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CONVERGENCE OF TIME-STEPPING METHOD FOR INITIAL AND BOUNDARY-VALUE FRICTIONAL COMPLIANT CONTACT PROBLEMS*

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Abstract. Beginning with a proof of the existence of a discrete-time trajectory, this paper establishes the convergence of a time-stepping method for solving continuous-time, boundary-value problems for dynamic systems with frictional contacts characterized by local compliance in the normal and tangential directions. Our investigation complements the analysis of the initial-value rigid-body model with one frictional contact encountering inelastic impacts by Stewart [Arch. Ration. Mech. Anal., 145 (1998), pp. 215–260] and the recent analysis by Anitescu [*Optimization-Based Simulation for Nonsmooth Rigid Multibody Dynamics*, Argonne National Laboratory, Argonne, IL, 2004] using the framework of measure differential inclusions. In contrast to the measure-theoretic approach of these authors, we follow a differential variational approach and address a broader class of problems with multiple elastic or inelastic impacts. Applicable to both initial and affine boundary-value problems, our main convergence result pertains to the case where the compliance in the normal direction is decoupled from the compliance in the tangential directions and where the friction coefficients are sufficiently small.

Key words. time-stepping methods, frictional contact problems, compliance models, differenctial complementarity problems, boundary-value problems

AMS subject classifications. 34B10, 90C33, 70E55, 70F40, 74H20

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1. Introduction.

This paper investigates the limiting properties of time-stepping methods for rigid-body dynamics problems with multiple contacts characterized by friction and local compliance. Comprehensive reviews on rigid-body models and their applications can be found in the monographs [5, 13] and the excellent survey [20]. The benefits of introducing contact compliance for analysis and numerical simulation have been discussed in previous work [23]. In particular, a compliant model eliminates the static indeterminacy that is inherent in a rigid body dynamic model with multiple contacts and the need to make assumptions about linear independence of the columns of the Jacobian matrix [3, 10, 19]. Most important, even when one makes the requisite assumptions for uniqueness and existence, it is not possible to analyze the boundary-value problem in a fully rigid-body model because of the presence of discontinuities in velocities during impacts.

The present paper is closest in spirit to the work of Stewart [19], who analyzed the convergence of a time-stepping method [21] for initial-value rigid-body problems

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with frictional contact. Stewart's analysis is the first of its kind in the rigid-body dynamics literature. However, his analysis is somewhat limiting in several respects. In particular, the main result of the paper [19], Theorem 1, pertains essentially to the case of one inelastic contact. Even for such a simplified case, the analysis relies on a Radon–Nikodym derivative with several technical restrictions. It is difficult to fully extend Stewart's analysis because of the intrinsic analytical difficulties associated with the rigid-body paradigm. This difficulty is acknowledged in the recent paper by Anitescu [1], who established the convergence of a sequential quadratic programming method to a solution of a measure differential inclusion for nonsmooth rigid multibody dynamics.

Our previous work was concerned with several analytical aspects of dynamic models with compliant frictional contacts. Comparisons between results obtained with and without local compliance with a singular perturbation analysis are included in [15]. Uniqueness and existence results for the discrete-time problem are presented in [17] under a semi-implicit discretization that permits the use of linear complementarity theory [6]. In this paper, we analyze the convergence of a broad scheme of time-stepping methods for solving frictional compliant contact problems. In contrast to [17], the discretization scheme employed here is more general, allowing in particular for nonlinearities in the state variables, thus going well beyond the previous analysis of existence and uniqueness that is based on a linear theory. Unlike the analysis in [19, 1], our main convergence result is not in terms of measure differential inclusions. Most importantly, our analysis is carried out in a broad setting that includes both initial-value and boundary-value problems with affine constraints on the initial and final state (see (13)). It should be noted that although boundary-value problems arise naturally in the design of mechanical systems governed by dynamics, previous literature on this subject addresses only initial-value problems and ours is the first attempt to study contact problems subject to boundary conditions.

This paper addresses neither the numerical implementation nor the order of convergence of the time-stepping methods. For details on practical implementation and computational results, see [16, 18]; see also [4] for a part-insertion application of a boundary-value planar rigid-body problem. The order of convergence analysis for frictional contact problems is a very difficult topic, even for initial-value problems. The discontinuity of the friction forces as a function of the system states is a main cause for such difficulty.

The organization of the rest of the paper is as follows. In the next section, we summarize the formulation of the continuous-time frictional compliant contact problem and formally define a concept of a weak solution to the problem. A numerical time-stepping scheme for computing such a solution is described in section 3. The convergence analysis of the numerical scheme begins in section 4, where we first investigate in detail the normal and tangential frictional conditions in the discrete-time subproblems, establishing in particular the existence of a discrete-time trajectory of the normal and tangential contact forces that are continuous functions of the state. We also establish the uniqueness of such a trajectory under a "small coefficient of friction" assumption; see Propositions 6 and 7. With the aid of the machinery of differential variational inequalities [11], and under the small-friction-coefficient assumption, we complete the convergence analysis of the time-stepping method for a compliant-body frictional contact problem in section 5. There, an existence result, Theorem 8, for the discrete-time boundary-value problem is first proved, which is followed by the main convergence theorem of the paper, Theorem 9. The small-friction assumption is the artifact of the nonlinear friction law that is in turn a characteristic of the discretization scheme that we employ. Such a nonlinear analysis is in contrast to previous analysis by Stewart and Anitescu, which is based on a polygonal approximation of the quadratic Coulomb cone.

2. Model formulation. The mathematical formulation of the frictional compliant contact problem has several components: (a) equations of motion, (b) compliance constitutive law, (c) contact and friction, and (d) boundary conditions. In what follows, we present only the essentials of the formulation and refer the reader to [14, 17] for the detailed explanation of the overall model.

Equations of motion. The dynamics equation of motion for a multibody system with frictional contacts can be written as

$$M(q)\dot{\nu} \;=\; f(t,q,\nu) + \mathbf{\Gamma}(q)^T \boldsymbol{\lambda}, \tag{1}$$

where q is the n_q-dimensional vector of generalized coordinates, ν is the n_ν-dimensional vector of the system velocities, $\dot{\nu}$ denotes the time derivative of ν (i.e., $\dot{\nu} = d\nu/dt$), $M(q)$ is the $n_\nu \times n_\nu$ symmetric positive definite mass-inertia matrix, $f(t,q,\nu)$ is the n_ν-dimensional external force vector (excluding contact forces),

$$\mathbf{\Gamma}(q)^T \;\equiv\; \left[\; \Gamma_{\rm n}(q)^T \;\; \Gamma_{\rm t}(q)^T \;\; \Gamma_{\rm o}(q)^T \;\right] \;\equiv\; G(q)^T \left[\; J\Psi_{\rm n}(q)^T \;\; J\Psi_{\rm t}(q)^T \;\; J\Psi_{\rm o}(q)^T \;\right]$$

is the transpose of the system Jacobian matrix, with $\Psi_{\rm n,t,o}(q)$ and $J\Psi_{\rm n,t,o}(q)$ being the constraint functions and their Jacobians for all possible contacts in the normal direction (labeled n) and the two tangential directions (labeled t and o), respectively, and $\boldsymbol{\lambda} \equiv (\lambda_{\rm n}, \lambda_{\rm t}, \lambda_{\rm o}) = \lambda_{\rm n,t,o}$ is the vector of contact forces in these directions. For compliant contact models, the dimensions of the contact forces and, accordingly, the orders of the associated Jacobian matrices, are related to the compliance constitutive model being used. The matrix $G(q)$ is a $n_q \times n_\nu$ parametrization matrix that allows us to use different parameterizations for the motion group via the the following kinematics equation:

$$\dot{q} \;=\; G(q)\nu, \tag{2}$$

where $\dot{q} \equiv dq/dt$ is the time-derivative of the system configuration. Together, (1) and (2) constitute the equations of motion governing the dynamics of the mechanical system.

Letting $T > 0$ be the terminal time of the problem, we postulate the following assumptions (A)–(C) on the above model functions. Notice that no rank assumption is imposed on $\mathbf{\Gamma}(q)$; this is a distinct advantage of a compliant model in that the number of contact points need not be restricted by the degrees of freedom of the bodies in contact.

(A) The function $f(t,q,\nu)$ is Lipschitz continuous on $[0,T] \times \Re^{n_q+n_\nu}$ with constant $L_f > 0$; thus,

$$\|f(t,q,\nu) - f(t',q',\nu')\| \le L_f[|t-t'| + \|q-q'\| + \|\nu-\nu'\|] \quad \forall\,(t,q,\nu),(t',q',\nu') \in [0,T] \times \Re^{n_q+n_\nu}.$$

(B) The functions $G(q)$ and $\mathbf{\Gamma}(q)$ are Lipschitz continuous and bounded on $\Re^{n_q}$; thus there exist positive constants L_G, L_W, η_G, and η_W such that for all q and q' in $\Re^{n_q}$,

$$\|\, G(q) - G(q') \,\| \;\le\; L_G \,\|\, q - q' \,\|, \quad \|\, \mathbf{\Gamma}(q) - \mathbf{\Gamma}(q') \,\| \;\le\; L_W \,\|\, q - q' \,\|,$$
$$\sup_{q \in \Re^{n_q}} \|\, G(q) \,\| \;\le\; \eta_G, \qquad \sup_{q \in \Re^{n_q}} \|\, \mathbf{\Gamma}(q) \,\| \;\le\; \eta_W;$$

moreover, the function $\Psi_\mathrm{n}(q)$ satisfies the limit condition

$$(3) \qquad \lim_{\|q-q'\|\to 0} \frac{\| \Psi_\mathrm{n}(q) - \Psi_\mathrm{n}(q') - J\Psi_\mathrm{n}(q')(q-q') \|}{\| q - q' \|} = 0,$$

or, equivalently, for every scalar $\varepsilon > 0$, a scalar $\varsigma > 0$ exists such that

$$(4) \qquad \| q - q' \| \le \varsigma \ \Rightarrow \ \| \Psi_\mathrm{n}(q) - \Psi_\mathrm{n}(q') - J\Psi_\mathrm{n}(q')(q-q') \| \le \varepsilon \| q - q' \|.$$

(C) The mass-inertia matrix $M(q)$ is Lipschitz continuous on $\Re^{n_q}$ with Lipschitz constant $L_M > 0$; moreover, positive constants σ_M and σ'_M exist such that

$$\inf_{q\in\Re^{n_q}} \min_{\|\nu\|=1} \nu^T M(q) \nu \ge \sigma_M \quad \text{and} \quad \sup_{q\in\Re^{n_q}} \max_{\|\nu\|=1} \nu^T M(q) \nu \le 1/\sigma'_M.$$

Condition (3) is clearly satisfied if $J\Psi_\mathrm{n}(q)$ is Lipschitz continuous. Unlike the treatment in [1], $\Psi_\mathrm{n}(q)$ is not assumed to be twice differentiable. (The squared distance function to a closed convex set—the obstacle set—is an example of a (scalar) function that is continuously differentiable with a Lipschitz gradient but is not twice differentiable.) Conditions (A), (B), and (C) have several immediate consequences, which will be used freely throughout the paper where appropriate.

A constitutive model for compliance. While there are many compliance models, we employ the distributed model described in [14, 17], to which we refer the reader for details and references. Specifically, this model postulates that the contact forces are linearly dependent on the body deformations and on the deformation rates:

$$(5) \qquad \boldsymbol{\lambda} = \mathbf{K}(q)\boldsymbol{\delta} + \mathbf{C}(q)\dot{\boldsymbol{\delta}}$$

where $\boldsymbol{\delta} \equiv (\delta_\mathrm{n}, \delta_\mathrm{t}, \delta_\mathrm{o}) = \delta_\mathrm{n,t,o}$ is the vector of body deformations in the normal (n) and the two tangential directions (t and o), $\dot{\boldsymbol{\delta}}$ denotes the vector of velocities of the deformations (i.e., $\dot{\boldsymbol{\delta}} = d\boldsymbol{\delta}/dt$); the stiffness matrix $\mathbf{K}(q)$ and the damping matrix $\mathbf{C}(q)$, which are partitioned as

$$\mathbf{K}(q) \equiv \begin{bmatrix} K_\mathrm{nn}(q) & K_\mathrm{nt}(q) & K_\mathrm{no}(q) \\ K_\mathrm{tn}(q) & K_\mathrm{tt}(q) & K_\mathrm{to}(q) \\ K_\mathrm{on}(q) & K_\mathrm{ot}(q) & K_\mathrm{oo}(q) \end{bmatrix} \text{ and } \mathbf{C}(q) \equiv \begin{bmatrix} C_\mathrm{nn}(q) & C_\mathrm{nt}(q) & C_\mathrm{no}(q) \\ C_\mathrm{tn}(q) & C_\mathrm{tt}(q) & C_\mathrm{to}(q) \\ C_\mathrm{on}(q) & C_\mathrm{ot}(q) & C_\mathrm{oo}(q) \end{bmatrix},$$

are each of order $3n_s^2 n_c$, with n_s^2 being the number of elements with lumped stiffness and damping properties that comprise a contact patch; each of the 18 block matrices (such as $K_\mathrm{nt}(q)$, etc.) in $\mathbf{K}(q)$ and $\mathbf{C}(q)$ is an $n_s^2 n_c$ block diagonal matrix with n_c diagonal blocks, one for each contact patch, and each such diagonal block is in turn a square matrix of order n_s^2. With $n_\delta \equiv n_s^2 n_c$, it follows that the vectors λ_n, λ_t, λ_o, δ_n, δ_t, and δ_o are each of dimension n_δ. We postulate the following condition:

(D) $\mathbf{K}(q)$ and $\mathbf{C}(q)$ are Lipschitz continuous symmetric positive definite matrix-valued functions of q; moreover, positive constants $\eta_K > 0$, σ_{KC} and η_{KC} exist such that $\sup_{q\in\Re^{n_q}} \|\mathbf{K}(q)\| \le \eta_K$, and, for all scalars $h > 0$ sufficiently small,

$$\inf_{q\in\Re^{n_q}} \min_{\|\boldsymbol{\delta}\|=1} \boldsymbol{\delta}^T [\, h\,\mathbf{K}(q) + \mathbf{C}(q)\,]^{-1} \boldsymbol{\delta} \ge \sigma_{KC} \text{ and } \sup_{q\in\Re^{n_q}} \left\| [\, h\,\mathbf{K}(q) + \mathbf{C}(q)\,]^{-1} \right\| \le \eta_{KC}.$$

Notice that the above implies $\sup_{q\in\Re^{n_q}} \|\mathbf{C}(q)\| \le 1/\sigma_{KC}$.

Contact and friction. Stated as a complementarity condition, the normal contact condition is

$$0 \leq \lambda_{\rm n} \perp \Psi_{\rm n}(q) + \delta_{\rm n} \geq 0, \tag{6}$$

where the notation $u \perp v$ means that the two vectors u and v are perpendicular. The tangential friction condition is expressed by a minimization principle over the cone of frictional forces: for each $i = 1, \ldots, n_\delta$,

$$(\lambda_{i\rm t}, \lambda_{i\rm o}) \in \operatorname{argmin} \left\{ s_{i\rm t}\tilde{\lambda}_{i\rm t} + s_{i\rm o}\tilde{\lambda}_{i\rm o} : (\tilde{\lambda}_{i\rm t}, \tilde{\lambda}_{i\rm o}) \in \mathcal{F}(\mu_i \lambda_{i\rm n}) \right\}, \tag{7}$$

where

$$\begin{aligned} s_{i\rm t} &\equiv \frac{d(\delta_{it} + \Psi_{it}(q))}{dt} = \dot{\delta}_{i\rm t} + \nabla\Psi_{i\rm t}(q)^T \dot{q}, \\ s_{i\rm o} &\equiv \frac{d(\delta_{i\rm o} + \Psi_{i\rm o}(q))}{dt} = \dot{\delta}_{i\rm o} + \nabla\Psi_{i\rm o}(q)\dot{q} \end{aligned} \tag{8}$$

are the tangential slip velocities at contact patch i, which depend on both the deformations of the compliant elements and the rigid body motions, and where $\mu_i \geq 0$ is the friction coefficient and

$$\mathcal{F}(\tau) \equiv \{ (a, b) \in \Re^2 : \sqrt{a^2 + b^2} \leq \tau \}, \quad \tau \geq 0,$$

is the standard Coulomb friction cone. From (7), it follows that

$$s_{i\rm t}\lambda_{i\rm t} + s_{i\rm o}\lambda_{i\rm o} = -\mu_i \lambda_{i\rm n} \sqrt{s_{i\rm t}^2 + s_{i\rm o}^2}. \tag{9}$$

Moreover, provided that $\mu_i\lambda_{i\rm n} > 0$, we have, with $r_i \equiv \sqrt{s_{i\rm t}^2 + s_{i\rm o}^2}$,

$$s_{i\rm t} + \frac{r_i \lambda_{i\rm t}}{\sqrt{\lambda_{i\rm t}^2 + \lambda_{i\rm o}^2}} = 0, \quad s_{\rm o} + \frac{r_i \lambda_{i\rm o}}{\sqrt{\lambda_{i\rm t}^2 + \lambda_{i\rm o}^2}} = 0,$$

$$0 \leq r_i \perp \mu_i \lambda_{i\rm n} - \sqrt{\lambda_{i\rm t}^2 + \lambda_{i\rm o}^2} \geq 0,$$

where we define $0/0$ to be 1. If we use polar coordinates to represent the pair $(s_{i\rm t}, s_{i\rm o})$, say,

$$s_{i\rm t} = r_i \cos\psi_i \quad \text{and} \quad s_{i\rm o} = r_i \sin\psi_i,$$

then there exists a scalar $\phi_i \in [-1, 1]$ satisfying $r_i > 0 \Rightarrow \phi_i = 1$ such that

$$\lambda_{i\rm t} = -\mu_i \lambda_{i\rm n} \phi_i \cos\psi_i \quad \text{and} \quad \lambda_{i\rm o} = -\mu_i \lambda_{i\rm n} \phi_i \sin\psi_i.$$

The latter representation of $(\lambda_{i\rm t}, \lambda_{i\rm o})$ remains valid when $\mu_i\lambda_{i\rm n} = 0$, by letting $\phi_i = 0$.

More on the compliance model. The constitutive law (5) can be used to eliminate the slip velocities (s_{it}, s_{io}) in the friction law (7), resulting in an expression of the latter in terms of the state variables $(q, \nu, \delta_{\rm n,t,o})$ and the normal force $\lambda_{\rm n}$. This reformulation of the friction law is significant because the slip velocities may behave discontinuously and lead to technical difficulties in the convergence analysis of a numerical method. From (5), we have $\dot{\boldsymbol{\delta}} = \mathbf{C}(q)^{-1}(\boldsymbol{\lambda} - \mathbf{K}(q)\boldsymbol{\delta})$. Writing

$$\mathbf{C}(q)^{-1} \equiv \begin{bmatrix} \widehat{C}_{\rm nn}(q) & \widehat{C}_{\rm nt}(q) & \widehat{C}_{\rm no}(q) \\ \widehat{C}_{\rm tn}(q) & \widehat{C}_{\rm tt}(q) & \widehat{C}_{\rm to}(q) \\ \widehat{C}_{\rm on}(q) & \widehat{C}_{\rm ot}(q) & \widehat{C}_{\rm oo}(q) \end{bmatrix},$$

we obtain

$$\begin{pmatrix} \dot{\delta}_{it} \\ \dot{\delta}_{io} \end{pmatrix} = \begin{bmatrix} \widehat{C}_{itn}(q) & \widehat{C}_{itt}(q) & \widehat{C}_{ito}(q) \\ \widehat{C}_{ion}(q) & \widehat{C}_{iot}(q) & \widehat{C}_{ioo}(q) \end{bmatrix} (\boldsymbol{\lambda} - \mathbf{K}(q)\boldsymbol{\delta}),$$

where $\widehat{C}_{itn}(q)$ denotes the ith row of the (sub)matrix $\widehat{C}_{tn}(q)$, and similarly for the other notation. Clearly, the friction condition (7) at contact i is equivalent to: for all $(\widetilde{\lambda}_{it}, \widetilde{\lambda}_{io}) \in \mathcal{F}(\mu_i \lambda_{in})$,

(10)

$$0 \leq \begin{pmatrix} \widetilde{\lambda}_{it} - \lambda_{it} \\ \widetilde{\lambda}_{io} - \lambda_{io} \end{pmatrix}^T \begin{pmatrix} s_{it} \\ s_{io} \end{pmatrix} = \begin{pmatrix} \widetilde{\lambda}_{it} - \lambda_{it} \\ \widetilde{\lambda}_{io} - \lambda_{io} \end{pmatrix}^T \left[\begin{pmatrix} \dot{\delta}_{it} \\ \dot{\delta}_{io} \end{pmatrix} + \begin{pmatrix} \Gamma_{it}(q) \\ \Gamma_{io}(q) \end{pmatrix} \nu \right]$$

$$= \begin{pmatrix} \widetilde{\lambda}_{it} - \lambda_{it} \\ \widetilde{\lambda}_{io} - \lambda_{io} \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{itn}(q) \; \widehat{C}_{itt}(q) \; \widehat{C}_{ito}(q) \\ \widehat{C}_{ion}(q) \; \widehat{C}_{iot}(q) \; \widehat{C}_{ioo}(q) \end{bmatrix} (\boldsymbol{\lambda} - \mathbf{K}(q)\boldsymbol{\delta}) + \begin{pmatrix} \Gamma_{it}(q) \\ \Gamma_{io}(q) \end{pmatrix} \nu \right\}.$$

Proposition 1 shows that under the constitutive compliance law (5), the tangential friction forces in a frictional compliant model can be characterized by the solution to a convex quadratic program.

PROPOSITION 1. *Given q, ν, λ_n, and $\boldsymbol{\delta}$, under* (5)*, the tangential forces* (λ_t, λ_o) *satisfy the minimum principle* (7) *if and only if* (λ_t, λ_o) *is the optimal solution, which must necessarily be unique, of the convex quadratic program:*

(11)

$$\textit{minimize} \quad \frac{1}{2} \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\lambda}_o \end{pmatrix} \begin{bmatrix} \widehat{C}_{tt}(q) & \widehat{C}_{to}(q) \\ \widehat{C}_{ot}(q) & \widehat{C}_{oo}(q) \end{bmatrix} \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\lambda}_o \end{pmatrix}$$

$$+ \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\lambda}_o \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{tn}(q) \\ \widehat{C}_{on}(q) \end{bmatrix} \lambda_n + \begin{bmatrix} \Gamma_t(q) \\ \Gamma_o(q) \end{bmatrix} \nu - \begin{bmatrix} \widehat{C}_{tn}(q) & \widehat{C}_{tt}(q) & \widehat{C}_{to}(q) \\ \widehat{C}_{on}(q) & \widehat{C}_{ot}(q) & \widehat{C}_{oo}(q) \end{bmatrix} \mathbf{K}(q)\boldsymbol{\delta} \right\}$$

$$\textit{subject to} \quad (\tilde{\lambda}_t, \tilde{\lambda}_o) \in \prod_{i=1}^{n_\delta} \mathcal{F}(\mu_i \lambda_{in}).$$

Proof. It suffices to note that the first-order optimality conditions of (11) are equivalent to the variational conditions (10). ∎

Boundary conditions. To complete the description of the model, we postulate a set of boundary conditions that connect the state variable $\mathbf{x} \equiv (q, \nu, \boldsymbol{\delta}) \in \Re^n$, where $n \equiv n_q + n_\nu + 3n_\delta$, at the initial and terminal times: $t = 0$ and $t = T$, respectively. The most general such conditions would be expressed by a nonlinear functional relation of the form $F(\mathbf{x}(0), \mathbf{x}(T)) = 0$. Nevertheless, such generality would make the analysis extremely difficult, if not impossible. In general, the boundary conditions should be consistent with the constraints; such consistency would require the initial pair (q^0, δ_n^0) to satisfy the feasibility condition:

(12) $$\Psi_n(q^0) + \delta_n^0 \geq 0.$$

Therefore, to both accommodate realistic applications and facilitate the mathematical analysis, we consider a class of boundary conditions where the initial configuration $q(0) = q^0$ and deformation $\boldsymbol{\delta}(0) = \boldsymbol{\delta}^0$ are known and satisfy (12); but the initial velocity $\nu(0)$ and terminal state $\mathbf{x}(T)$ are subject to a system of linear equations,

$$\mathbf{b} \,=\, \mathbf{M}_\nu \nu(0) + \mathbf{N}\mathbf{x}(T), \tag{13}$$

for some given vector $\mathbf{b} \in \Re^{n_\nu}$ and matrices $\mathbf{M}_\nu \in \Re^{n_\nu \times n_\nu}$ and $\mathbf{N} \in \Re^{n_\nu \times n}$. When $\mathbf{M}_\nu$ is the identity matrix and $\mathbf{N}$ is the zero matrix, we recover an initial-value problem with a known initial state $\mathbf{x}(0)$.

Weak solutions. The frictional compliant contact problem under study is to find a state trajectory $\mathbf{x} : [0,T] \to \Re^n$ and a force trajectory $\boldsymbol{\lambda} : [0,T] \to \Re^{3n_\delta}$ such that $q(0) = q^0$, $\boldsymbol{\delta}(0) = \boldsymbol{\delta}^0$, and the conditions (1), (2)–(8), (12), and (13) are satisfied. Ideally, we want these conditions to be satisfied at all times $t \in [0,T]$, but due to the possible discontinuity of the force trajectory $\boldsymbol{\lambda}$, this ideal goal is generally not attainable, especially when it pertains to the numerical solutions obtained by a time-stepping scheme, such as the one described in the next section; see [19, 20]. Therefore, we have to settle for a kind of weak solution that satisfies the dynamics equations and the contact and friction conditions in a weak sense. This is an inherent limitation of the model, particularly (5). It may be possible to get a strong solution by using a more sophisticated, nonlinear constitutive model. (See [15] for an example of such a model.) However, we refrain from pursuing such an extended consideration and restrict ourselves to the law (5), whose analysis is already fairly involved.

DEFINITION 2. *The pair of trajectories* $\mathbf{x} : [0,T] \to \Re^n$ *and* $\boldsymbol{\lambda} : [0,T] \to \Re^{3n_\delta}$ *is said to be a weak solution of the frictional compliant contact problem if*

(a) *(the state equations)* $\mathbf{x}(t)$ *is absolutely continuous on* $[0,T]$ *and satisfy for all* $\tau \leq \tau'$ *in* $[0,T]$,

$$\begin{aligned}
\nu(\tau') - \nu(\tau) &= \int_\tau^{\tau'} M(q(t))^{-1}[\, f(t, q(t), \nu(t)) + \boldsymbol{\Gamma}(q(t))^T \boldsymbol{\lambda}(t)\,]\, dt,\\
q(\tau') - q(\tau) &= \int_\tau^{\tau'} G(q(t))\nu(t)\, dt,\\
\boldsymbol{\delta}(\tau') - \boldsymbol{\delta}(\tau) &= \int_\tau^{\tau'} C(q(t))^{-1}[\, \boldsymbol{\lambda}(t) - \mathbf{K}(q(t))\boldsymbol{\delta}(t)\,]\, dt;
\end{aligned}$$

(b) *(the normal contact condition)* $\Psi_{\rm n}(q(t)) + \delta_{\rm n}(t) \geq 0$ *for all* $t \in [0,T]$, $\lambda_{\rm n}(t) \geq 0$ *for almost all* $t \in [0,T]$, *and*

$$\int_0^T \lambda_{\rm n}(t)^T [\, \Psi_{\rm n}(q(t)) + \delta_{\rm n}(t)\,]\, dt \,=\, 0;$$

(c) *(the friction condition) for every* $i = 1, \ldots, n_\delta$, $(\lambda_{it}(t), \lambda_{io}(t)) \in \mathcal{F}(\mu_i \lambda_{in}(t))$ *for almost all* $t \in [0,T]$ *and for every continuous function* $(\widetilde{\lambda}_{\rm t}, \widetilde{\lambda}_{\rm o}) : [0,T] \to \Re^{2n_\delta}$ *such that for every* $i = 1, \ldots, n_\delta$, $(\widetilde{\lambda}_{it}(t), \widetilde{\lambda}_{io}(t))$ *belongs to* $\mathcal{F}(\mu_i \lambda_{in}(t))$ *for almost all* $t \in [0,T]$, *it holds that*

$$\int_0^T \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \lambda_{\rm t}(t) \\ \widetilde{\lambda}_{\rm o}(t) - \lambda_{\rm o}(t) \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(q(t)) & \widehat{C}_{\rm to}(q(t)) \\ \widehat{C}_{\rm ot}(q(t)) & \widehat{C}_{\rm oo}(q(t)) \end{bmatrix} \left[\begin{pmatrix} \lambda_{\rm t}(t) \\ \lambda_{\rm o}(t) \end{pmatrix} \right.\right.$$
$$\left.\left. - \begin{bmatrix} K_{\rm tt}(q(t)) & K_{\rm to}(q(t)) \\ K_{\rm ot}(q(t)) & K_{\rm oo}(q(t)) \end{bmatrix} \begin{pmatrix} \delta_{\rm t}(t) \\ \delta_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(q(t)) \\ \Gamma_{\rm o}(q(t)) \end{pmatrix} \nu(t) \right\} dt \geq 0,$$

(d) *(initial and boundary conditions)* $q(0) = q^0$, $\boldsymbol{\delta}(0) = \boldsymbol{\delta}^0$, *and* (13) *hold.*

We make a couple remarks about the above definition. First, the slip velocities do not enter in the above definition; second, the tangential friction condition is stipulated to hold in an integral form that is an aggregation over all contacts. This is in contrast to requiring the condition to hold at every contact. In the special case where compliance is decoupled among the contacts, then the aggregated condition indeed decouples into separate conditions at each individual contact.

3. A time-stepping scheme. The kind of "semi-implicit" discretization methods described herein for computing a weak solution to the frictional compliant contact problems has been used extensively for solving initial-value rigid-body problems and, to a lesser extent, for compliant-body problems; see, e.g., [2, 3, 11, 21, 17, 18, 19]. Specifically, we divide the time interval $[0, T]$ into $N_h + 1$ subintervals each of equal length $h > 0$; thus $(N_h + 1)h = T$. The variables of the discrete-time system are

$$\{\, q^{h,0}, q^{h,1}, \ldots, q^{h,N_h+1} \,\},\ \{\, \nu^{h,0}, \nu^{h,1}, \ldots, \nu^{h,N_h+1} \,\},\ \{\, \delta^{h,0}_{\rm n,t,o}, \delta^{h,1}_{\rm n,t,o}, \ldots, \delta^{h,N_h+1}_{\rm n,t,o} \,\},$$
$$\{\, \lambda^{h,1}_{\rm n,t,o}, \lambda^{h,2}_{\rm n,t,o}, \ldots, \lambda^{h,N_h+1}_{\rm n,t,o} \,\},\ \text{and}\ \{\, s^{h,1}_{\rm t,o}, \ldots, s^{h,N_h+1}_{\rm t,o} \,\}. \tag{14}$$

We write $\mathbf{x}^{h,j} \equiv (q^{h,j}, \nu^{h,j}, \boldsymbol{\delta}^{h,j})$, $\boldsymbol{\delta}^{h,j} \equiv \delta^{h,j}_{\rm n,t,o}$, and $\boldsymbol{\lambda}^{h,j} \equiv \lambda^{h,j}_{\rm n,t,o}$. To derive the discrete-time system, we replace the time derivatives of the state variable $\mathbf{x} \equiv (q, \nu, \boldsymbol{\delta})$ by standard finite-difference quotients such as:

$$\dot{\mathbf{x}}(t) \,\approx\, \frac{\mathbf{x}(t+h) - \mathbf{x}(t)}{h}.$$

The right-hand expressions in the equation of motion (1) and in the kinematic equation (2) are approximated by a semi-implicit scheme that employs a θ-rule, whereby the differential variables q and ν are evaluated at some intermediate values in the respective subintervals determined by the scalar $\theta \in [0, 1]$. Specifically, with

$$q^{h,\theta_j} \,\equiv\, \theta\, q^{h,j} + (\, 1 - \theta\,)\, q^{h,j+1} \quad \text{and} \quad \nu^{h,\theta_j} \,\equiv\, \theta\, \nu^{h,j} + (\, 1 - \theta\,)\, \nu^{h,j+1},$$

the discrete-time dynamics and kinematics equations at time $t_{h,j+1}$ are

$$\begin{aligned} M(q^{h,j})(\, \nu^{h,j+1} - \nu^{h,j}\,) \,&=\, h\,[\, f(t_{h,j+1}, q^{h,\theta_j}, \nu^{h,\theta_j}) + \boldsymbol{\Gamma}(q^{h,j})^T \boldsymbol{\lambda}^{h,j+1}], \\ q^{h,j+1} - q^{h,j} \,&=\, h\, G(q^{h,j}) \nu^{h,\theta_j}. \end{aligned} \tag{15}$$

(More generally, we could use different θ-values in these two equations. For simplicity, we avoid this minor variation and use (15).) Since $s_{\rm t,o} = \dot{\delta}_{\rm t,o} + J\Psi_{\rm t,o}(q)\dot{q}$ by (8), we employ the following discrete-time approximation for the vector of tangential slip velocities $s_{\rm t,o}$:

$$s^{h,j+1}_{\rm t,o} = \frac{\delta^{h,j+1}_{\rm t,o} - \delta^{h,j}_{\rm t,o}}{h} + J\Psi_{\rm t,o}(q^{h,j}) \frac{q^{h,j+1} - q^{h,j}}{h} = \frac{\delta^{h,j+1}_{\rm t,o} - \delta^{h,j}_{\rm t,o}}{h} + \Gamma_{\rm t,o}(q^{h,j}) \nu^{h,\theta_j},$$

where we have used the discrete-time kinematics equation $q^{h,j+1} - q^{h,j} = hG(q^{h,j})\nu^{h,\theta_j}$ and the definition of $\Gamma_{\rm t,o}(q^{h,j}) \equiv J\Psi_{\rm t,o}(q^{h,j})G(q^{h,j})$ to obtain the second equality. In deriving the discrete-time normal contact condition, we employ the first-order approximation

$$\Psi_{\rm n}(q(t+h)) \,\approx\, \Psi_{\rm n}(q(t)) + h\, J\Psi_{\rm n}(q(t))\, \dot{q}(t),$$

which holds for all $h > 0$ sufficiently small, and approximate $\dot{q}(t)$ similarly.

Putting together all the above approximations, we arrive at the following discrete-time frictional compliant contact problem: given $(q^0, \boldsymbol{\delta}^0)$ satisfying (12), compute (14) such that the conditions below are satisfied for all $j = 0, 1, \ldots, N_h$,

$$
\begin{aligned}
& M(q^{h,j})(\,\nu^{h,j+1} - \nu^{h,j}\,) \,=\, h\,[\,f(t_{h,j+1}, q^{h,\theta_j}, \nu^{h,\theta_j}\,) + \boldsymbol{\Gamma}(q^{h,j})^T \boldsymbol{\lambda}^{h,j+1}],\\
& q^{h,j+1} - q^{h,j} \,=\, h\,G(q^{h,j})\nu^{h,\theta_j},\\
& \delta_{\rm t}^{h,j+1} - \delta_{\rm t}^{h,j} \,=\, h\,[s_{\rm t}^{h,j+1} - \Gamma_{\rm t}(q^{h,j})\nu^{h,\theta_j}\,],\\
& \delta_{\rm o}^{h,j+1} - \delta_{\rm o}^{h,j} \,=\, h\,[s_{\rm o}^{h,j+1} - \Gamma_{\rm o}(q^{h,j})\nu^{h,\theta_j}\,],\\
& 0 \,\leq\, \lambda_{\rm n}^{h,j+1} \,\perp\, \Psi_{\rm n}(q^{h,j}) + h\,\Gamma_{\rm n}(q^{h,j})\nu^{h,\theta_j} + \delta_{\rm n}^{h,j+1} \,\geq\, 0,\\
& \boldsymbol{\lambda}^{h,j+1} \,=\, \mathbf{K}(q^{h,j})\boldsymbol{\delta}^{h,j+1} + \frac{\mathbf{C}(q^{h,j})}{h}\,(\,\boldsymbol{\delta}^{h,j+1} - \boldsymbol{\delta}^{h,j}\,),\\
& \begin{pmatrix} \lambda_{i\rm t}^{h,j+1} \\ \lambda_{i\rm o}^{h,j+1} \end{pmatrix} \in \underset{(\tilde{\lambda}_{i\rm t},\tilde{\lambda}_{i\rm o}) \in \mathcal{F}(\mu_i\,\lambda_{i\rm n}^{h,j+1})}{\arg\min} \left\{ \begin{pmatrix} s_{i\rm t}^{h,j+1} \\ s_{i\rm o}^{h,j+1} \end{pmatrix}^T \begin{pmatrix} \tilde{\lambda}_{i\rm t} \\ \tilde{\lambda}_{i\rm o} \end{pmatrix} \right\},
\end{aligned}
\tag{16}
$$

$$
\mathbf{b} \,=\, \mathbf{M}_\nu \nu^{h,0} + \mathbf{N}\mathbf{x}^{h,N_h+1}, \quad \text{and} \quad (\,q^{h,0}, \boldsymbol{\delta}^{h,0}\,) \,=\, (\,q^0, \boldsymbol{\delta}^0\,).
$$

The inclusion of the parameter θ in selected terms raises the question of why it is not used consistently throughout the constraints. An answer to this question can be traced to the paper [21], where the intention was to use a linear complementarity solver to solve the subproblems. As seen from the subsequent paper [19], excluding θ from the matrices $M(q^{h,j})$, $\boldsymbol{\Gamma}(q^{h,j})$, and $G(q^{h,j})$ simplifies the analysis significantly. The Ph.D. thesis [24] contains an analysis of a fully implicit time-stepping method for an initial-value rigid-body model, which leads to subproblems that are nonlinear complementarity problems. A computational comparison between a fully implicit scheme versus a semi-implicit scheme can be found in [25]. Presumably, the use of the parameter θ is to induce a higher order of convergence; yet such a goal is hard to substantiate formally. The analysis below does not address this issue of order of convergence.

Beginning in the next section, we will analyze two fundamental issues associated with the above discrete-time system: (a) the existence of a solution to each discrete-time boundary-value subproblem, and (b) the convergence of such a discrete-time trajectory to a weak solution of the frictional compliant contact problem. Part of the challenge in the convergence analysis lies in the coupled nature of the individual time-step subproblems, which are linked by the boundary equation $\mathbf{b} = \mathbf{M}_\nu \nu^{h,0} + \mathbf{N}\mathbf{x}^{h,N_h+1}$. Briefly, the analysis consists of two major tasks. First, we show that for an arbitrary triple $\mathbf{x}^{h,j}$, a unique friction triple $\boldsymbol{\lambda}^{h,j+1}$ exists that has some desirable continuity and boundedness properties, provided that the friction coefficients μ_i are sufficiently small. These properties of the friction forces allow us to apply an argument used in [11] for a class of boundary-value differential variational inequalities to complete the convergence analysis of the discrete-time trajectory as the time step tends to zero.

Naturally, there is an important computational issue associated with the above numerical scheme; namely, how can the discrete-time system (16) be efficiently solved in practice? The proof of Theorem 8 suggests a fixed-point method. Yet, specialized complementarity methods [8] may prove to be more effective. Nevertheless, there is presently no formal study on the applicability of the latter methods. The numerical experiments in [17] employed the complementarity solver PATH [7], which produced satisfactory results. Despite such practical experience, which is somewhat limited, there is an urgent need for the development of some robust algorithms for solving (16) along with a rigorous proof of applicability.

4. Preliminary analysis: Initial-value problems. The analysis in this section is best considered as one for an initial-value problem, where $\nu^{h,0}$, in addition to $(q^{h,0}, \boldsymbol{\delta}^{h,0})$, is assumed to be fixed but arbitrary. (This is the case where $\mathbf{M}_\nu$ is the identity matrix and $\mathbf{N}$ is the zero matrix.) This analysis will be the basis for extension to the boundary-value problem where $\nu^{h,0}$ has to be determined to satisfy the boundary equation defined by a more general pair of boundary matrices $(\mathbf{M}_\nu, \mathbf{N})$. As the first step in the convergence analysis, we show that the discrete-time dynamics and kinematics equations (15) have a unique solution $(q^{h,j+1}, \nu^{h,j+1})$ for any $(q^{h,j}, \nu^{h,j})$ and $\boldsymbol{\lambda}^{h,j+1}$; moreover, such a solution, for fixed $(q^{h,j}, \nu^{h,j})$, has several desirable properties in $\boldsymbol{\lambda}^{h,j+1}$.

PROPOSITION 3. *Under conditions* (A)–(C), *for any* $\theta \in [0,1]$, *positive constants* h_0, η_q, L_q, *and* σ_ν *exist such that for every tuple* $y \equiv (t, q^{\rm ref}, \nu^{\rm ref}) \in [0,T] \times \Re^{n_q+n_\nu}$ *and every* h *in* $(0, h_0]$, *a bounded continuous function* $(q^h(\cdot;y), \nu^h(\cdot;y)) : \Re^{n_\nu} \to \Re^{n_q+n_\nu}$ *exists satisfying the following properties:*

(a) *for every vector* $e \in \Re^{n_\nu}$, $(q^h(e;y), \nu^h(e;y))$ *is the unique pair* (q^h, ν^h) *satisfying*

$$M(q^{\rm ref})(\nu^h - \nu^{\rm ref}) = h[f(t, q^{\rm ref} + (1-\theta)(q^h - q^{\rm ref}), \nu^{\rm ref} + (1-\theta)(\nu^h - \nu^{\rm ref})) + e],$$
$$q^h - q^{\rm ref} = h\, G(q^{\rm ref})[\, \nu^{\rm ref} + (\, 1-\theta\,)\,(\, \nu^h - \nu^{\rm ref}\,)\,];$$

moreover,

$$\|\, q^h - q^{\rm ref}\,\| + \|\, \nu^h - \nu^{\rm ref}\,\| \;\leq\; h\,\eta_q\, \left[\, 1 + \|\, q^{\rm ref}\,\| + \|\, \nu^{\rm ref}\,\| + \|\, e\,\|\,\right];$$

(b) $(q^h(\cdot;y), \nu^h(\cdot;y))$ *is Lipschitz continuous with constant* hL_q; *thus*

$$\|\, q^h(e^1;y) - q^h(e^2;y)\,\| + \|\, \nu^h(e^1;y) - \nu^h(e^2;y)\,\| \;\leq\; h\, L_q\, \|\, e^1 - e^2\,\| \quad \forall\, e^1, e^2 \,\in\, \Re^{n_\nu};$$

(c) *the function* $\nu^h(\cdot;y) : \Re^{n_\nu} \to \Re^{n_\nu}$ *is strongly monotone with constant* $h\sigma_\nu$; *thus,*

$$(\, \nu^h(e^1;y) - \nu^h(e^2;y)\,)^T(\, e^1 - e^2\,) \;\geq\; h\,\sigma_\nu\, \|\, e^1 - e^2\,\|^2 \quad \forall\, e^1, e^2 \,\in\, \Re^{n_\nu}.$$

Proof. For a given vector $e \in \Re^{n_\nu}$, it is easily seen that the map

$$\begin{pmatrix} \nu \\ q \end{pmatrix} \mapsto \begin{pmatrix} \nu^{\rm ref} + hM(q^{\rm ref})^{-1}\,[f(t,\, q^{\rm ref} + (1-\theta)(q - q^{\rm ref}),\, \nu^{\rm ref} + (1-\theta)(\nu - \nu^{\rm ref})) + e] \\ q^{\rm ref} + hG(q^{\rm ref})[\nu^{\rm ref} + (1-\theta)(\nu - \nu^{\rm ref})] \end{pmatrix}$$

is a contraction with a modulus that can be made as small as we want by choosing $h > 0$ sufficiently small. Moreover, the constant h_0 depends only on the constants L_f, θ, η_G, and σ_M. Therefore, the above map has a unique fixed point, which yields the existence and uniqueness of the pair $(q^h(e;y), \nu^h(e;y))$. The proof of the bound on $\|q^h - q^{\rm ref}\| + \|\, \nu^h - \nu^{\rm ref}\|$ is similar to that of (b); for this reason, we prove only the latter. For any two vectors e^1 and e^2, we have

$$\begin{aligned} &\|\, \nu^h(e^1;y) - \nu^h(e^2;y)\,\| \\ &\leq \frac{h}{\sigma_M}\,[\, L_f\,(\,1-\theta\,)\,(\,\|q^h(e^1;y) - q^h(e^2;y)\| + \|\nu^h(e^1;y) - \nu^h(e^2;y)\|\,) + \|e^1 - e^2\|\,] \\ &\leq \frac{h}{\sigma_M}\,[\, L_f\,(\,1-\theta\,)\,(\,1 + h\,(\,1-\theta\,)\,\eta_G\,)\,\|\,\nu^h(e^1;y) - \nu^h(e^2;y)\,\|\,) + \|e^1 - e^2\|\,], \end{aligned}$$

which implies

$$\| \nu^h(e^1;y) - \nu^h(e^2;y) \| \le \frac{h\,\sigma_M^{-1}\, \| e^1 - e^2 \|}{1 - h\,\sigma_M^{-1}\,[\, L_f\,(\,1-\theta\,)\,(\,1 + h\,(\,1-\theta\,)\,\eta_G\,)\,]}.$$

Hence,

$$\| q^h(e^1;y) - q^h(e^2;y) \| \le \frac{h^2\,(\,1-\theta\,)\,\eta_G\,\sigma_M^{-1}\, \| e^1 - e^2 \|}{1 - h\,\sigma_M^{-1}\,[\, L_f\,(\,1-\theta\,)\,(\,1 + h\,(\,1-\theta\,)\,\eta_G\,)\,]},$$

establishing the desired Lipschitz continuity of $(q^h(\cdot;y), \nu^h(\cdot;y))$. To prove (c), note that

$$\begin{aligned}&\sigma_M \| \nu^h(e^1;y) - \nu^h(e^2;y) \|^2 \le h\,(\,\nu^h(e^1;y) - \nu^h(e^2;y)\,)^T(\,e^1 - e^2\,)\\ &+ hL_f(1-\theta)\,\|\nu^h(e^1;y) - \nu^h(e^2;y)\|(\|q^h(e^1;y) - q^h(e^2;y)\| + \|\nu^h(e^1;y) - \nu^h(e^2;y)\|),\end{aligned}$$

which yields

$$\begin{aligned}&(\,\nu^h(e^1;y) - \nu^h(e^2;y)\,)^T(\,e^1 - e^2\,)\\ &\qquad \ge \frac{[\,\sigma_M - h\,L_f\,(\,1-\theta\,)\,(\,1 + h\,(\,1-\theta\,)\,\eta_G\,)\,]}{h}\, \| \nu^h(e^1;y) - \nu^h(e^2;y) \|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned} h\,\|\,e^1 - e^2\,\| &\le \eta_M\,\|\,\nu^h(e^1;y) - \nu^h(e^2;y)\,\| \\ &\quad + h\,L_f\,(\,1-\theta\,)\,(\|q^h(e^1;y) - q^h(e^2;y)\| + \|\nu^h(e^1;y) - \nu^h(e^2;y)\|) \\ &\le [\,\eta_M + h\,L_f\,(\,1-\theta\,)\,(\,1 + h\,(\,1-\theta\,)\,\eta_G\,)\,]\,\|\,\nu^h(e^1;y) - \nu^h(e^2;y)\|,\end{aligned}$$

which implies

$$\| \nu^h(e^1;y) - \nu^h(e^2;y)\| \ge \frac{h\,\|\,e^1 - e^2\,\|}{\eta_M + h\,L_f\,(\,1-\theta\,)\,(\,1 + h\,(\,1-\theta\,)\,\eta_G\,)}.$$

Consequently,

$$(\nu^h(e^1;y) - \nu^h(e^2;y))^T(e^1 - e^2) \ge \frac{h\,[\,\sigma_M - hL_f(1-\theta)\,(1 + h(1-\theta)\eta_G)\,]\,\|e^1 - e^2\|^2}{\eta_M + hL_f(1-\theta)\,(1 + h(1-\theta)\eta_G)},$$

which establishes the desired strong monotonicity of $\nu^h(\cdot;y)$. □

From the discrete-time compliance equation

$$\boldsymbol{\lambda}^{h,j+1} = \mathbf{K}(q^{h,j})\boldsymbol{\delta}^{h,j+1} + \frac{\mathbf{C}(q^{h,j})}{h}\,(\,\boldsymbol{\delta}^{h,j+1} - \boldsymbol{\delta}^{h,j}\,),$$

we deduce

$$\boldsymbol{\delta}^{h,j+1} - \boldsymbol{\delta}^{h,j} = h\,[\,h\,\mathbf{K}(q^{h,j}) + \mathbf{C}(q^{h,j})\,]^{-1}\,[\,\boldsymbol{\lambda}^{h,j+1} - \mathbf{K}(q^{h,j})\boldsymbol{\delta}^{h,j}\,]. \tag{17}$$

Considering the expression in the normal direction,

$$\Psi_{\rm n}(q^{h,j}) + h\Gamma_{\rm n}(q^{h,j})\nu^{h,\theta_j} + \delta_{\rm n}^{h,j+1} = \Psi_{\rm n}(q^{h,j}) + h\Gamma_{\rm n}(q^{h,j})\nu^{h,\theta_j} + \delta_{\rm n}^{h,j} + \delta_{\rm n}^{h,j+1} - \delta_{\rm n}^{h,j},$$

we define the discrete-time normal slip velocity,

$$s_{\rm n}^{h,j+1} \equiv \frac{\delta_{\rm n}^{h,j+1} - \delta_{\rm n}^{h,j}}{h} + \Gamma_{\rm n}(q^{h,j})\,\nu^{h,\theta_j},$$

which is consistent with the corresponding expressions for the discrete-time tangential velocities $s_{\rm t,o}^{h,j+1}$. Writing $\mathbf{s} \equiv s_{\rm n,t,o}$, we have

$$\begin{aligned}\mathbf{s}^{h,j+1} &= \frac{\boldsymbol{\delta}^{h,j+1} - \boldsymbol{\delta}^{h,j}}{h} + \boldsymbol{\Gamma}(q^{h,j})\,[\,\theta\,\nu^{h,j} + (\,1-\theta\,)\,\nu^{h,j+1}\,]\\ &= [\,h\mathbf{K}(q^{h,j}) + \mathbf{C}(q^{h,j})\,]^{-1}[\,\boldsymbol{\lambda}^{h,j+1} - \mathbf{K}(q^{h,j})\boldsymbol{\delta}^{h,j}\,]\\ &\quad + \boldsymbol{\Gamma}(q^{h,j})\nu^{h,j} + (1-\theta)\boldsymbol{\Gamma}(q^{h,j})(\nu^{h,j+1} - \nu^{h,j}).\end{aligned}$$

In view of the latter expression, we define, for fixed $\mathbf{y}^{\rm ref} = (t, q^{\rm ref}, \nu^{\rm ref}, \boldsymbol{\delta}^{\rm ref})$, the following function in $\boldsymbol{\lambda}$:

$$\begin{aligned}\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}^{\rm ref}) \equiv{}& [\,h\,\mathbf{K}(q^{\rm ref}) + \mathbf{C}(q^{\rm ref})\,]^{-1}[\boldsymbol{\lambda} - \mathbf{K}(q^{\rm ref})\boldsymbol{\delta}^{\rm ref}\,] + \boldsymbol{\Gamma}(q^{\rm ref})\nu^{\rm ref}\\ &+ (\,1-\theta\,)\,\boldsymbol{\Gamma}(q^{\rm ref})\,[\,\nu^{h,\rm ref}(\boldsymbol{\Gamma}(q^{\rm ref})^T\boldsymbol{\lambda}) - \nu^{\rm ref}\,],\end{aligned}$$

where $\nu^{h,\rm ref}(r) \equiv \nu^h(r;(t,q^{\rm ref},\nu^{\rm ref}))$. Since $\nu^{h,\rm ref}$ is strongly monotone (albeit nonlinear), it follows that the map $\boldsymbol{\lambda} \mapsto \boldsymbol{\Gamma}(q^{\rm ref})\,\nu^{h,\rm ref}(\boldsymbol{\Gamma}(q^{\rm ref})^T\boldsymbol{\lambda})$ is monotone. Consequently, by assumption (D), we deduce

$$(\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,)^T(\,\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}^{\rm ref}) - \mathbf{s}(\boldsymbol{\lambda}';\mathbf{y}^{\rm ref})\,) \ge \sigma_{_{KC}}\,\|\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,\|^2, \quad \forall\,\boldsymbol{\lambda},\boldsymbol{\lambda}' \in \Re^{3n_\delta};$$

that is, the function $\mathbf{s}(\cdot;\mathbf{y}^{\rm ref})$ is strongly monotone with a modulus that is independent of $\mathbf{y}^{\rm ref}$. Moreover, $\mathbf{s}(\cdot;\mathbf{y}^{\rm ref})$ is Lipschitz continuous with a modulus that is also independent of $\mathbf{y}^{\rm ref}$; indeed, by assumption (D) and part (b) of Proposition 3, we have

$$\|\,\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}^{\rm ref}) - \mathbf{s}(\boldsymbol{\lambda}';\mathbf{y}^{\rm ref})\,\| \le [\,\eta_{_{KC}} + h\,L_{_q}\,(\,1-\theta\,)\,\eta_{_W}^2\,]\,\|\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,\| \quad \forall\,\boldsymbol{\lambda},\boldsymbol{\lambda}' \in \Re^{3n_\delta}.$$

Furthermore,

$$\mathbf{s}(0;\mathbf{y}^{\rm ref}) = -[\,h\mathbf{K}(q^{\rm ref}) + \mathbf{C}(q^{\rm ref})\,]^{-1}\mathbf{K}(q^{\rm ref})\boldsymbol{\delta}^{\rm ref} + \theta\,\boldsymbol{\Gamma}(q^{\rm ref})\nu^{\rm ref} + (1-\theta)\,\boldsymbol{\Gamma}(q^{\rm ref})\,\nu^{h,\rm ref}(0).$$

From part (a) of Proposition 3, we obtain

$$\|\,\nu^{h,\rm ref}(0)\,\| \le \|\,\nu^{\rm ref}\,\| + h\,\eta_q\,[\,1 + \|\,q^{\rm ref}\,\| + \|\,\nu^{\rm ref}\,\|\,].$$

Consequently, we deduce that a constant $c_s > 0$ exists such that

(18)
$$\|\,\mathbf{s}(0;\mathbf{y}^{\rm ref})\,\| \le c_s\,[\,1 + \|\,q^{\rm ref}\,\| + \|\,\nu^{\rm ref}\,\| + \|\,\boldsymbol{\delta}^{\rm ref}\,\|\,] \quad \forall\,\mathbf{y}^{\rm ref} \equiv (\,t, q^{\rm ref}, \nu^{\rm ref}, \boldsymbol{\delta}^{\rm ref}\,).$$

This inequality will be used later; see Lemma 4. It is important to remark that the above constant c_s and the strong modulus and the Lipschitz constant of the function $\mathbf{s}(\cdot;\mathbf{y}^{\rm ref})$ are all independent of $\mathbf{y}^{\rm ref}$ and of $h > 0$, provided that the latter is sufficiently small.

In terms of the function $\mathbf{s}(\cdot;\mathbf{y}^{h,j})$, where $\mathbf{y}^{h,j} \equiv (t_{h,j+1}, \mathbf{x}^{h,j})$, the discrete frictional compliant contact problem at time step $t_{h,j+1}$, without the boundary condition, can be stated simply as the quasi-variational inequality of finding a triple $\boldsymbol{\lambda} \equiv \lambda_{\rm n,t,o} \in \Re^{3n_\delta}$ such that

$$0 \le \lambda_{\rm n} \perp \frac{\Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}}{h} + s_{\rm n}(\boldsymbol{\lambda};\mathbf{y}^{h,j}) \ge 0$$

and for all $i = 1, \ldots, n_\delta$,

$$(\lambda_{it}, \lambda_{io}) \in \operatorname{argmin} \left\{ s_{it}(\boldsymbol{\lambda}; \mathbf{y}^{h,j})\tilde{\lambda}_{it} + s_{io}(\boldsymbol{\lambda}; \mathbf{y}^{h,j})\tilde{\lambda}_{io} \,:\, (\tilde{\lambda}_{it}, \tilde{\lambda}_{io}) \in \mathcal{F}(\mu_i \lambda_{in}) \right\}.$$

We state and prove a lemma pertaining to the above contact and friction conditions. This lemma is the key to the entire convergence analysis of the time-stepping method.

LEMMA 4. *Let $\mathbf{s}(\boldsymbol{\lambda}; \mathbf{y})$ be a continuous function that is Lipschitz continuous and strongly monotone in $\boldsymbol{\lambda} \in \Re^{3n_\delta}$ uniformly in $\mathbf{y} \in \Re^m$; i.e., positive constants η_s and σ_s exist such that for all $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ and $\mathbf{y}$,*

$$(\boldsymbol{\lambda} - \boldsymbol{\lambda}')^T (\mathbf{s}(\boldsymbol{\lambda}; \mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}'; \mathbf{y})) \geq \sigma_s \|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|^2 \quad \text{and} \quad \|\mathbf{s}(\boldsymbol{\lambda}; \mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}'; \mathbf{y})\| \leq \eta_s \|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|.$$

Suppose further that a constant $c_s > 0$ exists such that $\|\mathbf{s}(0; \mathbf{y})\| \leq c_s \|\mathbf{y}\|$ for all $\mathbf{y} \in \Re^m$. There exists a positive scalar $\bar{\mu} > 0$ such that for every vector $\mu > 0$ satisfying $\max_{1 \leq i \leq n_\delta} \mu_i \leq \bar{\mu}$, a continuous function $\boldsymbol{\lambda}^\mu : \Re^m \to \Re^{3n_\delta}$ exists such that for every parameter $\mathbf{y}$, $\boldsymbol{\lambda}^\mu(\mathbf{y})$ is the unique triple $\lambda_{\mathrm{n,t,o}}$ satisfying

$$0 \leq \lambda_{\mathrm{n}} \perp s_{\mathrm{n}}(\lambda_{\mathrm{n,t,o}}; \mathbf{y}) \geq 0,$$

and, for every $i = 1, \ldots, n_\delta$,

$$\begin{pmatrix} \lambda_{it} \\ \lambda_{io} \end{pmatrix} \in \underset{(\tilde{\lambda}_{it}, \tilde{\lambda}_{io}) \in \mathcal{F}(\mu_i \lambda_{in})}{\arg\min} \left\{ \begin{pmatrix} \tilde{\lambda}_{it} \\ \tilde{\lambda}_{io} \end{pmatrix}^T \begin{pmatrix} s_{it}(\lambda_{\mathrm{n,t,o}}; \mathbf{y}) \\ s_{io}(\lambda_{\mathrm{n,t,o}}; \mathbf{y}) \end{pmatrix} \right\}.$$

Proof. There are several things to be proved: the existence of the scalar $\bar{\mu}$ and the existence, uniqueness, and continuity of $\lambda^\mu_{\mathrm{n,t,o}}(\mathbf{y})$ for all $\mu > 0$ as specified. Indeed, the existence of a triple $\boldsymbol{\lambda}$ satisfying the above friction conditions for every $\mu > 0$ is proved by invoking a general result from the theory of quasi-variational inequalities [8, Corollary 2.8.4], as done in several previous references, such as [12]. In what follows, we show the uniqueness of such a solution for all $\mu > 0$ sufficiently small.

Suppose that $\lambda^1_{\mathrm{n,t,o}}$ and $\lambda^2_{\mathrm{n,t,o}}$ are two solutions corresponding to a given $\mathbf{y}$. Write, for $j = 1, 2$, $s^j_{\mathrm{n,t,o}} \equiv s_{\mathrm{n,t,o}}(\lambda^j_{\mathrm{n,t,o}}; \mathbf{y})$. We may write, for every i,

$$s^j_{i\mathrm{t}} \equiv r^j_i \cos\psi^j_i \quad \text{and} \quad s^j_{i\mathrm{o}} \equiv r^j_i \sin\psi^j_i,$$

where $r^j_i \equiv \sqrt{(s^j_{i\mathrm{t}})^2 + (s^j_{i\mathrm{o}})^2}$. It then follows that $\phi^j_i \in [-1, 1]$ exist satisfying

$$r^j_i > 0 \;\Rightarrow\; \phi^j_i = 1 \tag{19}$$

(ϕ^j_i is not necessarily equal to 1 when $r^j_i = 0$) and

$$\lambda^j_{i\mathrm{t}} = -\mu_i \lambda^j_{i\mathrm{n}} \phi^j_i \cos\psi^j_i \quad \text{and} \quad \lambda^j_{i\mathrm{o}} = -\mu_i \lambda^j_{i\mathrm{n}} \phi^j_i \sin\psi^j_i.$$

We have

$$\begin{aligned} \lambda^1_{i\mathrm{t}} - \lambda^2_{i\mathrm{t}} &= -\mu_i \lambda^1_{i\mathrm{n}} \phi^1_i \cos\psi^1_i + \mu_i \lambda^2_{i\mathrm{n}} \phi^2_i \cos\psi^2_i \\ &= -(\mu_i \phi^1_i \cos\psi^1_i)(\lambda^1_{i\mathrm{n}} - \lambda^2_{i\mathrm{n}}) + \mu_i \lambda^2_{i\mathrm{n}} (\phi^2_i \cos\psi^2_i - \phi^1_i \cos\psi^1_i). \end{aligned}$$

Similarly,

$$\lambda_{io}^1 - \lambda_{io}^2 \;=\; -(\,\mu_i\,\phi_i^1\,\sin\psi_i^1\,)\,(\,\lambda_{in}^1 - \lambda_{in}^2\,) + \mu_i\,\lambda_{in}^2\,(\,\phi_i^2\,\sin\psi_i^2 - \phi_i^1\,\sin\psi_i^1\,).$$

Therefore, letting $D_{\rm t}$ and $D_{\rm o}$ be the diagonal matrices whose diagonal entries are $-\mu_i\phi_i^1\cos\psi_i^1$ and $-\mu_i\phi_i^1\sin\psi_i^1$, respectively, we can write

$$\begin{aligned}
\lambda_{\rm t}^1 - \lambda_{\rm t}^2 &= D_t(\,\lambda_{\rm n}^1 - \lambda_{\rm n}^2\,) + \mu\,\lambda_{\rm n}^2\,(\,\phi^2\cos\psi^2 - \phi^1\cos\psi^1\,),\\
\lambda_{\rm o}^1 - \lambda_{\rm o}^2 &= D_o(\,\lambda_{\rm n}^1 - \lambda_{\rm n}^2\,) + \mu\,\lambda_{\rm n}^2\,(\,\phi^2\sin\psi^2 - \phi^1\sin\psi^1\,),
\end{aligned}$$

where the notation in the second terms in the right-hand side of the above equations has an obvious componentwise meaning. Consequently,

$$\begin{pmatrix} \lambda_{\rm n}^1 - \lambda_{\rm n}^2 \\ \lambda_{\rm t}^1 - \lambda_{\rm t}^2 \\ \lambda_{\rm o}^1 - \lambda_{\rm o}^2 \end{pmatrix} = \begin{bmatrix} I & 0 & 0 \\ D_t & I & 0 \\ D_o & 0 & I \end{bmatrix} \begin{pmatrix} \lambda_{\rm n}^1 - \lambda_{\rm n}^2 \\ \mu\,\lambda_{\rm n}^2\,(\phi^2\cos\psi^2 - \phi^1\cos\psi^1) \\ \mu\,\lambda_{\rm n}^2\,(\phi^2\sin\psi^2 - \phi^1\sin\psi^1) \end{pmatrix}$$

or, equivalently,

$$\begin{aligned}
\begin{pmatrix} \lambda_{\rm n}^1 - \lambda_{\rm n}^2 \\ \mu\lambda_{\rm n}^2(\phi^2\cos\psi^2 - \phi^1\cos\psi^1) \\ \mu\lambda_{\rm n}^2(\phi^2\sin\psi^2 - \phi^1\sin\psi^1) \end{pmatrix} &= \begin{bmatrix} I & 0 & 0 \\ D_t & I & 0 \\ D_o & 0 & I \end{bmatrix}^{-1} \begin{pmatrix} \lambda_{\rm n}^1 - \lambda_{\rm n}^2 \\ \lambda_{\rm t}^1 - \lambda_{\rm t}^2 \\ \lambda_{\rm o}^1 - \lambda_{\rm o}^2 \end{pmatrix}\\
&= \begin{bmatrix} I & 0 & 0 \\ -D_t & I & 0 \\ -D_o & 0 & I \end{bmatrix} \begin{pmatrix} \lambda_{\rm n}^1 - \lambda_{\rm n}^2 \\ \lambda_{\rm t}^1 - \lambda_{\rm t}^2 \\ \lambda_{\rm o}^1 - \lambda_{\rm o}^2 \end{pmatrix}.
\end{aligned}$$

Writing

$$\mathbf{D}(\mu) \;\equiv\; \begin{bmatrix} I & 0 & 0 \\ -D_t & I & 0 \\ -D_o & 0 & I \end{bmatrix},$$

we claim that positive constants σ_s' and $\bar\mu$ exist such that for all $\mu > 0$ satisfying $\max\limits_{1\le i\le n_\delta} \mu_i \le \bar\mu$,

$$(\,\mathbf{D}(\mu)\boldsymbol{\lambda} - \mathbf{D}(\mu)\boldsymbol{\lambda}'\,)^T(\,\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}';\mathbf{y})\,) \;\ge\; \sigma_s'\,\|\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,\|^2$$

for all $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$. To establish the claim, we write

$$\begin{aligned}
&(\,\mathbf{D}(\mu)\boldsymbol{\lambda} - \mathbf{D}(\mu)\boldsymbol{\lambda}'\,)^T(\,\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}';\mathbf{y})\,)\\
&\quad = (\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,)^T(\,\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}';\mathbf{y})\,) - [\,(\,I - \mathbf{D}(\mu)\,)\,(\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,)\,]^T(\,\mathbf{s}(\boldsymbol{\lambda};\mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}';\mathbf{y})\,)\\
&\quad \ge [\,\sigma_s - \eta_s\,\|\,I - \mathbf{D}(\mu)\,\|\,]\,\|\,\boldsymbol{\lambda} - \boldsymbol{\lambda}'\,\|^2;
\end{aligned}$$

clearly, we can choose $\bar\mu > 0$ sufficiently small such that for all $\mu > 0$ satisfying $\max\limits_{1\le i\le n_\delta} \mu_i \le \bar\mu$, we have $\sigma_s - \eta_s\|I - \mathbf{D}(\mu)\| \ge \frac{1}{2}\sigma_s \equiv \sigma_s'$. This establishes the claim. Next, we show that

$$0 \;\ge\; (\,\mathbf{D}(\mu)\boldsymbol{\lambda}^1 - \mathbf{D}(\mu)\boldsymbol{\lambda}^2\,)^T(\,\mathbf{s}(\boldsymbol{\lambda}^1;\mathbf{y}) - \mathbf{s}(\boldsymbol{\lambda}^2;\mathbf{y})\,). \tag{20}$$

The right-hand side of the above inequality is equal to

$$\left(\begin{array}{c} \lambda_{\rm n}^1 - \lambda_{\rm n}^2 \\ \mu\,\lambda_{\rm n}^2\,(\phi^2\cos\psi^2 - \phi^1\cos\psi^1) \\ \mu\,\lambda_{\rm n}^2\,(\phi^2\sin\psi^2 - \phi^1\sin\psi^1) \end{array} \right)^T \left(\begin{array}{c} s_{\rm n}^1 - s_{\rm n}^2 \\ s_{\rm t}^1 - s_{\rm t}^2 \\ s_{\rm o}^1 - s_{\rm o}^2 \end{array} \right).$$

By complementarity, we have $(\lambda_{\rm n}^1 - \lambda_{\rm n}^2)^T(s_{\rm n}^1 - s_{\rm n}^2) \leq 0$. Furthermore,

$$\begin{aligned} &(\,\phi_i^2\cos\psi_i^2 - \phi_i^1\cos\psi_i^1\,)\,(\,s_{i\rm t}^1 - s_{i\rm t}^2\,) + (\,\phi_i^2\sin\psi_i^2 - \phi_i^1\sin\psi_i^1\,)\,(\,s_{i\rm o}^1 - s_{i\rm o}^2\,) \\ &= \,(\,\phi_i^2\cos\psi_i^2 - \phi_i^1\cos\psi_i^1\,)\,(\,r_i^1\,\cos\psi_i^1 - r_i^2\,\cos\psi_i^2\,) \\ &\quad + (\,\phi_i^2\sin\psi_i^2 - \phi_i^1\sin\psi_i^1\,)\,(\,r_i^1\,\sin\psi_i^1 - r_i^2\,\sin\psi_i^2\,) \\ &= \,-r_i^1\,\phi_i^1 - r_i^2\,\phi_i^2 + (\,r_i^1\,\phi_i^2 + r_i^2\,\phi_i^1\,)\cos(\psi_i^1 - \psi_i^2\,) \\ &= \,-r_i^1 - r_i^2 + (\,r_i^1\,\phi_i^2 + r_i^2\,\phi_i^1\,)\cos(\psi_i^1 - \psi_i^2\,)\,\leq\,0, \end{aligned}$$

where the last equality follows from (19) and the last inequality holds because $|\phi_j^{1,2}| \leq 1$. Consequently, the inequality (20) holds. In turn, this implies that $\lambda_{\rm n,t,o}^1 = \lambda_{\rm n,t,o}^2$. This establishes the uniqueness of $\boldsymbol{\lambda}^\mu(x)$ for all $\mu > 0$ sufficiently small.

In the rest of the proof, we fix an arbitrary $\mu > 0$ sufficiently small and drop the superscript μ in $\boldsymbol{\lambda}^\mu$. To show the continuity of $\lambda_{\rm n,t,o}(\mathbf{y})$, we first derive a bound for $\|\lambda_{\rm n,t,o}(\mathbf{y})\|$. We have

$$\begin{aligned} 0 \;&\geq\; \boldsymbol{\lambda}(\mathbf{y})^T\mathbf{s}(\boldsymbol{\lambda}(\mathbf{y});\mathbf{y}) \;=\; \boldsymbol{\lambda}(\mathbf{y})^T(\,\mathbf{s}(\boldsymbol{\lambda}(\mathbf{y});\mathbf{y}) - \mathbf{s}(0;\mathbf{y})\,) + \boldsymbol{\lambda}(\mathbf{y})^T\mathbf{s}(0;\mathbf{y}) \\ &\geq\; \sigma_s\,\|\,\boldsymbol{\lambda}(\mathbf{y})\,\|^2 - c_s\,\|\,\boldsymbol{\lambda}(\mathbf{y})\,\|\,\|\,\mathbf{y}\,\|, \end{aligned}$$

which implies $\|\boldsymbol{\lambda}(\mathbf{y})\| \leq c_s\|\mathbf{y}\|$. Let $\{\mathbf{y}^k\}$ be a sequence of parameters converging to $\mathbf{y}^\infty$. Write $\lambda_{\rm n,t,o}^k \equiv \lambda_{\rm n,t,o}(\mathbf{y}^k)$. Since the sequence $\{\lambda_{\rm n,t,o}^k\}$ is bounded, by what has just been shown, let $\lambda_{\rm n,t,o}^\infty$ be the limit of a convergent subsequence $\{\lambda_{\rm n,t,o}^k : k \in \kappa\}$, where κ is an infinite subset of $\{1, 2, \ldots\}$. It suffices to show that $\lambda_{\rm n,t,o}^\infty$ is a solution to the limiting system

$$0 \,\leq\, \lambda_{\rm n}^\infty \,\perp\, s_{\rm n}(\lambda_{\rm n,t,o}^\infty;\mathbf{y}^\infty) \,\geq\, 0 \tag{21}$$

and

$$\left(\begin{array}{c} \lambda_{i\rm t}^\infty \\ \lambda_{i\rm o}^\infty \end{array} \right) \in \underset{(\tilde\lambda_{i\rm t},\tilde\lambda_{i\rm o})\in\mathcal{F}(\mu_i\,\lambda_{i\rm n}^\infty)}{\arg\min} \left\{ \left(\begin{array}{c} \tilde\lambda_{i\rm t} \\ \tilde\lambda_{i\rm o} \end{array} \right)^T \left(\begin{array}{c} s_{i\rm t}(\lambda_{\rm n,t,o}^\infty;\mathbf{y}^\infty) \\ s_{i\rm o}(\lambda_{\rm n,t,o}^\infty;\mathbf{y}^\infty) \end{array} \right) \right\}.$$

Since $0 \leq \lambda_{\rm n}^k \perp s_{\rm n}(\boldsymbol{\lambda}^k;\mathbf{y}^k) \geq 0$ for all k, passing to the limit $k(\in\kappa) \to \infty$ yields (21). Similarly, since $(\lambda_{i\rm t}^k)^2 + (\lambda_{i\rm o}^k)^2 \leq \mu_i^2(\lambda_{i\rm n}^k)^2$ for all k, we deduce $(\lambda_{i\rm t}^\infty, \lambda_{i\rm o}^\infty) \in \mathcal{F}(\mu_i\lambda_{i\rm n}^\infty)$. Moreover, since

$$\lambda_{i\rm t}^k\,s_{i\rm t}(\boldsymbol{\lambda}^k;\mathbf{y}^k) + \lambda_{i\rm o}^k\,s_{i\rm o}(\boldsymbol{\lambda}^k;\mathbf{y}^k) \;=\; -\mu_i\,\lambda_{i\rm n}^k\sqrt{s_{i\rm t}(\boldsymbol{\lambda}^k;\mathbf{y}^k)^2 + s_{i\rm o}(\boldsymbol{\lambda}^k;\mathbf{y}^k)^2},$$

passing to the limit $k(\in\kappa) \to \infty$ easily completes the proof. ∎

Applying Lemma 4 to the friction and contact conditions, we conclude that for all $h > 0$ and sufficiently small and for all $\mu > 0$ not exceeding a certain upper bound $\bar\mu$, for a given triple $\mathbf{x}^{h,j} \equiv (q^{h,j}, \nu^{h,j}, \boldsymbol{\delta}^{h,j})$, a unique friction force triple $\lambda_{\rm n,t,o}^{h,j+1}$ exists at time step $t_{h,j+1}$ that is a continuous function of $\mathbf{x}^{h,j}$. In what follows, we derive a bound on $\|\boldsymbol{\lambda}^{h,j+1}\|$ that takes advantage of the normal contact condition at time step j. This improved bound is important for the subsequent analysis. (A straightforward application of the previous lemma would yield a bound of the order $1/h$, which tends to infinity as $h \downarrow 0$, and thus is not effective for small h. The bound obtained below stays finite as h tends to zero, as shown subsequently.)

LEMMA 5. *Let* $\boldsymbol{\lambda}^{h,j+1}$ *satisfy*

$$0 \,\leq\, \lambda_{\rm n}^{h,j+1} \,\perp\, \frac{\Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}}{h} + s_{\rm n}(\boldsymbol{\lambda}^{h,j+1};\mathbf{y}^{h,j}) \,\geq\, 0$$

and for all $i = 1, \ldots, n_\delta$,

$$(\lambda_{i\rm t}^{h,j+1}, \lambda_{i\rm o}^{h,j+1}) \in \mathop{\arg\min}_{(\tilde{\lambda}_{i\rm t},\tilde{\lambda}_{i\rm o})\in\mathcal{F}(\mu_i\lambda_{i\rm n}^{h,j+1})} \left\{ s_{i\rm t}(\boldsymbol{\lambda}^{h,j+1};\mathbf{y}^{h,j})\tilde{\lambda}_{i\rm t} + s_{i\rm o}(\boldsymbol{\lambda}^{h,j+1};\mathbf{y}^{h,j})\tilde{\lambda}_{i\rm o} \right\}.$$

A constant $\eta_\lambda > 0$*, which depends only the model functions, exists such that*

$$\|\boldsymbol{\lambda}^{h,j+1}\| \,\leq\, \eta_\lambda \left[\frac{\|\min(\,0, \Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}\,)\,\|}{h} + 1 + \|\mathbf{x}^{h,j}\| \right].$$

Proof. As in the proof of Lemma 4, we have

$$\begin{aligned} 0 \;&\geq\; (\,\lambda_{\rm n}^{h,j+1}\,)^T \frac{\Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}}{h} + (\,\boldsymbol{\lambda}^{h,j+1}\,)^T \mathbf{s}(\boldsymbol{\lambda}^{h,j+1};\mathbf{y}^{h,j}) \\ &\geq\; (\,\lambda_{\rm n}^{h,j+1}\,)^T \frac{\Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}}{h} + \|\,\boldsymbol{\lambda}^{h,j+1}\,\|^2 - c_s\,\|\,\boldsymbol{\lambda}^{h,j+1}\,\|\,[\,1 + \|\,\mathbf{x}^{h,j}\,\|\,] \\ &\geq\; (\,\lambda_{\rm n}^{h,j+1}\,)^T \min\left(0,\, \frac{\Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}}{h}\right) + \|\,\boldsymbol{\lambda}^{h,j+1}\,\|^2 - c_s\,\|\,\boldsymbol{\lambda}^{h,j+1}\,\|\,[\,1 + \|\,\mathbf{x}^{h,j}\,\|\,], \end{aligned}$$

where the last inequality holds because $\lambda_{\rm n}^{h,j+1} \geq 0$. Consequently, the desired bound on $\|\boldsymbol{\lambda}^{h,j+1}\|$ follows easily by rearranging terms and then applying the Cauchy–Schwartz inequality. ∎

Combining Proposition 3 and Lemmas 4 and 5, we obtain the following result, which brings us one step closer to the main existence and uniqueness for the discrete-time boundary value problem.

PROPOSITION 6. *Under conditions* (A)–(D)*, positive scalars* $\bar{\mu}$, h_0*, and* η_x *exist such that for every vector* $\mu > 0$ *satisfying* $\max_{1\leq i\leq n_\delta} \mu_i \leq \bar{\mu}$*, every scalar* $h \in (0, h_0]$*, and every tuple* $(q^{h,0}, \nu^{h,0}, \boldsymbol{\delta}^{h,0})$*, a unique discrete-time trajectory* (14) *exists satisfying* (16) *for every* $j = 0, 1, \ldots, N_h$ *but not necessarily* (13)*; moreover,*

$$\|\,\mathbf{x}^{h,j+1} - \mathbf{x}^{h,j}\,\| \,\leq\, h\,\eta_x\,[\,1 + \|\,\mathbf{x}^{h,j}\,\| + \|\,\boldsymbol{\lambda}^{h,j+1}\,\|\,]. \tag{22}$$

Finally, if $\Psi_{\rm n}(q^{h,0}) + \delta_{\rm n}^{h,0} \geq 0$*, then, for any scalar* $c_q > 0$*, the implication below holds for all* $j = 0, 1, \ldots, N_h$*, where* $q^{h,-1} \equiv q^{h,0}$*:*

$$\begin{aligned} &\|\min(0, \Psi_{\rm n}(q^{h,j}) - \Psi_{\rm n}(q^{h,j-1}) - J\Psi_{\rm n}(q^{h,j-1})(q^{h,j} - q^{h,j-1}))\| \,\leq\, c_q\,h \\ &\quad\Rightarrow\; \|\boldsymbol{\lambda}^{h,j+1}\| \,\leq\, \eta_\lambda\,(\,1 + c_q + \|\mathbf{x}^{h,j}\|\,). \end{aligned} \tag{23}$$

Proof. The bound for $\|\boldsymbol{\delta}^{h,j+1} - \boldsymbol{\delta}^{h,j}\|$, which is part of (22), follows from (17). Since

$$\Psi_{\rm n}(q^{h,j}) + \Gamma_{\rm n}(q^{h,j-1})(q^{h,j} - q^{h,j-1}) + \delta_{\rm n}^{h,j} \geq 0,$$

we have

$$\begin{aligned} 0 &\geq \min(\,0, \Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j}\,) \\ &\geq \min(\,0, \Psi_{\rm n}(q^{h,j}) - \Psi_{\rm n}(q^{h,j-1}) - \Gamma_{\rm n}(q^{h,j-1})(q^{h,j} - q^{h,j-1})\,) \\ &\quad + \min(\,0, \Psi_{\rm n}(q^{h,j-1}) + \Gamma_{\rm n}(q^{h,j-1})(q^{h,j} - q^{h,j-1}) + \delta_{\rm n}^{h,j}\,) \\ &= \min(\,0, \Psi_{\rm n}(q^{h,j}) - \Psi_{\rm n}(q^{h,j-1}) - \Gamma_{\rm n}(q^{h,j-1})(q^{h,j} - q^{h,j-1})\,). \end{aligned}$$

Taking norms, we obtain

$$\|\min(0, \Psi_{\rm n}(q^{h,j}) + \delta_{\rm n}^{h,j})\| \leq \|\min(\,0, \Psi_{\rm n}(q^{h,j}) - \Psi_{\rm n}(q^{h,j-1}) - \Gamma_{\rm n}(q^{h,j-1})(q^{h,j} - q^{h,j-1}))\|.$$

The bound (23) on $\|\boldsymbol{\lambda}^{h,j+1}\|$ follows readily from Lemma 5. ∎

So far, we have not used the limit condition (3) in proving the above results. This condition allows us to establish the boundedness of the state variables $\{\mathbf{x}^{h,j}\}$ and thus of the force variables $\{\boldsymbol{\lambda}^{h,j+1}\}$ also. We first state a technical fact, which can be proved by induction; see also [11, Lemma 7]. Namely, for every nonnegative integer $k \leq N_h$, if

$$\|\,\mathbf{x}^{h,j+1} - \mathbf{x}^{h,j}\,\| \,\leq\, h\,\psi_x\,(\,1 + \|\,\mathbf{x}^{h,j}\|\,) \quad \forall\, j \,=\, 0, 1, \ldots, k, \tag{24}$$

then (recalling that $T = (N_h + 1)h$),

$$\|\,\mathbf{x}^{h,j+1}\,\| \,\leq\, e^{T\,\psi_x}\,(\,1 + \|\,\mathbf{x}^{h,0}\,\|\,) - 1 \quad \forall\, j \,=\, 0, 1, \ldots, k. \tag{25}$$

PROPOSITION 7. *For any positive scalar c_q, let $\psi_x \equiv \eta_x(1 + \eta_\lambda(1 + c_q))$. For any scalar $R_0 > 0$, the scalar h_0 in Proposition 6 can be chosen such that (25) holds for $k = N_h$ for all $h \in (0, h_0]$ and for all $\mathbf{x}^{h,0}$ satisfying $\|\mathbf{x}^{h,0}\| \leq R_0$; moreover, for all $j = 0, 1, \ldots, N_h$,*

$$\|\,\boldsymbol{\lambda}^{h,j+1}\,\| \,\leq\, \eta_\lambda\,[\,c_q + e^{T\,\psi_x}\,(\,1 + \|\,\mathbf{x}^{h,0}\,\|\,)\,]. \tag{26}$$

Proof. Choose $\varepsilon > 0$ such that $\varepsilon\psi_x e^{T\psi_x}(1 + R_0) < c_q$. Corresponding to the chosen ε, let $\varsigma > 0$ be such that (3) holds. Let $h_0 > 0$ be sufficiently small such that $h_0\psi_x e^{T\psi_x}(1 + R_0) < \varsigma$. Let $\mathbf{x}^{h,0}$ be an arbitrary vector satisfying $\|\mathbf{x}^{h,0}\| \leq R_0$ and let $h \in (0, h_0]$ be arbitrary. It suffices to prove (24) for $k = N_h$. Clearly, (24) is valid for $k = 0$ because $\|\mathbf{x}^{h,1} - \mathbf{x}^{h,0}\| \leq h\eta_x(1 + \|\mathbf{x}^{h,0}\|) \leq h\psi_x(1 + \|\mathbf{x}^{h,0}\|)$. Assume that (24), and thus (25), holds for some $k \geq 0$. To complete the induction, we need to show

$$\|\,\mathbf{x}^{h,k+2} - \mathbf{x}^{h,k+1}\,\| \,\leq\, h\,\psi_x\,(\,1 + \|\,\mathbf{x}^{h,k+1}\,\|\,).$$

By the choice of h and $\|\mathbf{x}^{h,0}\|$, (24) with $j = k$ and (25) with $j = k - 1$ imply

$$\|\,\mathbf{x}^{h,k+1} - \mathbf{x}^{h,k}\,\| \,\leq\, h\,\psi_x\,e^{T\,\psi_x}\,(\,1 + \|\,\mathbf{x}^{h,0}\,\|\,) \,<\, \varsigma.$$

By (3) and the choice of c_q, it follows that

$$\begin{aligned} &\|\,\min(\,0, \Psi_{\rm n}(q^{h,k+1}) - \Psi_{\rm n}(q^{h,k}) - J\Psi_{\rm n}(q^{h,k})(q^{h,k+1} - q^{h,k})\,)\,\| \\ &\quad\leq\, \varepsilon\,\|\,q^{h,k+1} - q^{h,k}\,\| \,\leq\, \varepsilon\,h\,\psi_x\,e^{T\,\psi_x}\,(\,1 + \|\,\mathbf{x}^{h,0}\,\|\,) \,\leq\, h\,c_q. \end{aligned}$$

Consequently, by the implication (23), we obtain

$$\| \boldsymbol{\lambda}^{h,k+2} \| \leq \eta_{\lambda} (1 + c_q + \| \mathbf{x}^{h,k+1} \|) \leq \eta_{\lambda} (1 + c_q) (1 + \| \mathbf{x}^{h,k+1} \|).$$

Substituting this into (22) with $j = k + 1$ yields

$$\begin{aligned} \| \mathbf{x}^{h,k+2} - \mathbf{x}^{h,k+1} \| &\leq h \, \eta_x [1 + \| \mathbf{x}^{h,k+1} \| + \eta_{\lambda} (1 + c_q) (1 + \| \mathbf{x}^{h,k+1} \|)] \\ &= h \, \psi_x (1 + \| \mathbf{x}^{h,k+1} \|), \end{aligned}$$

completing the induction. The bound on $\|\boldsymbol{\lambda}^{h,j+1}\|$ holds by (23) and the bound on $\|\mathbf{x}^{h,j}\|$. $\square$

Based on Proposition 7, we can establish the convergence of the time-stepping method for an initial-value frictional compliant contact problem where $\mathbf{x}(0)$ is completely known. Since our treatment of the boundary-value problem will cover this case, we proceed directly to the latter.

5. Boundary-value analysis. Proposition 7 allows us to employ the line of proof in [11] to complete the convergence analysis of the time-stepping method. Needless to say, the boundary equation (13) will play a key role in this analysis. For this reason, we partition the boundary matrix $\mathbf{N}$ as

$$\mathbf{N} \equiv \left[\begin{array}{ccc} \mathbf{N}_q & \mathbf{N}_\nu & \mathbf{N}_\delta \end{array} \right],$$

where $\mathbf{N}_q \in \Re^{n_\nu \times n_q}$, $\mathbf{N}_\nu \in \Re^{n_\nu \times n_\nu}$, and $\mathbf{N}_\delta \in \Re^{n_\nu \times n_\delta}$, and we write the discrete-time boundary equation as

$$(\mathbf{M}_\nu + \mathbf{N}_\nu)\nu^{h,0} = \mathbf{b} + \mathbf{N}_\nu \nu^{h,0} - \mathbf{N}\mathbf{x}^{h,N_h+1} = \widehat{\mathbf{b}} - \mathbf{N}(\mathbf{x}^{h,N_h+1} - \mathbf{x}^{h,0}), \tag{27}$$

where $\widehat{\mathbf{b}} \equiv \mathbf{b} - \mathbf{N}_q q^0 - \mathbf{N}_\delta \boldsymbol{\delta}^0$. We are now ready to formally state and prove the two main results of this paper: Theorems 8 and 9. While the former establishes the existence of a solution to the discrete-time boundary system (16), including the boundary condition, the latter proves the convergence to a continuous-time trajectory.

THEOREM 8. *Assume conditions* (A)–(D) *and that* $\mathbf{M}_\nu + \mathbf{N}_\nu$ *is nonsingular. Let* ψ_x *be the constant obtained in Proposition* 7. *If*

$$e^{T\psi_x} < 1 + \frac{1}{\| (\mathbf{M}_\nu + \mathbf{N}_\nu)^{-1}\mathbf{N} \|}, \tag{28}$$

positive scalars $\bar{\mu}$, h_0, *and* ψ_x *exist such that for every vector* $\mu > 0$ *satisfying* $\max\limits_{1 \leq i \leq n_\delta} \mu_i \leq \bar{\mu}$, *every scalar* $h \in (0, h_0]$, *and every pair* $(q^{h,0}, \boldsymbol{\delta}^{h,0})$ *satisfying* (12), *a discrete-time trajectory* (14) *exists satisfying* (16) *for every* $j = 0, 1, \ldots, N_h$. *Moreover,* (24) *holds for* $k = N_h$ *and* (26) *holds for all* $j = 0, 1, \ldots, N_h$.

Proof. Throughout the proof below, the scalars h and μ_i are taken to be sufficiently small so that the previous results can all be applied. More specifically, with the constant r_0 chosen at the end of the proof (cf. (31)), the upper limits h_0 and $\bar{\mu}$ are then guaranteed by Proposition 7. The derivation below emphasizes the process of how the constant r_0 is obtained.

For $\mathbf{x}^{\rm ref} \equiv (q^{\rm ref}, \nu^{\rm ref}, \boldsymbol{\delta}^{\rm ref})$ in $\Re^n$, let $\nu^h(\mathbf{x}^{\rm ref})$ be the unique tuple $(q^h, \nu^h, \boldsymbol{\delta}^h)$, which, along with a (unique) triple of friction forces $\boldsymbol{\lambda}^h$, satisfies the following conditions:

$$\begin{aligned}
&M(q^{\rm ref})(\,\nu^h - \nu^{\rm ref}\,) \,=\, h\,[\,f(t_{h,j+1}, q^{h,\theta_{\rm ref}}, \nu^{h,\theta_{\rm ref}}) + \boldsymbol{\Gamma}(q^{\rm ref})^T \boldsymbol{\lambda}^h\,],\\
&q^h - q^{\rm ref} \,=\, h\,G(q^{\rm ref})\nu^{h,\theta_{\rm ref}},\\
&\delta^h_{\rm t} - \delta^{\rm ref}_{\rm t} \,=\, h\,[s^h_{\rm t} - \Gamma_{\rm t}(q^{\rm ref})\nu^{h,\theta_{\rm ref}}\,],\\
&\delta^h_{\rm o} - \delta^{\rm ref}_{\rm o} \,=\, h\,[s^h_{\rm o} - \Gamma_{\rm o}(q^{\rm ref})\nu^{h,\theta_{\rm ref}}\,],\\
&0 \,\le\, \lambda^h_{\rm n} \,\perp\, \Psi_{\rm n}(q^{\rm ref}) + h\,\Gamma_{\rm n}(q^{\rm ref})\nu^{h,\theta_{\rm ref}} + \delta^h_{\rm n} \,\ge\, 0,\\
&\boldsymbol{\lambda}^h \,=\, \mathbf{K}(q^{\rm ref})\boldsymbol{\delta}^h + \frac{\mathbf{C}(q^{\rm ref})}{h}\,(\,\boldsymbol{\delta}^h - \boldsymbol{\delta}^{\rm ref}\,),\\
&\begin{pmatrix} \lambda^h_{i{\rm t}} \\ \lambda^h_{i{\rm o}} \end{pmatrix} \in \mathop{\arg\min}_{(\tilde\lambda_{i{\rm t}},\tilde\lambda_{i{\rm o}})\in\mathcal{F}(\mu_i\,\lambda^h_{i{\rm n}})} \left\{ \begin{pmatrix} s^h_{i{\rm t}} \\ s^h_{i{\rm o}} \end{pmatrix}^T \begin{pmatrix} \tilde\lambda_{i{\rm t}} \\ \tilde\lambda_{i{\rm o}} \end{pmatrix} \right\},
\end{aligned}$$

where

$$q^{h,\theta_{\rm ref}} \,\equiv\, \theta\, q^{\rm ref} + (\,1-\theta\,)\,q^h \quad \text{and} \quad \nu^{h,\theta_{\rm ref}} \,\equiv\, \theta\,\nu^{\rm ref} + (\,1-\theta\,)\,\nu^h.$$

The well-definedness of $\nu^h(\mathbf{x}^{\rm ref})$ is ensured by Proposition 3 and Lemma 4; moreover, this map is continuous. For $j = 0, 1, \ldots, N_h$, define the maps $\Lambda^{h,j} : \Re^n \to \Re^n$ recursively by $\Lambda^{h,j+1}(\mathbf{x}) \equiv \nu^h(\Lambda^{h,j}(\mathbf{x}))$, where $\Lambda^{h,0}$ is the identity map. Define the auxiliary map $\Phi : \Re^{n_\nu} \to \Re^n$ by $\Phi(\nu) \equiv (q^{h,0}, \nu, \boldsymbol{\delta}^{h,0})$. In terms of these maps we can write the boundary equation (27) as a fixed-point equation: $\nu^{h,0} = \Upsilon(\nu^{h,0})$, where $\Upsilon : \Re^{n_\nu} \to \Re^{n_\nu}$ is the map defined by

$$\Upsilon(\nu) \,=\, (\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}[\,\widehat{\mathbf{b}} - \mathbf{N} \circ (\,\Lambda^{h,N_h+1} - I\,) \circ \Phi(\nu)\,],$$

which is continuous. We claim that a constant $r_0 > 0$ exists such that Υ maps the closed Euclidean ball with center at the origin and radius r_0 into itself. Once this claim is established, Brouwer's fixed-point theorem then shows that the discrete-time boundary system (16) has a solution.

By Proposition 7, we have, for $j = 0, 1, \ldots, N_h$,

$$(29) \quad \|\,\Lambda^{h,j+1}(\mathbf{x}^{\rm ref})\,\| \le R_{h,j+1} \;\text{ and }\; \|\,\Lambda^{h,j+1}(\mathbf{x}^{\rm ref}) - \Lambda^{h,j}(\mathbf{x}^{\rm ref})\,\| \le R_{h,j+1} - R_{h,j},$$

where $R_{h,j+1}$ satisfies the recursion

$$(30) \qquad R_{h,j+1} \,\equiv\, (\,1 + h\,\psi_x\,)\,R_{h,j} + h\,\psi_x, \quad j \,=\, 0, 1, \ldots, N_h,$$

with $R_{h,0} \ge \|\mathbf{x}^{\rm ref}\|$. Consequently, for any vector $\nu \in \Re^{n_\nu}$, letting $r_0 \ge \|\nu\|$ and $R_{h,0} \equiv r_0 + \|q^0\| + \|\boldsymbol{\delta}^0\|$, we have

$$\begin{aligned}
\|\Upsilon(\nu)\| &\le \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\widehat{\mathbf{b}}\,\| + \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\mathbf{N}\,\|\,\|\Lambda^{h,N_h+1}(\Phi(\nu)) - \Lambda^{h,0}(\Phi(\nu))\|\\
&\le \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\widehat{\mathbf{b}}\,\| + \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\mathbf{N}\,\|\,(\,R_{h,N_h+1} - R_{h,0}\,)\\
&\le \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\widehat{\mathbf{b}}\,\| + \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\mathbf{N}\,\|\,(\,e^{T\psi_x} - 1\,)(\,1 + R_{h,0}\,),
\end{aligned}$$

where the last inequality follows from (25), which gives $R_{h,N_h+1} \le e^{T\psi_x}(1 + R_{h,0}) - 1$. Consequently for any $r_0 \ge \|\nu\|$, we have

$$\|\Upsilon(\nu)\| \,\le\, \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\widehat{\mathbf{b}}\,\| + \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\mathbf{N}\,\|\,(\,e^{T\psi_x} - 1\,)(\,1 + \|\,q^0\,\| + r_0 + \|\,\boldsymbol{\delta}^0\,\|\,).$$

By (28), it follows that $1 > \|(\mathbf{M}_\nu + \mathbf{N}_\nu)^{-1}\mathbf{N}\|(e^{T\psi_x} - 1)$; hence if

$$r_0 \; > \; \frac{\|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\widehat{\mathbf{b}}\,\| + \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\mathbf{N}\,\|\,(\,e^{T\psi_x} - 1\,)(\,1 + \|\,q^0\,\| + \|\,\boldsymbol{\delta}^0\,\|\,)}{1 - \|\,(\,\mathbf{M}_\nu + \mathbf{N}_\nu\,)^{-1}\mathbf{N}\,\|\,(\,e^{T\psi_x} - 1\,)} \tag{31}$$

then $\|\Upsilon(\nu)\| < r_0$. □

5.1. Final convergence. The remaining issue to be dealt with is the convergence of the discrete-time trajectory to a weak solution of the continuous-time frictional compliant contact problem. To deal with this issue, we use the discrete-time iterates $\{\mathbf{x}^{h,0}, \mathbf{x}^{h,1}, \ldots, \mathbf{x}^{h,N_h+1}\}$ to construct a continuous-time state trajectory by linear interpolation. Specifically, define the affine function $\widehat{\mathbf{x}}^h : [0,T] \to \Re^n$ as follows:

$$\widehat{\mathbf{x}}^h(t) \;\equiv\; \mathbf{x}^{h,j} + \frac{t - t_{h,j}}{h}\,(\,\mathbf{x}^{h,j+1} - \mathbf{x}^{h,j}\,) \quad \forall\, t \in \,[\,t_{h,j}, t_{h,j+1}\,].$$

Let $\widehat{\boldsymbol{\lambda}}^h(t)$ be the (possibly discontinuous) piecewise constant interpolants of the families $\{\boldsymbol{\lambda}^{h,j+1}\}$, i.e., $\widehat{\boldsymbol{\lambda}}^h(t) \equiv \boldsymbol{\lambda}^{h,j+1}$ for $t \in (t_{h,j},\, t_{h,j+1}]$.

The following theorem is the main convergence result of this paper. Part (c) of the theorem assumes that the constitutive law of compliance for the normal forces is decoupled from that for the tangential forces. In this case, the submatrices $K_{\rm tn}(q)$, $K_{\rm on}(q)$, $\widehat{C}_{\rm tn}(q)$ and $\widehat{C}_{\rm on}(q)$ are zero, and the tangential friction QP becomes

$$\begin{array}{ll} \text{minimize} & \begin{pmatrix} \lambda_{\rm t} \\ \lambda_{\rm o} \end{pmatrix} \left\{ \dfrac{1}{2} \begin{bmatrix} \widehat{C}_{\rm tt}(q) & \widehat{C}_{\rm to}(q) \\ \widehat{C}_{\rm ot}(q) & \widehat{C}_{\rm oo}(q) \end{bmatrix} \begin{pmatrix} \lambda_{\rm t} \\ \lambda_{\rm o} \end{pmatrix} + \begin{bmatrix} \Gamma_{\rm t}(q) \\ \Gamma_{\rm o}(q) \end{bmatrix} \nu \right. \\ & \left. \quad - \begin{bmatrix} \widehat{C}_{\rm tt}(q) & \widehat{C}_{\rm to}(q) \\ \widehat{C}_{\rm ot}(q) & \widehat{C}_{\rm oo}(q) \end{bmatrix} \begin{bmatrix} K_{\rm tt}(q) & K_{\rm to}(q) \\ K_{\rm ot}(q) & K_{\rm oo}(q) \end{bmatrix} \begin{pmatrix} \delta_{\rm t} \\ \delta_{\rm o} \end{pmatrix} \right\} \\ \text{subject to} & (\,\lambda_{\rm t}, \lambda_{\rm o}\,) \,\in\, \displaystyle\prod_{i=1}^{n_\delta} \mathcal{F}(\mu_i\,\lambda_{i\rm n})\,. \end{array} \tag{32}$$

Theorem 9. *Under the setting of Theorem* 8, *the following statements hold:*

(a) *There is a sequence* $\{h_\ell\} \downarrow 0$ *such that* $\widehat{\mathbf{x}}^{h_\ell}$ *converges uniformly on* $[0,T]$ *to a Lipschitz function* $\widehat{\mathbf{x}}$, *and* $\widehat{\boldsymbol{\lambda}}^{h_\nu}$ *converge weakly to a function* $\widehat{\boldsymbol{\lambda}}$ *in* $L^2(0,T)$*; i.e.,*

$$\lim_{\ell\to\infty}\, \sup_{t\in[0,T]} \|\,\widehat{\mathbf{x}}(t) - \widehat{\mathbf{x}}^{h_\ell}(t)\,\| \;=\; 0$$

and, for any function $\varphi \in L^2(0,T)$,

$$\lim_{\ell\to\infty} \int_0^T \varphi(t)^T \lambda^{h_\ell}_{\rm n,t,o}(t)\,dt \;=\; \int_0^T \varphi(t)^T \widehat{\lambda}_{\rm n,t,o}(t)\,dt.$$

(b) *All such limits* $(\widehat{\mathbf{x}},\, \widehat{\boldsymbol{\lambda}})$ *satisfy properties* (a), (b), *and* (d) *in Definition* 2 *of a weak solution of the frictional compliant contact problem.*

(c) *If* $K_{\rm tn}(q)$, $K_{\rm on}(q)$, $C_{\rm tn}(q)$, *and* $C_{\rm on}(q)$ *are equal to zero for all* q, *then* $(\widehat{\mathbf{x}},\, \widehat{\boldsymbol{\lambda}})$ *also satisfies property* (c) *in Definition* 2 *and hence is a weak solution of the frictional compliant contact problem.*

Proof. Combining (24) and (25), we deduce

$$\| \widehat{\mathbf{x}}^h(t) - \mathbf{x}^{h,j} \| \leq \| \mathbf{x}^{h,j+1} - \mathbf{x}^{h,j} \| \leq h \, \psi_x \, e^{T\psi_x} \, (1 + r_0), \tag{33}$$

where r_0 satisfies (31). By the limit condition (3), we obtain

$$\lim_{h \downarrow 0} \max_{0 \leq j \leq N_h} \sup_{t \in [t_{h,j}, t_{h,j+1}]} \| \Psi_{\rm n}(\widehat{q}^h(t)) - \Psi_{\rm n}(q^{h,j}) - h \, \Gamma_{\rm n}(q^{h,j}) \nu^{h,\theta_j} \| = 0. \tag{34}$$

Moreover, the former inequalities show that the piecewise interpolants $\widehat{\mathbf{x}}^h$ are not only Lipschitz continuous on $[0,T]$, but the Lipschitz constant is independent of h. Hence there is a positive scalar h_0', which depends only on the model functions such that the family of functions $\{\widehat{\mathbf{x}}^h\}$ for h in $(0, h_0']$ is an equicontinuous family of functions. As in the proof of [11, Theorem 7.1], it follows from the Arzelá–Ascoli theorem (see, e.g., [22, p. 167] or [9, pp. 57–59]) that there is a sequence $\{h_\ell\} \downarrow 0$ such that $\{\widehat{\mathbf{x}}^{h_\ell}\}$ converges in the supremum (i.e., L^∞) norm to a Lipschitz function $\widehat{\mathbf{x}}$ on $[0,T]$. Since

$$\sup_{h \in (0,h_0']} \sup_{t \in [0,T]} \|\widehat{\mathbf{x}}^h(t)\| < \infty,$$

by (26), we deduce that

$$\sup_{h \in (0,h_0']} \sup_{t \in [0,T]} \|\widehat{\boldsymbol{\lambda}}^h(t)\| < \infty. \tag{35}$$

Moreover, by the same proof, it follows that, by working with an appropriate subsequence of $\{h_\ell\}$ if necessary and by invoking Alaoglu's theorem [9, pp. 71–72] and Mazur's theorem [9, p. 88], the sequence $\{\widehat{\boldsymbol{\lambda}}^{h_\ell}\}$ is weakly convergent with a weak* limit $\widehat{\boldsymbol{\lambda}}$, which satisfies $\widehat{\lambda}_{\rm n}(t) \geq 0$ and $(\widehat{\lambda}_{it}(t), \widehat{\lambda}_{io}(t)) \in \mathcal{F}(\mu_i \widehat{\lambda}_{in}(t))$ for almost all t. The proof of the latter frictional inclusion is based on the observation that a pair $(a,b) \in \mathcal{F}(\tau)$ if and only if the triple (a,b,τ) belongs to the closed convex graph of the friction map $\mathcal{F}$.

We need to verify the four properties (a)–(d) of a weak solution to the contact problem. The boundary equation (d) requires no verification, as it is a simple matter of passing to the limit in the discrete-time boundary equation (27). Hence we focus on the verification of (a)–(c). We first deal with the dynamics equations. We have

$$\begin{aligned} \nu^{h,j+1} - \nu^{h,j} &= h \, M(q^{h,j})^{-1} [f(t_{h,j+1}, q^{h,\theta_j}, \nu^{h,\theta_j}) + \boldsymbol{\Gamma}(q^{h,j})^T \boldsymbol{\lambda}^{h,j+1}] \\ &= \int_{t_{h,j}}^{t_{h,j+1}} M(\widehat{q}^h(t))^{-1} [\, f(t, \widehat{q}^h(t), \widehat{\nu}^h(t)) + \boldsymbol{\Gamma}(\widehat{q}^h(t))^T \widehat{\boldsymbol{\lambda}}^h(t) \,] \, dt + O(h^2). \end{aligned}$$

Hence for $0 \leq \tau \leq \tau' \leq T$, we obtain

$$\widehat{\nu}^h(\tau') - \widehat{\nu}^h(\tau) = \int_\tau^{\tau'} M(\widehat{q}^h(t))^{-1} [\, f(t, \widehat{q}^h(t), \widehat{\nu}^h(t)) + \boldsymbol{\Gamma}(\widehat{q}^h(t))^T \widehat{\boldsymbol{\lambda}}^h(t) \,] \, dt + O(h).$$

Restricted to the subsequence $\{h_\ell\}$, we have

$$\lim_{\ell \to \infty} \int_\tau^{\tau'} M(\widehat{q}^{h_\ell}(t))^{-1} \, f(t, \widehat{q}^{h_\ell}(t), \widehat{\nu}^{h_\ell}(t)) \, dt = \int_\tau^{\tau'} M(\widehat{q}(t))^{-1} \, f(t, \widehat{q}(t), \widehat{\nu}(t)) \, dt$$

by the uniform convergence of $(\widehat{q}^{h_\ell}, \widehat{\nu}^{h_\ell}) \to (\widehat{q}, \widehat{\nu})$. We also have

$$\begin{aligned}
&\left\| \int_\tau^{\tau'} \left[M(\widehat{q}^{h_\ell}(t))^{-1} \mathbf{\Gamma}(\widehat{q}^{h_\ell}(t))^T \boldsymbol{\lambda}^{h_\ell}(t) - M(\widehat{q}(t))^{-1} \mathbf{\Gamma}(\widehat{q}(t))^T \widehat{\boldsymbol{\lambda}}(t) \right] \right\| \\
&\quad \leq \int_\tau^{\tau'} \| M(\widehat{q}^{h_\ell}(t))^{-1} \mathbf{\Gamma}(\widehat{q}^{h_\ell}(t))^T - M(\widehat{q}(t))^{-1} \mathbf{\Gamma}(\widehat{q}(t))^T \| \, \| \widehat{\boldsymbol{\lambda}}^{h_\ell}(t) \| \, dt \\
&\qquad + \left\| \int_\tau^{\tau'} M(\widehat{q}(t))^{-1} \mathbf{\Gamma}(\widehat{q}(t))^T (\widehat{\boldsymbol{\lambda}}^{h_\ell}(t) - \widehat{\boldsymbol{\lambda}}(t)) \, dt \right\|
\end{aligned}$$

The first summand on the right-hand side converges to zero because $\{\widehat{q}^{h_\ell}\} \to \widehat{q}$ uniformly and $\widehat{\boldsymbol{\lambda}}^{h_\ell}$ is bounded; the second summand converges to zero because $\{\widehat{\boldsymbol{\lambda}}^{h_\nu}\}$ converges weakly in $L^2(0,T)$ to $\widehat{\boldsymbol{\lambda}}$. Consequently, we deduce

$$\widehat{\nu}(\tau') - \widehat{\nu}(\tau) = \lim_{\ell \to \infty} [\widehat{\nu}^{h_\ell}(\tau') - \widehat{\nu}^{h_\ell}(\tau)] = \int_\tau^{\tau'} M(\widehat{q}(t))^{-1} [f(t, \widehat{q}(t), \widehat{\nu}(t)) + \mathbf{\Gamma}(\widehat{q}(t))^T \widehat{\boldsymbol{\lambda}}(t)] dt.$$

Similarly, we can establish

$$\begin{aligned}
\widehat{q}(\tau') - \widehat{q}(\tau) &= \int_\tau^{\tau'} G(\widehat{q}(t)) \widehat{\nu}(t) \, dt \quad \text{and} \\
\widehat{\boldsymbol{\delta}}(\tau') - \widehat{\boldsymbol{\delta}}(\tau) &= \int_\tau^{\tau'} \mathbf{C}(\widehat{q}(t))^{-1} [\widehat{\boldsymbol{\lambda}}(t) - \mathbf{K}(\widehat{q}(t)) \widehat{\boldsymbol{\delta}}(t)] \, dt,
\end{aligned}$$

completing the proof of property (a) of a weak solution. We next address property (b). For t in $[t_{h,j}, t_{h,j+1}]$, we can write

$$\begin{aligned}
\Psi_{\rm n}(\widehat{q}^h(t)) + \widehat{\delta}^h_{\rm n}(t) = {} & \Psi_{\rm n}(\widehat{q}^h(t)) - \Psi_{\rm n}(q^{h,j}) + \widehat{\delta}^h_{\rm n}(t) - \delta^{h,j+1}_{\rm n} \\
& + \Psi_{\rm n}(q^{h,j}) + h \, \Gamma_{\rm n}(q^{h,j}) \nu^{h,\theta_j} + \delta^{h,j+1}_{\rm n} - h \Gamma_{\rm n}(q^{h,j}) \nu^{h,\theta_j};
\end{aligned}$$

since $\Psi_{\rm n}(q^{h,j}) + h \, \Gamma_{\rm n}(q^{h,j}) \nu^{h,\theta_j} + \delta^{h,j+1}_{\rm n} \geq 0$, we deduce

$$\Psi_{\rm n}(\widehat{q}^h(t)) + \widehat{\delta}^h_{\rm n}(t) \, \geq \, \Psi_{\rm n}(\widehat{q}^h(t)) - \Psi_{\rm n}(q^{h,j}) + \widehat{\delta}^h_{\rm n}(t) - \delta^{h,j+1}_{\rm n} - h \, \Gamma_{\rm n}(q^{h,j}) \nu^{h,\theta_j}.$$

Letting $\Phi^h_{\rm n}(t)$ be the right-hand expression, we deduce from (34) and (33) that $\|\Phi^h_{\rm n}(t)\|$ is bounded by a constant for all $h > 0$ sufficiently small and all t and $\Phi^h_{\rm n}(t) \to 0$ for all t as $h \downarrow 0$. Restricted to the subsequence $\{h_\ell\}$, the left-hand side converges uniformly to $\Psi_{\rm n}(\widehat{q}(t)) + \widehat{\delta}_{\rm n}(t)$; therefore, $\Psi_{\rm n}(\widehat{q}(t)) + \widehat{\delta}_{\rm n}(t) \geq 0$ for all $t \in [0,T]$. Next we show that

$$\int_0^T \widehat{\lambda}_{\rm n}(t)^T [\Psi_{\rm n}(\widehat{q}(t)) + \widehat{\delta}_{\rm n}(t)] \, dt \, = \, 0. \tag{36}$$

The left-hand side is equal to the limit

$$\lim_{\ell \to \infty} \int_0^T \widehat{\lambda}^{h_\ell}_{\rm n}(t)^T [\Psi_{\rm n}(\widehat{q}(t)) + \widehat{\delta}_{\rm n}(t)] \, dt \, = \, \lim_{\ell \to \infty} \int_0^T \widehat{\lambda}^{h_\ell}_{\rm n}(t)^T [\Psi_{\rm n}(\widehat{q}^{h_\ell}(t)) + \widehat{\delta}^{h_\ell}_{\rm n}(t)] \, dt.$$

For each $h > 0$, we have

$$\begin{aligned} \int_0^T \widehat{\lambda}_{\rm n}^h(t)^T [\, \Psi_{\rm n}(\widehat{q}^h(t)) + \widehat{\delta}_{\rm n}^h(t)\,]\, dt &= \sum_{j=0}^{N_h} \int_{t_{h,j}}^{t_{h,j+1}} \widehat{\lambda}_{\rm n}^h(t)^T [\, \Psi_{\rm n}(\widehat{q}^h(t)) + \widehat{\delta}_{\rm n}^h(t)\,]\, dt \\ &= \sum_{j=0}^{N_h} \int_{t_{h,j}}^{t_{h,j+1}} (\, \lambda^{h,j+1}\,)^T \Phi_{\rm n}^h(t)\, dt. \end{aligned} \tag{37}$$

Since $\{\lambda^{h,j+1}\}$ is bounded, by letting $h \downarrow 0$ in (37) along the subsequence $\{h_\ell\}$, (36) follows readily from the dominated convergence theorem, thereby completing the proof of property (b) of weak solution.

Finally, to prove property (c), let $(\widetilde{\lambda}_{\rm t}, \widetilde{\lambda}_{\rm o}) : [0,T] \to \Re^{2n_\delta}$ be continuous functions such that $(\widetilde{\lambda}_{it}(t), \widetilde{\lambda}_{io}(t)) \in \mathcal{F}(\mu_i \widehat{\lambda}_{in}(t))$ for almost all $t \in [0,T]$ and all $i = 1, \dots, n_\delta$. We need to verify

$$\int_0^T \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \widehat{\lambda}_{\rm t}(t) \\ \widetilde{\lambda}_{\rm o}(t) - \widehat{\lambda}_{\rm o}(t) \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \left[\begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix} \right.\right.$$
$$\left.\left. - \begin{bmatrix} K_{\rm tt}(\widehat{q}(t)) & K_{\rm to}(\widehat{q}(t)) \\ K_{\rm ot}(\widehat{q}(t)) & K_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\delta}_{\rm t}(t) \\ \widehat{\delta}_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(\widehat{q}(t)) \\ \Gamma_{\rm o}(\widehat{q}(t)) \end{pmatrix} \widehat{\nu}(t) \right\} dt \,\geq\, 0.$$

Since

$$\begin{aligned} &\int_0^T \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) \end{pmatrix}^T \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) \end{pmatrix} dt \\ &\quad = \int_0^T \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) - \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) - \widehat{\lambda}_{\rm o}(t) \end{pmatrix}^T \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) - \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) - \widehat{\lambda}_{\rm o}(t) \end{pmatrix} dt \\ &\qquad - 2\int_0^T \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) - \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) - \widehat{\lambda}_{\rm o}(t) \end{pmatrix}^T \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix} dt \\ &\qquad + \int_0^T \begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix}^T \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix} dt, \end{aligned}$$

and since the first integral on the right-hand side is nonnegative (by the positive semidefiniteness of the quadratic form), the second integral converges to zero because $\widehat{\lambda}_{\rm t,o}^{h_\ell}$ converge to $\widehat{\lambda}_{\rm t,o}$ in $L^2(0,T)$, we deduce

$$\begin{aligned} \infty \;&>\; \liminf_{\ell\to\infty} \int_0^T \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) \end{pmatrix}^T \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\lambda}_{\rm t}^{h_\ell}(t) \\ \widehat{\lambda}_{\rm o}^{h_\ell}(t) \end{pmatrix} dt \\ &\geq\; \int_0^T \begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix}^T \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix} dt, \end{aligned} \tag{38}$$

where the left-hand limit is finite by (35). Consequently, it follows that

$$\begin{aligned}
&\int_0^T \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \widehat{\lambda}_{\rm t}(t) \\ \widetilde{\lambda}_{\rm o}(t) - \widehat{\lambda}_{\rm o}(t) \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \left[\begin{pmatrix} \widehat{\lambda}_{\rm t}(t) \\ \widehat{\lambda}_{\rm o}(t) \end{pmatrix} \right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} K_{\rm tt}(\widehat{q}(t)) & K_{\rm to}(\widehat{q}(t)) \\ K_{\rm ot}(\widehat{q}(t)) & K_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\delta}_{\rm t}(t) \\ \widehat{\delta}_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(\widehat{q}(t)) \\ \Gamma_{\rm o}(\widehat{q}(t)) \end{pmatrix} \widehat{\nu}(t) \right\} dt \\
&\geq \limsup_{\ell\to\infty} \int_0^T \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \widehat{\lambda}^{h_\ell}_{\rm t}(t) \\ \widetilde{\lambda}_{\rm o}(t) - \widehat{\lambda}^{h_\ell}_{\rm o}(t) \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}(t)) & \widehat{C}_{\rm to}(\widehat{q}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}(t)) & \widehat{C}_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \left[\begin{pmatrix} \widehat{\lambda}^{h_\ell}_{\rm t}(t) \\ \widehat{\lambda}^{h_\ell}_{\rm o}(t) \end{pmatrix} \right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} K_{\rm tt}(\widehat{q}(t)) & K_{\rm to}(\widehat{q}(t)) \\ K_{\rm ot}(\widehat{q}(t)) & K_{\rm oo}(\widehat{q}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\delta}_{\rm t}(t) \\ \widehat{\delta}_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(\widehat{q}(t)) \\ \Gamma_{\rm o}(\widehat{q}(t)) \end{pmatrix} \widehat{\nu}(t) \right\} dt \quad \text{by (38)} \\
&\geq \limsup_{\ell\to\infty} \int_0^T \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \widehat{\lambda}^{h_\ell}_{\rm t}(t) \\ \widetilde{\lambda}_{\rm o}(t) - \widehat{\lambda}^{h_\ell}_{\rm o}(t) \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}^{h_\ell}(t)) & \widehat{C}_{\rm to}(\widehat{q}^{h_\ell}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}^{h_\ell}(t)) & \widehat{C}_{\rm oo}(\widehat{q}^{h_\ell}(t)) \end{bmatrix} \left[\begin{pmatrix} \widehat{\lambda}^{h_\ell}_{\rm t}(t) \\ \widehat{\lambda}^{h_\ell}_{\rm o}(t) \end{pmatrix} \right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} K_{\rm tt}(\widehat{q}^{h_\ell}(t)) & K_{\rm to}(\widehat{q}^{h_\ell}(t)) \\ K_{\rm ot}(\widehat{q}^{h_\ell}(t)) & K_{\rm oo}(\widehat{q}^{h_\ell}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\delta}^{h_\ell}_{\rm t}(t) \\ \widehat{\delta}^{h_\ell}_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(\widehat{q}^{h_\ell}(t)) \\ \Gamma_{\rm o}(\widehat{q}^{h_\ell}(t)) \end{pmatrix} \widehat{\nu}^{h_\ell}(t) \right\} dt,
\end{aligned}$$

where the second inequality holds because $\{(\widehat{q}^{h_\ell}, \widehat{\nu}^{h_\ell}, \widehat{\delta}^{h_\ell}_{\rm t,o})\}$ converges to $(\widehat{q}, \widehat{\nu}, \widehat{\delta}_{\rm t,o})$ uniformly. For each $h > 0$, we have

$$\begin{aligned}
&\int_0^T \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \widehat{\lambda}^{h}_{\rm t}(t) \\ \widetilde{\lambda}_{\rm o}(t) - \widehat{\lambda}^{h}_{\rm o}(t) \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}^{h}(t)) & \widehat{C}_{\rm to}(\widehat{q}^{h}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}^{h}(t)) & \widehat{C}_{\rm oo}(\widehat{q}^{h}(t)) \end{bmatrix} \left[\begin{pmatrix} \widehat{\lambda}^{h}_{\rm t}(t) \\ \widehat{\lambda}^{h}_{\rm o}(t) \end{pmatrix} \right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} K_{\rm tt}(\widehat{q}^{h}(t)) & K_{\rm to}(\widehat{q}^{h}(t)) \\ K_{\rm ot}(\widehat{q}^{h}(t)) & K_{\rm oo}(\widehat{q}^{h}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\delta}^{h}_{\rm t}(t) \\ \widehat{\delta}^{h}_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(\widehat{q}^{h}(t)) \\ \Gamma_{\rm o}(\widehat{q}^{h}(t)) \end{pmatrix} \widehat{\nu}^{h}(t) \right\} dt \\
&= \sum_{j=1}^{N_h} \int_{t_{h,j}}^{t_{h,j+1}} \begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \lambda^{h,j+1}_{\rm t} \\ \widetilde{\lambda}_{\rm o}(t) - \lambda^{h,j+1}_{\rm o} \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(\widehat{q}^{h}(t)) & \widehat{C}_{\rm to}(\widehat{q}^{h}(t)) \\ \widehat{C}_{\rm ot}(\widehat{q}^{h}(t)) & \widehat{C}_{\rm oo}(\widehat{q}^{h}(t)) \end{bmatrix} \left[\begin{pmatrix} \lambda^{h,j+1}_{\rm t} \\ \lambda^{h,j+1}_{\rm o} \end{pmatrix} \right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} K_{\rm tt}(\widehat{q}^{h}(t)) & K_{\rm to}(\widehat{q}^{h}(t)) \\ K_{\rm ot}(\widehat{q}^{h}(t)) & K_{\rm oo}(\widehat{q}^{h}(t)) \end{bmatrix} \begin{pmatrix} \widehat{\delta}^{h}_{\rm t}(t) \\ \widehat{\delta}^{h}_{\rm o}(t) \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(\widehat{q}^{h}(t)) \\ \Gamma_{\rm o}(\widehat{q}^{h}(t)) \end{pmatrix} \widehat{\nu}^{h}(t) \right\} dt.
\end{aligned}$$

Since for almost all $t \in (t_{h,j}, t_{h,j+1}]$, we have $(\widetilde{\lambda}_{it}(t), \widetilde{\lambda}_{io}(t)) \in \mathcal{F}(\mu_i \lambda^{h,j+1}_{\rm in})$, it follows that

$$\begin{pmatrix} \widetilde{\lambda}_{\rm t}(t) - \lambda_{\rm t}^{h,j+1} \\ \widetilde{\lambda}_{\rm o}(t) - \lambda_{\rm o}^{h,j+1} \end{pmatrix}^T \left\{ \begin{bmatrix} \widehat{C}_{\rm tt}(q^{h,j}) & \widehat{C}_{\rm to}(q^{h,j}) \\ \widehat{C}_{\rm ot}(q^{h,j}) & \widehat{C}_{\rm oo}(q^{h,j}) \end{bmatrix} \left[\begin{pmatrix} \lambda_{\rm t}^{h,j+1} \\ \lambda_{\rm o}^{h,j+1} \end{pmatrix} \right.\right.$$
$$\left.\left. - \begin{bmatrix} K_{\rm tt}(q^{h,j}) & K_{\rm to}(q^{h,j}) \\ K_{\rm ot}(q^{h,j}) & K_{\rm oo}(q^{h,j}) \end{bmatrix} \begin{pmatrix} \delta_{\rm t}^{h,j} \\ \delta_{\rm o}^{h,j} \end{pmatrix} \right] + \begin{pmatrix} \Gamma_{\rm t}(q^{h,j}) \\ \Gamma_{\rm o}(q^{h,j}) \end{pmatrix} \nu^{h,\theta_j} \right\} \geq 0$$

for almost all $t \in (t_{h,j}, t_{h,j+1}]$. Since the state variables satisfy

$$\lim_{\substack{\ell \to \infty \\ t \in (t_{h_\ell,j}, t_{h_\ell,j+1}]}} (\widehat{\mathbf{x}}^{h_\ell}(t) - \mathbf{x}^{h_\ell,j}) = 0$$

uniformly on $[0, T]$, we easily derive the desired limiting friction property (c). □

6. Conclusion and discussion. This paper provided an in-depth investigation of time-stepping methods for rigid body dynamics problems with multiple contacts characterized by friction and local compliance. The main results are (a) the existence of a discrete-time solution trajectory the boundary-value problem (Theorem 8), and (b) the convergence of such a solution to a weak solution of the corresponding continuous-time problem (Theorem 9). Whereas the convergence results obtained are in a sense stronger than those in [19, 1], it is worth noting that this is because of our choice of a phenomenologically correct model that explicitly characterizes the compliance at each contact. Even so, there are limitations in our investigation. First, the friction coefficients are required to be sufficiently small in the main results (this is the result of our discretization which respects the nonlinear friction conditions at all iterates $\boldsymbol{\lambda}^{h,\nu+1}$). Second, we are not able to establish convergence to a strong solution. This limitation begs the question of whether such a solution can be proved to exist in a continuous-time model under an appropriate compliance constitutive law. The key difficulty lies in the fact that the friction forces are not continuous functions of the system states with the model (5). This issue remains unresolved to date. Third, in our convergence analysis, the parameters of the compliance model (i.e., the stiffness and damping) are fixed. It would be very interesting to extend the analysis to allow these parameters to tend to infinity, with the goal of recovering a solution of some kind to a fully rigid-body model. Such an extended analysis is beyond the scope of this paper. In the previous paper [17], we considered, in a discrete-time framework with a fixed discretization step, the issue of convergence when the stiffness and damping both tend to infinity and obtained some positive results; nevertheless, such a convergence issue in a continuous-time model seems difficult and has not been studied.

In view of the unresolved issues associated with strong solutions, which are seemingly very difficult, our results are significant and provide a first step for a deeper analysis. Needless to say, we are interested in extending the analysis to models with nonlinear constitutive laws for which the existence of strong solutions to the continuous-time model could be shown and for which the convergence of a numerical time-stepping method to such a solution could be established. Our future work will address such extensions and the application of numerical methods for solving boundary value problems to the optimal design of manufacturing processes with frictional contacts.

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