_3\cdot\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4}\left(e^{\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}}-1\right)^n \\
&+b(j_4,s)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1}\cdot j_4\cdot\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}\left(e^{\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}}-1\right)^n\Bigg).
\end{aligned}
$$

By (10.13), we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{3,3}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
=&\sum_{k_1\neq 0} e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})(k_1,\beta)d\beta\right|\\
\leq& C_2\left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}.
\end{aligned}
$$

By Proposition 21 and (10.12), we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
=&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta\right|\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\cdot\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}\\
&\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|)(k_1)\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}\\
&\cdot\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|\right\|_{l_\nu^s}\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\|Q\|_{l_\nu^1}^n\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi^{j_4}\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(j_1+j_2)\\
&+\|Q\|_{l_\nu^s}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^{n-1}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi^{j_4}\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^n\left\|\phi^{j_4}\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\\
&+\left\|\phi^{j_4}\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^n\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\Bigg)\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3+j_4}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot(j_1+j_2)\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\cdot j_3\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\\
&+b(j_4+1,s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot j_4+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4}\right)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\Bigg).
\end{aligned}
$$

By Proposition 21 and (10.12), we obtain

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{5,1}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq & \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta) d\beta \right| \\
\leq & \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} \sum_{k_2,\ldots,k_{n+1} \in \mathbb{Z}} \prod_{d=1}^{n} |Q(k_d - k_{d+1})| \left| \mathcal{F}(\phi^{j_3})(k_{n+1}) \right| \\
\leq & \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} \left\| |Q| * \cdots * |Q| * \left| \mathcal{F}(\phi^{j_3}) \right| \right\|_{l_\nu^s} \\
\leq & \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} b(n+1,s) \left(\|Q\|_{l_\nu^s} \|Q\|_{l_\nu^1}^{n-1} \left\| \phi^{j_3} \right\|_{\mathcal{F}_\nu^{0,1}} \cdot n + \left\| \phi^{j_3} \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^{n} \right) \\
\leq & \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} b(n+1,s) \\
& \cdot \left(\left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \cdot n \right. \\
& \left. + b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n} \right).
\end{aligned}
$$

By Proposition 21 and (10.12), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{5,2}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
=&\sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta)d\beta\right|\\
\leq&\sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s\cdot\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\\
&\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1)\\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s}\\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\|Q\|_{l_\nu^1}^{n}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\cdot(j_1+j_2)\\
&+\|Q\|_{l_\nu^s}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^{n-1}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|Q\|_{l_\nu^1}^{n}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\Bigg)\\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\cdot(j_1+j_2)\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\cdot n\\
&+b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\cdot j_3\cdot(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\Bigg).
\end{aligned}$$

By (10.13), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{5,3}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
=&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta)d\beta\right|\\
\leq&\frac{C_1}{2}\left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}.
\end{aligned}$$

By Proposition 21 and (10.12), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{6,1}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
=&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B}_{6,1})(k_1,\beta)d\beta\right|\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|\\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\left(|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right)(k_1)\\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_{\nu}^{s}}\\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}b(n+1,s)\left(\|Q\|_{l_{\nu}^{s}}\|Q\|_{l_{\nu}^{1}}^{n-1}\left\|\phi^{j_3}\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot n+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|Q\|_{l_{\nu}^{1}}^{n}\right)\\
\leq&\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}b(n+1,s)\\
&\cdot\left(\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3}\cdot n\right.\\
&\left.+b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}\cdot j_3\cdot(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}\right).
\end{aligned}$$

By Proposition 21 and (10.12), we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{6,2}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta\right|\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}|k_1|^s\cdot\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\\
&\cdot\left(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right)(k_1)\\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s}\\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1}\|Q\|_{l_\nu^1}^{n}\left\|\phi^{j_3}\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot(j_1+j_2)\\
&+\|Q\|_{l_\nu^s}\|Q\|_{l_\nu^1}^{n-1}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}\left\|\phi^{j_3}\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot n\\
&+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^{n}\Bigg)\\
\leq&\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n\cdot(j_1+j_2)\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3}\cdot n\\
&+b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}\cdot j_3\cdot\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n\Bigg).
\end{aligned}
$$

By (10.13), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{6,3}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)d\beta\right|\\
\leq&\frac{C_1}{2}\left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}.
\end{aligned}$$

By Proposition 21 and (10.12), we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_7}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_7})(k_1,\beta)d\beta\right|\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\cdot\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\\
&\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1)\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_{\nu}^s}\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1}\left\|Q\right\|_{l_{\nu}^1}^{n}\left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot(j_1+j_2)\\
&+\left\|Q\right\|_{l_{\nu}^s}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}\left\|Q\right\|_{l_{\nu}^1}^{n-1}\left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot n+\left\|\phi^{j_3}\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}\left\|Q\right\|_{l_{\nu}^1}^{n}\Bigg)\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot(j_1+j_2)\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot n\\
&+b(j_3+1,s)(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\cdot j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3})\\
&\cdot\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n\Bigg).
\end{aligned}$$

By Proposition 21 and (10.12), we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_8}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta\right|\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\cdot\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\\
&\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1)\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s}\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\|Q\|_{l_\nu^1}^n\left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot(j_1+j_2)\\
&+\|Q\|_{l_\nu^s}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^{n-1}\left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n+\left\|\phi^{j_3}\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\|Q\|_{l_\nu^1}^n\Bigg)\\
\leq&\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot(j_1+j_2)\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+b(j_3+1,s)(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3})\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\Bigg).
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right|\\
\leq&\frac{\gamma}{4\pi}(1+\pi)\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|)(k_1)\\
\leq&\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|\right\|_{l_{\nu}^{s}}\\
\leq&\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}b(j_1+j_2,s)\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1}\cdot(j_1+j_2)\\
\leq&\frac{\gamma}{4\pi}(1+\pi)\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\sum_{j_1+j_2\geq 2}\frac{b(j_1+j_2,s)}{j_1!j_2!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1}\cdot(j_1+j_2).
\end{aligned}
$$

This completes the proof of Lemma 20.

CHAPTER 11

Estimating $(U_{\geq 2})_\alpha$

In Chapter 10, we derived that

$$U_{\geq 2}(\alpha) = \mathrm{Re}\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B(\alpha,\beta)d\beta\right),$$

where

$$B(\alpha,\beta) = \sum_{j=1}^{8} \widetilde{B_j}(\alpha,\beta) + \widetilde{B_{13}}(\alpha,\beta).$$

To estimate the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $(U_{\geq 2})_\alpha$, we differentiate the right hand side with respect to α. Recalling that

$$\widetilde{B_1}(\alpha,\beta) = \widetilde{B_{1,1}}(\alpha,\beta) + \widetilde{B_{1,2}}(\alpha,\beta) + \widetilde{B_{1,3}}(\alpha,\beta),$$

we note that

$$(\widetilde{B_{1,1}})_\alpha(\alpha,\beta) = \sum_{j=1}^{4} B_{1,1}^j(\alpha,\beta),$$
$$(\widetilde{B_{1,2}})_\alpha(\alpha,\beta) = \sum_{j=1}^{4} B_{1,2}^j(\alpha,\beta),$$

where

$$\begin{aligned}
B^1_{1,1}(\alpha,\beta) = &- \sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} j_1 \phi(\alpha-\beta)^{j_1-1}\phi_\alpha(\alpha-\beta)\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \\
B^2_{1,1}(\alpha,\beta) = &- \sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \phi(\alpha-\beta)^{j_1} j_2 \phi(\alpha)^{j_2-1}\phi_\alpha(\alpha) \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \\
B^3_{1,1}(\alpha,\beta) = &- \sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \\
B^4_{1,1}(\alpha,\beta) = &- \sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot n\left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}}\right. \\
&\left.\cdot \int_0^1 e^{-i(s-1)\beta}\frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\right)
\end{aligned} \tag{11.1}$$

and

$$
\begin{aligned}
B^1_{1,2}(\alpha,\beta) = &-\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} j_1 \phi(\alpha-\beta)^{j_1-1}\phi_\alpha(\alpha-\beta)\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right), \\
B^2_{1,2}(\alpha,\beta) = &-\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} j_2 \phi(\alpha)^{j_2-1}\phi_\alpha(\alpha) \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right), \\
B^3_{1,2}(\alpha,\beta) = &-\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} \phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s} j_3 \phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right), \\
B^4_{1,2}(\alpha,\beta) = &-\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} \phi(\alpha)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \\
&\cdot \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(\widetilde{B_{1,3}})_\alpha(\alpha,\beta) = -\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds \\
&\cdot \left(\sum_{m=2}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\right).
\end{aligned}
$$

We note that

$$(\widetilde{B_2})_\alpha(\alpha,\beta)=\sum_{j=1}^{5}B_2^j(\alpha,\beta),$$

where

$$\begin{aligned}
B_2^1(\alpha,\beta)=&-\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}j_1\phi(\alpha-\beta)^{j_1-1}\phi_\alpha(\alpha-\beta)\\
&\cdot\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_2^2(\alpha,\beta)=&-\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}j_2\phi(\alpha)^{j_2-1}\phi_\alpha(\alpha)\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_2^3(\alpha,\beta)=&-\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))\\
&\cdot(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_2^4(\alpha,\beta)=&-\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,
\end{aligned}$$

$$
\begin{aligned}
B_2^5(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} & \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} \phi(\alpha)^{j_2} \\
& \cdot \int_0^1 e^{-i\beta s} \phi(\alpha+\beta(-1+s))^{j_3} (-1+s) \phi'(\alpha+\beta(-1+s)) ds \\
& \cdot n \bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds \bigg)^{n-1} \\
& \cdot \bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \\
& \cdot \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1} (-i) \phi_\alpha(\alpha+(s-1)\beta)}{m!} ds \bigg).
\end{aligned}
$$

Recalling that

$$
\widetilde{B_3}(\alpha,\beta) = \widetilde{B_{3,1}}(\alpha,\beta) + \widetilde{B_{3,2}}(\alpha,\beta) + \widetilde{B_{3,3}}(\alpha,\beta),
$$

we note that

$$
\begin{aligned}
(\widetilde{B_{3,1}})_\alpha(\alpha,\beta) &= \sum_{j=1}^{3} B_{3,1}^j(\alpha,\beta), \\
(\widetilde{B_{3,2}})_\alpha(\alpha,\beta) &= \sum_{j=1}^{5} B_{3,2}^j(\alpha,\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
B_{3,1}^1(\alpha,\beta) = &\sum_{j_3+j_4+n\geq 2}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_3+j_4}(-1)^{j_3}}{2j_3!j_4!} \\
&\cdot\int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))ds \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \\
&\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \\
B_{3,1}^2(\alpha,\beta) = &\sum_{j_3+j_4+n\geq 2}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_3+j_4}(-1)^{j_3}}{2j_3!j_4!} \\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \\
&\cdot\int_0^1 e^{i\beta s} j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \\
B_{3,1}^3(\alpha,\beta) = &\sum_{j_3+j_4+n\geq 2}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_3+j_4}(-1)^{j_3}}{2j_3!j_4!} \\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \\
&\cdot n\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \\
&\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\right. \\
&\cdot\left.\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right),
\end{aligned}
$$

and

$$
\begin{aligned}
B^1_{3,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\\
&\cdot j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B^2_{3,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\\
&\cdot j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta)\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B^3_{3,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\\
&\cdot \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,
\end{aligned}
$$

$$B^4_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\cdot$$
$$\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\cdot$$
$$\int_0^1 e^{i\beta s} j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds\cdot$$
$$\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n,$$
$$B^5_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}$$
$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds$$
$$\cdot\, n\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^{n-1}$$
$$\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}$$
$$\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\Big).$$

Moreover,

$$(\widetilde{B_{3,3}})_\alpha(\alpha,\beta) = \frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds$$
$$\cdot\Big(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\Big).$$

We note that

$$(\widetilde{B_4})_\alpha(\alpha,\beta) = \sum_{j=1}^{6} B^j_4(\alpha,\beta),$$

where

$$
\begin{aligned}
B_4^1(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\\
&\cdot j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_4^2(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\\
&\cdot j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta)\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_4^3(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\\
&\cdot\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))ds\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\cdot\\
&\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,
\end{aligned}
$$

$$
\begin{aligned}
B_4^4(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} \\
&\cdot\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \\
&\cdot\int_0^1 e^{i\beta s}j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi_\alpha(\alpha+\beta(-1+s)) \\
&\cdot(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot\Bigg(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n, \\
B_4^5(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} \\
&\cdot\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi''(\alpha+\beta(-1+s))ds \\
&\cdot\Bigg(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n, \\
B_4^6(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot n\Bigg(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^{n-1} \\
&\cdot\Bigg(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}} \\
&\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\Bigg).
\end{aligned}
$$

Recalling that

$$
\widetilde{B_5}(\alpha,\beta) = \widetilde{B_{5,1}}(\alpha,\beta) + \widetilde{B_{5,2}}(\alpha,\beta) + \widetilde{B_{5,3}}(\alpha,\beta),
$$

we note that

$$
\begin{aligned}
(\widetilde{B_{5,1}})_\alpha(\alpha,\beta) &= \sum_{j=1}^{2} B_{5,1}^{j}(\alpha,\beta),\\
(\widetilde{B_{5,2}})_\alpha(\alpha,\beta) &= \sum_{j=1}^{4} B_{5,2}^{j}(\alpha,\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
B_{5,1}^{1}(\alpha,\beta) = &\sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_3}(-1)^{j_3}}{2j_3!}\\
&\cdot\int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n,\\
B_{5,1}^{2}(\alpha,\beta) = &\sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_3}(-1)^{j_3}}{2j_3!}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot n\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^{n-1}\\
&\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\\
&\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
B^1_{5,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n, \\
B^2_{5,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot \Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n, \\
B^3_{5,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} \phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot \Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n, \\
B^4_{5,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} \phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot n\Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^{n-1} \\
&\cdot \Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \\
&\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!} \phi_\alpha(\alpha+(s-1)\beta)ds\Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(\widetilde{B_{5,3}})_\alpha(\alpha,\beta) &= \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds \\
&\cdot\bigg(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\bigg).
\end{aligned}
$$

Recalling that

$$
\widetilde{B_6}(\alpha,\beta) = \widetilde{B_{6,1}}(\alpha,\beta) + \widetilde{B_{6,2}}(\alpha,\beta) + \widetilde{B_{6,3}}(\alpha,\beta),
$$

we note that

$$
\begin{aligned}
(\widetilde{B_{6,1}})_\alpha(\alpha,\beta) &= \sum_{j=1}^{2} B_{6,1}^j(\alpha,\beta), \\
(\widetilde{B_{6,2}})_\alpha(\alpha,\beta) &= \sum_{j=1}^{4} B_{6,2}^j(\alpha,\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
B_{6,1}^1(\alpha,\beta) = &\sum_{j_3+n\geq 2}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_3}}{2j_3!} \\
&\cdot\int_0^1 e^{i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n, \\
B_{6,1}^2(\alpha,\beta) = &\sum_{j_3+n\geq 2}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_3}}{2j_3!} \\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \\
&\cdot n\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^{n-1} \\
&\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}} \\
&\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
B^1_{6,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B^2_{6,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta)\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B^3_{6,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B^4_{6,2}(\alpha,\beta) = &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\\
&\cdot n\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1}\\
&\cdot\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\right.\\
&\left.\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right).
\end{aligned}
$$

Moreover,

$$(\widetilde{B_{6,3}})_\alpha(\alpha,\beta) = \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds$$
$$\cdot\left(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right).$$

We note that

$$(\widetilde{B_7})_\alpha(\alpha,\beta) = \sum_{j=1}^{5} B_7^j(\alpha,\beta),$$

where

$$
\begin{aligned}
B_7^1(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_7^2(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_7^3(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s)) \\
&\cdot (-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_7^4(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n,
\end{aligned}
$$

$$
\begin{aligned}
B_7^5(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} & \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
& \cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
& \cdot n\bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^{n-1} \\
& \cdot \bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \\
& \cdot \int_0^1 e^{i(s-1)\beta}\frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\bigg).
\end{aligned}
$$

We note that

$$
(\widetilde{B_8})_\alpha(\alpha,\beta) = \sum_{j=1}^{5} B_8^j(\alpha,\beta),
$$

where

$$
\begin{aligned}
B_8^1(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \\
& \cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_8^2(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \\
& \cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_8^3(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
& \cdot \int_0^1 e^{i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s)) \\
& \cdot (-1+s)\phi'(\alpha+\beta(-1+s))ds \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n,
\end{aligned}
$$

$$
\begin{aligned}
B_8^4(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_8^5(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \\
&\cdot n\left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^{n-1} \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}}\right. \\
&\left.\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\right).
\end{aligned}
$$

Lastly, we note that

$$
(\widetilde{B_{13}})_\alpha(\alpha,\beta) = \sum_{j=1}^{2} B_{13}^j(\alpha,\beta),
$$

where

$$
\begin{aligned}
B_{13}^1(\alpha,\beta) &= \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})} - \frac{1}{2}\right), \\
B_{13}^2(\alpha,\beta) &= \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta)\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})} - \frac{1}{2}\right).
\end{aligned}
$$

Of these terms, B_2^4, B_4^5, B_7^4, and B_8^4 contain the second derivative of ϕ, which need to be re-expressed in terms of lower-order derivatives of ϕ for the resulting estimate of the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $(U_{\geq 2})_\alpha$ to be useful. To re-express these terms, we use integration by parts to obtain

$$
\begin{aligned}
&\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \\
=&\int_0^1 e^{-i\beta s}(-1+s)\Bigg(\frac{\partial}{\partial s}\Bigg(\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\Bigg) \\
&-j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2\Bigg)ds \\
=&\int_0^1\Bigg(\frac{\partial}{\partial s}\Bigg(e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\Bigg) \\
&-\frac{\partial}{\partial s}\Bigg(e^{-i\beta s}(-1+s)\Bigg)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\Bigg)ds \\
&-\int_0^1 e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds \\
=&\frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta}+i\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \\
&-\int_0^1\frac{e^{-i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \\
&-\int_0^1 e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds\\
=&\int_0^1 e^{i\beta s}(-1+s)\bigg(\frac{\partial}{\partial s}\bigg(\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\bigg)\\
&-j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2\bigg)ds\\
=&\int_0^1\bigg(\frac{\partial}{\partial s}\bigg(e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\bigg)\\
&-\frac{\partial}{\partial s}\bigg(e^{i\beta s}(-1+s)\bigg)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\bigg)ds\\
&-\int_0^1 e^{i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds\\
=&\frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta}-i\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds\\
&-\int_0^1\frac{e^{i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds\\
&-\int_0^1 e^{i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds.
\end{aligned}
$$

Then we can write $B_2^4(\alpha,\beta) = \sum_{j=1}^4 B_2^{4,j}(\alpha,\beta)$, where

$$\begin{aligned}
B_2^{4,1}(\alpha,\beta) = &- \sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_2^{4,2}(\alpha,\beta) = &- \sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot i \int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s)) ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_2^{4,3}(\alpha,\beta) = &- \sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 \frac{-e^{-i\beta s}}{\beta} \phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s)) ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_2^{4,4}(\alpha,\beta) = &- \sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \\
&\cdot \int_0^1 -e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2 ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n.
\end{aligned}$$

Moreover, we can write $B_4^5(\alpha,\beta)=\sum_{j=1}^4 B_4^{5,j}(\alpha,\beta)$, where

$$
\begin{aligned}
B_4^{5,1}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\frac{\phi(\alpha-\beta)^{j_4}\phi'(\alpha-\beta)}{\beta}\\
&\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_4^{5,2}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 -ie^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_4}\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_4^{5,3}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 \frac{-e^{i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_4}\phi'(\alpha+\beta(-1+s))ds\\
&\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
B_4^{5,4}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\\
&\cdot\int_0^1 -e^{i\beta s}(-1+s)j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi'(\alpha+\beta(-1+s))^2ds\cdot\\
&\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n.
\end{aligned}
$$

Moreover, we can write $B_7^4(\alpha,\beta) = \sum_{j=1}^4 B_7^{4,j}(\alpha,\beta)$, where

$$\begin{aligned}
B_7^{4,1}(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} \phi(\alpha-\beta)^{j_2} \\
& \cdot \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)^n, \\
B_7^{4,2}(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} \phi(\alpha-\beta)^{j_2} \\
& \cdot \int_0^1 ie^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)^n, \\
B_7^{4,3}(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} \phi(\alpha-\beta)^{j_2} \\
& \cdot \int_0^1 \frac{-e^{-i\beta s}}{\beta} \phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)^n, \\
B_7^{4,4}(\alpha,\beta) = & \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} \phi(\alpha-\beta)^{j_2} \\
& \cdot \int_0^1 -e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2 ds \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)^n.
\end{aligned}$$

Lastly, we can write $B_8^4(\alpha,\beta) = \sum_{j=1}^4 B_8^{4,j}(\alpha,\beta)$, where

$$\begin{aligned}
B_8^{4,1}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_8^{4,2}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 -ie^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_8^{4,3}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 \frac{-e^{i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \\
B_8^{4,4}(\alpha,\beta) = &\sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \\
&\cdot \int_0^1 -e^{i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2 ds \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n.
\end{aligned}$$

11.1. Estimating Fourier Modes of $(U_{\geq 2})_\alpha$

We use arguments as in Section 10.1 to estimate the Fourier modes of $(U_{\geq 2})_\alpha$. First,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^{1})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+n\geq 1}\frac{j_1 C_n}{2j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\cdot\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$
$$\cdot\prod_{d=j_1+1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left|P(k_d-k_{d+1})\right|\cdot\left|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^{2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+n\geq 1}\frac{j_2 C_n}{2j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left|P(k_d-k_{d+1})\right|\cdot\left|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^{3})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left|P(k_d-k_{d+1})\right|\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^4)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+n\geq 1}\frac{nC_n}{2j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|P(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{m(-i)^m}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right|$$

$$\cdot\left|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,2}^1)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3\geq 1}\frac{j_1C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,2}^2)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3\geq 1}\frac{j_2C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|$$

$$\cdot|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^3_{1,2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3\geq 1}\frac{j_3C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\left|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+2})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^4_{1,2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\left|\sum_{m=1}^{\infty}\frac{m(-i)^m}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+1}-k_{j_1+j_2+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{(B_{1,3})}_\alpha)(k_1,\beta)d\beta\right| \leq \frac{C_1}{2}\left|\sum_{m=2}^{\infty}\frac{m(-i)^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^1_2)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_1C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$
$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^2)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2 C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^3)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^5)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{n C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|P(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{m(-i)^m}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Next,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^1_{3,1})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3 C_{n+1}}{2j_3!j_4!}\sum_{k_2,\ldots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{n+1}-k_{n+2})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_4})(k_{n+2})\right|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^2_{3,1})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4 C_{n+1}}{2j_3!j_4!}\sum_{k_2,\ldots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4-1}\phi')(k_{n+1}-k_{n+2})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{n+2})\right|.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^3_{3,1})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_3+j_4+n\geq 2}(n+1)\frac{n C_{n+1}}{2j_3!j_4!}\sum_{k_2,\ldots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n-1}|Q(k_d-k_{d+1})| \\
&\cdot\left|\sum_{m=1}^{\infty}\frac{m i^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_n-k_{n+1})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_4})(k_{n+1}-k_{n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{n+2})\right|.
\end{aligned}
$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^1_{3,2})(k_1,\beta)d\beta\right| \leq$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_1 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^2_{3,2})(k_1,\beta)d\beta\right| \leq$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|\cdot$$

$$\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^3_{3,2})(k_1,\beta)d\beta\right| \leq$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+2})\right|.$$

Moreover,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^4)(k_1,\beta)d\beta\right| \leq \\
&\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_4C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4-1}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^5)(k_1,\beta)d\beta\right| \leq \\
&\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.
\end{aligned}
$$

Next,

$$
\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{3,3}})_\alpha)(k_1,\beta)d\beta\right| \leq C_2\left|\sum_{m=2}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|.
$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^1)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_1 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$
$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^2)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^3)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+2})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^4)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi^{j_4-1}\phi'^2)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^6)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right|$$

$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,1}^1)(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,1}^2)(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n-1}|Q(k_d-k_{d+1})|$$

$$\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_n-k_{n+1})\right|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^1)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_1 C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$
$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^2)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_2 C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^3)(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_3 C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^{4})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1+j_2} |\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot \prod_{d=j_1+j_2+1}^{j_1+j_2+n-1} |Q(k_d-k_{d+1})| \left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right|$$
$$\cdot \left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{5,3}})_{\alpha})(k_1,\beta)d\beta\right| \leq \frac{1}{2}C_1\left|\sum_{m=2}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,1}^{1})(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|$$
$$\cdot \left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,1}^{2})(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n-1}|Q(k_d-k_{d+1})|$$
$$\cdot \left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_n-k_{n+1})\right|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^1_{6,2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$
$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^2_{6,2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_2C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B^3_{6,2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_3C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,2}^4)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{6,3}})_\alpha)(k_1,\beta)d\beta\right| \leq \frac{1}{2}C_1\left|\sum_{m=2}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^1)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^2)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^3)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^5)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{nC_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^1)(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right|$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^2)(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2 C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^3)(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^5)(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{n C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \\
&\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{13}})_\alpha)(k_1,\beta)d\beta\right| \leq \sum_{j_1+j_2\geq 2}\frac{j_2}{j_1!j_2!}$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2})\right|\cdot\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+3\pi\right).$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,1})(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+j_3+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_3}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$

$$\cdot\prod_{d=j_1+j_2+j_3+1}^{j_1+j_2+j_3+n}|P(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_3+n+1})\right|,$$

where

$$D_n=\frac{\gamma}{4\pi}\left(D+(n+1)\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi\right)$$

with D being an upper bound of $\left|\int_{-\pi}^{\pi}\frac{e^{i\beta x}}{\beta}d\beta\right|$, taken as a function of x. Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,2})(k_1,\beta)d\beta\right| \leq$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,3})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,4})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|.$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,1})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+j_4+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_4}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\prod_{d=j_1+j_2+j_4+1}^{j_1+j_2+j_4+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+j_4+n+1}-k_{j_1+j_2+j_4+n+2})\right|\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_4+n+2})\right|.$$

Moreover,

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,2})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{2j_1 j_2! j_3! j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2\in\mathbb{Z}}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\right|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,3})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1 j_2! j_3! j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2\in\mathbb{Z}}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\right|.
\end{aligned}$$

Lastly,

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,4})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1 j_2! j_3! j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2\in\mathbb{Z}}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_4-1}\phi'^2)(k_{j_1+j_2+n+2})\right|.
\end{aligned}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,1})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+j_3+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_3}|\mathcal{F}(\phi)(k_d-k_{d+1})|$$
$$\cdot\prod_{d=j_1+j_2+j_3+1}^{j_1+j_2+j_3+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_3+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,2})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,3})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.$$

Lastly,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,4})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,1})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+j_3+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_3}|\mathcal{F}(\phi)(k_d-k_{d+1})| \\
&\cdot\prod_{d=j_1+j_2+j_3+1}^{j_1+j_2+j_3+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_3+n+1})\right|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,2})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,3})(k_1,\beta)d\beta\right| \leq \\
&\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \\
&\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|.
\end{aligned}
$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,4})(k_1,\beta)d\beta\right| \leq$$
$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1\in\mathbb{Z}}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|$$
$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|.$$

11.2. Estimating $\|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}}$

We prove the following estimate for $\|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}}$, $s>0$.

Lemma 22. *For $s>0$,*

$$\begin{aligned}\|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\ & + R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\ & + R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}},\end{aligned}$$

where R_1, R_2, R_3, R_4, and R_5 are monotone increasing functions of $\|\phi\|_{\mathcal{F}_\nu^{0,1}}$.

We use estimates from Section 11.1 to prove Lemma 22. We take the notational convention that if a convolution of sequences on $\mathbb{Z}$ contains a sequence of the form $\left|\mathcal{F}(\phi^{j_3})\right|$ in which $j_3=0$, then we simply ignore that sequence in the convolution. For example, in (11.2), if $j_3=0$, then

$$\begin{aligned}&|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*\left|\mathcal{F}(\phi')\right|*|P|*\left|\mathcal{F}(\phi^{j_3})\right| \\ =&|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*\left|\mathcal{F}(\phi')\right|*|P|.\end{aligned}$$

We define

$$R(k) = \sum_{m=1}^{\infty} \frac{m(-i)^m}{m!} \mathcal{F}(\phi^{m-1}\phi')(k),$$
$$S(k) = \sum_{m=1}^{\infty} \frac{mi^{m-1}}{m!} \mathcal{F}(\phi^{m-1}\phi')(k),$$

and

$$D_n = \frac{\gamma}{4\pi}\left(D + (n+1)\left(\frac{\pi}{2}\right)^n \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot 2\pi\right),$$

where D is an upper bound of $\left|\int_{-\pi}^{\pi} \frac{e^{i\beta x}}{\beta} d\beta\right|$, uniform in $x \in \mathbb{R}$. First,

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{1,1}^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1 C_n}{2j_1!j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi')\right| \right\|_{l_\nu^s} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1 C_n}{2j_1!j_2!} b(j_1+j_2+n+1,s) \\
&\quad \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n (j_1+j_2) \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \cdot n \Bigg).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq & \sum_{j_1+j_2+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi')\right| \right\|_{l_\nu^s} \\
\leq & \sum_{j_1+j_2+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2!} b(j_1+j_2+n+1, s) \\
& \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n (j_1+j_2) \\
& + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
& + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \cdot n \Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq & \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2 j_1! j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi')\right| \right\|_{l_\nu^s} \\
\leq & \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2 j_1! j_2!} b(j_1+j_2+n+1, s) \\
& \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n (j_1+j_2) \\
& + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
& + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \cdot n \Bigg).
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^4_{1,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{nC_n}{2j_1!j_2!} \left\||\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |R|\right\|_{l^s_\nu} \\
&\leq \sum_{j_1+j_2+n\geq 1} \frac{nC_n}{2j_1!j_2!} b(j_1+j_2+n+1,s) \\
&\quad \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} (j_1+j_2+1) \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2+1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-2} \\
&\quad \cdot \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} (n-1) \\
&+ \left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu} + \left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \\
&\quad \cdot \|\phi\|^{j_1+j_2+1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \Bigg).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^1_{1,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
&\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_1C_1}{2j_1!j_2!j_3!} \left\||\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |P| * \left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l^s_\nu} \qquad (11.2) \\
&\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_1C_1}{2j_1!j_2!j_3!} b(j_1+j_2+2,s) \\
&\quad \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2-2}_{\mathcal{F}^{0,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1) \|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu} (j_1+j_2-1) \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1) \|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu} \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu} \\
&\quad + b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} \cdot j_3 \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1) \Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3\geq 1} \frac{j_2 C_1}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |P| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \\
\leq &\sum_{j_1+j_2+j_3\geq 1} \frac{j_2 C_1}{2 j_1! j_2! j_3!} b(j_1+j_2+2,s) \\
&\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (j_1+j_2-1) \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \\
&+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3\geq 1} \frac{j_3 C_1}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_\nu^s} \\
\leq &\sum_{j_1+j_2+j_3\geq 1} \frac{j_3 C_1}{2 j_1! j_2! j_3!} b(j_1+j_2+2,s) \\
&\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2) \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&+ b(j_3,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (j_3-1) + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \right) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \Bigg).
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^4(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq & \sum_{j_1+j_2+j_3\geq 1} \frac{C_1}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |R| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \\
\leq & \sum_{\substack{j_1,j_2\geq 0\\ j_3\geq 1}} \frac{C_1}{2j_1!j_2!j_3!} b(j_1+j_2+2,s) \\
& \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (j_1+j_2) \\
& + \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \\
& \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \\
& + b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \Bigg).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} ((\widetilde{B_{1,3}})_\alpha)(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq & \frac{1}{2} C_1 \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)(m-1)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} \\
& + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \Bigg).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\le &\sum_{j_1+j_2+j_3+n\ge 1}\frac{j_1 C_n}{2 j_1! j_2! j_3!}\\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s}\\
\le &\sum_{j_1+j_2+j_3+n\ge 1}\frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)\\
&\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1)\\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\\
&+ \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n\\
&+ b(j_3+1,s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right)\\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_2^2(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\leq&\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2C_n}{2j_1!j_2!j_3!}\\
&\cdot\left\|\left|\mathcal{F}(\phi)\right|*\cdots*\left|\mathcal{F}(\phi)\right|*\left|\mathcal{F}(\phi')\right|*\left|P\right|*\cdots*\left|P\right|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s}\\
\leq&\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(j_1+j_2-1)\\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+b(j_3+1,s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^3(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\leq &\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right|\right\|_{l_\nu^s}\\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\\
&\cdot\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\\
&\cdot\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)n(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}\\
&+\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s)(j_3-1)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^5(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq& \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |R| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
\leq& \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-2} \\
&+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \\
&\cdot \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{3,1}^{1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq & \sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3 C_{n+1}}{2j_3!j_4!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|*\left|\mathcal{F}(\phi^{j_4})\right|\right\|_{l_{\nu}^{s}} \\
\leq & \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\cdot\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3 C_{n+1}}{2j_3!j_4!}b(n+2,s)n(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3+j_4-1} \\
&+\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3 C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_3,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-2}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}(j_3-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}\right)(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4} \\
&+\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3 C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_4,s)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1} \\
&\cdot j_4(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{3,1}^{2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq & \sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4 C_{n+1}}{2j_3!j_4!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_4-1}\phi')\right|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_{\nu}^{s}} \\
\leq & \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\cdot\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4 C_{n+1}}{2j_3!j_4!}b(n+2,s)n(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3+j_4-1} \\
&+\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4 C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_4,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-2}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}(j_4-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1}\right)(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3} \\
&+\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4 C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}j_3 \\
&\cdot(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1}.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{3,1}^{3}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq & \sum_{j_3+j_4+n\geq 2} (n+1) \frac{nC_{n+1}}{2j_3!j_4!} \left\| |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_{\nu}^{s}} \\
\leq & \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}} \\
& \cdot \sum_{j_3+j_4+n\geq 2} (n+1) \frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s)(n-1)(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-2} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3+j_4} \\
& + \left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2} + \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1} \right) \\
& \cdot \sum_{j_3+j_4+n\geq 2} (n+1) \frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s)(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3+j_4} \\
& + \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}} \sum_{j_3+j_4+n\geq 2} (n+1) \frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s) b(j_4,s) \\
& \cdot \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1} j_4 (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3} \\
& + \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}} \sum_{j_3+j_4+n\geq 2} (n+1) \frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s) b(j_3,s) \\
& \cdot \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} j_3 (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4} .
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^1_{3,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
\leq &\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2-1) \\
&\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \\
&\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \\
&+ \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
&\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)n \\
&\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu} j_4 \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \\
&+ \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{3,2}^2(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le& \sum_{\substack{j_1+j_2\ge1\\ j_3+j_4+n\ge1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \\
\le& \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\ge1\\ j_3+j_4+n\ge1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2-1) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{\substack{j_1+j_2\ge1\\ j_3+j_4+n\ge1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\cdot \sum_{\substack{j_1+j_2\ge1\\ j_3+j_4+n\ge1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)n \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\ge1\\ j_3+j_4+n\ge1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} j_4 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\ge1\\ j_3+j_4+n\ge1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{3,2}^{3}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq& \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_{\nu}^{s}} \\
\leq& \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-2} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n} \\
&+ \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} n(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \\
&\cdot \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1} j_4 \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n} \\
&+ \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-2} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} (j_3-1) + \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} \right) \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{3,2}^{4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq &\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4-1}\phi')\right| \right\|_{l_{\nu}^{s}} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-2} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} n(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \\
&\cdot \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_4-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n} \\
&+ \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-2} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} (j_4-1) + \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1} \right) \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^5_{3,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}\\
&\le \sum_{\substack{j_1+j_2\ge 1\\ j_3+j_4+n\ge 1}} (n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!}\\
&\quad\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu}\\
&\le \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\ge 1\\ j_3+j_4+n\ge 1}} (n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2)\\
&\quad\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}\\
&\quad+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}\\
&\quad\cdot \sum_{\substack{j_1+j_2\ge 1\\ j_3+j_4+n\ge 1}} (n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(n-1)\\
&\quad\cdot \|\phi\|^{j_1+j_2+j_3+j_4}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-2}\\
&\quad+\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-2)!}\|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu}+\left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\\
&\quad\cdot \sum_{\substack{j_1+j_2\ge 1\\ j_3+j_4+n\ge 1}} (n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)\|\phi\|^{j_1+j_2+j_3+j_4}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}\\
&\quad+\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\ge 1\\ j_3+j_4+n\ge 1}} (n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)b(j_4,s)\\
&\quad\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu} j_4 \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}\\
&\quad+\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\ge 1\\ j_3+j_4+n\ge 1}} (n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)b(j_3,s)\\
&\quad\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2+j_4}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{3,3}})_{\alpha}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}$$
$$\leq C_2\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{m=2}^{\infty}\frac{b(m,s)(m-1)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2}+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right).$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_4^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}$$
$$\leq\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}$$
$$\cdot\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*\left|\mathcal{F}(\phi')\right|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_{\nu}^{s}}$$
$$\leq\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}^{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)(j_1+j_2-1)$$
$$\cdot\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-2}\left(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1\right)^{n}$$
$$+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)$$
$$\cdot\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-1}\left(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1\right)^{n}$$
$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}^{2}$$
$$\cdot\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)n$$
$$\cdot\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3+j_4-1}\left(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1\right)^{n-1}$$
$$+\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_4+1,s)$$
$$\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4-1}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}j_4+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_4}\right)\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1}\left(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1\right)^{n}$$
$$+\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}^{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_3,s)$$
$$\cdot\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}j_3\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_4-1}\left(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1\right)^{n}.$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^2(\alpha, \beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s)(j_1+j_2-1) \\
&\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) \\
&\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|^2_{\mathcal{F}_\nu^{0,1}} \\
&\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) n \\
&\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) b(j_4+1, s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-1}_{\mathcal{F}_\nu^{0,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_4 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
&+ \left\|\phi'\right\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) b(j_3, s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_4^3(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\leq& \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}\\
&\cdot\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|\right\|_{l_\nu^s}\\
\leq& \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)(j_1+j_2)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\\
&\cdot\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)n\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\\
&+\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_4+1,s)\\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}j_4+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\\
&+\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_3,s)\\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(j_3-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_4^4(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_4-1}\phi'^2)\right|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)n \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_4+1,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2(j_4-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot 2\right) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_3,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}j_3\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^6(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \\
&\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \\
&\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(n-1) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-2} \\
&+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \\
&\cdot \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!k_3!j_4!} b(j_1+j_2+n+2,s) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \\
&\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^1_{5,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
\leq &\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|\right\|_{l^s_\nu} \\
\leq &\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \\
&\cdot\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}b(n+1,s)n\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \\
&+\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}b(n+1,s)b(j_3,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-2}_{\mathcal{F}^{0,1}_\nu}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}(j_3-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}\right) \\
&\cdot(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^2_{5,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
\leq &\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\left\||Q|*\cdots*|Q|*|S|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l^s_\nu} \\
\leq &\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \\
&\cdot\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)(n-1)\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-2} \\
&+\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-2)!}\|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu}+\left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \\
&\cdot\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \\
&+\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}j_3(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{5,2}^{1}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
&\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_{\nu}^{s}} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1) \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \\
&\quad + \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \\
&\quad \cdot \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{5,2}^{2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
&\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_{\nu}^{s}} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1) \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n} \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \\
&\quad \cdot \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{5,2}^3(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\le \sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_\nu^s} \\
&\le \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\quad + \sum_{\substack{j_1+j_2\ge 1\\ j_3+n\ge 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \\
&\quad \cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(j_3-1) + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_{5,2}^{4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq &\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*|S|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_{\nu}^{s}}\\
\leq &\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2)\\
&\cdot\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1}(e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}\\
&\cdot\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(n-1)\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3}(e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-2}\\
&+\left(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-2)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2}+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\\
&\cdot\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3}(e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\\
&+\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3,s)\\
&\cdot\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}j_3\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}(e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{5,3}})_{\alpha}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\frac{1}{2}C_1\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{m=2}^{\infty}\frac{b(m,s)(m-1)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2}\\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\Bigg).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_{6,1}^{1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|\right\|_{l_{\nu}^{s}}\\
\leq&\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\\
&\cdot\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}b(n+1,s)n\,\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\\
&+\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}b(n+1,s)b(j_3,s)\\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-2}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}(j_3-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}\right)\\
&\cdot(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n}.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_{6,1}^{2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\\
\leq&\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\left\||Q|*\cdots*|Q|*|S|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_{\nu}^{s}}\\
\leq&\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}\\
&\cdot\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)(n-1)\,\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-2}\\
&+\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-2)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2}+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\\
&\cdot\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)\,\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}\\
&+\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)b(j_3,s)\,\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}j_3(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{6,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \\
&\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1) \|\phi\|^{j_1+j_2+j_3-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \\
&\quad + \left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \\
&\quad \cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{6,2}^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\quad \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\quad + \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \\
&\quad \cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{6,2}^{3}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
&\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|\right\|_{l_{\nu}^{s}} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \\
&\quad\cdot\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2)\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n \\
&\quad+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)n \\
&\quad\cdot\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n \\
&\quad+\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)b(j_3,s) \\
&\quad\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-2}\left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}}(j_3-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1}\right)\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{6,2}^{4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq & \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!}\left\|\left|\mathcal{F}(\phi)\right| * \cdots * \left|\mathcal{F}(\phi)\right| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_{\nu}^{s}} \\
\leq & \left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
& \cdot \left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \\
& + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right) \left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}} \\
& \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3} (e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-2} \\
& + \left(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2} + \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right) \\
& \cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3} (e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1} \\
& + \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \\
& \cdot \left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} j_3 \left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2} (e^{\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_{6,3}})_{\alpha}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \\
\leq & \frac{1}{2}C_1\Bigg(\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_{\nu}^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)(m-1)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-2} \\
& + \left\|\phi'\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\Bigg).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2! j_3!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \\
&\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n \\
&+ b(j_3+1,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^2(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\leq& \sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2 C_n}{2j_1!j_2!j_3!}\\
&\cdot \left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*\left|\mathcal{F}(\phi')\right|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s}\\
\leq& \sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2 C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(j_1+j_2-1)\\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+b(j_3+1,s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq& \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \\
\leq& \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} \\
&+ \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s)(j_3-1) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^5(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \\
&\quad \cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-2} \\
&\quad + \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \\
&\quad \cdot \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\quad + \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\quad \cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}.
\end{aligned}$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\\
\le& \sum_{j_1+j_2+j_3+n\ge 1}\frac{j_1 C_n}{2 j_1! j_2! j_3!}\\
&\cdot \left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*\left|\mathcal{F}(\phi')\right|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s}\\
\le& \sum_{j_1+j_2+j_3+n\ge 1}\frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)\\
&\cdot\Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(j_1+j_2-1)\\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n\\
&+b(j_3+1,s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \\
&\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n \\
&+ b(j_3+1,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \Bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^3(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right|\right\|_{l_\nu^s} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\quad \cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2)\,\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}\,(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\quad \cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\,\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} \\
&\quad + \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s)(j_3-1) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}\,(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}\,(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^5(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
\leq &\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|^2_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot \left\|\phi\right\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \left\|\phi\right\|^{m-1}_{\mathcal{F}_\nu^{0,1}}\right) \left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|^2_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \left\|\phi\right\|^{j_1+j_2+j_3}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-2} \\
&+ \left(\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \left\|\phi\right\|^{m-2}_{\mathcal{F}_\nu^{0,1}} + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \left\|\phi\right\|^{m-1}_{\mathcal{F}_\nu^{0,1}}\right) \\
&\cdot \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \left\|\phi\right\|^{j_1+j_2+j_3}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot \left(\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi\right\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi\right\|^{j_3}_{\mathcal{F}_\nu^{0,1}}\right) \left\|\phi\right\|^{j_1+j_2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_{13}})_\alpha(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+3\pi\right) \sum_{j_1+j_2\geq 2} \frac{j_2}{j_1!j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * \left|\mathcal{F}(\phi')\right| \right\|_{l_\nu^s} \\
\leq &\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+3\pi\right) \left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2\geq 2} \frac{j_2}{j_1!j_2!} b(j_1+j_2,s)(j_1+j_2-1) \\
&\cdot \left\|\phi\right\|^{j_1+j_2-2}_{\mathcal{F}_\nu^{0,1}} \\
&+ \left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+3\pi\right) \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2\geq 2} \frac{j_2}{j_1!j_2!} b(j_1+j_2,s) \left\|\phi\right\|^{j_1+j_2-1}_{\mathcal{F}_\nu^{0,1}}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^{4,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le &\sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi')\right|\right\|_{l_\nu^s} \\
\le &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+j_3+n+1,s)(j_1+j_2+j_3) \\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot\sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+j_3+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+j_3+n+1,s)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^{4,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le &\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s} \\
\le &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_2^{4,3}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\quad \cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\quad \cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\quad + \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\quad \cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_2^{4,4}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le & \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \\
\le & \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
& \cdot \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
& + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
& \cdot \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} \\
& + \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s)(j_3-1) \\
& \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
& + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
& \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_4^{5,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\quad \cdot \left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi')\right|\right\|_{l_\nu^s} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+j_4+n+2,s)(j_1+j_2+j_4) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\quad \cdot b(j_1+j_2+j_4+n+2,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&\quad + \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+j_4+n+2,s)b(j_3,s) \\
&\quad \cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}j_3\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&\quad + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+j_4+n+2,s) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^{5,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq & \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} \\
& \cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| \right\|_{l_\nu^s} \\
\leq & \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \\
& \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
& + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} \\
& \cdot b(j_1+j_2+n+2,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \\
& + \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
& \cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
& + \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \\
& \cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_4 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_4^{5,3}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| \right\|_{l_\nu^s} \\
\leq &\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \\
&\cdot \left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} \\
&\cdot b(j_1+j_2+n+2,s) n \left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\cdot \left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \\
&\cdot \left(\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_4 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_4}\right)\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^{5,4}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
&\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \\
&\quad \cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4-1}\phi'^2)\right| \right\|_{l_\nu^s} \\
&\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \\
&\quad \cdot b(j_1+j_2+n+2,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \\
&\quad + \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \\
&\quad \cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \\
&\quad + \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \\
&\quad \cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 (j_4-1) + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot 2 \right) \\
&\quad \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^{4,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le &\sum_{j_1+j_2+j_3+n\ge 1} \frac{D_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi')\right|\right\|_{l_\nu^s} \\
\le &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+j_3+n+1,s)(j_1+j_2+j_3) \\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot\sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+j_3+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\sum_{j_1+j_2+j_3+n\ge 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+j_3+n+1,s) \\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^{4,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le &\sum_{j_1+j_2+j_3+n\ge 1} \frac{C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s} \\
\le &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\sum_{j_1+j_2+j_3+n\ge 1}\frac{C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^{4,3}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}j_3+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^{4,4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le &\sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \\
\le &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\cdot \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 (j_3-1) + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot 2\right) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^{4,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi')\right|\right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s)(j_1+j_2+j_3) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} \\
&\cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^{4,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^{4,3}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\leq &\sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \\
\leq &\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \\
&\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}} j_3 + \left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_8^{4,4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\
\le& \sum_{j_1+j_2+j_3+n\ge 1} \frac{j_3 C_n}{2j_1!j_2!j_3!}\left\| |\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right|\right\|_{l_\nu^s} \\
\le& \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\ge 1}\frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \\
&\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \\
&+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2 \\
&\cdot \sum_{j_1+j_2+j_3+n\ge 1}\frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \\
&+\sum_{j_1+j_2+j_3+n\ge 1}\frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \\
&\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}^2(j_3-1)+\left\|\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\left\|\phi'\right\|_{\mathcal{F}_\nu^{0,1}}\cdot 2\right) \\
&\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n.
\end{aligned}
$$

This completes the proof of Lemma 22.

CHAPTER 12

Proof of the Main Theorem

12.1. Proof of the Main *a priori* Estimate

To complete the estimate for the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $\widetilde{\mathcal{N}}$, we let $s = 1$. Recalling (8.1), we can use Lemmas 15 and 16 and the estimates of the $\mathcal{F}_\nu^{0,1}$ norm of U_1 and $U_{\geq 2}$ in Sections 9.2 and 10.2, respectively, to obtain

$$\begin{aligned}
\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\leq \|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \|T_1 \phi_\alpha\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
&\leq \|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{1,1}} + (H_3 \|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4 \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}) \\
&\quad + (D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}}) \|\phi\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})(1 + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}) \\
&\quad \cdot (1 + \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}) \\
&\quad + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} (H_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4 \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}) \cdot (1 + \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}).
\end{aligned}$$

Using Lemma 22 and Proposition 1, we obtain

$$
\begin{aligned}
\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} \leq & R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}} \\
& + R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
& + R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}} \\
& + \left(H_3\|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\right) \\
& + \left(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(1 + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
& \cdot \left(1 + \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\right) \\
& + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\left(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(1 + \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\right) \\
\leq & \|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\Bigg(R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}} \\
& + R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \\
& + R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
& + 3\left(H_3\|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
& + 3\left(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(1 + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
& + \left(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\right)\left(1 + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
& + 6\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\left(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
& + 2\left(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\Bigg).
\end{aligned}
$$

Using this estimate for the $\dot{\mathcal{F}}_\nu^{1,1}$ norm of $\widetilde{\mathcal{N}}$, we obtain from (7.14)

$$
\begin{aligned}
&\frac{d}{dt}\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
\leq&\left(\nu'(t) - \frac{1}{2\Big(C_I(\left\|\phi\right\|_{\mathcal{F}^{0,1}})\left\|\phi\right\|_{\mathcal{F}^{0,1}}+1\Big)}\pi\frac{2}{R}\frac{\gamma}{4\pi}+\frac{\gamma}{4\pi}\frac{1}{R}A(\left\|\phi\right\|_{\mathcal{F}^{0,1}})\left\|\phi\right\|_{\mathcal{F}^{0,1}}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{2,1}} \\
&+\frac{1}{R}\frac{1}{A_1(\left\|\phi\right\|_{\mathcal{F}^{0,1}})}\Big(R_1(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}+R_2(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}} \\
&+R_3(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}+R_4(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}^2 \\
&+R_5(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} \\
&+3\left(H_3\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{0,1}}+H_4\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
&+3\left(D_1(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^2+D_2(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(1+2\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
&+\left(D_1(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}+D_2(\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}})\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}\right)\left(1+2\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
&+6\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\left(H_3\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}+H_4\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) \\
&+2\left(H_3\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}+H_4\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\Big)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{2,1}}.
\end{aligned}
$$

Since C_I, A, A_1^{-1}, R_1, R_2, R_3, R_4, R_5, D_1, and D_2 are all monotone increasing, we can use Proposition 1 to obtain

$$
\frac{d}{dt}\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} \leq -\Big(\Lambda(\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{1,1}})-\nu'(t)\Big)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{2,1}}, \tag{12.1}
$$

where

$$\begin{aligned}
\Lambda(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}) =& \frac{1}{2\left(C_I(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+1\right)}\pi\frac{2}{R}\frac{\gamma}{4\pi}-\frac{\gamma}{4\pi}\frac{1}{R}A(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \\
&-\frac{1}{R}\frac{1}{A_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})}\Bigg(R_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+R_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \\
&+R_3(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}+R_4(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \\
&+R_5(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \\
&+3\left(H_3\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+H_4\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \\
&+3\left(D_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}+D_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\left(1+2\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \\
&+\left(D_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+D_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\left(1+2\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \\
&+6\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\left(H_3\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+H_4\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \\
&+2\left(H_3\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+H_4\,\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\Bigg).
\end{aligned}$$

Integrating (12.1) with respect to time, we obtain

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu}+\int_0^t(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu})-\nu'(\tau))\,\|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau\leq\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}. \tag{12.2}$$

We choose the initial datum such that $\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})>0$. Then we let $\nu_0\in(0,\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}))$. From (2.2), it follows that for all $\tau\geq 0$,

$$0<\nu'(\tau)=\frac{\nu_0}{(1+\tau)^2}\leq\nu_0.$$

Then

$$\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu'(0)>0.$$

Let

$$T_1 = \sup\Big\{t_1 : \Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau) > 0 \text{ for all } \tau \in [0,t_1]\Big\}.$$

Since $\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0$ and $\Lambda(\|\phi(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\cdot)$ is a continuous function, we have $T_1 > 0$. For any $t_1 \in [0,T_1)$,

$$\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau) > 0 \text{ for all } \tau \in [0,t_1].$$

Then by (12.2) for all $t \in [0,t_1]$,

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}.$$

Fix $t_1 \in [0,T_1)$ and $t_2 \in [t_1,T_1)$. Then

$$\|\phi(t_2)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \int_{t_1}^{t_2}\Big(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau)\Big)\|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau \le \|\phi(t_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu}. \tag{12.3}$$

Since

$$\int_{t_1}^{t_2}\Big(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau)\Big)\|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau > 0,$$

it follows from (12.3) that $\|\phi(t_2)\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le \|\phi(t_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu}$. Since Λ is a monotone decreasing function of $\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, this means that $\Lambda(\|\phi(t_2)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \ge \Lambda(\|\phi(t_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu})$, i.e., $\Lambda(\|\phi(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu})$ is a monotone increasing function on $[0,T_1)$. Suppose for contradiction that $T_1 < \infty$. We note that $\Lambda(\|\phi(T_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(T_1) = 0$. Since $\Lambda(\|\phi(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu})$ is monotone increasing on $[0,T_1)$ and is continuous on $[0,T_1]$, it is monotone increasing on $[0,T_1]$. Then

$$\nu_0 = \nu'(0) < \Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \le \Lambda(\|\phi(T_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) = \nu'(T_1) = \frac{\nu_0}{(1+T_1)^2},$$

which is a contradiction. Hence, $T_1 = \infty$. Then for all $t \in [0, \infty)$,

$$\begin{aligned} \|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} &\leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \Big(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau) \Big) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \\ &\leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \Big(\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau. \end{aligned}$$

Therefore, for all $t \in [0, \infty)$,

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \Big(\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \int_0^t \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} . \tag{12.4}$$

12.2. Boundedness of $\mathcal{F}(\theta)(0)$

Using the *a priori* estimate for ϕ derived in Section 12.1, we now show that $\hat{\theta}(0)$ is bounded in time. To that end, we first take the zeroth Fourier mode of (2.19). Plugging into it

$$\mathcal{F}(U_\alpha)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U_\alpha(\alpha) d\alpha = \frac{1}{2\pi} (U(\pi) - U(-\pi)) = 0,$$

we obtain

$$\mathcal{F}(\theta)_t(0) = \frac{2\pi}{L(t)} (\mathcal{F}(T) * \mathcal{F}(1 + \theta_\alpha))(0).$$

Recalling that $T = T_1 + T_{\geq 2}$, we obtain

$$\begin{aligned} \hat{\theta}(0) - \hat{\theta}_0(0) = &\int_0^t \frac{2\pi}{L(\tau)} \mathcal{F}\Big(T_1(\alpha)(1 + \theta_\alpha(\alpha)) \Big)(0) d\tau \\ &+ \int_0^t \frac{2\pi}{L(t)} \mathcal{F}\Big(T_{\geq 2}(\alpha)(1 + \theta_\alpha(\alpha)) \Big)(0) d\tau. \end{aligned}$$

Using that

$$
\begin{aligned}
&\left|\mathcal{F}\left(T_1(\alpha)(1+\theta_\alpha(\alpha))\right)(0)\right| \\
=&\left|\sum_{k\in\mathbb{Z}}\mathcal{F}(T_1)(k)\mathcal{F}(1+\theta_\alpha(\alpha))(-k)\right| \\
\leq&\sum_{k\in\mathbb{Z}}|\mathcal{F}(T_1)(k)|\,|\mathcal{F}(1+\theta_\alpha(\alpha))(-k)| \\
\leq&\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}(T_1)(k)|\right)\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}(1+\theta_\alpha(\alpha))(-k)|\right) \\
=&\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}(T_1)(k)|\right)\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}(1+\theta_\alpha(\alpha))(k)|\right) \\
\leq&\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}(T_1)(k)|\right)\left(1+\sum_{k\in\mathbb{Z}}|\mathcal{F}(\theta_\alpha)(k)|\right) \\
=&\|T_1\|_{\mathcal{F}^{0,1}}\left(1+\sum_{k\neq 0}|k|\,|\mathcal{F}(\phi)(k)|\right) \\
=&\|T_1\|_{\mathcal{F}^{0,1}}\,(1+\|\phi\|_{\mathcal{F}^{1,1}}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&|\mathcal{F}(\theta)(0)| \\
=&\left|\mathcal{F}(\theta_0)(0) + \int_0^t \frac{2\pi}{L(t)}\mathcal{F}\Big(T_1(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)d\tau\right. \\
&\left. + \int_0^t \frac{2\pi}{L(\tau)}\mathcal{F}\Big(T_{\geq 2}(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)d\tau\right| \\
\leq& |\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\left|\mathcal{F}\Big(T_1(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)\right| d\tau \\
&+ \int_0^t \frac{2\pi}{L(\tau)}\left|\mathcal{F}\Big(T_{\geq 2}(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)\right| d\tau \\
\leq& |\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}(1+\|\phi\|_{\dot{\mathcal{F}}^{1,1}})d\tau \\
&+ \int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}(1+\|\phi\|_{\dot{\mathcal{F}}^{1,1}})d\tau \\
\leq& |\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}\,d\tau + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}\,d\tau \\
&+ \int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}\,d\tau + \int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}\,d\tau.
\end{aligned}
$$

Recall that

$$
\frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \geq \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)}}{R} \geq \frac{2\pi}{L(t)}.
$$

Hence,

$$\begin{aligned}
|\mathcal{F}(\theta)(0)| &\leq |\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)} \|T_1\|_{\mathcal{F}^{0,1}} \, d\tau + \int_0^t \frac{2\pi}{L(\tau)} \|T_1\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}} \, d\tau \\
&\quad + \int_0^t \frac{2\pi}{L(\tau)} \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \, d\tau + \int_0^t \frac{2\pi}{L(\tau)} \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}} \, d\tau \\
&\leq |\mathcal{F}(\theta_0)(0)| \\
&\quad + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_1\|_{\mathcal{F}^{0,1}} \, d\tau \\
&\quad + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_1\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}} \, d\tau \\
&\quad + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \, d\tau \\
&\quad + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}} \, d\tau.
\end{aligned}$$

Using that

$$\|T_1\|_{\mathcal{F}^{0,1}} \leq 2 \|U_1\|_{\dot{\mathcal{F}}_\nu^{0,1}} \leq 2 \|U_1\|_{\mathcal{F}_\nu^{0,1}} \leq 2(H_3 + H_4) \|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}$$

and that

$$
\begin{aligned}
\|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \leq & \|\mathcal{M}(U_{\geq 2})\|_{\mathcal{F}^{0,1}} + \|\mathcal{M}(\phi_\alpha U_{\geq 1})\|_{\mathcal{F}^{0,1}} \\
\leq & 2\bigg(\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \bigg) \\
\leq & 2\bigg(\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + \Big(\|\phi_\alpha\|_{\dot{\mathcal{F}}^{0,1}_\nu} \|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu} \|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \Big) \bigg) \\
\leq & 2\bigg(\|U_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} \bigg) \\
\leq & 2\bigg(\|U_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|U_1\|_{\mathcal{F}^{0,1}_\nu} + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|U_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} \bigg) \\
\leq & 2\bigg(D_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|^2_{\mathcal{F}^{0,1}_\nu} + D_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|_{\mathcal{F}^{0,1}_\nu} \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \\
& + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big(H_3 \|\phi\|_{\mathcal{F}^{0,1}_\nu} + H_4 \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big) \\
& + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big(D_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|^2_{\mathcal{F}^{0,1}_\nu} + D_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|_{\mathcal{F}^{0,1}_\nu} \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big) \bigg) \\
\leq & 2\bigg(D_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} + D_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \\
& + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big(H_3 \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} + H_4 \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big) \\
& + 2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} \Big(D_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} + D_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \Big) \bigg),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|\mathcal{F}(\theta)(0)| \leq & |\mathcal{F}(\theta_0)(0)| \\
& + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_1\|_{\mathcal{F}^{0,1}} \, d\tau \\
& + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_1\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}} \, d\tau \\
& + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \, d\tau \\
& + \frac{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}} - 1)}}{R} \int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}} \, d\tau
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left|\mathcal{F}(\theta_0)(0)\right| \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} 2(H_3+H_4)\int_0^t \|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\,d\tau \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \cdot 2(H_3+H_4)\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}\int_0^t \|\phi\|_{\dot{\mathcal{F}}^{2,1}}\,d\tau \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \\
& \cdot 2\Bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \\
& + 2\Bigg(H_3\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \Bigg) \\
& + 2\Bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \Bigg)\Bigg) \\
& \cdot \int_0^t \|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\,d\tau \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \\
& \cdot 2\Bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \\
& + 2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}\left(H_3\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \right) \\
& + 2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}\left(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \right)\Bigg) \\
& \cdot \int_0^t \|\phi\|_{\dot{\mathcal{F}}^{2,1}}\,d\tau \\
\leq & Y(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}),
\end{aligned}
$$

where

$$
\begin{aligned}
& Y(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \qquad (12.5)\\
=& \left|\mathcal{F}(\theta_0)(0)\right| \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} 2(H_3+H_4) \cdot \frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0} \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \cdot 2(H_3+H_4)\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \cdot \frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0} \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \\
& \cdot 2\bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \\
& + 2\bigg(H_3\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg) \\
& + 2\bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \bigg)\bigg) \\
& \cdot \frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0} \\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \\
& \cdot 2\bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \\
& + 2\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg(H_3\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg) \\
& + 2\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \bigg)\bigg) \\
& \cdot \frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0}.
\end{aligned}
$$

Hence, $\mathcal{F}(\theta)(0)$ is bounded in time.

12.3. Regularization Argument

In this Section, we present the details of the regularization argument that is necessary to construct a proof of our main theorem. First of all, based on the original equations for the dynamics of the interface, we define a collection of regularized equations for the dynamics of the interface, which is indexed by $\mathbb{N}$. The sequence of solutions to these regularized equations produces what turns out to be a solution to the original evolution equation for the interface. To obtain solutions to the regularized equations for the dynamics of the interface, we employ Picard's theorem in the Banach space setting as stated in Majda et al. (2002), i.e.,

Theorem 23. *Let $O \subseteq B$ be an open subset of a Banach space B with norm $\|\cdot\|_B$ and let $F : O \to B$ be a nonlinear operator satisfying the following conditions:*

1. *F maps O into B.*

2. *F is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subseteq O$ of X such that*

$$\left\|F(\tilde{X}) - F(\hat{X})\right\|_B \leq L \left\|\tilde{X} - \hat{X}\right\|_B$$

   for all $\tilde{X}, \hat{X} \in U_X$.

Then for any $X_0 \in O$, there exists a time T such that the ordinary differential equation

$$\frac{dX}{dt} = F(X)$$
$$X(0) = X_0 \in O$$

has a unique local solution $X \in C^1((-T,T); O)$. If F does not depend explicitly on time, then solutions to the above ODE can be continued until they leave the set O.

To obtain a candidate for a solution to the original equation for the dynamics of the interface, we use the Aubin-Lions lemma as stated in Gancedo et al. (2023a), i.e.,

Lemma 24. *Let X_0, X, and X_1 be Banach spaces such that*

$$X_0 \subseteq X \subseteq X_1,$$

with compact embedding $X_0 \hookrightarrow X$, and let $p \in (1, \infty]$. Let G be a set of functions mapping $[0,T]$ into X_1 such that G is bounded in $L^p([0,T];X) \cap L^1_{loc}([0,T];X_0)$ and $\partial_t G$ is bounded in $L^1_{loc}([0,T];X_1)$. Then G is relatively compact in $L^q([0,T];X)$, where $q \in [1,p)$.

12.3.1. Regularized Equations for Interface Dynamics

For each $N \in \mathbb{N}$, let us define regularized equations for the dynamics of the interface. We recall that, under HLS parametrization, the dynamics of the interface are governed by

$$\begin{aligned}\theta_t(\alpha) &= \frac{2\pi}{L(t)}(U_\alpha(\theta)(\alpha) + T(\theta)(\alpha)(1+\theta_\alpha(\alpha))), \\ L(t) &= 2\pi R\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta(\alpha)-\theta(\eta))^n d\eta d\alpha\right)^{-\frac{1}{2}}.\end{aligned} \tag{12.6}$$

As mentioned before, since the equations are written in terms of HLS parametrization, they satisfy the identity

$$\int_{-\pi}^{\pi} e^{i(\alpha+\phi(\alpha,t))}d\alpha = 0, \tag{12.7}$$

which constrains $\phi(\alpha,t)$ to have its ± 1 Fourier modes be completely determined by the rest of its nonzero Fourier modes at any given time. Therefore, we seek from the outset for a solution whose ± 1 Fourier modes remain zero in time. Throughout the rest of this Section, we exploit the fact that the analytical expressions for U and T written in terms of $\phi = \theta - \hat{\theta}(0)$ are identical to their respective analytical expressions written in terms of θ. This means that the analytical expressions for U and T in terms of θ are obtained by simply replacing ϕ with θ in the respective analytical expressions for U and T in terms of ϕ. For any fixed $N \in \mathbb{N}$, we define the regularized ordinary differential equation for the interface

$$\frac{d\theta_N}{dt} = (\mathcal{J}_N^1 \circ G_N)(\theta_N).$$

Here, $\mathcal{J}_N^1$ is the high frequency cut-off operator introduced in (2.9) and

$$\begin{aligned} G_N(\theta_N) =& R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N(\alpha)-\theta_N(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \\ &\cdot\left((U_\alpha)_N(\theta_N)+T_N(\theta_N)\Big(1+(\theta_N)_\alpha\Big)\right), \end{aligned}$$

where

$$\begin{aligned} (U_\alpha)_N(\theta_N)(\alpha) &= (\mathcal{J}_N\circ\mathrm{Re})\Big(W(\theta_N)(\alpha)\Big), \\ U_N(\theta_N)(\alpha) &= (\mathcal{J}_N\circ\mathrm{Re})\Big(V(\theta_N)(\alpha)\Big), \\ T_N(\theta_N)(\alpha) &= \mathcal{M}\bigg(\Big(1+(\theta_N)_\alpha(\alpha)\Big)U_N(\theta_N)(\alpha)\bigg). \end{aligned} \tag{12.8}$$

Here, $\mathcal{J}_N$ is the high frequency cut-off operator defined in (2.8). Recalling (4.8) and (4.9) as well as (9.2) and (10.3), we define

$$\begin{aligned} V(\theta_N)(\alpha) =& \sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_j(\theta_N)(\alpha,\beta)d\beta \\ &+\sum_{j=1}^{8}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_j}(\theta_N)(\alpha,\beta)d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\theta_N)(\alpha,\beta)d\beta. \end{aligned}$$

Lastly, we define

$$\begin{aligned} W(\theta_N)(\alpha) =& V_\alpha(\theta_N)(\alpha) \\ =& \sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N)(\alpha,\beta)d\beta \\ &+\sum_{j=1}^{8}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\theta_N)(\alpha,\beta)d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\theta_N)(\alpha,\beta)d\beta. \end{aligned}$$

In the expressions $(E_5)_\alpha(\theta_N)(\alpha,\beta)$ and $(E_6)_\alpha(\theta_N)(\alpha,\beta)$, the second derivative of θ_N shows up. We replace them with lower-order derivatives of θ_N by applying integration by parts. For example, recall that

$$\begin{aligned}(E_5)_\alpha(\theta_N)(\alpha,\beta) =& \frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2} \\ &\cdot \int_0^1 e^{-i\beta s}(-1+s)(\theta_N)_{\alpha\alpha}(\alpha+\beta(-1+s))ds.\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}&\int_0^1 e^{-i\beta s}(-1+s)(\theta_N)_{\alpha\alpha}(\alpha+\beta(-1+s))ds \\ =&\frac{(\theta_N)_\alpha(\alpha-\beta)}{\beta} - \int_0^1 \frac{1}{\beta}(e^{-i\beta s}(-i\beta)(-1+s)+e^{-i\beta s})(\theta_N)_\alpha(\alpha+\beta(-1+s))ds.\end{aligned}$$

12.3.2. Applying Picard's Theorem

We now specify an appropriate Banach space for Picard's theorem. For any $N \in \mathbb{N}$, let

$$H_N^m = \left\{ f \in H^m([-\pi,\pi)) : \operatorname{supp}(\hat{f}) \subseteq [-N,N],\ \hat{f}(\pm 1) = 0,\, \operatorname{Im}(f) = 0 \right\}.$$

The space H_N^m contains the requirement that the ± 1 Fourier modes be zero, because we intend to find a candidate for a solution to the original equations for the dynamics of the interface with this property. The following proposition states that H_N^m is indeed a Banach space.

Proposition 25. *H_N^m is a Banach space.*

Proof. It suffices to show that H_N^m is a closed $\mathbb{R}$-subspace of $H^m([-\pi,\pi))$. It is straightforward to check that H_N^m, which is nonempty because $0 \in H_N^m$, is closed under addition and scalar multiplication. To check that the subspace is closed in $H^m([-\pi,\pi))$, consider a sequence $\{f_n\}$ in H_N^m converging to f in the H^m norm. We show that $f \in H_N^m$. Since

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} (1+|k|^2)^m \left|\mathcal{F}(f_n - f)(k)\right|^2 = 0,$$

there exists a (non-relabeled) subsequence such that for all $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \mathcal{F}(f_n)(k) = \mathcal{F}(f)(k),$$

which implies that $\operatorname{supp}(\hat{f}) \subseteq [-N, N]$ and $\hat{f}(\pm 1) = 0$. For any $\alpha \in [-\pi, \pi)$,

$$\begin{aligned}
|f_n(\alpha) - f(\alpha)| &= \left|\sum_{k\in\mathbb{Z}} \mathcal{F}(f_n - f)(k)e^{ik\alpha}\right| \leq \sum_{k\in\mathbb{Z}} |\mathcal{F}(f_n - f)(k)| \\
&= \sum_{|k|\leq N} |\mathcal{F}(f_n - f)(k)| \leq \sum_{|k|\leq N} (1 + |k|^2)^m |\mathcal{F}(f_n - f)(k)| \\
&\leq \left(\sum_{|k|\leq N} (1 + |k|)^m\right)^{\frac{1}{2}} \left(\sum_{|k|\leq N} (1 + |k|^2)^m |\mathcal{F}(f_n - f)(k)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{|k|\leq N} (1 + |k|)^m\right)^{\frac{1}{2}} \|f_n - f\|_{H^m},
\end{aligned}$$

which shows that f_n converges to f pointwise. Thus,

$$\operatorname{Im}(f) = \lim_{n\to\infty} \operatorname{Im}(f_n) = 0.$$

Therefore, H^m_N is a closed $\mathbb{R}$-subspace of H^m_N, as needed. ∎

For any $M > 0$, let

$$O^M = \{f \in H^m_N : \|f\|_{H^m} < M\}.$$

We want to apply Picard's theorem by setting $B = H^m_N$, $O = O^M$, and $F = \mathcal{J}^1_N \circ G_N$. To check the first condition that $\mathcal{J}^1_N \circ G_N$ maps O^M into H^m_N, let $f \in O^M$. It is immediate from the definition of the regularized equations that $\operatorname{supp} \mathcal{F}((\mathcal{J}^1_N \circ G_N)(f)) \subseteq [-N, N]$ and $\mathcal{F}((\mathcal{J}^1_N \circ G_N)(f))(\pm 1) = 0$.

Since $G_N(f)$ is real,

$$\begin{aligned}
(\mathcal{J}_N^1 \circ G_N)(f)(\alpha) &= \sum_{\substack{|k|\le N \\ |k|\ne 1}} \mathcal{F}(G_N(f))(k)e^{ik\alpha} \\
&= \mathcal{F}(G_N(f))(0) + \sum_{j=2}^{N} \sum_{|k|=j} \mathcal{F}(G_N(f))(k)e^{ik\alpha} \\
&= \mathcal{F}(G_N(f))(0) + \sum_{j=2}^{N} \Big(\mathcal{F}(G_N(f))(-j)e^{-ij\alpha} + \mathcal{F}(G_N(f))(j)e^{ij\alpha} \Big) \\
&= \mathcal{F}(G_N(f))(0) + \sum_{j=2}^{N} \Big(\overline{\mathcal{F}(G_N(f))(j)e^{ij\alpha}} + \mathcal{F}(G_N(f))(j)e^{ij\alpha} \Big),
\end{aligned}$$

which is real. Hence, $\mathrm{Im}(\mathcal{J}_N^1 \circ G_N)(f) = 0$. To check that $(\mathcal{J}_N^1 \circ G_N)(f) \in H^m([-\pi,\pi))$, we note that it suffices to check the second condition that $\mathcal{J}_N^1 \circ G_N$ is locally Lipschitz continuous. In particular, we show that one can choose $U_X = O^M$ for any $X \in O^M$. Let $\theta_N^1, \theta_N^2 \in O^M$. Then

$$
\begin{aligned}
&G_N(\theta_N^1) - G_N(\theta_N^2)\\
=&R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^1)+T_N(\theta_N^1)\left(1+(\theta_N^1)_\alpha\right)\right)\\
&-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right)\\
=&R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^1)+T_N(\theta_N^1)\left(1+(\theta_N^1)_\alpha\right)\right)\\
&-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right)\\
&+R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right)\\
&-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
=&R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\
&\cdot\left((U_\alpha)_N(\theta_N^1)-(U_\alpha)_N(\theta_N^2)+T_N(\theta_N^1)\left(1+(\theta_N^1)_\alpha\right)-T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right)\\
&+\left(R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right.\\
&\left.-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right)\\
&\cdot\left((U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\|G_N(\theta_N^1) - G_N(\theta_N^2)\right\|_{H^m} \\
\leq & R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \\
&\cdot\left\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2) + T_N(\theta_N^1)\left(1 + (\theta_N^1)_\alpha\right) - T_N(\theta_N^2)\left(1 + (\theta_N^2)_\alpha\right)\right\|_{H^m} \\
&+ \left(R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right. \\
&\left. - R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha) - \theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right) \\
&\cdot\left\|(U_\alpha)_N(\theta_N^2) + T_N(\theta_N^2)\left(1 + (\theta_N^2)_\alpha\right)\right\|_{H^m} \\
\leq & R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} && (12.9) \\
&\cdot\left(\left\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2)\right\|_{H^m} + \left\|T_N(\theta_N^1) - T_N(\theta_N^2)\right\|_{H^m}\right. && (12.10) \\
&\left. + \left\|T_N(\theta_N^1)((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)\right\|_{H^m} + \left\|(T_N(\theta_N^1) - T_N(\theta_N^2))(\theta_N^2)_\alpha\right\|_{H^m}\right) && (12.11) \\
&+ \left(R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right. && (12.12) \\
&\left. - R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha) - \theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right) && (12.13) \\
&\cdot\left(\left\|(U_\alpha)_N(\theta_N^2)\right\|_{H^m} + \left\|T_N(\theta_N^2)\right\|_{H^m} + \left\|T_N(\theta_N^2)(\theta_N^2)_\alpha\right\|_{H^m}\right). && (12.14)
\end{aligned}
$$

To check the local Lipschitz continuity of $\mathcal{J}_N^1 \circ G_N$, we need to derive an appropriate estimate for the upper bound for $\left\|G_N(\theta_N^1) - G_N(\theta_N^2)\right\|_{H^m}$, shown in (12.9) through (12.14). We present in detail the process of deriving such estimates for a select few terms in this upper bound, which are typical of the terms making up the upper bound. These derivations will showcase all the techniques that are necessary to derive estimates for the rest of the terms in the upper bound. First of all, we consider the term $\left\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2)\right\|_{H^m}$, which appears in (12.10). Using the definition of

$(U_\alpha)_N$ in (12.8), we obtain

$$
\begin{aligned}
&\left\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2)\right\|_{H^m} && (12.15)\\
\leq &\left\|\mathcal{J}_N\Big(W(\theta_N^1)(\alpha) - W(\theta_N^2)(\alpha)\Big)\right\|_{H^m} \\
\leq &\sum_{j=1}^{7}\left\|\mathcal{J}_N\Big(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N^2)(\alpha,\beta)d\beta\Big)\right\|_{H^m} && (12.16)\\
&+\sum_{j=1}^{8}\left\|\mathcal{J}_N\Big(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\theta_N^2)(\alpha,\beta)d\beta\Big)\right\|_{H^m} && (12.17)\\
&+\left\|\mathcal{J}_N\Big(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\theta_N^2)(\alpha,\beta)d\beta\Big)\right\|_{H^m}. && (12.18)
\end{aligned}
$$

To obtain an appropriate estimate, it is necessary to build some groundwork. For any $m > 0$, we define for a sequence a defined on $k \in \mathbb{Z}$

$$\|a\|_{h^m} = \Big(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m\,|a(k)|^2\Big)^{1/2}.$$

Lemma 26. *Let $N \in \mathbb{N}$ and $m \geq 1$. If a and b are sequences on $\mathbb{Z}$, then*

$$
\begin{aligned}
&\left\|1_{|\cdot|\leq N}(a*b)\right\|_{h^m}\\
\leq &r(m,N)\left\|1_{|\cdot|\leq N}a\right\|_{h^m}\left\|1_{|\cdot|\leq N}b\right\|_{h^m},
\end{aligned}
$$

where

$$r(m,N) = 2^{2m}(1+N^2)^{m/2}\Big(\sum_{|k|\leq N}(1+|k|^2)^m\Big)^{1/2}.$$

Proof.

$$
\begin{aligned}
&\left\|1_{|\cdot|\leq N}(a*b)\right\|_{h^m}\\
=&\Bigg(\sum_{|k|\leq N}(1+|k|^2)^m\,|(a*b)(k)|^2\Bigg)^{1/2}\\
\leq&\Bigg(\sum_{|k|\leq N}(1+|k|^2)^m\left|\sum_{j\in\mathbb{Z}}|a(k-j)|\,|b(j)|\right|^2\Bigg)^{1/2}\\
\leq&\Bigg(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}(1+|k|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\Bigg)^{1/2}.
\end{aligned}
$$

Since $m\geq 1$, for any $k,j\in\mathbb{Z}$,

$$
\begin{aligned}
(1+|k|^2)^m &\leq \Big(1+2\,|k-j|^2+2\,|j|^2\Big)^m\\
&\leq \Big(2(1+|k-j|^2)+2(1+|j|^2)\Big)^m\\
&\leq 2^{m-1}\bigg(\Big(2(1+|k-j|^2)\Big)^m+\Big(2(1+|j|^2)\Big)^m\bigg)\\
&= 2^{2m-1}\Big((1+|k-j|^2)^m+(1+|j|^2)^m\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Bigg(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}(1+|k|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\Bigg)^{1/2}\\
\leq&\Bigg(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}\Big((1+|k-j|^2)^m+(1+|j|^2)^m\Big)\,|a(k-j)|\,|b(j)|\right|^2\Bigg)^{1/2}\\
\leq&\Bigg(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}(1+|k-j|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\Bigg)^{1/2}\\
&+\Bigg(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}(1+|j|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\Bigg)^{1/2}.
\end{aligned}
$$

Letting $(\mathfrak{F}a)(k) = (1+|k|^2)^m \cdot a(k)$, we obtain

$$\begin{aligned}
&\Bigg(\sum_{|k|\le N}\Bigg|\sum_{j\in\mathbb{Z}} 2^{2m-1}(1+|k-j|^2)^m\,|a(k-j)|\,|b(j)|\Bigg|^2\Bigg)^{1/2} \\
&\quad+\Bigg(\sum_{|k|\le N}\Bigg|\sum_{j\in\mathbb{Z}} 2^{2m-1}(1+|j|^2)^m\,|a(k-j)|\,|b(j)|\Bigg|^2\Bigg)^{1/2} \\
&\le 2^{2m-1}\Bigg(\sum_{|k|\le N}|(\mathfrak{F}\,|a|*|b|)(k)|^2\Bigg)^{1/2} + 2^{2m-1}\Bigg(\sum_{|k|\le N}|(\mathfrak{F}\,|b|*|a|)(k)|^2\Bigg)^{1/2}.
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}
&2^{2m-1}\Bigg(\sum_{|k|\le N}|(\mathfrak{F}\,|a|*|b|)(k)|^2\Bigg)^{1/2} + 2^{2m-1}\Bigg(\sum_{|k|\le N}|(\mathfrak{F}\,|b|*|a|)(k)|^2\Bigg)^{1/2} \\
\le& 2^{2m-1}\Bigg(\sum_{|k|\le N}|(\mathfrak{F}\,|a|)(k)|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{|k|\le N}|b(k)|\Bigg) \\
&+2^{2m-1}\Bigg(\sum_{|k|\le N}|(\mathfrak{F}\,|b|)(k)|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{|k|\le N}|a(k)|\Bigg) \\
=& 2^{2m-1}\Bigg(\sum_{|k|\le N}(1+|k|^2)^{2m}\,|a(k)|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\,|b(k)|\Bigg) \\
&+2^{2m-1}\Bigg(\sum_{|k|\le N}(1+|k|^2)^{2m}\,|b(k)|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\,|a(k)|\Bigg) \\
\le& 2^{2m-1}(1+N^2)^{m/2}\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\,|a(k)|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\Bigg)^{1/2} \\
&\cdot\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\,|b(k)|^2\Bigg)^{1/2} \\
&+2^{2m-1}(1+N^2)^{m/2}\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\,|b(k)|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\Bigg)^{1/2} \\
&\cdot\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\,|a(k)|^2\Bigg)^{1/2} \\
=& 2^{2m}(1+N^2)^{m/2}\Bigg(\sum_{|k|\le N}(1+|k|^2)^m\Bigg)^{1/2}\left\|1_{|\cdot|\le N}a\right\|_{h^m}\left\|1_{|\cdot|\le N}b\right\|_{h^m},
\end{aligned}$$

as needed. ∎

Lemma 27. *Let* $m \geq 0$*. If* f *is a periodic function such that* $\operatorname{supp} \mathcal{F}(f) \subseteq [-M, M]$*, then*

$$\|f_\alpha\|_{H^m} \leq \tilde{r}(M) \|f\|_{H^m},$$

where

$$\tilde{r}(M) = (1 + M^2)^{1/2}.$$

Proof.

$$\begin{aligned}
\|f_\alpha\|_{H^m} \leq & \left(\sum_{k \in \mathbb{Z}} (1 + |k|^2)^{m+1} |\mathcal{F}(f)(k)|^2 \right)^{1/2} \\
= & \left(\sum_{|k| \leq M} (1 + |k|^2)^{m+1} |\mathcal{F}(f)(k)|^2 \right)^{1/2} \\
\leq & (1 + M^2)^{1/2} \left(\sum_{k \in \mathbb{Z}} (1 + |k|^2)^m |\mathcal{F}(f)(k)|^2 \right)^{1/2} \\
= & (1 + M^2)^{1/2} \|f\|_{H^m},
\end{aligned}$$

as needed. ∎

Using these lemmas, we present the derivation of an estimate for

$$\left\| \mathcal{J}_N \left(\frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^1(\theta_N^1)(\alpha, \beta) d\beta - \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^1(\theta_N^2)(\alpha, \beta) d\beta \right) \right\|_{H^m},$$

which is one of the terms making up the first term in the sum appearing in (12.17). We recall that

$$\begin{aligned}
B^1_{1,1}(\theta_N)(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1} & \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} j_1 \theta_N(\alpha-\beta)^{j_1-1} \\
& \cdot (\theta_N)_\alpha(\alpha-\beta)\theta_N(\alpha)^{j_2} \cdot \\
& \int_0^1 e^{-i\beta s}\theta_N(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N(\alpha+(s-1)\beta))^m}{m!} ds\right)^n.
\end{aligned}$$

Using the telescoping sum, we obtain

$$\begin{aligned}
& B^1_{1,1}(\theta^1_N)(\alpha,\beta) - B^1_{1,1}(\theta^2_N)(\alpha,\beta) \\
= & -\sum_{j_1+j_2+n\geq 1} \Bigg(\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} j_1 \theta^1_N(\alpha-\beta)^{j_1-1} \\
& \cdot (\theta^1_N)_\alpha(\alpha-\beta)\theta^1_N(\alpha)^{j_2} \\
& \cdot \int_0^1 e^{-i\beta s}\theta^1_N(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta^1_N(\alpha+(s-1)\beta))^m}{m!} ds\right)^n \\
& - \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} j_1 \theta^2_N(\alpha-\beta)^{j_1-1} \\
& \cdot (\theta^2_N)_\alpha(\alpha-\beta)\theta^2_N(\alpha)^{j_2} \\
& \cdot \int_0^1 e^{-i\beta s}\theta^2_N(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta^2_N(\alpha+(s-1)\beta))^m}{m!} ds\right)^n \Bigg)
\end{aligned}$$

$$
\begin{aligned}
= -\sum_{j_1+j_2+n\geq 1} & \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2 j_1! j_2!} j_1 \\
& \cdot \Big((\theta_N^1 - \theta_N^2)(\alpha-\beta)\theta_N^1(\alpha-\beta)^{j_1-2} (\theta_N^1)_\alpha(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \qquad (12.19) \\
& \cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds \Big)^n \\
& + \cdots \\
& + \theta_N^2(\alpha-\beta)^{j_1-2}(\theta_N^1-\theta_N^2)(\alpha-\beta)(\theta_N^1)_\alpha(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \\
& \cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds \Big)^n \\
& + \theta_N^2(\alpha-\beta)^{j_1-1}((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \qquad (12.20) \\
& \cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds \Big)^n \\
& + \theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)(\theta_N^1-\theta_N^2)(\alpha)\theta_N^1(\alpha)^{j_2-1} \\
& \cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \\
& \cdot \Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds \Big)^n
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots\\
&+\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2-1}(\theta_N^1-\theta_N^2)(\alpha)\\
&\cdot\int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&+\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}(\theta_N^1-\theta_N^2)(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\\
&+\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2} \qquad (12.21)\\
&\cdot\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m-(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right)\\
&\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \qquad (12.22)\\
&+\cdots\\
&+\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1}\\
&\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m-(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right)\Bigg).
\end{aligned}
$$

Let us consider the term starting in (12.19), defined as

$$\begin{aligned} S_1(\alpha,\beta) =& (\theta_N^1-\theta_N^2)(\alpha-\beta)\theta_N^1(\alpha-\beta)^{j_1-2}(\theta_N^1)_\alpha(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \\ & \cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \\ & \cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n. \end{aligned} \tag{12.23}$$

Then

$$\begin{aligned} & \mathcal{F}(S_1(\cdot,\beta))(k_1) \\ =& \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \mathcal{F}((\theta_N^1-\theta_N^2)(\cdot-\beta))(k_1-k_2)\prod_{d=2}^{j_1-1}\mathcal{F}(\theta_N^1(\cdot-\beta))(k_d-k_{d+1}) \\ & \cdot \mathcal{F}((\theta_N^1)_\alpha(\cdot-\beta))(k_{j_1}-k_{j_1+1})\prod_{d=j_1+1}^{j_1+j_2}\mathcal{F}(\theta_N^1)(k_d-k_{d+1}) \\ & \cdot \prod_{d=j_1+j_2+1}^{j_1+j_2+n}\mathcal{F}\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\cdot+(s-1)\beta))^m}{m!}ds\right)(k_d-k_{d+1}) \\ & \cdot \mathcal{F}\left(\int_0^1 e^{-i\beta s}\theta_N^1(\cdot+\beta(-1+s))(-1+s)ds\right)(k_{j_1+j_2+n+1}) \\ =& \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \mathcal{F}(\theta_N^1-\theta_N^2)(k_1-k_2)\prod_{d=2}^{j_1-1}\mathcal{F}(\theta_N^1)(k_d-k_{d+1}) \\ & \cdot \mathcal{F}((\theta_N^1)_\alpha)(k_{j_1}-k_{j_1+1})\prod_{d=j_1+1}^{j_1+j_2}\mathcal{F}(\theta_N^1)(k_d-k_{d+1}) \\ & \cdot \prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left(\sum_{m=1}^\infty\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1)^m)(k_d-k_{d+1})\right)\mathcal{F}(\theta_N^1)(k_{j_1+j_2+n+1}) \\ & \cdot e^{-i\beta(k_1-k_2)}e^{-i\beta(k_2-k_{j_1})}e^{-i\beta(k_{j_1}-k_{j_1+1})} \\ & \cdot \prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\right) \\ & \cdot \int_0^1 e^{-i\beta s}(-1+s)e^{i\beta(-1+s)k_{j_1+j_2+n+1}}ds. \end{aligned}$$

We use arguments as in Section 10.1 to obtain

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_1(\cdot,\beta))(k_1)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \\
\le& C_n\Bigg(\left|\mathcal{F}(\theta_N^1-\theta_N^2)\right| * \left|\mathcal{F}(\theta_N^1)\right| * \cdots * \left|\mathcal{F}(\theta_N^1)\right| * \left|\mathcal{F}((\theta_N^1)_\alpha)\right| \\
&* \left|\mathcal{F}(\theta_N^1)\right| * \cdots * \left|\mathcal{F}(\theta_N^1)\right| * \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^1)\right|\Bigg)(k_1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^m} \\
=&\left(\sum_{k\in\mathbb{Z}}1_{|k|\le N}(1+|k|^2)^m\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_1(\cdot,\beta))(k)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right|^2\right)^{1/2} \\
\le& C_n\Big\|1_{|\cdot|\le N}\left|\mathcal{F}(\theta_N^1-\theta_N^2)\right| * \left|\mathcal{F}(\theta_N^1)\right| * \cdots * \left|\mathcal{F}(\theta_N^1)\right| * \left|\mathcal{F}((\theta_N^1)_\alpha)\right| \\
&* \left|\mathcal{F}(\theta_N^1)\right| * \cdots * \left|\mathcal{F}(\theta_N^1)\right| * \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^1)\right|\Big\|_{h^m}.
\end{aligned}$$

We can apply Lemma 26 to obtain

$$\begin{aligned}
&\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^m} \\
\le& C_n\cdot r(m,N)^{j_1+j_2+n} \\
&\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^m}\left\|\theta_N^1\right\|_{H^m}^{j_1+j_2-1}\left\|(\theta_N^1)_\alpha\right\|_{H^m}\left\|1_{|\cdot|\le N}P(\theta_N^1)\right\|_{h^m}^n.
\end{aligned}$$

Using Lemma 27, we obtain

$$\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{H^m} \\
\le& C_n\cdot r(m,N)^{j_1+j_2+n}\cdot\tilde{r}(N) \\
&\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^m}\left\|\theta_N^1\right\|_{H^m}^{j_1+j_2}\left\|1_{|\cdot|\le N}P(\theta_N^1)\right\|_{h^m}^n.
\end{aligned}$$

Now, let us consider the term starting in (12.22), defined as

$$
\begin{aligned}
&S_7(\alpha,\beta)\\
=&\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\\
&-\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\bigg)\\
&\cdot\bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^{n-1}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\mathcal{F}\bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\\
&-\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_1)\\
=&\mathcal{F}\bigg(\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(e^{-i\theta_N^1(\alpha+(s-1)\beta)}-1)ds\\
&-\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(e^{-i\theta_N^2(\alpha+(s-1)\beta)}-1)ds\bigg)(k_1)\\
=&\mathcal{F}\bigg(\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(e^{-i\theta_N^1(\alpha+(s-1)\beta)}-e^{-i\theta_N^2(\alpha+(s-1)\beta)})ds\bigg)(k_1),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{F}\bigg(e^{-i\theta_N^1(\alpha+(s-1)\beta)}-e^{-i\theta_N^2(\alpha+(s-1)\beta)}\bigg)(k_1)\\
=&\mathcal{F}\bigg(e^{-i\theta_N^2(\alpha+(s-1)\beta)}\Big(e^{-i(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta))}-1\Big)\bigg)(k_1)\\
=&\sum_{k_2\in\mathbb{Z}}\mathcal{F}\bigg(e^{-i\theta_N^2(\alpha+(s-1)\beta)}\bigg)(k_1-k_2)\mathcal{F}\bigg(e^{-i(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta))}-1\bigg)(k_2).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathcal{F}\left(e^{-i\theta_N^2(\alpha+(s-1)\beta)}\right)(k_1)\\
=&\mathcal{F}\left(\sum_{m=0}^{\infty}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}\right)(k_1)\\
=&\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\theta_N^2(\alpha+(s-1)\beta)^m)(k_1)\\
=&\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\ldots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\mathcal{F}(\theta_N^2(\alpha+(s-1)\beta))(k_d-k_{d+1})\mathcal{F}(\theta_N^2(\alpha+(s-1)\beta))(k_m)\\
=&\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\ldots,k_m\in\mathbb{Z}}e^{i(s-1)\beta(k_1-k_m)}\prod_{d=1}^{m-1}\mathcal{F}(\theta_N^2)(k_d-k_{d+1})\cdot e^{i(s-1)\beta k_m}\cdot\mathcal{F}(\theta_N^2)(k_m)\\
=&e^{i(s-1)\beta k_1}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(k_1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{F}\Big(e^{-i(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta))}-1\Big)(k_1)\\
=&\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{\Big(-i\Big(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\Big)\Big)^m}{m!}\Bigg)(k_1)\\
=&\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}\Big(\Big(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\Big)^m\Big)(k_1)\\
=&\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\dots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\mathcal{F}\Big(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\Big)(k_d-k_{d+1})\\
&\cdot\mathcal{F}\Big(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\Big)(k_m)\\
=&\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\dots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\Big(e^{i(s-1)\beta(k_d-k_{d+1})}\mathcal{F}(\theta_N^1)(k_d-k_{d+1})\\
&-e^{i(s-1)\beta(k_d-k_{d+1})}\mathcal{F}(\theta_N^2)(k_d-k_{d+1})\Big)\cdot e^{i(s-1)\beta k_m}\cdot(\mathcal{F}(\theta_N^1)(k_m)-\mathcal{F}(\theta_N^2)(k_m))\\
=&\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}e^{i(s-1)\beta k_1}\sum_{k_2,\dots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\mathcal{F}(\theta_N^1-\theta_N^2)(k_d-k_{d+1})\cdot\mathcal{F}(\theta_N^1-\theta_N^2)(k_m)\\
=&e^{i(s-1)\beta k_1}\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(k_1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\\
&-\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1)\\
=&\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\Bigg(\sum_{k_2\in\mathbb{Z}}e^{i(s-1)\beta(k_1-k_2)}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(k_1-k_2)\\
&\cdot e^{i(s-1)\beta k_2}\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(k_2)\Bigg)ds\\
=&\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}e^{i(s-1)\beta k_1}ds\\
&\cdot\Bigg(\bigg(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\bigg)*\bigg(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\bigg)\Bigg)(k_1).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathcal{F}(S_7(\cdot,\beta))(k_1) \\
=& \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1-1} \mathcal{F}(\theta_N^2(\cdot-\beta))(k_d-k_{d+1})\mathcal{F}((\theta_N^2)_\alpha(\cdot-\beta))(k_{j_1}-k_{j_1+1}) \\
&\cdot \prod_{d=j_1+1}^{j_1+j_2} \mathcal{F}(\theta_N^2)(k_d-k_{d+1}) \\
&\cdot \mathcal{F}\bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds \\
&- \sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!} ds\bigg) \\
&(k_{j_1+j_2+1}-k_{j_1+j_2+2}) \\
&\cdot \prod_{d=j_1+j_2+2}^{j_1+j_2+n} \mathcal{F}\bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\bigg)(k_d-k_{d+1}) \\
&\cdot \mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \\
=& \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1-1} \mathcal{F}(\theta_N^2)(k_d-k_{d+1})\mathcal{F}((\theta_N^2)_\alpha)(k_{j_1}-k_{j_1+1}) \\
&\cdot \prod_{d=j_1+1}^{j_1+j_2} \mathcal{F}(\theta_N^2)(k_d-k_{d+1}) \\
&\cdot \bigg(\bigg(\sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\bigg) * \bigg(\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\bigg)\bigg)(k_{j_1+j_2+1}-k_{j_1+j_2+2}) \\
&\cdot \prod_{d=j_1+j_2+2}^{j_1+j_2+n} \bigg(\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1)^m)(k_d-k_{d+1})\bigg)\mathcal{F}(\theta_N^2)(k_{j_1+j_2+n+1}) \\
&\cdot e^{-i\beta(k_1-k_{j_1})}e^{-i\beta(k_{j_1}-k_{j_1+1})} \prod_{d=j_1+j_2+1}^{j_1+j_2+n} \frac{i\beta e^{i\beta}}{1-e^{i\beta}} \int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds \\
&\cdot \int_0^1 e^{-i\beta s}(-1+s)e^{i\beta(-1+s)k_{j_1+j_2+n+1}}ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_7(\cdot,\beta))(k_1)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \\
\leq & C_n\Bigg(\left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| * \left|\mathcal{F}((\theta_N^2)_\alpha)\right| * \left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| \\
& * \left|\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)\right| \\
& * \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^2)\right|\Bigg)(k_1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_7(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^{m'}} \\
=&\left(\sum_{k\in\mathbb{Z}}1_{|k|\leq N}(1+|k|^2)^{m'}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_7(\cdot,\beta))(k)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right|^2\right)^{1/2} \\
\leq & C_n\Bigg\|1_{|\cdot|\leq N}\left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| * \left|\mathcal{F}((\theta_N^2)_\alpha)\right| * \left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| \\
& * \left|\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)\right| \\
& * \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^2)\right|\Bigg\|_{h^{m'}} \\
\leq & C_n\cdot r(m',N)^{j_1+j_2+n}\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2}\left\|(\theta_N^2)_\alpha\right\|_{H^{m'}} \\
& \cdot\left\|1_{|\cdot|\leq N}\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(\cdot)\right\|_{h^{m'}} \\
& \cdot\left\|1_{|\cdot|\leq N}P(\theta_N^1)\right\|_{h^{m'}}^{n-1} \\
\leq & C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2+1} \\
& \cdot\left\|1_{|\cdot|\leq N}\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(\cdot)\right\|_{h^{m'}} \\
& \cdot\left\|1_{|\cdot|\leq N}P(\theta_N^1)\right\|_{h^{m'}}^{n-1}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\left\|1_{|\cdot|\leq N}P(\theta_N^1)\right\|_{h^{m'}}\\
=&\left(\sum_{|k|\leq N}(1+|k|^2)^{m'}\left|P(\theta_N^1)(k)\right|^2\right)^{1/2}\\
=&\left(\sum_{|k|\leq N}(1+|k|^2)^{m'}\left|\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1)^m)(k)\right|^2\right)^{1/2}\\
\leq&\left(\sum_{|k|\leq N}(1+|k|^2)^{m'}\sum_{m=1}^{\infty}\left(\frac{\left|\mathcal{F}((\theta_N^1)^m)(k)\right|}{m!}\right)^2\right)^{1/2}\\
=&\left(\sum_{m=1}^{\infty}\frac{1}{(m!)^2}\sum_{|k|\leq N}(1+|k|^2)^{m'}\left|\mathcal{F}((\theta_N^1)^m)(k)\right|^2\right)^{1/2}\\
=&\left(\sum_{m=1}^{\infty}\frac{1}{(m!)^2}\left\|1_{|\cdot|\leq N}(\mathcal{F}(\theta_N^1)*\cdots*\mathcal{F}(\theta_N^1))\right\|_{h^{m'}}^2\right)^{1/2}\\
\leq&\left(\sum_{m=1}^{\infty}\frac{1}{(m!)^2}\left(r(m',N)^{m-1}\left\|\theta_N^1\right\|_{H^{m'}}^m\right)^2\right)^{1/2}\\
\leq&\left(\sum_{m=1}^{\infty}\frac{1}{(m!)^2}\left(r(m',N)^{m-1}M^m\right)^2\right)^{1/2}\\
\leq&\left(\frac{1}{r(m',N)^2}\sum_{m=1}^{\infty}\frac{(M\cdot r(m',N))^{2m}}{m!}\right)^{1/2}\\
=&\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|1_{|\cdot|\leq N}\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(\cdot)\right\|_{h^{m'}}\\
\leq&r(m',N)\left\|1_{|\cdot|\leq N}\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(\cdot)\right)\right\|_{h^{m'}}\\
&\cdot\left\|1_{|\cdot|\leq N}\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(\cdot)\right)\right\|_{h^{m'}}.
\end{aligned}
$$

We note that

$$\begin{aligned}
&\left\|1_{|\cdot|\le N}\Big(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(\cdot)\Big)\right\|_{h^{m'}}\\
=&\Big(\sum_{|k|\le N}(1+|k|^2)^{m'}\left|\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(k)\right|^2\Big)^{1/2}\\
\le&\Big(\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\Big(r(m',N)^{m-1}\left\|\theta_N^2\right\|_{H^{m'}}^m\Big)^2\Big)^{1/2}\\
\le&\frac{(e^{M^2r(m',N)^2})^{1/2}}{r(m',N)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|1_{|\cdot|\le N}\Big(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(\cdot)\Big)\right\|_{h^{m'}}\\
=&\Big(\sum_{|k|\le N}(1+|k|^2)^{m'}\left|\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(k)\right|^2\Big)^{1/2}\\
\le&\Big(\sum_{m=1}^{\infty}\frac{1}{(m!)^2}\Big(r(m',N)^{m-1}\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}^m\Big)^2\Big)^{1/2}\\
=&\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\Big(\sum_{m=1}^{\infty}\Big(\frac{(r(m',N)\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}})^{m-1}}{m!}\Big)^2\Big)^{1/2}\\
\le&\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\Big(e^{r(m',N)^2(2M)^2}-1\Big)^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|1_{|\cdot|\le N}\cdot\Big(\Big(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\Big)*\Big(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\Big)\Big)(\cdot)\right\|_{h^{m'}}\\
\le& r(m',N)\cdot\frac{(e^{M^2r(m',N)^2})^{1/2}}{r(m',N)}\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\Big(e^{r(m',N)^2(2M)^2}-1\Big)^{1/2}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_7(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^{m'}}\\
\leq& C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2+1}\cdot\left(\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}\right)^{n-1}\\
&\cdot r(m',N)\cdot\frac{(e^{M^2 r(m',N)^2})^{1/2}}{r(m',N)}\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\\
&\cdot\left(e^{r(m',N)^2(2M)^2}-1\right)^{1/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_{1,1}^1(\theta_N^1)(\alpha,\beta)d\beta-\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_{1,1}^1(\theta_N^2)(\alpha,\beta)d\beta\right)\right\|_{H^{m'}}\\
\le&\sum_{j_1+j_2+n\geq 1}\frac{j_1}{2j_1!j_2!}\left(\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^{m'}}+\cdots\right.\\
&\left.+\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_7(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^{m'}}+\cdots\right)\\
\le&\sum_{j_1+j_2+n\geq 1}\frac{j_1}{2j_1!j_2!}C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\\
&\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\left\|\theta_N^1\right\|_{H^{m'}}^{j_1+j_2}\left\|1_{|\cdot|\leq N}P(\theta_N^1)\right\|_{h^{m'}}^n\\
&+\cdots\\
&+\sum_{j_1+j_2+n\geq 1}\frac{j_1}{2j_1!j_2!}C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2+1}\\
&\cdot\left(\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}\right)^{n-1}\\
&\cdot r(m',N)\cdot\frac{(e^{M^2r(m',N)^2})^{1/2}}{r(m',N)}\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\left(e^{r(m',N)^2(2M)^2}-1\right)^{1/2}\\
&+\cdots\\
\le&\sum_{j_1+j_2+n\geq 1}\frac{j_1}{2j_1!j_2!}C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\\
&\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}M^{j_1+j_2}\left(\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}\right)^n\\
&+\cdots\\
&+\sum_{j_1+j_2+n\geq 1}\frac{j_1}{2j_1!j_2!}C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\cdot M^{j_1+j_2+1}\\
&\cdot\left(\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}\right)^{n-1}\\
&\cdot r(m',N)\cdot\frac{(e^{M^2r(m',N)^2})^{1/2}}{r(m',N)}\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\left(e^{r(m',N)^2(2M)^2}-1\right)^{1/2}\\
&+\cdots.
\end{aligned}
$$

We choose M sufficiently small so that all the geometric series contained in the expression above converge. We can similarly derive estimates for the rest of the terms represented by the $\cdots$ above.

In fact, using the techniques that have been showcased here, one can derive estimates for the rest of the terms making up the sum in (12.17) and the term in (12.18). Next, we present the derivation of an estimate for

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (E_1)_\alpha(\theta_N^1)(\alpha,\beta) d\beta - \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (E_1)_\alpha(\theta_N^2)(\alpha,\beta) d\beta \right\|_{H^m},$$

which is one of the terms making up the sum in (12.16). Recalling (9.3), we have

$$\begin{aligned} &(E_1)_\alpha(\theta_N^1)(\alpha,\beta) - (E_1)_\alpha(\theta_N^2)(\alpha,\beta) \\ =&\frac{-e^{i\beta}(-1+e^{i\beta})(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))}{2(-1+e^{i\beta})^2} \\ &\cdot \int_0^1 e^{-i\beta s}((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)(\alpha+\beta(-1+s)) ds. \end{aligned}$$

Then

$$\begin{aligned} &\mathcal{F}((E_1)_\alpha(\theta_N^1)(\cdot,\beta) - (E_1)_\alpha(\theta_N^2)(\cdot,\beta))(k) \\ =&\frac{-e^{i\beta}(-1+e^{i\beta})(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))}{2(-1+e^{i\beta})^2} \cdot \int_0^1 e^{-i\beta s} e^{ik\beta(-1+s)} ds \\ &\cdot \mathcal{F}((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)(k). \end{aligned}$$

Using the estimate in (9.4), we obtain

$$\begin{aligned} &\left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}((E_1)_\alpha(\theta_N^1)(\cdot,\beta) - (E_1)_\alpha(\theta_N^2)(\cdot,\beta))(k) d\beta \right| \\ =&\frac{\gamma}{4\pi} \left(2\pi + \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \frac{1}{2}\pi^2 \right) \left| \mathcal{F}((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)(k) \right|. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (E_1)_\alpha(\theta_N^1)(\cdot,\beta) - (E_1)_\alpha(\theta_N^2)(\cdot,\beta) d\beta \right\|_{H^m} \\
=&\left(\sum_{k\in\mathbb{Z}} (1+|k|^2)^m \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}((E_1)_\alpha(\theta_N^1)(\cdot,\beta) - (E_1)_\alpha(\theta_N^2)(\cdot,\beta))(k) d\beta \right|^2 \right)^{1/2} \\
\leq&\frac{\gamma}{4\pi} \left(2\pi + \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \frac{1}{2}\pi^2 \right) \left\| (\theta_N^1)_\alpha - (\theta_N^2)_\alpha \right\|_{H^m} \\
\leq&\frac{\gamma}{4\pi} \left(2\pi + \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \frac{1}{2}\pi^2 \right) \tilde{r}(N) \left\| \theta_N^1 - \theta_N^2 \right\|_{H^m}.
\end{aligned}
$$

We can similarly derive estimates for the rest of the terms in the sum in (12.16). To derive estimates for the rest of the terms appearing in the upper bound shown in (12.9) through (12.14), the following lemmas are helpful.

Lemma 28. *If θ_N has finite support, then $T_N(\theta_N)$ has finite support.*

Proof. Using (2.7), we obtain that for $k \neq 0$,

$$
\begin{aligned}
\mathcal{F}(T_N(\theta_N))(k) =& \mathcal{F}\left(\mathcal{M}\left((1+(\theta_N)_\alpha) U_N(\theta_N) \right) \right)(k) \\
=& -\frac{i}{k} \mathcal{F}\left((1+(\theta_N)_\alpha) U_N(\theta_N) \right)(k) \\
=& -\frac{i}{k} \left(\mathcal{F}(U_N(\theta_N))(k) + \mathcal{F}((\theta_N)_\alpha U_N(\theta_N))(k) \right).
\end{aligned}
$$

Since θ_N and $U_N(\theta_N)$ have finite support, the product $(\theta_N)_\alpha \cdot U_N(\theta_N)$ has finite support. Therefore, $T_N(\theta_N)$ has finite support as well, as needed. ■

Lemma 29. *If f is a periodic function such that* $\operatorname{supp} \hat{f} \subseteq [-M, M]$, *then*

$$
\|\mathcal{M}(f)\|_{H^m} \leq 2^M \|f\|_{H^m}.
$$

Proof.

$$
\begin{aligned}
&\|\mathcal{M}(f)\|_{H^m}\\
=&\left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m\,|\mathcal{F}(\mathcal{M}(f))(k)|^2\right)^{1/2}\\
=&\left(\left|\sum_{j\neq 0}\frac{i}{j}\mathcal{F}(f)(j)\right|^2+\sum_{k\neq 0}(1+|k|^2)^m\left|-\frac{i}{k}\mathcal{F}(f)(k)\right|^2\right)^{1/2}\\
\leq&\left(\left(\sum_{k\neq 0}\frac{1}{|k|}\,|\mathcal{F}(f)(k)|\right)^2+\sum_{k\neq 0}(1+|k|^2)^{m-1}\left(\frac{1}{|k|^2}+1\right)|\mathcal{F}(f)(k)|^2\right)^{1/2}\\
=&\left(\left(\sum_{\substack{|k|\leq M\\ k\neq 0}}\frac{1}{|k|}\,|\mathcal{F}(f)(k)|\right)^2+\sum_{\substack{|k|\leq M\\ k\neq 0}}(1+|k|^2)^{m-1}\left(\frac{1}{|k|^2}+1\right)|\mathcal{F}(f)(k)|^2\right)^{1/2}\\
\leq&\left(2^{2M-1}\sum_{\substack{|k|\leq M\\ k\neq 0}}\frac{1}{|k|^2}\,|\mathcal{F}(f)(k)|^2+\sum_{\substack{|k|\leq M\\ k\neq 0}}(1+|k|^2)^m\,|\mathcal{F}(f)(k)|^2\right)^{1/2}\\
\leq&\left(2^{2M}\sum_{|k|\leq M}(1+|k|^2)^m\,|\mathcal{F}(f)(k)|^2\right)^{1/2}\\
=&2^M\,\|f\|_{H^m}\,.
\end{aligned}
$$

■

Now, we consider the term $\left\|T_N(\theta_N^1)-T_N(\theta_N^2)\right\|_{H^m}$, which appears in (12.10). Using these lemmas, we observe that

$$\begin{aligned}
&\left\|T_N(\theta_N^1)-T_N(\theta_N^2)\right\|_{H^m}\\
=&\left\|\mathcal{M}\Big((1+(\theta_N^1)_\alpha)U_N(\theta_N^1)\Big)-\mathcal{M}\Big((1+(\theta_N^2)_\alpha)U_N(\theta_N^2)\Big)\right\|_{H^m}\\
=&\left\|\mathcal{M}\Big(U_N(\theta_N^1)-U_N(\theta_N^2)\Big)+\mathcal{M}\Big((\theta_N^1)_\alpha U_N(\theta_N^1)-(\theta_N^2)_\alpha U_N(\theta_N^2)\Big)\right\|_{H^m}\\
\leq&\left\|\mathcal{M}\Big(U_N(\theta_N^1)-U_N(\theta_N^2)\Big)\right\|_{H^m}\\
&+\left\|\mathcal{M}\Big((\theta_N^1)_\alpha U_N(\theta_N^1)-(\theta_N^1)_\alpha U_N(\theta_N^2)+(\theta_N^1)_\alpha U_N(\theta_N^2)-(\theta_N^2)_\alpha U_N(\theta_N^2)\Big)\right\|_{H^m}\\
\leq&\left\|\mathcal{M}\Big(U_N(\theta_N^1)-U_N(\theta_N^2)\Big)\right\|_{H^m}+\left\|\mathcal{M}\Big((\theta_N^1)_\alpha(U_N(\theta_N^1)-U_N(\theta_N^2))\Big)\right\|_{H^m}\\
&+\left\|\mathcal{M}\Big(((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)U_N(\theta_N^2)\Big)\right\|_{H^m}\\
\leq&2^N\left\|U_N(\theta_N^1)-U_N(\theta_N^2)\right\|_{H^m}+2^{l(N)}\left\|(\theta_N^1)_\alpha(U_N(\theta_N^1)-U_N(\theta_N^2))\right\|_{H^m}\\
&+2^{l(N)}\left\|((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)U_N(\theta_N^2)\right\|_{H^m}\\
\leq&2^N\left\|U_N(\theta_N^1)-U_N(\theta_N^2)\right\|_{H^m}+2^{l(N)}r(m,N)\tilde{r}(N)\left\|\theta_N^1\right\|_{H^m}\left\|U_N(\theta_N^1)-U_N(\theta_N^2)\right\|_{H^m}\\
\leq&2^N\left\|U_N(\theta_N^1)-U_N(\theta_N^2)\right\|_{H^m}+2^{l(N)}r(m,N)\tilde{r}(N)M\left\|U_N(\theta_N^1)-U_N(\theta_N^2)\right\|_{H^m}
\end{aligned}$$

for some function $l(N)$ of N. Hence, it suffices to find an estimate for $\left\|U_N(\theta_N^1)-U_N(\theta_N^2)\right\|_{H^m}$. To do so, we first observe that for any periodic function f,

$$\begin{aligned}
\mathcal{F}(\mathrm{Re}(f))(k)=&\mathcal{F}\left(\frac{1}{2}(f+\bar{f})\right)(k)\\
=&\frac{1}{2}\Big(\mathcal{F}(f)(k)+\mathcal{F}(\bar{f}(k))\Big)\\
=&\frac{1}{2}\Big(\mathcal{F}(f)(k)+\overline{\mathcal{F}(f)(-k)}\Big).
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\mathcal{F}\Big(\mathrm{Re}(V(\theta_N^1)) - \mathrm{Re}(V(\theta_N^2))\Big)(k)\right| \\
=&\left|\frac{1}{2}\Big(\mathcal{F}(V(\theta_N^1))(k) + \overline{\mathcal{F}(V(\theta_N^1))(-k)}\Big) - \frac{1}{2}\Big(\mathcal{F}(V(\theta_N^2))(k) + \overline{\mathcal{F}(V(\theta_N^2))(-k)}\Big)\right| \\
\leq&\frac{1}{2}\Big(\big|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(k)\big| + \big|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(-k)\big|\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\mathcal{F}\Big(\mathrm{Re}(V(\theta_N^1)) - \mathrm{Re}(V(\theta_N^2))\Big)(k)\right|^2 \\
=&\frac{1}{2}\Big(\big|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(k)\big|^2 + \big|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(-k)\big|^2\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big\|U_N(\theta_N^1) - U_N(\theta_N^2)\big\|_{H^m} \\
=&\Bigg(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m\left|\mathcal{F}\Big(U_N(\theta_N^1) - U_N(\theta_N^2)\Big)(k)\right|^2\Bigg)^{1/2} \\
=&\Bigg(\sum_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}\Big(\mathrm{Re}(V(\theta_N^1)) - \mathrm{Re}(V(\theta_N^2))\Big)(k)\right|^2\Bigg)^{1/2} \\
\leq&\Bigg(\frac{1}{2}\sum_{|k|\leq N}(1+|k|^2)^m(\big|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(k)\big|^2 \\
&+ \big|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(-k)\big|^2)\Bigg)^{1/2} \\
\leq&\big\|V(\theta_N^1) - V(\theta_N^2)\big\|_{H^m} \\
\leq&\sum_{j=1}^{7}\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m} \\
&+\sum_{j=1}^{8}\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_j}(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_j}(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m} \\
&+\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m}.
\end{aligned}$$

The derivation of an appropriate estimate for $\big\|U_N(\theta_N^1) - U_N(\theta_N^2)\big\|_{H^m}$ can be completed using the techniques that had been introduced earlier for $\left\|(U_\alpha)_N(\theta_N^1)-(U_\alpha)_N(\theta_N^2)\right\|_{H^m}$. Moreover, we note that using the techniques introduced thus far, appropriate estimates for the terms $\left\|T_N(\theta_N^1)((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)\right\|_{H^m}$ and $\left\|(T_N(\theta_N^1)-T_N(\theta_N^2))(\theta_N^2)_\alpha\right\|_{H^m}$ in (12.11) and the terms in (12.14) can be derived. Lastly, we derive an appropriate estimate for

$$\begin{aligned}&R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\\&-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}.\end{aligned}$$

which appears in (12.12) through (12.13). We note that for a concave function f,

$$f(y)-f(x)\leq f'(x)(y-x)$$

for all $x,y\in\mathbb{R}$. If $y>x$, then

$$\frac{f(y)-f(x)}{y-x}\leq f'(x).$$

If f is also monotone, then

$$\left|\frac{f(y)-f(x)}{y-x}\right|\leq f'(x).$$

Without loss of generality, we let

$$\begin{aligned}&\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\\>&\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha.\end{aligned}$$

Since the square root function is concave and monotone,

$$\left|y^{1/2}-x^{1/2}\right|\leq\frac{1}{2\sqrt{x}}\left|y-x\right|$$

for $y > x$. In particular,

$$
\begin{aligned}
&R^{-1}\Bigg|\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{1/2} \\
&\quad-\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{1/2}\Bigg| \\
\leq & R^{-1}\cdot\frac{1}{2}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} \\
&\quad\cdot\Bigg|\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}d\eta d\alpha \\
&\quad-\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))}d\eta d\alpha\Bigg| \\
\leq & R^{-1}\cdot\frac{1}{4\pi}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} \qquad (12.24) \\
&\quad\cdot\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right|.
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\\
=&\frac{1}{i}\int_{-\pi}^{\pi}\frac{\partial}{\partial\alpha}(e^{i\alpha})\int_0^{\alpha} e^{-i\eta}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\\
=&\frac{1}{i}\Bigg(\int_{-\pi}^{\pi}\frac{\partial}{\partial\alpha}\Bigg(e^{i\alpha}\int_0^{\alpha} e^{-i\eta}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta\Bigg)d\alpha\\
&-\int_{-\pi}^{\pi}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\alpha))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\alpha))})d\alpha\Bigg)\\
=&\frac{1}{i}\Bigg(e^{i\pi}\int_0^{\pi} e^{-i\eta}(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))})d\eta\\
&-e^{-i\pi}\int_0^{-\pi} e^{-i\eta}(e^{i(\theta_N^1(-\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(-\pi)-\theta_N^2(\eta))})d\eta\Bigg)\\
=&i\Bigg(\int_{-\pi}^{\pi} e^{-i\eta}(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))})d\eta\Bigg)\\
=&2\pi i\cdot\mathcal{F}\Bigg(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))}\Bigg)(1)\\
=&2\pi i\mathcal{F}\Bigg(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^1(\pi)-\theta_N^2(\eta))}+e^{i(\theta_N^1(\pi)-\theta_N^2(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))}\Bigg)(1)\\
=&2\pi i\mathcal{F}\Bigg(e^{i\theta_N^1(\pi)}\Bigg(e^{-i\theta_N^1(\eta)}-e^{-i\theta_N^2(\eta)}\Bigg)+e^{-i\theta_N^2(\eta)}\Bigg(e^{i\theta_N^1(\pi)}-e^{i\theta_N^2(\pi)}\Bigg)\Bigg)(1)\\
=&2\pi i\mathcal{F}\Bigg(e^{i\theta_N^1(\pi)}e^{-i\theta_N^2(\eta)}\Bigg(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1\Bigg)\Bigg)(1)\\
&+2\pi i\mathcal{F}\Bigg(e^{-i\theta_N^2(\eta)}e^{i\theta_N^2(\pi)}\Bigg(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1\Bigg)\Bigg)(1),
\end{aligned}
$$

where

$$
\begin{aligned}
&\left|\mathcal{F}\Bigg(e^{i\theta_N^1(\pi)}e^{-i\theta_N^2(\eta)}\Bigg(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1\Bigg)\Bigg)(1)\right|\\
=&\left|\sum_{k_2,k_3\in\mathbb{Z}}\mathcal{F}(e^{i\theta_N^1(\pi)})(1-k_2)\mathcal{F}(e^{-i\theta_N^2(\eta)})(k_2-k_3)\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\right|\\
=&\left|\sum_{k_3\in\mathbb{Z}}e^{i\theta_N^1(\pi)}\mathcal{F}(e^{-i\theta_N^2(\eta)})(1-k_3)\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\right|\\
\leq&\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathcal{F}\left(e^{-i\theta_N^2(\eta)}e^{i\theta_N^2(\pi)}\left(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1\right)\right)(1)\right| \\
=&\left|\sum_{k_2,k_3\in\mathbb{Z}}\mathcal{F}(e^{i\theta_N^2(\pi)})(1-k_2)\mathcal{F}(e^{-i\theta_N^2(\eta)})(k_2-k_3)\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right| \\
=&\left|\sum_{k_3\in\mathbb{Z}}e^{i\theta_N^2(\pi)}\mathcal{F}(e^{-i\theta_N^2(\eta)})(1-k_3)\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right| \\
\le&\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right|.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k)=&\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)e^{-ik\eta}d\eta \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}(\theta_N^1(\eta)-\theta_N^2(\eta))^n e^{-ik\eta}d\eta \\
=&\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\mathcal{F}((\theta_N^1(\eta)-\theta_N^2(\eta))^n)(k),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k)=&\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)e^{-ik\eta}d\eta \\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\frac{i^n}{n!}(\theta_N^1(\pi)-\theta_N^2(\pi))^n e^{-ik\eta}d\eta \\
=&\sum_{n=1}^{\infty}\frac{i^n}{n!}\mathcal{F}((\theta_N^1(\pi)-\theta_N^2(\pi))^n)(k),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right| \\
\leq& 2\pi\left|\mathcal{F}\left(e^{i\theta_N^1(\pi)}e^{-i\theta_N^2(\eta)}\left(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1\right)\right)(1)\right| \\
&+2\pi\left|\mathcal{F}\left(e^{-i\theta_N^2(\eta)}e^{i\theta_N^2(\pi)}\left(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1\right)\right)(1)\right| \\
\leq& 2\pi\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\right| \\
&+2\pi\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right| \\
\leq& 2\pi\sum_{n=1}^{\infty}\frac{\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}^n}{n!}+2\pi\sum_{n=1}^{\infty}\frac{\left\|\theta_N^1(\pi)-\theta_N^2(\pi)\right\|_{\mathcal{F}^{0,1}}^n}{n!} \\
\leq& 2\pi\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\left\|\theta_N^1\right\|_{\mathcal{F}^{0,1}}+\left\|\theta_N^2\right\|_{\mathcal{F}^{0,1}})^{n-1}}{n!} \\
&+2\pi\left\|\theta_N^1(\pi)-\theta_N^2(\pi)\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\left\|\theta_N^1(\pi)\right\|_{\mathcal{F}^{0,1}}+\left\|\theta_N^2(\pi)\right\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\left\|\theta_N^2\right\|_{\mathcal{F}^{0,1}}=&\sum_{k\in\mathbb{Z}}\left|\mathcal{F}(\theta_N^2)(k)\right| \\
\leq&\sum_{k\in\mathbb{Z}}1_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^2)(k)\right| \\
\leq&\left(\sum_{k\in\mathbb{Z}}\left(1_{|k|\leq N}(1+|k|^2)^{m/2}\right)^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}}\left((1+|k|^2)^{m/2}\left|\mathcal{F}(\theta_N^2)(k)\right|\right)^2\right)^{1/2} \\
\leq&\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left\|\theta_N^2\right\|_{H^m} \\
\leq&\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M,
\end{aligned}
\tag{12.25}
$$

and

$$\begin{aligned}
\left\|\theta_N^1(\pi)-\theta_N^2(\pi)\right\|_{\mathcal{F}^{0,1}} &= \sum_{k\in\mathbb{Z}}\left|\mathcal{F}(\theta_N^1(\pi)-\theta_N^2(\pi))(k)\right| \\
&= \left|\theta_N^1(\pi)-\theta_N^2(\pi)\right| \\
&= \left|\sum_{k\in\mathbb{Z}}\mathcal{F}(\theta_N^1-\theta_N^2)(k)e^{ik\pi}\right| \\
&\leq \left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}},
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right| \\
\leq& 2\pi\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!} \\
&+2\pi\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!}.
\end{aligned}$$

We note that

$$\begin{aligned}
\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}} &= \sum_{k\in\mathbb{Z}}\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right| \\
&\leq \sum_{k\in\mathbb{Z}}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right| \\
&\leq \sum_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right| \\
&\leq \left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left(\sum_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right|^2\right)^{1/2} \\
&\leq \left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left\|\theta_N^1-\theta_N^2\right\|_{H^m}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right| \\
\leq&2\pi\biggl(\sum_{|k|\leq N}(1+|k|^2)^m\biggr)^{1/2}\left\|\theta_N^1-\theta_N^2\right\|_{H^m} \\
&\cdot\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!} \\
&+2\pi\biggl(\sum_{|k|\leq N}(1+|k|^2)^m\biggr)^{1/2}\left\|\theta_N^1-\theta_N^2\right\|_{H^m} \\
&\cdot\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!}.
\end{aligned}
$$

As for

$$
\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2},
$$

which appears in (12.24), we use the estimate in (7.2) to obtain

$$
\begin{aligned}
&\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} \\
\leq&\left(1-\frac{\pi}{2}\left(e^{2\|\theta_N^2\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}.
\end{aligned}
$$

Using the estimate in (12.25), we obtain

$$
\begin{aligned}
&\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} \\
\leq&\left(1-\frac{\pi}{2}\left(e^{2\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M}-1\right)\right)^{-1/2}.
\end{aligned}
\tag{12.26}
$$

We choose M sufficiently small so that (12.26) is well-defined. We note that an appropriate estimate for the expression in (12.9) can be derived similarly. This completes the proof that the operator $\mathcal{J}_N^1 \circ G_N$ is indeed locally Lipschitz continuous. Therefore, by Picard's theorem, for any $\theta_{N,0} \in O^M$, there exists a time $T_N > 0$ such that the ordinary differential equation

$$\frac{d\theta_N}{dt} = (\mathcal{J}_N^1 \circ G_N)(\theta_N), \tag{12.27}$$

$$\theta_N(0) = \theta_{N,0} \in O^M \tag{12.28}$$

has a unique local solution $\theta_N \in C^1([0, T_N); O^M)$.

12.3.3. Derivation of an *a priori* Estimate

For every $n \in \mathbb{N}$, define $\phi_N(\alpha, t) = \theta_N(\alpha, t) - \hat{\theta}_N(0, t)$. We let

$$\begin{aligned}
(U_\alpha)_N(\theta_N) =& (U_\alpha)_{N,0}(\theta_N) + (U_\alpha)_{N,1}(\theta_N) + (U_\alpha)_{N,2}(\theta_N), \\
T_N(\theta_N) =& T_{N,0}(\theta_N) + T_{N,1}(\theta_N) + T_{N,2}(\theta_N),
\end{aligned}$$

where $(U_\alpha)_{N,0}(\theta_N)$, $(U_\alpha)_{N,1}(\theta_N)$, and $(U_\alpha)_{N,2}(\theta_N)$ are the parts of $(U_\alpha)_N(\theta_N)$ that are constant, linear, and superlinear in the variable θ_N; and $T_{N,0}(\theta_N)$, $T_{N,1}(\theta_N)$, and $T_{N,2}(\theta_N)$ are the parts of $T_N(\theta_N)$ that are constant, linear, and superlinear in the variable θ_N. We note that

$$\frac{d\theta_N}{dt} = \mathcal{L}_N(\theta_N) + \mathcal{N}_N(\theta_N),$$

where $\mathcal{L}_N(\theta_N)$ and $\mathcal{N}_N(\theta_N)$ are the parts of the right hand side of (12.27) which are linear and superlinear in the variable θ_N. In particular,

$$
\begin{aligned}
\mathcal{L}_N(\theta_N) =& \frac{2\pi}{L(\theta_N)(t)}\Big((U_\alpha)_{N,1}(\theta_N) + T_{N,0}(\theta_N)\cdot(\theta_N)_\alpha + T_{N,1}(\theta_N)\Big) \\
=& \frac{2\pi}{L(\theta_N)(t)}\Big((U_\alpha)_{N,1}(\theta_N) + T_{N,1}(\theta_N)\Big) \\
=& \frac{2\pi}{L(\theta_N)(t)}\Big((U_\alpha)_{N,1}(\theta_N) + \mathcal{M}\Big(U_{N,1}(\theta_N)(\alpha)\Big)\Big) \\
=& \frac{2\pi}{L(\theta_N)(t)}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N)(\alpha,\beta)d\beta\Big) \\
&+\mathcal{M}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N)(\alpha,\beta)d\beta\Big)\Big)\Big) \\
=& \frac{2\pi}{L(\phi_N)(t)}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\phi_N)(\alpha,\beta)d\beta\Big) \\
&+\mathcal{M}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\phi_N)(\alpha,\beta)d\beta\Big)\Big)\Big).
\end{aligned}
$$

Hence, for $k\neq 0$,

$$
\begin{aligned}
\mathcal{F}(\mathcal{L}_N(\phi_N))(k) =& \frac{2\pi}{L(\phi_N)(t)}\Big(1_{|k|\le N}\cdot\mathcal{F}\Big(\mathrm{Re}\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\phi_N)(\alpha,\beta)d\beta\Big)\Big)(k) \\
&- 1_{|k|\le N}\cdot\frac{i}{k}\mathcal{F}\Big(\mathrm{Re}\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\phi_N)(\alpha,\beta)d\beta\Big)\Big)(k)\Big).
\end{aligned}
$$

We note that this expression differs from that of $\mathcal{F}(\mathcal{L})(k)$ in (6.2) only by the presence of $1_{|k|\le N}$. This means that for $1\le|k|\le N$, the analogue of the expression in (6.26) holds, i.e.,

$$
\mathcal{F}(\mathcal{L}_N(\phi_N))(k) = \begin{cases} 0 & |k|>N, \\ -\frac{2\pi}{L(\phi_N)(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi_N)(k)\pi\,|k| & 1<|k|\le N, \\ 0 & |k|=1. \end{cases} \tag{12.29}
$$

Defining

$$\widetilde{\mathcal{L}}_N(\theta_N) = \frac{L(\theta_N)(t)}{2\pi}\mathcal{L}_N(\theta_N),$$
$$\widetilde{\mathcal{N}}_N(\theta_N) = \frac{L(\theta_N)(t)}{2\pi}\mathcal{N}_N(\theta_N),$$

we note that

$$\widetilde{\mathcal{N}}_N(\phi_N) = (U_\alpha)_{N,2}(\phi_N) + T_{N,2}(\phi_N)(1+(\phi_N)_\alpha) + T_{N,1}(\phi_N)\cdot(\phi_N)_\alpha.$$

The analogues of Lemmas 15 and 16 hold for $T_{N,2}(\phi_N)(1+(\phi_N)_\alpha)$ and $T_{N,1}(\phi_N)\cdot(\phi_N)_\alpha$, respectively. Hence, it suffices to derive estimates for the $\mathcal{F}^{0,1}_\nu$ and $\dot{\mathcal{F}}^{s,1}_\nu$ norms of $U_{N,1}(\phi_N)$ and $U_{N,2}(\phi_N)$, as well as the $\dot{\mathcal{F}}^{s,1}_\nu$ norm of $(U_\alpha)_{N,2}(\phi_N)$. For these norms, we can use the estimates presented in Chapters 9, 10, and 11. Using (12.29), we obtain for $0 < |k| \le N$,

$$\frac{\partial}{\partial t}\mathcal{F}(\phi_N)(k) = \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi_N)(k)(J_1(k)+J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}}_N(\phi_N))(k), \tag{12.30}$$

where J_1 and J_2 are the same as in (7.3). Since ϕ_N is real-valued, for $k > 0$,

$$\left|\hat{\phi_N}(-k)\right| = \left|\overline{\hat{\phi_N}(k)}\right| = \left|\hat{\phi_N}(k)\right|.$$

Then for $s > 0$,

$$\|\phi_N\|_{\dot{\mathcal{F}}^{s,1}_\nu} = \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s\left|\hat{\phi_N}(k)\right| = 2\sum_{k\ge 1} e^{\nu(t)k}k^s\left|\hat{\phi_N}(k)\right|.$$

The norm $\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}$ is differentiable with respect to time with

$$\begin{aligned}\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}} =&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k\cdot k^s\left|\hat{\phi_N}(k)\right| + e^{\nu(t)k}k^s\frac{\partial}{\partial t}\left|\hat{\phi_N}(k)\right| \\ =&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi_N}(k)\right| \\ &+ e^{\nu(t)k}k^s\frac{1}{\left|\hat{\phi_N}(k)\right|}\frac{1}{2}\left(\hat{\phi_N}(k)\overline{\frac{\partial}{\partial t}\hat{\phi_N}(k)} + \overline{\hat{\phi_N}(k)}\frac{\partial}{\partial t}\hat{\phi_N}(k)\right) \\ =&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi_N}(k)\right| \\ &+ 2\sum_{k\geq 1} e^{\nu(t)k}k^s\frac{\hat{\phi_N}(k)\overline{\frac{\partial}{\partial t}\hat{\phi_N}(k)} + \overline{\hat{\phi_N}(k)}\frac{\partial}{\partial t}\hat{\phi_N}(k)}{2\left|\hat{\phi_N}(k)\right|},\end{aligned}$$

where $\frac{\partial}{\partial t}\hat{\phi_N}(k)$ is given in (12.30). In particular, $\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}$ is continuous with respect to time. We can use the calculations presented in Chapter 7 to obtain

$$\begin{aligned}\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq&\nu'(t)\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi_N}(k)\right| + \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}_N\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\ &+ 2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi_N\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi_N}(k)\right|.\end{aligned}$$

Since $\hat{\phi_N}(1) = 0$, this is equal to

$$\begin{aligned}\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq&\nu'(t)\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} - \pi\frac{1}{R}\frac{\gamma}{4\pi}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} + \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}_N\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \\ &+ \frac{\gamma}{4\pi}\frac{1}{R}A\|\phi_N\|_{\mathcal{F}^{0,1}}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} \\ \leq&\left(\nu'(t) - \pi\frac{1}{R}\frac{\gamma}{4\pi} + \frac{\gamma}{4\pi}\frac{1}{R}A\|\phi_N\|_{\mathcal{F}^{0,1}}\right)\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} \\ &+ \frac{1}{R}\frac{1}{A_1}\left\|\widetilde{\mathcal{N}}_N\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}.\end{aligned}$$

Now, setting $s=1$ and using the estimates for the $\mathcal{F}^{0,1}_{\nu}$ and $\dot{\mathcal{F}}^{s,1}_{\nu}$ norms of $U_{N,1}(\phi_N)$ and $U_{N,2}(\phi_N)$, as well as the $\dot{\mathcal{F}}^{s,1}_{\nu}$ norm of $(U_\alpha)_{N,2}(\phi_N)$, we obtain

$$\frac{d}{dt}\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq -\bigg(\Lambda(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(t)\bigg)\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{2,1}_{\nu}},$$

where

$$\begin{aligned}
\Lambda(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) =& \pi\frac{2}{R}\frac{\gamma}{4\pi} - \frac{\gamma}{4\pi}\frac{1}{R}A(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \\
&- \frac{1}{R}\frac{1}{A_1(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})} \\
&\cdot\bigg(R_1(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + R_2(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \\
&+ R_3(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|^2_{\dot{\mathcal{F}}^{1,1}_{\nu}} + R_4(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|^2_{\dot{\mathcal{F}}^{1,1}_{\nu}} \\
&+ R_5(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \\
&+ 3\left(H_3\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + H_4\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&+ 3\left(D_1(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|^2_{\dot{\mathcal{F}}^{1,1}_{\nu}} + D_2(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|^2_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&\cdot\left(1 + 2\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&+ \left(D_1(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + D_2(\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&\cdot\left(1 + 2\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&+ 6\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\left(H_3\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + H_4\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right) \\
&+ 2\left(H_3\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + H_4\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\right)\bigg).
\end{aligned}$$

We note that the above expression is well-defined only when $\left\|\phi_N\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$ is small enough for all the geometric series with respect to n contained in this expression to converge. To ensure that this expression is well-defined for all time, we need to choose an appropriate initial datum. We let

$\theta^0 \in \dot{\mathcal{F}}^{1,1}$, $\mathrm{Im}(\theta^0) = 0$, such that $\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) > 0$, where

$$\begin{aligned}
\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) =& \pi \frac{2}{R}\frac{\gamma}{4\pi} - \frac{\gamma}{4\pi}\frac{1}{R} A(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} \\
&- \frac{1}{R}\frac{1}{A_1(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}})}\Big(R_1(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} + R_2(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} \\
&+ R_3(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}^2 + R_4(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}^2 \\
&+ R_5(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} \\
&+ 3\left(H_3 \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} + H_4 \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right) \\
&+ 3\Big(D_1(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}^2 + D_2(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}^2\Big)\left(1 + 2\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right) \\
&+ \left(D_1(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right)\left(1 + 2\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right) \\
&+ 6\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\left(H_3 \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} + H_4 \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right) \\
&+ 2\left(H_3 \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} + H_4 \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right)\Big).
\end{aligned} \tag{12.31}$$

To make $\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}})$ well-defined, we choose $\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}$ small enough so that all of the geometric series with respect to n contained in this expression converge. We further require that

$$\left|\mathcal{F}(\theta^0)(0)\right| + Y\Big(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\Big) + 2^{1/2}\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} < M,$$

where the function Y is defined in (12.5). For each $N \in \mathbb{N}$, let $\theta_{N,0} = \mathcal{J}_N^1 \theta^0 \in H_N^m$. Then

$$\begin{aligned}
\|\theta_{N,0}\|_{H^m} =& \Big(\sum_{k \in \mathbb{Z}} (1+|k|^2)^m \, |\mathcal{F}(\theta_{N,0})(k)|^2 \Big)^{1/2} \\
=& \Big(\sum_{|k| \le N} (1+|k|^2)^m \, |\mathcal{F}(\theta_{N,0})(k)|^2 \Big)^{1/2} \\
\le & \sum_{|k| \le N} (1+|k|^2)^{m/2} \, |\mathcal{F}(\theta_{N,0})(k)| \\
\le & |\mathcal{F}(\theta_{N,0})(0)| + 2^{m/2} \sum_{1 \le |k| \le N} |k|^m \, |\mathcal{F}(\theta_{N,0})(k)| \\
\le & \left|\mathcal{F}(\theta^0)(0)\right| + 2^{m/2} \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{m,1}} \\
\le & \left|\mathcal{F}(\theta^0)(0)\right| + Y\Big(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{m,1}} \Big) + 2^{m/2} \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{m,1}} .
\end{aligned}$$

Choosing $m = 1$, we obtain

$$\|\theta_{N,0}\|_{H^1} < M,$$

which ensures that the initial datum $\theta_{N,0}$ lies in O^M as prescribed by Picard's theorem. Let

$$\begin{aligned}
\phi^0 &= \theta^0 - \hat{\theta^0}(0), \\
\phi_{N,0} &= \theta_{N,0} - \hat{\theta_{N,0}}(0).
\end{aligned}$$

We note that $\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} \le \left\|\phi^0\right\|_{\dot{\mathcal{F}}^{1,1}}$. Since $\Lambda(\cdot)$ is monotone decreasing, for all $n \in \mathbb{N}$,

$$0 < \Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) = \Lambda(\left\|\phi^0\right\|_{\dot{\mathcal{F}}^{1,1}}) \le \Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}).$$

We choose ν_0 such that $0 < \nu_0 < \Lambda(\left\|\phi^0\right\|_{\dot{\mathcal{F}}^{1,1}}) < \Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}})$. From (2.2), it follows that for all $\tau \ge 0$,

$$0 < \nu'(\tau) = \frac{\nu_0}{(1+\tau)^2} \le \nu_0.$$

Then

$$\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0.$$

Let

$$T_{N,1} = \sup\left\{t_1 \in [0, T_N] : \Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau) > 0 \text{ for all } \tau \in [0, t_1]\right\}.$$

Since $\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0$ and $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\cdot)$ is a continuous function of time, we have $T_{N,1} > 0$. For any $\tau \in [0, T_{N,1})$,

$$\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau) > 0.$$

If $t_1 \in [0, T_{N,1}]$ and $t_2 \in [t_1, T_{N,1}]$, then

$$\|\phi_N(t_2)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \int_{t_1}^{t_2} \left(\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau)\right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \leq \|\phi_N(t_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu}. \tag{12.32}$$

Since

$$\int_{t_1}^{t_2} \left(\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau)\right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau > 0,$$

it follows from (12.32) that $\|\phi_N(t_2)\|_{\dot{\mathcal{F}}^{1,1}_\nu} \leq \|\phi_N(t_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu}$. Since Λ is a monotone decreasing function of $\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, this means that $\Lambda(\|\phi_N(t_2)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \geq \Lambda(\|\phi_N(t_1)\|_{\dot{\mathcal{F}}^{1,1}_\nu})$, i.e., $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu})$ is a monotone increasing function on $[0, T_{N,1}]$. Suppose for contradiction that $T_{N,1} < T_N$. If $\Lambda(\|\phi_N(T_{N,1})\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(T_{N,1}) \leq 0$, then since $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu})$ is monotone increasing on $[0, T_{N,1}]$,

$$\nu_0 = \nu'(0) < \Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) \leq \Lambda(\|\phi_N(T_{N,1})\|_{\dot{\mathcal{F}}^{1,1}_\nu}) \leq \nu'(T_{N,1}) = \frac{\nu_0}{(1 + T_{N,1})^2} < \nu_0,$$

which is a contradiction. If on the other hand $\Lambda(\|\phi_N(T_{N,1})\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(T_{N,1}) > 0$, then the function $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\cdot)$ is discontinuous at $T_{N,1} \in (0, T_N)$, a contradiction as well. Hence, we conclude that $T_{N,1} = T_N$. Thus, for all $t \in [0, T_N)$,

$$\begin{aligned}
\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} &\le \|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \Big(\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(\tau) \Big) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \\
&\le \|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \Big(\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \\
&\le \|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \Big(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \\
&\le \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \Big(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau.
\end{aligned}$$

Therefore, for all $t \in [0, T_N)$,

$$\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \Big(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \le \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} . \tag{12.33}$$

Moreover, for all $t \in [0, T_N)$,

$$\begin{aligned}
\frac{d}{dt} \|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} &\le - \Big(\Lambda(\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(t) \Big) \|\phi_N\|_{\dot{\mathcal{F}}^{2,1}_\nu} \\
&\le - \Big(\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \|\phi_N\|_{\dot{\mathcal{F}}^{2,1}_\nu} \\
&\le - \Big(\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \Big) \|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} ,
\end{aligned}$$

from which we deduce that $\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ decays exponentially on $[0, T_N)$.

12.3.4. A Remark on the Solution Being Global in Time

The global-in-time nature of the solution to the original equations for the dynamics of the interface is inherited from that of the solutions to the regularized equations. The latter is a consequence of the continuation property of Picard's theorem in the Banach space setting and the fact that the zeroth Fourier mode of θ_N is bounded in time. We fix $\epsilon > 0$ to be arbitrtarily small and let

$0_{new} = T_N - \epsilon$ be the new initial time. Then

$$
\begin{aligned}
\|\theta_{N,0_{new}}\|_{H^1} &\leq |\mathcal{F}(\theta_{N,0_{new}})(0)| + 2^{1/2} \|\theta_{N,0_{new}}\|_{\dot{\mathcal{F}}^{1,1}} \\
&= |\mathcal{F}(\theta_N(T_N - \epsilon))(0)| + 2^{1/2} \|\theta_N(T_{N-\epsilon})\|_{\dot{\mathcal{F}}^{1,1}} \\
&\leq |\mathcal{F}(\theta_N(T_N - \epsilon))(0)| + 2^{1/2} \|\theta_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} \\
&\leq |\mathcal{F}(\theta_N(T_N - \epsilon))(0)| + 2^{1/2} \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} \\
&\leq Y\left(\|\theta_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2} \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} \\
&\leq Y\left(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2} \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} \\
&\leq \left|\mathcal{F}(\theta^0)(0)\right| + Y\left(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2} \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} \\
&< M.
\end{aligned}
$$

This shows that the solution $\theta_N \in C^1([0, T_N); O^M)$ can be continued in time indefinitely due to the continuation property of Picard's theorem in the Banach space setting.

12.3.5. Applying Aubin-Lions' Lemma

To apply Aubin-Lions' lemma, we set $X_0 = \dot{\mathcal{F}}_\nu^{2,1}$, $X = \dot{\mathcal{F}}_\nu^{1,1}$, $X_1 = \dot{\mathcal{F}}_\nu^{0,1}$, $p = \infty$, and let

$$G = \{\theta_N : N \in \mathbb{N}\}.$$

Let $T > 0$. To show that G is uniformly bounded in $L^\infty([0, T]; \dot{\mathcal{F}}_\nu^{1,1}) \cap L^1_{loc}([0, T]; \dot{\mathcal{F}}_\nu^{2,1})$, we recall (12.33), i.e., for all $t \in [0, T]$,

$$
\begin{aligned}
&\|\theta_N(t)\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu_0\right) \int_0^t \|\theta_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau \\
&= \|\phi_N(t)\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu_0\right) \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau \\
&\leq \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}
$$

To show that $\partial_t G$ is uniformly bounded in $L^1_{loc}([0,T];\dot{\mathcal{F}}^{0,1}_\nu)$, we observe that

$$
\begin{aligned}
&\int_0^T \left\|(\mathcal{J}^1_N \circ G_N)(\theta_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
=&\int_0^T \left\|\mathcal{J}^1_N(G_N(\theta_N))\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
\leq&\int_0^T \frac{2\pi}{L(\tau)} \left\|\mathcal{J}^1_N\Big((U_\alpha)_N(\theta_N)\Big)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|\mathcal{J}^1_N\Big(T_N(\theta_N)\Big)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|\mathcal{J}^1_N\Big(T_N(\theta_N)\cdot(\theta_N)_\alpha\Big)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
\leq&\int_0^T \frac{2\pi}{L(\tau)} \left\|(U_\alpha)_N(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|T_N(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|T_N(\phi_N)\cdot(\phi_N)_\alpha\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
\leq&\int_0^T \frac{2\pi}{L(\tau)} \left\|W(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|\Big(1+(\phi_N)_\alpha\Big)U_N(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|\Big(1+(\phi_N)_\alpha\Big)U_N(\phi_N)\cdot(\phi_N)_\alpha\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
\leq&\int_0^T \frac{2\pi}{L(\tau)} \left\|W(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|V(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|(\phi_N)_\alpha\cdot U_N(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|U_N(\phi_N)\cdot(\phi_N)_\alpha\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)} \left\|U_N(\phi_N)\cdot(\phi_N)^2_\alpha\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau.
\end{aligned}
$$

Since

$$\|U_N(\phi_N)\cdot(\phi_N)_\alpha\|_{\mathcal{F}^{0,1}_\nu} \leq \|V(\phi_N)\|_{\mathcal{F}^{0,1}_\nu}\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}$$
$$\left\|U_N(\phi_N)\cdot(\phi_N)^2_\alpha\right\|_{\mathcal{F}^{0,1}_\nu} \leq \|V(\phi_N)\|_{\mathcal{F}^{0,1}_\nu}\|\phi_N\|^2_{\dot{\mathcal{F}}^{1,1}_\nu},$$

we obtain

$$\begin{aligned}
&\int_0^T \left\|(\mathcal{J}^1_N\circ G_N)(\theta_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
\leq &\int_0^T \frac{2\pi}{L(\tau)}\|W(\phi_N)\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)}\|V(\phi_N)\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \\
&+2\int_0^T \frac{2\pi}{L(\tau)}\|V(\phi_N)\|_{\mathcal{F}^{0,1}_\nu}\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} d\tau \\
&+\int_0^T \frac{2\pi}{L(\tau)}\|V(\phi_N)\|_{\mathcal{F}^{0,1}_\nu}\|\phi_N\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} d\tau.
\end{aligned}$$

Using estimates from Chapters 9, 10, and 11 and then (12.33), we see that

$$\|\partial_t\theta_N\|_{L^1_{loc}([0,T_N];\dot{\mathcal{F}}^{0,1})} = \int_0^T \left\|(\mathcal{J}^1_N\circ G_N)(\theta_N)\right\|_{\dot{\mathcal{F}}^{0,1}} d\tau$$

is indeed uniformly bounded. Therefore, by Aubin-Lions' lemma, G is relatively compact in $L^2([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$. This means that there exists a subsequence convergent in $L^2([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$. For notational convenience, we will continue to use θ_N to denote the subsequence. That is, there exists $\theta\in L^2([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$ such that $\theta_N\to\theta$ in $L^2([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$ as $N\to\infty$. It is crucial to bring to our attention that even though our application of Aubin-Lions' lemma provides a candidate for a solution to the original problem, it remains silent on the dynamics of $\mathcal{F}(\theta(t))(0)$. Part of our task to show that θ is a solution is to specify its dynamics. We first articulate the sense in which θ is to become a solution.

Definition 30. *We say that* $\theta\in L^\infty([0,T];\dot{\mathcal{F}}^{1,1}_\nu)\cap L^1([0,T];\dot{\mathcal{F}}^{2,1}_\nu)$ *is a weak solution of (12.6)* *through (12.7) if* $\mathcal{F}(\theta(t))(\pm 1) = 0$ *for almost every* $t \in [0,T]$ *and for any* $\psi \in C_0^\infty([-\pi,\pi) \times [0,T])$,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \theta(\alpha,T)\psi(\alpha,T)d\alpha - \int_{-\pi}^{\pi} \theta(\alpha,0)\psi(\alpha,0)d\alpha - \int_{-\pi}^{\pi}\int_0^T \theta(\alpha,t)\psi_t(\alpha,t)dtd\alpha \\
&= \int_{-\pi}^{\pi}\int_0^T R^{-1}\left(1 + \frac{1}{2\pi} Im \int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!}(\theta(\alpha,t) - \theta(\eta,t))^n d\eta d\alpha\right)^{1/2} \\
&\quad \cdot \Big(U_\alpha(\theta)(\alpha,t) + T(\theta)(\alpha,t)(1+\theta_\alpha(\alpha,t))\Big)\psi(\alpha,t)dtd\alpha.
\end{aligned}$$

To show that θ is a solution to the original problem in the sense of Definition 30, we use the following standard lemma from real analysis frequently.

Lemma 31. *For any sequence of measurable functions on a measure space,* L^p *convergence,* $p \geq 1$, *implies the existence of a subsequence convergent almost everywhere.*

Applying Lemma 31 to the fact that $\|\theta_N(\cdot) - \theta(\cdot)\|_{\dot{\mathcal{F}}_\nu^{1,1}} \to 0$ in L^2, we obtain a (non-relabeled) subsequence such that for almost every $t \in [0,T]$, $\lim_{N\to\infty} \|\theta_N(t) - \theta(t)\|_{\dot{\mathcal{F}}_\nu^{1,1}} = 0$. That is, for almost every $t \in [0,T]$, $\lim_{N\to\infty} \sum_{k\in\mathbb{Z}} |k|\,|\mathcal{F}(\theta_N(t) - \theta(t))(k)| = 0$. Applying Lemma 31 again to the fact that $a(k,t) = |k|\,|\mathcal{F}(\theta_N(t) - \theta(t))(k)| \to 0$ in l^2 for almost every $t \in [0,T]$, we obtain a (non-relabeled) subsequence such that for all $k \in \mathbb{Z} \setminus \{0\}$, $\lim_{N\to\infty} \mathcal{F}(\theta_N(t))(k) = \mathcal{F}(\theta(t))(k)$ for almost every $t \in [0,T]$. In particular, for almost every $t \in [0,T]$,

$$\mathcal{F}(\theta(t))(\pm 1) = \lim_{N\to\infty} \mathcal{F}(\theta_N(t))(k) = 0.$$

Let $\phi(t) = \theta(t) - \mathcal{F}(\theta(t))(0)$. Let us specify the dynamics of $\mathcal{F}(\theta)(0)$ by requiring that

$$\frac{d}{dt}\mathcal{F}(\theta)(0) = \mathcal{J}^1\left(\frac{2\pi}{L(\phi)}\Big(U_\alpha(\phi) + T(\phi)(1+\phi_\alpha)\Big)\right) - \frac{d}{dt}\phi \tag{12.34}$$

with the initial condition $\mathcal{F}(\theta(0))(0) = \mathcal{F}(\theta^0)(0)$. The initial condition is chosen this way because for all $N \in \mathbb{N}$, $\mathcal{F}(\theta_{N,0})(0) = \mathcal{F}(\mathcal{J}_N^1\theta^0)(0) = \mathcal{F}(\theta^0)(0)$. The dynamics equation (12.34) for $\mathcal{F}(\theta)(0)$

is equivalent to

$$\frac{d}{dt}\theta = \mathcal{J}^1\left(\frac{2\pi}{L(\phi)}\Big(U_\alpha(\phi) + T(\phi)(1+\phi_\alpha)\Big)\right),$$

which implies that θ is indeed a solution to the original problem in the sense of Definition 30.

12.3.6. Inheritance of the *a priori* Estimate

At the end of Subsection 12.3.3, we obtained that for all $t \in [0,T]$,

$$\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \Big(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\Big)\int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau \le \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}.$$

By Fatou's lemma, for any $t \in [0,T]$,

$$\int_0^t \liminf_{N\to\infty} \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau \le \liminf_{N\to\infty} \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau.$$

Then, using that

$$\begin{aligned}\liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} &= \|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu},\\ \liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu} &= \|\phi(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu},\end{aligned}$$

we obtain for all $t \in [0,T]$

$$\begin{aligned}&\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \Big(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\Big)\int_0^t \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau\\ &\le \liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \Big(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\Big)\liminf_{N\to\infty}\int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau\\ &\le \liminf_{N\to\infty}\left(\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \Big(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\Big)\int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau\right)\\ &\le \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}.\end{aligned}$$

In words, ϕ inherits the *a priori* estimate uniformly held for ϕ_N. As a consequence,

$$\theta \in L^\infty([0,T];\dot{\mathcal{F}}^{1,1}_\nu) \cap L^1([0,T];\dot{\mathcal{F}}^{2,1}_\nu)$$

and $\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ decays exponentially on $[0,T]$.

12.3.7. Continuity in Time

Now, we show that in fact $\theta \in C([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$. That is, letting $\mathfrak{G}(t) = \|\theta(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(t)}}$, we show that $\mathfrak{G}$ is continuous on $[0,T]$. We assume that $T > 0$ is arbitrarily small. This assumption can be made without loss of generality, because since the solution θ exists globally in time, we can imagine that the solution θ is continued over intervals of some arbitrary fixed small length. Let $\tau \in [0,T]$ and fix $\epsilon > 0$. We prove the continuity of $\mathfrak{G}$ at $\tau \in [0,T]$ by showing left- and right-continuity at that point. First, we suppose that $\tau' > \tau$. Then

$$\begin{aligned}
&\left|\mathfrak{G}(\tau') - \mathfrak{G}(\tau)\right| \\
=&\left| \left\|\theta(\tau')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right| \\
\leq&\left| \left\|\theta(\tau')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} \right| + \left| \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right| \\
\leq&\left\|\theta(\tau') - \theta(\tau)\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} + \left| \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right|.
\end{aligned}$$

First, we consider $\|\theta(\tau') - \theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}}$. We note that

$$
\begin{aligned}
&\left\|\theta(\tau') - \theta(\tau)\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} \\
\leq &\left\|\int_{\tau}^{\tau'} \partial_t \theta(\tau'') d\tau''\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} \\
\leq &\int_{\tau}^{\tau'} \left\|\partial_t \theta(\tau'')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} d\tau'' \\
= &\int_{\tau}^{\tau'} \left(\sum_{k\neq 0} e^{\nu(\tau')|k|} \left|k\right| \left|\mathcal{F}(\partial_t \theta(\tau''))(k)\right|\right) d\tau'' \\
= &\int_{\tau}^{\tau'} \left(\sum_{k\neq 0} e^{\nu_0 \frac{\tau'}{1+\tau'}|k|} \left|k\right| \left|\mathcal{F}(\partial_t \theta(\tau''))(k)\right|\right) d\tau'' \\
\leq &\int_{\tau}^{\tau'} \left(\sum_{k\neq 0} e^{\widetilde{\nu_0} \frac{\tau''}{1+\tau''}|k|} \left|k\right| \left|\mathcal{F}(\partial_t \theta(\tau''))(k)\right|\right) d\tau'' \\
= &\int_{\tau}^{\tau'} \left\|\partial_t \theta(\tau'')\right\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau'',
\end{aligned}
$$

where $\nu_0 < \widetilde{\nu_0}$ such that $\nu_0 \frac{\tau'}{1+\tau'} \leq \widetilde{\nu_0} \frac{\tau}{1+\tau}$ for all $\tau' \in [\tau, T]$. Since T is arbitrarily small, $\widetilde{\nu_0}$ can be chosen to be arbitrarily close to ν_0. We note that $\partial_t \theta \in L^1([0,T]; \dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}})$, in which $\widetilde{\nu}$ indicates the use of $\widetilde{\nu_0}$, instead of ν_0. For $n \in \mathbb{N}$, define

$$
a_n = \int_{[\tau, \tau + \frac{1}{n}]} \left\|\partial_t \theta(\tau'')\right\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau''.
$$

Since

$$
\left|1_{[\tau, \tau+\frac{1}{n}]} \left\|\partial_t \theta\right\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}}\right| \leq \left\|\partial_t \theta\right\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}}
$$

and

$$
\int_0^T \left\|\partial_t \theta\right\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau'' < \infty,
$$

by the dominated convergence theorem, we have

$$\lim_{n\to\infty} a_n = 0.$$

That is, there exists $N^* \in \mathbb{N}$ such that for all $N \geq N^*$,

$$\int_{[\tau,\tau+\frac{1}{n}]} \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\tilde{\nu}(\tau'')}}\, d\tau'' < \frac{\epsilon}{2}.$$

Hence, there exists $\delta > 0$ such that for all $|\tau' - \tau| < \delta$,

$$\int_{\tau}^{\tau'} \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\tilde{\nu}(\tau'')}}\, d\tau'' < \frac{\epsilon}{2}.$$

Next, we consider $\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right|$. We note that

$$\begin{aligned}
&\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right| \\
&= \left|\sum_{k\neq 0} e^{\nu(\tau')|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)| - \sum_{k\neq 0} e^{\nu(\tau)|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)|\right|.
\end{aligned}$$

We define for $\tau' \in [0,T]$

$$\mathfrak{H}(\tau') = \sum_{k\in\mathbb{Z}} e^{\nu(\tau')|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)|\,.$$

Then

$$\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right| = \left|\mathfrak{H}(\tau') - \mathfrak{H}(\tau)\right|.$$

Let $\mathfrak{h}_k(\tau') = e^{\nu(\tau')|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)|$. Since

$$\left|\mathfrak{h}_k(\tau')\right| = e^{\nu_0\frac{\tau'}{1+\tau'}|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)| \leq e^{\widetilde{\nu_0}\frac{\tau}{1+\tau}|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)|$$

and

$$\sum_{k\in\mathbb{Z}} e^{\widetilde{\nu_0}\frac{\tau}{1+\tau}|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)| = \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau)}} < \infty,$$

by the Weierstrass M-test,

$$\sum_{k\in\mathbb{Z}} e^{\nu(\tau')|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)|$$

converges absolutely and uniformly with respect to $\tau' \in [0,T]$. Since for each $k \in \mathbb{Z}$,

$$\mathfrak{h}_k(\tau') = e^{\nu(\tau')|k|}\,|k|\,|\mathcal{F}(\theta(\tau))(k)|$$

is continuous, so too is $\mathfrak{H}(\tau')$ on $[0,T]$. Hence, there exists $\delta > 0$ such that $|\tau' - \tau| < \delta$ implies that

$$\left|\mathfrak{H}(\tau') - \mathfrak{H}(\tau)\right| < \frac{\epsilon}{2}.$$

Now, suppose that $\tau' < \tau$. Then

$$\begin{aligned}
&\left|\mathfrak{G}(\tau') - \mathfrak{G}(\tau)\right| \\
&\le \left\|\theta(\tau) - \theta(\tau')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} + \left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right|
\end{aligned}$$

First, we consider $\|\theta(\tau) - \theta(\tau')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}}$. We observe that

$$\begin{aligned}
&\left\|\theta(\tau) - \theta(\tau')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} \\
&= \left\|\int_{\tau'}^{\tau} \partial_t\theta(\tau'')d\tau''\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} \\
&\le \int_{\tau'}^{\tau} \left\|\partial_t\theta(\tau'')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} d\tau'' \\
&\le \int_{\tau'}^{\tau} \left\|\partial_t\theta(\tau'')\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} d\tau''.
\end{aligned}$$

We note that $\partial_t \theta \in L^1([0,T]; \dot{\mathcal{F}}_{\nu}^{1,1})$. For $n \in \mathbb{N}$, define

$$b_n = \int_{[\tau - \frac{1}{n}, \tau]} \left\| \partial_t \theta(\tau'') \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} d\tau''.$$

Since

$$\left| 1_{[\tau - \frac{1}{n}, \tau]} \left\| \partial_t \theta(\tau'') \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} \right| \leq \left\| \partial_t \theta(\tau'') \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}$$

and

$$\int_0^T \left\| \partial_t \theta(\tau'') \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} d\tau'' < \infty,$$

by the dominated convergence theorem, we have

$$\lim_{n \to \infty} b_n = 0.$$

That is, there exists $N^{**} \in \mathbb{N}$ such that for all $N \geq N^{**}$,

$$\int_{[\tau - \frac{1}{n}, \tau]} \left\| \partial_t \theta(\tau'') \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} d\tau'' < \frac{\epsilon}{2}.$$

Hence, there exists $\delta > 0$ such that for all $|\tau - \tau'| < \delta$,

$$\int_{\tau'}^{\tau} \left\| \partial_t \theta(\tau'') \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} d\tau'' < \frac{\epsilon}{2}.$$

The second term $\left| \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right|$ can be estimated as in the case where $\tau' > \tau$. Therefore, we conclude that $\theta \in C([0,T]; \dot{\mathcal{F}}_{\nu}^{1,1})$.

12.3.8. Instantaneous Analyticity

Now, we show that θ is instantaneously analytic. To prove this, we use the following result from standard analysis.

Lemma 32. *The function f is analytic on $\mathbb{T}$ if and only if there exist constants $K > 0$ and $a > 0$ such that*

$$|\mathcal{F}(f)(j)| \leq Ke^{-a|j|}.$$

Let $t > 0$. We claim that there exist $C > 0$ and $k^* > 0$ such that for all $|k| \geq k^*$,

$$e^{\nu(t)|k|}\,|k|\,|\mathcal{F}(\phi(t))(k)| \leq C.$$

Suppose the contrary for contradiction. Then, for all $k^* > 0$, there exists $|k| \geq k^*$ such that

$$e^{\nu(t)|k|}\,|k|\,|\mathcal{F}(\phi(t))(k)| > 1.$$

This means that there is a sequence $\{k_j\}$ such that

$$e^{\nu(t)|k_j|}\,|k_j|\,|\mathcal{F}(\phi(t))(k_j)| > 1.$$

Hence,

$$\infty > \|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} = \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|\,|\mathcal{F}(\phi(t))(k)| \geq \sum_{j=1}^{\infty} e^{\nu(t)|k_j|}\,|k_j|\,|\mathcal{F}(\phi(t))(k_j)| = \infty,$$

a contradiction, as needed. Thus, there exist $C > 0$ and $k^* > 0$ such that for all $|k| \geq k^*$,

$$e^{\nu(t)|k|}\,|\mathcal{F}(\phi(t))(k)| \leq C\,|k|^{-1} \leq \frac{C}{k^*}.$$

Hence, for all $k \in \mathbb{Z}$,

$$e^{\nu(t)|k|}\,|\mathcal{F}(\theta(t))(k)| \leq \max\left\{\frac{C}{k^*}, \max_{|k|<k^*} e^{\nu(t)|k|}\,|\mathcal{F}(\theta(t))(k)|\right\}.$$

By Lemma 32, we can then conclude that θ is analytic.

CHAPTER 13

Uniqueness

Let θ_1 and θ_2 be two solutions to the original problem with the same initial datum, whose ± 1 Fourier modes remain zero in time. For $k > 0$,

$$\begin{aligned}\mathcal{F}(\theta_1 - \theta_2)(-k) =& \frac{1}{2\pi}\int_{-\pi}^{\pi}(\theta_1 - \theta_2)(\alpha)e^{ik\alpha}d\alpha \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{(\theta_1 - \theta_2)(\alpha)e^{-ik\alpha}}d\alpha \\ =& \overline{\mathcal{F}(\theta_1 - \theta_2)(k)}.\end{aligned}$$

Hence, we may write

$$\|\theta_1 - \theta_2\|_{\dot{\mathcal{F}}^{1,1}} = \sum_{k\neq 0}|k|\,|\mathcal{F}(\theta_1 - \theta_2)(k)| = 2\sum_{k>0}|k|\,|\mathcal{F}(\theta_1 - \theta_2)(k)|\,.$$

Then

$$\begin{aligned}&\frac{d}{dt}\|\theta_1 - \theta_2\|_{\dot{\mathcal{F}}^{1,1}} \qquad (13.1)\\ =&2\sum_{k>0}|k|\frac{d}{dt}|\mathcal{F}(\theta_1 - \theta_2)(k)| \\ =&2\sum_{k>0}|k|\frac{d}{dt}\left(\mathcal{F}(\theta_1 - \theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1 - \theta_2)(k)}\right)^{1/2} \\ =&\sum_{k>0}|k|\left(\mathcal{F}(\theta_1 - \theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1 - \theta_2)(k)}\right)^{-1/2} \\ &\cdot\left(\frac{d}{dt}\mathcal{F}(\theta_1 - \theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1 - \theta_2)(k)} + \mathcal{F}(\theta_1 - \theta_2)(k)\cdot\frac{d}{dt}\overline{\mathcal{F}(\theta_1 - \theta_2)(k)}\right) \\ =&\sum_{k>0}\frac{|k|}{|\mathcal{F}(\theta_1 - \theta_2)(k)|} \\ &\cdot\left(\frac{d}{dt}\mathcal{F}(\theta_1 - \theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1 - \theta_2)(k)} + \mathcal{F}(\theta_1 - \theta_2)(k)\cdot\overline{\frac{d}{dt}\mathcal{F}(\theta_1 - \theta_2)(k)}\right).\end{aligned}$$

Recalling that for a solution θ to the original problem, $\phi = \theta - \hat{\theta}(0)$ satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\phi)(k) =& \frac{1}{R}\cdot\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k)+J_2(k)) + \frac{2\pi}{L(\phi)}\mathcal{F}(\widetilde{N}(\phi))(k)\\ &+ \frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi)}\right),\end{aligned}$$

where J_1 and J_2 are the same as in (7.3), we have for $k > 0$

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\theta_1-\theta_2)(k) =& \frac{d}{dt}\mathcal{F}(\phi_1-\phi_2)(k)\\ =& \frac{1}{R}\cdot\frac{\gamma}{4\pi}\mathcal{F}(\phi_1-\phi_2)(k)(J_1(k)+J_2(k)) + \frac{2\pi}{L(\phi_1)}\mathcal{F}(\widetilde{N}(\phi_1))(k)\\ &- \frac{2\pi}{L(\phi_2)}\mathcal{F}(\widetilde{N}(\phi_2))(k)\\ &+ \frac{\gamma}{4\pi}\mathcal{F}(\phi_1)(k)(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\\ &- \frac{\gamma}{4\pi}\mathcal{F}(\phi_2)(k)(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_2)}\right)\\ =& \frac{1}{R}\cdot\frac{\gamma}{4\pi}\mathcal{F}(\phi_1-\phi_2)(k)(J_1(k)+J_2(k))\\ &+ \left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\mathcal{F}(\widetilde{N}(\phi_1))(k)\\ &+ \frac{2\pi}{L(\phi_2)}\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\\ &+ \frac{\gamma}{4\pi}\mathcal{F}(\phi_1-\phi_2)(k)(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\\ &- \frac{\gamma}{4\pi}\mathcal{F}(\phi_2)(k)(J_1(k)+J_2(k))\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right).\end{aligned}$$

Substituting this expression into (13.1), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\|\theta_1-\theta_2\right\|_{\dot{\mathcal{F}}^{1,1}} \\
=&\sum_{k>0}\frac{|k|}{|\mathcal{F}(\theta_1-\theta_2)(k)|}\Bigg(\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2\left(J_1(k)+J_2(k)\right) \\
&+\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\mathcal{F}(\widetilde{N}(\phi_1))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)} \\
&+\frac{2\pi}{L(\phi_2)}\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\cdot\overline{\mathcal{F}(\phi_1-\phi_2)(k)} \\
&+\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2\left(J_1(k)+J_2(k)\right)\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right) \\
&-\frac{\gamma}{4\pi}\mathcal{F}(\phi_2)(k)(J_1(k)+J_2(k))\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\overline{\mathcal{F}(\phi_1-\phi_2)(k)} \\
&+\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2\left(J_1(k)+J_2(k)\right) \\
&+\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\overline{\mathcal{F}(\widetilde{N}(\phi_1))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k) \\
&+\frac{2\pi}{L(\phi_2)}\overline{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k) \\
&+\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2\left(J_1(k)+J_2(k)\right)\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right) \\
&-\frac{\gamma}{4\pi}\overline{\mathcal{F}(\phi_2)(k)}(J_1(k)+J_2(k))\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\mathcal{F}(\phi_1-\phi_2)(k)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
=&2\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k>0}|k|\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|(J_1(k)+J_1(k)) \qquad (13.2)\\
&+\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\\
&\cdot\sum_{k>0}|k|\,\frac{\mathcal{F}(\widetilde{N}(\phi_1))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\\
&+\frac{2\pi}{L(\phi_2)}\sum_{k>0}|k|\\
&\cdot\frac{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\\
&+2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k>0}|k|\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|(J_1(k)+J_2(k))\\
&-\frac{\gamma}{4\pi}\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\\
&\cdot\sum_{k>0}|k|\,(J_1(k)+J_2(k))\frac{\mathcal{F}(\phi_2)(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\phi_2)(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}. \qquad (13.3)
\end{aligned}
$$

We will take a closer look at each of the five terms in (13.2) through (13.3) one by one. Since $\mathcal{F}(\phi_1)(\pm1)=\mathcal{F}(\phi_2)(\pm1)=0$, the first time can be written as

$$
\begin{aligned}
&2\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k>0}|k|\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|(J_1(k)+J_1(k))\\
=&-\pi\cdot2\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k\geq2}|k|^2\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|\\
=&-\pi\cdot2\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k>0}|k|^2\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|\\
=&-\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}.
\end{aligned}
$$

Next, the second term can be bounded above as follows.

$$
\begin{aligned}
&\left|\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\right| \\
&\cdot\left|\sum_{k>0}|k|\,\frac{\mathcal{F}(\widetilde{N}(\phi_1))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\right| \\
\leq&\left|\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\right|\cdot\sum_{k>0}|k|\cdot 2\left|\mathcal{F}(\widetilde{N}(\phi_1))(k)\right| \\
=&\left|\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\right|\cdot\left\|\widetilde{N}(\phi_1)\right\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}
$$

Similarly, the third term can be bounded above as follows.

$$
\begin{aligned}
&\left|\frac{2\pi}{L(\phi_2)}\right|\cdot\left|\sum_{k>0}|k|\right. \\
&\left.\cdot\,\frac{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\right| \\
\leq&\frac{2\pi}{L(\phi_2)}\cdot\sum_{k>0}|k|\cdot 2\left|\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\right| \\
=&\frac{2\pi}{L(\phi_2)}\left\|\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}
$$

The fourth term can be bounded above as follows.

$$
\begin{aligned}
&\left|2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k>0}|k|\,|\mathcal{F}(\phi_1-\phi_2)(k)|\,(J_1(k)+J_2(k))\right| \\
=&\left|-\pi\cdot 2\cdot\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k\geq 2}|k|^2\,|\mathcal{F}(\phi_1-\phi_2)(k)|\right| \\
=&\left|-\pi\cdot 2\cdot\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k>0}|k|^2\,|\mathcal{F}(\phi_1-\phi_2)(k)|\right| \\
\leq&\pi\cdot\frac{\gamma}{4\pi}\frac{1}{R}\cdot A(\|\phi_1\|_{\mathcal{F}^{0,1}})\,\|\phi_1\|_{\mathcal{F}^{0,1}}\,\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}.
\end{aligned}
$$

Lastly, the fifth term can be bounded above as follows.

$$
\begin{aligned}
&\left| -\frac{\gamma}{4\pi}\left(\frac{2\pi}{L(\phi_2)} - \frac{2\pi}{L(\phi_1)}\right)\right.\\
&\quad \left. \cdot \sum_{k>0} |k| \left(J_1(k) + J_2(k)\right) \frac{\mathcal{F}(\phi_2)(k)\overline{\mathcal{F}(\phi_1 - \phi_2)(k)} + \overline{\mathcal{F}(\phi_2)(k)}\mathcal{F}(\phi_1 - \phi_2)(k)}{|\mathcal{F}(\phi_1 - \phi_2)(k)|} \right|\\
=&\left| \pi \cdot \frac{\gamma}{4\pi}\left(\frac{2\pi}{L(\phi_2)} - \frac{2\pi}{L(\phi_1)}\right)\right.\\
&\quad \left. \cdot \sum_{k\geq 2} |k|^2 \, \frac{\mathcal{F}(\phi_2)(k)\overline{\mathcal{F}(\phi_1 - \phi_2)(k)} + \overline{\mathcal{F}(\phi_2)(k)}\mathcal{F}(\phi_1 - \phi_2)(k)}{|\mathcal{F}(\phi_1 - \phi_2)(k)|} \right|\\
\leq&\pi \cdot \frac{\gamma}{4\pi}\left|\frac{2\pi}{L(\phi_2)} - \frac{2\pi}{L(\phi_1)}\right| \sum_{k\geq 2} |k|^2 \cdot 2\left|\mathcal{F}(\phi_2)(k)\right|\\
=&\pi \cdot \frac{\gamma}{4\pi}\left|\frac{2\pi}{L(\phi_2)} - \frac{2\pi}{L(\phi_1)}\right| \cdot \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \, .
\end{aligned}
$$

We note that for a solution θ to the original problem,

$$
\widetilde{N}(\phi) = (U_{\geq 2})_{\alpha}(\phi) + T_{\geq 2}(\phi) \cdot (1 + \phi_\alpha) + T_1(\phi) \cdot \phi_\alpha,
$$

where $\phi = \theta - \hat{\theta}(0)$. Hence,

$$\begin{aligned}
&\widetilde{N}(\phi_1) - \widetilde{N}(\phi_2) \\
=&(U_{\geq 2})_\alpha(\phi_1) - (U_{\geq 2})_\alpha(\phi_2) + T_{\geq 2}(\phi_1)(1 + (\phi_1)_\alpha) \\
&- T_{\geq 2}(\phi_2)(1 + (\phi_2)_\alpha) + T_1(\phi_1)(\phi_1)_\alpha - T_1(\phi_2)(\phi_2)_\alpha \\
=&\mathrm{Re}\bigg(\sum_{j=1}^{8} \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_j})_\alpha(\phi_1)(\alpha,\beta)d\beta + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_{13}})_\alpha(\phi_1)(\alpha,\beta)d\beta\bigg) \\
&- \mathrm{Re}\bigg(\sum_{j=1}^{8} \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_j})_\alpha(\phi_2)(\alpha,\beta)d\beta + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_{13}})_\alpha(\phi_2)(\alpha,\beta)d\beta\bigg) \\
&+ T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2) + T_{\geq 2}(\phi_1)(\phi_1)_\alpha - T_{\geq 2}(\phi_1)(\phi_2)_\alpha \\
&+ T_{\geq 2}(\phi_1)(\phi_2)_\alpha - T_{\geq 2}(\phi_2)(\phi_2)_\alpha + T_1(\phi_1)(\phi_1)_\alpha - T_1(\phi_1)(\phi_2)_\alpha \\
&+ T_1(\phi_1)(\phi_2)_\alpha - T_1(\phi_2)(\phi_2)_\alpha \\
=&\sum_{j=1}^{8}\mathrm{Re}\bigg(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_j})_\alpha(\phi_1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_j})_\alpha(\phi_2)(\alpha,\beta)d\beta\bigg) \qquad (13.4)\\
&+ \mathrm{Re}\bigg(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_{13}})_\alpha(\phi_1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} (\widetilde{B_{13}})_\alpha(\phi_2)(\alpha,\beta)d\beta\bigg) \qquad (13.5)\\
&+ T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2) + T_{\geq 2}(\phi_1)((\phi_1)_\alpha - (\phi_2)_\alpha) + (\phi_2)_\alpha(T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2)) \\
&+ T_1(\phi_1)((\phi_1)_\alpha - (\phi_2)_\alpha) + (\phi_2)_\alpha(T_1(\phi_1) - T_1(\phi_2)).
\end{aligned}$$

To derive an estimate for this expression, we present in detail the process of deriving appropriate estimates for a few select terms that make up the expression. The techniques used to estimate such terms can be applied for the rest of the terms making up the expression. First, we consider the term

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{1,1}^1(\phi_1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{1,1}^1(\phi_2)(\alpha,\beta)d\beta,$$

which makes up one of the terms in the sum in (13.4) (to be precise, the $j = 1$ term in the sum). Let us derive an estimate for the integrand, i.e.,

$$B^1_{1,1}(\phi_1)(\alpha, \beta) - B^1_{1,1}(\phi_2)(\alpha, \beta).$$

In Subsection 12.3.2, we derived an estimate for an analogous expression, which is shown in (12.21). Borrowing notation used in that part of Section 12.3.2, we write

$$B^1_{1,1}(\phi_1)(\alpha, \beta) - B^1_{1,1}(\phi_2)(\alpha, \beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} \cdot i^{j_1+j_2+1}}{2j_1!j_2!} \cdot j_1$$

$$\cdot \left(S_1(\alpha, \beta) + \cdots + S_3(\alpha, \beta) + \cdots + S_7(\alpha, \beta) + \cdots\right),$$

where

$$
\begin{aligned}
&S_1(\alpha,\beta)\\
=&(\phi_1-\phi_2)(\alpha-\beta)\cdot\phi_1(\alpha-\beta)^{j_1-2}\cdot(\phi_1)_\alpha(\alpha-\beta)\cdot\phi_1(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi_1(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
&S_3(\alpha,\beta)\\
=&\phi_2(\alpha-\beta)^{j_1-1}\cdot((\phi_1)_\alpha-(\phi_2)_\alpha)(\alpha-\beta)\cdot\phi_1(\alpha)^{j_2}\\
&\cdot\int_0^1 e^{-i\beta s}\phi_1(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n,\\
&S_7(\alpha,\beta)\\
=&\phi_2(\alpha-\beta)^{j_1-1}(\phi_2)_\alpha(\alpha-\beta)\phi_2(\alpha)^{j_2}\cdot\int_0^1 e^{-i\beta s}\phi_2(\alpha+\beta(-1+s))(-1+s)ds\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right.\\
&\left.-\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_2(\alpha+(s-1)\beta))^m}{m!}ds\right)\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1},
\end{aligned}
$$

and the $\cdots$ represents the other finitely many terms making up $B^1_{1,1}(\phi_1)(\alpha,\beta)-B^1_{1,1}(\phi_2)(\alpha,\beta)$. First, we study $S_1(\alpha,\beta)$ and $S_7(\alpha,\beta)$ and then turn the attention to $S_3(\alpha,\beta)$. We note that

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_1(\cdot,\beta))(k_1)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right|\\
\leq&C_n\Big(|\mathcal{F}(\phi_1-\phi_2)|*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)|*|\mathcal{F}((\phi_1)_\alpha)|\\
&*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)|*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_1)|\Big)(k_1).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_1(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}}\\
\le& C_n\sum_{k\neq 0}|k|\Bigg(|\mathcal{F}(\phi_1-\phi_2)|*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)|*|\mathcal{F}((\phi_1)_\alpha)|\\
&*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)|*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_1)|\Bigg)(k).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_7(\cdot,\beta))(k_1)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right|\\
\le& C_n\Bigg(|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|*|\mathcal{F}((\phi_2)_\alpha)|\\
&*|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|\\
&*\left|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right)\right|\\
&*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_2)|\Bigg)(k_1),
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_7(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}}\\
\le& C_n\sum_{k\neq 0}|k|\Bigg(|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|*|\mathcal{F}((\phi_2)_\alpha)|\\
&*|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|\\
&*\left|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right)\right|\\
&*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_2)|\Bigg)(k).
\end{aligned}
$$

For a sequence a defined on $\mathbb{Z}$, we define for $s \geq 0$

$$\|a\|_{l^{s,1}} = \sum_{k\in\mathbb{Z}} |k|^s \, |a(k)| \, .$$

Lemma 33. *For sequences $a_1, \ldots, a_n$ defined on $\mathbb{Z}$,*

$$\|a_1 * \cdots * a_n\|_{l^{1,1}} \leq \sum_{j=1}^{n} \|a_j\|_{l^{1,1}} \prod_{\substack{k=1\\k\neq j}}^{n} \|a_k\|_{l^{0,1}} \, .$$

Proof. This lemma can be proved by modifying the proof of Proposition 21. ■

Using Lemma 33, we obtain

$$\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} S_1(\cdot, \beta) \cdot \frac{-i\beta e^{2i\beta}}{1 - e^{i\beta}} d\beta \right\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq & C_n \bigg(\|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{0,1}} \|P(\phi_1)\|_{l^{0,1}}^n \\
& + \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-2} \|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{0,1}} \|P(\phi_1)\|_{l^{0,1}}^n \cdot (j_1 + j_2 - 1) \\
& + \|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|P(\phi_1)\|_{l^{0,1}}^n \\
& + \|P(\phi_1)\|_{l^{1,1}} \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{0,1}} \|P(\phi_1)\|_{l^{0,1}}^{n-1} \cdot n \bigg)
\end{aligned}$$

and

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_7(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}}\\
\leq &C_n\Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1}\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\\
&\cdot\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right)\right\|_{l^{0,1}}\\
&\cdot\|P(\phi_1)\|_{l^{0,1}}^{n-1}\cdot(j_1+j_2)\\
&+\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2}\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1\phi_2)^m)\right)\right\|_{l^{0,1}}\\
&\cdot\|P(\phi_1)\|_{l^{0,1}}^{n-1}\\
&+\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1\phi_2)^m)\right)\right\|_{l^{1,1}}\\
&\cdot\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2}\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|P(\phi_1)\|_{l^{0,1}}^{n-1}\\
&+\|P(\phi_1)\|_{l^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2}\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\\
&\cdot\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right)\right\|_{l^{0,1}}\\
&\cdot\|P(\phi_1)\|_{\dot{\mathcal{F}}^{0,1}}^{n-2}\cdot(n-1)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\leq C_n\Bigg(& \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot (j_1+j_2) \\
& + \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right)(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \\
& + \Bigg(\left\|\sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right\|_{l^{1,1}} \cdot \left\|\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right\|_{l^{0,1}} \\
& + \left\|\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right\|_{l^{1,1}} \cdot \left\|\sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right\|_{l^{0,1}}\Bigg) \\
& \cdot \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \\
& + \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \\
& \cdot \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-2} \cdot (n-1)\Bigg) \\
\leq C_n\Bigg(& \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \\
& \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot (j_1+j_2) \\
& + \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right)(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \\
& + \Bigg(\|\phi_2\|_{\mathcal{F}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \\
& + \|\phi_1-\phi_2\|_{\mathcal{F}^{1,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{(m-1)!}\right) e^{\|\phi_2\|_{\mathcal{F}^{0,1}}}\Bigg) \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \\
& \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \\
& + \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \\
& \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-2} \cdot (n-1)\Bigg).
\end{aligned}
$$

Now, we consider $S_3(\alpha,\beta)$. We note that

$$
\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_3(\cdot,\beta))(k_1)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \\
\leq &C_n\Big(|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|*|\mathcal{F}((\phi_1)_\alpha-(\phi_2)_\alpha)|*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)| \\
&*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_1)|\Big)(k_1)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_3(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq &C_n\sum_{k\neq 0}|k|\Big(|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|*|\mathcal{F}((\phi_1)_\alpha-(\phi_2)_\alpha)|*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)| \\
&*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_1)|\Big)(k) \\
\leq &C_n\Big(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-2}\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}\|P(\phi_1)\|_{l^{0,1}}^{n}\cdot(j_1-1) \\
&+\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}\|P(\phi_1)\|_{l^{0,1}}^{n} \\
&+\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2}\|P(\phi_1)\|_{l^{0,1}}^{n}\cdot(j_2+1) \\
&+\|P(\phi_1)\|_{l^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}\|P(\phi_1)\|_{l^{0,1}}^{n-1}\cdot n\Big) \\
\leq &C_n\Big(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-2}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n\cdot(j_1-1) \\
&+\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \\
&+\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n\cdot(j_2+1) \\
&+\|P(\phi_1)\|_{l^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1}\cdot n\Big).
\end{aligned}
$$

Combining these results, we obtain

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{1,1}(\phi_1)(\alpha,\beta) d\beta - \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{1,1}(\phi_2)(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq &\sum_{j_1+j_2+n\geq 1} \frac{j_1}{2 j_1! j_2!} \Bigg(C_n \Bigg(\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|^{j_1+j_2-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \\
&+ \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|^{j_1+j_2-2}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \cdot (j_1+j_2-1) \\
&+ \|\phi_1\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|^{j_1+j_2-1}_{\dot{\mathcal{F}}^{0,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \\
&+ \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|^{j_1+j_2-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot n \Bigg) \\
&+ \cdots \\
&+ C_n \Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|^{j_1-2}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|^{j_2+1}_{\dot{\mathcal{F}}^{0,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \cdot (j_1-1) \\
&+ \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|^{j_1-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|^{j_2+1}_{\dot{\mathcal{F}}^{0,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \\
&+ \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|^{j_1-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|^{j_2}_{\dot{\mathcal{F}}^{0,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \cdot (j_2+1) \\
&+ \|P(\phi_1)\|_{l^{1,1}} \|\phi_2\|^{j_1-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|^{j_2+1}_{\dot{\mathcal{F}}^{0,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot n \Bigg) \\
&+ \cdots \\
&+ C_n \Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|^{j_1+j_2-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|^{m-1}_{\mathcal{F}^{0,1}}}{m!} \right) \\
&\cdot \left(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} - 1 \right)^{n-1} \cdot (j_1+j_2) \\
&+ \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|^{j_1+j_2}_{\dot{\mathcal{F}}^{0,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|^{m-1}_{\mathcal{F}^{0,1}}}{m!} \right) (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \\
&+ \Bigg(\|\phi_2\|_{\mathcal{F}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|^{m-1}_{\mathcal{F}^{0,1}}}{m!} \right) \\
&+ \|\phi_1-\phi_2\|_{\mathcal{F}^{1,1}} \Big(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|^{m-1}_{\mathcal{F}^{0,1}}}{(m-1)!} \Big) e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \Bigg) \cdot \|\phi_2\|^{j_1+j_2}_{\dot{\mathcal{F}}^{0,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \\
&+ \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_2\|^{j_1+j_2}_{\dot{\mathcal{F}}^{0,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \\
&\cdot \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|^{m-1}_{\mathcal{F}^{0,1}}}{m!} \right) (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-2} \cdot (n-1) \Bigg) \\
&+ \cdots \Bigg),
\end{aligned}
$$

where the $\cdots$ represents the finitely many terms making up $B^1_{1,1}(\phi_1)(\alpha,\beta) - B^1_{1,1}(\phi_2)(\alpha,\beta)$ besides $S_1(\alpha,\beta)$, $S_3(\alpha,\beta)$, and $S_7(\alpha,\beta)$. If we collected all the coefficients for $\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}$ among the shown terms, then we obtain as its coefficient

$$\sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} C_n \|\phi_2\|^{j_1-1}_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|^{j_2+1}_{\dot{\mathcal{F}}^{0,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} - 1)^n.$$

We note that if we summed the coefficients for $\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}$ across all the terms appearing in (13.4) and (13.5), then we can choose $\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}$ small enough such that the sum is smaller than $\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}$. Next, we consider the term

$$T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2),$$

where

$$T(\phi) = \mathcal{M}((1+\phi_\alpha)U(\phi)).$$

Then

$$\begin{aligned} T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2) =& \mathcal{M}\Big(U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\Big) + \mathcal{M}\Big((\phi_1)_\alpha \cdot U_{\geq 1}(\phi_1) - (\phi_2)_\alpha \cdot U_{\geq 1}(\phi_2)\Big) \\ =& \mathcal{M}\Big(U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\Big) + \mathcal{M}\Big((\phi_1)_\alpha \cdot (U_{\geq 1}(\phi_1) - U_{\geq 1}(\phi_2))\Big) \\ &+ \mathcal{M}\Big(U_{\geq 1}(\phi_2)\cdot((\phi_1)_\alpha - (\phi_2)_\alpha)\Big). \end{aligned}$$

Hence,

$$
\begin{aligned}
&\|T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2)\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq &\|U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\|_{\mathcal{F}^{0,1}} + \|(\phi_1)_\alpha \cdot (U_{\geq 1}(\phi_1) - U_{\geq 1}(\phi_2))\|_{\mathcal{F}^{0,1}} \\
&+ \|U_{\geq 1}(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\|_{\mathcal{F}^{0,1}} \\
\leq &\|U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\|_{\mathcal{F}^{0,1}} \\
&+ \|\phi_1\|_{\mathcal{F}^{1,1}} \left(\|U_1(\phi_1) - U_1(\phi_2)\|_{\mathcal{F}^{0,1}} + \|U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\|_{\mathcal{F}^{0,1}} \right) \\
&+ \|U_{\geq 1}(\phi_2)\|_{\mathcal{F}^{0,1}} \cdot \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}}.
\end{aligned}
$$

Next, we consider the term

$$T(\phi_1) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha).$$

We have

$$
\begin{aligned}
&\|T(\phi_1) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq &\|T(\phi_1)\|_{\dot{\mathcal{F}}^{1,1}} \|(\phi_1)_\alpha - (\phi_2)_\alpha\|_{\mathcal{F}^{0,1}} + \|(\phi_1)_\alpha - (\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}} \|T(\phi_1)\|_{\mathcal{F}^{0,1}} \\
\leq &\|(1 + (\phi_1)_\alpha) \cdot U(\phi_1)\|_{\mathcal{F}^{0,1}} \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} + \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|(1 + (\phi_1)_\alpha) \cdot U(\phi_1)\|_{\mathcal{F}^{0,1}} \\
\leq &\left(\|U(\phi_1)\|_{\mathcal{F}^{0,1}} + \|\phi_1\|_{\mathcal{F}^{1,1}} \|U(\phi_1)\|_{\mathcal{F}^{0,1}} \right) \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} \\
&+ \left(\|U(\phi_1)\|_{\mathcal{F}^{0,1}} + \|\phi_1\|_{\mathcal{F}^{1,1}} \cdot \|U(\phi_1)\|_{\mathcal{F}^{0,1}} \right) \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{2,1}}.
\end{aligned}
$$

Next, we consider the term

$$(\phi_2)_\alpha \cdot (T(\phi_1) - T(\phi_2)).$$

We have

$$
\begin{aligned}
&\left\|(\phi_2)_\alpha \cdot (T(\phi_1) - T(\phi_2))\right\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq &\left\|(\phi_2)_\alpha\right\|_{\dot{\mathcal{F}}^{1,1}} \left\|T(\phi_1) - T(\phi_2)\right\|_{\mathcal{F}^{0,1}} + \left\|T(\phi_1) - T(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}} \left\|(\phi_2)_\alpha\right\|_{\mathcal{F}^{0,1}} \\
= &\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}} \left\|T(\phi_1) - T(\phi_2)\right\|_{\mathcal{F}^{0,1}} + \left\|T(\phi_1) - T(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}} \left\|\phi_2\right\|_{\mathcal{F}^{1,1}}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
&T(\phi_1) - T(\phi_2) \\
=&\mathcal{M}\Big((1 + (\phi_1)_\alpha) \cdot U(\phi_1)\Big) - \mathcal{M}\Big((1 + (\phi_2)_\alpha) \cdot U(\phi_2)\Big) \\
=&\mathcal{M}\Big(U(\phi_1) - U(\phi_2)\Big) \\
&+ \mathcal{M}\Big((\phi_1)_\alpha \cdot U(\phi_1) - (\phi_1)_\alpha \cdot U(\phi_2) + (\phi_1)_\alpha \cdot U(\phi_2) - (\phi_2)_\alpha \cdot U(\phi_2)\Big) \\
=&\mathcal{M}\Big(U(\phi_1) - U(\phi_2)\Big) + \mathcal{M}\Big((\phi_1)_\alpha \cdot (U(\phi_1) - U(\phi_2))\Big) \\
&+ \mathcal{M}\Big(U(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\Big).
\end{aligned}
$$

Since

$$
\begin{aligned}
\left\|\mathcal{M}(f)\right\|_{\mathcal{F}^{0,1}} =&\sum_{k\in\mathbb{Z}} \left|\mathcal{F}(\mathcal{M}(f))(k)\right| \\
=&\left|\mathcal{F}(\mathcal{M}(f))(0)\right| + \sum_{k\neq 0} |k|^{-1} \left|\mathcal{F}(f)(k)\right| \\
=&2\sum_{k\neq 0} |k|^{-1} \left|\mathcal{F}(f)(k)\right| \\
\leq&2\sum_{k\in\mathbb{Z}} \left|\mathcal{F}(f)(k)\right| \\
=&2\left\|f\right\|_{\mathcal{F}^{0,1}},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\|T(\phi_1)-T(\phi_2)\|_{\mathcal{F}^{0,1}}\\
\le&2\,\|U(\phi_1)-U(\phi_2)\|_{\mathcal{F}^{0,1}}+2\,\|(\phi_1)_\alpha\cdot(U(\phi_1)-U(\phi_2))\|_{\mathcal{F}^{0,1}}\\
&+2\,\|U(\phi_2)\cdot((\phi_1)_\alpha-(\phi_2)_\alpha)\|_{\mathcal{F}^{0,1}}\\
\le&2\,\|U(\phi_1)-U(\phi_2)\|_{\mathcal{F}^{0,1}}+2\,\|\phi_1\|_{\mathcal{F}^{1,1}}\,\|U(\phi_1)-U(\phi_2)\|_{\mathcal{F}^{0,1}}\\
&+2\,\|U(\phi_2)\|_{\mathcal{F}^{0,1}}\,\|\phi_1-\phi_2\|_{\mathcal{F}^{1,1}}\,.
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
\|\mathcal{M}(f)\|_{\dot{\mathcal{F}}^{1,1}}&=\sum_{k\neq 0}|k|\,|\mathcal{F}(\mathcal{M}(f))(k)|\\
&=\sum_{k\neq 0}|k|\cdot|k|^{-1}\,|\mathcal{F}(f)(k)|\\
&\le\sum_{k\in\mathbb{Z}}|\mathcal{F}(f)(k)|\\
&=\|f\|_{\mathcal{F}^{0,1}}\,,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|T(\phi_1)-T(\phi_2)\|_{\dot{\mathcal{F}}^{1,1}}\le&\|U(\phi_1)-U(\phi_2)\|_{\mathcal{F}^{0,1}}+\|(\phi_1)_\alpha\cdot(U(\phi_1)-U(\phi_2))\|_{\mathcal{F}^{0,1}}\\
&+\|U(\phi_2)\cdot((\phi_1)_\alpha-(\phi_2)_\alpha)\|_{\mathcal{F}^{0,1}}\\
\le&\|U(\phi_1)-U(\phi_2)\|_{\mathcal{F}^{0,1}}+\|\phi_1\|_{\mathcal{F}^{1,1}}\,\|U(\phi_1)-U(\phi_2)\|_{\mathcal{F}^{0,1}}\\
&+\|U(\phi_2)\|_{\mathcal{F}^{0,1}}\,\|\phi_1-\phi_2\|_{\mathcal{F}^{1,1}}\,.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\left\|(\phi_2)_\alpha \cdot (T(\phi_1) - T(\phi_2))\right\|_{\dot{\mathcal{F}}^{1,1}} \\
\leq &\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}} \Bigg(2\left\|U(\phi_1) - U(\phi_2)\right\|_{\mathcal{F}^{0,1}} + 2\left\|\phi_1\right\|_{\mathcal{F}^{1,1}} \left\|U(\phi_1) - U(\phi_2)\right\|_{\mathcal{F}^{0,1}} \\
&+ 2\left\|U(\phi_2)\right\|_{\mathcal{F}^{0,1}} \left\|\phi_1 - \phi_2\right\|_{\mathcal{F}^{1,1}}\Bigg) \\
&+ \left\|\phi_2\right\|_{\mathcal{F}^{1,1}} \Bigg(\left\|U(\phi_1) - U(\phi_2)\right\|_{\mathcal{F}^{0,1}} + \left\|\phi_1\right\|_{\mathcal{F}^{1,1}} \left\|U(\phi_1) - U(\phi_2)\right\|_{\mathcal{F}^{0,1}} \\
&+ \left\|U(\phi_2)\right\|_{\mathcal{F}^{0,1}} \left\|\phi_1 - \phi_2\right\|_{\mathcal{F}^{1,1}}\Bigg).
\end{aligned}$$

Now, consider the expression

$$\left|\frac{2\pi}{L(\phi_1)} - \frac{2\pi}{L(\phi_2)}\right|.$$

Without loss of generality, let

$$\begin{aligned}
&\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!}(\phi_1(\alpha) - \phi_1(\eta))^n d\eta d\alpha \\
>&\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!}(\phi_2(\alpha) - \phi_2(\eta))^n d\eta d\alpha.
\end{aligned}$$

Then

$$
\begin{aligned}
&\left|\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right| \\
\leq &R^{-1}\cdot\frac{1}{4\pi}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi_2(\alpha)-\phi_2(\eta))^n d\eta d\alpha\right)^{-1/2} \\
&\cdot\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}(e^{i(\phi_1(\alpha)-\phi_1(\eta))}-e^{i(\phi_2(\alpha)-\phi_2(\eta))})d\eta d\alpha\right| \\
\leq &R^{-1}\cdot\frac{1}{4\pi}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi_2(\alpha)-\phi_2(\eta))^n d\eta d\alpha\right)^{-1/2} \\
&\cdot\Bigg(2\pi\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\phi_1\|_{\mathcal{F}^{0,1}}+\|\phi_2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!} \\
&+2\pi\left\|\phi_1(\pi)-\phi_2(\pi)\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\phi_1(\pi)\|_{\mathcal{F}^{0,1}}+\|\phi_2(\pi)\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\Bigg) \\
\leq &R^{-1}\left(1-\frac{\pi}{2}\left(e^{2\|\phi_2\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\phi_1\|_{\mathcal{F}^{0,1}}+\|\phi_2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}.
\end{aligned}
$$

Combining these results, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\|\theta_1-\theta_2\right\|_{\dot{\mathcal{F}}^{1,1}}\\
\leq& -\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}+\left|\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right|\left\|\widetilde{N}(\phi_1)\right\|_{\dot{\mathcal{F}}^{1,1}}\\
&+\frac{2\pi}{L(\phi_2)}\left\|\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}}+\pi\cdot\frac{\gamma}{4\pi}\cdot\frac{1}{R}A(\left\|\phi_1\right\|_{\mathcal{F}^{0,1}})\left\|\phi_1\right\|_{\mathcal{F}^{0,1}}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}\\
&+\pi\cdot\frac{\gamma}{4\pi}\left|\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right|\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}\\
\leq& -\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}\\
&+R^{-1}\left(1-\frac{\pi}{2}\left(e^{2\left\|\phi_2\right\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}\left(\sum_{n=1}^{\infty}\frac{(\left\|\phi_1\right\|_{\mathcal{F}^{0,1}}+\left\|\phi_2\right\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\right)\\
&\cdot\left\|\widetilde{N}(\phi_1)\right\|_{\dot{\mathcal{F}}^{1,1}}\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}}\\
&+\left(\frac{1+\frac{\pi}{2}\left(e^{2\left\|\phi_2\right\|_{\mathcal{F}^{0,1}}}-1\right)}{R^2}\right)^{1/2}\left\|\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}}\\
&+\pi\cdot\frac{\gamma}{4\pi}\cdot\frac{1}{R}A(\left\|\phi_1\right\|_{\mathcal{F}^{0,1}})\left\|\phi_1\right\|_{\mathcal{F}^{0,1}}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}\\
&+\pi\cdot\frac{\gamma}{4\pi}\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}R^{-1}\left(1-\frac{\pi}{2}\left(e^{2\left\|\phi_2\right\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}\\
&\cdot\left(\sum_{m=1}^{\infty}\frac{(\left\|\phi_1\right\|_{\mathcal{F}^{0,1}}+\left\|\phi_2\right\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\right)\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}}.
\end{aligned}
$$

Ultimately, for sufficiently small $\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}$,

$$
\frac{d}{dt}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{1,1}}\leq\mathcal{E}\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{1,1}},
$$

where $\mathcal{E}$ is a coefficient that may depend on $\left\|\phi_1\right\|_{\dot{\mathcal{F}}^{1,1}}$, $\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{1,1}}$, $\left\|\phi_1\right\|_{\dot{\mathcal{F}}^{2,1}}$, and $\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}$ in such a way that it is integrable in time. By Grönwall's inequality, since the two solutions share the same initial datum, $\phi_1=\phi_2$. Since the dynamics of $\mathcal{F}(\theta_1)(0)$ and $\mathcal{F}(\theta_2)(0)$ are determined completely by ϕ_1 and ϕ_2, respectively, with the shared initial condition $\mathcal{F}(\theta^0)(0)$, we conclude that $\mathcal{F}(\theta_1)(0)=\mathcal{F}(\theta_2)(0)$ as well.

CHAPTER 14

Numerical Verification

In this Chapter, we introduce a numerical scheme for our model based on the boundary integral formulation (2.10) of the interfacial fluid velocity. In the end, we see that implementation of the numerical scheme is in line with the analytical well-posedness result contained in this work.

For the purposes of this Chapter, we rewrite (2.10) in vector notation as

$$\partial_t \boldsymbol{X}(\theta,t) = \frac{1}{4\pi}\int_{\Gamma} G(\boldsymbol{X}(\theta,t)-\boldsymbol{X}(s,t))(-\gamma\cdot\kappa(s)\cdot\boldsymbol{n}(s))ds, \tag{14.1}$$

where $\boldsymbol{X}(\theta,t)$ denotes the interface at time t in HLS parametrization. Computing interfacial fluid flow based on such a boundary integral is known as a boundary integral (BI) method. Boundary integral methods are among the most popular numerical methods for computing the evolution of interfaces in water waves, Stokes flow, Hele-Shaw flow, and flows exhibiting Kelvin-Helmholtz and Rayleigh-Taylor instabilities. The popularity of BI methods is mainly due to the fact that their implementation entails only quantities that describe the interface, which effectively reduces the dimension of the model under study. The reduction of the dimension facilitates the handling of complex geometries and lowers the number of discretization points. The popularity of BI methods is also in part ascribed to their high accuracy. Since a BI method is based on the boundary integral formulation of the model under study, it is able to accurately account for intricate interfacial phenomena, such as discontinuity in normal stress due to surface tension. A number of highly accurate numerical schemes based on BI methods have been introduced and implemented for two-dimensional Stokes bubbles, two-dimensional elastic capsules, and even three-dimensional axisymmetric flow problems.

14.1. Preliminary Work

Let N be an even integer. For a discrete periodic function w, let us define the discrete Fourier transform by

$$(\mathcal{F}_N w)_k = \sum_{l=0}^{N-1} e^{-\frac{2\pi ikl}{N}} w_l$$

and the discrete inverse Fourier transform by

$$(\mathcal{F}_N^{-1} w)_k = \frac{1}{N} \sum_{l=0}^{N-1} e^{\frac{2\pi ikl}{N}} (\mathcal{F}_N w)_l,$$

where $k = 0, 1, \ldots, N-1$. The discrete differential operator $\mathcal{D}_N$ is defined by $\mathcal{D}_N w = \mathcal{F}_N^{-1} \tilde{w}$, where

$$\tilde{w}_k = \begin{cases} ik(\mathcal{F}_N w)_k & k \neq \frac{N}{2}, \\ 0 & k = \frac{N}{2}. \end{cases}$$

For a function w defined on S^1, the Hilbert transform of w is defined by

$$(\mathcal{H}w)(\theta) = \frac{1}{2\pi} \mathrm{pv} \int_{S^1} \cot\left(\frac{\theta - \theta'}{2}\right) w(\theta')\theta'.$$

For a discrete periodic function w, the discrete Hilbert operator $\mathcal{H}_N$ is defined by $\mathcal{H}_N w = \mathcal{F}_N^{-1} \bar{w}$, where

$$\bar{w}_k = \begin{cases} -i \cdot \mathrm{sgn}(k) \cdot (\mathcal{F}_N w)_k & k \neq \frac{N}{2}, \\ 0 & k = \frac{N}{2}. \end{cases}$$

14.2. The Numerical Scheme

Our numerical scheme for (14.1) requires both spatial and temporal discretization. We discretize the interface with N points for some fixed even integer N. For a fixed time step size $dt > 0$, we let $\boldsymbol{X}_m^n$ denote the position of the mth point of the discretized interface at time $n \cdot dt$, where

$m = 0, 1, \ldots, N-1$, and let $\boldsymbol{X}^n = (\boldsymbol{X}_0^n, \boldsymbol{X}_1^n, \ldots, \boldsymbol{X}_{N-1}^n)$. Let $\gamma = 1$. For any fixed position $\boldsymbol{X}_c$ of the discretized interface, the boundary integral (14.1) can be written as

$$\begin{aligned}\partial_t \boldsymbol{X}(\theta, t) = &-\frac{1}{4\,|\partial_\theta \boldsymbol{X}_c|}\mathcal{H}(\partial_\theta \boldsymbol{X}) - \frac{1}{4}\mathcal{H}\left(\left(\frac{1}{|\partial_\theta \boldsymbol{X}|} - \frac{1}{|\partial_\theta \boldsymbol{X}_c|}\right)\partial_\theta \boldsymbol{X}\right)(\theta)\\ &-\frac{1}{4\pi}\int_{S^1}\partial_{\theta'}\left(-\log\left(\frac{|\Delta \boldsymbol{X}|}{2\left|\sin(\frac{\theta-\theta'}{2})\right|}\right)I + \frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\cdot\frac{\partial_{\theta'}\boldsymbol{X}}{|\partial_{\theta'}\boldsymbol{X}|}d\theta',\end{aligned}$$

where $\Delta\boldsymbol{X} = \boldsymbol{X}(\theta, t) - \boldsymbol{X}(s, t)$. This formulation of the boundary integral serves as a template for our numerical scheme.

Suppose that the initial position $\boldsymbol{X}^0$ of the interface is provided. We define our numerical scheme recursively. Given $\boldsymbol{X}^n$, we first ensure that any adjacent pair of the N points constituting the interface have the same chordal length, which yields a discrete analog of HLS parametrization. Next, we compute $\boldsymbol{X}^{n+1}$ by numerically solving

$$\begin{aligned}\frac{\boldsymbol{X}^{n+1/2} - \boldsymbol{X}^n}{\Delta t/2} &= -\frac{1}{4\,|\mathcal{D}_N \boldsymbol{X}^n|}\mathcal{H}_N\left(\mathcal{D}_N \boldsymbol{X}^{n+1/2}\right) + R_2(\boldsymbol{X}^n)\\ \frac{\boldsymbol{X}^{n+1} - \boldsymbol{X}^n}{\Delta t} &= -\frac{1}{4\,|\mathcal{D}_N \boldsymbol{X}^n|}\mathcal{H}_N\left(\frac{\mathcal{D}_N\boldsymbol{X}^n + \mathcal{D}_N\boldsymbol{X}^{n+1}}{2}\right) + R_1(\boldsymbol{X}^{n+1/2}, \boldsymbol{X}^n)\\ &\quad + R_2(\boldsymbol{X}^{n+1/2}),\end{aligned}$$

where

$$R_1(\boldsymbol{X}^{n+1/2}, \boldsymbol{X}^n) = \frac{1}{4}\mathcal{H}_N\left(\frac{\mathcal{D}_N(\boldsymbol{X}^{n+1/2} - \boldsymbol{X}^n)\cdot\mathcal{D}_N(\boldsymbol{X}^{n+1/2} + \boldsymbol{X}^n)}{\left|\mathcal{D}_N\boldsymbol{X}^{n+1/2}\right|\cdot|\mathcal{D}_N\boldsymbol{X}^n|\cdot\left(\left|\mathcal{D}_N\boldsymbol{X}^{n+1/2}\right| + |\mathcal{D}_N\boldsymbol{X}^n|\right)}\mathcal{D}_N\boldsymbol{X}^{n+1/2}\right)$$

and $R_2(\boldsymbol{X})$ is a numerical computation of the integral

$$-\frac{1}{4\pi}\int_{S^1}\partial_{\theta'}\left(-\log\left(\frac{|\Delta \boldsymbol{X}|}{2\left|\sin(\frac{\theta-\theta'}{2})\right|}\right)I + \frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\cdot\frac{\partial_{\theta'}\boldsymbol{X}}{|\partial_{\theta'}\boldsymbol{X}|}d\theta',$$

where spatial variables θ and θ' are discretized as $\theta_j = \theta'_j = j\cdot\frac{2\pi}{N}$ for $j = 0, 1, \ldots, N-1$, the spatial derivatives inside the integrand are replaced by the discrete differential operator $\mathcal{D}_N$, and the quadrature rule

$$\int_{S^1} \boldsymbol{W}(\theta,\theta')d\theta' \approx \sum_{j=0}^{N-1} \boldsymbol{W}_{k,j} \cdot \frac{2\pi}{N}$$

is used to compute the integral. Now, using recursion, we can deduce the position $\boldsymbol{X}^n$ of the discretized interface for any $n \in \mathbb{N}$.

14.3. Computational Verification of Analytical Results

We recall that, according to our analytical result, the interfacial perturbation about a circular steady-state solution decays at an exponential rate. In this Section, we implement our numerical scheme for some computational evidence in line with our analytical result and some insight into the magnitude of the exponential rate of decay. After the nth iteration of our numerical scheme, we obtain the numerical position of the interface at time $n \cdot dt$, where any adjacent pair of the N points that make up the interface have the same chordal length. To compute the perturbation, we need to devise a way to "project away" circles from the interface. To that end, we parametrize a circle of radius $A^2 + B^2 > 0$ centered at (C_1, C_2) by

$$\boldsymbol{X}(\theta) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + A \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + B \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Since $|\partial_\theta \boldsymbol{X}| = \sqrt{A^2 + B^2}$ is independent of θ, the points $\boldsymbol{X}(k \cdot \frac{2\pi}{N})$ for $k = 0, 1, \ldots, N-1$ that make up the discretized circle will be uniformly spaced as in the case of our numerical scheme. For discrete periodic functions $\boldsymbol{V}$ and $\boldsymbol{W}$, the discrete inner product is defined by

$$\langle \boldsymbol{V}, \boldsymbol{W} \rangle_N = \sum_{k=0}^{N-1} (\boldsymbol{V}_k \cdot \boldsymbol{W}_k) \cdot \frac{2\pi}{N}.$$

Let $\mathbf{e_1^N}$, $\mathbf{e_2^N}$, $\mathbf{e_3^N}$, and $\mathbf{e_4^N}$ be the vectors

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{e_4} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

evaluated at $\theta_k = k \cdot \frac{2\pi}{N}$ for $k = 0, 1, \ldots, N-1$, respectively. We define the discrete perturbation operator by

$$\Pi_N \boldsymbol{V} = \boldsymbol{V} - \mathcal{P}_N \boldsymbol{V},$$

where

$$\mathcal{P}_N \boldsymbol{V} = \frac{1}{2\pi}\Big(\left\langle \boldsymbol{V}, \mathbf{e_1^N} \right\rangle_N \mathbf{e_1} + \left\langle \mathbf{V}, \mathbf{e_2^N} \right\rangle_{\mathbf{N}} \mathbf{e_2} + \left\langle \mathbf{V}, \mathbf{e_3^N} \right\rangle_{\mathbf{N}} \mathbf{e_3} + \left\langle \mathbf{V}, \mathbf{e_4^N} \right\rangle_{\mathbf{N}} \mathbf{e_4} \Big).$$

We measure the size of the perturbation using the discrete L^∞ norm

$$\|\boldsymbol{V}\|_\infty = \sup_k |\boldsymbol{V}_k|.$$

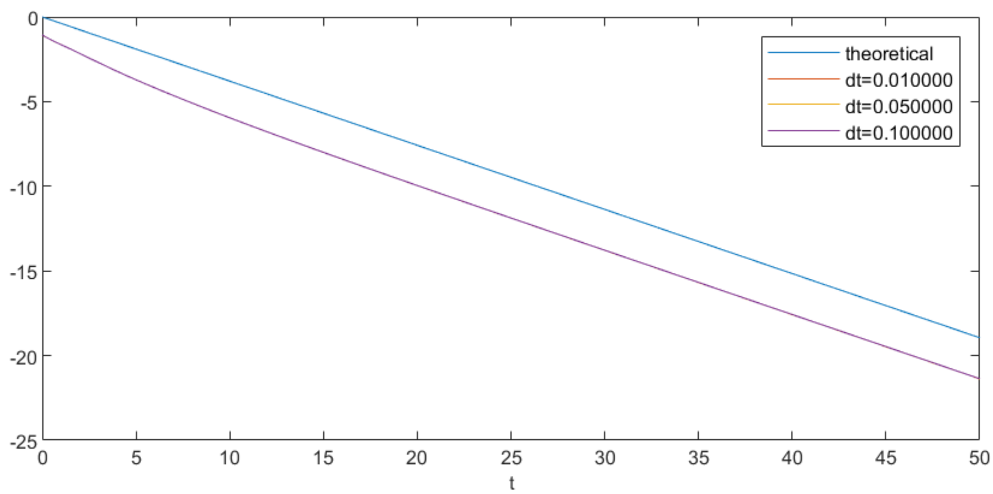

Figure 14.1: The plot of $\log \|\Pi_{100}(\boldsymbol{X}^n)\|_\infty$ against n for $dt = 0.1$, 0.05, and 0.01, up to $t = 50$.

In Figure 14.1, we plot $\log \|\Pi_{100}(\boldsymbol{X}^n)\|_\infty$ against n for $dt = 0.1$, 0.05, and 0.01 up to $t = 50$ for the initial position of the interface specified by

$$\boldsymbol{X}^0 = \begin{pmatrix} \left(1 + \frac{e^{\cos(3\theta)}}{4}\right) \cos\theta \\ \left(1 + \frac{e^{\cos(4\theta)}}{4}\right) \sin\theta \end{pmatrix}.$$

The blue "theoretical" line has a slope of $-\frac{\sqrt{\pi}}{2\sqrt{A}}$, where A is the area enclosed by the initial interface. The plot in Figure 14.1 then suggests that the perturbation decays at an exponential rate of $-\frac{\sqrt{\pi}}{2\sqrt{A}}$.

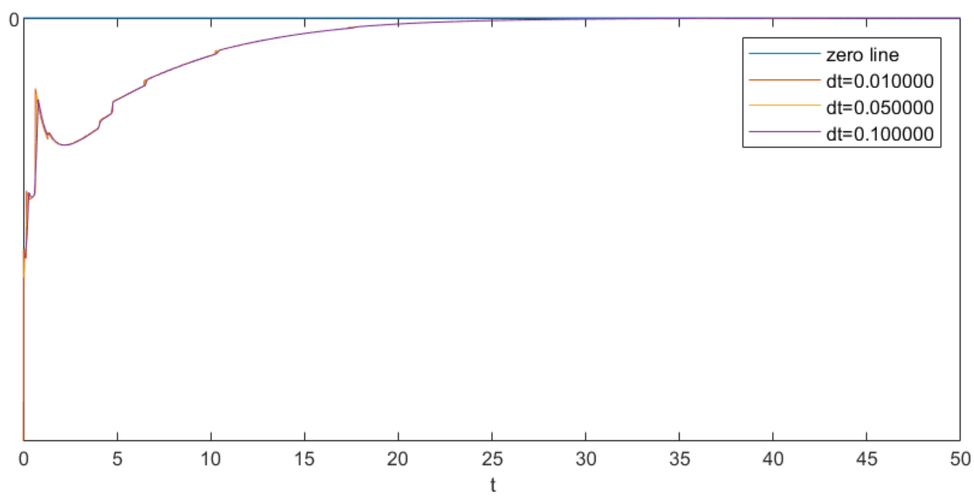

Figure 14.2: The plot of $\frac{1}{dt} \cdot \log \frac{\|\Pi_{100}(\boldsymbol{X}^n)\|_\infty}{\|\Pi_{100}(\boldsymbol{X}^{n-1})\|_\infty} - \left(-\frac{\sqrt{\pi}}{2\sqrt{A}}\right)$ against $n \cdot dt$ for $dt = 0.1$, 0.05, and 0.01, up to $t = 50$.

This observation can be made more explicitly in Figure 14.2, where the difference between the difference quotient of $\log \|\Pi_{100}(\boldsymbol{X}^n)\|_\infty$ and $-\frac{\sqrt{\pi}}{2\sqrt{A}}$ is plotted. What is the significance of $-\frac{\sqrt{\pi}}{2\sqrt{A}}$? According to our analytical result, the interfacial perturbation θ about a circular steady-state solution is governed by (2.19). The evolution of this perturbation is largely captured by the principal linear operator $\mathcal{L}$, whose ± 1 Fourier modes are 0 and whose higher Fourier modes coincide with those of the Hilbert transform acting on the first spatial derivative. According to (6.26), Fourier modes of $\mathcal{L}$ with a nonzero Fourier multiplier that is smallest in magnitude are

$$\mathcal{F}(\mathcal{L})(\theta)(\pm 2) = -\frac{\pi}{L(t)}\mathcal{F}(\theta)(\pm 2) = -\frac{\sqrt{\pi}}{2\sqrt{A}}\mathcal{F}(\theta)(\pm 2).$$

If we treated the perturbation as being governed exclusively by the operator $\mathcal{L}$, then all nonzero Fourier modes of the perturbation decay exponentially, with the ± 2 Fourier modes decaying the slowest and therefore setting the rate at which the perturbation decays as a whole. The same phenomenon is observed for the Peskin problem (Mori et al., 2019).

14.4. The Order of the Numerical Scheme

In Section 14.3, we generated computational evidence that the interfacial perturbation about a circular steady-state solution decays at an exponential rate, as predicted by our analytical result. In fact, the numerics there indicate that the exponential rate of decay corresponds to the smallest nonzero Fourier mode of the principal linear operator $\mathcal{L}$ that governs the dynamics of the perturbation. In this Section, we provide computational evidence that our numerical scheme is first order in time, provided that the boundary integral formulation of the interfacial velocity admits a unique solution and our numerical scheme converges to it.

Let $\boldsymbol{X}_{dt}^{N,T}$ be the collection of N points that constitute the interface at time T computed by our numerical scheme with time step size dt. Suppose that for sufficiently large $n \in \mathbb{N}$,

$$E_n^{N,T} = \left\| \boldsymbol{X}_{2^{-(n-1)}}^{N,T} - \boldsymbol{X}_{2^{-n}}^{N,T} \right\|_\infty \leq C \cdot 2^{-nk}$$

for some constants $C > 0$ and $k > 0$. If $\boldsymbol{X}^T$ is the unique analytical solution at time T sampled at

N points such that the adjacent pairs of such points have the same chordal length, then

$$\begin{aligned}
\left\|\boldsymbol{X}^{N,T}_{2^{-(n-1)}} - \boldsymbol{X}^{T}\right\|_\infty &\leq \left\|\boldsymbol{X}^{N,T}_{2^{-(n-1)}} - \boldsymbol{X}^{N,T}_{2^{-n}}\right\|_\infty + \left\|\boldsymbol{X}^{N,T}_{2^{-n}} - \boldsymbol{X}^{T}\right\|_\infty \\
&\leq \left\|\boldsymbol{X}^{N,T}_{2^{-(n-1)}} - \boldsymbol{X}^{N,T}_{2^{-n}}\right\|_\infty + \left\|\boldsymbol{X}^{N,T}_{2^{-n}} - \boldsymbol{X}^{N,T}_{2^{-(n+1)}}\right\|_\infty + \cdots \\
&\leq C\left(2^{-nk} + 2^{-(n+1)k} + \cdots\right) \\
&= \frac{C}{1-2^{-k}} \cdot 2^{-nk}.
\end{aligned}$$

Therefore, if we can fit a line through the points generated by plotting $\log_2 E_n^{N,T}$ against n, then its slope would correspond to $-k$, where k is the order of convergence in time for our numerical scheme.

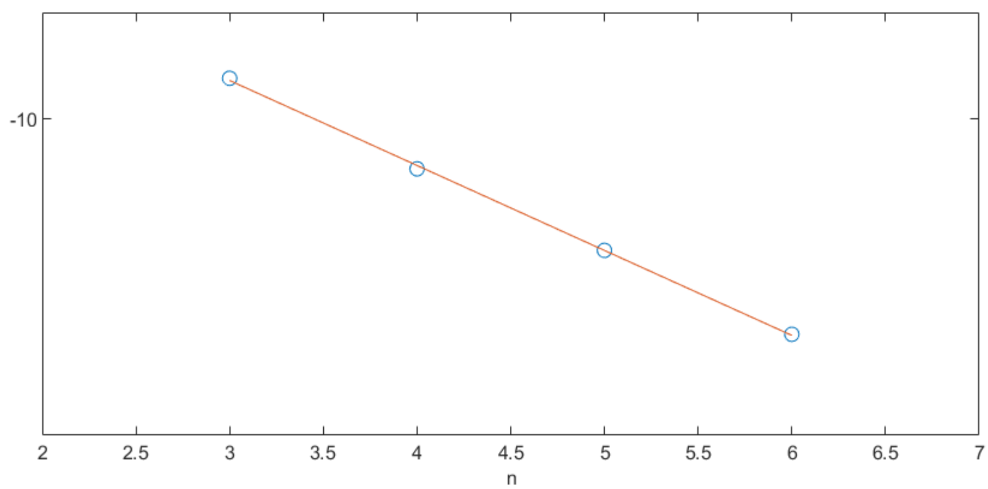

Figure 14.3: The plot of $\log_2 E_n^{100,40}$ against n for $n = 3, 4, 5, 6$.

Figure 14.3 shows that the line of best fit has a slope of -0.8037, which suggests that our numerical scheme is first order in time.

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