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A NOTE ON CONDITIONAL EXPONENTIAL MOMENTS AND ONSAGER-MACHLUP FUNCTIONALS

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It is proven that, for any deterministic $L^2[0, 1]$ function $\phi(t)$,

$$E\left(\exp \int_0^1 \phi(t)\, dw_t \,\middle|\, \|w\| < \varepsilon\right) \to 1 \quad \text{as } \varepsilon \to 0,$$

where w_t is a standard Brownian motion and $\|\cdot\|$ is any "reasonable" norm on $C_0[0, 1]$. Applications to the computation of Onsager-Machlup functionals are pointed out.

Let $\phi \in L^2[0, 1]$ be a given deterministic function. Let w_t be a standard Brownian motion, and define $I(\phi) \triangleq \int_0^1 \phi_t\, dw_t$. Denote by $|||\cdot|||$ the supremum norm on $[0, 1]$, and by $\|\cdot\|_2$ the L^2 norm on $[0, 1]$. In [2], it is shown that if for a given deterministic path ϕ_t, $t \in [0, 1]$, one has that

$$E(\exp I(\phi) \mid |||w||| < \varepsilon) \to 1 \quad \text{as } \varepsilon \to 0, \tag{1}$$

then the Onsager-Machlup functional computation for $\hat{\phi}_t = \int_0^t \phi_t\, dt$ (cf. [3], chapter 6, and [2]) follows, that is, if x_t is a diffusion satisfying

$$dx_t = f(x_t)\, dt + dw_t,$$

with f bounded and having two continuous bounded derivatives, then

$$\frac{\text{Prob}\{|||x - \hat{\phi}||| < \varepsilon\}}{\text{Prob}\{|||w||| < \varepsilon\}} \to \exp\left\{-\frac{1}{2}\int_0^1 \left[\left(\phi_t - f\left(\hat{\phi}_t\right)\right)^2 + f'\left(\hat{\phi}_t\right)\right] dt\right\} \quad \text{as } \varepsilon \to 0.$$

Further applications to construction of the skeleton of Wiener functionals follow and will be described elsewhere.

In [2], (1) is proven for $\phi \in C^\alpha[0, 1]$, for any $\alpha > 0$. The proof, which uses analytic methods, breaks down in the case of $\phi \in L^2[0, 1]$ and, moreover, is quite complex. In this note, we bring a new simple proof to (1), based on different methods, which allows for $\phi \in L^2[0, 1]$.

Denote by $\|\cdot\|$ any norm on $[0, 1]$ which dominates the L^1 norm, that is, for all measurable ψ on $[0, 1]$,

$$\int_0^1 |\psi|\, dt \le K\|\psi\|. \tag{2}$$

In particular, we wish to consider the case of the supremum norm. We claim the following.

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THEOREM. *Assume that, for any deterministic constant c,*

$$E(\exp cw_1 \mid \|w\| < \varepsilon) \to 1 \quad \text{as } \varepsilon \to 0. \tag{3}$$

Then

$$E(\exp I(\phi) \mid \|w\| < \varepsilon) \to 1 \quad \text{as } \varepsilon \to 0. \tag{4}$$

REMARK. (2) and (3) are trivially satisfied for "reasonable" norms, such as L^p norms, supremum norm and so on.

PROOF OF THE THEOREM. First, by Jensen's inequality, using the fact that the conditioning is symmetric and that $I(-\phi) = -I(\phi)$,

$$E(\exp I(\phi) \mid \|w\| < \varepsilon) \geq 1. \tag{5}$$

Let ϕ^δ be a $C^1[0,1]$ function such that $\|\phi - \phi^\delta\|_2 < \delta$. Note that by an integration by parts,

$$E\big(\exp(I(\phi^\delta))\big| \|w\| < \varepsilon\big) = E\left(\exp\left(\phi_1 w_1 - \int_0^1 w_t \phi'(t)\, dt\right)\middle| \|w\| < \varepsilon\right) \to 1 \quad \text{as } \varepsilon \to 0, \tag{6}$$

where the last limit follows by (2) and (3). Using now Schwartz's inequality, one has that

$$\begin{aligned} E(\exp I(\phi) \mid \|w\| < \varepsilon) \leq{}& E^{1/2}\big(\exp 2I(\phi^\delta)\big| \|w\| < \varepsilon\big) \\ &\times E^{1/2}\big(\exp 2\big(I(\phi) - I(\phi^\delta)\big)\big| \|w\| < \varepsilon\big). \end{aligned} \tag{7}$$

Due to (6), the first term in the right-hand side of (7) converges to 1, and due to (5), the theorem will hold true for any $\phi \in L^2[0,1]$ once we show that, for any $\psi \in L^2[0,1]$,

$$E(\exp I(\psi) \mid \|w\| < \varepsilon) \leq E(\exp|I(\psi)|). \tag{8}$$

Indeed, using the fact that $I(\psi)$ is a Gaussian random variable with zero mean and variance $\sigma_\psi^2 = \|\psi\|_2^2$, one has

$$\begin{aligned} E(\exp|I(\psi)|) &= \frac{2}{\sqrt{2\pi}\,\sigma_\psi}\int_0^\infty \exp\left(x - \frac{x^2}{2\sigma_\psi^2}\right) \\ &= \exp\left(\frac{1}{2}\sigma_\psi^2\right)\frac{2}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{(x-\sigma_\psi)^2}{2}\right)dx \to 1 \quad \text{as } \sigma_\psi \to 0. \end{aligned} \tag{9}$$

Using now $\psi = 2(\phi - \phi^\delta)$, noting that $\sigma_\psi^2 \leq \delta^2 \to 0$ as $\delta \to 0$ and substituting (8) and (9) into (6),

$$\lim_{\varepsilon \to 0} E(\exp I(\phi) \mid \|w\| < \varepsilon) \leq k(\delta),$$

where $k(\delta) \to 0$ as $\delta \to 0$, which together with (5) proves the theorem.

To prove (8), we will actually prove that, for any $\eta > 0$ and $\varepsilon > 0$,

$$Pr(|I(\psi)| < \eta \mid \|w\| < \varepsilon) \geq Pr(|I(\psi)| < \eta), \tag{10}$$

from which (8) follows immediately.

To see (10), let e_i denote an orthonormal base for $L^2[0,1]$ such that $e_1 = \psi$. Let $x_i \triangleq \int_0^1 e_i(t)\,dw_t$, and denote by $\mathbf{x}$ the infinite vector whose components are x_i. Note that the x_i are standard, i.i.d. normal random variables. By Theorem 2.1 of [1], one has that for any convex, symmetric set $C_k \subset R^k$,

$$(11)\quad Pr(|x_1| < \delta, (x_1, \ldots, x_k) \in C_k) \geq Pr(|x_1| < \delta) Pr((x_1, \ldots, x_k) \in C_k).$$

Let C be a convex, symmetric set in R^∞, and denote by C_k its projection on R^k. Clearly, C_k is convex and symmetric and therefore (11) applies to it. Using dominated convergence, one concludes that

$$(12)\qquad Pr(|x_1| < \delta, \mathbf{x} \in C) \geq Pr(|x_1| < \delta) Pr(\mathbf{x} \in C).$$

Therefore, choosing $C \triangleq \{\mathbf{x} | \|w\| < \varepsilon\}$, and noting that C is both symmetric and convex, we get from (12) that (10) holds true. Inequality (8), and therefore the theorem, follow. □

REMARK 1. Note that (10) is not a direct consequence of the FKG inequalities, except in the particular case when $\|\cdot\|$ is the L^2 norm. In other cases, the particular properties of elliptically contoured measures are needed for Theorem 2.1 of [1] to hold true, and then the properties of the Gaussian measure are needed to translate that into a statement concerning conditional probabilities as in (10).

REMARK 2. One could ask whether (4) holds true for functionals which are more complex than the Wiener integral. For an L^2 norm conditioning, it follows by the FKG inequalities that, for a class of symmetric and convex functionals, the answer is positive. This approach may be extended for some classes of functionals to other norm conditionings. To use the method presented previously, one would need the extension of Theorem 2.1 of [1] to the situation where the set $\{|x_1| < \delta\}$ is replaced by a general convex and symmetric set. This is stated as an open conjecture in [1] and, to our knowledge, is still an open problem.

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