

Randomly diluted xy and resistor networks near the percolation threshold

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(Received 7 August 1986)

A formulation based on that of Stephen for randomly diluted systems near the percolation threshold is analyzed in detail. By careful consideration of various limiting procedures, a treatment of xy spin models and resistor networks is given which shows that previous calculations (which indicate that these systems having continuous symmetry have the same crossover exponents as the Ising model) are in error. By studying the limit wherein the energy gap goes to zero, we exhibit the mathematical mechanism which leads to qualitatively different results for xy -like as contrasted to Ising-like systems. A distinctive feature of the results is that there is an *infinite* sequence of crossover exponents needed to completely describe the probability distribution for $R(\mathbf{x}, \mathbf{x}')$, the resistance between sites \mathbf{x} and \mathbf{x}' . Because of the difference in symmetry between the xy model and the resistor network, the former has an infinite sequence of crossover exponents in addition to those of the resistor network. The first crossover exponent $\phi_1 = 1 + \epsilon/42$ governs the scaling behavior of $R(\mathbf{x}, \mathbf{x}')$ with $|\mathbf{x} - \mathbf{x}'| \equiv r$: $[R(\mathbf{x}, \mathbf{x}')]_c \sim r^{\phi_1/\nu}$, where $[\]_c$ indicates a conditional average, subject to \mathbf{x} and \mathbf{x}' being in the same cluster, ν is the correlation length exponent for percolation, and $\epsilon = 6 - d$, where d is the spatial dimensionality. We give a detailed analysis of the scaling properties of the bulk conductivity and the anomalous diffusion constant introduced by Gefen *et al.* Our results show conclusively that the Alexander-Orbach conjecture, while numerically quite accurate, is not exact, at least in high spatial dimension. We also evaluate various amplitude ratios associated with susceptibilities, χ_n involving the n th power of the resistance $R(\mathbf{x}, \mathbf{x}')$, e.g., $\lim_{p \rightarrow p_c} \chi_2 \chi_0 / \chi_1^2 = 2[1 + (19\epsilon/420)]$. In an appendix we outline how the calculation can be extended to treat the diluted m -component spin model for $m > 2$. As expected, the results for ϕ_1 remain valid for $m > 2$. The techniques described here have led to several recent calculations of various infinite families of exponents.

I. INTRODUCTION

The properties of a randomly diluted network of resistors near the percolation threshold has been the object of much study over the last decade or so.¹⁻⁷ Most attention has been given to the exponent³ t which describes how the conductivity $\Sigma(p)$ behaves for concentration, p , near the percolation threshold at $p = p_c$:

$$\Sigma(p) \sim |p - p_c|^t. \quad (1.1)$$

Early studies⁸ seemed to indicate that for spatial dimensionality, $d = 2$, the exponent t was close to unity. A heuristic argument given independently by Skal and Shklovskii² and by deGennes⁵ yielded the prediction that

$$t = (d - 2)\nu + \phi_1, \quad (1.2)$$

where ν is the correlation-length ξ exponent for percolation: $\xi(p) \sim |p - p_c|^{-\nu}$, and ϕ_1 is a crossover exponent which governs the growth of the configurationally averaged resistance $[R(r)]$ between two nodes which are separated by a distance r and which are known to be in the same cluster: $[R(r)] \sim r^{\phi_1/\nu}$. The relation (1.2) is commonly believed to be a consequence of the node-link picture,^{2,5} but in reality it is a result of scaling and hydrodynamics,⁶ and as such, should be true even though now the node-link picture has been superseded by the node-link-blobs model.⁹ One result of the present paper is to give a firm scaling argument for Eq. (1.2). The first ϵ -

expansion treatments of the resistor network were developed almost simultaneously but based on very different formalisms, one on the Potts model^{10,11} and the other on the xy model.¹² Both formalisms gave results in agreement with a general symmetry argument of Wallace and Young,¹³ which asserted that ϕ_1 was unity to all order in ϵ .

As of 1978, then, all results seemed to be in rough agreement with one another, except perhaps a series determination⁶ of ϕ_1 for $d = 2$ which gave $\phi = 1.4$, very different from unity, indeed. Since then, however, the values of ϕ_1 from numerical simulations began to increase and eventually the value $\phi_1 = 1$ for $d = 2$ became excluded.¹⁴⁻¹⁷ Also experimental data,¹⁸ and a very convincing physical argument of Coniglio,¹⁹ showed that indeed ϕ_1 was not expected to be the same for both the Ising model (where $\phi_1 = 1$ has been accepted for some time²⁰) and the Heisenberg model (which is known to be equivalent to the resistor network). It remained to resolve the final question as to why the ϵ -expansion treatments giving $\phi_1 = 1$ were in error.

As reported previously,^{21,22} we have located the difficulty in the previous ϵ -expansion treatments. Here we give the analysis based on the xy model as first presented by Stephen,¹⁴ but slightly modified here. In the following paper²³ we give the analysis based on the s -state Potts model in the limit $s \rightarrow 0$. That these two quite different looking calculations should give the same nontrivial result for an infinite family of crossover exponents (of which ϕ_1

is the first member), is a strong indication that the ϵ -expansion recursion relations are correct. Together with the second-order calculations of Lubensky and Wang,²⁴ the ϵ expansion indicates that ϕ_1 is a very weak function of d . This conclusion comports quite well with the fact that $\phi_1 = 1.3$ for $d = 2$ is not too different from unity.

Of the two formulations the present one, based on the xy model, has been by far the more fruitful. One reason for presenting the technique in as much detail as we do here, is that it has been used for a number of seemingly disparate crossover calculations for diluted systems near the percolation threshold. We mention the calculations of (a) ϕ_1 for a network having a singular distribution of resistances,²⁵ (b) various crossover exponents for a diluted network of Josephson junctions,²⁶ (c) the noise exponents (related to moments of the probability distribution of currents) in the random-resistor network,²⁷ and (d) noise and resistance exponents for the nonlinear resistor network.²⁸ It seems entirely possible that more calculations of this type will appear in the near future.

The formulation in terms of the xy model affords a direct calculation of the configurational average of arbitrary powers of the resistance, $R(\mathbf{x}, \mathbf{x}')$ between two sites \mathbf{x} and \mathbf{x}' . In fact, we will give results, both within mean-field theory and to first order in ϵ for the probability distribution governing the stochastic variable $R(\mathbf{x}, \mathbf{x}')$. As we shall see, in spatial dimensionalities d for which scaling holds, i.e., for $d < 6$, this distribution involves a scaling function in which only the exponent ϕ_1 sets the scale of resistances.^{22,29,30} We find an infinite hierarchy of exponents, $\{\phi_n\}$, of which ϕ_1 is the first member. The ϕ_n 's for $n > 1$ describe corrections to scaling in the probability distribution function for $R(\mathbf{x}, \mathbf{x}')$. The calculation of these additional exponents involves some unusual subtleties. As we discuss, the calculation of these correction-to-scaling exponents given previously^{21,22} is not the appropriate one for the random-resistor network. As this calculation may have repercussions for other problems we describe it in some detail here.

Briefly this paper is organized as follows. In Sec. II we summarize the principal results of this work. Among these are the scaling behavior of the various resistive and xy correlation functions and the macroscopic conductivity. In Sec. III we develop the field theory by the standard technique involving the Hubbard-Stratanovich transformation. Section IV is devoted to the derivation of the ϵ -expansion results for the family of crossover exponents of which ϕ_1 is the leading member. In Sec. V we give a detailed treatment of the ϵ -expansion recursion relations from which we identify the nonlinear scaling fields, and which allows us to calculate explicitly certain universal amplitude ratios of the resistive susceptibilities. Finally in Sec. VI our results are summarized. The analogous calculation for ϕ_1 for the randomly diluted m -component model is outlined in Appendix D.

II. SUMMARY OF RESULTS

A. The model systems

Both the xy model and the resistor network on a lattice with sites \mathbf{x} can be described in terms of the reduced Hamiltonian

$$H(\theta) = - \sum_{\langle \mathbf{x}, \mathbf{x}' \rangle} U[\theta(\mathbf{x}) - \theta(\mathbf{x}')] - \sum_{\mathbf{x}} h[\theta(\mathbf{x})] \quad (2.1)$$

where $\langle \mathbf{x}, \mathbf{x}' \rangle$ denotes a nearest-neighbor bond of the lattice and $\theta(\mathbf{x})$ is a continuous dynamical variable defined on some interval. For the xy -model, θ is the angle ϕ which specifies the orientation of the spin at site \mathbf{x} and is defined on the interval $[-\pi, \pi]$ and the potentials

$$U(\phi) = K \cos \phi, \quad (2.2a)$$

$$h(\phi) = h \cos \phi \quad (2.2b)$$

have periods of 2π . Here, $K = J/T$ and $h = \mathcal{H}/T$ where J is the usual exchange integral, \mathcal{H} the external magnetic field, and T the temperature. Other potentials periodic in θ , such as those introduced by Villain,³¹ could equally well be considered. For the resistor network, we replace θ by the voltage V defined on the interval $[-\infty, \infty]$, and we have

$$U(V) = -\frac{1}{2} \sigma V^2, \quad (2.3a)$$

$$h(V) = \frac{1}{2} i \omega V^2, \quad (2.3b)$$

where σ is the conductance of a bond and ω is a frequency. Canonical averages with weight $\exp(-H)$ yield the usual thermodynamic functions for the xy model and the resistive correlation functions for the resistor network. The use of Eqs. (2.1) and (2.3) for the resistor network are reviewed in Appendix A.

In the randomly diluted lattices of interest to us here, bonds \mathbf{x}, \mathbf{x}' are present with probability p and absent with probability $1-p$. The Hamiltonian H , thus, depends on the configuration c of occupied bonds. We will denote thermodynamic averages with respect to the Hamiltonian $H(\theta; c)$ associated with a configuration c by angular brackets. For example, averages of the order parameter

$$\psi_k(\mathbf{x}) = e^{ik\theta(\mathbf{x})} \quad (2.4)$$

are written

$$\langle \psi_k(\mathbf{x}) \rangle = \frac{1}{Z(c)} \int \mathcal{D}\theta(\mathbf{x}) e^{-H(\theta; c)} e^{ik\theta(\mathbf{x})}, \quad (2.5)$$

where $\mathcal{D}\theta$ indicates an integration over the set of variables $\{\theta(\mathbf{x})\}$ for all \mathbf{x} : $\mathcal{D}\theta \equiv \prod_{\mathbf{x}} \mathcal{D}\theta(\mathbf{x})$ and

$$Z(c) = \int \mathcal{D}\theta e^{-H(\theta; c)}. \quad (2.6)$$

Quenched averages over the random configurations of bonds will be denoted by square brackets $[\]_{\text{av}}$. For example, the averaged order parameter and order-parameter correlation function are, respectively,

$$M_k(\mathbf{x}) = [\langle \psi_k(\mathbf{x}) \rangle]_{\text{av}} \quad (2.7a)$$

$$= [\langle e^{ik\theta(\mathbf{x})} \rangle]_{\text{av}} = M_k, \quad (2.7b)$$

$$\chi_k(\mathbf{x}, \mathbf{x}') = [\langle \psi_k(\mathbf{x}) \psi_{-k}(\mathbf{x}') \rangle]_{\text{av}} \quad (2.8a)$$

$$= [\langle e^{ik[\theta(\mathbf{x}) - \theta(\mathbf{x}')]} \rangle]_{\text{av}} \\ = \chi_k(\mathbf{x} - \mathbf{x}'), \quad (2.8b)$$

where we used the translational invariance of averages to write the final equalities in Eqs. (2.7b) and (2.8b). When the external field $h(\theta)$ is zero, the Hamiltonian is invari-

ant with respect to a "rotation" $[\theta(\mathbf{x}) \rightarrow \theta(\mathbf{x}) + d\theta(\mathbf{x})]$, for which $d\theta(\mathbf{x})$ is uniform within each cluster of sites connected by occupied bonds. Consequently $M_{\mathbf{k}}$ and $\langle \psi_{\mathbf{k}}(\mathbf{x}) \psi_{-\mathbf{k}}(\mathbf{x}') \rangle$ vanish in zero field unless \mathbf{x} and \mathbf{x}' are part of the same cluster of sites connected by occupied bonds. Thus an expression equivalent to Eq. (2.8) is

$$\chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = [\nu(\mathbf{x}, \mathbf{x}'; c) \langle \psi_{\mathbf{k}}(\mathbf{x}) \psi_{-\mathbf{k}}(\mathbf{x}') \rangle]_{\text{av}} \quad (h=0), \quad (2.9)$$

where $\nu(\mathbf{x}, \mathbf{x}'; c)$ is an indicator variable which is unity if \mathbf{x} and \mathbf{x}' are in the same cluster for the configuration c and is zero otherwise. Equation (2.9) is useful in that it emphasizes the information about connectivity contained in $\chi_{\mathbf{k}}$.

We can also consider configurational averages of products of thermal averages of the type encountered in the study of spin glasses.³² Therefore, as generalizations of Eqs. (2.8) and (2.9) we define

$$M_{\mathbf{k}}(\mathbf{x}) = \left[\prod_{\alpha} \langle \psi_{k_{\alpha}}(\mathbf{x}) \rangle \right]_{\text{av}} \quad (2.10a)$$

$$= \left[\prod_{\alpha} \langle e^{ik_{\alpha}\theta(\mathbf{x})} \rangle \right]_{\text{av}} = M_{\mathbf{k}} \quad (2.10b)$$

and

$$\chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = \left[\prod_{\alpha} \langle \psi_{k_{\alpha}}(\mathbf{x}) \psi_{-k_{\alpha}}(\mathbf{x}') \rangle \right]_{\text{av}} \quad (2.11a)$$

$$= \left[\prod_{\alpha} \langle e^{ik_{\alpha}[\theta(\mathbf{x}) - \theta(\mathbf{x}')] } \rangle \right]_{\text{av}} \\ = \chi_{\mathbf{k}}(\mathbf{x} - \mathbf{x}'), \quad (2.11b)$$

where \mathbf{k} is a vector of arbitrary dimension n whose α th component is k_{α} . As we shall see later, quenched averages are obtained using the replica trick in which the formal limit $n \rightarrow 0$ is employed.

Before discussing our predictions regarding the functions $M_{\mathbf{k}}$ and $\chi_{\mathbf{k}}$, it is useful to discuss the type of information contained in these functions. When applied to Eq. (2.7b), the general expression for the average of an exponential in terms of cumulants (denoted $\langle \rangle_c$) yields

$$[\langle \psi_{\mathbf{k}}(\mathbf{x}) \rangle]_{\text{av}} = \exp \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} k_{\alpha}^{2n} \langle \theta^{2n}(\mathbf{x}) \rangle_c \right]_{\text{av}}, \quad (2.12)$$

where, $k^2 = \sum_{\alpha} k_{\alpha}^2$ and, for instance, the lowest non-vanishing cumulants of θ (with respect to the average $\langle \rangle$) in zero field h are

$$\langle \theta^2(\mathbf{x}) \rangle_c = \langle \theta^2(\mathbf{x}) \rangle, \quad (2.13a)$$

$$\langle \theta^4(\mathbf{x}) \rangle_c = \langle \theta^4(\mathbf{x}) \rangle - 3 \langle \theta^2(\mathbf{x}) \rangle^2. \quad (2.13b)$$

$$M_{\mathbf{k}} = |\Delta p|^{\beta} \mathcal{L}(\{w_{n\Gamma} k_{\Gamma}^{2n} / |\Delta p|^{\phi_{n\Gamma}}\}, h / |\Delta p|^{\Delta}) \quad (2.16a)$$

$$\chi_{\mathbf{k}}(\mathbf{x}) = x^{-(d-2+\eta)} \mathcal{Y}(x/\xi, \{w_{n\Gamma} k_{\Gamma}^{2n} / |\Delta p|^{\phi_{n\Gamma}}\}, h / |\Delta p|^{\Delta}), \quad (2.16b)$$

where $\xi \sim |\Delta p|^{-\nu}$ is the percolation correlation length and where β , $\Delta \equiv \beta + \gamma = d\nu - \beta$, and η are, respectively, the order-parameter, gap, and anomalous-dimension exponents for the percolation problem. Here and below cap-

ital script letters [e.g., \mathcal{L} in Eq. (2.16a) and \mathcal{Y} in Eq. (2.16b)] denote scaling functions which are universal functions of their arguments. Also $\{A\}$ as an argument indicates dependence on the family of variables whose typical

$$M_{\mathbf{k}} = [e^{-k^2 \langle V^2(\mathbf{x}) \rangle / 2}]_{\text{av}} = [e^{-k^2 R_{\infty}(\mathbf{x}) / 4}]_{\text{av}}, \quad (2.14a)$$

$$\chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = [e^{-k^2 \langle [V(\mathbf{x}) - V(\mathbf{x}')]^2 \rangle / 2}]_{\text{av}} \\ = [e^{-k^2 R(\mathbf{x}, \mathbf{x}') / 2}]_{\text{av}}, \quad (2.14b)$$

where as discussed in Appendix A, $R_{\infty}(\mathbf{x})$ is the resistance from \mathbf{x} to ∞ , and $R(\mathbf{x}, \mathbf{x}')$ is the resistance between the points \mathbf{x} and \mathbf{x}' . Since the right-hand sides of Eqs. (2.14) are the generating functions for the probability distributions for $R_{\infty}(\mathbf{x})$ and $R(\mathbf{x}, \mathbf{x}')$, these distributions (which we discuss in Sec. II C) can be calculated if $M_{\mathbf{k}}$ and $\chi_{\mathbf{k}}$ can be evaluated. Note that Eq. (2.14b) implies that $\prod_{\alpha} \langle e^{ik_{\alpha} V(\mathbf{x})} e^{-ik_{\alpha} V(\mathbf{x}')} \rangle$ vanishes if \mathbf{x} and \mathbf{x}' are not in the same cluster, because in that case, $R(\mathbf{x}, \mathbf{x}')$ is infinite.

In contrast, the Hamiltonian for the xy model is non-Gaussian. Then cumulants of arbitrarily high order are nonzero and contribute to Eq. (2.12). For the xy model we therefore write

$$M_{\mathbf{k}} = \left[\exp \left[\sum_{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} k_{\alpha}^{2n} \langle \phi^{2n}(\mathbf{x}) \rangle_c \right] \right]_{\text{av}}. \quad (2.15)$$

Thus $M_{\mathbf{k}}$ is a function of all of the hypercubic invariants of \mathbf{k} rather than of k^2 only (as was the case for the resistor network). In particular, $M_{\mathbf{k}}$ depends on $\sum_{\alpha} k_{\alpha}^4$ as well as $k^4 = (\sum_{\alpha} k_{\alpha}^2)^2$.

B. Crossover and scaling

Our primary concern in this paper is the behavior of averaged order parameters and correlation functions in the vicinity of the percolation threshold and $T=0$ or $\sigma^{-1}=0$. This behavior is characterized by critical exponents and scaling functions. An important result of our analysis is that there are an infinite number of crossover exponents ϕ_n which become unity at the upper critical dimension $d_c=6$. The first of these, ϕ_1 is related to the macroscopic conductivity of the random network or the spin-wave stiffness of the xy model. The other exponents represent corrections to scaling²² and are not trivially related to observables that we have been able to identify.

We find that near the percolation threshold the order-parameter and correlation functions have the following scaling form:

member is A . In the resistor network problem, the external field $h = i\omega$ describes the finite frequency ω response when there are unit capacitances to ground at each site. The index Γ in $w_{n\Gamma}$ refers to symmetry of the $2n$ th-order polynomial in k with which it is associated. For the isotropic combinations we will omit the subscript Γ . Also we point out that w_n is proportional to T^n for the xy model and to σ^{-n} for the resistor network. No anisotropic potentials appear for the resistor network.

Note that there is an independent exponent for each n and Γ . We have calculated the isotropic exponents to first order in $\epsilon = 6 - d$ and find

$$\phi_n = 1 + c_n \epsilon / 14, \quad (2.17)$$

where $c_1 = -1/3$, $c_2 = 0$, and $c_3 = -1/35$. A general expression for c_n is given in Eq. (4.29b). In addition, we have calculated the exponent associated with the cubic invariant $\sum_{\alpha} k_{\alpha}^4$:

$$\phi_{2c} = 1 + \epsilon / 105. \quad (2.18)$$

We have not calculated exponents associated with higher-order nonspherical invariants, though there is no reason, in principle, why they cannot be calculated.

It is worth discussing some of the implications of Eq. (2.16). The most obvious implications concern the low-temperature properties of the magnetization M and the susceptibility χ of the xy model. Expanding the appropriate parts of Eq. (2.16), we find

$$\begin{aligned} M &= \text{Re}[\langle \psi_1(\mathbf{x}) \rangle]_{\text{av}} \\ &= P(p)(1 + a_1 T |\Delta p|^{-\phi_1} + b_1 T^2 |\Delta p|^{-2\phi_1} \\ &\quad + a_2 T^2 |\Delta p|^{-\phi_2} + a_{2c} T^3 |\Delta p|^{-\phi_{2c}} + \dots), \end{aligned} \quad (2.19a)$$

$$\begin{aligned} \chi T &= \sum_{\mathbf{x}} \chi_1(\mathbf{x}) \\ &= S(p)(1 + e_1 T |\Delta p|^{-\phi_1} + f_1 T^2 |\Delta p|^{-2\phi_1} \\ &\quad + e_2 T^2 |\Delta p|^{-\phi_2} + e_{2c} T^3 |\Delta p|^{-\phi_{2c}} + \dots), \end{aligned} \quad (2.19b)$$

where

$$P(p) \sim |\Delta p|^{\beta}, \quad S(p) \sim |\Delta p|^{-\gamma} \quad (2.20)$$

are respectively the probability that a site is in the infinite cluster and the mean-square cluster size. The first terms in these expressions can easily be identified with the leading corrections from thermally excited spin excitations (spin waves when there is an infinite cluster). The higher-order terms are more difficult to characterize.

The expansion in Eq. (2.19) is valid for $T \ll |\Delta p|^{\phi_1}$. For $T \sim |\Delta p|^{\phi_1}$, near the paramagnetic-ferromagnetic phase boundary for example, it is appropriate to express the scaling relations in Eq. (2.16) so that T appears in one place

$$\begin{aligned} M_{\mathbf{k}} &= |\Delta p|^{\beta} \tilde{\mathcal{F}}(Tk^2 / |\Delta p|^{\phi_1}, \{ |\Delta p|^{n\phi_1 - \phi_n} \}, \\ &\quad |\Delta p|^{3\phi_1 - \phi_{2c}}, \dots, h / |\Delta p|^{\Delta}). \end{aligned} \quad (2.21)$$

From this we conclude that the paramagnetic-ferromagnetic transition temperature satisfies

$$T_c(p) \sim |\Delta p|^{\phi_1} \quad (2.22)$$

near p_c . Since $\phi_1 > 1$, this implies that the phase boundary approaches zero with zero slope as shown in Fig. 1.

We now turn to the implications of Eq. (2.16) for the resistor network. An important consequence of the analysis of the next section is that the voltage variable of the resistor network is merely a component of the percolation order parameter to which ω couples linearly, as in Eq. (3.22), below. This allows us to conclude that the voltage-voltage correlation function satisfies a scaling relation of the form

$$\begin{aligned} g(\mathbf{q}, \omega) &\equiv \sum_{\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}} [\langle V(\mathbf{x})V(0) \rangle]_{\text{av}} \\ &= |\Delta p|^{-\gamma} \mathcal{Z}(q\xi, \sigma / |\Delta p|^{\phi_1}, \{ |\Delta p|^{n\phi_1 - \phi_n} \}, \\ &\quad i\omega / |\Delta p|^{-\Delta}), \end{aligned} \quad (2.23)$$

where, as in our discussion of $T_c(p)$, we have chosen variables so that σ appears only in one place. An additional scaling feature of the voltage correlation function is evident from the Gaussian form of the voltage Hamiltonian, Eqs. (2.1) to (2.3). The transformation $V \rightarrow b^{-1/2}V$ implies transformations $\omega \rightarrow b^{-1}\omega$ and $\sigma \rightarrow b^{-1}\sigma$. These invariances are a manifestation of the fact that the scale of time can be chosen arbitrarily. They imply the following scaling law for g :

$$g(\mathbf{q}, \omega, \sigma) = b^{-1} g(\mathbf{q}, b^{-1}\omega, b^{-1}\sigma), \quad (2.24)$$

or, with the choice $b = \sigma$, that g is a function of ω/σ only,

$$g(\mathbf{q}, \omega, \sigma) = \sigma^{-1} f(\mathbf{q}, \omega/\sigma). \quad (2.25)$$

Combining this with the previous scaling expression, we obtain

$$\begin{aligned} g(\mathbf{q}, \omega) &= \sigma^{-1} |\Delta p|^{-\gamma - \phi_1} \tilde{\mathcal{F}}(q\xi, (\omega/\sigma) / |\Delta p|^{-\Delta - \phi_1}, \\ &\quad \{ |\Delta p|^{n\phi_1 - \phi_n} \}), \end{aligned} \quad (2.26)$$

where $\tilde{\mathcal{F}}$ is some scaling function. Since $n\phi_1 - \phi_n$ is of order $n - 1$ for $n > 1$, one sees that the variables involving these exponents represent corrections to the dominant scaling behavior. The macroscopic conductivity can be obtained from $g(\mathbf{q}, \omega)$ via the Kubo formula¹²

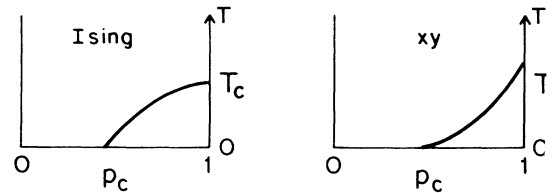


FIG. 1. Phase diagram for the dilute Heisenberg or xy model (right) contrasted to that for the Ising model (left).

$$\Sigma = \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \frac{\omega^2}{q^2} \text{Reg}(\mathbf{q}, \omega). \quad (2.27)$$

To obtain a finite nonzero result for Σ one sees that the dominant behavior of $\tilde{\mathcal{F}}$ must be

$$\begin{aligned} \tilde{\mathcal{F}}(q\xi, (\omega/\sigma) |\Delta p|^{-\Delta-\phi_1}, \{|\Delta p|^{n\phi_1-\phi_n}\}) \\ \sim (q\xi)^2 [(\omega/\sigma) |\Delta p|^{-\Delta-\phi_1}]^{-2}, \end{aligned} \quad (2.28)$$

in which case we find that

$$\begin{aligned} \Sigma \sim \sigma \xi^2 |\Delta p|^{2\Delta+\phi_1-\gamma} \sim \sigma |\Delta p|^{(d-2)\nu+\phi_1} \\ \equiv \sigma |\Delta p|^t, \end{aligned} \quad (2.29)$$

where the last equality defines the conductivity exponent,³ t . We thereby obtain the familiar equation^{2,5}

$$t = (d-2)\nu + \phi_1, \quad (2.30)$$

for the conductivity exponent. Note that the required linear dependence of Σ on σ emerges as a natural result of our scaling analysis and does not need to be “put in by hand.” It is worth emphasizing that although the relation (2.30) is commonly derived using the node-links model,^{2,5} it in fact has a firmer basis⁶ within a hydrodynamic treatment of spin-waves,³³ which in essence is the physical content of the present derivation.

When the correction-to-scaling terms proportional to $|\Delta p|^{n\phi_1-\phi_n}$ are neglected, Eq. (2.26) leads to the scaling forms discussed by Gefen *et al.*³⁴ for the frequency dependent conductivity, average distance traveled by a diffusing particle and diffusion constant, as we now verify. Using Eq. (2.30), we obtain

$$\begin{aligned} \Delta + \phi_1 &= (t - \beta) + 2\nu \\ &= (2 + \theta)\nu, \end{aligned} \quad (2.31)$$

where

$$\theta = (t - \beta) / \nu \quad (2.32)$$

is the diffusion constant exponent introduced by Gefen *et al.*³⁴ Using Eq. (2.31) and extending the definition of Σ to finite frequency and wave number, we obtain

$$\Sigma(\mathbf{q}, \omega) = \sigma |\Delta p|^t \mathcal{L}(q\xi, (\omega/\sigma)\xi^{2+\theta}) \quad (2.33)$$

in agreement with Refs. 34 to 36. The voltage satisfies the same diffusion equation as the probability that a diffusing particle is at position \mathbf{x} at time τ . Thus,

$$g(\mathbf{x}, \tau) = \int \frac{d^2q}{(2\pi)^d} e^{i\mathbf{q}\cdot\mathbf{x}} e^{-i\omega\tau} g(\mathbf{q}, \omega) \quad (2.34)$$

is the probability averaged over clusters that a diffusing particle is at \mathbf{x} at time τ given that it was at the origin at time zero. The average squared displacement then satisfies

$$\begin{aligned} \langle r^2(\tau) \rangle &= \int d^d\mathbf{x} x^2 g(\mathbf{x}, \tau) = - \frac{d}{dq^2} g(\mathbf{q}, \tau) \Big|_{q=0} \\ &= \xi^2 F(\sigma\tau\xi^{-(2+\theta)}). \end{aligned} \quad (2.35)$$

C. Probability distribution for the resistance

A useful characterization of the properties of a random-resistor network can be obtained via the probability $P(R, \mathbf{x} - \mathbf{x}') dR$ that two sites \mathbf{x} and \mathbf{x}' be in the same cluster and have a resistance in the interval $[R, R + dR]$. The distribution function $P(R, \mathbf{x} - \mathbf{x}')$ can be expressed as

$$P(R, \mathbf{x} - \mathbf{x}') = [v(\mathbf{x}, \mathbf{x}'; c) \delta(R - R(\mathbf{x}, \mathbf{x}'; c))]_{av}. \quad (2.36)$$

This quantity might be measured as follows: Fix the two probes of an ohmmeter to have a separation \mathbf{x} . Measure the resistance between all pairs of lattice points separated by \mathbf{x} , and prepare a histogram of the values of the measured resistance. On occasion, the resistance will be infinite indicating that the two probes of the ohmmeter are in different clusters. These measurements should be placed in a separate bin of the histogram as shown in Fig. 2. Then $P(R, \mathbf{x}) dR$ is the number of points in the histogram between R and $R + dR$ divided by the total number of measurements made. A related and sometimes more useful distribution is the conditional probability $P_c(R, \mathbf{x} - \mathbf{x}')$ that the resistance between \mathbf{x} and \mathbf{x}' is R given that \mathbf{x} and \mathbf{x}' are in the same cluster. This is simply

$$\begin{aligned} P_c(R, \mathbf{x} - \mathbf{x}') &= \frac{[v(\mathbf{x}, \mathbf{x}'; c) \delta(R - R(\mathbf{x}, \mathbf{x}'; c))]_{av}}{[v(\mathbf{x}, \mathbf{x}'; c)]_{av}} \\ &\equiv \frac{P(R, \mathbf{x} - \mathbf{x}')}{[v(\mathbf{x}, \mathbf{x}'; c)]_{av}}. \end{aligned} \quad (2.37)$$

$P_c(R, \mathbf{x})$ can be obtained from the histogram above by dividing the number of measurements with resistance between R and $R + dR$ by the total number of measurements with finite resistance.

From the above we see that $\chi_k(\mathbf{x})$ (evaluated at zero field, h) is the Laplace transform with respect to $k^2/2$ of the distribution function $P(R, \mathbf{x})$:

$$\chi_k(\mathbf{x}) \Big|_{k^2=2\lambda} = \int_0^\infty e^{-\lambda R} P(R, \mathbf{x}). \quad (2.38a)$$

Thus $P(R, \mathbf{x})$ can be obtained from $\chi_k(\mathbf{x})$ by the inverse transform,

$$\begin{aligned} P(R, \mathbf{x}) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda e^{\lambda R} \chi_k(\mathbf{x}) \Big|_{k^2=2\lambda} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda e^{\lambda R} [e^{-\lambda R(\mathbf{x}, \mathbf{x}')}]_{av}. \end{aligned} \quad (2.38b)$$

Thus, using the scaling relations of Eq. (2.16), one can predict the scaling form of $P(R, \mathbf{x})$:

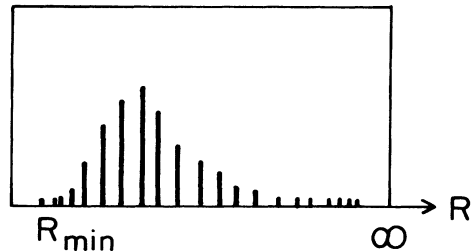


FIG. 2. Schematic representation of a histogram of resistances, $R(\mathbf{x}, \mathbf{x} + \mathbf{s})$, found by an ohmmeter with its terminals at \mathbf{x} and $\mathbf{x} + \mathbf{s}$ for an ensemble of values of \mathbf{x} for fixed separation \mathbf{s} in a random sample.

$$P(R, \mathbf{x}) = \frac{1}{R} x^{-(d-2+\eta)} \mathcal{L}(\sigma R x^{-\phi_1/\nu}, \{(\sigma R)^{-n+\phi_n/\phi_1}\}, x/\xi) \tag{2.39a}$$

$$\equiv \text{const} \times x^{-(d-2+\eta)} P_c(R, \mathbf{x}), \tag{2.39b}$$

where $x \equiv |\mathbf{x}|$. Several results are evident from Eq. (2.39). The first involves the conditional moments of R ,

$$\{R^n(\mathbf{x})\} \equiv \int dR R^n P_c(R, \mathbf{x}). \tag{2.40}$$

When these moments exist [as they will for $n < (d-2+\eta)\nu/\phi_1$, as indicated by Eq. (2.47), below], they satisfy

$$\{R^n(\mathbf{x})\} \sim \sigma^{-n} x^{n\phi_1/\nu} [1 + O(x^{-(2\phi_1-\phi_2)/\nu})]. \tag{2.41}$$

at $p = p_c$. For $d > 6$, $\phi_1 = 1$ and $\nu = 1/2$, and this equation says that the average resistance between two sites in the same cluster is proportional to x^2 . The interpretation is that the dominant paths between points separated by \mathbf{x} are random walks of x^2 steps. Since each step carries a resistance σ^{-1} and resistances add in series, this gives a resistance proportional to x^2 . The resistive susceptibility introduced by Fisch and Harris⁶ is merely the first moment of R with respect to P :

$$\begin{aligned} \chi^{(1)}(\mathbf{x}, \mathbf{x}') &\equiv [\nu(\mathbf{x}, \mathbf{x}'; c) R(\mathbf{x}, \mathbf{x}'; c)]_{\text{av}} = \int dR R P(R, \mathbf{x} - \mathbf{x}') \\ &\sim |\mathbf{x} - \mathbf{x}'|^{-(d-2+\eta)+\phi_1/\nu} f(x/\xi, \{|\Delta p|^{-n+\phi_n/\phi_1}\}). \end{aligned} \tag{2.42}$$

We can obtain higher-order resistive susceptibilities corresponding to averages of higher moments of the two-point resistance by differentiation with respect to the parameter λ in Eq. (2.38a):

$$\chi^{(l)}(\mathbf{x}, \mathbf{x}') \equiv [\nu(\mathbf{x}, \mathbf{x}'; c) R^l(\mathbf{x}, \mathbf{x}'; c)]_{\text{av}} \tag{2.43a}$$

$$= \left[\frac{\partial^l}{\partial (-k^2/2)^l} \right] G_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') \Big|_{k^2=0}. \tag{2.43b}$$

In series work^{6,37-40} it is convenient to evaluate the zero wave-vector component of the spatial Fourier transform of $\chi^{(n)}$, which we write as

$$\chi_n(p) \equiv \sum_{\mathbf{x}} \chi^{(n)}(\mathbf{x}, \mathbf{x}'), \tag{2.44}$$

and it is easily established that this quantity diverges as

$$\chi_n(p) \sim |p - p_c|^{-\gamma - n\phi_1}. \tag{2.45}$$

Of some interest are various universal amplitude ratios, which can be expressed in terms of the quantities ρ_s , defined as

$$\rho_s \equiv \chi_0^{s-1}(p) \chi_s(p) / \chi_1^s(p), \quad p \rightarrow p_c. \tag{2.46}$$

To order ϵ we find

$$\rho_2 = 2 \left[1 - \frac{19\epsilon}{420} \right]$$

and

$$\rho_3 = 6 \left[1 - \frac{601\epsilon}{5880} \right].$$

Results for arbitrary s can in principle be calculated.

We now discuss the behavior of $P(R, \mathbf{x})$ as a function of R . The large- R behavior of $P(R, x)$ can be obtained with the aid of the observation that for $\sigma R \gg x^{\phi_1/\nu}$, and

$\sigma R \gg 1$, the probability of having a resistance R should be independent of x . Thus Eq. (2.39) yields

$$P(R, x) \sim R^{-1-(d-2+\eta)\nu/\phi_1}, \tag{2.47}$$

indicating that there is a long tail in the distribution at large R as shown in Fig. 3. For $d > 6$, $\nu = 1/2$, $\phi_1 = 1$, and $\eta = 0$, so that

$$P(R, x) \sim R^{-d/2}. \tag{2.48}$$

This result can again be interpreted in terms of random walks. The probability of having a resistance $\sigma R \gg x^2$ is merely the probability that a walk will return to the origin

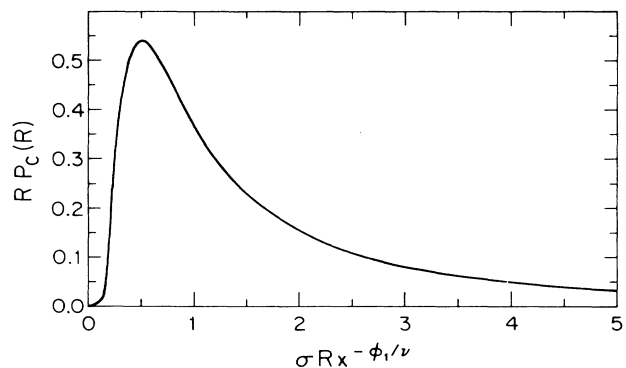


FIG. 3. Schematic representation of $RP_c(R, \mathbf{x})$ for $x \gg 1$ and $\sigma R/x^{\phi_1/\nu}$ bounded away from zero. To draw this figure we used the mean-field result of Eq. (2.52) for $d = 6$, arbitrarily setting $a = 4$. The lower cutoff which must occur when R is equal to its value for separation x on a pure lattice is not shown, as it corresponds to an infinitesimally small value of $\sigma R/x^{\phi_1/\nu}$.

after R steps.

The opposite limit of $\sigma R \ll x^{\phi_1/\nu}$ (but $\sigma R \gg 1$, still) is more difficult to discuss. For a fixed separation x we know that the minimum possible resistance is that for a pure lattice, $R_{\min}(x)$. The behavior of $P(R, x)$ for $R \sim R_{\min}$ requires an evaluation of $\chi_k(\mathbf{x})$ for general \mathbf{k}

and \mathbf{x} and is not considered here. Within mean-field theory it is straightforward to calculate $P(R, x)$ for $(\sigma R/x^2) \gg \delta$ and $x \gg 1$, where δ is a fixed nonzero number. (This restriction eliminates the regime where $R \sim R_{\min}$.) With Eqs. (3.28a) and 3.27 for χ_k , Eq. (2.38b) becomes

$$P(R, \mathbf{x}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda e^{\lambda R} \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{q^2 + a\lambda/\sigma + b(\lambda/\sigma)^2 + \dots}, \quad (2.49)$$

where a and b are unimportant constants and where higher-order terms in (λ/σ) have been neglected. The changes of variables $\lambda = y/R$ and $\mathbf{k} = x\mathbf{q}$ lead to

$$P(R, \mathbf{x}) = \frac{1}{R x^{d-2}} \int_{c-i\infty}^{c+i\infty} \frac{dy}{2\pi i} e^y \int_0^\infty \frac{d^d k}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{e}}}{k^2 + a(yx^2/\sigma R) + b x^{-2}(yx^2/\sigma R)^2 + \dots}, \quad (2.50)$$

where $\mathbf{e} = \mathbf{x}/|\mathbf{x}|$. Thus for $\sigma R/x^2$ greater than some fixed number δ and $x^2 \gg 1$, the terms of order y^2 and higher in the denominator can be neglected. The integral over y and the subsequent integral over \mathbf{k} are easily carried out yielding

$$P(R, \mathbf{x}) \sim \frac{\sigma}{a} \left[\frac{a}{\sigma R} \right]^{d/2} e^{-ax^2/(4\sigma R)}. \quad (2.51)$$

Since $[\nu(0, \mathbf{x})]_{\text{av}} \sim x^{2-d}$ in mean field-theory, we may use this result to express the conditional probability distribution as

$$P_c(R, \mathbf{x}) = \frac{1}{R} \frac{1}{[(d-4)/2]!} \left[\frac{ax^2}{4\sigma R} \right]^{(d-2)/2} \exp[-ax^2/(4\sigma R)], \quad (2.52)$$

where we used the normalization to fix some of the constants. Thus, the mean-field result indicates that $P(R, \mathbf{x})$ is a peaked function of the scaling variable $\sigma R/x^2$ as shown in Fig. 3 and in accordance with the more general result of Eq. (2.39). The regime where σR is of order unity for large x arises from clusters having many parallel paths. To treat this regime in mean-field theory, it is

necessary to include all powers of λ in the denominator of Eq. (2.49). In the critical theory, the contribution in Eq. (2.39) involving all crossover exponents ϕ_n are presumably of equal importance though we have not done any explicit calculations to verify this.

III. FIELD THEORY

In this section, we will derive the field theory for the quenched averages for the randomly diluted xy and resistor network models introduced in the preceding section. Much of this derivation is similar to that presented by Stephen¹² in his discussion of the resistor network. However, in order to facilitate the calculation of critical exponents in an ϵ expansion to be presented in the next section, we will be somewhat more precise about the limiting processes than was Stephen.

We begin with a brief review of how the replica procedure is used to generate quenched random averages. In its simplest form, the replica procedure involves the introduction of factors Z^n (which become unity when $n \rightarrow 0$) into functions to be configurationally averaged. For example, the correlation function introduced in the preceding section can be written as

$$\chi_k(\mathbf{x}, \mathbf{x}') = \lim_{n \rightarrow 0} [Z(c)^{n-1} \int \mathcal{D}\theta e^{-H(\theta, c)} \psi_k(\mathbf{x}) \psi_{-k}(\mathbf{x}')]_{\text{av}} / [Z(c)^n]_{\text{av}} \quad (3.1a)$$

$$= \lim_{n \rightarrow 0} \left[\left\{ \int \mathcal{D}\theta e^{-H(\theta, c)} \right\}^{n-1} \int \mathcal{D}\theta e^{-H(\theta, c)} \psi_k(\mathbf{x}) \psi_{-k}(\mathbf{x}') \right]_{\text{av}} / [Z(c)^n]_{\text{av}}. \quad (3.1b)$$

The averages in Eq. (3.1) can be interpreted as thermal averages with respect to a replica Hamiltonian H_n defined so as to satisfy

$$\exp(-H_n) \equiv \left[\exp \left[- \sum_{\alpha} H(\theta_{\alpha}, c) \right] \right]_{\text{av}}. \quad (3.2)$$

Then, introducing replicated variables θ_{α} , $\alpha = 1, \dots, n$ and setting

$$\psi_k(\mathbf{x}) = e^{i\mathbf{k}\cdot\theta} \quad (3.3)$$

where $\mathbf{k}\cdot\theta = \sum_{\alpha=1}^n k_{\alpha} \theta_{\alpha}$, we may generalize Eq. (3.1) as

$$\chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = \lim_{n \rightarrow 0} \int \mathcal{D}\theta e^{-H_n} \psi_{\mathbf{k}}(\mathbf{x}) \psi_{-\mathbf{k}}(\mathbf{x}') / \int \mathcal{D}\theta e^{-H_n}, \quad (3.4a)$$

where $\mathcal{D}\theta = \prod_{\alpha} \mathcal{D}\theta_{\alpha}$. For notational simplicity we will henceforth write averages with respect to the replica Hamiltonian, like that in Eq. (3.4a), as

$$\chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = \langle \psi_{\mathbf{k}}(\mathbf{x}) \psi_{-\mathbf{k}}(\mathbf{x}') \rangle_{H_n}, \quad (3.4b)$$

with the limit $n \rightarrow 0$ left implicit. The important step, $\lim_{n \rightarrow 0} Z^n(c) = 1$, of the replica procedure only works if $Z(c)$ is finite and positive. In the xy model, the partition function $Z(c)$ for a configuration c is simply $e^{Nf(c)}$, where N is the number of sites and $f(c)$ is the free energy per site which is bounded. Thus, one satisfies the boundedness condition by taking first the limit $n \rightarrow 0$ followed by $N \rightarrow \infty$. For the resistor network, however, the partition function is infinite in the absence of the external fields (capacitors to ground) because the integral over the mode in each cluster for which the voltage is the same at all sites in the cluster diverges. We will circumvent this problem by restricting the values of the voltage to lie in the interval $[-\theta_M, \theta_M]$ and allow $\theta_M \rightarrow \infty$ after the limit $n \rightarrow 0$ is taken. In this way, $\lim_{\theta_M \rightarrow \infty} \lim_{n \rightarrow 0} Z^n = 1$ as required. As usual, the limit $N \rightarrow \infty$ is to be taken last of all.

We should point out that problems with the replica procedure can arise for the $s = 0$ state Potts model which also provides a description of the resistor network. Here again $Z(c) \rightarrow 0$ for $s \rightarrow 0$. In this case, the proper procedure is to take the limit $n \rightarrow 0$ before letting $s \rightarrow 0$ as discussed in Refs. 10, 11, and 23.

Although one can in principle study the diluted xy model directly in terms of replicas, we find it convenient, in order to make contact with the Potts model formulation of percolation,⁴¹ to discretize the values of θ . To this end, we allow θ to take on $s = 2M + 1$ discrete values on the interval $[-\theta_M, \theta_M]$. For the xy model, $\theta_M = \pi$ whereas for the resistor network, $\theta_M \rightarrow \infty$ after $n \rightarrow 0$. The interval between successive values of θ is

$$\Delta\theta = \theta_M / M. \quad (3.5)$$

The continuum limit is obtained by letting $\Delta\theta \rightarrow 0$ (or $s \rightarrow \infty$). For the resistor network, one can let θ_M and $s \rightarrow \infty$ simultaneously by setting $M = m^2$, $\theta_M = V_0 m$ and taking the limit $m \rightarrow \infty$. Conjugate to θ is the variable

$$k = \frac{2\pi}{s\Delta\theta} q, \quad (3.6)$$

where q is an integer that can take on the s values in the interval $[-M, M]$. The usual completeness and orthogonality relations apply to θ and k :

$$\sum_{\theta} \frac{\Delta k \Delta \theta}{2\pi} e^{ik\theta} = \delta_{k,0}, \quad \sum_k \frac{\Delta k \Delta \theta}{2\pi} e^{-ik\theta} = \delta_{\theta,0}, \quad (3.7)$$

where we see that $\Delta k = 2\pi / (s\Delta\theta)$ from Eq. (3.6). Since k takes on s different values, there are $s^n - 1$ distinct fields $\psi_{\mathbf{k}}$ provided the trivial component with $\mathbf{k} = 0$ is excluded.

Using the relations Eq. (3.7), H_n can be expressed as a quadratic form in $\psi_{\mathbf{k}}$. First we perform the average over

the diluted lattice to obtain

$$H_n = - \sum_{\langle \mathbf{x}, \mathbf{x}' \rangle} \ln \left[1 + v \exp \left[\sum_{\alpha} U[\theta_{\alpha}(\mathbf{x}) - \theta_{\alpha}(\mathbf{x}')] \right] \right]_{\text{av}} - \sum_{\alpha} h[\theta_{\alpha}(\mathbf{x})] - N_B \ln(1-p), \quad (3.8)$$

where $v \equiv p / (1-p)$ and N_B is the number of bonds in the undiluted lattice. The constant term $N_B \ln(1-p)$ will be dropped in subsequent discussions. Using Eq. (3.7) to Fourier transform H_n , we obtain

$$H_n = - \frac{1}{2} \sum_{\mathbf{x}, \mathbf{x}' \neq 0} Dk B_{\mathbf{k}} \gamma_{\mathbf{x}, \mathbf{x}'} \psi_{\mathbf{k}}(\mathbf{x}) \psi_{-\mathbf{k}}(\mathbf{x}') - \sum_{\mathbf{x}, \mathbf{k} \neq 0} Dk h_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{x}), \quad (3.9)$$

where $Dk = (\Delta k / 2\pi)^n$ and $\gamma_{\mathbf{x}, \mathbf{x}'}$ is defined to be unity if \mathbf{x} and \mathbf{x}' are nearest neighbors and zero otherwise. Note that $\mathbf{k} = 0$ is excluded from the sums because $\psi_{\mathbf{k}=0}$ is trivially unity. The coefficients $B_{\mathbf{k}}$ and $h_{\mathbf{k}}$ are easily calculated:

$$B_{\mathbf{k}} = \sum_{\theta} (\Delta\theta)^n e^{-ik\theta} \ln \left[1 + v \exp \left[\sum_{\alpha} U(\theta_{\alpha}) \right] \right]_{\text{av}} = \sum_l \frac{(-1)^{l+1}}{l} v^l F_l(\mathbf{k}), \quad (3.10)$$

where

$$F_l(\mathbf{k}) = \sum_{\theta} (\Delta\theta)^n e^{-ik\theta} \exp \left[l \sum_{\alpha} U(\theta_{\alpha}) \right]. \quad (3.11)$$

Similarly,

$$h_{\mathbf{k}} = \sum_{\theta} (\Delta\theta)^n e^{-ik\theta} \sum_{\alpha} h(\theta_{\alpha}). \quad (3.12)$$

In the continuum limit, the sums can be replaced by integrals. In the case of the resistor network, $F_l(\mathbf{k})$ is simply a Gaussian integral,

$$F_l(\mathbf{k}) = \lim_{n \rightarrow 0} \int_{-\infty}^{\infty} \frac{d^n V}{(2\pi)^n} e^{-ik \cdot V} e^{-l\sigma V^2/2} = e^{-k^2/(2l\sigma)}. \quad (3.13)$$

For the xy model, $U(\theta)$ is not quadratic in θ , and $F_l(\mathbf{k})$ will not be a function of k^2 only. In the large- K (i.e., low-temperature: $K = J/T$) limit,

$$F_l(\mathbf{k}) = \lim_{n \rightarrow 0} \int_{-\pi}^{\pi} \frac{d^n \theta}{(2\pi)^n} e^{-ik \cdot \theta} e^{IK \sum_{\alpha} \cos \theta_{\alpha}} \rightarrow e^{-k^2/(2lK)} \exp \left[\sum_{j=2}^{\infty} a_j \left[\sum_{\alpha} k_{\alpha}^{2j} \right] \right] \quad (3.14)$$

where $a_2 = (IK)^{-3}/4!$, and in general $a_j \sim (IK)^{-(2j-1)}$. Equation (3.14) gives the coefficient, correct to leading order in powers of $(IK)^{-1}$, of each of the invariant combinations of k_{α}^2 in an expansion of $F_l(\mathbf{k})$ in powers of k_{α}^2 . Note that the coefficient of k^4 in the above expansion is of order $(IK)^{-2}$ whereas the coefficient of $\sum_{\alpha} k_{\alpha}^4$ is of order $(IK)^{-3}$. In view of Eqs. (3.13) and (3.14), $B_{\mathbf{k}}$ can be expanded in a power series in K^{-1} (or, for the resistor network, σ^{-1}) as

$$B_k = \ln(1-p)^{-1} + \sum_{j=1}^{\infty} b_j k^{2j} + b_{2c} \sum_{\alpha} k_{\alpha}^4 + \dots, \quad (3.15)$$

where b_{2c} is zero for the resistor network and is proportional to K^{-3} in the xy model and where

$$b_j = (-1)^j K^{-j} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^{j+1}} v^l, \quad (3.16a)$$

for the xy model and for the resistor network

$$b_j = (-1)^j \sigma^{-j} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^{j+1}} v^l, \quad (3.16b)$$

Thus when $K^{-1} = \sigma^{-1} = 0$, $B_k = \ln(1-p)^{-1}$ is independent of \mathbf{k} and H_n reduces to the s^n -state Potts model which describes percolation in the $n \rightarrow 0$ limit:

$$H_n(K^{-1}=0, h=0) = - \sum_{\langle \mathbf{x}, \mathbf{x}' \rangle} [\ln(1-p)^{-1}] (2\theta_M)^{-n} \times (s^n \delta_{\theta(\mathbf{x}), \theta(\mathbf{x}')} - 1). \quad (3.17)$$

This clearly shows that by taking the limit $n \rightarrow 0$ before $s \rightarrow \infty$ we will regain percolation at $K^{-1} = 0$ ($\sigma = \infty$).

As pointed out by Stephen,¹² it is on occasion useful to consider not the order parameter ψ_k but its Fourier transform,

$$\begin{aligned} \psi_w(\mathbf{x}) &\equiv \sum_{\mathbf{k} \neq 0} Dk \psi_k e^{-i\mathbf{k} \cdot \mathbf{w}}, \\ &= (\Delta\theta)^{-n} \delta_{\mathbf{w}, \theta} - (2\theta_M)^{-n}. \end{aligned} \quad (3.18)$$

Since the term with $\mathbf{k} = 0$ is excluded from the sum in Eq. (3.18), the Fourier transformed order parameter satisfies

$$\sum_{\mathbf{w}} (\Delta\theta)^n \psi_w(\mathbf{x}) = 0 \quad (3.19)$$

provided the limit $n \rightarrow 0$ is taken before the limit $\theta_M \rightarrow \infty$. Note that ψ_w is a form of the Potts-model order parameter commonly encountered in the literature.^{42,43} When

$$\Xi = [Z^n] = \int D\phi_k(\mathbf{x}) e^{-L[\phi_k(\mathbf{x})]} \exp\left(-\frac{1}{2} \sum Dk h_k B_k^{-1}(\mathbf{x}) \gamma_{\mathbf{x}, \mathbf{x}'}^{-1} h_{-\mathbf{k}}(\mathbf{x}')\right), \quad (3.23)$$

where $\gamma_{\mathbf{x}, \mathbf{x}'}^{-1}$ is the matrix inverse of $\gamma_{\mathbf{x}, \mathbf{x}'}$, $D\phi_k(\mathbf{x}) = \prod_{\mathbf{k}, \mathbf{x}} d\phi_k(\mathbf{x})$, and

$$L[\phi_k(\mathbf{x})] = \frac{1}{2} \sum_{\mathbf{x}, \mathbf{x}', \mathbf{k}} Dk B_k^{-1} \gamma_{\mathbf{x}, \mathbf{x}'}^{-1} \phi_k(\mathbf{x}) \phi_{-\mathbf{k}}(\mathbf{x}') - \sum_{\mathbf{x}, \mathbf{k}} Dk \phi_k(\mathbf{x}) B_k^{-1} \gamma_{\mathbf{x}, \mathbf{x}'}^{-1} h_{-\mathbf{k}}(\mathbf{x}') - \sum_{\mathbf{x}} \ln \left[\int \mathcal{D}\theta \exp \left[\sum_{\mathbf{k}} Dk \phi_k(\mathbf{x}) \psi_{-\mathbf{k}}(\mathbf{x}) \right] \right]. \quad (3.24)$$

From this, it is easy to verify that

$$\langle \psi_k(\mathbf{x}) \rangle = B_k^{-1} \sum_y \gamma_{\mathbf{x}, \mathbf{y}}^{-1} [\langle \phi_k(\mathbf{y}) \rangle_L - h_k(\mathbf{y})] \quad (3.25a)$$

$$\langle \psi_{k_1}(\mathbf{x}) \psi_{k_2}(\mathbf{x}') \rangle_{H_n} - \langle \psi_{k_1} \rangle_{H_n} \langle \psi_{k_2}(\mathbf{x}') \rangle_{H_n} = B_{k_1}^{-1} B_{k_2}^{-1} \sum_{\mathbf{y}, \mathbf{y}'} \gamma_{\mathbf{x}, \mathbf{y}}^{-1} \gamma_{\mathbf{y}, \mathbf{y}'}^{-1} \langle \phi_{k_1}(\mathbf{y}) \phi_{k_2}(\mathbf{y}') \rangle_L + B_{k_1}^{-1} \delta_{k_1, -k_2} \gamma_{\mathbf{x}, \mathbf{x}'}^{-1} (Dk)^{-1}, \quad (3.25b)$$

where $\langle \rangle_L$ indicates an average with respect to $\exp\{-L[\phi_k(\mathbf{x})]\}$. Equation (3.24) defines our fundamental field theory. In the evaluation of critical exponents in the vicinity of six dimensions, we can expand L in powers of $\psi_k(\mathbf{x})$ and use a continuum limit. In addition, we can expand B_k^{-1} in powers of k^2 . Also, since we will always take the limit $n \rightarrow 0$ before $\Delta k \rightarrow 0$, we can replace the factors Dk by unity. For $h(\theta) = 0$, the resulting Hamiltonian, which we will study in the next section, can then be expressed as

$$L = \int d^d x \left[\frac{1}{2} \sum_{\mathbf{k} \neq 0} r_k \phi_k(\mathbf{x}) \phi_{-\mathbf{k}}(\mathbf{x}) + \nabla \phi_k(\mathbf{x}) \cdot \nabla \phi_{-\mathbf{k}}(\mathbf{x}) - \frac{1}{3!} u_3 \int d^d x \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_1 + \mathbf{k}_2 \neq 0} \phi_{\mathbf{k}_1}(\mathbf{x}) \phi_{\mathbf{k}_2}(\mathbf{x}) \phi_{-\mathbf{k}_1 - \mathbf{k}_2}(\mathbf{x}) \right] + \dots, \quad (3.26)$$

$\Delta\theta \rightarrow 0$, $\psi_w(\mathbf{x})$ contains a Dirac rather than a Kronecker delta:

$$\psi_w(\mathbf{x}) \rightarrow \delta(\mathbf{w} - \theta(\mathbf{x})) - (2\theta_M)^{-n}. \quad (3.20)$$

In this limit, the integral, rather than the sum, of ψ_w over \mathbf{w} is zero. When the limit $\theta_M \rightarrow \infty$ is taken, ψ_w is simply the Dirac delta function setting \mathbf{w} equal to θ with the constraint that its integral over \mathbf{w} be zero. This is the form of the order parameter used by Stephen.¹² Equation (3.20) clearly establishes that the voltage is merely a component of the order parameter for the resistor network since we can write

$$\mathbf{V} = \sum_{\mathbf{w}} (\Delta\theta)^n [\psi_w(\mathbf{x}) + (2\theta_M)^{-n}] \mathbf{w}. \quad (3.21)$$

A similar expression applies for the angle variable of the xy model.

The external field warrants further comment. In the resistor network, $h = -i\omega V^2$, and the potential arising from the capacitive coupling to ground can be expressed as

$$H_{\text{ext}} = \sum_{\mathbf{x}} i\omega V^2(\mathbf{x}) = i\omega \sum_{\mathbf{x}, \mathbf{w}} (\Delta\theta)^n w^2 [\psi_w(\mathbf{x}) + (2\theta_M)^{-n}]. \quad (3.22)$$

Thus, there is a positive energy associated with all $\mathbf{V} \neq 0$, i.e., the unique state with $\mathbf{V} = 0$ is favored by the capacitance to ground. This is analogous to but not identical to the situation encountered in more familiar treatments of the Potts model where the external field favors one of the s possible states, but the other $s - 1$ states are still equivalent. For the xy model, the external fields favor order that is distributed unequally among the vectors.

The Hamiltonian of Eq. (3.9) is bilinear in the ψ_k 's and can be converted to a continuum field theory in the standard way using a Hubbard-Stratanovich transformation. We obtain

where

$$r_{\mathbf{k}} = r + \sum_j w_j k^{2j} + w_{2c} \sum_{\alpha} k_{\alpha}^4 + \dots \quad (3.27)$$

with $r \sim p_c - p$. The coefficients w_j can be obtained from the expansions of $B_{\mathbf{k}}$ in powers of k^2 presented in Eqs. (3.13) and (3.14).

The Hamiltonian in Eq. (3.26) forms the starting point for detailed calculations. Within mean-field theory one neglects the fluctuations implied by the higher-than-quadratic terms in L . In this case one has the mean-field propagator

$$G_0(\mathbf{q}, r_p) = (q^2 + r_p)^{-1} \quad (3.28a)$$

in terms of the Fourier wave vector \mathbf{q} . This propagator is an approximation to the exact propagator

$$G_{\mathbf{k}}(\mathbf{q}) \equiv \int d^d x e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \phi_{\mathbf{k}}(\mathbf{x}) \phi_{-\mathbf{k}}(\mathbf{x}') \rangle_L \quad (3.28b)$$

in terms of which the resistive correlation functions are given via Eq. (3.25). The distinction between correlation functions involving the ψ 's and those in terms of the ϕ 's is unimportant, because at long wavelength, they differ only by a constant.

IV. ϵ EXPANSION

In this section, we will use momentum shell renormalization-group⁴⁴ recursion relations to calculate critical exponents to first order in $\epsilon = 6 - d$ for the models discussed in the preceding sections. We proceed in the usual way by removing degrees of freedom from the Hamiltonian of Eq. (3.26) with wave number of magnitude q lying between the upper cutoff $\Lambda = 1$ and $e^{-l}\Lambda$ followed by a rescaling of ϕ ,

$$\phi(e^{-l}\mathbf{q}) \rightarrow e^{(d+2-\eta)l/2} \phi(\mathbf{q}) \quad (4.1)$$

with η chosen to keep the coefficient of $\nabla \phi_{\mathbf{k}} \cdot \nabla \phi_{-\mathbf{k}}$ constant. By removing only an infinitesimal shell in wave number space at a time, we obtain differential equations for the potentials $r_{\mathbf{k}}(l)$ and $u_3(l)$. For d near 6, potentials associated with terms of higher than third order in the ϕ 's are strongly irrelevant and may be neglected.

We begin by studying the case when $w_{\mathbf{k}\Gamma} = 0$ for all \mathbf{k} and Γ . This case corresponds to $\sigma^{-1} = 0$ or $T = 0$ and should, therefore, reduce to the familiar results for the pure percolation problem.^{42,43,45,46} Near six dimensions the only relevant diagram for the self-energy is that shown in Fig. 4. By requiring the coefficient of $|\nabla \phi_{\mathbf{k}}|^2$ to remain equal to unity we find that $\eta(l)$ is given by

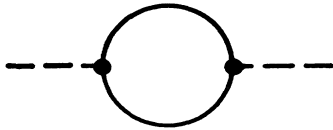


FIG. 4. Diagram which contributes to the recursion relation Eq. (4.4) for $r(l)$ for d near 6.

$$\eta(l) = (s^n - 2)g(l) \frac{2 + 3r(l)}{12[1 + r(l)]^4} \quad (4.2a)$$

$$\rightarrow -\frac{1}{6}g^* \quad (s^n \rightarrow 1; r \rightarrow r^*), \quad (4.2b)$$

where

$$g(l) \equiv K_d u_3^2(l), \quad (4.3)$$

where $K_d = \Gamma(d/2)/(2\pi)^{d/2}$ is the phase-space element in d dimensions. The factor $s^n - 2$ arises in Eq. (4.2a) because there are s^n possible values of \mathbf{k} in the sum over internal legs in Fig. 4, two of which are prohibited because propagators with $\mathbf{k} = 0$ are excluded. Similarly, the recursion relations for $r(l)$ and $g(l)$ are obtained from Figs. 4 and 5, respectively, as

$$\frac{dr}{dl} = [2 - \eta(l)]r - \frac{1}{2}(s^n - 2)g(l) \frac{1}{[1 + r(l)]^2}, \quad (4.4)$$

$$\frac{dg}{dl} = [\epsilon - 3\eta(l)]g(l) + (s^n - 3) \frac{2g^2(l)}{[1 + r(l)]^3}. \quad (4.5)$$

Again the factors of $s^n - 2$ and $s^n - 3$ arise because there are s^n values of \mathbf{k} , but in the sum for the internal legs the constraint $\mathbf{k} \neq 0$ excludes two values for Fig. 4 and three for Fig. 5. If the limit $n \rightarrow 0$ is taken *before* $s \rightarrow \infty$, the usual recursion relations in $6 - \epsilon$ dimensions for the percolation problem are retrieved. At the fixed point (for $s^n \rightarrow 1$), to order ϵ the potentials assume the values

$$g^* = 2\epsilon/7, \quad (4.6a)$$

$$r^* = -\epsilon/14, \quad (4.6b)$$

and the correlation length and critical-point exponents the values⁴²

$$\nu = (2 - \eta - g^*)^{-1} = \frac{1}{2} + 5\epsilon/84, \quad (4.6c)$$

$$\eta = -\epsilon/21. \quad (4.6d)$$

We emphasize again the importance of the order of limits in the above analysis.

To obtain the crossover properties associated with nonzero $w_{\mathbf{k}}$, it suffices to set $g(l) = g^*$. As shown by Rudnick and Nelson,⁴⁴ inclusion of the l dependence of g leads to corrections to scaling which we will not consider explicitly. The zero wave-vector part of Fig. 4 yields

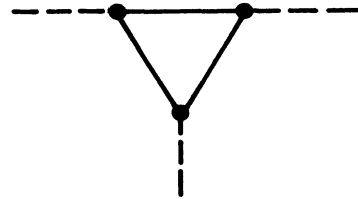


FIG. 5. Diagram which contributes to the recursion relation Eq. (4.5) for the third-order potential, $g(l)$ for d near 6.

$$\frac{dr_{\mathbf{k}}}{dl} = [2 - \eta(l)]r_{\mathbf{k}} - \frac{1}{2}g^*\Pi_{\mathbf{k}}, \tag{4.7}$$

where

$$\Pi_{\mathbf{k}} = \sum_{\mathbf{p}: \mathbf{p} \neq 0, \mathbf{p} + \mathbf{k} \neq 0} D(\mathbf{p} + \mathbf{k})D(\mathbf{p}), \tag{4.8}$$

where $D(\mathbf{p})$ is the mean-field propagator evaluated at $q^2 = 1$:

$$D^{-1}(\mathbf{p}) = 1 + r_{\mathbf{p}} \equiv G_0^{-1}(q^2 = 1, r_{\mathbf{p}}). \tag{4.9}$$

In terms of an unrestricted sum we may write

$$\Pi_{\mathbf{k}} = -2D(\mathbf{k})D(0) + \sum_{\mathbf{p}} D(\mathbf{p} + \mathbf{k})D(\mathbf{p}) \tag{4.10a}$$

$$= \Pi_{\mathbf{k}}^{(0)} + \delta\Pi_{\mathbf{k}}, \tag{4.10b}$$

where we have separated Π into a “normal” part $\Pi_{\mathbf{k}}^{(0)}$ which will give all crossover exponents equal to unity:

$$\Pi_{\mathbf{k}}^{(0)} = D^2(0) - 2D(\mathbf{k})D(0) \tag{4.11a}$$

and an “anomalous” part given by

$$\delta\Pi_{\mathbf{k}} = -D^2(0) + \sum_{\mathbf{p}} D(\mathbf{p} + \mathbf{k})D(\mathbf{p}) \tag{4.11b}$$

$$= \sum_{\mathbf{p}} D(\mathbf{p})[D(\mathbf{p} + \mathbf{k}) - D(\mathbf{p})] \tag{4.11c}$$

$$= -\frac{1}{2} \sum_{\mathbf{p}} [D(\mathbf{p} + \mathbf{k}) - D(\mathbf{p})]^2. \tag{4.11d}$$

In writing the above results we used the relation

$$\sum_{\mathbf{p}} F(\mathbf{p}) = F(0) \tag{4.12}$$

which can be established in the limit $n \rightarrow 0$ for convergent sums over functions $F(\mathbf{p})$ [here $F(\mathbf{p}) = D^2(\mathbf{p})$] which are invariant with respect to the symmetry operations of an n -dimensional hypercube and which have a series expansion in powers of \mathbf{p} .

Up to this point our treatment has been general enough to cover the xy model for which there occur nonspherically symmetric terms as in Eq. (2.16). We now specialize to the resistor network for which the propagators and vertex functions are functions of k^2 . (In Appendix E we consider the calculation of the crossover exponent associated with the lowest-order nonspherically symmetric potential.) To analyze Eq. (4.7) we expand both sides in powers of k^2 :

$$r_{\mathbf{k}} = r + \sum_{j=1}^{\infty} w_j k^{2j}, \tag{4.13a}$$

$$\begin{aligned} \Pi_{\mathbf{k}}^{(0)} &= (1+r)^{-2} - 2(1+r)^{-1} \left[1+r + \sum_j w_j k^{2j} \right]^{-1} \\ &= \sum_{j=0}^{\infty} \Pi_j^{(0)} k^{2j}, \end{aligned} \tag{4.13b}$$

$$\delta\Pi_{\mathbf{k}} = \sum_{j=1}^{\infty} \delta\Pi_j k^{2j}. \tag{4.13c}$$

in which case Eq. (4.7) may be written as

$$\frac{dw_j(l)}{dl} = [2 - \eta(l)]w_j(l) - \frac{1}{2}g^*\Pi_j^{(0)} - \frac{1}{2}g^*\delta\Pi_j. \tag{4.14}$$

To gain some insight into the implications of this recursion relation, we start by ignoring completely $\delta\Pi$. The coefficients $\Pi_j^{(0)}$ in Eq. (4.13b) are

$$\Pi_0^{(0)} = -\frac{1}{[1+r(l)]^2} = -1 + \frac{2r(l) + r^2(l)}{[1+r(l)]^2}, \tag{4.15a}$$

$$\Pi_1^{(0)} = \frac{2w_1(l)}{[1+r(l)]^3}, \tag{4.15b}$$

$$\Pi_2^{(0)} = \frac{2w_2(l)}{[1+r(l)]^3} - \frac{2w_1^2(l)}{[1+r(l)]^4}, \tag{4.15c}$$

$$\Pi_t^{(0)} = \frac{2w_t(l)}{[1+r(l)]^3} \cdots + (-1)^{t-1} \frac{2w_1^t(l)}{[1+r(l)]^{t+2}}, \tag{4.15d}$$

$t > 1.$

We have indicated that in Eq. (4.15d) we have not written explicitly the terms on the right-hand side which are products of from 2 to $t-1$ factors of w_k 's. Among such omitted terms are those for general t proportional to $w_1^{t-2}w_2$, $w_1^{t-3}w_3$, and so forth. To calculate the crossover exponent associated with w_t we only need the term in $\Pi_t^{(0)}$ linear in $w_t(l)$, i.e., is the first one on the right-hand side of Eq. (4.15d). Keeping only this term the recursion relation is

$$\frac{dw_t(l)}{dl} = [2 - \eta(l)]w_t(l) - g^* \frac{w_t(l)}{[1+r(l)]^3} \approx \lambda_t w_t(l), \tag{4.16}$$

with

$$\lambda_t \equiv \phi_t / \nu = 2 - \eta - g^*, \tag{4.17}$$

so that when we neglect $\delta\Pi$, we have

$$\phi_t = 1 \tag{4.18}$$

for all t . The terms of order higher than linear in the recursion relations do not affect the crossover exponents, but they do determine the nonlinear scaling fields, as we shall see later. In calculating the amplitude ratios we need only the highest-order nonlinear terms in the scaling fields. For later use we have written these terms [of order $w_1^t(l)$] explicitly in Eq. (4.15d). The other terms will not be needed.

Now we return to Eq. (4.14) and study the effect of the anomalous term, $\delta\Pi$. Equation (4.11d) indicates that each term in the sum over \mathbf{p} is at least quadratic in the w_j 's. To see the implication of this result, let us, for the moment, assume that all w_j 's are zero except for w_1 . Then Eq. (4.11d) can be written as

$$\delta\Pi_{\mathbf{k}} = -\frac{1}{2}w_1^2 \sum_{\mathbf{p}} \left[\frac{2(\mathbf{p} \cdot \mathbf{k}) + k^2}{[1+w_1 p^2][1+w_1(\mathbf{p} + \mathbf{k})^2]} \right]^2. \tag{4.19}$$

If the sum over \mathbf{p} converged when $w_1 = 0$, we could then conclude that $\delta\Pi_{\mathbf{k}}$ would be of order w_1^2 and would thus not contribute to the crossover exponent for w_1 . In this case, as we have just seen, all the crossover exponents ϕ_t would be unity. This analysis is valid so long as s is not infinite, because for finite s the sum consists of a finite

number of finite terms, and as such is trivially convergent. The result, Eq. (4.18), is in agreement with previous calculations for dilute Ising models²⁰ and with the general arguments of Wallace and Young¹³ and Coniglio,¹⁹ which are expected to hold for discrete-spin models.

When $s \rightarrow \infty$, however, the sum over \mathbf{p} in Eq. (4.19) diverges unless the dependence of the summand on w_1 is properly taken into account. If this dependence is improperly neglected, one has

$$\delta\Pi_{\mathbf{k}} = -2w_1^2 k^2 \sum_{\mathbf{p}} p_1^2 + O(k^4) \tag{4.20a}$$

$$= -\frac{2}{3}w_1^2 k^2 M^2, \tag{4.20b}$$

where $M \rightarrow \infty$ was introduced above Eq. (3.5). Thus, the coefficient of w_1^2 diverges when $M \rightarrow \infty$. Of course, a proper calculation should include the fact that this divergence is removed by the w_1 dependence of the denominators in Eq. (4.19). In effect, the denominators restrict the effective number of terms in the sum over \mathbf{p} to be of order w_1^{-1} , and $\delta\Pi_{\mathbf{k}}$ becomes proportional to w_1 . To see this explicitly for the case when only w_1 is present, we can write Eq. (4.19) as

$$\delta\Pi_{\mathbf{k}} = -\frac{1}{2}w_1^2 \sum_{\alpha,\beta} k_\alpha k_\beta \sum_{\mathbf{p}} \int d^n p \frac{4p_\alpha p_\beta}{(1+w_1 p^2)^4}, \tag{4.21}$$

$$\delta\Pi_t = \frac{(-1)^t}{t!(t-1)!} \int_0^\infty dy \left[\frac{d^t}{dy^t} \left[1+r(l) + \sum_j w_j(l)y^j \right]^{-1} \right]^2 y^{t-1}. \tag{4.25}$$

With the change of variables $y = z/w_1(l)$ this becomes

$$\delta\Pi_t = \frac{(-1)^t w_1^t(l)}{t!(t-1)!} \int_0^\infty dz \left\{ \frac{d^t}{dz^t} \left[1+r(l) + \sum_{j=1}^\infty \left[\frac{w_j(l)}{w_1^j(l)} \right] z^j \right]^{-1} \right\}^2 z^{t-1}. \tag{4.26}$$

To calculate the crossover exponents for w_t we need to extract from $\delta\Pi_t$ the term that is linear in w_t for large l . It is clear from Eq. (4.25) that in addition to this term there are many other terms [as in Eq. (4.15d)] which involve powers of w_s for $s \neq t$. We will show in detail in the next section that these terms affect the nonlinear scaling fields but *not* the crossover exponents. The term linear in w_t in $\delta\Pi_t$ depends on the initial conditions. Note from Eq. (3.16) that for the xy model, $w_t(0) \sim T^t$ with $T \rightarrow 0$ and for the resistor network, $w_t(0) \sim \sigma^{-t}$ with $\sigma^{-1} \rightarrow 0$. Thus in both cases, $w_t(0) \sim w_1^t(0)$, so that w_t/w_1^t is initially of order unity. To lowest order in ϵ , $w_t(l) \sim e^{2l} w_t(0)$ for all t , so that $w_t(l)/w_1^t(l) \sim e^{-2(t-1)l} w_t(0)/w_1^t(0)$. Thus, for the physically relevant initial conditions, the large l behavior of $\delta\Pi_t$ can be obtained by an expansion in

$$c_t(r) = \frac{2(-1)^t}{t!(t-1)!} \int_0^\infty dz \left[\frac{d^t}{dz^t} \frac{1}{1+r+z} \right] \left[\frac{d^t}{dz^t} \frac{z^t}{(1+r+z)^2} \right] z^{t-1} \tag{4.29a}$$

$$= (-1)^t \frac{2(t-2)[(t+1)!]^2}{(2t+2)!(1+r)^3} \equiv \frac{c_t}{(1+r)^3}. \tag{4.29b}$$

where we have neglected terms of order k^4 . Using the fact that the integral must be isotropic and $d^n p = \Omega_n p^{n-1} dp$ where Ω_n is the solid angle subtended by an n -dimensional sphere, we obtain in the limit $n \rightarrow 0$

$$\delta\Pi_{\mathbf{k}} = -w_1^2 k^2 \int_0^\infty \frac{2p dp}{(1+w_1 p^2)^4} = -\frac{1}{3}w_1 k^2. \tag{4.22}$$

Thus, the recursion relation for w_1 when $r=0$ is

$$\frac{dw_1}{dl} = (2-\eta)w_1 - \frac{1}{2}g^*(2-\frac{1}{3})w_1 \equiv \lambda_1 w_1 \tag{4.23}$$

from which we obtain

$$\phi_1 = \lambda_1 \nu = 1 + \frac{1}{6}g^* \frac{1}{2-\eta-g^*} = 1 + \frac{\epsilon}{42} + O(\epsilon^2). \tag{4.24}$$

Thus the continuous symmetry of the xy model and the resistor network brought about by the limit $s \rightarrow \infty$ leads to order ϵ corrections to the exponent ϕ_1 . This subtlety was not addressed in the early calculations of Dasgupta *et al.*¹⁰ and Stephen,¹² both of whom gave $\phi_1 = 1$, but was later corrected.^{21,22} Apparently, the general discussion of Wallace and Young¹³ breaks down for the continuous symmetry case for this reason.

In order to calculate the crossover exponents associated with w_t for $t > 1$, it is necessary to consider $\delta\Pi_t$ for general t . It is convenient to manipulate (as done in Appendix B) $\delta\Pi_t$ into the form

powers of the small parameters $[w_s(l)/w_1^s(l)]$, for $s=2, 3, \dots$. Keeping the same type of terms as in Eq. (4.15d) we find, in this limit,

$$\delta\Pi_t = (-1)^t w_1^t(l) a_t(r_t) \cdots - w_t(l) c_t(r_t) + \cdots \tag{4.27}$$

with

$$a_t(r) = \frac{1}{t!(t-1)!} \int_0^\infty dz \left[\frac{d^t}{dz^t} (1+r+z)^{-1} \right]^2 z^{t-1} \tag{4.28a}$$

$$= \frac{t!(t+1)!}{(2t+1)!} \frac{1}{(1+r)^{t+2}} \equiv a_t \frac{1}{(1+r)^{t+2}} \tag{4.28b}$$

and

In Eq. (4.27) some of the terms omitted involve multinomials of positive powers of the w 's as in Eq. (4.15), but there are others involving arbitrarily large positive powers of the variables $w_s(l)/w_1^s(l)$. Substituting Eqs. (4.15d) and (4.27) into Eq. (4.14) we have

$$\frac{dw_t(l)}{dl} = \frac{w_t(l)}{\nu} + \frac{1}{2}g^*c_t \frac{w_t(l)}{[1+r(l)]^3} + \cdots + \frac{1}{2}g^*(-1)^t \frac{(2-a_t)w_1^t(l)}{[1+r(l)]^{2+t}} \quad (4.30)$$

with ν as in Eq. (4.6c). The term linear in w_t gives the crossover exponent [defined in Eq. (4.17)] as

$$\phi_t \equiv \lambda_t \nu = 1 + \frac{1}{4}g^*c_t = 1 + \frac{\epsilon}{14}c_t, \quad (4.31)$$

where $c_1 = 1/3$, $c_2 = 0$, $c_3 = -1/35$, and so forth. As before, the omitted terms in Eq. (4.30) determine the nonlinear scaling field associated with ϕ_t . These will be considered in more detail in the next section.

For $t > 1$ the exponents quoted in Eq. (4.31) differ from those of Refs. 21 and 22. In those references the crossover exponent associated with w_t was calculated using the boundary condition that all w_s for $s \neq t$ were zero at $l = 0$. (This is the normal procedure in determining crossover exponents.) In that case the expansion in powers of w_t/w_1^t was inappropriate. The term linear in w_t was obtained by the change of variables $w_t y^t = z$, in terms of which

$$\delta \Pi_t = -\bar{c}_t [r(l)] w_t(l) \quad (4.32)$$

with

$$\bar{c}_t(0) = \frac{(-1)^t}{t!(t-1)!} \int_0^\infty dy y^{t-1} \left[\frac{d^t}{dy^t} \frac{1}{1+y^t} \right]^2. \quad (4.33)$$

This yields $\phi_t = 1 + (\epsilon/14)\bar{c}_t$ as given previously.^{21,22} We should emphasize that this result does *not* apply to the randomly diluted xy model or resistor network for which the boundary conditions are that w_t/w_1^t is initially of order unity. There exist a multitude of other crossover exponents which refer to the boundary condition for which $(w_{k_i})^l / (w_{l_i})^{k_i}$ is of order unity for some set of k 's and l 's. These cases have no obvious physical application.

V. AMPLITUDE RATIOS AND CROSSOVER FUNCTIONS

The resistive susceptibilities introduced in Ref. 6 and discussed in Sec. II can be evaluated by series expansions.^{6,37-40} As in the case of pure percolation,⁴⁷ there are a number of invariant amplitude ratios⁴⁸ that can be constructed from the various resistive susceptibilities. Among these are the ratios

$$\rho_s \equiv \chi_0(p)^s - 1 \chi_s(p) / \chi_1(p)^s, \quad p \rightarrow p_c. \quad (5.1)$$

(In this section we will always be concerned with the limit $p \rightarrow p_c$, which we do not always indicate explicitly in what follows.) General ratios of the form

$$\prod_s \chi_s^{b_s} / \prod_{s'} \chi_{s'}^{b_{s'}} \quad \text{with} \quad \sum_s b_s = \sum_{s'} b_{s'} \quad \text{and} \quad \sum_s s b_s = \sum_{s'} s' b_{s'} \quad (5.2)$$

such as $\chi_1 \chi_3 / \chi_2^2 = \rho_3 / \rho_2^2$ can be constructed from the ρ_s . In this section we explicitly evaluate ρ_s for $s = 2, 3$ and indicate how the calculation can be performed for general s . As a by-product we obtain general expressions for the nonlinear scaling fields which scale with the crossover exponents ϕ_t calculated in the preceding section.

The spatial Fourier transform of the inverse correlation functions, at zero wave vector \mathbf{q} , denoted $G_{\mathbf{k}}(q=0)$ introduced in Eq. (3.28b) can be expanded in powers of the replica vector \mathbf{k} as

$$G_{\mathbf{k}}^{-1}(q=0) = \sum_{j=0}^{\infty} \Gamma_j k^{2j}, \quad (5.3)$$

where $\Gamma_0 \equiv \chi_0^{-1}$ is the inverse of the percolation susceptibility, i.e., the inverse of the mean-square cluster size. From the discussion in Sec. II, one sees that the resistive susceptibilities χ_s can be obtained from $\chi_{\mathbf{k}}$ by differentiation:

$$\chi_1 = \frac{d}{d(-\frac{1}{2}k^2)} \chi_{\mathbf{k}} = 2\chi_0^2 \Gamma_1, \quad (5.4a)$$

$$\chi_2 = \frac{d^2}{d(-\frac{1}{2}k^2)^2} \chi_{\mathbf{k}} = 8\chi_0^2 (\chi_0 \Gamma_1^2 - \Gamma_2), \quad (5.4b)$$

$$\chi_3 = \frac{d^3}{d(-\frac{1}{2}k^2)^3} \chi_{\mathbf{k}} = 48\chi_0^2 (\chi_0^2 \Gamma_1^3 - 2\chi_0 \Gamma_1 \Gamma_2 + \Gamma_3), \quad (5.4c)$$

where the derivatives are evaluated at $\mathbf{k} = 0$. In principle, we should relate the $\chi_{\mathbf{k}}$ to $G_{\mathbf{k}}$ as in Eq. (3.25). However, that equation shows that the spatially Fourier-transformed versions of these functions differ only by a constant at long wavelength ($\mathbf{q} \rightarrow 0$). Thus we treat $G_{\mathbf{k}}$ and $\chi_{\mathbf{k}}$ as completely equivalent. Using relations such as the above, one can express any ρ_s in terms of the coefficients Γ_k , e.g.,

$$\rho_2 = \frac{\chi_0 \chi_2}{\chi_1^2} = 2 \left[1 - \frac{\Gamma_0 \Gamma_2}{\Gamma_1^2} \right], \quad (5.5a)$$

$$\rho_3 = \frac{\chi_0^2 \chi_3}{\chi_1^3} = 6 \left[1 - 2 \frac{\Gamma_0 \Gamma_2}{\Gamma_1^2} + \frac{\Gamma_0^2 \Gamma_3}{\Gamma_1^3} \right]. \quad (5.5b)$$

Within mean-field theory, $\Gamma_0 \sim p_c - p$ and $\Gamma_s \sim \text{const}$, for $s > 1$, as $p \rightarrow p_c$, in which case $\rho_s \rightarrow s!$ as $p \rightarrow p_c$.

Under renormalization all the vertices rescale with the same power of e^l :

$$\Gamma_s(r(0), \{w_t(0)\}) = e^{-\int_0^l [2 - \eta(l)] dl} \Gamma_s(r(l), \{w_t(l)\}), \quad (5.6)$$

so that ρ_s can be evaluated using the potentials $\Gamma_s(l) \equiv \Gamma_s(r(l), \{w_s(l)\})$ for any value of l including the

matching point l^* defined by $r(l^*)=1$. In Appendix C we give a simplified version of the calculation to follow which explicitly shows that the vertex functions $\Gamma_s(r(0), \{w_s(0)\})$ are independent of the choice of l^* , but which does not identify the nonlinear scaling fields.

To one-loop order we have

$$\Gamma_0(r) = r + \frac{1}{2}g \int_0^1 \frac{q^{d-1} dq}{(q^2+r)^2}. \quad (5.7)$$

Here and below the construction of the vertex functions may be accomplished by noting that they only differ from the corresponding Eqs. (4.13) and (4.25) by extending the range of integration over q over the full range from $q=0$ to $q=1$. To do this, it is only necessary to replace $q^2=1$ in those equations by q^2 and integrate over q , noting that

the phase-space factor K_d has been incorporated into the definition of g . To lowest order in $\epsilon=6-d$, we obtain from Eq. (4.5) $\Gamma_0(l^*)=t(l^*)$, where

$$t(l^*) = r(l^*) - r^* = e^{v^{-1}l^*} t(0) \quad (5.8)$$

near the critical point. At the matching point $t(l^*)=\Gamma_0(l^*)=1$, i.e., to order ϵ $r(l^*)=1$, so that $e^{v^{-1}l^*} \sim p_c - p$. As mentioned above, the one-loop contributions to the vertex functions Γ_j are found by generalizing Eqs. (4.13) and (4.25). We write the results as

$$\Gamma_j = w_j - \frac{1}{2}gw_j J_j \equiv w_j - \frac{1}{2}gw_j J_j^{(0)} - \frac{1}{2}gw_j \delta J_j, \quad (5.9)$$

where J_j has a normal part [as in Eq. (4.15)]

$$J_j^{(0)} = -\frac{2}{w_j} \int_0^1 q^{d-1} dq \frac{1}{j!} \left[\frac{d^j}{d(k^2)^j} G_0(q, r_k) \right]_{k=0} G_0(q, r_0) \quad (5.10a)$$

$$= -\frac{r}{w_j} \left[\frac{w_1}{r} \right]^j \frac{1}{j!} \int_0^{r^{-1}} \frac{x^2 dx}{1+x} \left[\frac{d}{dz} \right]^j \left[1+x+z + \sum_{t=2}^{\infty} w_t r^{t-1} w_1^{-t} z^t \right]^{-1} \Big|_{z=0} \quad (5.10b)$$

and an anomalous part [as in Eq. (4.25)]

$$\delta J_j = \frac{(-1)^j}{w_j j!(j-1)!} \int_0^1 q^{d-1} dq \int_0^{\infty} dy \left[\frac{d^j}{dy^j} \frac{1}{r+q^2 + \sum_{t=1}^{\infty} w_t y^t} \right]^2 y^{j-1} \quad (5.11a)$$

$$= \frac{r}{2w_j} \left[\frac{w_1}{r} \right]^j \frac{(-1)^j}{j!(j-1)!} \int_0^{r^{-1}} x^2 dx \int_0^{\infty} dz \left[\frac{d}{dz} \right]^j \frac{1}{1+x+z + \sum_{t=2}^{\infty} w_t w_1^{-t} r^{t-1} z^t} \Big|_{z=0}^2 z^{j-1}, \quad (5.11b)$$

where $w_j \equiv w_j(l)$ and $r \equiv r(l)$ and where we evaluated the integrals at $d=6$.

To calculate $\Gamma_j(l^*)$, we need to know $w_j(l^*)$. The recursion relations for w_j in Eq. (4.14) can be expressed as

$$\frac{dw_j(l)}{dl} = [2 - \eta(l)]w_j(l) - \frac{1}{2}gw_j(l)K_j(l), \quad (5.12)$$

where $K_j(l) = K_j^{(0)}(l) + \delta K_j(l)$ has a normal part, $K_j^{(0)}(l) = \Pi_j^{(0)}/w_j(l)$ and an anomalous part $\delta K_j(l) = \delta \Pi_j/w_j(l)$, so that $K_j(l)$ depends on $r(l)$ and all the potentials $w_j(l)$ in a complicated way. Nevertheless, at least formally, Eq. (5.12) can be integrated using the fact that to lowest order in ϵ , $r(l) \sim r(0)e^{2l}$ and $w_j(l) \sim w_j(0)e^{2l}$ so that, as indicated, K_j can be regarded as a function of l :

$$\frac{w_j(l)}{w_j(0)} = \exp \left[\int_0^l [2 - \eta(l')] dl' \right] \times \exp \left[-\frac{1}{2} \int_0^l g(l') K_j(l') dl' \right]. \quad (5.13)$$

Note that $K_j(l')$ represents the variation of w_j as integration over successive shells in momentum space are per-

formed from $l=0$ to the matching point, $l=l^*$, whereas $J_j(l)$ represents the final "one-shot" integration over momentum for $l=l^*$. The condition that Γ_j be independent of l^* is

$$-\frac{dJ_j(l)}{dl} = K_j(l), \quad (5.14)$$

and to leading order in ϵ the expressions given here satisfy this relation. To see this, we evaluate $-dJ_j/dl$ using Eqs. (5.10b) and (5.11b). In so doing, note that $d[w_1(l)/r(l)]/dl \sim O(\epsilon)$ and $d[w_t(l)r(l)^{t-1}w_1(l)^{-t}]/dl \sim O(\epsilon)$. Thus in evaluating the derivatives with respect to l we only need to take explicit account of the l dependence of the upper limit of the integrals over x . The resulting expression for $-dJ_j/dl$ can be shown to be equivalent to $K_j(l) = \Pi_j(l)/w_j(0)$ as given by Eqs. (4.13) and (4.26).

We now turn to an analysis of $J_j(l)$. We will show that it can be written in the form

$$J_j(l) = -(1 - \frac{1}{2}c_j) \ln r(l) + J_j^{\text{sing}} + J_j^{\text{reg}}(l), \quad (5.15)$$

where J_j^{sing} consists of terms proportional to inverse powers of $r(l)$ and $J_j^{\text{reg}}(l)$ remains finite in the limit

$r(l) \rightarrow 0$ with $w_j(l)$ fixed. (This limit is a formal one: it has nothing to do with actual renormalization-group trajectories, along which $d[r(l)/w_j(l)]/dl$ is of order ϵ .) In order to make the decomposition indicated in Eq. (5.15) we set $z = (1+x)\rho$ in Eqs. (5.10b) and (5.11b) to obtain

$$J_j^{(0)} = \frac{r}{w_j} \left[\frac{w_1}{r} \right]^j \frac{1}{j!} \int_0^{r^{-1}} \frac{x^2 dx}{(1+x)^{j+2}} \Phi_j^{(0)}[r(1+x)], \tag{5.16a}$$

where

$$\Phi_j^{(0)}(y) = - \left[\frac{d}{d\rho} \right]^j \left[1 + \rho + \sum_{t=2}^{\infty} w_t w_1^{-t} y^{t-1} \rho^t \right]^{-1} \Big|_{\rho=0} \tag{5.16b}$$

and

$$\delta J_j = \frac{r}{w_j} \left[\frac{w_1}{r} \right]^j \frac{1}{j!} \int_0^{r^{-1}} \frac{x^2 dx}{(1+x)^{j+2}} \delta \Phi_j[r(1+x)], \tag{5.17a}$$

where

$$\delta \Phi_j(y) = \frac{1}{2} \frac{(-1)^j}{(j-1)!} \int_0^{\infty} d\rho \left[\left[\frac{d}{d\rho} \right]^j \frac{1}{1 + \rho + \sum_{t=2}^{\infty} w_t w_1^{-t} y^{t-1} \rho^t} \right]^2 \rho^{j-1}. \tag{5.17b}$$

Thus we see that $J_j(l)$ is of the form

$$J_j(l) = \frac{r(l)}{w_j(l)} \left[\frac{w_1(l)}{r(l)} \right]^j \frac{1}{j!} \int_0^{r^{-1}(l)} \frac{x^2 dx}{(1+x)^{j+2}} \sum_0^{\infty} b_n(l) [r(l)(1+x)]^n, \tag{5.18}$$

where $b_n(l)$ are the coefficients in the Laurent expansion of the rational function $\Phi_j^{(0)}(y) + \delta \Phi_j(y)$, in powers of y , which can be obtained from Eq. (5.17). The $b_n(l)$'s depend on l through their dependence on the $w(l)$'s. If we integrate the series in Eq. (5.18) term by term, we will get results corresponding to the decomposition in Eq. (5.15). In doing this integration it is convenient to replace the upper limit in the integral over x by $r_c^{-1} \equiv r^{-1}(l)$. We thereby separate the dependence on r into two types: one from the r 's that appear in the integrand (which are always accompanied by the appropriate factors of w_j such that to leading order in ϵ the integrand has no dependence on l), and the other from the factor r_c appearing in the upper limit of the integral. Thus $b_n(l)$ depends on the w 's in such a way that $b_n(l)r^n(l)$ is independent of l . This implies that the l dependence of $J_j(l)$ arises only from the l dependence of r_c . Explicitly, the term proportional to $b_n(l)$ is

$$b_n(l)r(l)^n \frac{r(l)}{w_j(l)} \left[\frac{w_j(l)}{r(l)} \right]^j \frac{1}{j!} \left[\frac{1}{j-n-1} \left[1 - \frac{1}{(1+r_c^{-1})^{j-n-1}} \right] - \frac{2}{j-n} \left[1 - \frac{1}{(1+r_c^{-1})^{j-n}} \right] + \frac{1}{j-n+1} \left[1 - \frac{1}{(1+r_c^{-1})^{j-n+1}} \right] \right]. \tag{5.19}$$

For $n > j-1$, this term is regular as $r \rightarrow 0$, at fixed w_j . For $n = j-1$, Eq. (5.19) yields the logarithmic term in Eq. (5.15) with additional regular contributions. Finally, for $n < j-1$, the terms in Eq. (5.19) which are independent of r_c yield the "singular" contributions in Eq. (5.15). As mentioned, the fact that these terms are independent of r_c implies that they are independent of l , to order ϵ . That is

$$J_j^{\text{sing}}(l) = J_j^{\text{sing}}(0) \equiv J_j^{\text{sing}}. \tag{5.20}$$

For $n < j-1$ we also see terms in Eq. (5.19) which depend on r_c and which therefore are dependent on l . However, counting powers of r indicates that these are regular terms. It is clear from this discussion that a calculation of the regular terms is too complicated to be feasible. The logarithmic term and the most singular of the singular terms were in essence calculated in the preceding section.

In view of Eqs. (5.14) and (5.20) we have to leading order in ϵ

$$\begin{aligned} - \int_0^l K_j(l') dl' &= J_j(l) - J_j(0) \\ &= -(2-c_j)l + J_j^{\text{reg}}(l) - J_j^{\text{reg}}(0). \end{aligned} \tag{5.21a}$$

Also for $g = g^*$

$$\int_0^l [2 - \eta(l')] dl' = (2 - \eta)l + \frac{1}{2} g^* [H(l) - H(0)], \tag{5.21b}$$

where

$$\begin{aligned} H(l) &= -\frac{1}{36} [1+r(l)]^{-3} + \frac{1}{12} [1+r(l)]^{-2} \\ &\quad + \frac{1}{6} [1+r(l)]^{-1} - \frac{1}{6} \ln [1+r(l)]. \end{aligned} \tag{5.22}$$

Thus for $g = g^*$, Eq. (5.13) can be written as

$$w_j(l)\{1 - \frac{1}{2}g^*[J_j^{\text{reg}}(l) + H(l)]\} = e^{(\phi_j/\nu)l} w_j(0)\{1 - \frac{1}{2}g^*[J_j^{\text{reg}}(0) + H(0)]\}, \quad (5.23)$$

where we expanded the exponentials for small $g^* \sim \epsilon$ and

identified $2 - \eta - g^* + \frac{1}{2}g^*c_j$ as $\lambda_j = \phi_j/\nu$. The combination on the left-hand side of this equation is the nonlinear scaling field associated with w_j and ϕ_j . It was necessary to explicitly remove J_j^{sing} in order to obtain a scaling field which remains well defined in the limit $r \rightarrow 0$.

To determine $\Gamma_j(l^*)$, we rewrite Eq. (5.9) as

$$\Gamma_j(l^*) = w_j(l^*) - \frac{1}{2}g^*w_j(l^*)\{J_j^{\text{sing}} - (1 - \frac{1}{2}c_j)\ln[r(l^*)] + J_j^{\text{reg}}(l^*)\}. \quad (5.24a)$$

Using Eq. (5.23) and setting $r(l^*) = 1$ we have

$$\Gamma_j(l^*) = e^{(\phi_j/\nu)l^*} w_j(0)\{1 + \frac{1}{2}g^*[H(l^*) - H(0)]\}[1 - \frac{1}{2}g^*J_j^{\text{reg}}(0)] - \frac{1}{2}g^*w_j(l^*)J_j^{\text{sing}}. \quad (5.24b)$$

To evaluate the amplitude ratios of Eq. (5.5) we only need the most singular part of J_j^{sing} which has contributions from both the normal and anomalous terms:

$$J_j^{\text{sing}} \sim -\frac{r(l)}{w_j(l)} \left[\frac{w_j(l)}{r(l)} \right]^j (A_j^{(0)} + \delta A_j), \quad j > 1, \quad (5.25a)$$

$$= -\frac{w_1(l^*)^j}{w_j(l^*)} (A_j^{(0)} + \delta A_j), \quad (5.25b)$$

where

$$A_j^{(0)} = \frac{-1}{j!} \int_0^\infty \frac{x^2 dx}{1+x} \left[\frac{d}{dz} \right]^j \frac{1}{1+x+z} \Big|_{z=0} = (-1)^{j+1} \frac{2}{(j^2-1)j}, \quad (5.26a)$$

$$\delta A_j = \frac{1}{2} \frac{(-1)^j}{j!(j-1)!} \int_0^\infty x^2 dx \int_0^\infty dz \left[\left[\frac{d}{dz} \right]^j \frac{1}{1+x+z} \right]^2 z^{j-1} \quad (5.26b)$$

$$= \frac{(-1)^j}{(j^2-1)j} a_j. \quad (5.26c)$$

There are no singular terms for $j = 1$. According to Eqs. (5.5) and (5.6) the invariant amplitude ratios are given in terms of the ratios

$$\frac{\Gamma_j(l^*)\Gamma_0^{j-1}(l^*)}{\Gamma_1^j(l^*)} = \frac{\Gamma_j(l^*)}{\Gamma_1^j(l^*)} \exp\{\frac{1}{2}g^*(j-1)[H(l^*) - H(0)]\} \quad (5.27a)$$

$$= e^{l^*(\phi_j - j\phi_1)/\nu} \frac{w_j(0)[1 - \frac{1}{2}g^*J_j^{\text{reg}}(0)]}{w_1(0)^j[1 - \frac{1}{2}g^*J_1^{\text{reg}}(0)]^j} - \frac{1}{2}g^* \frac{w_j(l^*)J_j^{\text{sing}}\{1 + \frac{1}{2}g^*(j-1)[H(l^*) - H(0)]\}}{w_1(l^*)^j[1 - \frac{1}{2}g^*J_1^{\text{reg}}(l^*)]^j} \quad (5.27b)$$

$$\rightarrow -\frac{1}{2}g^*w_j(l^*)J_j^{\text{sing}}/w_1(l^*)^j \quad (5.27c)$$

$$= \frac{\epsilon}{7} (-1)^j \left[\frac{2 - a_j}{(j^2 - 1)j} \right], \quad (5.27d)$$

to order ϵ for $p - p_c \rightarrow 0$ ($e^{l^*} \rightarrow \infty$). In the above, we used $\Gamma_0(l^*) = \exp\{\frac{1}{2}g^*[H(l^*) - H(0)]\}$ and expanded for small g^* where appropriate. From the above, we find

$$\rho_2 = 2 \left[1 - \frac{19\epsilon}{420} \right], \quad (5.28a)$$

$$\rho_3 = 6 \left[1 - \frac{601\epsilon}{5880} \right], \quad (5.28b)$$

and so forth. For $d > 6$, the correction terms of order ϵ in Eq. (5.28) become corrections to scaling so that for $d = 6 + \epsilon$

$$\rho_2 = 2(1 + \text{const} \times |p - p_c|^\epsilon) \rightarrow 2 \text{ as } p \rightarrow p_c. \quad (5.29)$$

VI. SUMMARY

We may summarize this paper as follows.

(1) We have given a detailed calculation of the crossover exponents, scaling fields, and amplitude ratios at the percolation threshold using the xy formalism proposed by Stephen.¹² We introduced some refinements in the formalism in order to recover the usual results for the percolation problem.⁴² We discussed the necessity of taking the replica limit ($n \rightarrow 0$) before the limit $s \rightarrow \infty$, in which continuous spin symmetry is recovered. Indeed our results reproduce the expected behavior: for s finite the crossover exponents are unity,^{13,19,20} whereas for $s \rightarrow \infty$ their values depart from unity.

(2) We have introduced order-parameter correlation functions which are the generating functions for the probability distribution governing the resistance between two points known to be in the same cluster.

(3) We have shown how the theory naturally gives a number of previously accepted scaling results, in particular, the well-known result^{2,5,6} for the conductivity exponent³ $t = (d-2)\nu + \phi_1$ and that for anomalous diffusion,³⁴ $\theta = (t - \beta)/\nu$.

(4) We have evaluated the infinite set of crossover exponents ϕ_n of which the first, ϕ_1 , sets the scale of the probability distribution for two-point resistances.^{22,29,30} The fact that $\phi_1 = (\beta + \gamma)2$ is *not* satisfied indicates that the Alexander-Orbach conjecture⁴⁹ cannot be rigorously true.

(5) We have given an algorithm to determine the scaling fields associated with each ϕ_n . Explicit calculations of these scaling fields is out of the question, however, since they depend on an infinite set of potentials.

(6) Perhaps the most concrete result of this paper, apart from the crossover exponents, is the calculation (in Sec. V) of the universal amplitudes associated with appropriate ratios of resistive susceptibilities corresponding to various moments of the two-point resistance. These quantities are currently being investigated by series techniques.

Note added. Series results of Y. Meir, J. Adler, A.B. Harris, and A. Aharony (unpublished) give $\rho_2 = 2(1 - \epsilon/21)$ in excellent agreement with Eq. (5.28a).

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. DMR85-19216. We would like to thank A. Aharony for stimulating our interest in the amplitude ratios.

APPENDIX A. RESISTANCE TO "INFINITY"

In zero field the Hamiltonian for the resistor network in a given configuration is the quadratic form in the voltages,

$$H = \frac{1}{2} \sum_{\mathbf{x}, \mathbf{x}'} D(\mathbf{x}, \mathbf{x}') V_{\mathbf{x}} V_{\mathbf{x}'}, \quad (\text{A1})$$

where \vec{D} is the matrix obtained when Eq. (2.3) is substituted into Eq. (2.1). If unit current is put into the circuit at node \mathbf{x} and taken out at node \mathbf{x}' , then the voltage at site \mathbf{y} is given by

$$V_{\mathbf{y}} = D^{-1}(\mathbf{y}, \mathbf{x}) - D^{-1}(\mathbf{y}, \mathbf{x}'). \quad (\text{A2})$$

This result expresses the well-known fact that D^{-1} is the Green's function for the system of linear equations determining the voltages of the network. The resistance, $R(\mathbf{x}, \mathbf{x}')$, between sites \mathbf{x} and \mathbf{x}' in the presence of the unit imposed current is equal to the difference in voltage between the source and the sink:

$$R(\mathbf{x}, \mathbf{x}') = D^{-1}(\mathbf{x}, \mathbf{x}) + D^{-1}(\mathbf{x}', \mathbf{x}') - 2D^{-1}(\mathbf{x}, \mathbf{x}'). \quad (\text{A3})$$

The matrix inverses on the right-hand side of Eq. (A3) may be generated by Gaussian averages with respect to the weight function $\exp(-H)$, with H as in Eq. (1). Thus we conclude that

$$\langle [V(\mathbf{x}) - V(\mathbf{x}')]^2 \rangle = R(\mathbf{x}, \mathbf{x}'). \quad (\text{A4})$$

This relation leads to Eq. (2.14b).

Now we consider the behavior of $G_{\mathbf{k}}(\mathbf{x}, \mathbf{x}')$ in the limit when the sites \mathbf{x} and \mathbf{x}' are infinitely far apart. We expect, and it is true here, that in this limit the correlation function is the square of the order parameter. In this limit $D^{-1}(\mathbf{x}, \mathbf{x}')$ vanishes, so that

$$\chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') \rightarrow [\exp(-\frac{1}{2}k^2 D^{-1}(\mathbf{x}, \mathbf{x})) - \frac{1}{2}k^2 D^{-1}(\mathbf{x}', \mathbf{x}')]_{\text{av}}, \quad |\mathbf{x} - \mathbf{x}'| \rightarrow \infty. \quad (\text{A5})$$

Since widely separated points have independent environments we have

$$\begin{aligned} \chi_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') &\rightarrow [\exp(-\frac{1}{2}k^2 D^{-1}(\mathbf{x}, \mathbf{x}))]_{\text{av}} \\ &\times [\exp(-\frac{1}{2}k^2 D^{-1}(\mathbf{x}', \mathbf{x}'))]_{\text{av}} \\ &\rightarrow M_{\mathbf{k}}^2, \quad |\mathbf{x} - \mathbf{x}'| \rightarrow \infty. \end{aligned} \quad (\text{A6})$$

The second equality in Eq. (A6) follows from comparison with an analysis of $M_{\mathbf{k}}$ in Eq. (2.10) which is similar to that given above for $G_{\mathbf{k}}$. It remains to identify the physical interpretation of $M_{\mathbf{k}}$. For infinitely separated points we have $R(\mathbf{x}, \mathbf{x}') = D^{-1}(\mathbf{x}, \mathbf{x}) + D^{-1}(\mathbf{x}', \mathbf{x}')$, and in a homogeneous system these diagonal Green's functions would be independent of position. [In a finite system, $D^{-1}(\mathbf{x}, \mathbf{x})$ is infinite, of course. In the thermodynamic limit this quantity is well-defined in spatial dimension greater than two.] Thus for a homogeneous system we would identify $2D^{-1}(\mathbf{x}, \mathbf{x})$ as being the resistance from site \mathbf{x} to a site infinitely far away. We do that here, noting that this quantity has a probability distribution of finite width, in view of the fact that the local environment of site \mathbf{x} influences the value of the resistance from this site to infinity. Thus we write

$$M_{\mathbf{k}} = [\exp(-\frac{1}{2}k^2 D^{-1}(\mathbf{x}, \mathbf{x}))]_{\text{av}} = [\exp(-\frac{1}{4}k^2 R_{\infty}(\mathbf{x}))]_{\text{av}}, \quad (\text{A7})$$

where this can be taken as a definition of what is meant by the resistance from \mathbf{x} to infinity, denoted $R_{\infty}(\mathbf{x})$. For $\mathbf{k} \rightarrow 0$, $M_{\mathbf{k}}$ becomes the percolation order parameter,⁴² the probability of being in the infinite cluster. Note that the resistance between two widely separated sites obeys

$$[R(\mathbf{x}, \mathbf{x}')]_{\text{av}} = [R_{\infty}(\mathbf{x}) + R_{\infty}(\mathbf{x}')]_{\text{av}}/2. \quad (\text{A8})$$

APPENDIX B. DERIVATION OF Eq. (4.25)

In this appendix we derive Eq. (4.25). We start by deriving the following lemma:

$$x(p^2x)^l = p^l x^{l+1} p^l, \quad (\text{B1})$$

where $p \equiv d/dx$. In our notation the commutation relations are

$$px^k - x^k p = kx^{k-1}, \quad (\text{B2a})$$

$$p^k x - x p^k = k p^{k-1}. \quad (\text{B2b})$$

It is easy to explicitly verify that Eq. (B1) is true for $l=0$ and $l=1$. Now we prove by induction that this relation holds for general l . Thus we assume that Eq. (B1) holds for $l=n-1$ and use this hypothesis to show that this equation holds for $l=n$. Thus we write

$$C_n \equiv x(p^2x)^n = xp^2[x(p^2x)^{n-1}] = xp^2 C_{n-1}, \quad (\text{B3})$$

which, by the inductive hypothesis is

$$C_n = xp^2(p^{n-1}x^n p^{n-1}) = (xp^{n+1})x^n p^{n-1}. \quad (\text{B4})$$

Using Eq. (B2b) we have

$$\begin{aligned} C_n &= [p^{n+1}x - (n+1)p^n]x^n p^{n-1} \\ &= p^n [px^{n+1} - (n+1)x^n] p^{n-1}. \end{aligned} \quad (\text{B5})$$

Now applying Eq. (B2a) we obtain

$$C_n = p^n (x^{n+1} p) p^{n-1}, \quad (\text{B6})$$

which is the desired relation for $l=n$. Thus, by induction, Eq. (B1) is true for general l .

We now turn to Eq. (4.11c) and expand in the right-hand side in powers of \mathbf{k} :

$$\delta\Pi_{\mathbf{k}} = \sum_t \delta\Pi_t k^{2t} = \sum_{p,t} \frac{1}{2t!} D(p^2)(\mathbf{k} \cdot \nabla_p)^{2t} D(p^2), \quad (\text{B7})$$

where we explicitly assume that the propagator is a function of p^2 . It is clear that $\delta\Pi$ is a rotationally invariant function of \mathbf{k} . We therefore can replace the right-hand side of Eq. (B7) by its average (denoted $\langle \rangle_{\theta}^{(n)}$) over orientations of \mathbf{k} . For an arbitrary vector \mathbf{x} , we know that

$$\langle (\mathbf{k} \cdot \mathbf{x})^{2t} \rangle_{\theta}^{(n)} = (k^2 x^2)^t \langle \cos^{2t}\theta \rangle_{\theta}^{(n)}, \quad (\text{B8})$$

where $\langle \cos^{2t}\theta \rangle_{\theta}^{(n)} \sim \text{const}/n + O(1)$, for small n . Since ∇_p transforms as a vector, the Wigner-Eckart theorem⁵⁰ allows us to write

$$\langle (\mathbf{k} \cdot \nabla_p)^{2t} \rangle_{\theta}^{(n)} = k^{2t} (\nabla_p^2)^t \langle \cos^{2t}\theta \rangle_{\theta}^{(n)}. \quad (\text{B9})$$

We now set

$$\nabla_p^2 = \frac{\partial^2}{\partial p^2} + \frac{n-1}{p} \frac{\partial}{\partial p} = 4p^2 \left[\frac{\partial}{\partial p^2} \right]^2 + 2n \frac{\partial}{\partial p^2} \quad (\text{B10})$$

and, for integration of angle-independent integrands,

$$\sum_p = K_n \int_0^\infty p^{n-1} dp, \quad (\text{B11})$$

where $K_n \sim n$ is the surface area of an n -dimensional hypersphere for $n \rightarrow 0$. Thus

$$\delta\Pi_t = \frac{1}{2t!} \langle \cos^{2t}\theta \rangle_{\theta}^{(n)} K_n I_t(n), \quad (\text{B12})$$

where, with $x = p^2$,

$$I_t(n) = \int_0^\infty \frac{1}{2} x^{n/2} \frac{dx}{x} D(x) \left[4x \frac{d^2}{dx^2} + 2n \frac{d}{dx} \right]^t D(x). \quad (\text{B13})$$

As $n \rightarrow 0$, the factor $\langle \cos^{2t}\theta \rangle_{\theta}^{(n)} K_n$ is finite and nonzero, so we consider $I_t(n)$ in this limit. The term $2nd/dx$ is only relevant when it produces the constant term within the square brackets. Otherwise, the integration is non-singular in the $n \rightarrow 0$ limit and the result is proportional to n . In the expansion of the t th power of the Laplacian in Eq. (B13) one sees that only the single term where the factor $2nd/dx$ occurs once and, in that case, as the left-most part of the differential operator (so as to remove the x factor from the operator $4xd^2/dx^2$) is relevant. Thus for $n \rightarrow 0$ we may write

$$\begin{aligned} I_t(n) &= \int_0^\infty dx \left[D(x) \frac{1}{2x} (4xp^2)^t D(x) \right. \\ &\quad \left. + nx^{(n-2)/2} D(x) p (4xp^2)^{t-1} D(x) \right], \end{aligned} \quad (\text{B14})$$

where p is again d/dx . Note that we cannot yet let $n \rightarrow 0$ in the second term in the integrand of Eq. (B14). We now integrate this term by parts [$nx^{n/2} dx = 2d(x^{n/2})$] and note that for $n > 0$ the boundary contributions at $x=0$ and at $x=\infty$ drop out, the former due to the factor $x^{n/2} \rightarrow 0$ and the latter since we assume that the integral converges for large x . After this operation, we can let $n \rightarrow 0$ and we obtain the result

$$I_t(n) = -2 \int_0^\infty dx [pD(x)][p(4xp^2)^{t-1} D(x)], \quad (\text{B15a})$$

where the derivatives are restricted to act within the square brackets. Applying the lemma of Eq. (B1), we obtain

$$I_t(n) = -(2)^{2t-1} \int_0^\infty dx [pD(x)][p^{t-1} x^{t-1} p^t D(x)]. \quad (\text{B15b})$$

Now we integrate by parts $t-1$ times, again noting no contribution from boundary terms, so that

$$I_t(n) = (-1)^{t-1} \int_0^\infty dx [d^t D(x)/dx^t]^2 (x)^{t-1}. \quad (\text{B16})$$

It remains only to evaluate the other factors in Eq. (B12). For $n \rightarrow 0$ we have

$$\langle \cos^{2t}\theta \rangle_{\theta}^{(n)} K_n \rightarrow 2t!(t!)^{-1}(t-1)!^{-1} 2^{1-2t}. \quad (\text{B17})$$

Substituting Eqs. (B16) and (B17) into Eq. (B12) yields finally

$$\delta\Pi_t = \frac{(-1)^t}{t!(t-1)!} \int_0^\infty [d^t D(x)/dx^t]^2(x) x^{t-1} dx, \quad (\text{B18})$$

which is Eq. (4.25) of the text when the explicit form of the propagator is used for $D(x)$.

APPENDIX C. DIRECT CALCULATION OF VERTEX FUNCTIONS

In this appendix we give a simplified calculation which leads to the results of Eq. (5.27) without identifying the nonlinear scaling fields. We use Eqs. (5.6), (5.10b), and (5.11b) to write

$$\Gamma_j(r(0), \{w_t(0)\}) = \exp \left[- \int_0^{l^*} [2 - \eta(l)] dl \right] w_j(l^*) \left[1 - \frac{g^*}{4} \int_0^{r^{-1}(l^*)} \frac{x^2 dx}{(1+x)^3} \bar{K}_j[w_t(l^*) r^{t-1}(l^*) w_1^{-t}(l^*) (1+x)^{t-1}] \right], \quad (\text{C1})$$

where $\bar{K}_j = [1 + r(l)]^3 \Pi_j / w_j$ is a function of the arguments indicated. In terms of Π_j we can write

$$w_j(l^*) = w_j(0) \exp \left[\int_0^{l^*} [2 - \eta(l)] dl - \frac{1}{2} g^* \int_0^{l^*} \frac{dl}{[1 + r(l)]^3} \bar{K}_j \{ w_t(l) w_1^{-t}(l) [1 + r(l)]^{t-1} \} \right]. \quad (\text{C2})$$

We now convert the integral over l in Eq. (C2) into one over the variable $x = 1/r$ from $1/r(l^*)$ to $1/r(0)$. To order ϵ the term of order ϵ in Γ_j can be exponentiated. Then, combining Eqs. (C1) and (C2) gives

$$\Gamma_j(r(0), \{w_t(0)\}) = w_j(0) \exp \left[- \frac{g^*}{4} \int_0^{r^{-1}(0)} \frac{x^2 dx}{(1+x)^3} \bar{K}_j [w_t(0) r^{t-1}(0) w_1^{-t}(0) (1+x)^{t-1}] \right]. \quad (\text{C3})$$

In writing this equation we also used the fact that to order ϵ , $w_t(l) r^{t-1}(l) / w_1^t(l) = w_t(0) r^{t-1}(0) / w_1^t(0)$.

Note that this form shows explicitly that the result for $\Gamma_j(0)$ does not depend on the choice of l^* . The most singular part of Γ_j comes from the term in F_j of order w_1^j / w_j , which is obtained from Eqs. (4.15d) and (4.28b) as

$$F_j^{\text{sing}} \sim (-1)^j (a_j - 2) w_1^j / [w_j r^{j-1} (1+x)^{j-1}], \quad (\text{C4})$$

where a_j was given in Eq. (4.28b). In this way we obtain the dominant contribution to Γ_j (for $j > 1$) as

$$\Gamma_j(r(0), \{w_t(0)\}) \sim \frac{1}{2} g^* \frac{(2 - a_j)}{(j^2 - 1)j} \left[\frac{-w_1(0)}{r(0)} \right]^j r(0), \quad j > 1, \quad (\text{C5})$$

where we have omitted subdominant singular terms. To order ϵ this result is equivalent to Eq. (5.26c).

APPENDIX D. GENERALIZATION TO m -COMPONENT SPIN SYSTEMS

To start it is useful to note that Eq. (3.9) can be viewed as an expansion in orthogonal polynomials in the two components, x_1 and x_2 , of a vector constrained to have unit magnitude. To see this write

$$Q_0^0 = 1, \quad (\text{D1a})$$

$$Q_{\pm 1}^1 = x_1 \pm i x_2 \equiv e^{\pm i\theta}, \quad (\text{D1b})$$

and generally

$$Q_{\pm 1}^k = (x_1 \pm i x_2)^k = e^{\pm i k \theta}. \quad (\text{D1c})$$

For two-component vectors there are just two eigenvectors of angular momentum of magnitude k . To generalize to three components we should expand in spherical harmonics. However, the analysis can be carried out for a general number of spin components, m . In this case the expansion will be in terms of k th-order polynomials in the variables x_1, x_2, \dots, x_m which are orthogonal on the unit sphere. As will become evident, we do not have to explicitly construct this family of polynomials. We will label the $n(k)$ k th-order polynomials as

$$Q_\mu^{(k)}(\Omega), \quad \mu = 1, 2, \dots, n(k), \quad (\text{D2})$$

where Ω is the set of $m-1$ spherical angles needed to specify the vector x_i . Each $Q_\mu^{(k)}$ is a homogeneous polynomial of degree k which satisfies

$$\nabla^2 Q_\mu^{(k)} = 0. \quad (\text{D3})$$

We will assume that these polynomials are *real* and have been orthogonalized, but may not be normalized. However, for a given value of k they have equivalent normalization so that under rotations in m -component space, the functions $Q_\mu^{(k)}$ transform into one another by orthogonal matrices. The normalized functions, $\hat{Q}_\mu^{(k)} \equiv b_k Q_\mu^{(k)}$ will also be used. We can choose one angle θ via $\cos\theta = x_m$, and there will be a single k th-order polynomial which remains nonzero for $\cos\theta = 1$, which we label $Q_0^{(k)}$, in analogy with the spherical harmonic Y_k^0 . Thus Eq. (3.9) will be generalized by replacing $\psi_{k_\alpha}(\mathbf{x})$ by $Q_{\mu_\alpha}^{(k)}(\mathbf{x})$.

We therefore wish to determine the expansion coefficient $F_l(\mathbf{k}, \mu)$ analogous to that in Eq. (3.11). For this purpose we do not worry about discretizing the angular phase space. We set

$$F_l(\mathbf{k}, \boldsymbol{\mu}) = \prod_{\alpha=1}^n f_l(k_\alpha, \mu_\alpha)$$

and we study $f_l(k_\alpha, \mu_\alpha)$ for a single replica. Let $\hat{\mathbf{v}}_{\mathbf{x}}$ and $\hat{\mathbf{v}}_{\mathbf{x}'}$ be spin vectors associated with two sites \mathbf{x} and \mathbf{x}' as in Eq. (3.9). We expand the interaction term as

$$e^{K \cos \theta} \equiv e^{K \hat{\mathbf{v}}_{\mathbf{x}} \cdot \hat{\mathbf{v}}_{\mathbf{x}'}} = \sum_k f_k Q_0^{(k)}(\cos \theta). \quad (\text{D4})$$

Using orthogonality we determine the expansion coefficients as

$$f_k = \frac{\int e^{K \cos \theta} Q_0^{(k)}(\cos \theta) d\Omega^{(m)}}{\int [Q_0^{(k)}(\cos \theta)]^2 d\Omega^{(m)}}, \quad (\text{D5})$$

where $d\Omega^{(m)}$ is the angular measure on the m -dimensional unit sphere. We obtain $Q_0^{(k)}$ from the generating function solution to Laplace's equation in m dimensions:

$$(1 - 2x \cos \theta + x^2)^{(2-m)/2} = \sum_k x^k Q_0^{(k)}(\cos \theta). \quad (\text{D6})$$

Consider the relation

$$Q_0^{(k)}(\cos \theta) Q_0^{(k)}(1) = \sum_{\boldsymbol{\mu}} Q_{\boldsymbol{\mu}}^{(k)}(\hat{\mathbf{v}}_{\mathbf{x}}) Q_{\boldsymbol{\mu}}^{(k)}(\hat{\mathbf{v}}_{\mathbf{x}'}), \quad (\text{D7})$$

where from Eq. (D6) we have $Q_0^{(k)}(1) = (m-3+k)! / [(m-3)!k!]$. The right-hand side of Eq. (D7) is summed over all the $n(k)$ orthogonal polynomials of degree k . Using the fact that under rotations in m dimensions they transform according to orthogonal matrices, one can show that the right-hand side of Eq. (D7) is a rotational invariant which can be evaluated in a special coordinate system to give the result on the left-hand side. Also note that the functions $Q_{\boldsymbol{\mu}}^k$ can be normalized by

$$\hat{Q}_{\boldsymbol{\mu}}^{(k)} = Q_{\boldsymbol{\mu}}^{(k)} / \left[\int [Q_0^{(k)}(\cos \theta)] d\Omega^{(m)} \right]^{1/2}, \quad (\text{D8a})$$

$$= Q_{\boldsymbol{\mu}}^{(k)} \left[\frac{2\pi^{m/2}}{(m-3)!} \frac{(m+k-3)!}{k!(\frac{1}{2}m-2)!(\frac{1}{2}m+k-1)} \right]^{-1/2}. \quad (\text{D8b})$$

Thus

$$e^{K \cos \theta} = \sum_{k, \boldsymbol{\mu}} \hat{f}_k(K) \hat{Q}_{\boldsymbol{\mu}}^{(k)}(\hat{\mathbf{v}}_{\mathbf{x}}) \hat{Q}_{\boldsymbol{\mu}}^{(k)}(\hat{\mathbf{v}}_{\mathbf{x}'}), \quad (\text{D9})$$

where

$$\hat{f}_k(K) = \int e^{K \cos \theta} d\Omega^{(m)} [Q_0^{(k)}(\cos \theta) / Q_0^{(k)}(1)]. \quad (\text{D10})$$

We will eventually need the product over replicas of \hat{f} 's: $F_l(\mathbf{k}, \boldsymbol{\mu}) = \prod_{\alpha} f_{k_{\alpha}}(lK)$. However, we do not wish to analyze the nonspherical terms which are generalizations of that proportional to b_{2c} in Eq. (3.15). Therefore it suffices to make an evaluation to leading order in $1/K$. For this purpose we set $\cos \theta = 1 - \tau$, where $\tau \sim K^{-1}$. Also $d\Omega^{(m)} \sim \sin^{m-2} \theta$, so that we may write

$$\frac{\hat{f}_k}{\hat{f}_0} = \frac{(m-3)!k!}{(m-3+k)!} \frac{\int_0^{\infty} e^{-K\tau} \tau^{(m-3)/2} Q_0^{(k)}(1-\tau) d\tau}{\int_0^{\infty} e^{-K\tau} \tau^{(m-3)/2} d\tau}. \quad (\text{D11})$$

This expression is best evaluated using the generating function of Eq. (D6). The result is

$$\frac{\hat{f}_k}{\hat{f}_0} = \sum_{r=0}^k \left[-\frac{1}{2K} \right]^r \frac{k!}{r!(k-r)!} \frac{(m+k-r-3)!}{(m+k-3)!} \quad (\text{D12a})$$

$$= 1 + \sum_{r=1}^k \left[-\frac{1}{2K} \right]^r \frac{1}{r!} \prod_{p=1}^r [L^2 + (m+p-3)(1-p)] \quad (\text{D12b})$$

$$\approx \exp \left[-\frac{L^2}{2K} \right], \quad (\text{D12c})$$

where L^2 is the angular momentum in m dimensions:

$$L^2 = k(k+m-2). \quad (\text{D13})$$

In going from Eq. (D12a) to (D12c) we kept only the leading term in $1/K$ corresponding to a given power of L^2 .

We let \mathbf{k} and $\boldsymbol{\mu}$ be the m -component vectors k_1, k_2, \dots, k_m and $\mu_1, \mu_2, \dots, \mu_m$. Following Sec. III, we introduce fields $\phi_{\boldsymbol{\mu}}^{\mathbf{k}}(\mathbf{x})$ conjugate to $\prod_{\alpha} Q_{\mu_{\alpha}}^{(k_{\alpha})}(\hat{\mathbf{v}}_{\mathbf{x}})$. Then the mean-field propagator takes the form

$$G_{\mathbf{k}, \boldsymbol{\mu}}^{-1} = 1 + r + \sum_{t=1}^{\infty} w_t L^{2t}, \quad (\text{D14})$$

where L^{2t} is

$$L^{2t} = \left[\sum_{\alpha=1}^n L_{\alpha}^2 \right]^t. \quad (\text{D15})$$

As we have mentioned, the above analysis is confined to the "spherical" terms, i.e., those involving powers of L^2 .

We now investigate the recursion relations and will confine the analysis to w_1 . Note that the third-order coupling of the ϕ 's is

$$-\frac{1}{3!} u_3 \sum_{k_1, \mu_1} \sum_{k_2, \mu_2} \sum_{k_3, \mu_3} \sum_{\mathbf{x}} \int d\mathbf{x} \phi_{\mu_1}^{k_1}(\mathbf{x}) \phi_{\mu_2}^{k_2}(\mathbf{x}) \phi_{\mu_3}^{k_3}(\mathbf{x}) \prod_{\alpha=1}^n \int d\Omega^{(m)} Q_{\mu_{1\alpha}}^{(k_{1\alpha})}(\Omega) Q_{\mu_{2\alpha}}^{(k_{2\alpha})}(\Omega) Q_{\mu_{3\alpha}}^{(k_{3\alpha})}(\Omega). \quad (\text{D16})$$

Here $\sum_{\mathbf{k}, \boldsymbol{\mu}} = \prod_{\alpha=1}^n \left[\sum_{k_{\alpha}, \mu_{\alpha}} \right]$, where μ_{α} is summed over $n(k_{\alpha})$ values and k_{α} is summed from 0 to ∞ . Using this interaction we find the contribution to the recursion relation of Eq. (4.7) from Fig. 4 as

$$\begin{aligned} \Pi(k, \mu; k', \mu') = \sum''_{\mathbf{k}_1, \mathbf{k}_2} \sum''_{\mu_1, \mu_2} G(\mathbf{k}_1, \mu_1) G(\mathbf{k}_2, \mu_2) \prod_{\alpha=1}^n \left[\int d\Omega^{(m)} \widehat{Q}_{\mu_\alpha}^{(k_\alpha)}(\Omega) \widehat{Q}_{\mu_{1,\alpha}}^{(k_{1,\alpha})}(\Omega) \widehat{Q}_{\mu_{2,\alpha}}^{(k_{2,\alpha})}(\Omega) \right. \\ \left. \times \int d\Omega'^{(m)} \widehat{Q}_{\mu'_\alpha}^{(k'_\alpha)}(\Omega') \widehat{Q}_{\mu_{1,\alpha}}^{(k_{1,\alpha})}(\Omega') \widehat{Q}_{\mu_{2,\alpha}}^{(k_{2,\alpha})}(\Omega') \right], \end{aligned} \quad (\text{D17})$$

where the primes on the summation indicate that the values $\mathbf{k}_1=0$ and $\mathbf{k}_2=0$ are excluded. One can show that symmetry requires Π to be diagonal in both \mathbf{k} and μ , and that it be independent of μ . Summing over μ , repeatedly using Eq. (D6), and noting that $\widehat{Q}_0^{(k)}(1)^2 \int d\Omega^{(m)} = n(k)$, we get

$$\Pi_{\mathbf{k}} = \sum''_{\mathbf{k}_1, \mathbf{k}_2} G(\mathbf{k}_1) G(\mathbf{k}_2) \prod_{\alpha=1}^n \left[\int d\Omega^{(m)} \widehat{Q}_0^{k_\alpha}(\cos\theta) \widehat{Q}_0^{k_{1,\alpha}}(\cos\theta) \widehat{Q}_0^{k_{2,\alpha}}(\cos\theta) [\widehat{Q}_0^{k_{1,\alpha}}(1) \widehat{Q}_0^{k_{2,\alpha}}(1) / \widehat{Q}_0^{k_\alpha}(1)] \right]. \quad (\text{D18})$$

In writing this result we used the fact that G also does not depend on μ . Note that the excluded terms simply give $2G(\mathbf{k})G(0)$ as in the xy model. If we neglected the dependence of the G 's on \mathbf{k}_1 and \mathbf{k}_2 , then the sums over these variables would produce a δ function setting $\cos\theta=1$. When this dependence is not neglected, the integrand will still be dominated by $\cos\theta \approx 1$. We set $\cos\theta = 1 - \tau$, with $\tau \sim 1/K$ and find, using Eq. (D6)

$$[\widehat{Q}_0^{(k)}(1 - \tau) / \widehat{Q}_0^{(k)}(1)] = 1 - \frac{\tau L^2}{(m-1)} + O(\tau^2), \quad (\text{D19})$$

This result will suffice to give the part of $\Pi_{\mathbf{k}}$ of order L^2 correct to leading order in $1/K$ as required for the recursion relation for w_1 . To evaluate the integral in Eq. (D18) we use the following recursion relation obtained by differentiating Eq. (D6) with respect to x :

$$(k+1)Q_0^{(k+1)}(y) + (k+m-3)Q_0^{(k-1)}(y) = (m+2k-2)yQ_0^{(k)}(y). \quad (\text{D20})$$

Then, taking the normalization constant from Eq. (D8b), we find

$$\int \cos\theta \widehat{Q}_0^{(k)}(\cos\theta) \widehat{Q}_0^{(k+1)}(\cos\theta) d\Omega^{(m)} = \left[\frac{(k+1)(m+k-2)}{(m+2k)(m+2k-2)} \right]^{1/2}. \quad (\text{D21})$$

Note that τ has matrix elements in which k is increased or decreased by one unit. Also since $\tau = 1 - \cos\theta$ has a unit constant term, it gives rise to a unit diagonal term in the matrix element of Eq. (D18). Thus to order L^2 the angular integrations in Eq. (D18) yield

$$\begin{aligned} \Pi_{\mathbf{k}} = & -2G(\mathbf{k})G(0) + G^2(0) - \frac{L^2}{m-1} \sum_{\mathbf{p}} \left[G(\mathbf{p}) \prod_{\alpha=1}^n \widehat{Q}_0^{(p_\alpha)}(1) \right]^2 \\ & + \frac{L^2}{m-1} \sum_{\mathbf{p}} G(\mathbf{p}) G(\mathbf{p}^+) \prod_{\alpha=1}^n \left[\widehat{Q}_0^{(p_\alpha)}(1) \widehat{Q}_0^{(p_\alpha^+)}(1) \left[\frac{(p_1+1)(m+p_1-2)}{(m+2p_1-2)(m+2p_1)} \right]^{1/2} \right] \\ & + \frac{L^2}{m-1} \sum_{\mathbf{p}} G(\mathbf{p}) G(\mathbf{p}^-) \prod_{\alpha=1}^n \left[\widehat{Q}_0^{(p_\alpha)}(1) \widehat{Q}_0^{(p_\alpha^-)}(1) \left[\frac{(p_1)(m+p_1-3)}{(m+2p_1-4)(m+2p_1-2)} \right]^{1/2} \right], \end{aligned} \quad (\text{D22})$$

where $p_\alpha^\pm = p_\alpha$ for $\alpha > 1$ and $p_1^\pm = p_1 \pm 1$. Substituting into Eq. (D22) we obtain $\Pi_{\mathbf{k}} = \Pi_{\mathbf{k}}^{(0)} + \delta\Pi_{\mathbf{k}}$ as in Eqs. (4.10b), with $\Pi_{\mathbf{k}}^{(0)}$ as in Eq. (4.11a) and

$$\delta\Pi_{\mathbf{k}} = \frac{L^2}{m-1} \sum_{\mathbf{p}} G(\mathbf{p}) \left[\prod_{\alpha=1}^n \frac{(m+p_\alpha-3)!}{(m-3)!p_\alpha!} \left(\frac{1}{2}m+p_\alpha-1 \right) \right] \left[\left[\frac{m+p_1-2}{2p_1m-2} \right] G(\mathbf{p}^+) - G(\mathbf{p}) + \left[\frac{p_1}{2p_1+m-2} G(\mathbf{p}^-) \right] \right]. \quad (\text{D23})$$

The integral is dominated by large values of p_α , so that $(m+p_\alpha-3)!/(p_\alpha)! \approx p_\alpha^{m-3}$. Also we use Eq. (D14) dropping terms with $t > 1$. Then we have

$$\delta\Pi_{\mathbf{k}} = \frac{L^2}{m-1} \sum_{\mathbf{p}} \left[\prod_{\alpha=1}^n p_\alpha^{m-3} (2p_\alpha+m-2) \right] [-(m-1)w_1 G^3(\mathbf{p}) + w_1^2 (2p_1+m-2)^2 G^4(\mathbf{p})], \quad (\text{D24})$$

where we have dropped terms which lead to contributions which are higher than linear in w_1 . To proceed we replace the sums by integrals and set

$$G^n(\mathbf{p}) = \frac{1}{(n-1)!} \int_0^\infty \lambda^{n-1} e^{-\lambda(1+r+w_1L^2)} d\lambda. \quad (\text{D25})$$

In this way the contribution to $\delta\Pi_{\mathbf{k}}$ from the term in Eq. (D24) proportional to $G^3(\mathbf{p})$ is found to be $-\mathbf{L}^2 w_1 G^3(0)$, and the other term is

$$\frac{\mathbf{L}^2}{m-1} \frac{w_1^2}{3!} \int_0^\infty \lambda^3 d\lambda e^{-\lambda(1+r)} \frac{\int_0^\infty p_1^{m-3} (2p_1+m-2)^3 e^{-\lambda w_1 p_1 (p_1+m-2)} dp_1}{\int_0^\infty p_1^{m-3} (2p_1+m-2) e^{-\lambda w_1 p_1 (p_1+m-2)} dp_1}, \quad (\text{D26})$$

where the denominator comes from $n-1 \rightarrow -1$ identical integrals over replicas $2, 3, \dots, n$. These integrals are easily evaluated for small w_1 . In this way we find

$$\delta\Pi_{\mathbf{k}} = w_1 \mathbf{L}^2 G^3(0) \left(-1 + \frac{2}{3}\right) + \mathcal{O}(w_1^{3/2}). \quad (\text{D27})$$

Comparison with Eq. (4.22) indicates that we have obtained the same result for ϕ_1 for general m as for the ($m=2$) xy model. This is expected, since the equations for the transverse spin components for $m \geq 2$ is equivalent to the circuit equations for the corresponding resistor network.

APPENDIX E. CALCULATION OF THE EXPONENT ϕ_{2c}

In this appendix we calculate the exponent ϕ_{2c} that distinguishes the xy model from the resistor network. To do this we need to find the contribution to $\Pi_{\mathbf{k}}$ that is linear in both w_{2c} and $\sum_{\alpha} k_{\alpha}^4$. Accordingly, we may set all potentials other than w_1 and w_{2c} equal to zero. We expand $\Pi_{\mathbf{k}}$ in powers of w_{2c} :

$$\delta\Pi_{\mathbf{k}} = \sum_{\mathbf{p}} D(\mathbf{p} + \frac{1}{2}\mathbf{k}) D(\mathbf{p} - \frac{1}{2}\mathbf{k}) \quad (\text{E1a})$$

$$\sim \sum_{\mathbf{p}} \left[D_0(\mathbf{p} + \frac{1}{2}\mathbf{k}) - w_{2c} D_0^2(\mathbf{p} + \frac{1}{2}\mathbf{k}) \sum_{\alpha} (p_{\alpha} + \frac{1}{2}k_{\alpha})^4 \right] \left[D_0(\mathbf{p} - \frac{1}{2}\mathbf{k}) - w_{2c} D_0^2(\mathbf{p} - \frac{1}{2}\mathbf{k}) \sum_{\alpha} (p_{\alpha} - \frac{1}{2}k_{\alpha})^4 \right], \quad (\text{E1b})$$

where $D_0^{-1}(\mathbf{p}) = 1 + w_1 p^2$. Denoting the part of $\delta\Pi_{\mathbf{k}}$ linear in w_{2c} by $\delta\Pi_{\mathbf{k},1}$, we have

$$\delta\Pi_{\mathbf{k},1} = -2w_{2c} \sum_{\mathbf{p}} \left[\sum_{\alpha} (p_{\alpha} + \frac{1}{2}k_{\alpha})^4 \right] D_0^2(\mathbf{p} + \frac{1}{2}\mathbf{k}) D_0(\mathbf{p} - \frac{1}{2}\mathbf{k}). \quad (\text{E2})$$

We use the representation

$$D_0^k(\mathbf{p}) = \frac{1}{(k-1)!} \int_0^\infty \lambda^{k-1} d\lambda e^{-\lambda(1+w_1 p^2)}, \quad (\text{E3})$$

so that

$$\delta\Pi_{\mathbf{k},1} = -2w_{2c} \int_0^\infty d\lambda e^{-\lambda} \lambda \int_0^\infty d\mu e^{-\mu} \sum_{\mathbf{p}} \exp[-w_1(\lambda+\mu)(p^2 + \frac{1}{4}k^2) - w_1(\lambda-\mu)\mathbf{p}\cdot\mathbf{k}] \left[\sum_{\alpha} (p_{\alpha} + \frac{1}{2}k_{\alpha})^4 \right]. \quad (\text{E4})$$

We set $p_{\alpha} = \phi_{\alpha} + \frac{1}{2}(\mu-\lambda)k_{\alpha}/(\mu+\lambda)$, so that

$$\delta\Pi_{\mathbf{k},1} = -2w_{2c} \int_0^\infty d\lambda e^{-\lambda} \lambda \int_0^\infty d\mu e^{-\mu} \left\{ \sum_{\phi} e^{-w_1(\lambda+\mu)\phi^2} \exp[-w_1\lambda\mu k^2/(\lambda+\mu)] \left[\sum_{\alpha} \left[\phi_{\alpha} + \frac{\mu}{\mu+\lambda} k_{\alpha} \right]^4 \right] \right\}. \quad (\text{E5})$$

The contribution linear in $Q \equiv \sum_{\alpha} k_{\alpha}^4$ (which we denote $\delta\Pi_{2c}$) is thus

$$\delta\Pi_{2c} = -2w_{2c} Q \int_0^\infty d\lambda e^{-\lambda} \lambda \int_0^\infty d\mu e^{-\mu} \left[\frac{\mu}{\lambda+\mu} \right]^4 \left[\sum_{\phi} e^{-w_1(\lambda+\mu)\phi^2} \right] \quad (\text{E6a})$$

$$= -2w_{2c} Q \int_0^\infty d\lambda e^{-\lambda} \lambda \int_0^\infty d\mu e^{-\mu} \left[\frac{\mu}{\lambda+\mu} \right]^4 \quad (\text{E6b})$$

$$= -\frac{2}{15} w_{2c} \sum_{\alpha} k_{\alpha}^4, \quad (\text{E6c})$$

Remembering that $\Pi^{(0)}$ contributes a term to the recursion relation for w_{2c} that has the same coefficient as that for all of the other potentials, we obtain

$$\phi_{2c} = 1 - \frac{1}{4} \left(\frac{2}{15} \right) \mathbf{g}^* \quad (\text{E7a})$$

$$= 1 + \frac{1}{105} \epsilon. \quad (\text{E7b})$$

This answer, as well as the calculation leading thereto, is essentially identical to that for the lowest-order noise crossover exponent.²⁷

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