

New Product Diffusion with Two Interacting Segments or Products

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Abstract

We study the diffusion of a product in two customer segments where the acceptance level in one segment affects the diffusion rate not only in that same segment, but also in the other. The inter-segment influence can be positive or negative, i.e., the acceptance level of the product in one segment can reinforce or impede its diffusion in the other. The model set-up also applies to the diffusion of two products, with independent market potential, in a single population. Since the diffusion system we study does not have a closed-form solution, we use phase plane analysis to identify the equilibrium points of the joint diffusion process and to characterize their stability properties. Further, we provide a means to identify the regions with different convergence behavior, i.e., to identify boundaries for regions within which all trajectories converge to a particular equilibrium point. For the cases of asymmetric influence (+/-) and mutually impeding influence (-/-), we also provide conditions under which both products can achieve full market potential in equilibrium. Finally, we provide managerial insights into the effectiveness of two strategies in the context of asymmetric (+/-) interaction between two customer segments: (1) “seeding,” i.e., using free samples to support the launch of a product in one segment being harmed by the adoption in the other, and (2) “demand control,” i.e., purposely limiting market potential for the customer segment harming product diffusion in the other segment.

Key words: diffusion of innovations; innovation; marketing strategy; new product research; social contagion; word-of-mouth.

1. Introduction

Product diffusion with a group of customers rarely occurs in isolation and is often influenced by the presence of other products or customer groups in the market. Multiproduct interactions go well beyond the standard argument that innovation diffusion is a substitution process from an old to a new technology, product or practice, and include a vast set of competitive, complementary, and asymmetric interactions among jointly diffusing innovations (Bayus, Kim and Shocker 2000). Examples of the joint diffusion of *competing* products are rife: gas vs. electric stove in kitchen appliances, desktop vs. laptop PCs in the computer industry, DSL vs. cable broadband in the telecommunications industry, and so on. Of particular interest, in the last ten years or so, have been cases where both competing products benefit from installed base effects or direct network effects, resulting in “get big first” races. More recently, indirect network effects such as those in the diffusion of complementary hardware and software have received considerable attention (e.g., Gupta, Jain and Sawhney 1999; Parker and Van Alstyne 2005; Stremersch et al. 2007). Examples of the joint diffusion of *complementary* products include that of microwave ovens and microwave-ready meals, that of clothes washers and dryers (Peterson and Mahajan 1978), that of legal and pirated software (Givon, Mahajan and Muller 1995; Nascimento and Vanhonacker 1988), and that of retail scanners and UPC barcodes (Bucklin and Sengupta 1993). Multiproduct interactions are not limited to only mutually impeding (-/-), as with substitutes, or mutually facilitating or symbiotic (+/+), as with complements. Various types of *asymmetric* interactions are possible as well, including “predator-prey” (+/-) and facilitating or “commensalistic” (+/0) interactions (Bauer and Castillo-Chávez 2001; Bayus et al. 2000).

Interactions among diffusion processes are of interest not only in settings involving multiple products diffusing in one and the same population, but also in settings involving a single product diffusing in multiple sets of customer that interact with each other. The latter may involve the diffusion across multiple countries, states, or other geographical areas (e.g., Kumar and Krishnan 2002; Putsis et al. 1997) or the diffusion of a single product in multiple, interacting market segments (e.g., Berger and Heath 2007; Lehmann and Esteban-Bravo 2006; Van den Bulte and Joshi 2007). This last type of interacting diffusion processes has received considerable research attention lately, as it pertains to phenomena of great theoretical and managerial interest. These include not only opinion leader-follower dynamics and competition for status (e.g., Van den Bulte and Joshi 2007; Watts and Dodds 2007), but also social identity dynamics between segments that present important challenges to the development of brands with strong iconic content, including Burberry, Diesel Jeans, Porsche, Red Bull, Vans, and Tommy Hilfiger (e.g., Grigorian and Chandon 2004; Kumar, Linguri and Tavassoli 2005; Moon and Kiron 2002; Moon et al. 2003). Managing the growth of such brands has proven particularly tricky as it features asymmetric “predator-prey” (+/-) influences among segments. These brands first become popular among one set of

customers. That success in the original core segment subsequently spills over to other customers who use the latter as reference or aspiration group. The success among such “wannabes,” however, detracts from the appeal of the brand among the “originals” and may ultimately lead all of them to drop the brand and chose another, less “overexposed” and still “authentic,” iconic brand to signify their identity (e.g., Berger and Heath 2008; Clunas 2004; Thornton 1996). As aspirants imitate originals who seek to protect their distinctiveness, the latter disadopt the products, brands, and cultural practices they helped make popular in the first place (Bourdieu 1984; Simmel 1971). Mutually impeding (-/-) influences have also long been documented among sub-cultures, the classic example being the rivalry between “mods” and “rockers” affecting the diffusion of the Vespa and other scooters in Great Britain in the 1960s (Hebdige 1979; 1988). When new products and brands diffuse in markets characterized by the existence of multiple sub-cultures (e.g., “geeks,” “jocks” and “Goths” in US high schools) and these products or brands get picked as social signifiers, then their market acceptance becomes very hard to predict.

Given their importance in the success and downfall of iconic brands and identity-related products and their importance in cultural dynamics more generally, negative cross-segment effects have become a topic of considerable interest to consumer researchers interested in social identity issues (Berger and Heath 2007, 2008; Berger and Rand 2008; Escalas and Bettman 2005; White and Dahl 2006, 2007). The issue has also become the subject of analytic modeling in marketing. Amaldoss and Jain (2005a, 2005b) examine how desires for uniqueness and conformism affect firm strategy. Their one-period game-theoretic framework, however, does not examine diffusion trajectories, and considers consumers who wish to be different from all other consumers or wish to be similar to all other consumers rather than group-specific attraction and repulsion. In more recent work, Amaldoss and Jain (2008) explicitly study asymmetric (+/-) effects between reference groups, but focus on identifying optimal price and product strategy in a two-period game-theoretic setting rather than on characterizing diffusion trajectories over time. Joshi, Reibstein and Zhang (2006) analyze the optimal timing of entry in two segments with asymmetric (+/-) contagion effects between them. They study the issue both in a two-period and in a continuous-time set-up, and can determine the optimal entry time for the latter only numerically. Hence, while the question how negative cross-segment interaction affects new product acceptance has emerged as a topic of great interest to both managers and researchers, recent research offers only limited analytic modeling insights about diffusion trajectories in settings with negative cross-segment effects.

Theoretical progress in the area of new product diffusion featuring multiproduct or multisegment interactions has been hampered by the difficulty to develop closed-form solutions. Modeling efforts have eschewed formal analytics in favor of numerical analysis (e.g., Lehmann and Esteban-Bravo 2006), addressed only one-way influence between influentials and followers (Van den Bulte and Joshi 2007), or have had to make other restrictive assumptions on the nature of the inter-product or inter-market

interactions in order to derive closed-form solutions (Kumar and Krishnan 2002; Peterson and Mahajan 1978). Fortunately, managerially useful analytical understanding of a multiproduct or multisegment diffusion process can be gained even when closed-form solutions for the full diffusion curve do not exist.

Consider the example of a new prestige product like the iPhone diffusing across two interacting population segments, the “originals” and the “wannabes,” who exert an asymmetric influence on each other, as discussed earlier. The marketer would be interested in determining the possible equilibrium outcomes of the diffusion process based on current installed base and the nature of inter-segment interaction. For instance, will the “originals” stay with the product or will they start disadopting once the product becomes popular among wannabees? In the latter case, how many of the originals will be using the product in equilibrium? Is it possible to end up in a situation where none of the originals use the product yet all wannabes do? To detect possible problems early on, the marketer would also like to know which levels of acceptance in both segments are likely to lead to complete acceptance by both segments and which levels pose the risk of evolving into a situation where the full market potential is not reached. Other issues of interest include identifying whether and how strategies such as providing free samples to jumpstart the diffusion in the impeded segment of originals or limiting the market potential in the impeding segment of wannabes can influence the equilibrium outcomes and overall profits of the firm. So, even when closed-form solutions to the co-diffusion process cannot be derived, an analytical characterization of the possible equilibrium outcomes and their stability properties can be quite useful if they allow a manager to determine whether or not some type of intervention is required. Once this is known, optimal policies such as the level of sampling can be determined numerically.

We study the diffusion system of two interacting products or segments; and apply phase plane analysis to a model specification that was proposed by Peterson and Mahajan (1978), Nascimento and Vanhonacker (1993), Bucklin and Sengupta (1993) and Geroski (2000), and that nests the model specification studied recently by Van den Bulte and Joshi (2007). In the model we work with, each product or segment has its own independent market potential, and the user base of a product or segment affects the diffusion rate of not only itself, but also the other product or segment. The latter effect may be either positive or negative. Specifically, the model consists of two interlinked differential equations:

$$\frac{dN_1}{dt} = (a_1 + b_1 N_1 + c_1 N_2)(m_1 - N_1) \quad (1a)$$

$$\frac{dN_2}{dt} = (a_2 + b_2 N_2 + c_2 N_1)(m_2 - N_2), \quad (1b)$$

where the subscript i refers to one of two segments or products, and $a_i > 0$, $b_i > 0$, $m_i > 0$, $0 \leq N_i \leq m_i$.

Table 1 summarizes the notation, using the terminology for two segments rather than two products.

Table 1. Glossary of Terms

N_i	number of users in segment i at time t (shorthand for $N_i(t)$)
m_i	market potential for segment i
a_i	coefficient of innovation for segment i
b_i	coefficient of imitation or contagion for segment i
c_i	coefficient of inter-segment interaction (contagion effect from segment j on segment i)

The c parameters capture the interaction between the two segments and can be positive, zero, or negative. We assume that the rate $(a_i + b_i N_i + c_i N_j)$ can be negative, i.e., that the product can be disadopted. However, the constraints $N_1 \geq 0$ and $N_2 \geq 0$ ensure that disadoption cannot reduce the number of current users to less than zero. Allowing for disadoption is a feature that is absent from most diffusion models but that is important to appropriately representing the full consequences of negative cross-segment effects (Berger and Heath 2008). The system of equations does not explicitly model the impact of disadoption in a segment on the rate of diffusion in the same segment beyond the effect of having a lower user base. This is because, in our setting, the reason for disadoption in segment i is the negative influence exerted by segment j (when c_i is strictly negative) rather than any negative word of mouth within segment i . Our model captures the antecedent detrimental cross-segment effect but ignores any additional mediating effect of negative within-segment word-of-mouth. Since this system of simultaneous differential equations does not have a closed-form solution when both cross-effects are at work (Peterson and Mahajan 1978), we investigate the equilibrium values and trajectories of N_1 and N_2 by *phase plane analysis*.

Our research makes three contributions. First, we analytically determine equilibrium points of the joint diffusion process and characterize their stability properties. We find that irrespective of the nature (positive or negative) of inter-segment interaction in a two-segment diffusion model, the stable equilibrium outcomes belong to the set $\{(0, m_2), (m_1, 0), (m_1, m_2)\}$, where m_1 and m_2 are the market potential for segments 1 and 2 respectively. Candidate equilibrium outcomes with intermediate values of market penetration do arise, but are always found to be unstable.

Second, we show how to clearly demarcate regions within which all trajectories converge to a particular equilibrium point, which in turn can be used to identify the need for early managerial interventions. The boundaries of these regions are referred to as *separatrices*. We also point the readers to a computational approach to plot the separatrices. An important implication of identifying the separatrix

is that, in the region in the vicinity of the separatrix, small shifts in the installed base can lead to dramatically different equilibrium outcomes of the diffusion process.

Finally, we illustrate the insight-generating potential of this methodology by assessing the effectiveness of two strategies in the context of asymmetric (+/-) interaction between customer segments: (1) “seeding,” i.e., using free samples to support the launch of a product in a customer segment that is being harmed by product adoption in another customer segment, and (2) “demand control,” i.e., purposely limiting product distribution in the customer segment that is harming the diffusion of the product in the other segment. The analyses reveal several counter-intuitive results and demonstrate that ignoring joint diffusion in environments with strong inter-segment interactions can generate incorrect policy recommendations. For example, while prior research has revealed that the optimal sampling level is non-decreasing with the coefficient of imitation, we find that the optimal sampling level is decreasing in the coefficient of imitation when there is a strong impeding influence from the other segment. The rest of the paper is organized as follows. Section 2 contains a description of our research strategy which is mainly centered on the use of phase plane analysis. Section 3 deals with the Asymmetric Influence (+/-) Model, and includes an illustrative discussion of managerial implications in this setting which is of particular interest to marketers of products with strong status or social identity value. Section 4 describes the Mutually Impeding (-/-) Model; and Section 5 deals with the Symbiotic Influence (+/+) Model. Finally, Section 6 discusses our main results and their implications.

2. Research Strategy

Since the model does not have a closed-form solution when both cross-effects are at work, we study its behavior by identifying the possible equilibrium outcomes and trajectories of N_1 and N_2 . We do so using *phase plane analysis*. The key tool in such analysis is the *phase diagram*, which is the path of diffusion process in the (t, N_1, N_2) space projected onto the N_1 - N_2 plane. In other words, it plots $N_1(t)$ versus $N_2(t)$ and is what one would see if one stood high on the time axis and looked down upon on the N_1 - N_2 plane, sometimes referred to as the *phase plane* (Hubbard and West 1995). Such phase diagrams provide a convenient visual tool for studying the properties of a dynamical system. We adopt the convention (without loss of generality) that N_1 is measured along the horizontal axis and N_2 on the vertical axis of the phase diagram. Other papers in marketing that have used phase diagrams include Heiman et al. (2001) and Muller (1983).

We illustrate the technique of phase plane analysis by applying it to the system of equations described in (1a) and (1b), subject to the conditions $c_1 < 0$, $c_2 \geq 0$. The properties of the same system are analyzed and interpreted more comprehensively in Section 3.1. There are five main steps in the phase plane analysis (Hubbard and West 1995), as required for our purposes:

1. Identify the isoclines, i.e., the curves in the phase plane representing points at which

$$\frac{dN_1}{dt} = N_1' = 0 \text{ or } \frac{dN_2}{dt} = N_2' = 0. \text{ The isoclines for } N_1 \text{ are identified by the condition}$$

$N_1' = 0$. Substituting this into equations (1a) and (1b), we get

$$N_1 = m_1 \tag{2}$$

and

$$(a_1 + b_1 N_1 + c_1 N_2) = 0 \tag{3}$$

Since $c_1 < 0$, equation (3) can be rewritten as

$$N_2 = (a_1 + b_1 N_1) / |c_1| \tag{4}$$

The isoclines for N_2 are identified by the condition $N_2' = 0$. Since $c_2 \geq 0$, the only isocline satisfying the required condition is $N_2 = m_2$.

2. Sketch the trajectories to visually illustrate the behavior of the dynamical system. Any point on the isoclines represented by equations (2) and (4) cannot undergo a change in N_1 (since $N_1' = 0$). Thus, from any point on those isoclines, the $N_1 N_2$ trajectory must either remain at that point (if $N_2' = 0$), move up (if $N_2' > 0$), or move down (if $N_2' < 0$) in the phase plane. Further, since $c_2 \geq 0$, it follows that $N_2' > 0$, for all $N_2 < m_2$. Thus, from any point on the isoclines represented by (2) and (4), the $N_1 N_2$ trajectory must move upwards as long as $N_2 < m_2$, as shown in Figure 1. Points on the isocline $N_2' = 0$ can only move to the right or to the left, depending on whether $N_1' > 0$ or $N_1' < 0$ respectively.
3. Identify the singular points, i.e., the points where the isoclines $N_1' = 0$ and $N_2' = 0$ cross each other. Such points are of special interest as they may, but need not, be stable equilibrium outcomes of the diffusion process. The two singular points in our system are (m_1, m_2)

and $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2 \right)$. The latter can lie within the feasible region, or outside. For the

purposes of this illustrative discussion, we focus our attention on the case in which the point

lies within the feasible region ($0 < \frac{(m_2 |c_1| - a_1)}{b_1} < m_1$). In Section 3.1 we consider both

cases. The two singular points are represented as points A and B respectively in Figure 1.

Even though the boundary points $(0, m_2)$ and $(m_1, 0)$ are not singular points as they do not lie at

the intersection of isoclines, they can nevertheless be candidate equilibrium outcomes due to the impact of the constraints $N_1 \leq m_1$ and $N_2 \leq m_2$.

4. Formally characterize the stability properties of the relevant singular and boundary points.

One achieves this by slightly perturbing the system in the neighborhood of the point of interest, and then checking the sign of the derivatives N_1' and N_2' to determine if the system returns to the same point (stable), or not (unstable). For the cases dealt with in this paper, a singular point is either a stable *sink* or an unstable *saddle point*. Sinks are stable singular points into which infinitely many trajectories converge. A sink represents a stable equilibrium because a small perturbation will cause the system to return to the same equilibrium. Saddle points, in contrast, are unstable singular points into which precisely two trajectories will converge. A saddle point represents an unstable equilibrium because a small perturbation can change the resulting equilibrium. In Section 3.1 we show rigorously that (m_1, m_2) is stable

while $\left(\frac{(m_2 | c_1 | - a_1)}{b_1}, m_2 \right)$ is unstable. The boundary point $(0, m_2)$ is a stable equilibrium

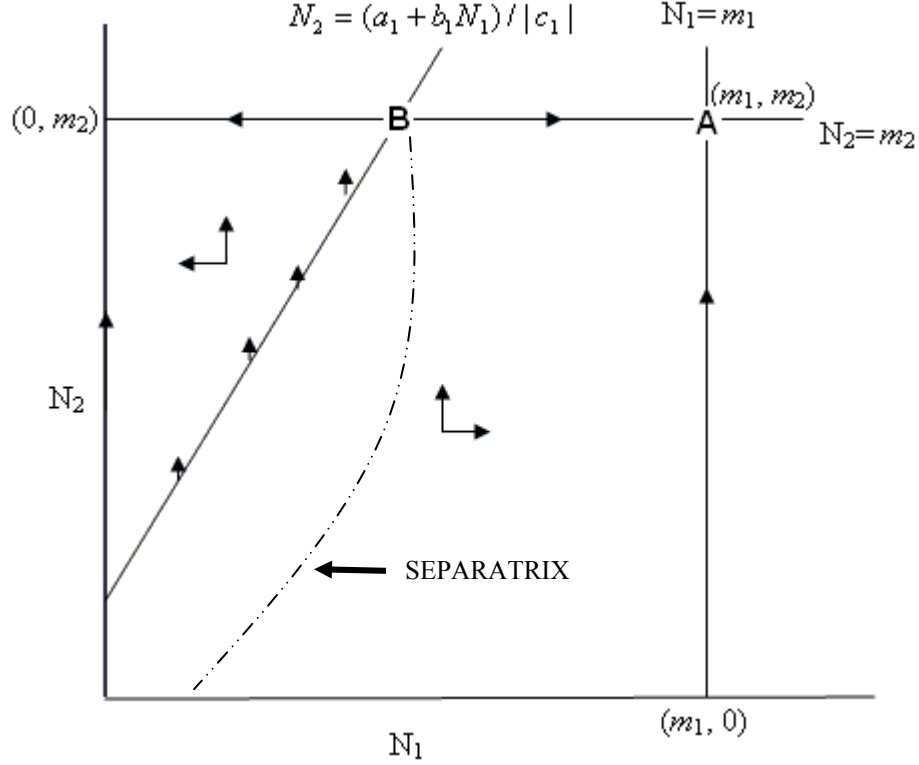
outcome also.

5. Finally, the phase plane is segmented into regions within which all trajectories converge to the same stable equilibrium outcome. We provide an algorithm to plot the boundary between any two such adjacent regions (also known as the *separatrix*) in Appendix C. The following is a heuristic explanation for the observed convergence behavior. Consider the

isocline $N_2 = (a_1 + b_1 N_1) / |c_1|$. Points on this line have $N_1' = 0$. We know that $N_2' > 0$ for all $N_2 < m_2$. Thus, from points on this line, the $N_1 N_2$ curve will move vertically upwards as shown by the arrows in Figure 1. Points to the left of this isocline have $N_1' < 0$ and $N_2' > 0$ (see step 2) and thus the $N_1 N_2$ curve passing through points in this region will move left and upwards as shown by the arrows. Thus, if we start at any point in this region, then N_1 will keep decreasing and N_2 increasing until we reach the boundary $(0, m_2)$. Thus, all trajectories passing through points to the left of the isocline $N_2 = (a_1 + b_1 N_1) / |c_1|$ or on the isocline itself, will eventually converge to $(0, m_2)$. For points to the right of the isocline we have $N_1' > 0$ and $N_2' > 0$. Thus, the trajectory will move right and upwards as shown by the arrows. Points that are close to point A will converge to (m_1, m_2) . However, that cannot be said of all points to the right of the isocline because a trajectory can potentially

cross the isocline. The curve which separates the trajectories into two regions, depending on the final equilibrium outcome, is the separatrix (see Figure 1).

Figure 1. Phase Diagram



We now introduce some terminology which is helpful in classifying the different types of inter-segment interaction that we will be studying in this paper. We define the concept of a *limiting hazard rate* (LHR_i) for product i , where $i, j \in \{1, 2\}$ and $i \neq j$:

$$LHR_i = \lim_{\substack{N_i \rightarrow m_i \\ N_j \rightarrow m_j}} \frac{1}{(m_i - N_i)} \frac{dN_i}{dt} = \lim_{\substack{N_i \rightarrow m_i \\ N_j \rightarrow m_j}} (a_i + b_i N_i + c_i N_j)$$

LHR_i is the hazard rate of the diffusion process at the upper extreme of the feasible region, i.e., $(N_i, N_j) \rightarrow (m_i, m_j)$. In the limit $(N_i, N_j) \rightarrow (m_i, m_j)$, a negative value for LHR_i is indicative of a very strong negative influence of segment j on segment i , while a positive LHR_i is indicative of a mild negative influence of segment j . In all future reference to LHR_i , it is implied that the term is defined in the limit.

Table 2 provides definitions of the main analytical concepts used in this paper.

Table 2. Definitions of Technical Terms

Technical Term	Definition
Phase Diagram	For a mathematical system which describes how two variables $N_1(t)$ and $N_2(t)$ evolve with time, the phase diagram is the solution in the (t, N_1, N_2) space projected onto the N_1 - N_2 plane. The plot is what one would see if one stood high on the time axis and looked down upon on the N_1 - N_2 plane.
Isoclines	For a mathematical systems evolving with time, isoclines are curves representing points at which $\frac{dN_i}{dt} = 0$, or the time rate of change of variable N_i is 0.
Singular Points	Points at which the time rate of change of all variables in a mathematical system is equal to 0.
Sinks	Stable singular points into which infinitely many trajectories converge. A sink represents a stable equilibrium because a small perturbation will once again result in the same equilibrium outcome.
Saddle Points	Unstable singular points into which precisely two trajectories converge. A saddle point represents an unstable equilibrium because a small perturbation can change the resulting equilibrium.
Separatrices	Boundaries for regions in the phase plane that display different convergence behavior, i.e., regions within which all trajectories converge to a particular singular point.
Limiting Hazard Rate	For the system of differential equations in R^2 , as described in (1a) & (1b), the limiting hazard rate is the rate of decrease (percentage) in the population yet to adopt product i , at the upper extreme of the feasible region, i.e., $(N_i, N_j) \rightarrow (m_i, m_j)$.

In the next three sections, we present phase plane analyses of the following three cases: Asymmetric influence ($c_1 < 0, c_2 \geq 0$), Mutually impeding influence ($c_1 < 0, c_2 < 0$), and Symbiotic influence ($c_1 \geq 0, c_2 \geq 0$). A special case of symbiotic (one-way) influence ($c_1 = 0, c_2 > 0$) for which closed-form solutions do exist, has been analyzed recently by Van den Bulte and Joshi (2007).

3. Asymmetric Influence Model (+ / -)

We first consider an asymmetric influence model in which the success in segment 1 fosters the diffusion in segment 2, but success in segment 2 adversely affects the diffusion in segment 1 ($c_1 < 0, c_2 \geq 0$). The case with $c_1 \geq 0, c_2 < 0$ is analogous in its treatment. These cases correspond to the situation faced by many brands with high status or reference group appeal, like Burberry, Red Bull

and Vans. Another instance, in which product form and segment status are intertwined, is that of the interactions between Porsche sports cars bought by driving enthusiasts and Porsche SUVs bought by more mainstream drivers, including soccer moms. While the SUV sales in all likelihood benefited from the success of the Porsche 911 and 959 as high performance cars, the latter's drivers were frustrated by seeing soccer moms on the road in Porsche Cayenne SUVs (Joshi, Reibstein and Zhang 2007). We analyze the various equilibrium outcomes that might be attained under such cross-segment dynamics, and identify conditions under which both segments can achieve their full market potential.

The two singular points for our system, identified by determining the points of intersection of the isoclines (as represented by (2), (4), and $N_2 = m_2$), are (m_1, m_2) and $\left(\frac{(m_2 | c_1 | -a_1)}{b_1}, m_2\right)$. When LHR_I is positive (i.e., $a_1 + b_1 m_1 + c_1 m_2 > 0$, which corresponds to mild negative influence of segment 2 on segment 1) then $\frac{(m_2 | c_1 | -a_1)}{b_1} < m_1$ and thus the singular point $\left(\frac{(m_2 | c_1 | -a_1)}{b_1}, m_2\right)$ lies within the feasible region. Conversely, when LHR_I is negative (which corresponds to strong negative influence of segment 2 on segment 1), the point lies outside the feasible region. Accordingly, we consider two cases below.

3.1. Mild Negative Influence (Positive LHR_I)

When $0 < \frac{(m_2 | c_1 | -a_1)}{b_1} < m_1$ (i.e., mild negative influence of segment 2 on 1, or positive LHR_I), there are two singular points within the feasible region. These are labeled A and B in Figure 1. In addition, $(0, m_2)$ is a boundary point of interest.

As described in Section 2, (m_1, m_2) is a sink or a stable equilibrium and the other singular point $\left(\frac{(m_2 | c_1 | -a_1)}{b_1}, m_2\right)$ is a saddle point. Therefore, the system has only two long-term stable outcomes, namely $(0, m_2)$ and (m_1, m_2) . This result is formally stated in Proposition 1.

Proposition 1: For the Asymmetric Influence model with a positive value for LHR_I , the only stable equilibrium outcomes possible are $(0, m_2)$ and (m_1, m_2) .

Proof: See Appendix B.

Proposition 1 states that all trajectories in the phase plane converge to either $(0, m_2)$ or (m_1, m_2) . Given the two equilibrium outcomes, it is additionally possible to separate the phase plane into two

regions, one within which all trajectories converge to $(0, m_2)$ and another within which all trajectories converge to (m_1, m_2) .

A powerful result to this end is that the trajectories that converge to the saddle point, the so-called *separatrices*, separate the phase plane into regions demonstrating different convergence behavior (Hubbard and West 1995). These separatrices can be computed numerically for any system as described in detail in Appendix C. A spreadsheet implementation of the algorithm for computation of the Separatrix for the scenario described in this section is also available..

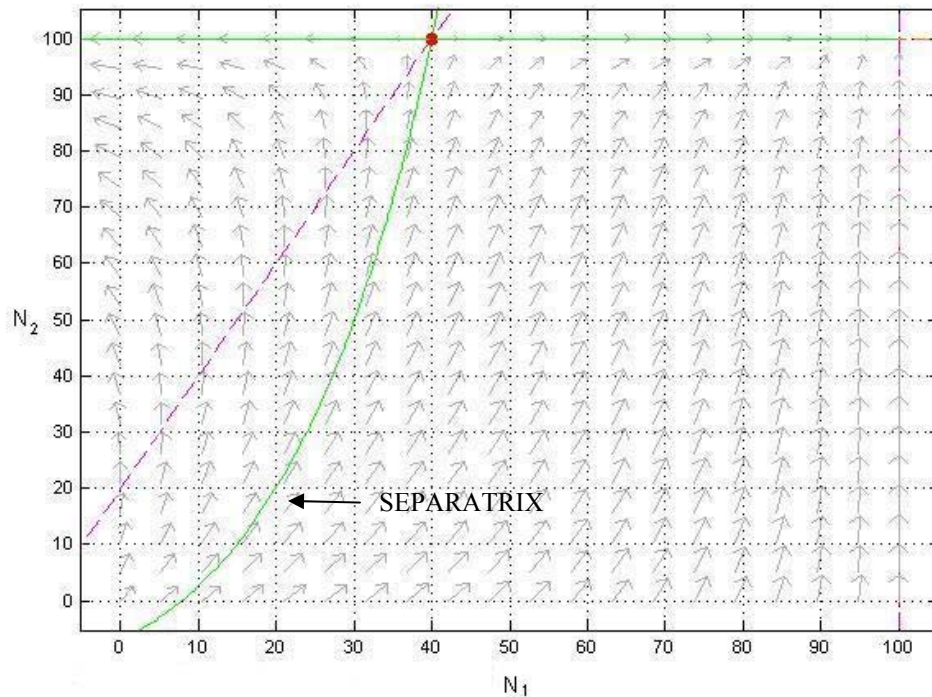
Figure 2 shows a plot of the phase diagram with the separatrix and several sample trajectories for the following system:

$$\begin{aligned} \frac{dN_1}{dt} &= (0.02 + 0.002N_1 - 0.001N_2)(100 - N_1) \\ \frac{dN_2}{dt} &= (0.03 + 0.002N_2 + 0.001N_1)(100 - N_2) \end{aligned} \quad (5)$$

$$0 \leq N_1 \leq 100$$

$$0 \leq N_2 \leq 100$$

Figure 2. Phase Diagram with Separatrix for the diffusion system in (5)



The order of magnitude of the parameters in (5) is consistent with prior research (e.g., Bucklin and Sengupta 1993; Joshi et al. 2006; Van den Bulte and Stremersch 2004).

The trajectories to the left of the separatrix lie in a region where the relative size of the installed base in segment 2 to that in segment 1 is such that the negative influence exerted by segment 2 is dominant in segment 1's diffusion rate. As a result, the users in segment 1 will eventually disadopt, leading to a final equilibrium outcome $(0, m_2)$.

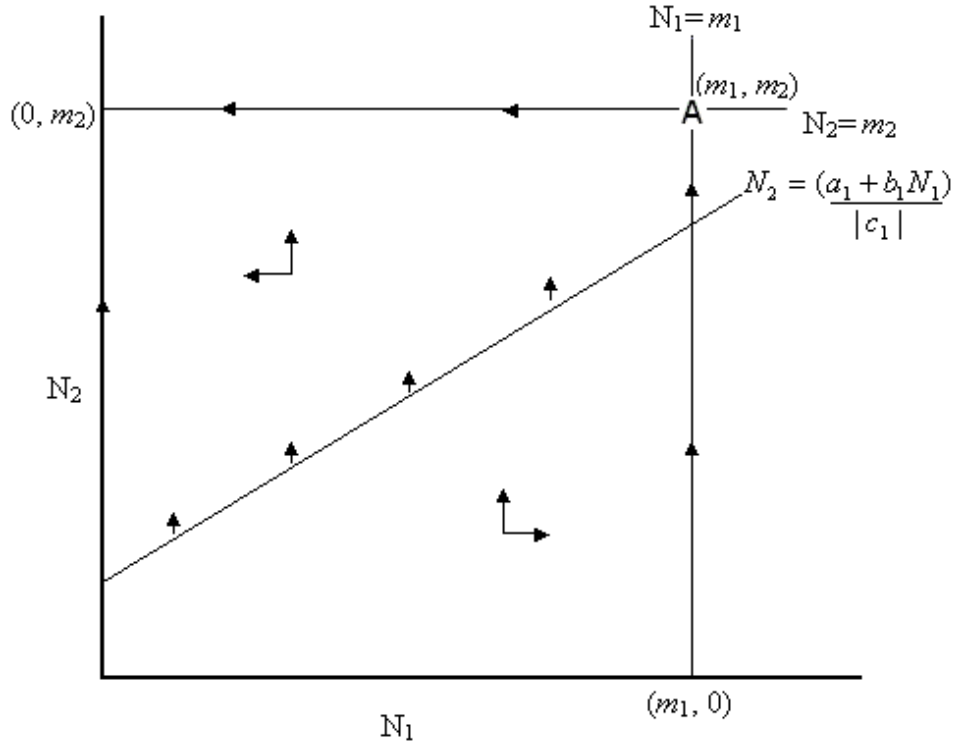
The identification of the stable equilibrium outcomes and the separatrix can together answer several questions. For example, starting from $(0,0)$, can the system converge to (m_1, m_2) ? For the system in (5), this is clearly impossible. However, distributing free samples of the product in segment 1 can help jump-start the diffusion from an initial point to the right of the separatrix. How many free samples are needed to change the long-term equilibrium? This can also be determined for a system with known diffusion parameters. We discuss some of these managerial implications in greater detail in Section 3.3. However, before doing so, we first proceed to analyze the second case for the Asymmetric Influence model, where $\frac{(m_2 |c_1|-a_1)}{b_1} > m_1$. (The degenerate case in which $\frac{(m_2 |c_1|-a_1)}{b_1} = m_1$ leads to a line of unstable equilibria.)

3.2 Strong Negative Influence (Negative LHR_I)

When $\frac{(m_2 |c_1|-a_1)}{b_1} > m_1$ (i.e., strong negative influence of segment 2 on 1, or negative LHR_I), the singular point $\left(\frac{(m_2 |c_1|-a_1)}{b_1}, m_2\right)$ is outside the feasible region and thus, is not of interest. The only singular point in the feasible region is (m_1, m_2) . A small perturbation of the system around the singular point (m_1, m_2) will result in a negative value for dN_1/dt (by virtue of negative LHR_I), and thus the trajectory will not return to (m_1, m_2) . This suggests that the singular point (m_1, m_2) may not be a stable equilibrium point.

The phase diagram for this case can be constructed following the steps described in Section 2, and is shown in Figure 3. Since $N_2' > 0$ for all $N_2 < m_2$, a trajectory starting from any point in the feasible region will eventually be above the isocline $N_2 = (a_1 + b_1 N_1) / |c_1|$. Once the trajectory is above the isocline, we have $N_1' < 0$. Thus, N_1 will decrease until it reaches zero. The only possible long-term outcome is $(0, m_2)$. In the long-term, segment 1 consumers will disadopt completely. This is stated formally in our next result.

Figure 3: Phase diagram for Strong Negative Influence (Asymmetric Interaction)



Proposition 2: For the Asymmetric Influence model with a negative value for LHR_1 , the only stable equilibrium outcome possible is $(0, m_2)$.

Proof: See Appendix B.

Using the technique described above, a manager can determine early on, whether or not the product will diffuse to its market potential in both consumer segments. If the diffusion parameters result in a phase diagram as in Figure 3, the full-diffusion outcome (m_1, m_2) will never materialize. If, in contrast, the situation is as in Figure 2, full diffusion is possible but not guaranteed. For example, a process that starts at $(10,0)$ will achieve full diffusion but one that starts at $(0,0)$ will not. So, phase plane analysis can determine whether or not some intervention, such as seeding the process using free samples, will be necessary to achieve full diffusion. Phase plane analysis may also avoid managers becoming lulled in a false sense of complacency. Consider the diffusion path of the product as captured in Figure 3. For a trajectory in the region below the isocline (where $N_1' > 0$), diffusion in segment 1 may seem to be proceeding quite smoothly early on, but will reverse once adoption in segment 2 is sufficiently high. Thus, initial diffusion data can mislead managers into believing that the product adoption will continue in a smooth manner. Under such circumstances, managers and analysts may need to explore other options, such as reducing the impeding contagion effect (c_1) from segment 2 by launching different product

variants or brands in the two segments or restricting the potential market size of segment 2. We investigate and assess some managerial interventions next.

3.3. Managerial Implications for the Asymmetric Influence case

In this section we illustrate the ability of phase plane analysis to generate managerial insights, by studying the following two strategies: (1) “seeding,” i.e., using free samples to support the launch of a product in a consumer segment being harmed by the adoption in another consumer segment, and (2) “demand control,” i.e., purposely limiting market potential for the customer segment harming product diffusion in the other segment.

3.3.1. Optimal Seeding

The phase diagram in Figure 1 shows that a diffusion process starting with zero initial penetration in both segments will naturally evolve to an equilibrium in which segment 2 reaches full penetration but segment 1 has no adopters. However, if the process were to start to the right side of the separatrix, full penetration would be achieved in both segments. One way to achieve this outcome is for the marketer to seed segment 1 with enough samples at launch.

Product sampling is usually recommended when product benefits cannot be fully conveyed by advertising, the product has new features that need to be appreciated to overcome adoption risks, or when WOM effects play a critical role in product diffusion. Jain, Mahajan and Muller (1995) have investigated the optimal sampling level to offer in a single product, single segment setting and found that optimal sampling levels are high for products with a low coefficient of innovation, or a high coefficient of imitation. Lehman and Esteban-Bravo (2006) investigated a setting with one-way (+/0) influence, and found that the optimal sampling level in the affected segment decreases as its coefficient of innovation increases but first decreases and then increases as its coefficient of imitation increases. We extend these analyses by investigating the problem of optimal sampling for a two-segment diffusion model with asymmetric negative (-/0) interaction between the segments.

The firm chooses the optimal number of free samples for consumers in segment 1 (whose diffusion is impeded by adoption in segment 2), in order to maximize its total discounted profit. The decision problem is:

$$\max_{N_1^S} \pi = \max_{N_1^S} \sum_{t=1}^{\infty} \delta^{t-1} (p_1 - u_1) (dN_1 / dt) dt + \sum_{t=1}^{\infty} \delta^{t-1} (p_2 - u_2) (dN_2 / dt) dt - (s_1 N_1^S) \quad (6)$$

where N_1^S is the number of free samples of product offered in segment 1, p_i is the price charged to segment i , u_i is the unit cost of manufacturing the product, δ is a discount factor, and s_j is the unit cost of

offering a free sample of the product to segment 1. Note that the above formulation is the same as in Jain, Mahajan and Muller (1995), with the exception that there are two consumer segments in our formulation. We investigate the solution numerically using parameters within the same range as those used in Jain, Mahajan and Muller (1995). We assume that $c_2 = 0$, in order to focus on the impact of c_1 . The parameter settings are summarized in Table 2. The discount factor is $\delta = 1/1.08 = 0.926$, and $s_i = u_i$. Even though u_i is set to 0 (for convenience, and without loss of generality since p_i can be interpreted as profit margin instead of price) in our numerical computations, sampling is still expensive due to the lost revenue of the sampled product. In order to evaluate the value of dN_1 / dt and dN_2 / dt we use the Euler approximation with step size 1.¹

Table 2: Parameter Settings

Parameter	Segment 1	Segment 2
Coefficient of Innovation (a)	0.005:0.03	0.03
Coefficient of Imitation (b)	0.002	0.0018
Coefficient of Inter-product Interaction (c)	0:-0.002	0
Market Potential (m)	100	100
Price (p)	1	1
Unit Cost (u)	0	0

¹ Even though more sophisticated numerical techniques are available, they do not change the qualitative nature of our results. Some representative calculations in the parameter range of interest show that a 90% reduction in step size lead to less than 1.75% change in undiscounted revenue.

Table 3: Optimal Sampling levels

Parameters		c_1				
		0	-0.0005	-0.001	-0.0015	-0.002
a_1	0.005	6	11	16	26	0
	0.006	6	10	16	25	0
	0.007	5	10	15	24	0
	0.00766	5	9	15	24	0
	0.00767	5	9	15	24	21
	0.008	5	9	15	24	21
	0.009	4	9	14	23	21
	0.01	4	8	14	22	21
	0.015	1	6	11	19	19
	0.02	0	3	8	16	17
	0.025	0	1	6	13	15
	0.03	0	0	3	10	14

Table 3 provides the results for the optimal sampling level for various values of a_1 and c_1 . For any coefficient of innovation (a_1), we observe that the optimal number of samples initially increases as c_1 becomes more negative. This is because more samples are needed in order for the diffusion trajectory to cross the separatrix in Figure 2. However, if the impeding influence of segment 2 is very strong and the coefficient of innovation for segment 1 is very low, sampling becomes prohibitively expensive and the optimal number of samples can drop to zero (for example, see top row of Table 3).

For $c_1 = 0$, we find that the optimal sampling levels decrease with the coefficient of innovation, as reported in Jain, Mahajan and Muller (1995). However, for highly negative values of c_1 shown in the far right column of Table 3, optimal sampling levels may initially be zero, when the coefficient of innovation is low, then jump quite markedly to a high level beyond a threshold value for the coefficient of innovation, and finally decline again as the coefficient of innovation increases further. This occurs because, initially (for low values of a_1 , and highly negative values of c_1), a large number of free samples may be needed in order to change the equilibrium outcomes. As a result, it may be prohibitively expensive to use seeding to attain market potential for both products. Hence, it may be suboptimal to seed the product in segment 1. However, beyond a certain threshold value for a_1 , we find that it is once again profitable to provide free samples targeted at consumers in segment 1. In summary, unlike the findings

reported in Jain, Mahajan and Muller (1995) in the single product case and Lehmann and Esteban-Bravo (2006) in the (+/0) case, optimal samples need not always decrease with the coefficient of innovation in settings with negative interaction.

Next, we investigate the impact of the coefficient of imitation (b_1) on the optimal number of samples. The parameters for the simulation are in Table 4 and the results are in Table 5. For a given value of the coefficient of imitation, the optimal number of samples initially increases as c_1 becomes more negative. However, if the negative influence from segment 2 is strong and the word-of-mouth (WOM) effect within segment 1 is weak, then sampling is undesirable. This observation is consistent with the results in Table 3 also. For $c_1 = 0$, the optimal sampling level is non-decreasing with the coefficient of imitation, as reported by Jain, Mahajan and Muller (1995). This is because the initial samples help seed the market and WOM then helps speed the diffusion. However, the optimal number of samples is decreasing in the coefficient of imitation when there is an impeding influence from segment 2. When WOM effects are weak and the coefficient of inter-segment interaction is negative and significant, a large number of samples are needed to overcome the negative influence from segment 2. For example, the number of samples needed to cross the separatrix in Figure 2 can be very high when c_1 is highly negative but b_1 is small. However, when WOM effects are strong, the negative inter-segment interaction is less relevant once the installed base attains a certain level of market penetration. Thus, the firm no longer needs a large sampling level. Unlike the results reported by Lehmann and Esteban-Bravo (2006) for the (+/0) case, we find that the optimal level of sampling may be decreasing smoothly, or show abrupt upward jump points followed by smooth declines, as the coefficient of imitation increases.

Table 4: Parameter Setting

Parameter	Product 1	Product 2
Coefficient of Innovation (a)	0.02	0.03
Coefficient of Imitation (b)	0.001: 0.005	0.002
Coefficient of Inter-product Interaction (c)	0:-0.002	0
Market Potential (m)	100	100
Price (p)	1	1
Unit Cost (u)	0	0

Table 5: Optimal Sampling levels

Parameters		c_1				
		0	-0.0005	-0.001	-0.0015	-0.002
b_1	0.001	0	7	0	0	0
	0.0015	0	4	15	0	0
	0.002	0	3	9	17	17
	0.0025	0	3	6	10	16
	0.003	1	3	5	8	11
	0.0035	1	3	4	6	8
	0.004	2	3	4	5	6
	0.0045	2	3	4	4	5
	0.005	2	3	3	4	5

In summary, a very important potential role of sampling under negative inter-segment interaction is that of modifying the long term equilibrium. As a result, the optimal level of sampling can be much higher with inter-segment interaction, than when diffusion is analyzed in a homogeneous consumer population. For example, Jain, Mahajan and Muller (1995) report that the maximum sampling level observed in their analysis was never higher than 9%. However, for similar parameters, we find that the maximum sampling levels in the presence of strong inter-segment interaction can exceed 20%. This is because the role of sampling is not only to encourage early adoption when innovation is low or when imitation is high, but also to mitigate any negative influence from the other segment.

3.3.2. Demand Control

In Section 3.2 we showed that when LHR_1 is negative, the only stable equilibrium outcome is $(0, m_2)$. In such cases, seeding will not be helpful in achieving (m_1, m_2) as a stable outcome. An alternative strategy available to a marketing manager is to limit the diffusion ceiling for the segment exerting the negative influence (i.e., to restrict m_2 to m'_2 , where $m'_2 < m_2$ in order to make (m_1, m'_2) a stable equilibrium outcome). Such a strategy can be implemented in many ways, including limiting the distribution of the product to select channels rarely patronized by the impeding segment. For instance, Diesel and Burberry have limited the effective access of their products to “wannabes” by setting a high enough price and restricting distribution to certain exclusive channels. Offering limited editions is another

way to control the availability of the product to the impeding segment, as noted by Amaldoss and Jain (2008).

We provide a numerical illustration. Table 6 lists the parameter values for the two segments. The discount factor is set to $\delta = 0.926$. Table 7 reports the results. Segment 2 exerts a negative influence on segment 1. In the absence of any intervention, the only stable equilibrium is (0,100). Reducing the market potential for segment 2 from 100 to 50 increases the total discounted profit by 22.1% (from 54.42 to 66.46). The new equilibrium is (100,50) and thus the consumers in segment 1 do not disadopt the product in the long run.

Table 6: Parameter Settings

Parameter	Segment 1	Segment 2
Coefficient of Innovation (a)	0.02	0.03
Coefficient of Imitation (b)	0.003	0.001
Coefficient of Inter-Product Interaction (c)	-0.002	0.002
Market Potential (m)	100	100
Price (p)	1	1
Unit Cost (u)	0	0

Table 7: Results for Reduced Market Potential

	Base Case	Demand Control
Market Potential (m_2)	100	50
Is (m_1, m_2) stable?	No	Yes
Profit	54.42	66.46

4. Mutually Impeding Influence Model (-/-)

We now turn to the case in which $c_1 < 0, c_2 < 0$. For example, cellular service providers often observe such negative interaction between the mainstream and youth segments (Maier 2003). If the brand is primarily perceived as one for teenagers, it draws many of the mainstream customers, especially

business customers, away from the service. Similarly, increased adoption by business users impedes its diffusion among the teenager segment.

In the analysis that follows, we identify conditions under which the two segments (or products) can achieve their market potential despite the negative interaction. Intervention strategies like seeding and demand control are still relevant in the context of this model, though there are some differences from the Asymmetric Influence Model. For example, an additional decision that has to be made is whether to seed both segments (products) or only one, and—in the latter case—which of the two to seed.

As before, we first identify the isoclines, or curves along which $N_1' = 0$ or $N_2' = 0$. Using the condition $N_1' = 0$ and equations (1a) and (1b), we obtain the isoclines of N_1 as $N_1 = m_1$ and $N_2 = (a_1 + b_1 N_1) / |c_1|$. Similarly, the isoclines for N_2 are $N_2 = m_2$ and $N_1 = (a_2 + b_2 N_2) / |c_2|$. The latter equation can be rewritten as $N_2 = (N_1 |c_2| - a_2) / b_2$. We refer to the isocline $N_2 = (a_1 + b_1 N_1) / |c_1|$ as I1 and the isocline $N_2 = (N_1 |c_2| - a_2) / b_2$ as I2.

The isoclines $N_1 = m_1$ and $N_2 = m_2$ intersect at (m_1, m_2) . We denote this singular point as A. The isoclines $N_2 = m_2$ and I1 intersect at $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2 \right)$, a singular point which we label B and which

lies in the feasible region if $0 < \frac{(m_2 |c_1| - a_1)}{b_1} < m_1$. The isoclines $N_1 = m_1$ and I2 intersect at

$\left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2} \right)$, a singular point which we label C and which lies in the feasible region if $0 <$

$\frac{(m_1 |c_2| - a_2)}{b_2} < m_2$. Finally, the isoclines I1 and I2 intersect at point D which has the coordinates

$$\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) \right).$$

Based on whether or not B, C and D lie within the feasible region, one obtains six possible cases. We describe two of those cases below in Sections 4.1 and 4.2. The scenario described in Section 4.1 is the only case in which the product can stably attain its market potential in both segments, while the scenario in Section 4.2 is fairly representative of the kind of results obtained for the remaining four cases described in complete detail in Appendix A.

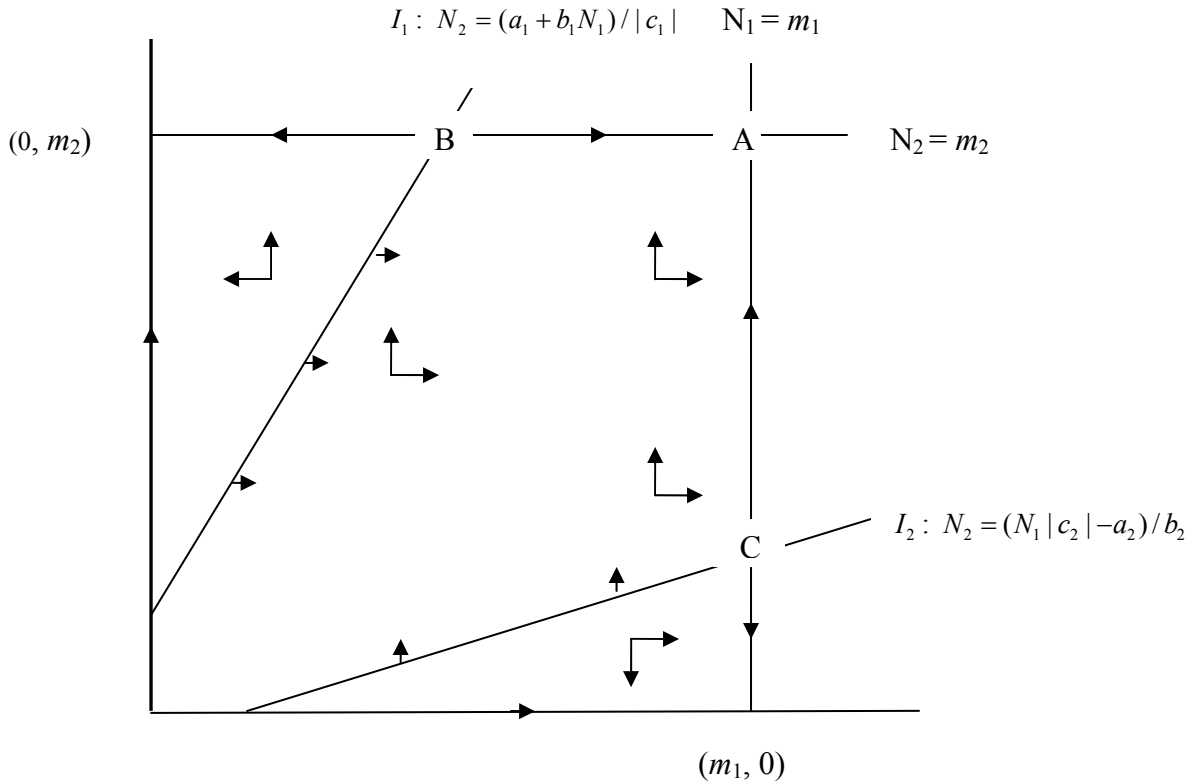
4.1. Mild Mutually Impeding Influence (Positive LHR_1 and Positive LHR_2)

When LHR_1 and LHR_2 are both positive (i.e., $0 < \frac{(m_2 |c_1| - a_1)}{b_1} < m_1$ and $0 < \frac{(m_1 |c_2| - a_2)}{b_2} < m_2$), both

B and C lie within the feasible region. When these conditions are satisfied, it can be verified that the two isoclines I_1 and I_2 cannot intersect in the feasible region, implying that singular point D lies outside the feasible region and is therefore, irrelevant to our analysis (see Figure 4).

There exist three singular points in the feasible region: (m_1, m_2) , $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right)$ and $\left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2}\right)$ denoted by A, B and C respectively in Figure 4. Trajectories passing through points above isocline I_1 always move up and to the left until they converge to $(0, m_2)$. Similarly, trajectories passing through points below I_2 move down and to the right until they converge to the boundary point $(m_1, 0)$. Trajectories in the region between isoclines I_1 and I_2 move up and to the right. Thus, trajectories close to singular point A will eventually converge to (m_1, m_2) . Proposition 3 formalizes these observations.

Figure 4. Phase Diagram for Mild Mutually Impeding Influence



Proposition 3: For the Mutually Impeding Influence model with positive values for LHR_1 and LHR_2 , the only stable equilibrium outcomes possible are $(0, m_2)$, $(m_1, 0)$ and (m_1, m_2) .

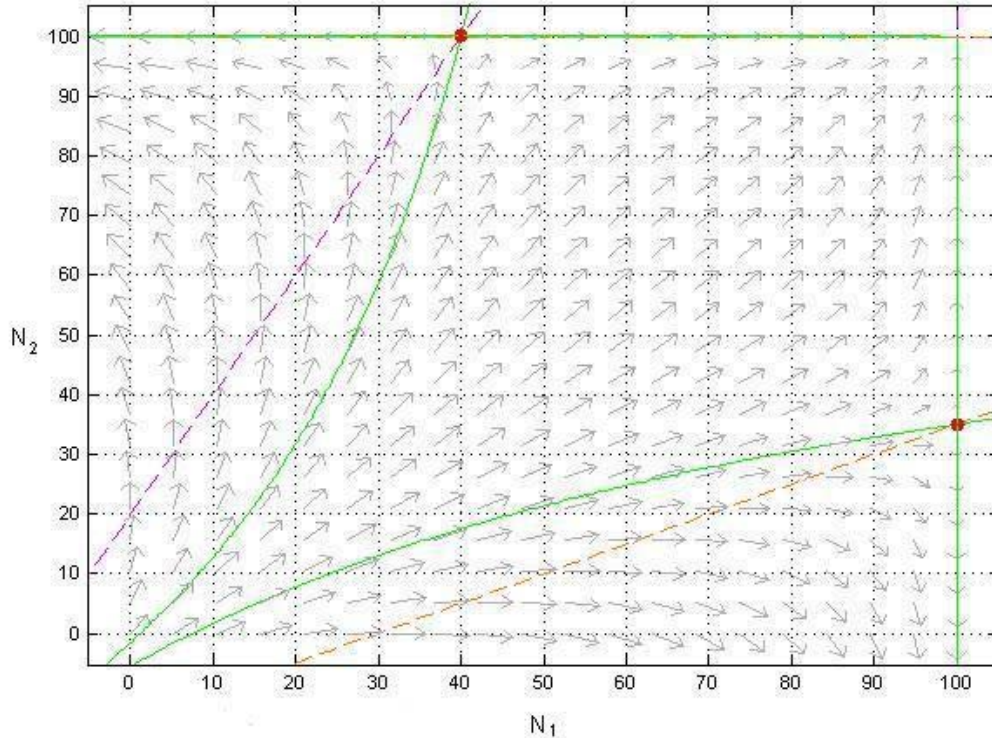
Proof: See Appendix B.

In Lemma 1 of Appendix B we show that points B and C are saddle points. Therefore there exist two corresponding separatrices that demarcate the phase plane into three non-overlapping regions. All trajectories belonging to a region will converge to the same equilibrium outcome. In Figure 5, we plot separatrices and sample trajectories for the system:

$$\begin{aligned} \frac{dN_1}{dt} &= (0.02 + 0.002N_1 - 0.001N_2)(100 - N_1) \\ \frac{dN_2}{dt} &= (0.03 + 0.002N_2 - 0.001N_1)(100 - N_2) \end{aligned} \quad (7)$$

$0 \leq N_1 \leq 100$
 $0 \leq N_2 \leq 100$

Figure 5. Phase Diagram with Separatrices for (7)



The separatrices are easy to compute numerically, as explained in Appendix C. Once the separatrices are identified, a manager can assess the need for various intervention strategies based on the long-term equilibrium associated with any given starting point. Interestingly, observe in Figure 5 that the

region wherein trajectories converge to (m_1, m_2) is very narrow initially. Even if the marketer can seed the market to ensure that the diffusion begins in this region, small shocks in the environment can potentially push the trajectory into one of the two surrounding regions. Thus, it can be rare for both segments (products) to reach their market potential. Considerable care will be needed during the early phases of product diffusion to ensure that the diffusion trajectory stays in the desirable region.

4.2. Strong Mutually Negative Influence (Negative LHR_1 and Negative LHR_2)

In this case, isocline I1 intersects $N_2 = m_2$ outside the feasible region. Similarly, isocline I2 intersects $N_1 = m_1$ outside the feasible region. This implies that the two isoclines I1 and I2 necessarily intersect in the feasible region as shown in Figure 6. Singular point D will then lie in the feasible region.

Thus, we get two singular points (m_1, m_2) and $\left(\frac{|c_1|a_2 + a_1b_2}{|c_1||c_2| - b_1b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1|a_2 + a_1b_2}{|c_1||c_2| - b_1b_2} \right) \right)$, labeled A and D respectively in Figure 6. If we consider the non-negativity constraints, then $(m_1, 0)$ and $(0, m_2)$ behave like additional sinks. The stable equilibrium points are identified below.

Proposition 4: For the Mutually Impeding Influence model with negative values for LHR_1 and LHR_2 , the only stable equilibrium outcomes possible are $(0, m_2)$ and $(m_1, 0)$.

Proof: See Appendix B.

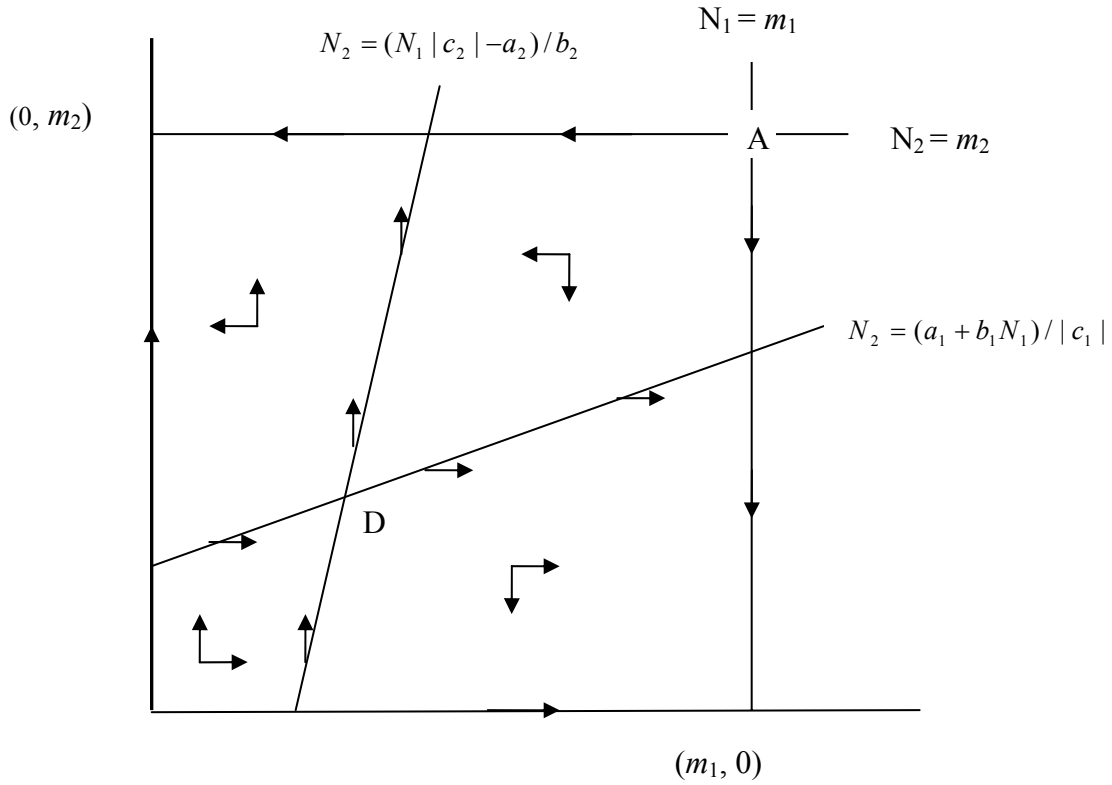
A significant implication is that the two segments (products) cannot co-exist in equilibrium in this case.

A summary of the stable equilibrium outcomes for all possible cases in the Mutually Impeding Influence model is provided in Table 8. Both segments can co-exist in equilibrium only when the impeding influences from the other customer segment are mild, i.e., if $0 < \frac{(m_2|c_1| - a_1)}{b_1} < m_1$ and

$0 < \frac{(m_1|c_2| - a_2)}{b_2} < m_2$. Otherwise, adoption in only one segment will eventually reach its market potential

and adoption in the other segment will fade away. Further, even in the case in which (m_1, m_2) is stable, the region in which trajectories converge to (m_1, m_2) can be narrow early on. Thus, it can be very challenging to achieve full market potential for both segments (products) under mutually impeding diffusion.

Figure 6. Phase Diagram for Strong Mutually Impeding Influence



**Table 8. Summary of equilibrium outcomes and their stability properties:
Mutually Impeding Influence Model**

Section	Scenario			Candidate Equilibrium Outcomes					
	LHR_1	LHR_2	D feasible?	(m_1, m_2)	B	C	D	$(0, m_2)$	$(m_1, 0)$
4.1	+ve	+ve	-	Stable	Unstable	Unstable	-	Stable	Stable
A.1	-ve	+ve	Yes	Unstable	-	Unstable	Unstable	Stable	Stable
A.2	-ve	+ve	No	Unstable	-	Unstable	-	Stable	Stable
A.3	+ve	-ve	Yes	Unstable	Unstable	-	Unstable	Stable	Stable
A.4	+ve	-ve	No	Unstable	Unstable	-	-	Stable	Stable
4.2	-ve	-ve	Yes	Unstable	-	-	Unstable	Stable	Stable

Note: Sections A.1 through A.4 can be found in Appendix A

Due to this difficulty, firms may choose to disassociate or decouple the two segments or products even if it imposes additional costs (see also Amaldoss and Jain 2008). This is typically achieved by targeting each segment with distinct product versions or even distinct brands. For example, Nextel (now merged with Sprint) responded to the negative interaction between the teenager and mainstream segments by marketing to teenagers through a new brand. Nextel partnered with Boost, a firm that operated in Australia and was relatively unknown in the US at that time. “Most Boost subscribers in America likely don't realize that their calls are carried by Nextel ... and a company whose brand name, by its own admission, is a total loser with the young,” (Maier 2003). Another way to decouple segments is simply to reduce the visibility of the product. For instance, in trying to reduce the appeal of the Burberry brand among British hooligans, the firm started to make its distinctive and iconic tartan check pattern much less prominent on its apparel and accessories.

5. Symbiotic Influence Model (+/+)

The case of symbiotic influence is equally applicable to the context of two customer segments or two products. The discussion here is in the context of two complementary products. Several products are known to have positive influence on each other. For example, consumers are more likely to adopt cellular data services when more content is available on the platform. Simultaneously, content developers are more likely to develop content for the platform when there are a large number of consumers using it. Similar two-sided network effects are commonly observed in a number of information markets such as those tied to technology platforms. When products have positive interaction (i.e., $c_1 \geq 0, c_2 \geq 0$), the hazard rate is always positive. Thus, both products will reach their market potential in equilibrium.

Nonetheless, marketing managers face unique issues raised by the inter-product interactions. For example, what is the fastest or most profitable way to reach the equilibrium (m_1, m_2) ? In the asymmetric influence model in Section 3 we discussed seeding as a strategy to help achieve (m_1, m_2) as a stable equilibrium outcome. In the symbiotic influence model, (m_1, m_2) is the only singular point and it is a stable equilibrium outcome. Nonetheless, seeding can help speed the diffusion process towards (m_1, m_2) . Seeding a product not only helps speed its own diffusion but also contributes to faster diffusion of the other product. Hence, the analysis of optimal sampling is very relevant, the main idea being quite similar to that of focusing one's marketing efforts on customers with more than average social influence to gain maximum leverage. Due to space constraints, we do not delve into the details.

6. Discussion and Conclusions

We have studied the diffusion of a product in two customer segments where the acceptance level in one segment affects the diffusion rate not only in that same segment, but also in the other. In doing so, we have focused on the cases of asymmetric (+/-) and mutually impeding (-/-) influence. Both types of cross-segment interaction are quite important for products and brands that act as social signifiers, and have become of considerable interest to both consumer researchers and marketing scientists (e.g., Amaldoss and Jain 2008; Berger and Heath 2007, 2008; Joshi et al. 2006). The analytical approach we use and our key results apply also to the diffusion of two interacting products in a single population, a more established area of research where analytical results have been hard to obtain (Bayus et al. 2000). Since the diffusion system we study does not have a closed-form solution, we use phase plane analysis to determine equilibrium points of the joint diffusion process and characterize their stability properties.

A rather surprising result is that, even in situations with symmetric or asymmetric negative influence, stable equilibrium outcomes do not include partial penetration. In other words, in a stable equilibrium, each segment has adopted either entirely or not at all. This strong result, however, is likely to hinge on the assumption that disadoption is possible. If this were not so, then trajectories in the phase plane would likely “freeze” at a particular level of, say, N_1 , without reverting back to the origin and settling at $(0, m_2)$.

An important practical contribution consists in using separatrices to identify, for each possible combination of acceptance level in each segment (or, in a two-product setting, for each level of installed base of each product) which particular equilibrium point the diffusion trajectory will converge to. Importantly, we find at most three such regions with different convergence behavior in all of our analyses, making it easy to make such inferences. For the cases of asymmetric influence (+/-) and mutually impeding influence (-/-) of central interest, we also identify the conditions under which both products can achieve full market potential in equilibrium. Our results indicate that considerable care is needed in the early stages of product launch. For instance, we analyzed a setting with mild mutually negative influence across segments that may easily induce a false sense of complacency. While the odds of reaching full penetration in both segments (m_1, m_2) might seem high intuitively, a closer analysis (as in Figure 5) reveals that the “funnel” of trajectories that eventually end up at (m_1, m_2) can be very narrow early on in the lifecycle of a product. Such early bifurcation-like behavior in the model may explain why predicting market success is especially difficult for products with strong iconic and social identity appeal (Farrell 1998; Lieberman 2000).

We find that for many realistic parameter combinations, full penetration in both segments will not be achieved without specific intervention. So, our results provide an explanation for several marketing practices for new products appealing to segments between which asymmetric or mutually impeding influence operates: (1) “seeding,” i.e., using free samples to support the launch of a product in one

segment being harmed by the adoption in the other, (2) “demand control,” i.e., purposely limiting market potential for the customer segment harming product diffusion in the other segment by making the product more selective or by launching targeted limited editions, and (3) “decoupling,” i.e., purposely decreasing the amount of (negative) cross-segment influence, typically achieved by targeting distinct offerings to different segments (versioning) or by decreasing the visibility of the product.

We provide managerial insights into the effectiveness of seeding and demand control in the context of asymmetric (+/-) interaction. The key novel finding about seeding is that, unlike cases without negative cross-segment interaction studied previously, the optimal level of sampling may show abrupt jumps followed by smooth declines as the coefficients of innovation and imitation increase. A simple numerical analysis on the effectiveness of demand control confirms that, in situations with asymmetric influence, “less can be more”, hence validating common practices among marketers of brands with strong social identity value.

Separatrix analysis can be quite valuable to firms as it allows managers to identify whether or not the diffusion process will evolve to full penetration in both segments without special managerial intervention. These separatrices can be numerically computed for any system, as described in detail in Appendix C. MATLAB code is available (e.g., Polking and Arnold 2003; <http://math.rice.edu/~polking/odesoft/>), and the algorithm for computing the separatrix can also be implemented in a spreadsheet. An Excel implementation for the scenario described in section 3.1 is available.

The presence of symmetric and asymmetric negative cross-segment and positive cross-product interactions can have a critical impact on diffusion outcomes and can significantly affect the effectiveness of particular marketing actions. Yet, much research remains to be done.

As already mentioned, the result that any stable equilibrium has each segment either fully adopted or not at all, need not hold when disadoption is not possible. This raises the managerially interesting question whether policies allowing for disadoption (such as leasing rather than selling equipment) may, in some cases not lead to more desirable equilibrium diffusion outcomes.

Other model variations could be analyzed. For instance, we analyzed situations where cross-segment or cross-product interactions operate through the diffusion rate rather than through the market potential. If one were interested specifically in substitution effects across products rather than in cross-segment dynamics, then having the effect operate via the market potential might be intuitively more appealing, and it is not clear to what extent this would lead to results different from the ones presented here. For instance, the asymmetric influence (+/-) specification of such a model would be similar—though not identical—to the standard Lotka-Volterra predator-prey model which can produce cyclical patterns.

Given our finding that prior results on optimal seeding may not be applicable when significant inter-product/inter-segment interactions exist, it may be useful to investigate how pricing and advertising

strategies are affected by such interactions. Such research on how firms can effectively balance social forces of distinction and emulation across segments could prove a valuable complement to recent game-theoretic research on the same question that does not explicitly focus on diffusion trajectories.

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Online Appendices

Appendix A: Four Remaining Cases: Mutually Impeding Influence Model

A.1. Negative LHR_1 and Positive LHR_2 ; D within feasible region.

In this case there are three singular points - (m_1, m_2) , $\left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2}\right)$ and

$\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}\right)\right)$ in the feasible region. These are labeled A, C and D respectively

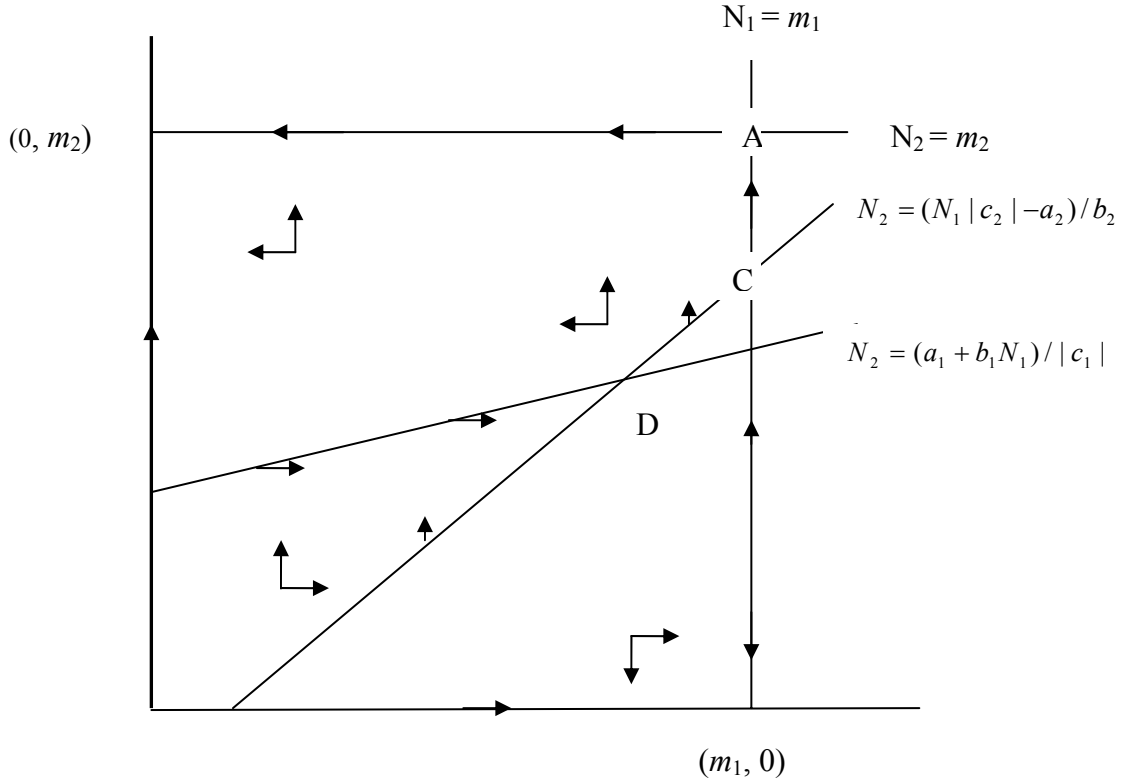
in Figure A1. Trajectories passing through points above both isoclines I1 and I2 move up and to the left until they converge to $(0, m_2)$. Similarly, trajectories passing through points below both I1 and I2 move down and to the right until they converge to the boundary point $(m_1, 0)$. Trajectories in the remaining two regions cross over into one of the two regions described above (see Figure A1). The equilibrium outcomes can be characterized as follows:

Proposition 5: For the Mutually Impeding Influence model with a negative value for LHR_1 , but positive LHR_2 , and D lying within the feasible region, the only stable equilibrium outcomes are $(0, m_2)$ and $(m_1, 0)$.

Proof: See Appendix B.

Thus, in this system, both products cannot exist in equilibrium.

**Figure A1. Phase Diagram with A, C and D as singular points
(Mutually Impeding Influence)**



A.2. Negative LHR_1 and Positive LHR_2 ; D outside feasible region.

We now consider the case in which there are two singular points - (m_1, m_2) and $\left(m_1, \frac{(m_1 | c_2 | -a_2)}{b_2}\right)$ in the feasible region. These are labeled A and C respectively in Figure A2. The intersection of isclines I1 and I2 (point D) lies outside the feasible region.

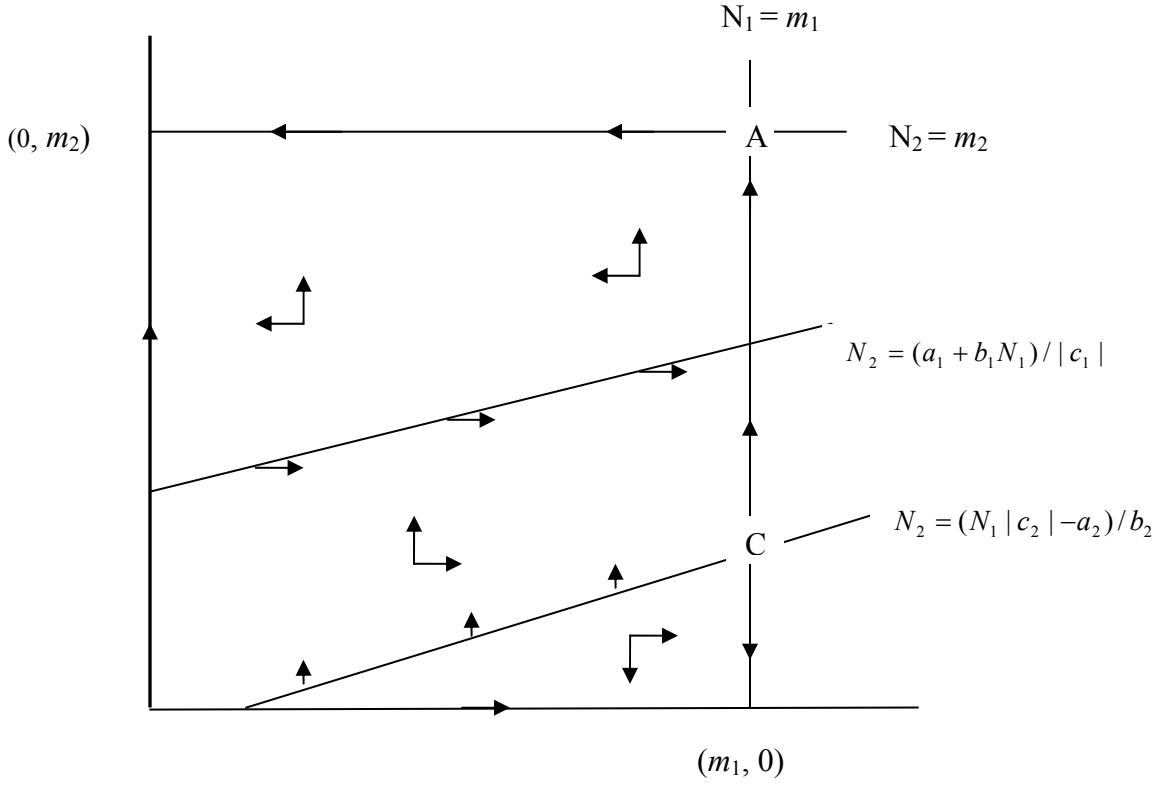
Trajectories passing through points above isocline I1 move up and to the left until they converge to $(0, m_2)$. Similarly, trajectories passing through points below I2 move down and to the right until they converge to the boundary point $(m_1, 0)$. The equilibrium outcomes can be characterized as follows:

Proposition 6: For the Mutually Impeding Influence model with a negative value for LHR_1 , but positive LHR_2 , and D lying outside the feasible region, the only stable equilibrium outcomes possible are $(0, m_2)$ and $(m_1, 0)$.

Proof: See Appendix B.

Thus, in this system, both products cannot exist in equilibrium.

**Figure A2. Phase Diagram with A and C as singular points
(Mutually Impeding Influence)**



A.3. Positive LHR_1 and Negative LHR_2 ; D in feasible region.

In this case we get three singular points - (m_1, m_2) , $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right)$ and

$\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| c_2 - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| c_2 - b_1 b_2}\right)\right)$. These are labeled A, B and D respectively in Figure A3.

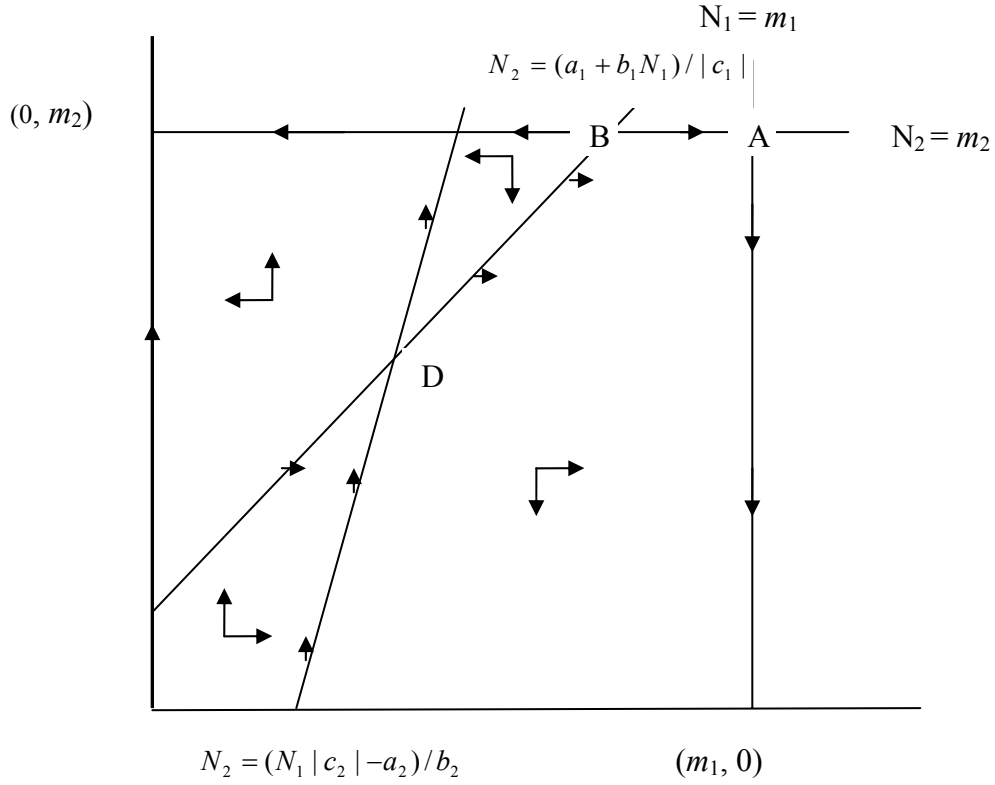
Thus, the equilibrium outcomes can be characterized as follows:

Proposition 7: Under mutually negative interaction between products (i.e. $c_1 < 0, c_2 < 0$) with negative LHR_1 and positive LHR_2 and D lying within the feasible region, the only stable equilibrium outcomes possible are $(0, m_2)$ and $(m_1, 0)$.

Proof: See Appendix B.

Thus, in this case as well, both products cannot co-exist in equilibrium.

**Figure A3. Phase Diagram with A, B and D as singular points
(Mutually Impeding Influence)**



A.4. Positive LHR_1 and Negative LHR_2 ; D outside feasible region.

In this case we get two singular points - (m_1, m_2) and $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right)$. These are labeled A, and B

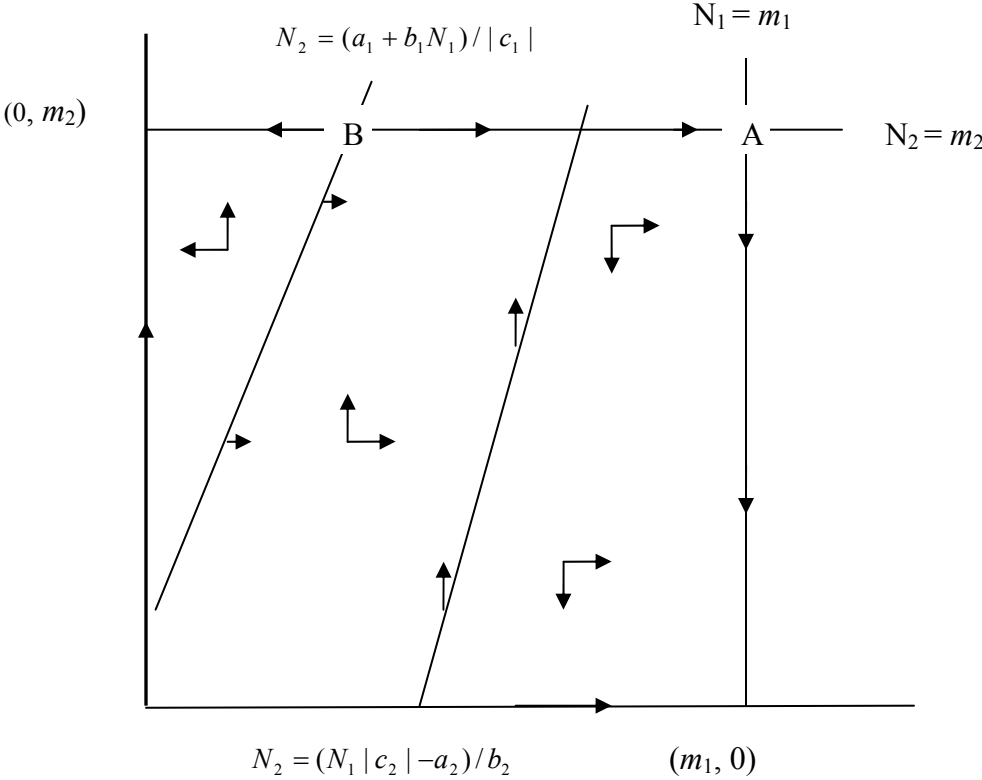
respectively in Figure A4. The equilibrium outcomes can be characterized as follows:

Proposition 8: Under mutually negative interaction between products (i.e. $c_1 < 0, c_2 < 0$), with negative LHR_1 and positive LHR_2 and D lying outside the feasible region, the only stable equilibrium outcomes possible are $(0, m_2)$ and $(m_1, 0)$.

Proof: See Appendix B.

Thus, in this case as well, both products cannot co-exist in equilibrium.

**Figure A4. Phase Diagram with A and B as singular points
(Mutually Impeding Influence)**



Appendix B: Proofs

Lemma 1: For the dynamical systems described in (1a) and (1b), stability analyses of the singular points yield the following:

- (i) Point A, or (m_1, m_2) , is a singular point for any set of values of the diffusion parameters. It is stable if and only if both LHR_1 and LHR_2 are positive, and $LHR_1 \neq LHR_2$.
- (ii) Point B, or $\left(-\frac{a_1 + m_2 c_1}{b_1}, m_2\right)$ with $c_1 < 0$, is a singular point if and only if LHR_1 is positive. Furthermore, it is a saddle point, whenever it qualifies as a singular point.
- (iii) Point C, or $\left(m_1, -\frac{a_2 + c_2 m_1}{b_2}\right)$ with $c_2 < 0$, is a singular point if and only if LHR_2 is positive. Furthermore, it is a saddle point, whenever it qualifies as a singular point.
- (iv) Point D, or $\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}\right)\right)$ with $c_1 < 0$ and $c_2 < 0$, is a singular point whenever it is within the feasible region (i.e., $0 < \frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} < m_1$ and $0 < \left(\frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}\right)\right) < m_2$). Furthermore, it is a saddle point, whenever it qualifies as a singular point.

Proof:

The system of equations (1a) and (1b) can be represented as:

$$\frac{dN_1}{dt} = f(N_1, N_2)$$

$$\frac{dN_2}{dt} = g(N_1, N_2)$$

$$N_1 \geq 0; N_2 \geq 0$$

In order to study the stability properties of the singular points of this non-linear system of equations, we will have to linearize it in the region around the singular point. Then we can use standard techniques to analyze the stability of singularities in linear systems. We do this as follows. Since f and g are twice continuously differentiable, we can expand them in a Taylor polynomial around the singular point (N_1^0, N_2^0) :

$$f(N_1, N_2) = \frac{\partial f}{\partial N_1} \Big|_{(N_1^0, N_2^0)} (N_1 - N_1^0) + \frac{\partial f}{\partial N_2} \Big|_{(N_1^0, N_2^0)} (N_2 - N_2^0) + P(N_1 - N_1^0, N_2 - N_2^0)$$

$$g(N_1, N_2) = \frac{\partial g}{\partial N_1} \Big|_{(N_1^0, N_2^0)} (N_1 - N_1^0) + \frac{\partial g}{\partial N_2} \Big|_{(N_1^0, N_2^0)} (N_2 - N_2^0) + Q(N_1 - N_1^0, N_2 - N_2^0)$$

where $P(N_1 - N_1^0, N_2 - N_2^0)$ and $Q(N_1 - N_1^0, N_2 - N_2^0)$ are functions that comprise terms that are at least quadratic, or higher order, in the arguments $(N_1 - N_1^0)$ and $(N_2 - N_2^0)$. Near the singular point, these terms will be negligibly small compared to the linear terms, and hence we can study the behavior of the non-linear system near its singular points, by making the following change of coordinates:

$$x_1 = N_1 - N_1^0; x_2 = N_2 - N_2^0 \text{ to get}$$

$$\frac{dx_1}{dt} = \frac{\partial f}{\partial N_1} \Big|_{(N_1^0, N_2^0)} x_1 + \frac{\partial f}{\partial N_2} \Big|_{(N_1^0, N_2^0)} x_2$$

$$\frac{dx_2}{dt} = \frac{\partial g}{\partial N_1} \Big|_{(N_1^0, N_2^0)} x_1 + \frac{\partial g}{\partial N_2} \Big|_{(N_1^0, N_2^0)} x_2$$

The standard form of a linear system of differential equations is $\mathbf{dx}/dt = \mathbf{Ax}$, for vector \mathbf{x} . Then

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f}{\partial N_1} \Big|_{(N_1^0, N_2^0)} & \frac{\partial f}{\partial N_2} \Big|_{(N_1^0, N_2^0)} \\ \frac{\partial g}{\partial N_1} \Big|_{(N_1^0, N_2^0)} & \frac{\partial g}{\partial N_2} \Big|_{(N_1^0, N_2^0)} \end{bmatrix}$$

In the above matrix:

$$\frac{\partial f}{\partial N_1} \Big|_{(N_1^0, N_2^0)} = b_1 m_1 - a_1 - c_1 N_2^0 - 2b_1 N_1^0; \frac{\partial f}{\partial N_2} \Big|_{(N_1^0, N_2^0)} = c_1 (m_1 - N_1^0)$$

$$\frac{\partial g}{\partial N_1} \Big|_{(N_1^0, N_2^0)} = c_2 (m_2 - N_2^0); \frac{\partial g}{\partial N_2} \Big|_{(N_1^0, N_2^0)} = b_2 m_2 - a_2 - c_2 N_1^0 - 2b_2 N_2^0$$

Next, we compute the eigenvalues of the matrix \mathbf{A} by solving $Det(\mathbf{A} - \lambda \mathbf{I}) = 0$. This gives us the quadratic equation:

$$\lambda^2 - \lambda[b_1 m_1 - a_1 + b_2 m_2 - a_2 - (c_2 + 2b_1)N_1^0 - (c_1 + 2b_2)N_2^0] +$$

$$(b_1 m_1 - a_1 - c_1 N_2^0 - 2b_1 N_1^0)(b_2 m_2 - a_2 - c_2 N_1^0 - 2b_2 N_2^0) - c_1 c_2 (m_1 - N_1^0)(m_2 - N_2^0) = 0$$

Let the eigenvalues corresponding to a singular point be denoted by λ_1 and λ_2 . A singular point is a saddle point if the eigenvalues are of opposite sign. The singular point is a sink if the eigenvalues are distinct and negative (Hubbard and West 1995). We now determine the eigenvalues associated with each of the 4 identified singular points.

i) Stability analysis of A, or (m_1, m_2) :

Solving the quadratic equation at $(N_1^0, N_2^0) = (m_1, m_2)$, we get the following eigenvalues:

$$\lambda_1 = -(a_1 + b_1 m_1 + c_1 m_2) = -LHR_1$$

$$\lambda_2 = -(a_2 + b_2 m_2 + c_2 m_1) = -LHR_2$$

Under the conditions $0 < \frac{(m_2 | c_1 | -a_1)}{b_1} < m_1$ and $0 < \left(\frac{|c_2| m_1 - a_2}{b_2} \right) < m_2$, which are also implied by

a positive value for LHR_1 and LHR_2 , we can easily see both eigenvalues are strictly negative, and if we assume them to be distinct then we can conclude that (m_1, m_2) is a sink. (Equal eigenvalues imply a degenerate node.) At the same time, when either LHR_1 or LHR_2 is negative (at least one eigenvalue is positive) then (m_1, m_2) is a saddle point.

ii) Stability analysis of B, or $\left(-\frac{a_1 + m_2 c_1}{b_1}, m_2 \right)$:

When $c_1 > 0$, this singular point does not lie in the feasible region and is thus not relevant. We therefore focus on the case in which $c_1 < 0$. The singular point can be rewritten as $\left(\frac{(m_2 | c_1 | -a_1)}{b_1}, m_2 \right)$.

After solving the quadratic equation, the eigenvalues are:

$$\lambda_1 = b_1 m_1 + a_1 - |c_1| m_2;$$

$$\lambda_2 = -[b_2 m_2 + a_2 + \frac{c_2}{b_1} (-a_1 + |c_1| m_2)]$$

Under the condition $0 < \frac{m_2 | c_1 | -a_1}{b_1} < m_1$ (i.e., singular point B is in the feasible region), we can clearly

see that $\lambda_2 < 0 < \lambda_1$. The condition for B to lie in the feasible region is also implied by a positive LHR_1 .

Hence, the singular point, if it exists in the feasible region, is always a saddle point.

iii) Stability analysis of C, or $\left(m_1, -\frac{a_2 + c_2 m_1}{b_2} \right)$

When $c_2 > 0$, this singular point does not lie in the feasible region and is thus not relevant. We therefore focus on the case in which $c_2 < 0$. The singular point can be rewritten as $\left(m_1, \frac{|c_2| m_1 - a_2}{b_2}\right)$. Solving the

quadratic equation, the eigenvalues corresponding to this singular point are:

$$\lambda_1 = b_2 m_2 + a_2 - |c_2| m_1;$$

$$\lambda_2 = -[b_1 m_1 + a_1 + \frac{c_1}{b_2}(-a_2 + |c_2| m_1)]$$

Under the condition $0 < \frac{m_1 |c_2| - a_2}{b_2} < m_2$ (i.e., singular point C is in the feasible region), we can

clearly see that $\lambda_2 < 0 < \lambda_1$. The condition for C to lie in the feasible region is also implied by a positive LHR_2 . Hence, the singular point C , if it exists in the feasible region, is always a saddle point.

iv) Stability analysis of D , or $\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}\right)\right)$

For this singular point to take on positive coordinate values we necessarily require that

$|c_1| |c_2| - b_1 b_2 > 0$. The quadratic equation that is to be solved to determine the eigenvalues, takes on the form:

$$\lambda^2 - \lambda[b_1(m_1 - N_1^0) + b_2(m_2 - N_2^0)] + (b_1 b_2 - |c_1| |c_2|)(m_1 - N_1^0)(m_2 - N_2^0) = 0$$

We observe that the coefficient of λ^2 is 1 and the independent term in the equation is negative. The basics of the theory of quadratic equations then imply that the eigenvalues (or roots of the quadratic equation) are real and of opposite sign. Hence, this singular point is a saddle point. QED.

Lemma 2: For the asymmetric influence model, if $|c_1| < (m_1 b_1 + a_1)/m_2$, then (m_1, m_2) is a stable

equilibrium point (a sink) and $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right)$ is an unstable equilibrium point (a saddle point) for

the dynamic system represented by (1a) and (1b).

Proof: Follows from Lemma 1-(i) and Lemma 1-(ii). QED.

Lemma 3: For asymmetric interaction between the segments (i.e., $c_1 < 0, c_2 \geq 0$), if

$|c_1| < (m_1 b_1 + a_1)/m_2$, then $\lim_{t \rightarrow \infty} (N_1(t), N_2(t)) \in \left\{ (m_1, m_2), (0, m_2), \left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right) \right\}$. Further,

because $((m_2 | c_1 | -a_1) / b_1, m_2)$ is a saddle point, the only stable equilibrium outcomes are $(0, m_2)$ and (m_1, m_2) .

Proof: At $(0, m_2)$, and in its vicinity, the sign of dN_1 / dt is negative and that of dN_2 / dt is positive. As a result, a slight perturbation in location from $(0, m_2)$ to a nearby location in within the feasible region on the phase plane will result in the trajectory returning to $(0, m_2)$. So even though $(0, m_2)$ is not a singular point, the constraints $N_1 \geq 0$ and $N_2 \leq m_2$, ensure that it behaves like one. All trajectories in the neighborhood of $(0, m_2)$ will converge to it in the limit as $t \rightarrow \infty$. This observation, along with Lemma 2 proves the result. QED.

Proposition 1: For the Asymmetric Influence model with a positive value for LHR_I , the only stable equilibrium outcomes possible are $(0, m_2)$ and (m_1, m_2) .

Proof: Follows immediately from Lemma 2 and Lemma 3. QED.

Here, and in the subsequent analyses, we don't go into the details of the *degenerate* boundary case, i.e., when $\frac{(m_2 | c_1 | -a_1)}{b_1} = m_1$. This is because it leads to a zero eigenvalue at the singular point (m_1, m_2) . This

is a degenerate case because it results in a line of equilibria. With the slightest perturbation, this whole line of equilibria disappears, and new equilibrium points are obtained (Hubbard and West 1995).

Lemma 4: For asymmetric influence between the segments or products (i.e., $c_1 < 0, c_2 \geq 0$), if $|c_1| > (m_1 b_1 + a_1) / m_2$, then (m_1, m_2) is a saddle point for the system represented by (1a) and (1b).

Proof: Follows from Lemma 1-(i). QED.

Lemma 5: For asymmetric influence between the segments or products (i.e., $c_1 < 0, c_2 \geq 0$), if $|c_1| > (m_1 b_1 + a_1) / m_2$, then $\lim_{t \rightarrow \infty} (N_1(t), N_2(t)) \in \{(m_1, m_2), (0, m_2)\}$. Further, because (m_1, m_2) is a saddle point, the only stable equilibrium outcome is $(0, m_2)$.

Proof: All trajectories in neighborhood of $(0, m_2)$ will converge to it in the limit as $t \rightarrow \infty$, by a reasoning similar to the one presented in the proof of Lemma 3. This observation, along with Lemma 4, proves the result. QED.

Proposition 2: For the Asymmetric Influence model with a negative value for LHR_I , the only stable equilibrium outcome possible is $(0, m_2)$.

Proof: Follows immediately from Lemma 4 and Lemma 5. QED.

Lemma 6: For the case with mutually negative interaction between segments or products (i.e. $c_1 < 0$, $c_2 < 0$), if the conditions $0 < \frac{(m_2 |c_1| - a_1)}{b_1} < m_1$ and $0 < \frac{(m_1 |c_2| - a_2)}{b_2} < m_2$ are satisfied, then (m_1, m_2) is a sink while $\left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right)$ and $\left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2}\right)$ are saddle points.

Proof: Follows from Lemma 1-(i), (ii) and (iii).

Lemma 7: When there is mutually negative interaction between segments or products (i.e. $c_1 < 0$, $c_2 < 0$), and the conditions on parameters: $0 < \frac{(m_2 |c_1| - a_1)}{b_1} < m_1$ and $0 < \frac{(m_1 |c_2| - a_2)}{b_2} < m_2$ are satisfied, then:

$$\lim_{t \rightarrow \infty} (N_1(t), N_2(t)) \in \left\{ (0, m_2), (m_1, 0), (m_1, m_2), \left(\frac{(m_2 |c_1| - a_1)}{b_1}, m_2\right), \left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2}\right) \right\}.$$

Further, the only stable equilibrium outcomes are $(0, m_2)$, $(m_1, 0)$ and (m_1, m_2) .

Proof: All trajectories in neighborhood of $(0, m_2)$ will converge to it in the limit as $t \rightarrow \infty$, by a reasoning similar to the one presented in the proof of Lemma 3. Also all trajectories in neighborhood of $(m_1, 0)$ will converge to it in the limit as $t \rightarrow \infty$. This is because at $(m_1, 0)$, and in its vicinity, the sign of dN_1 / dt is positive and that of dN_2 / dt is negative. Even though $(m_1, 0)$ is not a singular point, the constraints $N_1 \leq m_1$ and $N_2 \geq 0$, ensure that it behaves like one. These observations along with Lemma 6 prove the result. QED.

Proposition 3: For the Mutually Impeding Influence model with positive values for LHR_1 and LHR_2 , the only stable equilibrium outcomes possible are $(0, m_2)$, $(m_1, 0)$ and (m_1, m_2) .

Proof: Follows directly from Lemma 6 and Lemma 7. QED.

Lemma 8: If the conditions $\frac{(m_2 |c_1| - a_1)}{b_1} > m_1$ and $\frac{(m_1 |c_2| - a_2)}{b_2} > m_2$ are satisfied, then (m_1, m_2) and

$$\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| c_2 - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| c_2 - b_1 b_2} \right) \right) \text{ are saddle points.}$$

Proof: Follows from Lemma 1 – (i) and (iv). QED.

Lemma 9: Under mutually negative interaction between products (i.e. $c_1 < 0, c_2 < 0$), if the conditions on diffusion parameters: $\frac{(m_2 |c_1| - a_1)}{b_1} > m_1$ and $\frac{(m_1 |c_2| - a_2)}{b_2} > m_2$ are satisfied, then

$\lim_{t \rightarrow \infty} (N_1(t), N_2(t)) \in \left\{ (0, m_2), (m_1, 0), (m_1, m_2), \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) \right) \right\}$. Further the only stable equilibrium outcomes are $(0, m_2)$ and $(m_1, 0)$.

Proof: Follows from Lemma 8 and the logic described in Lemma 7. QED.

Proposition 4: For the Mutually Impeding Influence model with negative values for LHR_1 and LHR_2 , the only stable equilibrium outcomes possible are $(0, m_2), (m_1, 0)$.

Proof: Follows immediately from Lemma 8 and Lemma 9. QED.

Lemma 10: If the conditions $\frac{(m_2 |c_1| - a_1)}{b_1} > m_1$ and $0 < \frac{(m_1 |c_2| - a_2)}{b_2} < m_2$ are satisfied, and D is in the

feasible region (i.e., $0 < \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) < m_1$ and $0 < \left(\frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) \right) < m_2$), then (m_1, m_2) ,

$\left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2} \right)$ and $\left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) \right)$ are saddle points.

Proof: Follows from Lemma 1 – (i), (iii) and (iv). QED.

Lemma 11: For segments or products which exert a mutually negative influence on each other during the diffusion process (i.e. $c_1 < 0, c_2 < 0$), if $\frac{(m_2 |c_1| - a_1)}{b_1} > m_1$; $0 < \frac{(m_1 |c_2| - a_2)}{b_2} < m_2$, and D is in the feasible

region (i.e., $0 < \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) < m_1$ and $0 < \left(\frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) \right) < m_2$) then we get:

$\lim_{t \rightarrow \infty} (N_1(t), N_2(t)) \in \left\{ (0, m_2), (m_1, 0), (m_1, m_2), \left(m_1, \frac{(m_1 |c_2| - a_2)}{b_2} \right), \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2}, \frac{a_1}{|c_1|} + \frac{b_1}{|c_1|} \left(\frac{|c_1| a_2 + a_1 b_2}{|c_1| |c_2| - b_1 b_2} \right) \right) \right\}$

Further the only stable equilibrium outcomes are $(0, m_2)$ and $(m_1, 0)$.

Proof: Follows from Lemma 10 and the logic described in Lemma 7. QED.

Proposition 5: For the Mutually Impeding Influence model with a negative value for LHR_1 , but positive LHR_2 , and D lying within the feasible region, the only stable equilibrium outcomes are $(0, m_2), (m_1, 0)$.

Proof: Follows immediately from Lemma 10 and Lemma 11. QED.

Proposition 6: For the Mutually Impeding Influence model with a negative value for LHR_1 , but positive LHR_2 , and D lying outside the feasible region, the only stable equilibrium outcomes possible are $(0, m_2), (m_1, 0)$.

Proof: The proof is identical to that for Proposition 4, with segments 1 and 2 interchanged.

Proposition 7: Under mutually negative interaction between products (i.e. $c_1 < 0, c_2 < 0$) with negative LHR_1 and positive LHR_2 and D lying within the feasible region, the only stable equilibrium outcomes possible are $(0, m_2), (m_1, 0)$.

Proof: The proof is identical to that for Proposition 5, with products 1 and 2 interchanged.

Lemma 12: If the conditions $\frac{(m_2 | c_1 | -a_1)}{b_1} > m_1$ and $0 < \frac{(m_1 | c_2 | -a_2)}{b_2} < m_2$ are satisfied, and D is outside the feasible region (i.e., either $0 < \left(\frac{|c_1 | a_2 + a_1 b_2}{|c_1 | |c_2 | - b_1 b_2} \right) < m_1$ or $0 < \left(\frac{a_1}{|c_1 |} + \frac{b_1}{|c_1 |} \left(\frac{|c_1 | a_2 + a_1 b_2}{|c_1 | |c_2 | - b_1 b_2} \right) \right) < m_2$ is not satisfied) then (m_1, m_2) and $\left(m_1, \frac{(m_1 | c_2 | -a_2)}{b_2} \right)$ are saddle points.

Proof: Follows from Lemma 1-(i) and (ii). QED.

Lemma 13: For segments or products which exert a mutually negative influence on each other during the diffusion process (i.e. $c_1 < 0, c_2 < 0$), if $\frac{(m_2 | c_1 | -a_1)}{b_1} > m_1$; $0 < \frac{(m_1 | c_2 | -a_2)}{b_2} < m_2$, and D is outside the

feasible region (i.e., either $0 < \left(\frac{|c_1 | a_2 + a_1 b_2}{|c_1 | |c_2 | - b_1 b_2} \right) < m_1$ or $0 < \left(\frac{a_1}{|c_1 |} + \frac{b_1}{|c_1 |} \left(\frac{|c_1 | a_2 + a_1 b_2}{|c_1 | |c_2 | - b_1 b_2} \right) \right) < m_2$ is not satisfied)

then we get

$$\lim_{t \rightarrow \infty} (N_1(t), N_2(t)) \in \left\{ (0, m_2), (m_1, 0), (m_1, m_2), \left(m_1, \frac{(m_1 | c_2 | -a_2)}{b_2} \right), \left(\frac{|c_1 | a_2 + a_1 b_2}{|c_1 | |c_2 | - b_1 b_2}, \frac{a_1}{|c_1 |} + \frac{b_1}{|c_1 |} \left(\frac{|c_1 | a_2 + a_1 b_2}{|c_1 | |c_2 | - b_1 b_2} \right) \right) \right\}$$

Further the only stable equilibrium outcomes are $(0, m_2)$ and $(m_1, 0)$.

Proof: Follows from Lemma 12 and the logic described in Lemma 7. QED.

Proposition 8: Under mutually negative interaction between products (i.e. $c_1 < 0$, $c_2 < 0$), with negative LHR_1 and positive LHR_2 and D lying outside the feasible region, the only stable equilibrium outcomes possible are $(0, m_2)$, $(m_1, 0)$.

Proof: Follows immediately from Lemma 12 and 13.

Appendix C. Numerical Computation of the Separatrix

Exactly two N_1N_2 trajectories pass through a saddle point, which is an unstable singular point. The paths traced out by the incoming trajectories corresponding to the negative eigenvalues are the Separatrices for the dynamical system under study. Below, we outline an algorithm to plot the Separatrices associated with a given saddle point. We also illustrate the steps for the following dynamical system corresponding to the Asymmetric Influence Model described in Section 3.1:

$$\frac{dN_1}{dt} = (0.02 + 0.002N_1 - 0.001N_2)(100 - N_1)$$

$$\frac{dN_2}{dt} = (0.03 + 0.002N_2 + 0.001N_1)(100 - N_2)$$

The complete implementation is in the attached spreadsheet.

1. Calculate the eigenvalues associated with the saddle point, (N_1^0, N_2^0) , being analyzed, using the quadratic equation described in the proof of Lemma 1:

$$\lambda^2 - \lambda[b_1m_1 - a_1 + b_2m_2 - a_2 - (c_2 + 2b_1)N_1^0 - (c_1 + 2b_2)N_2^0] + (b_1m_1 - a_1 - c_1N_2^0 - 2b_1N_1^0)(b_2m_2 - a_2 - c_2N_1^0 - 2b_2N_2^0) - c_1c_2(m_1 - N_1^0)(m_2 - N_2^0) = 0$$

For the above system, the saddle point is (40,100). The two eigenvalues are 0.12 and -0.27 and the corresponding eigenvectors are (1,0) and (0.1, 0.65).

2. Identify a ‘starting point’ which is slightly displaced from the saddle point in the direction of the eigenvector corresponding to the negative eigenvalue. Recall that for a saddle point $\lambda_1 > 0 > \lambda_2$. For the above system, we identify a starting point which is slightly displaced from (40,100) in the direction (0.1, 0.65) while ensuring we are within the feasible region. This gives us the starting point (39.9,99.35)
3. Using the discretized version of the system of equations described in (1a) and (1b), and the ‘starting point’ identified in step 2, trace out the path (N_1N_2 trajectory) obtained by moving in positive time, as well as that obtained by moving in negative time. For the former, the displacement of the starting point should be along the direction of the eigenvector obtained in step 2, while for the latter the displacement should be in the opposite direction. The path obtained is the Separatrix of interest.

The accompanying spreadsheet demonstrates the detailed implementation of these steps for the above dynamical system. MATLAB code is available as well (e.g., Polking and Arnold 2003;

<http://math.rice.edu/~polking/odesoft/>).