Kelvin–Helmholtz instability for parallel flow in porous media: A linear theory

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Two fluid layers in fully-saturated porous media are considered. The lighter fluid is above the heavier one so that in the absence of motion the arrangement is stable and the interface is flat. It is shown that when the fluids are moving parallel to each other at different velocities, the interface may become unstable (the Kelvin–Helmholtz instability). The corresponding conditions for marginal stability are derived for Darcian and non-Darcian flows. In both cases, the velocities should exceed some critical values in order for the instability to manifest itself. In the case of Darcy's flow, however, an additional condition, involving the fluids' viscosity and density ratios, is required.

I. INTRODUCTION

Many technological processes involve the parallel flow of fluids of different viscosity and density through porous media. Such parallel flows exist in packed bed reactors in the chemical industry, in petroleum production engineering, in boiling in porous media (countercurrent flow of liquid and vapor), and in many other processes as well. Should the interface between the two fluids become unstable, a substantial increase in the resistance to the flow will result. This increase in resistance, in turn, may cause flooding in countercurrent packed chemical reactors and dryout in boiling in porous media. In the same vein, in petroleum production engineering, such instabilities may lead to emulsion formation. Hence, knowledge of the conditions for the onset of instability will enable us to predict the limiting operating conditions of the above processes. The purpose of this paper is to establish the condition for the onset of instability.

In the comparable case of parallel flow of continuum fluids (not through porous media), an instability of the interface may arise when the two fluids are in relative motion. This is known as the Kelvin–Helmholtz (KH) instability. The KH instability has been studied extensively for continuum, inviscid flows. A review of the classical work, which states conditions for marginal stability, is given in Chandrasekhar.[1] More recent contributions by Nayfeh[2] and Drazin[3] include the study of nonlinear effects.

In contrast, the KH instability for flow in porous media has attracted little attention in the scientific literature. Raghaven and Mardsen[4] have studied this problem for Darcy-type flow. They used linear stability analysis to obtain a characteristic equation for the the growth rate of the disturbance and then solved this equation numerically. They concluded that KH instability is possible only if the heavier fluid is overlying the light one (statically unstable situation).

This paper focuses on a statically stable case in which the lighter fluid is above the heavier one. We begin by analyzing the stability of a Darcy type flow (Sec. II). In contrast to Raghaven and Marsden[4], we obtain a closed form expression for the marginal stability, which indicates that KH instability may develop under certain conditions. The critical velocity predicted herein may be, however, somewhat high and above the range of validity of Darcy's law. Therefore, we extend the analysis (Sec. III) to non-Darcian flows by using the Forchheimer's equation,[5] which includes a term proportional to the velocity square. It is shown that this additional term has a destabilizing effect.

II. DARCY FLOW

Consider parallel flow of two immiscible fluids in an infinite, fully saturated, uniform, homogeneous and isotropic porous media with both constant porosity (ϕ) and constant permeability (λ). The two fluids are incompressible and have constant properties. The interface between the fluids is assumed to be well-defined and initially flat. In fact, a sharp interface between the two fluids may not exist. Rather, there is an ill-defined transition region in which the two fluids intermix. The width of this transition zone is usually small compared with the other characteristic length of the motion; hence, for the purposes of the mathematical analysis, we shall assume that the fluids are separated by a sharp interface. The interface between the two fluids forms angle α with the horizontal, and both fluids have uniform velocities U_1, U_2 parallel to the interface (Fig. 1). Assuming that Darcy's law is valid, we obtain the following continuity and momentum equations:

$$\nabla\cdot\mathbf{q}_i = 0, \tag{1}$$

$$\frac{\partial p_i}{\partial x} = -\frac{\rho_i}{\phi}\frac{\partial q_{x,i}}{\partial t} - \frac{\mu_i}{\lambda} q_{x,i} + \rho_i g \sin\alpha, \tag{2}$$

where $q_{x,i}$ denotes the x component of the Darcian velocity vector $\mathbf{q}_i$, p the pressure, x a coordinate parallel to the interface (Fig. 1), ρ the density, t the time, μ the viscosity, and g the gravitational acceleration. The suffix i designates the upper ($i = 1$) and the lower ($i = 2$) fluids.

We note in passing that the convective term, which appears in the Navier–Stokes equations $\mathbf{q}\cdot\nabla\mathbf{q}$ is absent in Eq. (2). This is because of the averaging process through which Darcy's equation (2) has been derived.[6] Indeed, Beck[7] shows that the inclusion of such a convective term is inconsistent with the slip boundary condition. It appears that if any nonlinear inertia term should be included at all, it will be of the form $|q|\mathbf{q}$ (i.e., Irmay[5]). For low Reynolds number flows 0(1),

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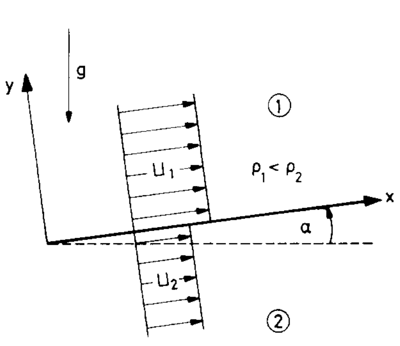

FIG. 1. Schematic statement of the problem.

the nonlinear term can be neglected. For higher Reynolds numbers, this term should be included as, in fact, we do in Sec. III.

The arrangement discussed herein is statically stable with the lighter fluid overlying the heavier one ($\rho_1 < \rho_2$). Also, the interface is initially stationary. That is, the pressure gradients are the same on both sides of the interface. This condition yields:

$$U_1 \mu_1 - U_2 \mu_2 = \lambda (\rho_2 - \rho_1) g \sin \alpha. \tag{3}$$

Next, we assume that there is an irrotational perturbation of the base flow (U_1, U_2), which causes an elevation of the interface to a new position $y = \eta(x, t)$, so that the velocity

$$\mathbf{q}_i = \nabla(U_i x + \Phi_i) \quad (i = 1 \text{ for } y > \eta,\ i = 2 \text{ for } y < \eta). \tag{4}$$

The perturbation velocity potential (Φ_i) satisfies

$$\nabla^2 \Phi_i = 0, \tag{5}$$

$$\nabla \Phi_i \to 0 \quad \text{as } y \to \mp \infty. \tag{6}$$

The kinematic and dynamic boundary conditions at the interface are:

$$\phi \frac{\partial \eta}{\partial t} + \left(U_i + \frac{\partial \phi_i}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{\partial \Phi_i}{\partial y} \quad (y = \eta), \tag{7}$$

$$\left(\frac{\rho_1}{\phi}\frac{\partial}{\partial t} + \frac{\mu_1}{\lambda}\right)\Phi_1 - \left(\frac{\rho_2}{\phi}\frac{\partial}{\partial t} + \frac{\mu_2}{\lambda}\right)\Phi_2 + \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} - g(\rho_2 - \rho_1)\eta \cos \alpha = 0 \quad (y = \eta), \tag{8}$$

where σ is the interfacial surface tension, and subscript x in Eq. (8) designates a derivative with respect to x.

We consider only disturbances parallel to the basic flow since we assume that, as in the case of Squire's theorem, these disturbances will grow most rapidly.

In order to reduce the interfacial conditions at $y = \eta$ to those at $y = 0$, we introduce (Drazin[3]) the operator

$$Q = \frac{\eta}{1!}\frac{\partial}{\partial y} + \frac{\eta^2}{2!}\frac{\partial^2}{\partial y^2} + \frac{\eta^3}{3!}\frac{\partial^3}{\partial y^3} + \ldots, \tag{9}$$

thereby getting

$$-\frac{\partial \Phi_i}{\partial y} + \phi \frac{\partial \eta}{\partial t} + U_i \frac{\partial \eta}{\partial x} = Q \frac{\partial \Phi_i}{\partial y} - \frac{\partial \eta}{\partial x}(1 + Q)\frac{\partial \Phi_i}{\partial x} \quad (y = 0), \tag{10}$$

$$\frac{1}{\phi}\frac{\partial}{\partial t}(\rho_1 \Phi_1 - \rho_2 \Phi_2) + \frac{1}{\lambda}(\mu_1 \Phi_1 - \mu_2 \Phi_2) + \sigma \eta_{xx} - g(\rho_2 - \rho_1)\eta \cos \alpha = \sigma \eta_{xx}\left(\frac{3}{2}\eta_x^2 - \frac{15}{8}\eta_x^4 + \cdots\right) - Q\left(\frac{1}{\phi}\frac{\partial}{\partial t}(\rho_1 \Phi_1 - \rho_2 \Phi_2) + \frac{1}{\lambda}(\mu_1 \Phi_1 - \mu_2 \Phi_2)\right) \quad (y = 0). \tag{11}$$

The linearized terms have been put on the left-hand side of Eqs. (10) and (11). When the nonlinear terms are neglected, Eqs. (10) and (11) become:

$$-\frac{\partial \Phi_i}{\partial y} + \phi \frac{\partial \eta}{\partial t} + U_i \frac{\partial \eta}{\partial x} = 0, \quad (y = 0), \tag{12}$$

$$\frac{1}{\phi}\frac{\partial}{\partial t}(\rho_1 \Phi_1 - \rho_2 \Phi_2) + \frac{1}{\lambda}(\mu_1 \Phi_1 - \mu_2 \Phi_2) + \sigma \eta_{xx} - g(\rho_2 - \rho_1)\eta \cos \alpha = 0, \quad (y = 0). \tag{13}$$

The general solution of linearized equations (5), (6), (12), and (13) for a typical normal mode of wavenumber k in the x direction is the real part of

$$\Phi_1 = -\eta_0 \frac{\phi\omega + ikU_1}{k} \exp(ikx - ky + \omega t), \quad y > 0,$$

$$\Phi_2 = \eta_0 \frac{\phi\omega + ikU_2}{k} \exp(ikx + ky + \omega t), \quad y < 0, \tag{14}$$

$$\eta = \eta_0 \exp(ikx + \omega t),$$

where η_0 is the amplitude of the disturbance and ω is the growth rate. That is, if the real part of ω is positive, the disturbances will grow in time and the base flow (U_1, U_2) will be unstable. On the other hand, if the real part of ω is negative, the disturbances will decay and the base flow (U_1, U_2) will be stable.

The resulting equation for ω is obtained by substituting Eq. (14) into Eqs. (12) and (13):

$$\omega^2(\rho_1 + \rho_2) + \frac{\phi\omega}{\lambda}(\mu_1 + \mu_2) + \frac{i\omega k}{\phi}(\rho_1 U + \rho_2 U_2) + (ik/\lambda)(\mu_1 U_1 + \mu_2 U_2) + k^2[\sigma k + (1/k) g(\rho_2 - \rho_1)\cos \alpha] = 0. \tag{15}$$

Equation (15) is slightly more general than the characteristic equation obtained by Raghavan and Marsden.[4] Raghavan and Marsden investigate the root locus of this equation numerically. In contrast, we establish the conditions for marginal stability analytically. Equation (15) can be rewritten in a somewhat more compact form as:

$$\omega^2 A + \omega B + i\omega k C + ikD + k^2 E = 0, \tag{16}$$

where the significance of the coefficients A, B, C, D, and E is clear from the context, and ω in general is the complex number $\omega_R + i\omega_I$.

There are various ways to investigate the locus of the roots of Eq. (16). Probably the simplest method consists of decomposing Eq. (16) into real and imaginary parts and then constructing a fourth-order equation for ω_R in the form:

$$4A^3\omega_R^4 + 8A^2B\omega_R^3 + \omega_R^2(5AB^2 + k^2AC^2 + 4A^2Ek^2) + \omega_R(B^3 + BC^2k^2 + 4ABEk^2) - k^2(AD^2 - BDC - B^2E) = 0. \quad (17)$$

For ω_R to be always negative, the condition

$$(AD^2 - BDC - B^2E)/4A^3 < 0 \quad (18)$$

should be satisfied.[8] In terms of the physical parameters of our system, the stability criteria become:

$$\sigma k + \frac{g(\rho_2 - \rho_1)\cos\alpha}{k} \geqslant \frac{\rho_1 + \rho_2}{\phi^2}\frac{\mu_1 U_1 + \mu_2 U_2}{\mu_1 + \mu_2} \times \left(\frac{\mu_1 U_1 + \mu_2 U_2}{\mu_1 + \mu_2} - \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}\right), \quad (19)$$

where U_1 and U_2 satisfy Eq. (3).

Before proceeding any further, it is convenient to introduce nondimensional variables. We use $[\sigma/g(\rho_2 - \rho_1)]^{1/2}$ as a length scale and $\{\phi^4[g\sigma(\rho_2 - \rho_1)/(\rho_1 + \rho_2)]\}^{1/4}$ as a velocity scale. As will be shown later, this length scale corresponds to the most dangerous wave length. Additionally, we define $\mu = \mu_1/\mu_2$ and $\rho = \rho_1/\rho_2 < 1$ as the viscosity and density ratios, respectively. Equation (19) in nondimensional form becomes:

$$k^* + \frac{\cos\alpha}{k^*} \geqslant \frac{(\mu - \rho)(U_1^* - U_2^*)(\mu U_1^* + U_2^*)}{(1+\mu)^2(1+\rho)}, \quad (20)$$

where the nondimensional variables are identified by a superscript star (*). Clearly, the gravity and the surface tension [the left-hand side of Eq. (20)] have a stabilizing effect. It is not clear, however, whether the flow may become unstable. If the right-hand side of Eq. (20) were negative, the flow would be unconditionally stable regardless of the magnitude of the velocity (assuming that Darcy's law is still valid). For example, for $U_1^* > U_2^* > 0$, we need $\mu > \rho$. Hence, for the instability to grow, the fluid properties have to meet certain conditions. To investigate this point more closely, let us focus on the special case of a horizontal interface ($\alpha = 0$). Substitution of Eq. (3) into Eq. (20) yields the stability criteria

$$k^* + \frac{1}{k^*} \geqslant \frac{2(U_1^*)^2\mu(\mu - \rho)(1-\mu)}{(1+\mu)^2(1+\rho)} \quad (\alpha = 0). \quad (21)$$

Clearly, in order for an unstable wave to develop, we need

$$\rho < \mu < 1 \quad (\alpha = 0). \quad (22)$$

Similar conditions can be derived for $\alpha \neq 0$. We conclude that for KH instability to develop in a Darcian flow, the fluid properties must meet certain conditions [i.e., Eq. (22)]. In this respect, the KH instability for flow in porous media differs from the instability in continuum flow where the occurrence of the instability depends on the magnitude of the velocities alone. This difference apparently arises because of the important role the viscosity plays in our case, whereas the classical continuum model assumes inviscid fluids.

We note also that, in the case of a horizontal interface ($\alpha = 0$) and a homogeneous medium, the marginal stability condition is independent of the medium permeability.

The fluid velocity $\widetilde{U}_1(k)$ at marginal stability can be readily calculated from Eq. (21):

$$\widetilde{U}_1^*(k) = \frac{1+\mu}{\sqrt{2}}\left(\frac{[k^* + (1/k^*)](1+\rho)}{\mu(1-\mu)(\mu-\rho)}\right)^{1/2} \quad (\rho < \mu < 1, \alpha = 0). \quad (23)$$

The most dangerous case is for $k^* = 1$ {or in dimensional form $k = [g(\rho_2 - \rho_1)/\sigma]^{1/2}$}. The corresponding critical velocity

$$\widetilde{U}_{1c}^* = (1+\mu)\left(\frac{(1+\rho)}{\mu(1-\mu)(\mu-\rho)}\right)^{1/2} \quad (\alpha = 0). \quad (24)$$

The marginally stable wave in the linear problem is not steady; rather it is a progressive wave with phase velocity:

$$v = -\frac{\omega_I}{k} = \frac{\mu_1 U_1 + \mu_2 U_2}{\phi(\mu_1 + \mu_2)}. \quad (25)$$

Hence, at conditions close to the marginal stability, the disturbances have the form:

$$\eta = \eta_0(t)\cos k(x - vt),$$
$$\Phi_1 = \eta_0(t)U_{1c}[(1-\mu)/(1+\mu)]\, e^{-ky}\sin k(x - vt) \quad (y > 0),$$
$$\Phi_2 = \eta_0(t)U_{2c}[(1-\mu)/(1+\mu)]\, e^{ky}\sin k(x - vt) \quad (y < 0), \quad (26)$$

where U_{1c} and U_{2c} are the critical velocities at the marginal stability.

Next, we calculate the magnitude of the critical velocity for a particular case, say, e.g. if $\sigma = 30 \times 10^{-3}$ N/m, $\rho_2 = 10^3$ kg/m^3, $\rho = 0.1$, $\mu = 0.5$, $\alpha = 0$, and $\phi = 0.3$, then $U_{1c} = 0.18$ m/s.

We see that the magnitude of the critical velocity may be quite high and may exceed the range of validity of Darcy's law. Therefore, we turn now to the investigation of the influence of non-Darcian effects.

III. NON-DARCIAN FLOW

In this section we investigate the effects of deviations from Darcy's law on the KH instability. The motion of higher velocity fluids in porous medium can be described by Forchheimer's equation (Irmay[5]):

$$\frac{\partial p_i}{\partial x} = -\left(\frac{\rho_i}{\phi}\frac{\partial q_{x,i}}{\partial t} + \frac{\mu_i}{\lambda}q_{x,i} + \frac{b\rho_i}{\lambda}|q|q_{x,i}\right) + \rho_i g\sin\alpha. \quad (27)$$

The Forchheimer's parameter b is either evaluated experimentally or calculated approximately (Ergun[9]) from $b \sim 0.012\, d/(1-\phi)$, where d is the grain size. The condition (3) for an initially steady interface becomes:

$$(\mu_2 U_2 + b\rho_2|U_2|U_2) - (\mu_1 U_1 + b\rho_1|U_1|U_1) = \lambda(\rho_2 - \rho_1)g\sin\alpha. \quad (28)$$

Assuming that both velocities U_1, $U_2 > 0$, the derivation proceeds in similar manner to Sec. I.

The stability criteria (19) becomes

$$\sigma k + \frac{g(\rho_2 - \rho_1)}{k}\cos\alpha \geqslant \frac{\rho_1 + \rho_2}{\phi^2}\frac{\mu_1 U_1 + \mu_2 U_2 + b(\rho_1 U_1^2 + \rho_2 U_2^2)}{\mu_1 + \mu_2 + b(\rho_1 U_1 + \rho_2 U_2)}$$

$$\times\left(\frac{\mu_1 U_1+\mu_2 U_2+b(\rho_1 U_1^2+\rho_2 U_2^2)}{\mu_1+\mu_2+b(\rho_1 U_1+\rho_2 U_2)} - \frac{\rho_1 U_1+\rho_2 U_2}{\rho_1+\rho_2}\right). \tag{29}$$

Also here, the marginally stable wave is traveling with phase velocity

$$v=\frac{\mu_1 U_1+\mu_2 U_2+b(\rho_1 U_1^2+\rho_2 U_2^2)}{[\mu_1+\mu_2+b(\rho_1 U_1+\rho_2 U_2)]}. \tag{30}$$

Note that the expression the the square brackets on the right-hand side of Eq. (29) can always be made positive given a sufficiently high velocity. Hence, for non-Darcian flow the instability may always occur. This compares favorably to the case of continuum flow but contrasts with that of Darcian flow (Sec. II) where the fluids' properties have to satisfy special conditions [i.e., Eq. (22)] in order to allow the instability to develop.

To amplify this point, let us focus on the special case of a horizontal interface ($\alpha=0$) with sufficiently high velocities U_1 and U_2 so that viscous effects can be neglected. Under these conditions, Eqs. (28) and (29) yield the marginal stability criteria

$$\sigma k+\frac{g(\rho_2-\rho_1)}{k} \geqslant 2\rho_1 U_1^2 \frac{\rho_1\rho_2(\sqrt{\rho_2}-\sqrt{\rho_1})^2}{(\rho_1\sqrt{\rho_2}+\rho_2\sqrt{\rho_1})^2} \tag{31}$$

through which we can observe that instability always may occur for sufficiently high velocity U_1.

IV. CONCLUSION

Herein we demonstrate that KH instability may develop in the parallel flow of two statically stable fluids through porous media. The Darcian formulation allows for the instability to develop only if the viscosity ratio of the two fluids satisfies some special conditions. The critical velocities predicted in this case are relatively high and may be beyond the range of validity of Darcy's law. Consequently, we extend the analysis to include non-Darcian effects and we show that KH instability may always develop given sufficiently high velocities.

The derivation of the characteristic Eq. (15) is similar to that of Raghavan and Marsden.[4] However, Raghavan and Marsden[4] investigate the root-locus of Eq. (15) numerically. Naturally, the numerical work is limited to a finite number of parameters and, therefore, their work does not reveal the instability discussed in this paper.

To date there is no experimental evidence of the KH instability in porous media. The natural extension of the work reported here is the procural of such evidence.

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