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Some Simple Supermoduli Spaces

Philip Nelson[†]

Department of Physics
Boston University
Boston MA 02215, USA

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5/88

[†] Address after 1 September 1988: Department of Physics, University of Pennsylvania, Philadelphia, PA.

This talk outlines some recent work by Giddings, Rothstein, Vafa and myself on the geometry of supermoduli space.[1] Specifically we were looking for concrete examples which were so simple that one can describe completely the global geometry of the space, including for example the issue of splitness, after compactification. The example to be described below is the moduli superspace $\widehat{\mathcal{M}}_{0,4,0}$ of spheres with four Neveu-Schwarz punctures, and also briefly $\widehat{\mathcal{M}}_{1,2,0}$.[2] The former space was chosen for scrutiny because its dimension is $1|2$ and its body is an ordinary sphere. Thus it has the potential to be isomorphic to the simplest nonsplit space, discussed in Giddings' talk. In fact we will see that this space is split — but not in a "nice " way to be defined below. Indeed no "nice" splitting exists, a fact with implications for the consistency of string amplitudes.

First let's recall the bosonic situation. The moduli space $\mathcal{M}_0$ is of course trivial by the uniformization theorem: the sphere is conformally rigid. If we introduce punctures, however — ultimately the locations of vertex operators — then not all configurations are related by conformal automorphisms. In our case any configuration can be brought to one where in the z-plane punctures P_1, P_2, P_3 are located at $z = -1, 1, \infty$ while P_4 is at some arbitrary point $z = w$. To be somewhat pedantic we can say that we have a trivial family of Riemann surfaces of the form $C = \mathsf{P}^1 \times \mathsf{P}^1$ over a parameter space P^1; coordinates for the two spheres are z and w. C is called the universal curve; the parameter space is the moduli space. In addition we have four sections $s_i : \mathsf{P}^1 \to C$, of which three are at $z(w)$ =constant while the last is the diagonal, $z(w) = w$.

Conformal field theory tells us to be careful when two punctures collide. To avoid double poles we should perform another conformal map, from a family where $P_4 \to P_i$ to one where the P's stay fixed but a pinch develops:

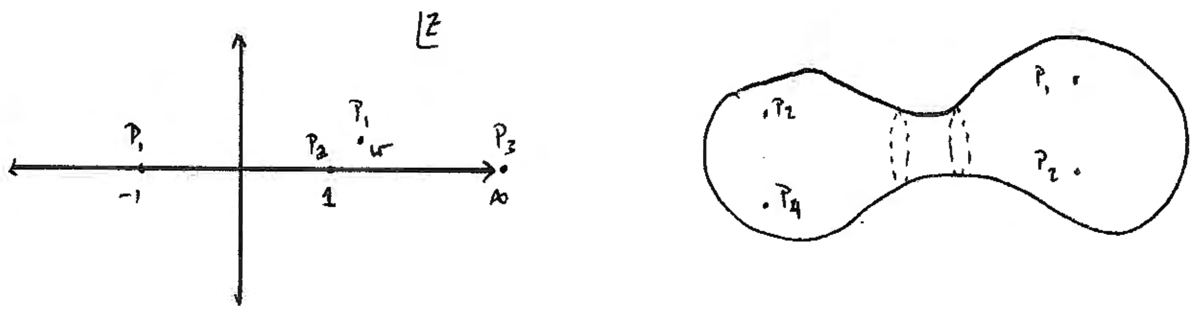

Fig. 1a,b

[1] General references on super Riemann surfaces (SRS) include [1]–[3] and references therein, as well as Giddings' talks in this volume.

[2] $\widehat{\mathcal{M}}_{g,p,q}$ denotes the moduli superspace of super Riemann surfaces (SRS) of genus g with p super and q spin punctures.

We can say this precisely. Within the parameter space P^1 we define an open set $\bar{U} = \mathsf{P}^1 \backslash \{\pm 1, \infty\}$. Over $\overline{U}$ we can put the product family and mark points as in Fig. 1a. Next we define little open sets U_-, U_+, U_∞, each of them a small disk with coordinate q_-, q_+, q_∞ centered at $w = \pm 1, \infty$ respectively. Over U_+, for example, we now build a family of pinching curves as in Fig. 2 (see *e.g.* [4]). Each curve will be built from a fixed disk on the left with coordinate x and two fixed punctures at $x = -1, \infty$ and similarly a disk on the right with coordinate y and fixed punctures at $y = 1, \infty$, joined by a central tube. We take the latter to be the standard pinching family, or "plumbing fixture":

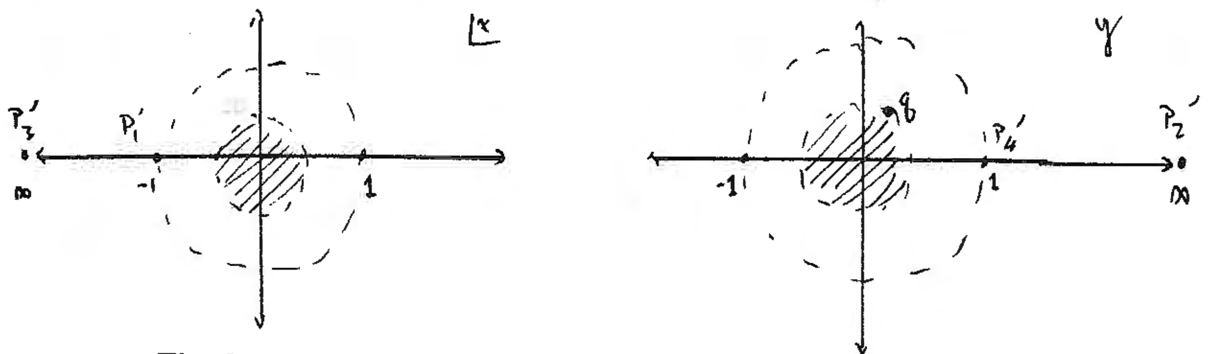

Fig. 2

Fig. 2 depicts one fiber of the family $C_+ \to U_+$; for each $q_+ \in U_+$ the shaded region is excluded while the annular regions are glued via $y = x^{-1} q_+$.

Having defined a family C_+ over U_+ we must now proceed to glue it to $\overline{C} = \overline{U} \times \mathsf{P}^1$ over $\overline{U}$. The gluing map must satisfy very restrictive conditions:

a) It glues fibers isomorphically to fibers. Thus $q_+(w; z)$ is independent of z, and $x(w; z)$ or $y(w; z)$ give a conformal automorphism of fig 1a to fig 2 for each fixed w near 1, $w \neq 1$.

b) It sends the four punctures P_i onto the P_i' in order.

It's easy to see that these requirements fix the gluing map uniquely:

$$x = \frac{1}{2}(z-1)$$
$$q_+ = \frac{1}{2}(w-1) \quad .$$

Since q_+ is a regular function of w all the way down to $w = 1$, we see that we may as well forget about q_+ and continue to use w as our coordinate throughout $\overline{U} \cup U_+$. Similarly w is a good coordinate near $w = -1$. In the neighborhood of $w = \infty$, however we find that the good pinching coordinate q_∞ is related to w by

$$q_\infty = w^{-1} \quad .$$

Thus $\mathcal{M}_{0,4}$ is covered by two patches with transition functions which make it into a sphere — as we guessed all along.

Matters become somewhat more interesting in superspace. Consider the fixed SRS $\mathsf{P}^{1|1}$ built from P^1 with its spin bundle, and the moduli space $\widehat{\mathcal{M}}_{0,4,0}$ with four NS punctures.[3] Now we find that automorphisms can fix the bosonic coordinates of 3 punctures, but the fermionic coordinates of just 2. We will depict a choice of coordinates by a picture:

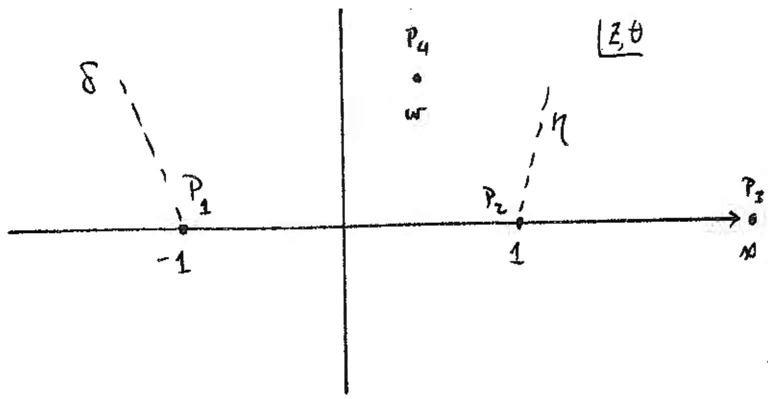

Fig. 3

The dashed lines indicate that P_1, P_2 are free to move in the odd directions while P_3, P_4 are fixed. Thus the points are at $(z, \theta) = (-1, \delta), (1, \eta), (\infty, 0), (w, 0)$. More precisely if we let $\widetilde{z} = -z^{-1}, \widetilde{\theta} = z^{-1}\theta$ then P_3 has $(\widetilde{z}, \widetilde{\theta}) = (0, 0)$. We thus get a trivial family $\widehat{\overline{C}} = \widehat{\overline{U}} \times \mathsf{P}^{1|1}$ of SRS with a nontrivial family of punctures, over the big open set $\widehat{\overline{U}} = \overline{U} \times \mathsf{C}^{0|2}$.

When we try to glue in $\widehat{U}_{\pm 1}, \widehat{U}_{\infty}$, however, we find a surprise. Following [6]–[8] the universal degeneration is now parametrized by $t_+; \gamma_+, \zeta_+$ with picture

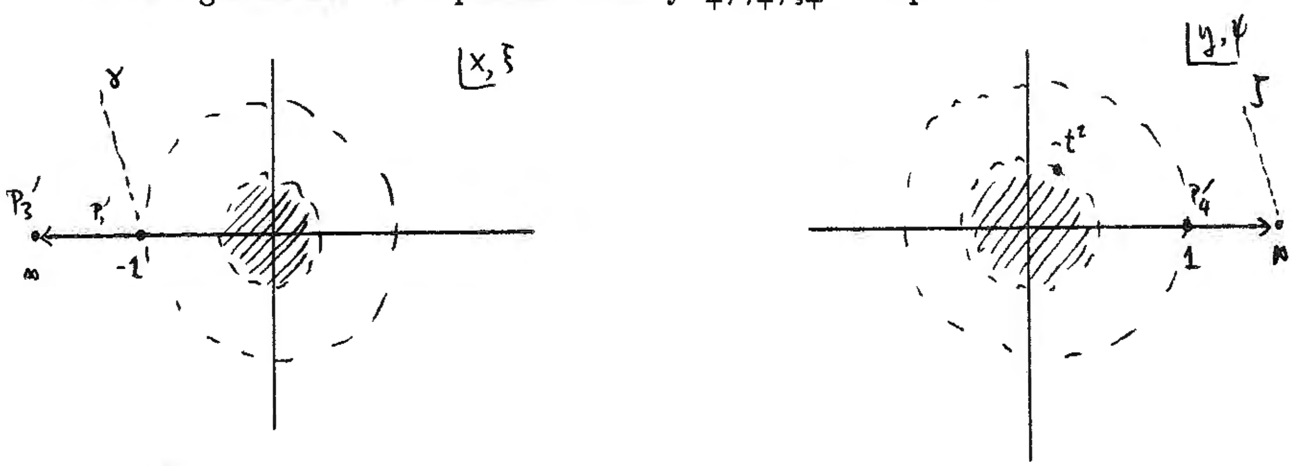

Fig. 4

[3] Punctures on SRS are discussed *e.g.* in [5].

Again this says P_1' is at $(x,\xi) = (-1,\gamma_+)$ etc. Now however the gluing map is:

$$xy = -t_+^2 \quad ; \; x\psi = -t_+\xi \quad ; \; y\xi = t_+\psi \quad ; \; \xi\psi = 0 \quad .$$

The surprise is that t_+^2, not q_+, parametrizes the pinch. This is necessary in order that ψ and ξ, which are spinors, should be single-valued with respect to each other as we walk around the "boundary" $t_+ = 0$ of supermoduli space, or equivalently that spinors on one side should have a good meromorphic extension to the whole family. Thus $\widehat{\mathcal{M}}$ must *ramify* over its 3 points at infinity! [7] Let r be a coordinate for this covering, with $w = \frac{1}{2}(r^2+r^{-2})$. Then the r-plane covers the w-plane four times, except at the points at infinity. The latter are located now at $w = \pm 1, \pm i, 0, \infty$.

Again we can glue the pinching family $\widehat{C}_{+1}$ to the big set $\widehat{\widehat{C}}$, requiring the map to satisfy

a′) It is a superconformal automorphism fiber by fiber on the overlap.

b′) It sends the super punctures to their counterparts.

This map is again unique. To describe it we first redefine δ, η to $\widetilde{\delta} = \left(r+r^{-1}\right)^{-1}\delta$, $\widetilde{\eta} = \left(r-r^{-1}\right)^{-1}\eta$. This is permissible since on $\widehat{\widehat{U}}$ the extra factors just introduced do not blow up. These factors rescale δ, η into coordinates regular at some of the points at infinity. Indeed, solving the above conditions one finds

$$\begin{aligned} z &= 2x+1 \\ \theta &= \sqrt{2}\xi \\ t_+ &= \frac{1}{2}\left(r-r^{-1}\right) \\ \zeta_+ &= -i\sqrt{2}\widetilde{\eta} \\ \gamma_+ &= \frac{1}{\sqrt{2}}\left(r+r^{-1}\right)\widetilde{\delta} \quad . \end{aligned} \tag{1}$$

Note that again $t_+; \zeta_+, \gamma_+$ are regular functions of $r; \widetilde{\eta}, \widetilde{\delta}$ as $r \to \pm 1$, so again on $\widehat{U}_{\pm 1}$ we can simply use the same coordinates as on the big patch $\widehat{\widehat{U}}$. A similar argument works on $\widehat{U}_{\pm i}$. We note from (1) that the new coordinates $r, \widetilde{n}, \widetilde{\delta}$ are related to the natural ones t, γ, ζ by a split transformation.

As before the last kind of pinch (or "channel") causes troubles. (There is of course nothing significant in the fact that one channel is singled out, since we have all along been treating the pinches asymmetrically.) The problem is that the coordinates depicted by

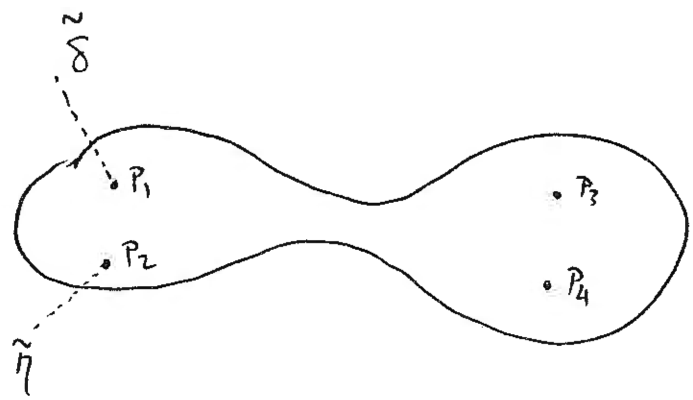

Fig. 5

become singular as $r \to 0, \infty$; unlike the previous cases the two free coordinates are on the same side, which is impossible at the pinch. Thus the unique gluing map for r close but not equal to 0 must include a super-Mobius transform taking one free odd coordinate to the other side. This yields

$$\begin{aligned} r - r^{-1} &= \frac{2i}{t}(1 - \frac{1}{2}\gamma_0\zeta_0(t_0^{-1} - t_0)) \\ \widetilde{\eta} &= \frac{1}{\sqrt{2}}\zeta_0 \\ \widetilde{\delta} &= \frac{1}{\sqrt{2}}\frac{r + r^{-1}}{r - r^{-1}}\zeta_0 + \frac{i\sqrt{2}}{r + r^{-1}}\gamma_0 \quad . \end{aligned}$$

In particular r is a *nonsplit* function of the natural coordinates $t_0; \gamma_0, \zeta_0$, and similarly at $r \to \infty$.

This does not mean that $\widehat{\mathcal{M}}_{0,4,0}$ cannot be split! Indeed we see that we can once again use r itself as the even coordinate on U_0, and $\widetilde{r} = r^{-1}$ on U_∞. Since r is a regular function of $t_0; \gamma_0, \zeta_0$ throughout $\widehat{U}_0$, we see that $r; \gamma_0, \zeta_0$ are good coordinates for $\widehat{U}_0$, and similarly near ∞; this suffices to show that $\widehat{\mathcal{M}}_{0,4,0}$ can be split. But something important has been lost: this time the new coordinates, while regular, are related to the natural ones by nonsplit transformations. In this sense the splitting given is not "good". A completely analogous situation obtains for $\widehat{\mathcal{M}}_{1,2,0}$, at least when the ordinary modulus is suppressed.

We can say this is an intrinsic way. A splitting, or just a projection, provides a map $\widehat{\mathcal{M}}_{0,4,0} \xrightarrow{\pi} \mathcal{M}_{0,4}$. The fibers of this map are divisors, subvarieties of codimension $1|0$ in $\widehat{\mathcal{M}}$. But $\widehat{\mathcal{M}}$ has more structure than just a supermanifold: it also comes equipped with a special divisor, the locus $\widehat{\Delta}$ of pinched SRS. Similarly $\mathcal{M}$ has a divisor Δ of pinched Riemann surfaces. The projection π defined by taking $r, \widetilde{r}$ to be the good even coordinates *conflicts* with this extra structure in the sense that $\widehat{\Delta} \neq \pi^{-1}(\Delta)$. For example at $r \to 0$

the divisor is given by $\{t_0 = 0\}$, while the fiber is given by $\{r = 0\}$. Since for small r we have $r \sim \frac{i}{2}(t_0 + \frac{1}{2}\gamma_0\zeta_0)$, these two divisors differ.

I should briefly discuss another approach to this problem. As on the ordinary sphere we can define a cross ratio

$$p = \frac{z_{13}z_{42}}{z_{14}z_{32}} \quad ; \; z_{ij} = z_i - z_j - \theta_i\theta_j \quad .$$

The quantity p is invariant under super-Mobius transformations. Naively it seems that we can take p and $\widetilde{p} = p^{-1}$ to be good even coordinates on all of $\widehat{\mathcal{M}}_{0,4,0}$, and clearly $\widetilde{p}$ is a split function of p and the odd coordinates — i.e. a function only of p. Moreover at each of the points at infinity as $t \to 0$ we have that p or $\widetilde{p}$ approach 1 or 0, independently of the odd parameters. Doesn't this mean that the even functions $p, \widetilde{p}$ define a splitting compatible with the divisor at infinity? Alas, no. Both p and $\widetilde{p}$ are bad coordinates at infinity. For example, as $t_+ \to 0$ we have $\widetilde{p} = \frac{3+t_+\gamma_+\zeta_+}{t_+^2+3}$, so $1 - \widetilde{p} \sim t_+^2$+nilpotent. A good coordinate would be $(1-\widetilde{p})^{1/2}$, but at $t_+ = 0$ this is not longer zero; instead $(1-\widetilde{p})^{1/2} \to -\gamma_+\zeta_+/2\sqrt{3}$. Again the "natural" choice of splitting doesn't match the one we want at the boundary.

There is a great deal of rigidity in an analytic object such as a splitting. Many supermanifolds admit no splitting at all; while $\mathcal{M}_{0,4,0}$ is not of this type, still it seems clear that a splitting compatible with its extra structure cannot be found. A real proof of this assertion requires that we find the space of all possible splittings and check them all. Briefly, the splittings form an affine space modeled on $H^0(\mathcal{T} \otimes \wedge^2\mathcal{E})$, where $\mathcal{T}$ is the tangent space to $\mathcal{M}$ and $\mathcal{E}$ is the bundle where the odd variables live[9]. In the present case this is a vector space of dimension five. But an element taking our bad projection to a good one must satisfy one condition at each of the six boundary points, and so no good choice exists.

Why does this matter? To compute 4-point string amplitudes we must integrate a density over $\widehat{\mathcal{M}}$. Even after the GSO projection this density will in general blow up at infinity, and so in general one needs to pick up a residue there. But in superspace the definition of residue is not so obvious; unlike a SRS $\widehat{\mathcal{M}}$ has no superconformal structure to help us out. We can always expand in powers of the pinch coordinate t — but only if the splitting used to integrate on the rest of $\widehat{\mathcal{M}}$ is compatible with the natural t. If this cannot be arranged, as happens even in this simple case, then corrections must be added to the integral, similar to the ones discovered by Green and Seiberg [10]. We are studying the general geometry of such terms and their relation to the integration ambiguity.

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