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Spin-1/2 Bosonization on Compact Surfaces

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We extend an existing Bose-Fermi equivalence formula to two-dimensional Euclidean spacetimes with arbitrary compact topology. The result relates the nonchiral Dirac partition function to that of a scalar field, times a theta function. The proof is a short application of methods from complex geometry and Quillen's determinant constructions.

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1. Introduction

Two-dimensional field theories exhibit so many bizarre and amusing special features that at times they seem not to be field theories at all, but rather something simpler and more algebraic. At the classical level we see this in the trivial observation that in Minkowski spacetime massless particles divide into left- and right-movers which do not mix, even in the presence of background gauge fields. In Euclidean language this becomes the statement that the wave operators for fields of any spin are Cauchy-Riemann $\bar{\partial}$ operators. Zero modes of such operators, for example, are analytic functions on spacetime X, or more generally holomorphic sections of some bundle on X.

In the past few months, however, a much deeper connection to complex analysis has emerged at the *quantum* level. This began with the paper of Quillen[1], and has continued for example in [2][3][4][5][6][7]. Some of these papers were concerned with string theory, but the techniques apply to 2d field theory in general.

Quillen observed that usually we are interested not in a single wave operator $\bar{\partial}$ but in a *family* of such operators, and that frequently this family depends holomorphically on some complex parameter space, so that complex analyticity plays a double role. For example, the gauged scalar operator $\bar{D}_A \equiv \partial_{\bar{z}} + A_{\bar{z}}$ depends on the complex vector potential $A_{\bar{z}} = A_1 + iA_2$ but not on $\overline{A_{\bar{z}}}$. Quillen showed that this second analytic structure can survive after quantization. Powerful tools from complex geometry are then at our disposal to help investigate these quantum theories.

The main point of this letter is to give an example of such tools at work. We will prove a formula for the Dirac partition function which was derived assuming bosonization by Alvarez-Gaumé, Moore, and Vafa[5]. Turning this around, the present derivation provides a rigorous justification for spin-1/2 bosonization on surfaces of arbitrary compact topology. This relation also provides one of the steps in the general bosonization proof announced in an earlier letter[8].The key principle we will use is the well-known fact that on a compact complex manifold the only global analytic functions are the constants. To show that two things are equal up to a constant we thus need only show that they differ by an analytic function. The steps we actually take will rely on some complicated constructions, but these are used only to maneuver us into position to apply the above simple observation. Many more interesting identities needed in string theory and elsewhere can also be proved along these lines[8], in particular using the deep methods of [1] and [9].

Recently Friedan, Martinec, and Shenker have also obtained bosonization results from a different approach[10]. Analyticity also plays the key role in the recent paper of Friedan and Shenker[11].

2. Bosonization

Bosonization refers to the complete quantum-mechanical equivalence of a Bose field theory with a Fermi theory, and in particular to a mapping between all the observables of the two. the Here we will consider the less ambitious project of showing that a Fermi partition function when summed over spin structures equals that of a Bose theory[1] :

$$Z_{BOSE} \stackrel{?}{=} \sum_{\text{spin str.}} Z_{FERMI} \quad . \tag{2.1}$$

Both quantities are functionals of a given background 2-metric, so we do have something very nontrivial to check.

Specifically, consider a single boson with values in the circle $U(1)$, on a compact surface X with a fixed number of handles g. This field can wind

[1] The more general analysis of [8] is able to bosonize a *single* spin structure.

various times around the $2g$ noncontractible loops of Euclidean spacetime, and we know that to get a Fermi equivalence we will need to sum over all possible windings, *i.e.* all instanton sectors. To prove Bose-Fermi equivalence, we will quote an expression for the Bose partition function, derive another expression for the Fermi system, and note that they are the same.

In [5] the instanton-sector sum for the Bose theory was done carefully, yielding

$$Z_{BOSE} = \left(\frac{\det' \bar\partial^\dagger \bar\partial}{\det(\omega^i, \omega^j) \int_X \sqrt{g}}\right)^{-\frac{1}{2}} \sum_{\epsilon_1, \epsilon_2} \left|\vartheta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(0|\tau)\right|^2 \quad . \tag{2.2}$$

Briefly the notation is as follows (see [5] and [12]): We have chosen a slice, that is a representative 2-metric g on X for each gauge-equivalence class of metrics. These are parametrized by a finite-dimensional space of equivalence classes, the "moduli space" $\mathcal{M}$ of surfaces of genus g. For each metric $\int_X \sqrt{g}$ is the corresponding area of X. We have also chosen a reference set of $2g$ noncontractible loops on X. These define: (i)local coordinates τ_{ij} for $\mathcal{M}$ analogous to the one complex parameter τ on a torus; (ii) a preferred basis $\omega_z^1, \cdots, \omega_z^g$ of holomorphic 1-forms on X, whose inner products we have taken; and (iii) a Riemann theta function ϑ. The vectors (ϵ_1, ϵ_2) run over a discrete set of 4^g points, generalizing the four different ϑ-functions on a torus, where $g = 1$. After this sum is performed (2.2) is modular invariant.

The functional determinant in (2.2) has periodic boundary conditions. All determinants are understood to be ζ-function regulated.

On the other hand, in [5] it was also proved that for a single Dirac fermion with fixed boundary conditions (spin structure) one has

$$Z_{FERMI} \equiv \det(\partial\!\!\!/_R \partial\!\!\!/_L)_{\epsilon_1,\epsilon_2} = |c(\tau)|^2 \left|\vartheta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(0|\tau)\right|^2 \quad . \tag{2.3}$$

Here ϵ_1, ϵ_2 describe the spin structure. c is some function of the background metric independent of ϵ_1, ϵ_2.

Equations (2.2) and (2.3) are certainly compatible with the bosonization statement (2.1). Indeed (2.1) says precisely that (2.3) can be refined to[5]

$$\left[\det(\partial\!\!\!/_R \partial\!\!\!/_L)_{\epsilon_1,\epsilon_2}\right]^2 = \left(\frac{\det' \bar\partial^\dagger \bar\partial}{\det(\omega^i, \omega^j) \cdot \int_X \sqrt{g}}\right)^{-1} \left|\vartheta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(0|\tau)\right|^4 \quad . \tag{2.4}$$

We will prove (2.4), and hence in particular (2.1), up to an overall multiplicative constant.

Eqn. (2.4) is very plausible. It is exactly true on the torus[5]. Another check is that the anomaly we get when we vary the metric slice is the same for each side[5]. Even though anomalies are local on the world sheet, and so seem insensitive to the global topology, we will see how this observation is crucial to extending (2.4) to arbitrary genus.

3. Proof of (2.4)

Quillen's essential construction involved a line bundle over moduli space $\mathcal{M}$ with a special metric, which we will briefly recall.

The family of operators $\bar\partial$ takes functions on X to (0,1)-forms. It depends parametrically on the complex structure we have chosen for X, *i.e.* on $\mathcal{M}$. We will need two key facts about $\mathcal{M}$:

a) $\mathcal{M}$ is itself a connected complex space, and $\bar\partial$ varies holomorphically (see the review [13]).

b) For genus $g > 2$ any globally defined analytic function on $\mathcal{M}$ is constant[14]. In this sense, $\mathcal{M}$ is "almost compact." This comes from the fact that $\mathcal{M}$ can be embedded into a compact space $\tilde{\mathcal{M}}$ by adding a set of points of complex dimension less than $(dim\mathcal{M}) - 1$. Roughly speaking this means that any analytic (or pluriharmonic) function on $\mathcal{M}$ extends to the compact $\tilde{\mathcal{M}}$ and so is constant [15].

While these two facts are hard to prove, they are easy to apply. They are at the heart of the simplicity of 2d field theory.

We are interested in operators like $\bar\partial$ or $\partial\!\!\!/_L$ which map from one bundle to another. Such a family gives rise to a determinant line bundle $\mathrm{DET}\,\bar\partial$ over $\mathcal{M}$ [16], whose fiber over any point $X \in \mathcal{M}$ is the vector space $\wedge^{\max}(\ker\bar\partial)^{-1} \otimes \wedge^{\max}(\ker\bar\partial^\dagger)$. The inverse refers to the dual line bundle. In the present setting this bundle is holomorphic[1][17].

Similarly we obtain $\mathrm{DET}\partial\!\!\!/_L$, with one new feature: we must now replace $\mathcal{M}$ by the space $\mathcal{S}$ of Riemann surfaces with spin structure. Since (2.3) is trivially zero for some spin structures (the "odd" ones[5]), we will consider only the remaining "even" ones. Then $\mathcal{S}$ is a covering space of $\mathcal{M}$, for which properties (a) and (b) continue to hold. We will trivially lift the $\bar\partial$ family to $\mathcal{S}$ as well in order to compare it to $\partial\!\!\!/_L$.

We thus get a family of spin bundles $\mathcal{L}$ over $\mathcal{S}$. By definition $\mathcal{L}$ is a square root of the cotangent space $\mathcal{T}$, in the sense that over any surface X, $\mathcal{T} \cong \mathcal{L}\otimes\mathcal{L}$. We also have that the left-handed derivative $\partial\!\!\!/_L$ is just the $\bar\partial$ operator coupled to $\mathcal{L}$, *i.e.* $\partial\!\!\!/_L = \bar\partial_{\mathcal{L}}$ takes sections of $\mathcal{L}$ to spinors of the opposite chirality, in $\mathcal{T}\otimes\mathcal{L}$.

The index theorem for families lets one compare various determinant bundles, just as in the analysis of chiral anomalies[16][2] . In our case an easy verification shows that

$$(\mathrm{DET}\,\bar\partial_{\mathcal{L}})^{\otimes 2} \cong (\mathrm{DET}\,\bar\partial)^{-1} \quad . \tag{3.1}$$

The computation is the same as the one which shows that the anomalies match in (2.4)[3] .

[2] Actually the version we need is the older Riemann-Roch-Grothendieck theorem.

[3] Strictly speaking the RRG theorem gives (3.1) up to torsion. This subtlety doesn't matter for nonchiral bosonization.

As it stands, (3.1) is of no use in proving (2.4), since it says nothing about functional determinants! The DET bundles are simply built up from zero modes of $\bar\partial_{\mathcal{L}}$, $\bar\partial$, and their adjoints. Quillen introduced determinants into the game by defining a *metric* on a general DET bundle. For example, if $\omega^1_z, \cdots \omega^g_z$ are a basis of holomorphic 1-forms on X then $\bar\omega^i_{\bar z}$ span the kernel of $\bar\partial^\dagger$. Also the kernel of $\bar\partial$ just contains the constant function $\varphi_0(x) \equiv 1$, so that $\sigma \equiv \varphi_0^{-1} \otimes (\bar\omega^1 \wedge \cdots \wedge \bar\omega^g)$ is a vector in $\mathrm{DET}\bar\partial$. (The inverse refers to the dual section.) Declare its norm to be $\|\sigma\|^2_Q \equiv \|\varphi_0\|^{-2}\,\|\bar\omega^1 \wedge \cdots \wedge \bar\omega^g\|^2 \cdot \det{}'\,\bar\partial^\dagger\bar\partial$, where on the right we use ordinary metric norms.

Thus

$$\|\sigma\|^2_Q = \left(\textstyle\int_X \sqrt{g}\right)^{-1} \det(\omega^i,\omega^j)\,\det{}'\,\bar\partial^\dagger\bar\partial \quad . \tag{3.2}$$

On the other hand, the Dirac operator usually has no zero modes at all. The highest exterior power of a trivial vector space is just a copy of the complex numbers C, so we can consider the section s which for most X equals $1 \in$ C. For this section we then have

$$\|1\|^2_Q = \det{}'\bar\partial^\dagger_{\mathcal{L}}\bar\partial_{\mathcal{L}} \quad . \tag{3.3}$$

Quillen showed that (3.2)-(3.3) make sense everywhere. Furthermore, his norm has the remarkable property that isomorphic bundles given by the Riemann-Roch theorem, such as (3.1), actually have the *same curvature* in his metric connection[1][2][3][4][5].

We thus have a pleasant confluence of the mathematical desire to introduce good norms with nice properties, and our own need to say something about field theory: Quillen's norm accomplishes both. If we can show that the isomorphism I in (3.1) is an *isometry* then we will relate the two partition functions. Already we can see that (3.1) has the correct powers to give (2.4).

To establish the isometry, use the fact (b) above that spinmoduli space $\mathcal{S}$ is "almost" compact. Given any section s of $\mathrm{DET}\bar\partial_{\mathcal{L}}$ we know that the curvatures

$\partial\bar\partial \log \|s \otimes s\|_Q^2$ and $\partial\bar\partial \log \|I(s \otimes s)\|_Q^2$ agree. Thus $\|s \otimes s\|_Q^2$ and $\|I(s \otimes s)\|_Q^2$ must themselves agree up to an analytic function[4] , and so they agree up to a constant. A similar argument shows that in fact I itself is unique up to a constant, since any other I' is of the form $f \cdot I$, for an analytic function f.

We now know that (3.1) is an isometry, and so to finish the proof of (2.3) we need only find the explicit form of the isomorphism I. This can be found for example in section 6 of [9]. The idea is that given a section s of $\mathrm{DET}\,\bar\partial_{\mathcal{L}}$ we can multiply it by a Riemann theta function, which vanishes exactly when $\det \not\partial_R \not\partial_L$ does. This almost suffices to make $s \cdot \vartheta$ an ordinary function on $\mathcal{S}$, but now we have introduced a dependence on the basis ω^i of 1-forms used in ϑ. Faltings notes that this dependence is just right to make $(s \cdot \vartheta)^2$ actually transform like $(\bar\omega^1 \wedge \cdots \wedge \bar\omega^g)^{-1}$, and so it gives a section of $(\mathrm{DET}\,\bar\partial)^{-1}$.

For example, consider again the section $s \equiv 1$ of $\mathrm{DET}\,\bar\partial_{\mathcal{L}}$ when $\not\partial_L$ has no zero modes. Then the precise formula in our notation says that[5]

$$I(1 \otimes 1) = \left(\vartheta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(0|\tau)\right)^2 \cdot \det(\omega^i, \omega^j) \cdot \varphi_0 \otimes (\bar\omega^1 \wedge \cdots \wedge \bar\omega^g)^{-1} \quad , \qquad (3.4)$$

where ϵ_1, ϵ_2 label the spin structure and again φ_0 is just the constant function on X. Since we know $\|s \otimes s\|_Q^2 = \|I(s \otimes s)\|_Q^2$, we collect (3.2),(3.3),(3.4) and recall that $\bar\partial_{\mathcal{L}} = \not\partial_L$ to obtain

$$[\det(\not\partial_R \not\partial_L)_{\epsilon_1,\epsilon_2}]^2 = \left(\frac{\det' \bar\partial^\dagger \bar\partial}{\det(\omega^i, \omega^j) \cdot \int_X \sqrt{g}}\right)^{-1} \left|\vartheta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(0|\tau)\right|^4 \quad ,$$

as was to be shown.

In fact we have only shown (2.4) up to a multiplicative constant K_g which could depend on the genus. In a string theory, where we work with various different genera, it may be possible to relate all of the K_g to the known $K_1 = 1$ by letting the Riemann surface X degenerate.

[4] Strictly speaking, up to the exponential of a pluriharmonic function.
[5] Again up to torsion, which again does not matter.

4. Remarks

Actually in [5] the authors proved something stronger than (2.3): they gave the fermions *arbitrary* twists, then found (2.3) still held with ϵ_1, ϵ_2 not necessarily at the 4^g places describing untwisted spin structures. It is not hard to generalize our discussion to give the corresponding extension of (2.4).

It is important to have bosonization formulæ which apply to spins other than 1/2, in order to get at the ghost determinants of the superstring [10][5]. In [8] the approach of [1] and [9] is used to relate the determinants of $\bar\partial$ coupled to quite arbitrary bundles.

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