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Introduction to Sigma Model Anomalies

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INTRODUCTION TO SIGMA MODEL ANOMALIES †

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I'd like to tell you about two different kinds of results which A. Manohar, G. Moore, and I have recently been studying. [1][2][3]In each case the point is that some nonlinear sigma models coupled to chiral fermions are just bad. You can't just write them down at random and assume that they make quantum mechanical sense, just as you can't do this for chiral gauge theories. In each case a theory which was perfectly well-defined classically turns out to have quantum problems due to an anomaly.

One of our results deals with the specific case in which a sigma model with fermions arises as a result of dynamical symmetry breakdown from some group G to a subgroup H[3]. H may contain chiral symmetries protecting some fermions in a representation ρ_H of H. In this case it may be impossible to quantize the remaining fermions in a way which reproduces the anomalous Ward identities of *any* underlying strongly-interacting gauge theory, and so we conclude that this pattern of symmetry breakdown will not be realized in any such theory. The criterion in this case is *local* in character, since roughly speaking once we know how the sigma model in question behaves for field configurations close to the vacuum base point in the internal manifold, we can use symmetry to find how it behaves everywhere else.

If we drop the assumption that the internal space is homogeneous, then no particular local behavior can be excluded. Accordingly, in this broader class of models we get a weaker condition based only on the *global* topology of the configuration space. The analysis of this case is harder than the special case. An example of this type of situation is the compactified heterotic string[4], in which the internal space need have no symmetries at all. We then find that permissible compactifications obey a topological condition. I'll begin with this global argument, since in fact it came first.

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Recall that in gauge theory the problem with anomalies is that the fermion effective action $\Gamma_f[A]$ is not gauge-invariant on the space $\mathcal{A}$ of vector potentials. We can rephrase this by saying that $\Gamma_f[A]$ doesn't define a function $\Gamma_f[\bar{A}]$ on the true configuration space $\mathcal{C} \equiv \mathcal{A}/\mathcal{G}$ of vector potentials modulo gauge transformations. But it does come close. In fact the effective action does have a geometrical interpretation on $\mathcal{C}$, but only as a *section* of a (possibly twisted) line bundle over it[5][6][7]. In other words, there is an obstruction to defining $\Gamma_f[\bar{A}]$ which is like the obstruction to thinking of the wavesection for a Schrödinger particle near a magnetic monopole as an ordinary function. Since only bona fide functions can be integrated over $\mathcal{C}$, we cannot quantize the bosons in this theory. But since line bundles are classified by H^2, very little can go wrong: either the theory has a problem visible on some two-sphere in $\mathcal{C}$, or else it has none at all. Therefore we can without loss of generality consider not all of $\mathcal{C}$, but only two-parameter families in it.

If we can think of the gauge anomaly as an obstruction rather than as the failure of some kind of symmetry, then we can consider other theories, perhaps with no symmetries at all, which have complicated configuration spaces. These too will in general have anomalies obstructing their quantization.

Sigma models are examples of such theories. A bosonic nonlinear sigma model has degrees of freedom φ which are maps from (compactified) spacetime S^4 into some manifold M. To couple these "pions" to fermions in a geometrical way, consider the special case of supersymmetry. Here for every point x in spacetime $\psi(x)$ is a tangent vector to M at $\varphi(x)$, in order for the transformation law $\delta\varphi \propto \bar{\epsilon}\psi$ to make sense. For a given boson configuration φ, then, we paste together all the tangent spaces $TM|_{\varphi(x)}$ into one bundle called the *pullback* $\varphi^*(TM)$ over spacetime.

To complete this picture we must tensor in the spin bundle over spacetime. We also need to replace φ by a two-parameter family $\hat{\varphi}$, ie a function from $S^2 \times S^4 \to M$. Finally, we generalize TM, which was appropriate for supersymmetry, to an arbitrary bundle B over M. For example, in the CCWZ prescription[8] for writing down phenomenological lagrangians we will use $M = G/H$ and B will be associated to the bundle $G \to M$ by the representation ρ_H. Whatever the origin of B, we can now proceed to define a Dirac operator for the fermions in the presence of a background boson field φ, giving us a well-defined classical sigma model with fermions.

To quantize this system we begin with the fermions. Thus we have to make sense out of the functional integral $\exp[-\Gamma_f[\varphi]] = \int d\bar{\psi}\, d\psi\, \exp[-\int \bar{\psi} \not{D}_\varphi \psi]$. As usual we turn the functional integral into countably many ordinary integrals by choosing an eigenmode expansion $\{u^i_\varphi(x)\}$ for $\not{D}^\dagger \not{D}$ and $\{v^i_\varphi(x)\}$ for $\not{D}\not{D}^\dagger$. Thus u_φ's are an orthonormal *frame* for the Hilbert space $\mathcal{H}^+_\varphi$ of positive-chirality wavefunctions in the background φ; similarly the v's frame $\mathcal{H}^-_\varphi$.

But what frame are we to use? If we were to choose some other orthonormal $u'^i_\varphi = U^{ij}(\varphi) u^j_\varphi$, $v'^i = V^{ij} v^j$, then the value we computed for $\exp[-\Gamma_f[\varphi]]$ would change by the phase $\det(UV^{-1})$. [1] Indeed, all that is happening here is that we are trying to find the (functional) determinant of an operator $\not{D}_\varphi$ which maps one space $\mathcal{H}^+_\varphi$ to a *different* space $\mathcal{H}^-_\varphi$ and so has no determinant. What we have done amounts to framing both spaces and taking the determinant of the *matrix* representing our operator; while this works fine at one fixed φ, it leaves open the possibility that we may paint ourselves into a corner trying to extend the framing smoothly throughout $\mathcal{C}$.

Mathematically, this means that $\exp[-\Gamma_f[\varphi]]$ must be regarded as a section of a line bundle $\mathcal{D} = \mathrm{DET}\mathcal{H}^+ - \mathrm{DET}\mathcal{H}^-$, where $\mathrm{DET}\mathcal{H}$ means the bundle whose transition functions are the determinants of those of $\mathcal{H}$, and the subtraction indicates that the second factor's transition functions are to be inverted as above. $\mathcal{D}$ can be rigorously defined using a cutoff, whereupon it is called the "index of the family of Dirac operators" parametrized by $\mathcal{C}$ (or in our case, by a two-sphere in $\mathcal{C}$). Now the computation of the twist of $\mathcal{D}$ by the above definition may seem a hopeless task. Incredibly, however, it turns out to be a topological invariant given by the Atiyah-Singer index theorem [10][5][11][2] as

$$(\text{anomaly}) = \int_{S^2 \times S^4} ch_3\, \hat{\varphi}^*(B).$$

For details on how to compute and interpret this invariant, see [2]. In particular, the het-

[1] This infinite-dimensional determinant turns out to be a local functional of φ — a Wess-Zumino term[9].

erotic string will be safe if its background Yang-Mills field strength equals its background Riemann curvature, since then the curvature form used to compute the Chern character has two cancelling pieces.

Thus the topology of the bundle B combines with that of the configuration space $\mathcal{C}$ to determine whether quantized fermions can exist at all in the given model.

Turning now to the case of dynamical symmetry breakdown to $M = G/H$, we now can and should ask more of a quantization than that it exist at all. Not only must we choose framings of $\mathcal{H}^\pm$ which *exist* globally (up to unimodular changes U, V), we must now require of them the right *variation* under symmetry transformations. We can phrase this in more familiar language if we work locally, so that there is no problem finding some framing of $\mathcal{H}^\pm$. We then choose any old framing, for example the one used in ordinary perturbation theory, and seek to fix it up. By the previous footnote, this amounts to finding a Wess-Zumino-type counterterm which modifies the anomalous current algebra of our model until it coincides with that of some underlying "preon" model.

Specifically, suppose the underlying model has fermions ("preons") transforming linearly in some representation ρ_G of G. Suppose our putative effective theory is a nonlinear sigma model with internal space G/H and residual fermions ("composite quarks") transforming in the representation ρ_H of H and constructed by the recipe of [8]. By the 't Hooft matching condition[12] we know that the H-anomalies of ρ_H must match those of $\rho_G|_H$, the restriction of ρ_G to H.

But what about the anomalies of G? In general the effective theory as we have quantized it will get them wrong. In general no local counterterm can be built from the "pion" fields which reproduces them. Can these two *conspire* to produce the correct Ward identities for all of G?

They can. Whenever the 't Hooft condition is satisfied we can write down a Wess-Zumino counterterm taking the G-variation of $\Gamma_f(\rho_H)$ to the desired variation of $\Gamma_f^{linear}(\rho_G)$ [3]. What is more, while the counterterm isn't globally defined, neither is $\Gamma_f(\rho_H)$, in such a way that the *sum* is global. Hence whenever the local condition applies and is satisfied, the global obstruction of the previous section vanishes too.

There is a quick and dirty argument for this result. To define the effective theory locally, we can use the prescription of [8] for the fermion effective action $\Gamma_f(\rho_H)$. We can conveniently describe the anomalous Ward identities for $\Gamma_f(\rho_H)$ by coupling the theory to external "flavor" gauge fields for G and discussing the anomalous variation of Γ_f under local G-transformations. Now consider adding the H-representation $\rho_G|_H$, and its complex conjugate. This does not change the anomalies, so we have

$$\Gamma_f(\rho_H) \doteq \Gamma_f(\rho_H + \bar{\rho}_G|_H) + \Gamma_f(\rho_G|_H) \ ,$$

where the symbol $\doteq$ means "has the same anomalous variation under G."

Recall that in the CCWZ prescription we begin by choosing a local section $s : G/H \to G$. For example, in [8] the section near the identity is $s_0(p) = e^{\xi \cdot X}$ where ξ are the normal coordinates of p and X^r are the broken generators. Define the function $h(p; g_0)$ by

$$g_0 s(p) = s(g_0 \cdot p) h(p; g_0)$$

where $h(p; g_0) \in H$. For the choice in [8] we thus have $h(p; g_0) = e^{u' \cdot T}$ where $u'(p; g_0)$ is defined by $g_0 e^{\xi \cdot X} = e^{\xi' \cdot X} e^{u' \cdot T}$ and T^a are the unbroken generators. Then under the transformation by g_0 the fermions rotate by $\rho_H(h(\varphi(x), g_0))$. But $\rho_H + \bar{\rho}_G|_H$ has no H-anomaly by hypothesis so the first term on the right hand side has no G-variation.

Moving on to the second term on the right, there the fermions correspond to those in the effective action of the underlying linear theory, $\Gamma_f^{linear}(\rho_G)$. Before we can conclude that $\Gamma_f(\rho_H) \doteq \Gamma_f^{linear}(\rho_G)$, however, we must investigate the transformation from the Fermi fields ψ in the CCWZ basis to fields Ψ transforming linearly under G. Following [8] this is

$$\rho_G(s(\varphi(x)))\,\psi(x) = \Psi(x) \ . \qquad (1)$$

Since the fields Ψ transform in the same way as the underlying fermions the flavor anomalies are the same. As shown in [9] in the case of the chiral quark model, this change of variables contributes an anomalous Jacobian factor to Γ_f which can be compensated by the addition of a local counterterm, and so we expect

$$\Gamma_f(\rho_H) + F \doteq \Gamma_f^{linear}(\rho_G) \ . \qquad (2)$$

Hence the addition of a local counterterm F should be sufficient to realize the G symmetry properly.

In general, the map s above will be defined only on a neighborhood $\mathcal{U} \subseteq G/H$, so we need to use a collection of maps $\{s_\alpha\}$ on patches $\{\mathcal{U}_\alpha\}$ covering G/H. These define a set of Γ_f^α, and by (1) a corresponding set of F^α. Since the right hand side of equation (2) is independent of s_α, we have that $\exp\left(-\Gamma_f^\alpha - F^\alpha\right)$ regarded as a section of a line bundle has trivial transition functions. Thus the global anomaly of [1] is absent.

A more convincing argument would simply construct F. This is what we do in [3]. We show that appropriate F exist for any of the various lagrangians constructed in [8], as long as the 't Hooft condition is satisfied. In particular there is one choice which makes it plain that the global obstruction vanishes.

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