On the stability of electric arc discharges

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The stability of electric arc discharges has been explored through the use of an energy balance coupled with charge conservation. In order to facilitate this analysis, a new model for the electrical conductivity function has been proposed. Asymptotic solutions for the arc governing equations have been obtained. Stability criteria have been developed from both the linear theory (infinitesimal size disturbance) and from a minimizing solution point of view for finite size disturbances. The results delineate an open region in the stability diagram where arc instabilities may be possible.

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I. INTRODUCTION

An idealized description of electric arc phenomena centers on an energy balance between conduction heat transfer and dissipation by Ohmic heating. In this paper, we shall base a model for arc stability on this energy balance and as a result obtain stability criteria in terms of both arc plasma and circuit parameters. The model allows for variations in the transport properties. Other energy loss mechanisms, such as forced convection and radiation, are often important in practical applications. These are neglected here partly because idealized situations can be constructed in which they are negligible, but more importantly, because we are formulating a new procedure to obtain stability criteria and we have chosen to start with the simplest possible nontrivial problem. We intend to include the effect of convection and radiation in subsequent work.

Two features of the present study are noteworthy. First, the study provides a new model for the electrical conductivity function that reproduces all gross features of experimentally obtained characteristics and yet is suitable for easy mathematical manipulation. Second, a systematic inquiry into asymptotic limiting solutions of the governing equations has been shown to aid the development of the stability theories. Interesting estimates of voltage and current extremal values stemming from the solutions of scaled equations have been provided and compared with existing results where available.

Finally, the stability criteria are obtained by investigating the perturbations of the steady solutions of the governing equations. This investigation has been undertaken from both the linear theory (disturbance size infinitesimal) and from a minimizing solution point of view for finite size disturbances. The results complement each other in the sense that taken together they provide the necessary and sufficient conditions for instability for an arc. It is seen that they delineate an open region in the stability diagram wherein arc instabilities may be possible. Since the finite size disturbance theory solutions correspond to minimizing solutions, no definitive conclusions can be drawn with regard to subcritical instabilities, and it is hoped that experimental results (unavailable at present) will shed further light on this matter.

II. THEORETICAL FORMULATION

Consider a planar arc between planar electrodes, essentially as a conducting slab which is bounded by isothermal side walls as shown in Fig. 1. If we neglect the effects of radiation and convection, energy conservation reduces to a balance between conduction heat transfer and Ohmic heating. Furthermore, with neglect of axial variations (see Ref. 1 for two-dimensional effects), we obtain the following steady-state Elenbass-Heller equation for energy conservation, in terms of the heat flux potential

$$\frac{d^2S}{dx^2} + E^2\sigma(S) = 0, \quad -x_w \le x \le x_w, \tag{1}$$

with $S(T) = \int_0^T k(T')\,dT'$,

subject to the boundary condition

$$\frac{dS}{dx} = 0 \quad \text{at } x=0; \quad S=S_w \quad \text{at } x=x_w. \tag{2}$$

In Eq. (1) we have used Ohm's law,

$$\mathbf{j} = \sigma\mathbf{E}. \tag{3}$$

The current I in the y direction can be obtained from

$$I = 2wE\int_0^{x_w} \sigma(S)\,dx. \tag{4}$$

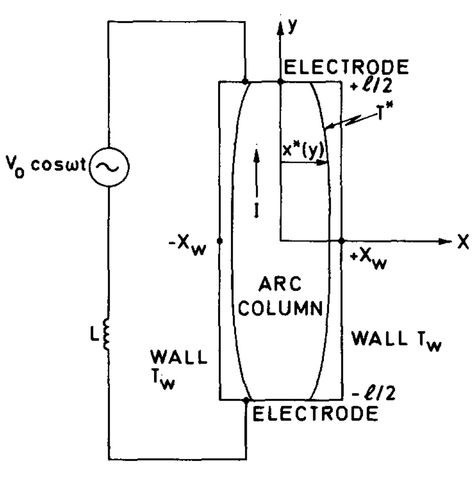

FIG. 1. The arc geometry and arc as element of a circuit.

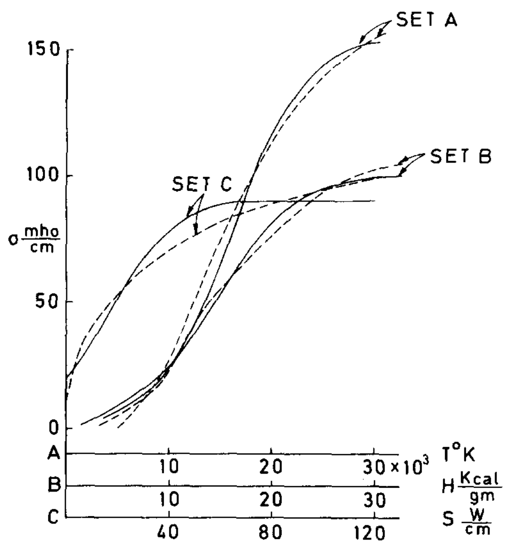

FIG. 2. Comparison of data for air with hyperbolic tangent conductivity model

$\sigma(Z) = \frac{1}{2}\sigma_*\{1 + \tanh[c(Z/Z_* - 1)]\}$.

Curves A: $\sigma(T)$, air at 10 atm. (Yos, Ref. 4), dashed line. $Z = T$, $Z_* = T_* = 16\,000$°K, $\sigma_* = 156$ mho/cm, $c = 2.3$, solid line. Curves B: $\sigma(h)$ (h, enthalpy), air at 10 atm. (Yos, Ref. 4), dashed line. $Z = h$, $Z_* = h_* = 14.1$ kcal/g, $\sigma_* = 100$ mho/cm, $c = 2.0$, solid line. Curves C: $\sigma(S)$, air at 1 atm. (Devoto, Ref. 5), dashed line. $Z = S$, $Z_* = S_* = 15.5$ W/cm, $\sigma_* = 90$ mho/cm, $c = 0.6$, solid line. No special attempt has been made to achieve a "best fit" to the data.

In the above, $\mathbf{E}$, σ, k, T, $\mathbf{j}$, and w are the electric field intensity, electrical conductivity, thermal conductivity, plasma temperature, current density, and width (in z direction) of the planar arc, respectively. The arc energy equation used for the analysis contains only a thermal-conduction energy loss. It must be emphasized that in general the full range of energy transfer processes (radiation, convection, and conduction including the effects of turbulent transport) have importance in the energy conservation equation for an arc. These are neglected here partly because idealized situations can be constructed in which they are negligible, but more importantly, because we are formulating a new procedure to obtain stability criteria and we have chosen to start with the simplest possible nontrivial problem. We intend to include the effects of convection and radiation in subsequent work.

The temperature variation of the transport properties and the quadratic manner in which the electric field enters the problem introduce strong nonlinearities, and any solution procedure other than a complete numerical scheme requires some simplifications. Here, through the use of the heat flux potential S, variation of the thermal conductivity function has already been taken into account. We must then make a suitable choice for approximately representing the electrical conductivity. In two previous papers (Refs. 2 and 3) we have obtained arc-column electric-field—vs—arc-current characteristics which showed sensitivity for large currents to the manner in which the electrical conductivity varied with temperature. However, with any "switch-on" electrical conductivity that is identically zero for temperatures less than a "critical" temperature T_*, the characteristic behavior for small currents is the same—E increases when $I \to 0$ as $1/I$. For the purpose of examining arc stability, the neighborhood of zero current is of crucial importance. A careful examination of our previous models showed that the infinite electric field at zero current was a direct consequence of the discontinuity in the derivative $d\sigma(T)/dT$ at T_*. To provide a more realistic model of the arc for small currents it is proposed in this paper that $\sigma(S)$ be represented by the relation

$$\sigma(S) = \tfrac{1}{2}\sigma_* \{1 + \tanh[a(S - S_*)]\}, \tag{5}$$

where σ_*, T_*, and a are material constants. For high temperatures, this "tanh conductivity" model reproduces the features of the constant property arc model, but, since $\sigma(S)$ and all its derivates are everywhere continuous, the arc characteristic has no singularities and reproduces all gross features of experimentally obtained characteristics. In this regard, an examination of the electrical conductivity in equilibrium air[4,5] indicates that on a linear plot, it is well fitted by a tanh function. (See Fig. 2.). Moreover, the one-dimensional analysis yields current-electric field characteristics shown in Fig. 3 which are linear scale representations of gas discharge characteristics (for example, see Fig. 1 of Ref. 6). Since our emphasis is on arcs, all phenomena on current scales much smaller than the arc scale are forced to the vertical axis. These phenomena have been treated in detail in Refs. 7 and 8.

Now the problem can be made dimensionless by defining

$$f = (S - S_*)/(S_* - S_w), \quad \mu = a(S_* - S_w), \quad \xi = x/x_w,$$

and by measuring E and I in units of

$$[(S_* - S_w)/\sigma_* x_w^2]^{1/2} \quad \text{and} \quad w[4\sigma_*(S_* - S_w)]^{1/2},$$

respectively. Then from Equations (1)—(4),

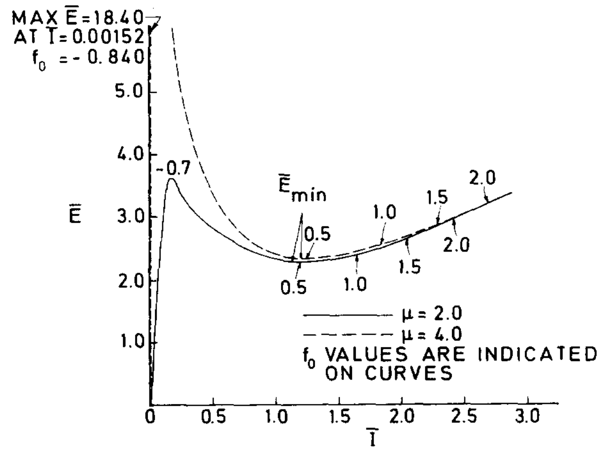

FIG. 3. Current-electric field characteristic—the tanh conductivity arc. E is given by Eq. (9) and I is given by Eq. (10), through the parameter f_0. A few representative values of f_0 are indicated on the figure.

$$\frac{d^2f}{d\xi^2} + \tfrac{1}{2}E^2[1+\tanh(\mu f)] = 0 \quad (6)$$

and

$$I = \tfrac{1}{2}E\int_0^1[1+\tanh(\mu f)]\,d\xi, \quad (7)$$

subject to the boundary conditions

$$\frac{df}{d\xi} = 0 \quad \text{at } \xi = 0; \quad f = -1 \quad \text{at } \xi = 1. \quad (8)$$

III. ASYMPTOTIC SOLUTIONS AND ESTIMATES

Let f_0 be the value of f at $\xi = 0$. This is the heat flux potential at the centerline of the arc and can be used as a measure of the centerline temperature. Then, from integrating Eq. (6) once, we get the relation between the electric field and this measure of the centerline temperature as

$$E = \frac{1}{\mu^{1/2}}\int_{-\mu}^{\mu f_0}\frac{d\zeta}{[\mu f_0 + \ln\cosh(\mu f_0) - \zeta - \ln\cosh\zeta]^{1/2}} \quad (9)$$

and through the use of Eq. (7), we can write

$$I = \frac{1}{2\mu^{1/2}}\int_{-\mu}^{\mu f_0}\frac{(1+\tanh\zeta)\,d\zeta}{[\mu f_0 + \ln\cosh\mu f_0 - \zeta - \ln\cosh\zeta]^{1/2}} \quad (10)$$

The set of solutions represented by Eqs. (9) and (10) is the current-electric field characteristic of the arc in parametric form with f_0 as the parameter. A calculation for $\mu = 2$ and 4 is shown in Fig. 3. An analytical evaluation of the integrals above is too complicated although solution estimates appropriate to the cases of large f_0 and f_0 nearly equal to -1 (with μ "large") can be obtained by suitable order-of-magnitude arguments. These estimates aid in the development of stability theories.

With $f_0 \to \infty$, we write

$$E \sim \frac{1}{\mu^{1/2}}\int_A^{\mu f_0}\frac{d\zeta}{2^{1/2}(\mu f_0 - \zeta)^{1/2}} + \frac{1}{\mu^{1/2}}\int_{-\mu}^{A}\frac{d\zeta}{[\mu f_0 + \ln\cosh(\mu f_0) - \zeta - \ln\cosh\zeta]^{1/2}}, \quad (11)$$

where A is a number of O(1) but numerically large enough to validate the approximation,

$$A + \ln\cosh A \sim 2A - \ln 2 . \quad (12)$$

The value of E in this limit can be obtained in major part from the first integral in Eq. (11), and to dominant order it is given by

$$E = (2f_0)^{1/2} + \frac{1}{(2f_0)^{1/2}} + O\,(f_0^{-3/2}). \quad (13)$$

With the same level of approximation

$$I = (2f_0)^{1/2} - \frac{\ln 2 - \mu + \ln\cosh\mu}{2\mu(2f_0)^{1/2}} + O(f_0^{-3/2}) \quad (14)$$

or, for the characteristic in dimensionless form, we have

$$E = I + \left(1 + \frac{\ln 2 - \mu + \ln\cosh\mu}{2\mu}\right)\frac{1}{I} + O(I^{-3}). \quad (15)$$

Thus, for high centerline temperatures, the arc looks like a linear resistor as nearly all of the gap is filled with high-conductivity material ($\sigma \sim \sigma_*$ for most all of $|x| \le x_w$). On the other hand, for f_0 close to -1 (actually $\mu f_0 \to -\infty$), we have

$$E \sim \frac{1}{\mu^{1/2}}\int_{-\mu}^{\mu f_0}\frac{d\zeta}{[\exp(2\mu f_0) - \exp(2\zeta)]^{1/2}}, \quad (16)$$

where $\zeta + \ln\cosh\zeta$ has been approximated by $-\ln 2 + \exp(2\zeta)$, for $\zeta \to \infty$. Transforming Eq. (16), we can produce a standard form,

$$E \sim (1/\mu^{1/2})\exp(-\mu f_0)\tanh^{-1}\{1 - \exp[-2\mu(1+f_0)]\}^{1/2}. \quad (17)$$

The function represented by Eq. (17) has a maximum and and furthermore, since

$$\frac{dE}{dI} = \left(\frac{dE}{df_0}\right)\left(\frac{dI}{df_0}\right)^{-1},$$

the maximum corresponding to dE/df_0 is a maximum of the characteristic. This last criterion yields,

$$\tanh\{1 - \exp[-2\mu(1+f_0)]\}^{1/2} = \{1 - \exp[-2\mu(1+f_0)]\}^{-1/2}, \quad (18)$$

which has a unique solution

$$f_0 = -1 + \frac{1}{2\mu}\ln\left(\frac{1}{1-b^2}\right), \quad (19)$$

where

$$b = \{1 - \exp[-2\mu(1+f_0)]\}^{1/2} = 0.8336. \quad (20)$$

For this value of f_0,

$$E_{\max} \sim \frac{(1-b^2)^{1/2}}{b}\frac{\exp(\mu)}{\mu^{1/2}} . \quad (21)$$

By similar arguments,

$$I \sim \frac{\exp(\mu f_0)}{\mu^{1/2}}\{1 - \exp[-2\mu(1+f_0)]\}^{1/2} \quad (22)$$

and the current at $E_{\max}$ is therefore

$$I_{\max} \sim \frac{b}{(1-b^2)^{1/2}}\frac{\exp(-\mu)}{\mu^{1/2}} . \quad (23)$$

As mentioned previously, these results will prove useful in developing stability criteria.

IV. DEVELOPMENT OF STABILITY CRITERIA

In order to develop suitable stability criteria, consider the arc to form part of a circuit of inductance L and driving voltage $V_0\cos\omega t$ as shown in Fig. 1. Equation (1) has to be modified to include time dependence and must be supplemented by the circuit equation. Thus

$$\frac{\partial^2 S}{\partial x^2} + E^2\sigma(S) = \kappa\frac{\partial S}{\partial t} \quad (24)$$

and

$$L\frac{dI}{dt} + El = V_0\cos\omega t. \quad (25)$$

Here κ and ω are the reciprocal thermal diffusivity and the dimensionless driving frequency, respectively. In addition, l is the distance between electrodes so that in Eq. (25) El is the arc voltage drop. We are therefore neglecting in this formulation the effect of the contraction regions as well as the cathode and anode sheaths on the over-all voltage drop. The above equations must be considered together with the current conservation equation (4) to form the required system of governing equations. Introducting $\tau = \omega t$, $\omega^* = \omega\kappa x_w^2$, $\Delta = 2x_w w\sigma_* L\omega/l$, and $\lambda = (V_0 x_w/l)[\sigma_*/(S_* - S_w)]^{1/2}$, these equations become

$$\frac{\partial^2 f}{\partial \xi^2} + \tfrac{1}{2}E^2[1+\tanh(\mu f)] = \omega^* \frac{\partial f}{\partial \tau}, \quad (26)$$

$$I = E\int_0^1 \tfrac{1}{2}[1+\tanh(\mu f)]d\xi, \quad (27)$$

and

$$\Delta \frac{dI}{d\tau} + E = \lambda \cos\tau. \quad (28)$$

We shall restrict our considerations to cases in which ω^* is much less than unity. Then the arc thermal inertia term is negligible so long as time variations occur on the scale τ and therefore the arc is quasisteady with current-electric field characteristics given by our previous result [Eqs. (21) and (23)]. Substituting this into Eq. (28) then produces a nonlinear ordinary differential equation for the current $I(\tau)$. However, for $\mu \gg 1$, our previous result [see Eqs. (21) and (23)] indicates that when the current is small $O[\mu^{-1/2}\exp(-\mu)]$, the arc voltage is large $O[\mu^{-1/2}\exp(\mu)]$ and the maximum temperature is not very different from the wall temperature, $f_{max}+1 = O(\mu^{-1})$. Then in Eq. (28) the inductive voltage drop must balance the arc voltage drop and this requires that $d/d\tau$ be large. This argument shows that in a region of small current ("current-zero") time variations occur on a much shorter scale than τ and hence the arc thermal inertia is not a perturbation quantity. In order to investigate this region we are thus led to scale the equations as follows:

$$I = \tilde{I}\,\mu^{-1/2}\exp(-\mu), \quad E = \tilde{E}\,\mu^{-1/2}\exp(\mu)$$

$$f = -1 + g/\mu, \quad \tau = \beta + \omega^*\tilde{\tau}.$$

The f scaling yields

$$\tfrac{1}{2}[1+\tanh(\mu f)] \sim \exp(-2\mu)\exp(2g),$$

so that to leading order in μ and ω^* we obtain

$$\frac{\partial^2 g}{\partial \xi^2} + \tilde{E}^2\exp(2g) \approx \frac{\partial g}{\partial \tilde{\tau}}, \quad (29)$$

$$\tilde{I} \approx \tilde{E}\int_0^1 \exp(2g)\,d\xi, \quad (30)$$

and

$$\delta\frac{d\tilde{I}}{d\tilde{\tau}} + \tilde{E} \approx \Lambda, \quad (31)$$

where we have introduced $\delta = \Delta\exp(-2\mu)/\omega^*$ and $\Lambda = \lambda\exp(-\mu)\mu^{1/2}\cos\beta$. The quantity β is a measure of the time at which the "current-zero" occurs.

There are two time scales in this set of equations. One is the time scale on which the temperature changes; this is the basic scale of the problem. The other is the scale on which the arc voltage recovers to the instantaneous value of the source voltage $\Delta\exp(-2\mu)/\omega^*$ which depends on the circuit and arc parameters.

Now time does not appear explicitly in the coefficients of Eqs. (29)–(31) and therefore we can obtain a steady-state solution. This is given by (overbars indicate steady-state solutions)

$$\bar{E} = \Lambda \quad (32)$$

$$\bar{I} = \Lambda\int_0^1 \exp(2\bar{g})\,d\xi \quad (33)$$

and from Eq. (29)

$$\frac{\partial^2 \bar{g}}{\partial \xi^2} + \Lambda^2\exp(2\bar{g}) = 0, \quad (34)$$

subject to

$$\frac{\partial \bar{g}}{\partial \xi} = 0 \quad \text{at } \xi = 0; \quad \bar{g} = 0 \quad \text{at } \xi = 1. \quad (35)$$

Therefore,

$$\bar{g} = g_0 - \ln(\cosh\{\xi\tanh^{-1}[1-\exp(-2g_0)]^{1/2}\}). \quad (36)$$

We note that for a given g_0, the value of Λ is unique, while for a given Λ two different values for g_0 are possible. Finally,

$$\bar{I} = \Lambda\int_0^1 \exp(2g_0)\,\mathrm{sech}^2\{\xi\tanh^{-1}[1-\exp(-2g_0)]^{1/2}\}d\xi$$
$$= \exp(g_0)[1-\exp(-2g_0)]^{1/2}. \quad (37)$$

If we find that some steady solution given above is a stable one, then subsequently the arc current will remain small and we can say that the arc is extinguished. On the other hand, if a steady-state solution given above is unstable, the arc current will again grow to finite values and we would conclude that the arc has reignited. Thus an investigation of the perturbation of these steady-state solutions will enable us to obtain criteria governing stability. We shall approach the stability aspects of this problem from both the linear theory and the nonlinear theory view points.

V. LINEAR STABILITY THEORY

Consider letting $\tilde{g} = \bar{g} + g'$, $\tilde{E} = \bar{E} + E'$, and $\tilde{I} = \bar{I} + I'$, where the prime denotes perturbation quantities. The perturbation equations for a linear theory become

$$\frac{\partial^2 g'}{\partial \xi^2} + 2\Lambda^2\exp(2\bar{g})g' + 2\Lambda\exp(2\bar{g})E' = \frac{\partial g'}{\partial \tilde{\tau}}, \quad (38)$$

$$I' = E'\int_0^1 \exp(2\bar{g})\,d\xi + 2\Lambda\int_0^1 \exp(2\bar{g})g'\,d\xi \quad (39)$$

and

$$\delta\frac{dI'}{d\tilde{\tau}} + E' = 0. \quad (40)$$

Solution of these equations is greatly facilitated by reducing this system to a single equation in g'. From Eq. (38)

$$2\Lambda\int_0^1 \exp(2\bar{g})g'\,d\xi = \frac{1}{\Lambda}\int_0^1 \frac{\partial g'}{\partial \tilde{\tau}}d\xi - 2E'\int_0^1 \exp(2\bar{g})\,d\xi - \frac{1}{\Lambda}\frac{\partial g'}{\partial \xi}.$$

Now with Eqs. (39) and (33),

$$\frac{-E'}{\delta} \approx -\frac{\bar{I}}{\Lambda}\frac{dE'}{d\tilde{\tau}} - \frac{1}{\Lambda}\frac{\partial^2 g'}{\partial \tilde{\tau}\,\partial \xi} + \frac{1}{\Lambda}\int_0^1 \frac{\partial^2 g'}{\partial \tilde{\tau}^2}\,d\xi.$$

Using these relations, after some manipulation, we get from Eq. (38),

$$\frac{\partial^3 g'}{\partial \xi^2\,\partial \tilde{\tau}} + 2\Lambda^2\exp(2\bar{g})\frac{\partial g'}{\partial \tilde{\tau}} + \frac{\Lambda}{\delta\bar{g}}\left(\frac{\partial g'}{\partial \tilde{\tau}} - \frac{\partial^2 g'}{\partial \xi^2} - 2\Lambda^2\exp(2\bar{g})g'\right)$$
$$+ \frac{2\Lambda\exp(2\bar{g})}{\bar{I}}\left(-\frac{\partial^2 g'}{\partial \tilde{\tau}\,\partial \xi} + \int_0^1 \frac{\partial^2 g'}{\partial \tilde{\tau}^2}d\xi\right) = \frac{\partial^2 g'}{\partial \tilde{\tau}^2}. \quad (41)$$

Letting, $g' = \hat{g}(\xi)\exp(\hat{\sigma}\tilde{\tau})$, where $\hat{\sigma}$ is the frequency of the perturbation,

$$\frac{\partial^2\hat{g}}{\partial\xi^2}\hat{\sigma} + 2\Lambda^2\exp(2\overline{g})\hat{g}\hat{\sigma} + \frac{\Lambda}{\delta\overline{I}}\left(\hat{\sigma}\hat{g} - \frac{\partial^2\hat{g}}{\partial\xi^2} - 2\Lambda^2\exp(2\overline{g})\hat{g}\right) + \frac{2\Lambda\exp(2\overline{g})}{\overline{I}}\left(-\frac{\partial\hat{g}}{\partial\xi}\hat{\sigma} + \int_0^1 \hat{\sigma}^2\hat{g}\,d\xi\right) = \hat{\sigma}^2\hat{g}. \quad (42)$$

We wish to inquire into mathematical conditions that correspond to a total physical failure or the complete extinguishing of the arc. We will assume that this physical situation corresponds to a time-independent instability process and therefore the mathematical solution corresponds to the neutral stability problem. The neutral state is governed by the stationary solution to

$$\frac{d^2\hat{g}}{d\xi^2} + 2\Lambda^2\exp(2\overline{g})\hat{g} = 0. \quad (43)$$

Setting,

$$\tanh^{-1}\{[1-\exp(-2g_0)]^{1/2}\} = r, \quad (44)$$

we need to solve

$$\frac{d^2\hat{g}}{d\xi^2} + 2r^2\operatorname{sech}^2(\xi r)\hat{g} = 0 \quad (45)$$

subject to

$$\frac{d\hat{g}}{d\xi} = 0 \quad \text{at } \xi = 0; \quad \hat{g} = 0 \quad \text{at } \xi = 1. \quad (46)$$

Equation (45) together with conditions (46) can be converted into an initial-value problem and integrated numerically. The minimum r is found to be 1.1997. It is of interest to note that this value of r corresponds to the maximum of the current-voltage characteristic as obtained from Eq. (18) and shown in Fig. 3. The result obtained here is valid for infinitesimal disturbances from equilibrium and is the sufficient condition for arc reignition. For values of g_0, the scaled dimensionless centerline heat flux potential, above those given by $[1-\exp(-2g_0)]^{1/2} = 0.8336$, small size disturbances will grow and hence the current will ultimately grow large. We note at this point that this result, although interesting, is of limited practical value since the initial state in approaching the current-zero region is far removed from the equilibrium state described by the source voltage. This observation naturally leads to the finite amplitude disturbance problem. With this in mind, in order to examine whether "subcritical" instabilities may exist or not, we now proceed to the formualtion and solution of the nonlinear stability problem through a variational approach.

VI. NONLINEAR THEORY AND NECESSARY CONDITION FOR INSTABILITY

The perturbation equations for a nonlinear theory are

$$\frac{\partial^2 g'}{\partial\xi^2} + 2\overline{E}E'\exp(2\overline{g})\exp(2g') + \overline{E}^2\exp(2\overline{g})[\exp(2g')-1] + E'^2\exp(2\overline{g})\exp(2g') = \frac{\partial g'}{\partial\tau}, \quad (47)$$

$$I' = \overline{E}\left\{\int_0^1 \exp(2\overline{g})[\exp(2g')-1]\,d\xi\right\} + E'\left[\int_0^1 \exp(2\overline{g})\exp(2g')\,d\xi\right\} \quad (48)$$

and

$$\delta\frac{dI'}{d\tilde{\tau}} + E' = 0\ . \quad (49)$$

The necessary conditions for the instability of the arc can be obtained by investigating the criteria

$$\frac{d}{d\tilde{\tau}}\frac{g'^2}{2}\,d\xi = 0 \text{ and } \int\left(\frac{d}{d\tilde{\tau}}\ \frac{g'^2}{2}\right)d\xi \leqslant 0. \quad (50)$$

We note that I'^2 and g'^2 are the squares of the current and temperature disturbance amplitude, respectively. The joint requirement demanded by inequalities (50) guarantees stability with respect to all possible disturbances. For the purposes of clarity, consider the latter requirement. If

$$\int\frac{d}{d\tilde{\tau}}\frac{g'^2}{2}\,d\xi < 0,$$

then temperature disturbances decay. When

$$\int\frac{d}{d\tilde{\tau}}\ \frac{g'^2}{2}\,d\xi = 0$$

we have the stability boundary: neutral stability. A similar explanation holds for the former requirement. Thus we have from inequalities (50), and from Eq. (49)

$$\delta + E'I' \geqslant 0 \quad (51)$$

and

$$-\int\frac{dg'}{d\xi}\frac{dg'}{d\xi}\,d\xi + \int 2\overline{E}E'g'\exp(2\overline{g})\exp(2g')\,d\xi + \int\overline{E}^2\exp(2\overline{g})[\exp(2g')-1]g'\,d\xi + \int E'^2 g'\exp[2(\overline{g}+g')]\,d\xi \leqslant 0. \quad (52)$$

Now consider taking variations of Eqs. (51) and (52) with respect to I' and g', respectively. In extremalizing the requirements posed by inequalities (50), we are finding conditions for the minimum value, i.e., the maximum negative value. This is because there is no finite maximum of either

$$\int\frac{d}{d\tilde{\tau}}\frac{g'^2}{2}\,d\xi \quad \text{or} \quad \frac{d}{d\tilde{\tau}}\ \frac{I'^2}{2}\ \ ;$$

when positive, they grow to infinity. Thus we are finding conditions for the maximum decay rates for spontaneous disturbances about equilibrium for an arc. These lead to the most conservative nontrivial criterion for stability. From Eq. (51), a variational formulation requires that E' be zero as appropriate to a nonzero I'. This condition can, at best, be only interpreted as representative of situations when the decay rates for nonzero E' are very much faster than those for I'. Nevertheless, the formulation is mathematically correct and solutions will be proper minimizing functions for the problem posed. With $E'=0$, from Eq. (52) we obtain

$$\frac{d^2g'}{d\xi^2} + \tfrac{1}{2}\overline{E}^2\exp(2\overline{g})\exp(2g')(1+2g') - \tfrac{1}{2}\overline{E}^2\exp(2\overline{g}) = 0. \quad (53)$$

Inserting the known values for $\overline{E}$ and $\overline{g}$, the variational equation governing nonlinear stability becomes

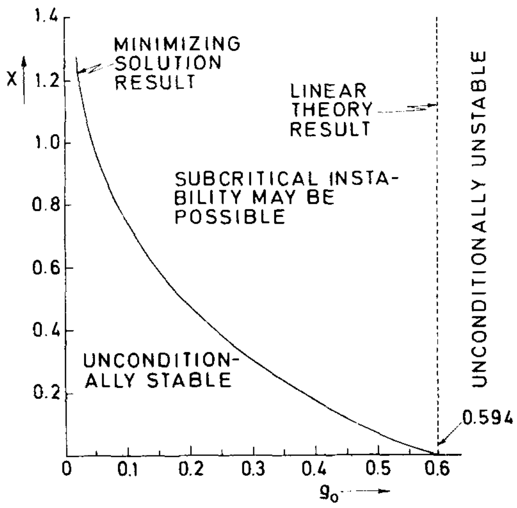

FIG. 4. Stability boundary for a slender arc.

$$\frac{d^2g'}{d\xi^2} + \tfrac{1}{2}r^2 \operatorname{sech}^2 (r\xi) [\exp(2g') + \exp(2g')2g' - 1] = 0 \tag{54}$$

subject to the boundary conditions,

$$\frac{dg'}{d\xi} = 0 \quad \text{at } \xi = 0; \quad g' = 0 \quad \text{at } \xi = 1. \tag{55}$$

With the transformation $\eta = r\xi$, we can convert the above boundary-eigenvalue problem to an initial value problem as follows:

$$\frac{d^2g'}{d\eta^2} + \tfrac{1}{2} \operatorname{sech}^2 \eta [\exp(2g') + \exp(2g')2g' - 1] = 0 \tag{56}$$

with

$$\frac{dg'}{d\eta} = 0 \text{ and } g' = g_0' \quad \text{at } \eta = 0; \quad g' = 0 \quad \text{at } \eta = r. \tag{57}$$

In order to facilitate a discussion of the solution to this initial-value problem, we introduce the weight

$$\int_0^1 g' \, d\xi = \chi \tag{58}$$

as a measure of the size of the disturbance. It is then enough to examine the behavior of χ as a function of g_0, the scaled centerline heat flux potential. Figure 4 shows this characteristic. The curve delineates an open region in the χ-g_0 space where subcritical instabilities *may* be possible. Since the results correspond to minimizing solutions, the area enclosed by the curve and the χ axis corresponds to the region of unconditional stability. Arc instabilities are not possible here. It is interesting to note that the size of the disturbance and the dimensionless centerline heat flux potential bear an inverse relation in the region of absolute stability, although an explicit relationship cannot be written. Note that small g_0 corresponds to small currents in Fig. 3. From Fig. 4 we see that with an increasing g_0, the size of the disturbance that can be accommodated by a stable arc correspondingly decreases. Definitive conclusions, however, about the existence of subcritical instabilites in the open region between the nonlinear and linear characteristics cannot be drawn in the absence of any experimental results. It is seen that the "nonlinear" result approaches the linear critical value as the disturbance size becomes smaller, approaching the "global" value $g_0 = 0.594$.

VII. CONCLUDING REMARKS

It is seen from a linear theory study that a variable property, slender arc is unconditionally unstable with respect to infinitesimal size disturbances when the dimensionless arc centerline heat flux potential exceeds a given numerical constant, viz., 0.584. Based on this result, however, no conclusion can be drawn regarding the existence of regions of arc stability. The question then arises, under what conditions can we expect an arc to be stable, and in particular, with respect to finite size disturbances. We are therefore led to the following queries: Do there exist combinations of finite size disturbances and centerline heat flux potential values that correspond to arc stability, and if there are, can we chart the domain of their existence on the stability diagram. It appears that these queries can be answered by investigating the complementary problem that provides estimates for the necessary conditions for instability. Clearly, the criterion stemming from this latter analysis describes a region where the arc is always stable, From the results presented in here, we see that with increasing values of the centerline heat flux potential, the size of the disturbance that can be accomodated by a stable arc correspondingly decreases. With the size of the disturbance becoming infinitesimally small, the value of g_0 predicted by the linear theory is properly recovered. Thus the two results complement each other. One must, however, be cautious in interpreting this minimizing solution result. Although the result is rigorously valid, based on this result no definitive conclusions can be drawn as to the existence of subcritical instabilites.

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